Seeing How It Goes: Paper-and-Pencil Reasoning in Mathematical Practice

Article *in* Philosophia Mathematica · February 2011

DOI: 10.1093/philmat/nkr006

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Paper-and-Pencil Reasoning in Mathematical Practice*

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What is the role of writing in mathematics? If one thinks of a sufficiently tedious problem in arithmetic—say, that of dividing forty-three thousand eight hundred and seventy-three by nine hundred seventeen million six hundred eighty-nine thousand three hundred and eleven—the writing seems essential insofar as, although practically anyone can solve this problem, most (all?) of us can solve it *only* in the positional system of Arabic numeration. One simply cannot calculate in English, or any other natural language, as one can in Arabic numeration; and again, for most of us, there is just no other way to solve arithmetical problems of any degree of difficulty.[1] In other cases the notation, even any writing at all, seems utterly irrelevant. Although I cannot *say* a calculation in Arabic numeration but only show it, or describe what it would look like if one performed it, one *does say*, for instance, the proof, known already to the ancient Greeks, that there is no largest prime. Here it is.

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* An earlier version of this paper was presented at the conference, "From Practice to Results in Logic and Mathematics", organized by the PratiScienS group and held at the University of Nancy 2, Nancy, France, June 21-23, 2010. I am grateful for comments received at that conference, and also for those of two anonymous reviewers.

[1] As Whitehead once remarked, "probably nothing in the modern world would have more astonished a Greek mathematician than to learn that, under the influence of compulsory education, the whole population of Western Europe, from the highest to the lowest, could perform the operation of division for the largest numbers. This fact would have seemed to him a sheer impossibility". Alfred North Whitehead, *Introduction to Mathematics* (Barnes and Noble Books, 2005), pp. 32-3; first published in 1911.

Suppose that there are only finitely many primes, and that we have an ordered list of all of them. Now consider the number that is the product of all these primes plus one. Either this new number is prime or it is not. If it is prime then we have a prime number that is larger than all those originally listed; and if it is not prime then, because none of the numbers on our list divide this new number without remainder (because it is the product of all those primes plus one), this new number must have a prime divisor larger than any of the primes on our list. Either way there is a prime number larger than any with which we began. Q.E.D.

This proof clearly does not rely on any system of written signs. It depends not on the capacity to write but on the capacity to reflect on ideas, and to think, that is, to reason or infer.[2]

A calculation in Arabic numeration is essentially written—though of course the writing can be merely imaginatively performed rather than actually performed. The proof that there is no largest prime is not. Although the words clearly do convey the line of reasoning, the proof is not *in* the words (whether spoken or written); it is not the words that one attends to in the proof that there is no largest prime, but instead the relevant ideas, central among them the idea of a number that is the product of a collection of primes plus one. The task of the proof is to think through what follows in the case of such a number. It can furthermore seem that this is a *paradigm* case of reasoning in mathematics, that the various systems of written marks that have been devised for mathematics are merely useful devices that simplify the work of mathematics but are in no way essential to it. In one way this is obviously right: the

[2] As Kant might think of it, whereas a calculation in Arabic numeration involves an intuitive use of reason, a construction (in pure intuition), the reasoning involved in the ancient proof that there is no largest prime instead makes a discursive use of reason directly from concepts. See the first *Critique*, first section of the first chapter of the Transcendental Doctrine of Method (especially A712/B740-A723/B751).

systems of written signs that have been devised for mathematics were devised for mathematics that already existed. It would be impossible to design a notation for mathematics without already knowing at least some of the mathematics the notation was designed to capture. But in another way it is not right, not if it is taken to mean that the systems of signs that have been devised for various sorts of mathematics are merely a convenience, a kind of shorthand writing. The system of Arabic numeration is not shorthand for something one could also do in the longhand of written natural language. And yet it is often assumed both by mathematicians and by philosophers that mathematical languages, that is, the systems of written signs we devise in mathematics, are exactly that: convenient shorthand.[3]

Jourdain takes a different view:

> it is important to realize that the long and strenuous work of the most gifted minds was necessary to provide us with simple and expressive notation which, in nearly all parts of mathematics, enables even the less gifted of us to reproduce theorems which needed the greatest genius to discover. Each improvement in notation seems, to the uninitiated, but a small thing; and yet, in a calculation, the pen sometimes seems to be more intelligent than the user.[4]

Jourdain claims that a good—that is, simple and expressive—mathematical

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[3] We read, for example, in a recent college-level mathematics textbook—Carol Schumacher's *Chapter Zero: Fundamental Notions of Abstract Mathematics* (Addison-Wesley, 2001), p. 5—that "the symbols are simply a convenience: It is easier to write 'x^2' than 'the square of x,' and '$x \in A$' is more compact than 'x is an element of the set A' . . . mathematics is written using a variety of English". Brian Rotman, in *Mathematics as Sign: Writing, Imagining, Counting* (Stanford: Stanford University Press, 2000), p. 48, calls this the documentist view of mathematical writing. It is, he suggests, very prevalent among philosophers, and also mistaken: "mathematical signs do not record or code or transcribe any language prior to themselves. They certainly do not arise as abbreviations or symbolic transcriptions of words in some natural language" (p. 44).

[4] Philip E. B. Jourdain, *The Nature of Mathematics* (London: T. C. and E. C. Jack, and New York: Dodge Publishing Co., 1912), p. 16.

notation, although not necessary to the practice of mathematics, enables even the less gifted of us to reproduce theorems. But if so, then collections of signs in mathematics do not merely *record* results; they actually *embody* the relevant bits of mathematical reasoning. They put the reasoning itself before our eyes in a way that is simply impossible in written natural language. And this seems right, at least for some cases: a calculation in Arabic numeration, for instance, embodies the reasoning. It *shows how it goes* in a way that the words we use to convey the proof that there is no largest prime do not.

Arabic numeration provides a paradigm of a system of written signs within which to work in mathematics. But a calculation in Arabic numeration, because it is algorithmic, is not very interesting as mathematics. Significant mathematics is not algorithmic and often intellectually very challenging. And yet, Jourdain claims, a good mathematical notation can make it accessible even to the less gifted of us. My aim is to clarify how this works, what is required of a mathematical notation within which to reason to significant results in mathematical practice.

I. Some Preliminary Reflections on Notation

The words of natural language are obviously different from the things those words are used to refer to and talk about (excepting, of course, words such as 'word'). This difference in turn suggests a distinction between written natural language, which is first and foremost a record of speech, of the myriad utterances that speakers of natural language are capable of making, and systems of written notation that more directly express, record, or picture the things about which we speak. The written word 'four' traces the sounds speakers make in uttering the

English word for a particular number; a collection of four strokes, ////, directly pictures a collection of four things. The written word ‘circle’ traces sounds; a drawn circle provides an instance of the shape the word names. But although this distinction between the two sorts of writing—on the one hand, written natural language, and on the other, systems of written marks that directly trace or map the things we talk about—is quite obvious and natural, it is, I think, unable to provide us with any real insight into the role of writing in mathematical practice.

The first problem with the distinction is that even natural language, whether spoken or written, can be conceived as a means of picturing states of affairs. This is, for instance, Wittgenstein’s view in the *Tractatus*. To say or write that (say) Andy is taller than Bill is, on this view, to picture a certain state of affairs that obtains between two objects, the same state of affairs as can be expressed or pictured in a convenient shorthand of standard logical notation: aTb, with ‘a’ standing in for Andy, ‘b’ for Bill, and the manner of writing those names, namely, to the left and right of the letter ‘T’, respectively, showing how things stand with the objects so designated.[5] If we conceive natural language in this way then it will be quite natural to think that the signs we introduce in mathematics are similarly convenient shorthand for words of natural language.

And there is another problem as well. Suppose that we accept the distinction between the words of natural language, and their written counterparts, on the one hand, and systems of notation that more directly express, record, or picture that

[5] Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, trans. D. F. Pears and B. F. McGuinness (London and Henley: Routledge and Kegan Paul, 1961); see especially §§ 3.1432 and 4.0311.

about which we speak, on the other. What then are we to say about the signs of Arabic numeration or about equations in the symbolic language of arithmetic and algebra more generally? The Arabic numeral '4' neither records the sounds speakers of a natural language make nor pictures any collection of things. The familiar equation for a circle traced out in Cartesian coordinates about the origin, '$x^2 + y^2 = r^2$', although it can be read aloud in a sense, is not an abbreviation of a sentence of English—as becomes evident when one has a speaker of another natural language, say, Japanese or French, read it aloud. The sounds such speakers will make are very different from the sounds a speaker of English makes in reading that equation. But nor does it seem helpful to say that the equation pictures something that the words of natural language only name. Indeed, we will see, this is true even of a drawn figure, say, a circle, in a Euclidean demonstration; a drawn circle in Euclid is not usefully thought of as giving us a picture or instance of the thing that the word 'circle' names.

But although the simple and obvious distinction between (written) natural language and systems of signs that more directly picture or map the things about which we speak is not very illuminating, other, related distinctions are, we will see, more useful. The first is suggested by differences between the Roman and the Arabic systems of numeration.

Roman numeration is a natural extension of the idea of using individual marks to stand for the objects in a collection so as to record how many of them there are. It provides a convenient shorthand: ‘V’ in place of ‘IIIII’, ‘X’ in place of ‘VV’, and so on.[6] Like the more primitive use of marks, one for each object in a collection, the system of Roman numeration enables one to record how many. The positional system of Arabic numeration is essentially different insofar as it serves not merely to record how many but to enable paper-and-pencil calculations. And it can be so used because it formulates the arithmetical content of numbers in a way that enables one to break a problem down into a series of small, directly solvable steps. Unlike the system of Roman numeration, the Arabic system provides even the less gifted of us with a simple and expressive notation within which to solve arithmetical problems of arbitrary difficulty.

Perhaps it will be objected that one can calculate also in Roman numeration.[7] To divide, say, twenty-seven by three, one first writes the number: XXVII. Then one looks for signs that have three or more occurrences. Because in our example none do, we rewrite, putting ‘VV’ for each ‘X’ and ‘IIIII’ for ‘V’: VVVVIIIIIII. Now we can separate out three ‘V’ and three collections of two ‘I’, leaving VI, which we again break down to give IIIIII. Because this latter collection can be divided into three collections of two ‘I’ each, we can see that our original collection can be regarded also as three collections, each VIIII. We have our answer: twenty-seven divides into three collections of nine. But have we in this case *calculated*, reasoned *in* the system

[6] Roman numeration today includes also a subtractive convention according to which, for example, ‘IV’ stands for a collection of four things. This was not a part of the original system devised by the Romans but a much later addition.
[7] See D. Schlimm and H. Neth, “Modeling Ancient and Modern Arithmetical Practices: Addition and Multiplication with Arabic and Roman Numerals”, in B. Love, K. McRae, and V. Sloutsky, eds., *Proceedings of the 30th Annual Meeting of the Cognitive Science Society* (Austin, Texas: Cognitive Science Society, 2008), pp. 2097-2102.

of signs? It seems to me clear that we have not. What we have done is to manipulate the signs in a way that mimics the way one would manipulate collections of objects, and we can so manipulate the signs precisely because Roman numeration serves directly to picture collections of objects. (Think of 'V' as like a bag of five apples, apples that, if necessary, can be taken out and manipulated directly. 'X' is a bag of two such bags, and so on.) The system of Arabic numeration is different. It is a positional, and in particular a decimal, system that does not directly picture collections of things (as Roman numeration does) but instead *formulates* arithmetical content in a mathematically tractable way, in a way that enables calculations *in* the system of signs. On does not operate *on* the signs of Arabic numeration in a calculation as above we operated on signs of Roman numeration. Rather one reasons in the language; the language is a vehicle or medium of reasoning.

This intuitive distinction between directly picturing something and formulating content in a mathematically tractable way, in a way enabling reasoning in the system, applies also to various sorts of picture proofs, on the one hand, and a Euclidean demonstration, on the other. There are, for example, many picture proofs of the Pythagorean theorem, or of a special case of it (as in Plato's *Meno*). One very familiar one is this.

One can (with a little thought) see in this display that the Pythagorean theorem holds, and similar picture proofs can be devised for other cases. In this picture proof, one directly displays or pictures the relevant areas, and so can manipulate those depicted areas much as we manipulated the signs of Roman numeration (depicting how many) above. Similarly, an Euler or Venn diagram directly pictures the information contained in pairs of categorical propositions and provides thereby a means of testing the validity of various categorical syllogisms, that is, whether the information in the conclusion is contained already in the two premises taken together. Diagrams in knot theory likewise directly picture knots that can then be manipulated, in Reidemeister moves, essentially as one would manipulate an actual knot. In all these cases, as in the case of Roman numeration, one directly pictures something and then can manipulate the picture as one might manipulate that which it pictures.[8] A Euclidean diagram, we will see, is different insofar as it (like a numeral of Arabic numeration) does not merely picture something but instead formulates the content of something—in this case, the contents of concepts of various plane figures—in a mathematically tractable way, in a way that enables reasoning *in* the system of signs. Although one cannot *picture* something that is impossible (say, that a circle cuts a circle at more than two points), one *can* formulate the content of such a claim for the purposes of a *reductio* demonstration in Euclid. The possibility of *reductio* reasoning in Euclid's system of signs shows that

[8] The manipulation will be either a matter of rewriting (as in a diagram in knot theory), or a matter of seeing the display in a new way (as in a Venn diagram).

a Euclidean diagram does not—as a Venn or Euler diagram does—picture states of affairs.

Another distinction, more exactly, a pair of distinctions, that will be helpful in the discussion to follow, is that between, on the one hand, reasoning intra-configurationally and reasoning trans-configurationally, and on the other, signs that function (as we will say) graphically and signs that function symbolically.[9] The first of these distinctions is straightforward. Some picture proofs, Euler and Venn diagrams, and Euclidean diagrammatic demonstration are intra-configurational insofar as the reasoning "stays within the diagram". It is by looking at the drawn diagram in different ways, sometimes in an ordered series of steps, that one sees that what the diagram shows is so. Reasoning in the notation of arithmetic and algebra (and in knot theory) is instead trans-configurational insofar as the steps of reasoning in these systems require new writing.[10] The second distinction is subtler insofar as it is not so much a distinction between different systems of written signs as it is a distinction between different sorts of uses to which such signs can be put. Roughly speaking, a notation functions *symbolically* just if each primitive sign in the system has its meaning independent of any context of use; in a *graphical* system it is

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[9] The distinction between intra-configurational and trans-configurational reasoning borrows from Marco Panza's discussion in "On the Notion of Algebra in Early Modern Mathematics and its Relations with Analysis: Some Reflections on Bos' Definitions", which was presented at a conference on varieties of analysis at St. Catherine's College, Oxford, in the spring of 2005. Although my account of the distinction between graphic and symbolic notations is somewhat different from hers, the distinction itself is due to Sun-Joo Shin in *The Iconic Logic of Peirce's Graphs* (Cambridge Mass.: The MIT Press, 2002), Chapter Four.
[10] Here I use the word 'reasoning' loosely, in a way that applies to all cases under consideration. The distinction between operating *on* a collection of signs (as in our division using Roman numerals) and operating *in* signs, that is, reasoning in the language, is orthogonal to the distinction we are here concerned with.

only in a context of use and relative to some particular way of regarding a collection of signs that the primitive signs, or collections of them, have any designation or meaning.[11]

We learn, for example, Peirce's system of alpha graphs by first treating its primitive signs symbolically (in our technical sense). There are two sorts of written signs in the system, sentential signs, 'A', 'B', 'C', and so on, and circles that enclose (simple and complex) sentential signs, which are called "cuts" and function to negate what they enclose. There is also a convention according to which the juxtaposition of signs, whether simple or complex, is read as conjunction. For example, let 'A' signify the proposition that apples are red and 'B' the proposition that berries are blue. To juxtapose the two signs on a page is to express their conjunction, that apples are red and berries are blue. Because a cut negates whatever it encloses, juxtaposing an encircled 'A' and an encircled 'B' thus (using brackets to mark cuts for ease of typing):

(A) (B)

expresses the thought that it is not the case that apples are red and not the case that berries are blue. Similarly, enclosing both letters within a single cut thus:

(A   B)

expresses the thought that it is not the case both that apples are red and that berries are blue.

[11] We will see that there can be intermediate cases, for instance, that of signs that designate only in the context of a proposition, but not relative to any particular way of regarding them in that context. In this case, in effect, an analysis or particular way of regarding the collection of signs, is always already assumed.

Having learned in this way to read the system of signs symbolically (juxtaposition always meaning conjunction, a cut always meaning negation), we can now learn to read it graphically. Instead of starting with the sentential symbols and reading outward, each cut as negation and juxtapositions as conjunction, we now simply regard the whole graph as a given totality that is amenable to a variety of analyses, among them those that accord with the following two principles:

1. A sentential sign is negated if it is enclosed within an odd number of cuts; it is left unnegated otherwise.
2. The juxtaposition of two sentential signs, whether simple or complex, is a disjunction if both are enclosed within a single cut, as conjunction otherwise.

Although on our first way of reading, the graph '((A) (B))' expresses the negation of a conjunction of negations, we can also read it according to the above principles as expressing the disjunction of A and B. Read graphically rather than symbolically, this expression in Peirce's system of alpha graphs does *not* express either a negated conjunction or a disjunction independent of an analysis of it. Independent of an analysis it simply presents A and B in a logical relationship; to say *which* logical relationship, one has to regard it in some particular way, provide it with an analysis.

In standard logical notation, as it is generally read, there is always a main connective. In Pierce's system of alpha graphs, read graphically, one can identify a main connective (in all but the simplest cases) only relative to some one reading of a given graph. The same is true of Frege's two-dimensional *Begriffsschrift* notation. The formula

for example, can be read either as presenting the content P on two conditions, Q and R, or as presenting a conditional, P-on-condition-that-Q, all on condition that R. On the first reading, the main connective is between P, on the one hand, and Q and R, on the other, that is, in standard notation, $(Q \& R) \supset P$. On the second reading, it is instead between P and Q, on the one hand, and R, on the other, that is, in standard notation, $R \supset (Q \supset P)$. One can read the formula either way. Independent of any reading the notation merely displays a particular logical relation. The notation functions graphically, not symbolically.[12]

Indeed, Frege shows, even equations in arithmetic can be read graphically, that is, as displays that can be carved up into function and argument in various ways. His example is the equation ‘$2^4 = 16$’.[13] As we usually read this equation it is an arithmetical identity, that two raised to the fourth power is equal to sixteen. But having learned so to read it, one can learn to read the language differently, the primitive signs as only expressing a (Fregean) sense independent of a context of use. So understood the primitive signs are combined in an equation which then expresses a thought that is a function of the senses of the primitive signs and the manner of their combination. That thought can then be variously analyzed into

[12] See my *Frege’s Logic* (Cambridge, Mass.: Harvard University Press, 2005), §2.2, for further development of the point.

[13] Gottlob Frege, “Boole’s Logical Calculus and the Concept-script”, in *Posthumous Writings*, trans. Peter Long and Roger White, and ed. Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach (Chicago: University of Chicago Press, 1979), pp. 16-17.

function and argument; one can carve up the expression '$2^4 = 16$' in various different ways to give signs, simple or complex, that designate or mean various concepts and objects.

Suppose, for example, that we take the numeral '2' in our equation to mark the argument place, in effect, the thing we are talking about; the rest of the equation, that is, the complex sign '$\xi^4 = 16$', then at once expresses the arithmetically articulated sense of the concept *fourth root of sixteen* and designates that concept. But we can also read the formula differently. If, for example, we instead take the numeral '4' to mark the argument place then the remainder, '$2^{\xi} = 16$', exhibits the arithmetically articulated content of the concept *logarithm of sixteen to the base two* and also designates that concept. And other analyses are possible as well. In order to *learn* the language one must first understand the signs symbolically, as having their own designations or meanings independent of a context of use. But once one *has* grasped this meaning, one can learn to read sentences of the language in Frege's new way, as expressing thoughts that can be variously analyzed. Subsentential expressions of the language understood as Frege teaches us to understand them, whether they are simple or complex, designate only in a context of use and relative to an analysis. Again, as we will use the terms, a system of signs functions *symbolically* if the primitive signs of the language designate prior to and independent of a context of use. A system of signs functions *graphically* if the primitive signs only express a (Fregean) sense prior to and independent of a context of use, if they, and relevant complexes of them, designate something only in a context of use and relative to an analysis.

We have distinguished between reasoning intra-configurationally and reasoning trans-configurationally, and also between a system of signs functioning symbolically and functioning graphically. It is not hard to see that these two distinctions interact in interesting ways. First, and most obviously, if a system of signs is functioning symbolically, each primitive sign having its meaning prior to and independent of any context of use, then, assuming that the system of signs supports any reasoning at all, that reasoning must be trans-configurational. Because the meanings of the signs are fixed, the only way to move from one thought to another in such a system is by rewriting. Similarly, and for essentially the same reason, intra-configurational reasoning is possible only in a system of signs that functions graphically. If one's reasoning stays within the diagram, as it does in a Euclidean demonstration, that can only be because it is possible to regard the collection of signs now in one way now in another. There can be no intra-configurational reasoning in a system of signs that functions symbolically. What is not ruled out, at least in principle, is a system of notation in which the signs function graphically and the reasoning is trans-configurational.[14]

We turn now, first, to the paradigm example of a system of signs, a mathematical notation, that functions graphically and intra-configurationally,

[14] The relationship between our two distinctions is thus isomorphic to that between Kant's two distinctions, between the analytic/synthetic distinction and the a priori/a posteriori distinction. What is analytic must be knowable a priori (much as signs functioning symbolically can support only trans-configurational reasoning). What is known a posteriori must be synthetic (as intra-configurational reasoning requires signs that function graphically). There can then be no analytic a posteriori judgments (as there cannot be intra-configurational reasoning in a system of signs functioning symbolically), though the possibility remains that there might be synthetic apriori judgments (as there might be trans-configurational reasoning in a system of signs functioning graphically).

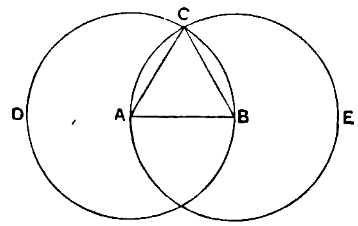
namely, the system of Euclidean diagrammatic reasoning.[15] We will then consider a system that involves trans-configurational reasoning in a system of signs that functions (largely) symbolically, namely, the familiar language of elementary algebra. Our discussion concludes with some reflections on Frege's much less familiar system, the signs of which, we will see, function graphically, though the reasoning is trans-configurational. In all three cases, we will see, the notation serves not directly to picture some things but instead to formulate the contents of things—in particular, the contents of concepts and functions—in a way that enables reasoning, on the basis of those contents, through to new and significant results regarding the concepts and functions whose contents they are.

II. Diagrammatic Reasoning in Euclid's *Elements*

Consider Euclid's first proposition, I.1, which is a construction problem: to construct an equilateral triangle on a given straight line. We are given a finite straight line. A circle is then drawn with the given line as radius and one endpoint as center, as licensed by one of Euclid's postulates, and then another circle is drawn with the given line again as radius and the other endpoint as center. Two further lines are drawn, as licensed by another postulate, one from a point of intersection of the two circles to one endpoint of the given line, and the other from that point of

[15] It should not be assumed that all the demonstrations in Euclid's *Elements* are diagrammatic. Many are not, for example, those concerning numbers in Books VII through IX. Our focus here is on the plane geometry of the early books. As we will see, it is in plane geometry in particular that the contents of the relevant concepts can be exhibited in diagrams in a way that enables one to reason in the diagram.

intersection to the other endpoint of the given line. The result is this:



Now we reason through the diagram. Because AB and AC are radii of circle BCD, they are equal in length. Similarly, BA and BC are equal in length because they are both radii of circle ACE. It follows (from the fact that things equal to the same are equal to each other) that all three lines, AB, AC, and BC are equal in length. The triangle ABC, constructed on the given line, is equilateral. The desired construction has been achieved. Euclid similarly shows us how to construct a parallel given a line and a point not on the line, a square on a line, and so on. All such constructions function as derived postulates in Euclid's system; they enable the construction of the diagrams that are needed in demonstrations in more complicated cases.

As just indicated, one reasons *in* the diagram in a Euclidean demonstration: one begins by constructing the needed diagram, and then one reasons through it in an ordered series of steps to the desired conclusion. We need to understand (if only in outline here) how this works.[16]

Consider first the fact that lines that are at one stage in the reasoning regarded as radii of a circle—as they must be to determine that they are equal in length—are later regarded as sides of a triangle, as they must be if we are to conclude that we

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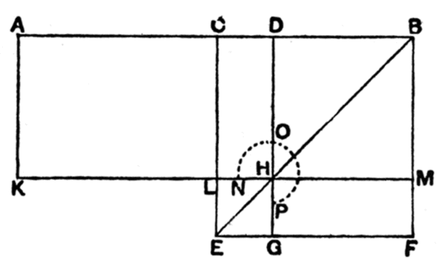
[16] A much more detailed analysis can be found in my "Diagrammatic Reasoning in Euclid's *Elements*", in *Philosophical Perspectives on Mathematical Practice*, *Texts in Philosophy*, vol. 12, ed. Bart Van Kerkhove, Jonas De Vuyst, and Jean Paul Van Bendegem (London: College Publications, 2010).

have constructed the desired equilateral triangle. One and the same lines are now regarded as parts of a circle and later as parts of a triangle. The diagram as a whole, then, is functioning graphically; what a given line means is a function of how it is regarded in relation to other parts of the diagram. Also, and equally importantly, the diagram has three clearly discernable levels of articulation. First, there are the primitive parts out of which everything is composed: points, lines, angles, and areas. Then there are the geometrical figures that are composed of those primitive parts, that form the subject matter of geometry, and that can be discerned in the diagram: circles with their centers, circumferences, radii, and areas; triangles with their sides, angles, and areas; squares with their sides, angles, and areas, and so on. And finally, there is the whole diagram, the whole collection of lines, points, angles, and areas, whose various proper parts can be seen now this way and now that. It is precisely because the figures of interest—those at the second, middle level—both *have* (primitive) parts and *are* parts of the diagram as a whole that one can, for instance, introduce circles and radii *into* a diagram and then take *out* of it an equilateral triangle. The demonstration is fruitful, a real extension of our knowledge for just this reason: because we are able perceptually to take parts of one whole and combine them with parts of another whole to form a new whole, we are able to discover something that was simply not there, even implicitly, in the materials with which we began.

Euclid's first little demonstration in the *Elements* shows that an equilateral triangle can be constructed on a given straight line by actually generating a triangle with, demonstrably, the requisite features. But not all problems in Euclid are construction problems. In some cases the task is instead to demonstrate the truth of a theorem, for instance, this (Prop. II.5): if a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the line between the points of section is equal to the square in the half. Again we begin by constructing the needed diagram.

We set out a line that is cut into equal and unequal segments as required by the problem. Then we draw a square on the half, and add the diagonal. The next step is to draw a series of parallel lines: one parallel to the side from the point of the unequal cut, one parallel to the original line through the point of intersection of the diagonal and the first line, and another again parallel to the side from the left end of the original line. The diagram that results is this:



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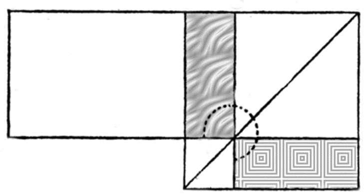
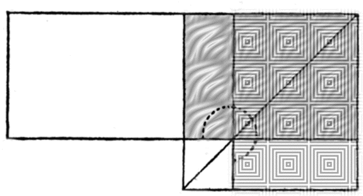
Notice that here again we regard the various lines in multiple ways depending on the context we consider them in; just as in our earlier example, we take them to designate now this and now that depending on the way we regard some proper part of the diagram. The line CB, for example, is first taken to be the half of line AB, but then as a side of square CEFB. BM is a line equal in length to DB but also a part of the line BF, which is another side of that same square. It is in virtue of these various relations of parts iconically presented in the diagram, together with the possibility of various reconfigurings of its parts, that the diagram enables one to show that the theorem is true.

1.



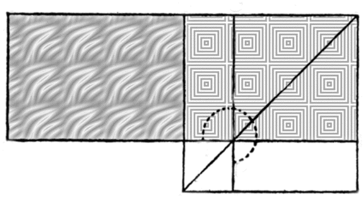
These (differently shaded) areas are equal, as is shown by Prop. I.43: the complements of a parallelogram about the diameter (the two differently shaded areas) are equal to one another.

2.



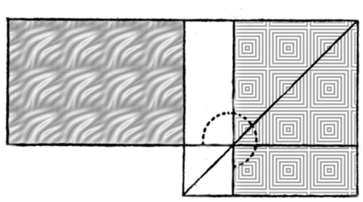
It follows from (1) that these (now overlapping, differently shaded) areas are equal, because the same has been added to the same.

3.

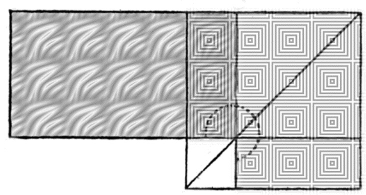


These two (differently shaded) areas are known to be equal on the basis of what we know about the relationships that obtain among the lines that form the boundaries of the two shaded areas.

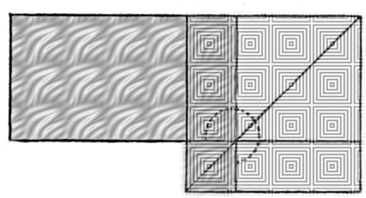
4.



It follows from (2) and (3) that these areas are equal because things equal to the same are equal to each other.

5. 

It follows from (4) that these areas are equal because the same (namely, the doubly shaded area) has been added to equals, that is, to the two areas shaded in (4).

6. 

It follows from (5) that these areas are equal because the same, namely, the little square left unshaded in (4), has been added to equals.

But that is just what we wanted to show: that the square on the half is equal to the rectangle contained by the unequal segments plus the square on the line between the points of section. By construing various aspects of the diagram in these various ways in the appropriate sequence one comes to see that the theorem is, indeed, must be, true.

It is, I think, obvious that one calculates *in* the Arabic numeration system. Similarly, though less obviously, one reasons *in* the diagram in Euclid. In Euclid, the reasoning is not merely diagram-based; it is diagrammatic. One reasons in the system of signs just as one does in Arabic numeration.[17] A Euclidean demonstration is, however, unlike a calculation in arithmetic insofar as there is no algorithm for finding the diagram that will mediate one's passage from one's starting point to the desired endpoint. Once one has been shown the diagram, and has been shown how

[17] Again, this is argued at length in my "Diagrammatic Reasoning in Euclid's *Elements*".

to use it to make the passage from the starting point to the endpoint, one can see how the diagram serves to establish the result, and one can reproduce that result. The hard part, the part that can take real mathematical genius, is finding the diagram.

Diagrams in Euclid provide a mathematical language within which to reason in geometry; they enable one to formulate content in a mathematically tractable way through a system of written signs. And this is possible in virtue of the way the system works overall. First, as we have seen, the notation, the system of written signs, functions graphically insofar as what a particular written mark, say, a line, signifies, whether, say, a radius of a circle or a side of a triangle or a part of a larger line, is determined only in the context of a diagram and relative to a way of regarding it. Because the notation functions graphically, the reasoning can be, and is, intra-configurational. It stays within the diagram. Furthermore, as our examples aimed to illustrate, written marks in Euclid do not merely record or picture something as a picture proof does. Instead they *formulate content*—even, in the case of a reductio proof, contradictory content—in a way that enables reasoning in the system.[18] Drawn figures in Euclid do not picture various geometrical figures (any more than Arabic numerals picture collections of things); instead they display the contents of the concepts of figures in plane geometry, themselves understood in

[18] For example, in proposition III.10 Euclid shows by reductio that a circle does not cut a circle at more than two points. He begins by assuming that a circle cuts a circle at four points, and the needed diagram is drawn. What the diagram appears to depict is a circle cut at four points by something that looks like an oval. As the subsequent reasoning shows, what is actually formulated in the diagram, for the purposes of that reasoning, is the (false) assumption. This case is further discussed in my "Diagrammatic Reasoning in Euclid's *Elements*".

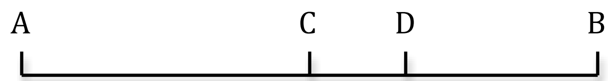
terms of relations of parts, in a mathematically tractable way. A drawn circle in Euclid is not a picture or instance of a circle but instead an iconic display of the relation of parts that is constitutive of something being a circle. Because the contents of concepts are displayed in this way in a Euclidean diagram, the diagram can serve as a vehicle of reasoning that enables even the less gifted of us to reproduce significant results about the logical relations that obtain among the various concepts of that geometry. Euclidean diagrammatic practice, for two millennia the paradigm of mathematical practice, uses a simple and expressive notation that does just what Jourdain says any good mathematical notation does. It enables one to reason in the system so as to reproduce results, and this works in virtue of the peculiar way the notation works to formulate content as it matters to mathematical reasoning.

III. Reasoning in the Formula Language of Algebra

The second system of mathematical signs that we will consider—the familiar symbolic language of elementary algebra that was first introduced by Descartes[19]—functions in a very different way from the way Euclid's system works. Nevertheless, we will see that it too enables one not merely to record but to *reproduce* mathematical results, and this in virtue of the fact that it formulates content rather than merely picturing things.

[19] I argue in "Viète, Descartes, and the Emergence of Modern Mathematics", *Graduate Faculty Philosophy Journal* 25 (2004): 87-117, that it is Descartes, rather than Viète, who should been seen as the first to use symbols in the way to be described.

Think again of the second problem we considered from Euclid's *Elements*, but this time, following Descartes, from the perspective of algebra. We are given a line ACDB that is cut into equal segments at C and unequal segments at D:



We first assign names to the three lengths, say, a to AC, b to CD, and c to DB. We know, then, that $a = b + c$; that is, we interpret the claim that a line is cut into equal and unequal segments as a claim about an *arithmetical* relationship between the three segments that are generated by the two cuts. What is to be shown is similarly interpreted. The idea of a rectangle contained by the unequal segments is interpreted as $(a + b)c$; the square on the line between the points of section is b^2; and the square on the half is a^2. What is to be shown, then, is that $(a + b)c + b^2 = a^2$, given that $a = b + c$. This is easily done: simply replace all occurrences of 'a' in what is to be shown by '$b + c$', and using the familiar rewrite rules of algebra, perform the appropriate symbol manipulations until the expressions on both sides of the equation are the same:

$$((b + c) + b)c + b^2 = (b + c)^2,$$

$$(2b + c)c + b^2 = (b + c)^2,$$

$$2bc + c^2 + b^2 = b^2 + 2bc + c^2,$$

$$b^2 + 2bc + c^2 = b^2 + 2bc + c^2.$$

But how is it that we come to interpret an expression such as 'the rectangle contained by the unequal segments' *arithmetically*, that is, as '$(a + b)c$'? A rectangle is an *object*, a geometrical figure, that is, certain parts in a particular (spatial) relation that as a whole has a characteristic look. How does such an object come to be conceived as something expressible in the formula language of arithmetic? As obvious and natural as it may seem to us, this use of symbols was not at all easy to achieve.

In Euclid's practice, diagrams formulate the contents of geometrical concepts. A drawn circle iconically presents the relations of parts that are constitutive of a circle; a drawn triangle iconically presents the relation of parts that are constitutive of a triangle; and so on. Descartes learns to see such drawings differently. For him a visual display of lines, for instance, a drawing of a right triangle, is not a drawing of an *object* but is instead a graphic representation of an arithmetical relation, of one way—in the case of a right triangle, an especially interesting and revealing way—that measurable quantities can be related one to another.[20] For Descartes the drawn right triangle formulates not geometrical content but instead *arithmetical* content, content that can as well be exhibited in symbols: $a^2 + b^2 = c^2$, where c is the length of the hypotenuse and a and b the lengths of the other two sides. A circle similarly is, for Descartes, not an object but instead the path traced out by a moving point governed by the law that $x^2 + y^2 = r^2$. After Descartes, the science of mathematics becomes a science not of objects as it had been for the ancient Greeks, but of arithmetical relations and the patterns that are expressible in such laws as that $x^2 + y^2 = r^2$. Once again, we will see, it is the notation, the mathematical language, that enables those of us less gifted to follow where mathematical geniuses such as Descartes have led.

[20] See my "Viète, Descartes, and the Emergence of Modern Mathematics" for further discussion of what this new way of seeing involves.

Consider this problem. It is given that the sum of a number and its reciprocal is equal to one; the task is to find the sum of the cube of that number and the reciprocal of the cube. I assume that a mathematical genius could discover the solution merely by reflecting on the relevant ideas as we did in the case of the ancient proof that there is no largest prime. For those of us less gifted, the language of elementary algebra provides a much easier route. As one would do in a calculation in Arabic numeration, we begin by formulating the problem in the symbolic language: we are given that $x + 1/x = 1$, and must find the value of $x^3 + 1/x^3$. An obvious strategy is to try cubing both sides of the given equation: $(x + 1/x)^3 = 1^3$. So $x^3 + 3x^2(1/x) + 3x(1/x)^2 + 1/x^3 = 1$. That is, $x^3 + 3x + 3/x + 1/x^3 = 1$, or $x^3 + 3(x + 1/x) + 1/x^3 = 1$. But we know that $x + 1/x = 1$; so, putting equals for equals, $x^3 + 3(1) + 1/x^3 = 1$. And we have our answer: $x^3 + 1/x^3 = -2$.[21]

The problem we just solved is a kind of construction problem; the task was to produce a number meeting certain specifications. But one also can prove theorems in the formula language of algebra, for instance, Euler's Theorem that $e^{ix} = \cos(x) + i\sin(x)$—widely regarded as, in Feynman's words, "one of the most remarkable, almost astonishing, formulas in all of mathematics".[22]

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[21] An reviewer offered this more elegant solution. First multiply the original equation through by x to yield $x^2 + 1 = x$. From the original equation we also have $x = 1 - 1/x$. So, putting equals for equals gives $x^2 + 1 = 1 - 1/x$, that is, $x^2 = -1/x$, which if we multiply through by x again gives $x^3 = -1$. So $x^3 + 1/x^3 = -2$.

[22] Richard Feynman, *The Feynman Lectures on Physics*, vol. I (Addison-Wesley, 1977), pp. 21-2. It is also the theorem that is the basis for Euler's famous equation that $e^{i\pi} + 1 = 0$. Euler proves the theorem in Chapter 8 of his *Introductio in analysin infinitorum* (1748), though not in the way to be shown here. The proof sketched here follows J. O. Smith, *Mathematics of the Discrete Fourier Transform (DFT) with*

The number e is the number, whatever it is, such that the derivative of e^x just is e^x. (It can be shown that there is such a number.) Knowing only that e^x is its own derivative, and that $e^0 = 1$, as well as familiar basic rewrite rules of calculus such as that the derivative of x^n is nx^{n-1}, we can express the function $f(x) = e^x$ as a power series:

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots + x^n/n! + \ldots$$

And we can see that this has to be right because taking the derivative of this series just gives the same series again. The derivative of the first term, 1, is 0; the derivative of the second term, x, is 1, that is, the original first term; the derivative of the third term, $x^2/2!$, is $2x/2!$, that is, simply x, which is the original second term; the derivative of the fourth term is the third term, and so on. Because the derivative of each subsequent term is the previous term and there are infinitely many terms, to take the derivative of the whole series is to get the same series again. So we know that our infinite series is equal to the function e^x.

We can also write the trigonometric functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ as power series; indeed, they can be defined by the following series expansions:

$$\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \ldots$$

$$\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$$

This is very suggestive. Between the two trigonometric functions, expressed as power series, we have precisely the terms we find in the power series expansion of e^x. Only the signs are different. We get around this as follows. Because the number i is, by definition, such that i^2 is equal to minus one, it follows that $i^3 = -i$, that $i^4 = 1$,

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Audio Applications, Second Edition, http://ccrma.stanford.edu/~jos/mdft/, 2007, online book, accessed 1 October 2010, third chapter, “Proof of Euler’s Identity”.

that $i^5 = i$, and then the cycle repeats beginning with i^2. Suppose now that we replace 'x' with 'ix' in our function e^x. That gives us:

$$e^{ix} = 1 + ix + (ix)^2/2! + (ix)^3/3! + (ix)^4/4! + (ix)^5/5! + \ldots$$

Now we do some standard algebraic manipulations to get:

$$e^{ix} = 1 + ix - x^2/2! - ix^3/3! + x^4/4! + ix^5/5! - x^6/6! - \ldots$$

Rearranging things a bit, by collecting together the terms that contain i, gives:

$$e^{ix} = (1 - x^2/2! + x^4/4! - x^6/6! + \ldots) + i(x - x^3/3! + x^5/5! - \ldots).$$

And now we can see that the first of the bracketed series is the power series of cos(x), and the second is that of sin(x). So putting equals for equals gives us Euler's Theorem. "All" we had to do to see that Euler's Theorem is true was to think of the power series expansions of our functions—on the one hand, the logarithmic function e^x, and on the other, the trigonometric functions sin(x) and cos(x)—and replace 'x' with 'ix', and we could *see* that they are related in a certain arithmetical way. Of course it takes the genius of Euler to *discover* the result. But once it has been discovered, anyone knowing the symbolic language can reproduce it.

We have looked at three examples of reasoning in algebra. The first example, in which we proved Euclid's Proposition II.5 algebraically, reveals a fundamental connection between Euclidean diagrammatic reasoning and algebraic reasoning. The second example, to find the sum of a cube and its reciprocal given that the sum of the root and its reciprocal equals one, aimed to supply a simple and clear illustration of Jourdain's thesis about the role of writing in mathematics. Others could have been given instead insofar as we know, as a matter of historical fact, that many algebraic results were discovered by gifted mathematicians before any adequate notation of algebra was devised. But as historians of mathematics are well aware, without the notation it is often *very* difficult to understand how the reasoning goes, why the relevant theorem is true. With it, almost anyone can understand the proof. Our third example, the (here, rather sketchy) proof of Euler's theorem, provides some indication of the enormous power of this system of notation. As this result indicates, the mathematician's genius does not invariably lie in what can be discovered without the help of any notation but instead in what one can reveal given that notation, given, in particular, the way it formulates content—in our example, in a particular infinite series—in written signs.

When reasoning in the symbolic language of arithmetic and algebra one does not, as one does in Euclid, write a bunch of stuff down and only after one has finished writing reason through what one has written. Rather the writing is itself the reasoning in the sense that the steps of reasoning are at the same time steps of writing. Whereas in Euclid the reasoning is intra-configurational insofar as the reasoning stays within the diagram and involves only a *perceptual* reconfiguring of parts, in arithmetic and algebra the reasoning is trans-configurational insofar as the successive steps of reasoning require new writing. We have seen already that intra-configurational reasoning is possible only given a system of signs that functions graphically, but also that, at least in principle, trans-configurational reasoning can involve signs that function either graphically or symbolically. In fact, what our three algebraic examples, most obviously that of Euler's theorem, suggest is that the symbolic/graphic distinction does not cut finely enough. Whereas in our first example the signs seem to function symbolically in a quite straightforward sense, in the case of Euler's theorem the primitive signs are used to formulate complex names for functions and to do so in a way that enables one to show the identity of functions. What Euler shows is that what might appear to be names for radically different functions are in fact names for one and the same function. As Frege would put the point, Euler shows that although the function expressions 'e^{ix}' and '$\cos(x) + i\sin(x)$' express radically different senses, they nonetheless designate one and the same function; they differ in the *Sinn* expressed but not in their *Bedeutung*. And if that is right, then one can, in this system of signs, form complex names for functions out of the primitive signs of the language, and thereby formulate their contents in mathematically useful ways. But in order to designate individual functions using *collections* of signs—for example, the sine function not merely as '$\sin(x)$', which is little more than a label, but as a power series with a great deal of internal articulation—it must be the case that the primitive signs do not designate independent of a context of use. And that suggests that the signs are functioning graphically; one can use whole collections of signs to form names for particular, individual functions. Nevertheless, in algebra, at least in the sorts of cases we have considered, one does not, as one does in Euclid, perceptually reconfigure such collections of signs in the course of reasoning.[23] To see an expression in a new way

[23] In an arithmetical calculation (for instance, in multiplying 375 by 62 in the usual paper-and-pencil way), one does regard various proper parts of the array now this way and now that in the course of the calculation; both the horizontal and the vertical expanse of the page are utilized in the calculation. And the same can happen in algebra, as Descartes shows already in Book III of his *Geometry*, for instance, in his discussion of how to increase or decrease the roots of an equation by some particular number. This feature of the notation can be important, but it is not immediately relevant to the aspects of systems of signs we are concerned with here.

one generally writes it in a new way, for instance, now as a product of sums, say, as '(a + b)(a + b)', and now as a sum of products, '$a^2 + 2ab + b^2$'.

Reasoning in algebra is trans-configurational and the notation functions in a way that is symbolic, at least in the sense that nothing like the perceptual reconfiguring of parts that we find in Euclid is usually necessary, or often even possible. Nevertheless, in the formula language of algebra as in a Euclidean diagram, one does not merely record or picture something; instead one formulates the content of something and does so in a mathematically tractable way. As a drawn figure in Euclid formulates the contents of the concepts of geometry in a way that enables diagrammatic reasoning about those concepts, so an algebraic expression of a function can formulate arithmetical content in a way enabling one to demonstrate truths about the functions so designated. It is, for example, the *particular* complexity of the expression '$1 + x + x^2/2! + x^3/3! + \ldots x^n/n! + \ldots$' that enables us to conclude that this series designates precisely the same function as the expression 'e^x'. Without that result we could not establish Euler's Theorem.

We saw that Euclidean diagrammatic reasoning is fruitful because one can perceptually reconfigure parts of different wholes within a diagram into a new whole. Parts of wholes that are at first conceived separately, as parts of different geometrical figures, are later conceived together, as parts of one and the same geometrical figure. In algebraic reasoning one instead combines content by putting equals for equals. And much as the genius required for Euclidean mathematical practice lies in finding the diagram that will provide the vehicle of reasoning from one's starting point to the desired conclusion, so the genius that is required to discover, say, Euler's Theorem lies in discovering just what will serve as what we can think of as the *middle* that will show that two interestingly different, and apparently unrelated, expressions for a function are in fact expressions for one and the same function. The expression 'e^{ix}' does not look anything like the expression '$\cos(x) + i\sin(x)$', and the functions they designate appear to have no mathematical relation, and yet we have shown that they are alternative formulations for one and the same function by showing that they can be, each of them, equated with a certain power series. This power series serves as the middle connecting the two apparently unrelated functions. In algebraic reasoning, then, there are two very different sorts of rewritings, both the relatively mechanical rewriting according to the familiar rules of algebra, and the creative combining of content (by putting equals for equals) that is made possible by such rewriting. Reasoning in algebra can extend our knowledge for just this reason.

IV. Trans-configurational Reasoning in a Graphical System of Signs

As it had been in the seventeenth century, mathematical practice was again transformed in the nineteenth. Rather than trying to compute, that is, construct, the function that is wanted by paper-and-pencil manipulations as Euler had done, mathematicians such as Riemann sought to *describe* the essential properties of the desired function, and to infer deductively what must be true of a function so described. For Riemann, "the objects of mathematics were no longer formulae but not yet sets. They were concepts."[24] The task was (in Dedekind's words) "to draw

[24] Detlef Laugwitz, *Bernhard Riemann 1826-1866: Turning Points in the Conception of Mathematics*, trans. Abe Shenitzer (Basil, Berlin, and Boston: Birkhäuser, 1999), p. 305.

the demonstrations, no longer from calculations, but directly from the characteristic fundamental concepts, and to construct the theory in such a way that it will . . . be in a position to predict the results of the calculation".[25]

Chains of deductive reasoning from defined concepts in this new mathematical practice could be recorded, reported, in natural language in much the way we reported the proof that there is no largest prime. But there was, at least at first, no system of written signs within which to reproduce such reasoning. Perhaps one could be devised. Perhaps there could be a kind of concept-writing or concept-script, a *Begriffsschrift* that like the earlier languages of mathematics would enable one to reason in mathematics—in this instance, deductively from defined concepts—*in* the system of signs. In 1879, Gottlob Frege, a Jena mathematician of the Riemann school[26], published a little monograph introducing just such a language.[27] Modeled on the formula language of arithmetic, Frege's *Begriffsschrift* was to enable one to exhibit the contents of mathematical concepts in a way enabling one to reason deductively from those contents. It was to exhibit the (inferentially articulated) contents of the concepts of concern to nineteenth century mathematicians much as Arabic numeration exhibits the (arithmetical) contents of

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[25] Quoted in Howard Stein, "*Logos*, Logic, and *Logistiké*: Some Philosophical Remarks on Nineteenth-Century Transformations of Mathematics", in *History and Philosophy of Modern Mathematics*, ed. William Aspray and Philip Kitcher, *Minnesota Studies in the Philosophy of Science*, vol. XI (Minneapolis: Minnesota University Press, 1988), p. 241.
[26] See Jamie Tappenden, "The Riemannian Background to Frege's Philosophy", in *The Architecture of Modern Mathematics: Essays in History and Philosophy*, ed. J. Ferreirós and J. J. Gray (Oxford: Oxford University Press, 2006).
[27] Gottlob Frege, *Conceptual Notation, a formula language of pure thought modeled upon the formula language of arithmetic*, in *Conceptual Notation and Related Articles*, trans. and ed. T. W. Bynum (Cambridge, Mass.: Harvard University Press, 1970).

numbers, Euclidean drawings exhibit the (spatially articulated) contents of geometrical concepts conceived as relations of parts, and the language of algebra exhibits the (arithmetically articulated) contents of functions. Frege's system of signs, we will see, functions graphically although the reasoning it enables is trans-configurational.[28]

A drawing of a geometrical figure in Euclid displays the content of the concept of that figure, what it is to be, say, a circle or triangle conceived as a relation of parts. An equation in the symbolic language of Descartes and Euler is different insofar as it instead exhibits arithmetical relations, for instance, that which holds among the length of the hypotenuse and of the other two sides of a right triangle, or, more subtly, that which holds between argument and value in a particular function. Frege in effect combines these two ideas: he will exhibit the (now inferentially articulated) contents of mathematical concepts and he will do so by displaying the (now logical) relations that obtain among the constituents of those concepts. To do that, however, he needs to learn to read the symbolic language in a radically new way; he needs to learn to read it *graphically*, that is, as like a Euclidean diagram whose parts can be conceived now one way and now another.

Much as Descartes taught us to see a drawn right triangle not as a drawing of a particular sort of geometrical object but instead as expressing an arithmetical relation among arbitrary quantities—one that could equally well be expressed in symbols—so, we have already seen, Frege teaches us to see an equation such as '$2^4 = 16$' differently, as merely presenting what he calls a sense (*Sinn*), one that can be

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[28] Although it is not there described in such terms, the reading of Frege's notation that is followed here is introduced, developed, and defended in my *Frege's Logic*.

regarded in various ways, that is, carved into function and argument in various ways. The primitive signs of the language so conceived function, in other words, in a way that is analogous to the way the marks for points, lines, angles, and areas function in Euclid; they function graphically. In the language as Frege conceives it, the primitive signs only express a sense independent of a context of use. Only in a whole formula and relative to an analysis into function and argument do the sub-sentential expressions of the language, whether simple or complex, serve to designate anything. Because the notation functions graphically, Frege can exhibit the contents of mathematical concepts in a way enabling deductive reasoning on the basis of those contents.

In the mathematical practice of concern to Frege, theorems are established by reasoning deductively from explicitly formulated definitions, where a definition is a stipulation that some newly introduced simple, unanalyzable sign is to have precisely the same meaning as some complex expression formed, in the first instance, out of the primitive signs of the language. The definition exhibits in the definiens the inferentially articulated content of the concept being defined, and introduces in the definiendum a simple sign that has, by stipulation, precisely the same designation. In Frege's 1879 logic, Part III—which is, Frege says (§23), "meant to give a general idea of how to handle" his system of written signs—Frege provides four definitions on the basis of which to prove a theorem in the theory of sequences. The task of the proof is to find a path *from* Frege's four definitions *to* theorem 133 displaying a particular logical relation among three of Frege's four defined signs. Somehow the defined signs that originally occur in three different definitions are to be brought together and combined in one formula. That is, much as three different functions—the logarithmic function and the sine and cosine functions—are brought together, combined in Euler's theorem, so three different concepts are brought together, combined in Frege's theorem. Our interest lies in seeing, at least in outline, how this goes.[29]

We saw that in Euclidean diagrammatic reasoning one perceptually reconfigures various parts of a diagram in an ordered sequence of steps so as to discover something new. The reasoning is intra-configurational; it stays within the diagram. The mathematical reasoning of Descartes and Euler is instead trans-configurational; one reasons in that case by writing and rewriting according to rules. And here again, the reasoning can enable one to discover something new, something that was not implicit already in one's starting points. Reasoning in Frege's concept-script incorporates elements of both systems, and like those other systems enables extensions of our knowledge. Although formulae in his language can be regarded (that is, analyzed into function and argument) in various ways much as the various lines in a Euclidean diagram can be regarded in different ways, the reasoning, like reasoning in Descartes and Euler, is trans-configurational, a matter of writing and rewriting. And just as in the reasoning of Descartes and Euler, there are two different sorts of rewritings in Frege's system. There are simple rewritings of formulae (in what I call *linear* inferences) according to rules that are analogous to the simple transformation (rewrite) rules of elementary algebra, and there are rewritings (in *joining* inferences) that combine content from two different formulae

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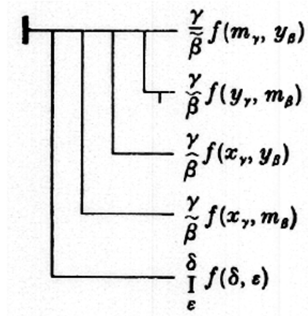
[29] More details can be found in my "Diagrammatic Reasoning in Frege's *Begriffsschrift*", *Synthese*, forthcoming.

thereby extending our knowledge. But whereas rewritings that combine content in, say, Euler involve a middle that enables one to put equals for equals, rewritings that combine content in Frege instead connect antecedent and consequent using some version of hypothetical syllogism: if it can be shown that P if Q and that Q if R, then it may be concluded that P if R.

The proof begins with four definitions each of which formulates (in the definiens) the inferentially articulated sense of a concept word and stipulates that a simple sign, newly introduced, is to have the same meaning (*Bedeutung*) as the complex of signs on the definiens. And as already indicated, the contents of concepts *can* be exhibited in this way in Frege's notation in virtue of the fact that the notation functions graphically: independent of an analysis, a particular way of regarding it, a *Begriffsschrift* formula only expresses a sense, a Fregean thought. In a definition, the definiens is a concept word that, on the analysis that is stipulated by the definition, exhibits the sense of the relevant concept word; it formulates the content of that concept in a mathematically (that is, in this case, deductively) tractable way. The definiendum is a concept word for that same concept, but unlike the definiens it is a simple sign. Because it is simple, it cannot in the context of a judgment be variously analyzed.

What we want to construct is theorem 133:



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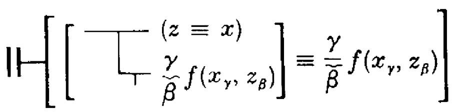
The task of the proof is to join the various defined signs that occur in this theorem in the way shown in the theorem. And, again, as in Euler, there are two stages to the process, both the stage of preparation and the stage at which contents suitably prepared can be joined. We consider here only one small illustrative example.

We begin with definition 99 of belonging to a sequence:



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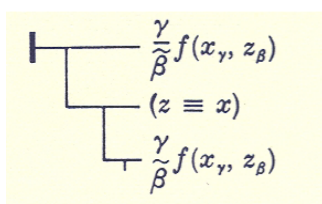
The first step in the preparation is to convert this definition, more exactly the judgment that immediately derives from it, into a conditional judgment, theorem 105:



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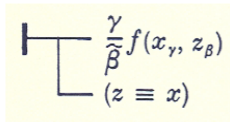
That is, we make the definiens (the complex of signs on the left in the definition) the condition on the content that is the definiendum of the original definition. Now we rewrite according to the rule that if something is true on a condition that has a condition then it is true on that condition alone to yield theorem 112:



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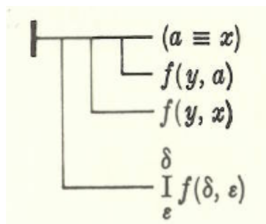
The preparation is complete. We assume a similar preparation for the second formula needed in the join, derived through a series of linear inferences from the definition of being a single-valued function, namely, theorem 120:



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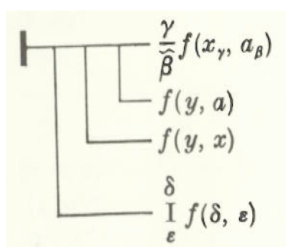
Because the condition in theorem 112, with *a* for *z*, is identical to the conditioned judgment in theorem 120, read as a judgment on three conditions, together these two formulae yield, by hypothetical syllogism, theorem 122:



,

which joins in one formula two of our defined signs (plus some extra conditions). It is in just this way that, much as in reasoning through a diagram in Euclid one (perceptually) joins parts of different wholes into a new whole, and in reasoning in Euler one (actually) joins parts of different wholes into a new whole by putting equals for equals, so here one (actually) joins parts of different wholes into a new whole by hypothetical syllogism. The rest of the proof is essentially similar.

Frege devised his concept-script as a notation within which to reason deductively from defined concepts. The notation was to do for the new form of mathematical practice that emerged over the course of the nineteenth century in the work of Riemann and others what the system of Euclidean diagrams does for ancient mathematical practice and what Descartes' symbolic language does for algebraic practice. Like reasoning in the formula language of algebra, reasoning in *Begriffsschrift* is trans-configurational. But like the signs in Euclid, the primitive signs of *Begriffsschrift* designate only in the context of a whole proposition and relative to an analysis. The signs function graphically and because they do Frege is able to exhibit the inferentially articulated contents of concepts in his language in a way that enables deductive reasoning from those contents.

Frege's notation was not understood; it was read, as it is still today, not graphically but symbolically, as a mere notational variant of our own logical notations. As a result, the enormous expressive power of Frege's *Begriffsschrift* was never tapped by mathematicians. Though in fact Frege's notation provides it, we have even today no simple and expressive notation enabling one to reproduce the theorems that are the fruits of our current mathematical practice of reasoning deductively on the basis of defined concepts.

V. Conclusion

We have seen that in Euclidean reasoning one constructs a diagram that formulates the contents of various geometrical concepts in a way enabling one to reason through the diagram to the desired conclusion. In the course of that reasoning, we saw, one perceptually configures and reconfigures parts of the diagram in ways that effectively combine parts of different wholes into new wholes. In Descartes and Euler the reasoning is also in two parts, though in this case both require new writing. At one stage one simply rewrites what one has in new ways according to rules. At the other one combines content from two different formulae by putting equals for equals. Here again, though in a different way, one combines parts of different wholes into a new whole. The same is true, I have suggested, in the case of reasoning in Frege. In *Begriffsschrift*, one formulates the contents of concepts in definitions on the basis of which to reason deductively. In linear inferences one rewrites what one has, beginning with a definition, in a sequence of new ways according to rules. In joining inferences, one combines content from two different formulae (derived ultimately from two definitions) as mediated by a syllogistic middle. In each of the three cases one discovers thereby something new; one extends one's knowledge.

Jourdain claims that a good mathematical notation enables even the less gifted of us to reproduce theorems. It follows that a good mathematical notation serves not merely to record something but to embody the reasoning, to put the reasoning itself before our eyes. What we have seen here is that this works because the notation functions not merely to depict, record, or picture some things or relations among things; it formulates the *contents* of significant mathematical concepts and functions, and does so in a way enabling reasoning on the basis of those contents. As Arabic numerals formulate the contents of numbers, so drawings of figures in ancient geometry formulate the contents of concepts of geometrical figures (conceived as parts in relation), a formula in early modern algebra formulates the contents of various functions, and a definition in Frege formulates the inferentially articulated contents of mathematical concepts generally. This content can then be manipulated, either perceptually or by rewriting; and more significantly, different contents, or parts of contents, can be joined. It is in just this way that in all these cases the chain of reasoning to some significant result is *embodied* in the writing, put before our eyes. To have a demonstration in Euclid, an algebraic proof in the symbolic language of arithmetic, and a deductive proof in Frege's *Begriffsschrift*, is literally to see how it goes, and this works because in each case, as in the case of Arabic numeration, the contents of mathematically significant concepts and functions are formulated in the various systems of signs in a mathematically tractable way, in a way that enables the mathematical demonstration of significant results.