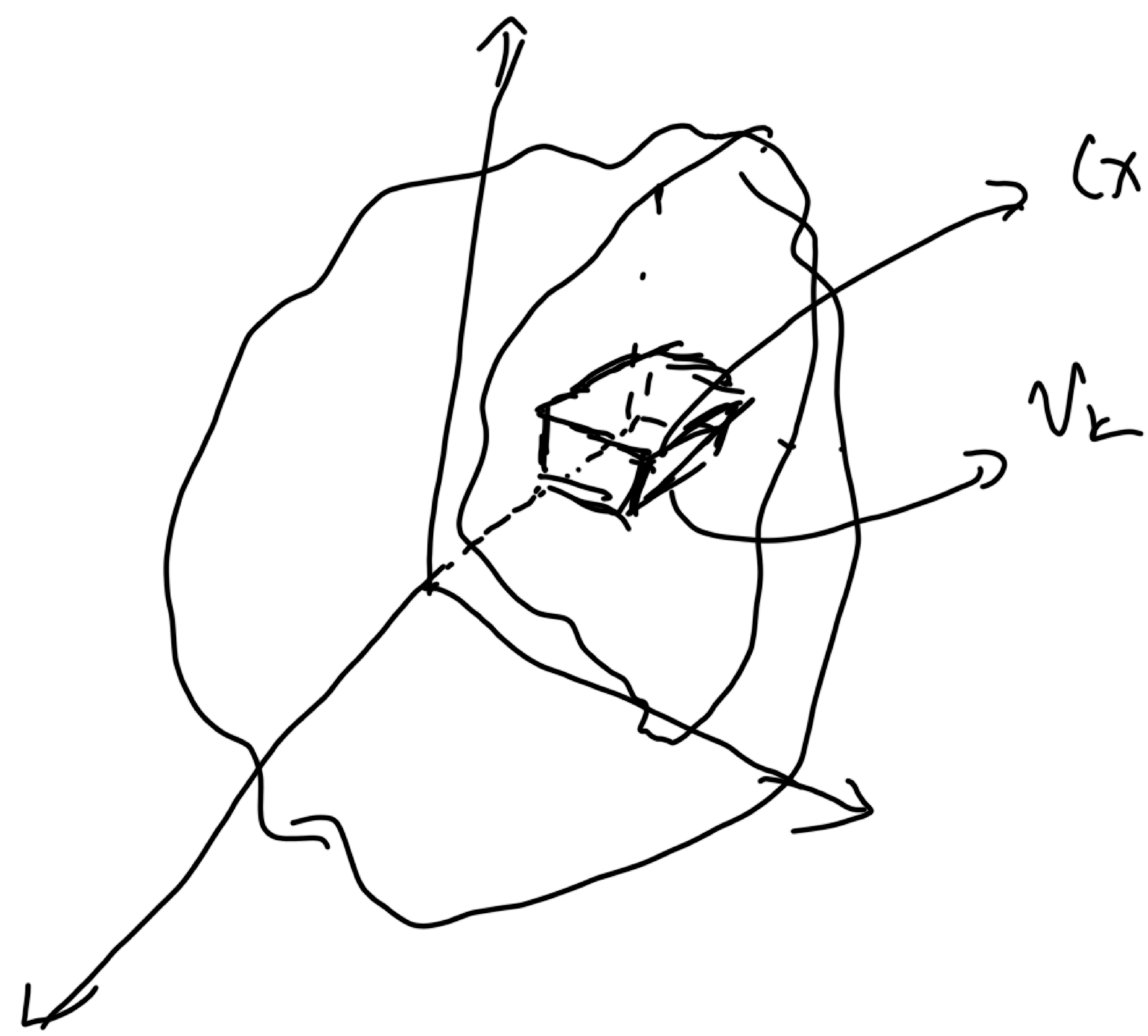
Triple in Cartesian Coordinates:

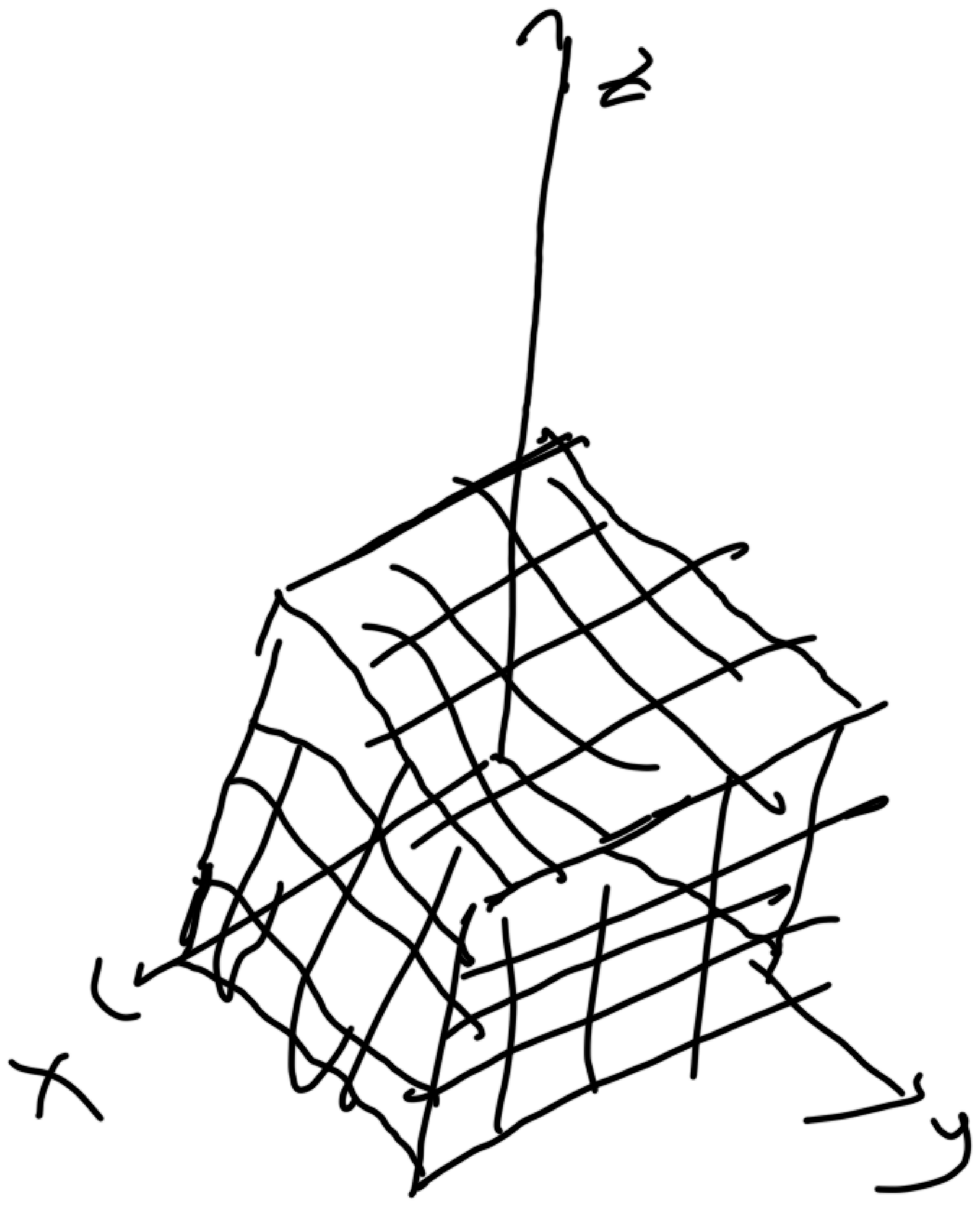
The integral of $f(x,y,z)$ over D may be defined in the following way. we partition a rectangular box-like region containing D into rectangular cell by planes parallel to the coordinate axes (fig 1). we number the cells that lie completely inside D from 1 to n in some order. The k^{th} cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V = \Delta x_k \cdot \Delta y_k \cdot \Delta z_k$. we choose a point (x_k, y_k, z_k) in each cell and form the sum:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

D: $a \le x \le b$, $c \le y \le d$, $p \le z \le q$

f: continous in all points over D





$D_{ijk}\,(x_i, y_j, z_k)$

$$\left.\begin{array}{l} x_{i-1} \le x \le x_i \\ y_{j-1} \le y \le y_j \\ z_{k-1} \le z \le z_k \end{array}\right\} \Delta V_{ijk} = \\ = \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$$

$$\Delta x_i = x_i - x_{i-1} \quad ; \quad i = 1, 2, 3 \ldots m$$

$$\Delta y_j = y_j - y_{j-1} \quad ; \quad j = 1, 2, 3 \ldots n$$

$$\Delta z_k = z_k - z_{k-1} \quad ; \quad k = 1, 2, 3 \ldots r$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} f(x_i, y_j, z_k) \cdot \Delta V_{ijk} \Longrightarrow \lim_{\substack{n, m, r \to \infty \\ (\|P\| \to 0)}} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{r} f(x_i, y_j, z_k) \cdot \Delta V_{ijk}$$

$$= \iiint f(x, y, z) \, dV$$

$$\underline{dV = dz\,dy\,dx}$$

Volume of a Region in Space:

Definition: The volume of a closed bounded region D in space is

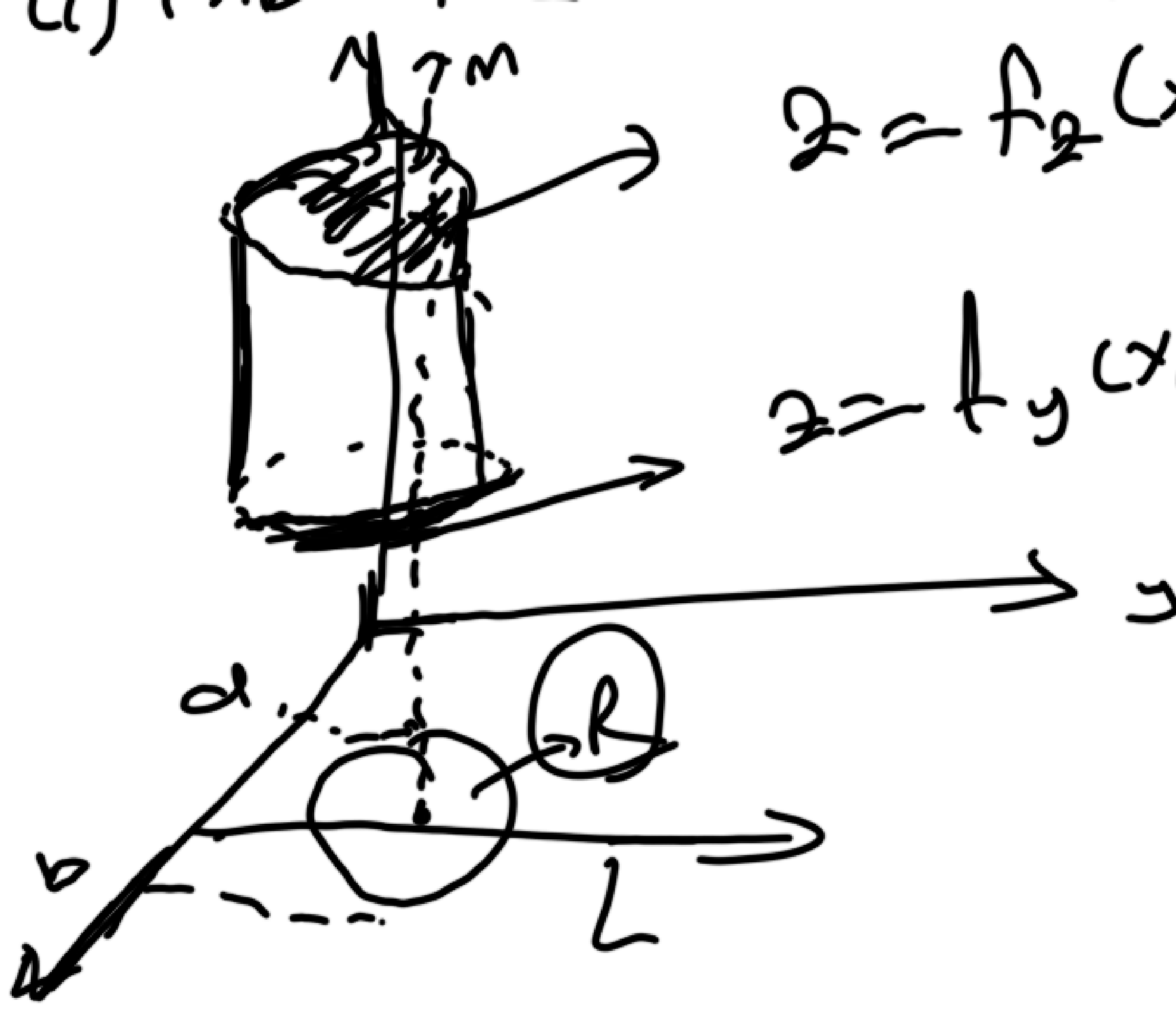
yüksekliğin sonuçları verir

$$\boxed{V = \iiint_D dV}, \quad F(x,y,z) = 1.$$

Region integration olur

Finding the limits of integration in the order $dz\,dy\,dx$:

1) Sketch the region
2) Find the z-limits of integration (line M)
3) Find the y-limits of integration (line (2))
4) Find the x-limits of integration ($x=a$, $x=b$)



$z = f_2(x,y)$ (leaves at)

$z = f_1(x,y)$ (enters at)

$$\iiint f(x,y,z)\,dV = \left[\int_{x=a}^{b}\left[\int_{y}^{g(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\right)dy\right]dx\right]$$

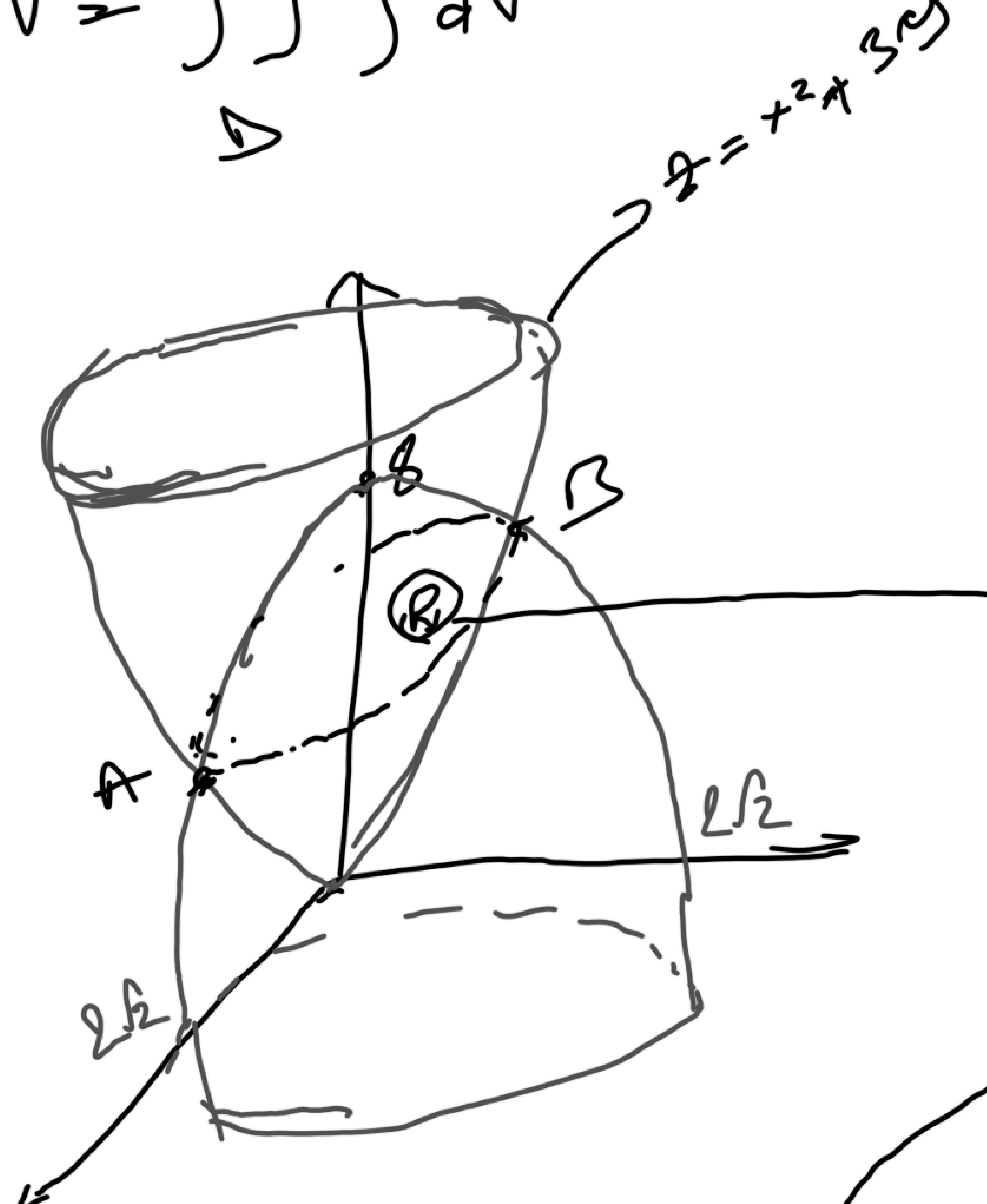
R

Ex: find the volume of the region D enclosed by surfaces $\underline{z = x^2+3y^2}$ and $\underline{z = 8-x^2-y^2}$

$z = x^2+3y^2$ → elliptical paraboloid

$z = 8-x^2-y^2$ → Paraboloid

$$V = \iiint_D dV$$



intersection points:

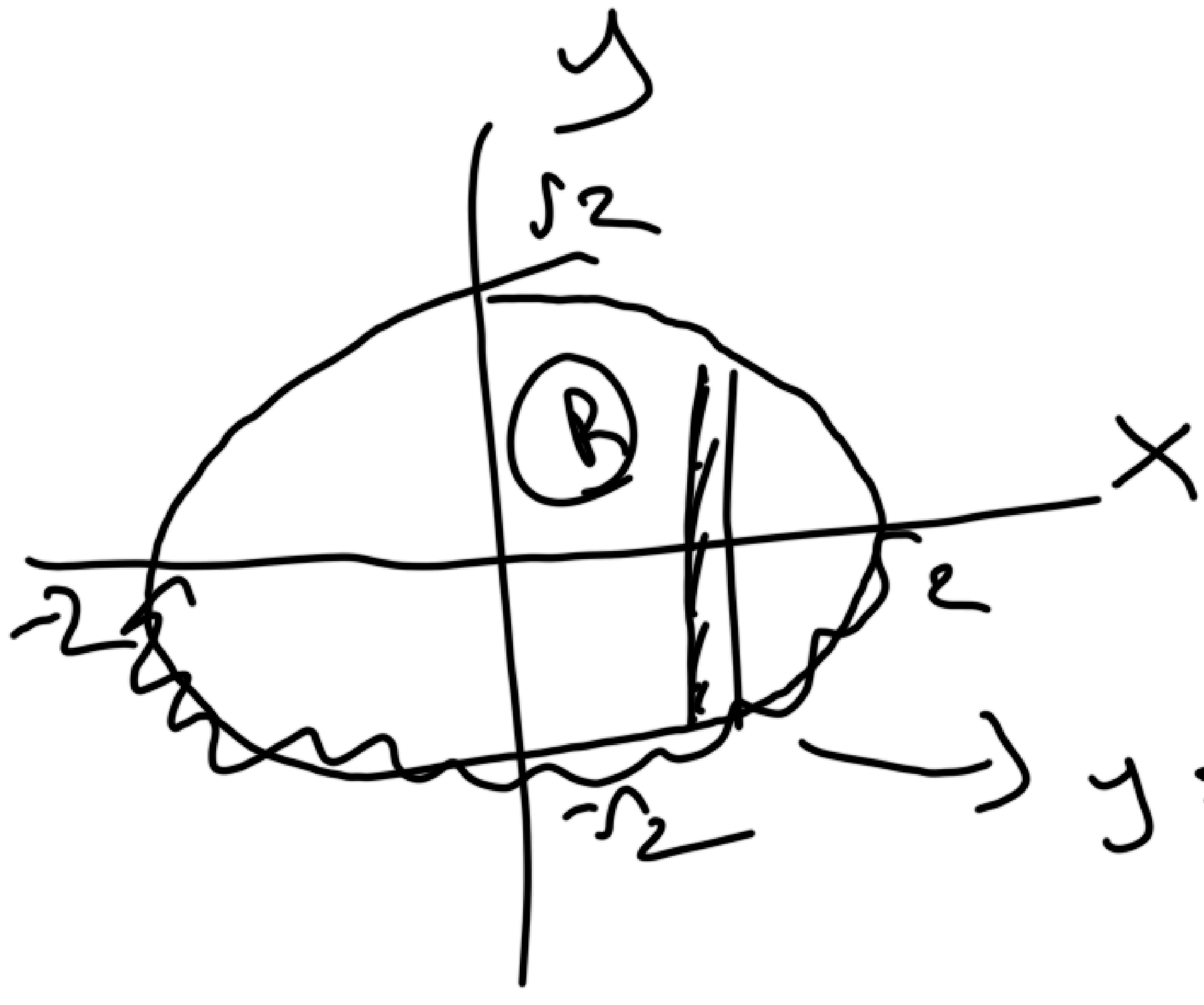
$x^2+3y^2 = 8-x^2-y^2$

$2x^2+4y^2 = 8$

$\frac{x^2}{4}+\frac{y^2}{2} = 1$ (Ellipse)

$\frac{x^2}{2^2}+\frac{y^2}{(\sqrt{2})^2} = 1$

$a=2 \qquad b=\sqrt{2}$



$y = -\sqrt{\frac{8-2x^2}{4}}$

$$V = \iiint_{R} \int_{z=x^2+3y^2}^{z=8-x^2-y^2} dz\, dx\, da = \int_{x=-2}^{2} \int_{y=-\sqrt{\frac{8-2x^2}{4}}}^{\sqrt{\frac{8-2x^2}{4}}} \int_{z=x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

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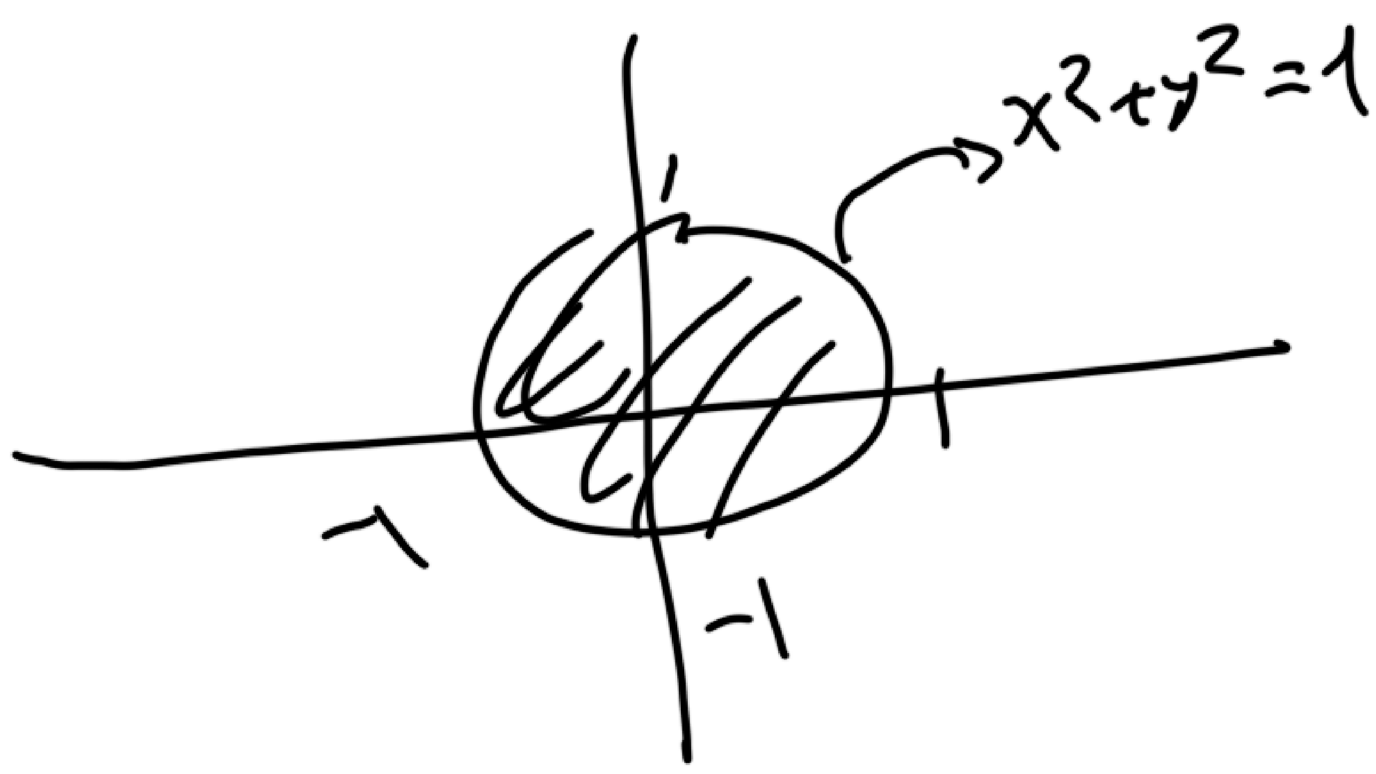
use transformation

$$\begin{cases} x = 2r\cos\theta \\ y = \sqrt{2}\, r\sin\theta \end{cases} \Rightarrow r^2 = 1 \Rightarrow r = 1$$

$B1$ circle



$$V = \iint_{R} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$j(r,\theta) = \frac{\partial(x,\theta)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

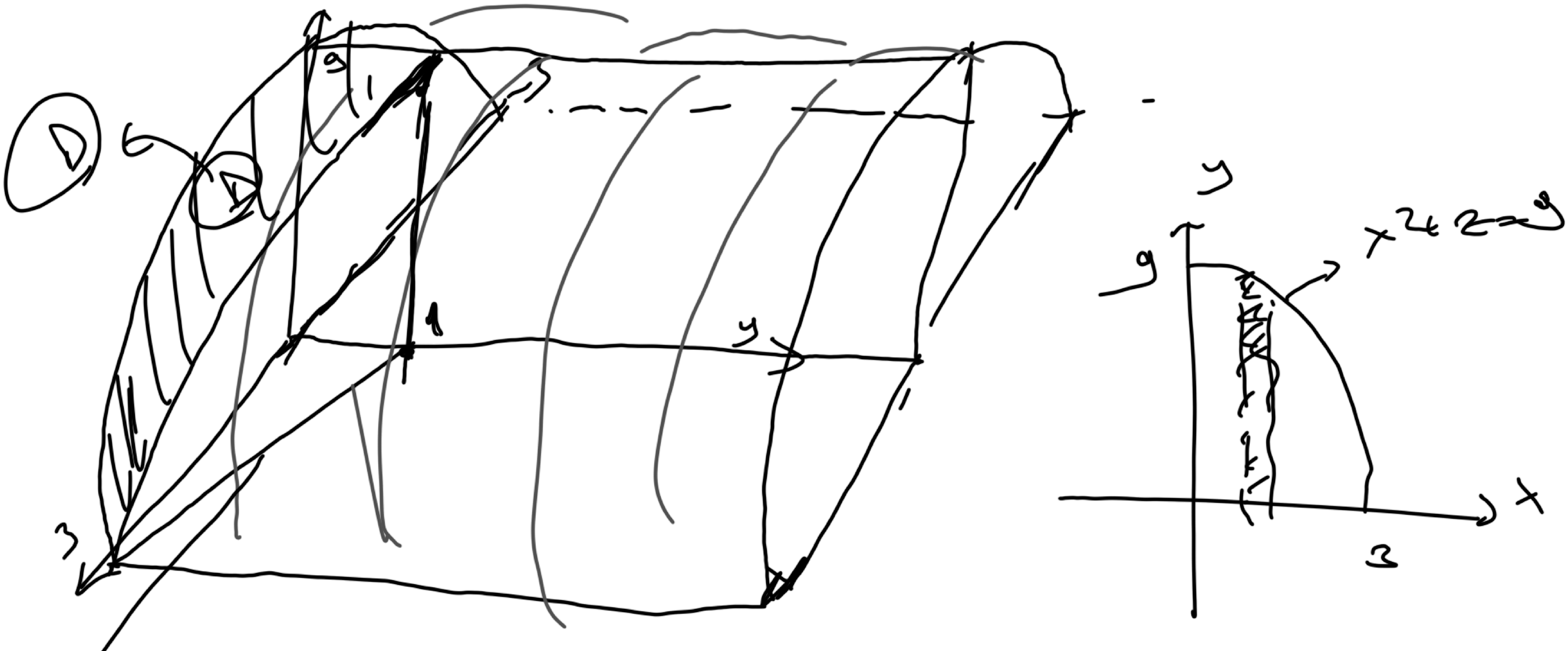
$$= \begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sqrt{2}\sin\theta & \sqrt{2}\,r\cos\theta \end{vmatrix} = 2\sqrt{2}\cdot r$$

$$V = \iint_R (8-2x^2-4y^2)\,dy\,dx = \iint_{(R)} \overbrace{\left(8-2(2r\cos\theta)^2-4(\sqrt{2}\cdot r\cdot\sin\theta)^2\right]}^{f(u,v)} \underline{|2\sqrt{2}\cdot r|\,dr\,d\theta}$$

$$|j(u,v)|\,du\,dv$$

$$= \int_{\theta=\pi}^{2\pi}\int_{r=0}^{1} 8-2(2r\cos\theta)^2 - 4(\sqrt{2}\,r\sin\theta)^2\Big)\,2\sqrt{2}\,r\,dr\Big)\,d\theta$$

Ex Find the volume of the solid enclosed by the parabolic cylinder $x^2 + z = 9$ and the plane $x + 9y = 3$ in the first octant.



$x+3y=3$

$$V = \iiint dV = \iint_{R_{xz}} \int_{y} dy\, dx\, dz$$

$$= \int_{x=0}^{3} \left(\int_{z=0}^{9-x^2} \left(\int_{y=0}^{\frac{3-x}{3}} dy \right) dz \right) dx = \frac{45}{4} \text{ cubic units}$$

* Triple Integrals in Cylindricsel and Spherical Coordinaties

Cylindricsel Coordinates:

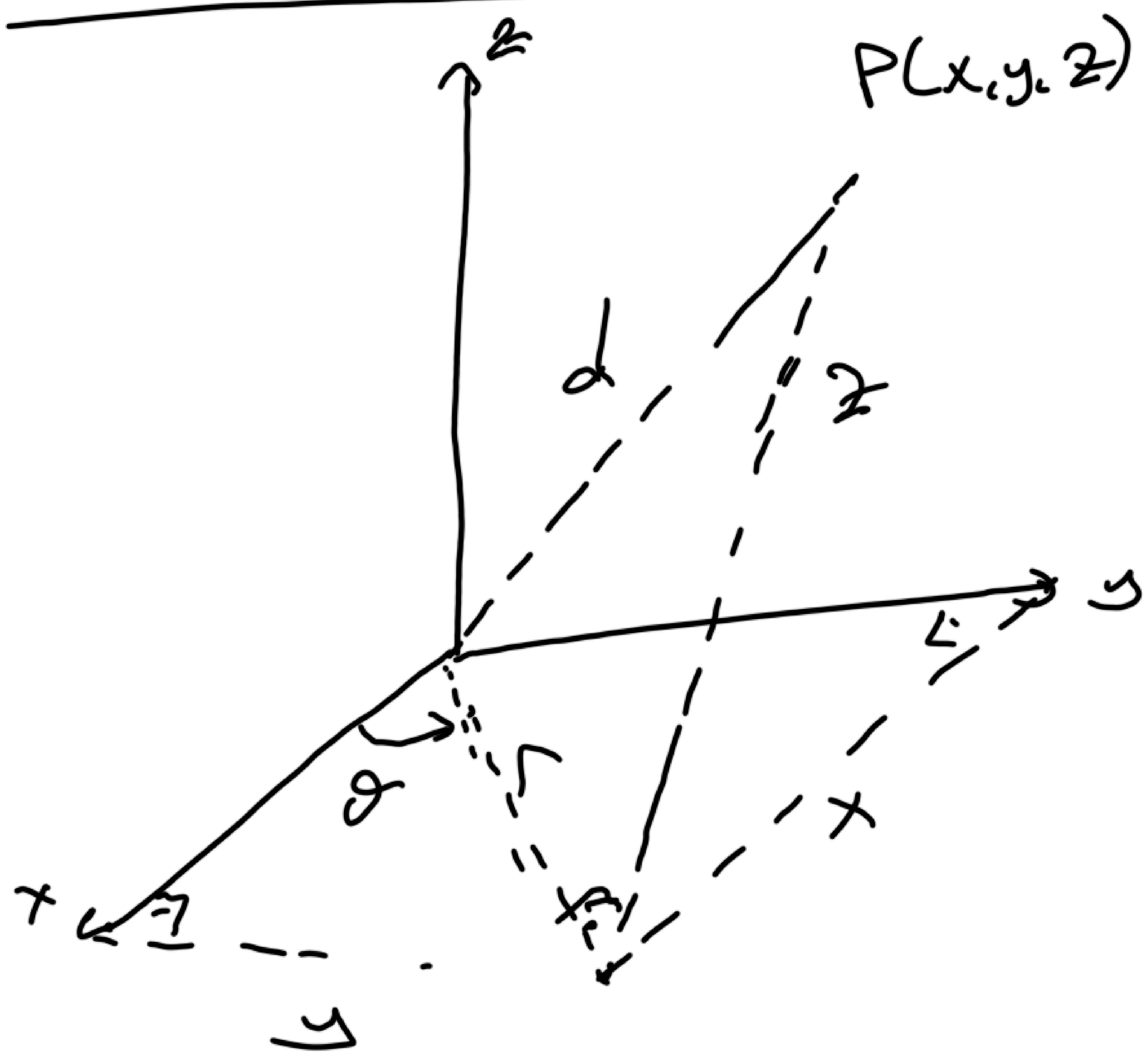


Figure The cylindricsel coordinates of a point

Cylindricsel Coord. Transform

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dV = |J(r,\theta,z)|\,dr\,d\theta\,dz$

$= r\,dr\,d\theta\,dz$

$x^2+y^2=r^2$, $\tan\theta = \frac{y}{x}$, $r \geq 0$, $0 \leq \theta \leq 2\pi$, $-\infty \leq z \leq \infty$

$$(x,y,z) \xrightarrow[J(r,\theta,z)\neq 0]{} (r,\theta,z)$$

$$J(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} =$$

önemli: Kaw

Tekrar Et

$$\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \underbrace{(-1)\cdot(-1)}_{1} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 8$$

Ex Evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\frac{1}{\sqrt{x^2+y^2}}}^{1} dz\,dy\,dx$ by ......

cylindirical coordination

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dV = r\,dr\,d\theta\,dz$

$x^2+y^2=r^2$

$x=-1$ | $y=0$ | $y\geq 0$ | $z=\dfrac{1}{\sqrt{x^2+y^2}} \Rightarrow z=\dfrac{1}{r}$

$x=1$ | $y=\sqrt{1-x^2}$ | $z=1 \Rightarrow z=1$

$x^2+y^2=1$

$x^2+y^2=1$

$y=\pm\sqrt{1-x^2}$

$\theta=0$

$y=0$

$0\leq\theta\leq\pi$

Arka sayfaya

$r\leq 1$

$$I=\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\frac{1}{\sqrt{x^2+r}}}^{1} dz\,dy\,dx=\int_{\theta=0}^{\pi}\int_{r=0}^{1}\int_{z=\frac{1}{r}}^{1} dz$$

$$=\frac{\pi}{12}$$

sonuç bu olmalı

$$y=\pm\sqrt{1-x^2}$$

$0\leq\theta\leq\pi$ $\qquad = x^2+y^2=1$

$0\leq r\leq 1$ $\qquad r^2=1 \qquad r\neq-1 \qquad r=1$

Bir de bunun grafiği var. ...

üşendim çizmeye.