Linear Partial Differential Equations

Semi-Linear Partial Differential Equations (First Order)

Quasi-Linear Partial Differential Equations (First Order)

Non-Linear Partial Differential Equations (First Order)

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u)$$

$$a(x,y,u) + b(x,y,u) = c(x,y,u)$$

$$f(x,y,u,u_x,u_y) = 0$$

characteristic equations for

$$\frac{dx}{dt} = f_p, \quad \frac{dy}{dt} = f_q, \quad \frac{du}{dt} = p f_p + q f_q$$

$$\frac{dp}{dt} = -f_x - p f_u, \quad \frac{dq}{dt} = -f_y - q f_u,$$

$$p := u_x = \frac{\partial u}{\partial x}, \quad q := u_y = \frac{\partial u}{\partial y}$$

Non-Linear PDE

How get PDE from first order equation?

$$z_x = \frac{\partial z}{dx} = p$$
$$z_y = \frac{dz}{dy} = q$$

Example-1

$2z = (ax+y)^2 + b$ Let us ~~get~~ get PDE

$\cancel{2}.z_x = \cancel{2}a(ax+y) \rightarrow$

$\cancel{2}.z_y = \cancel{2}(ax+y)$

$z_y = ax+y$

$a = \frac{z_y - y}{x}$

I $z_x = a.(ax+y)$

II $z_x = a.z_y$

III $z_x = \left(\frac{z_y - y}{x}\right).z_y$

IV $z_x = \frac{z_y^2 - y.z_y}{x}$

V $x.z_x - z_y^2 + y.z_y = 0$

VI $px - q^2 + qy = 0$

non-linear equation

Example 2 $\quad z = xy + f(x^2+y^2)$

I $\quad z_x = y + f' \cdot 2x$

II $\quad z_y = x + f' \cdot 2y$

$$f' = \frac{z_y - x}{2y}$$

I $\quad z_x = y + \frac{z_y - x}{\cancel{2}y} \cdot \cancel{2}x$

II. $\quad z_x \cdot y = y^2 + (z_y - x) \cdot x$

III $\quad z_x \cdot y = y^2 + z_y x - x^2$

IV $\quad z_x \cdot y - z_y \cdot x = y^2 - x^2$

Example 1

Let us consider the following equation

$$2u = (ax + y)^2 + b$$

$$2U_x = 2.(ax+y).a$$

$$\Rightarrow \cancel{2}U_x = \cancel{2}.a.U_y$$

$$\Rightarrow U_x = a.\underline{U_y}$$

$$2u_y = 2.(ax+y)$$

$$(ax+y) = \frac{\cancel{2}u_y}{2}$$

Example - 2

Consider the following equation

$$u = xy + \varphi(x^2+y^2)$$

x/ $$\frac{\partial u}{\partial x} = y + \varphi' \cdot 2x$$

y/ $$\frac{\partial u}{\partial y} = x + \varphi' \cdot 2y$$

U_y

$$\frac{U_y - x}{2y} = \varphi'$$

$$\frac{\partial u}{\partial x} = y + 2x \cdot \frac{\frac{\partial u}{\partial y} - x}{2y}$$

$$\frac{\partial u}{\partial x} \cdot y = y^2 + \frac{\partial u}{\partial y} \cdot x - x^2$$

$$y \cdot \frac{\partial u}{\partial x} - x \cdot \frac{\partial u}{\partial y} = y^2 - x^2$$

Example-3

Consider the relation

$$\varphi(x^2+y^2+u^2,\, u^2-2xy)=0$$

$$\boxed{f(k,L)=0}$$

$$x^2+y^2+u^2=k \qquad u^2-2xy=L$$

$x/$ $$\frac{\partial\phi}{\partial k_x}\cdot\frac{\partial k_x}{\partial x}+\frac{\partial\phi}{\partial L_x}\cdot\frac{\partial L_x}{\partial \cancel{x}}=0$$

y $$\frac{\partial\phi}{\partial k_y}\cdot\frac{\partial k}{\partial x}+\frac{\partial\phi}{\partial L}\cdot\frac{\partial L}{\partial x}=0$$

$$\begin{vmatrix} \frac{\partial\phi}{\partial k}\cdot 2x - \frac{\partial\phi}{\partial L}\cdot 2y \\ \frac{\partial\phi}{\partial k}\cdot 2x - \frac{\partial\phi}{\partial L}\cdot 2y \end{vmatrix}=0$$

Semi-linear or Quasi-linear PDE