Surfaces and Area

We have defined curves in the plane in three different forms:

✓ Explicit form: $y=f(x)$ $(x=f(y))$

✓ Implicit form: $F(x,y)=0$

✓ Parametric vector form: $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$, $a\le t\le b$.

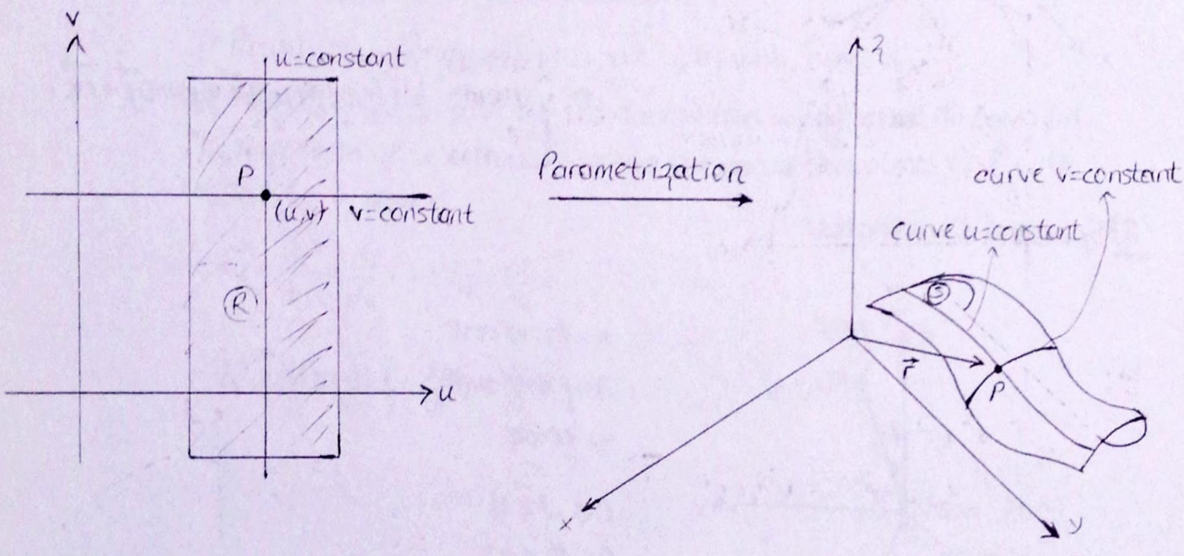
We have analogous definitions of surfaces in space:

✓ Explicit form: $z=f(x,y)$

✓ Implicit form: $F(x,y,z)=0$

Parametrization of Surfaces

Suppose $\vec{r}(u,v)=f(u,v)\vec{i}+g(u,v)\vec{j}+h(u,v)\vec{k}$ (1) is a continuous vector function that is defined on a region R in the uv-plane and one-to-one interior of R (Fig 1)



(Figure 1)

Equation 1 and the domain R constitute a parametrization of the surface. The variables u and v are the parameters, and R is the parameter domain we take R to be a rectangle defined by the inequalities of the form $a\le u\le b$, $c\le v\le d$. The requirement that $\vec{r}$ be one to one on the interior of R ensures that S does not cross itself.

$$x=f(u,v),\quad y=g(u,v),\quad z=h(u,v).$$

Examples

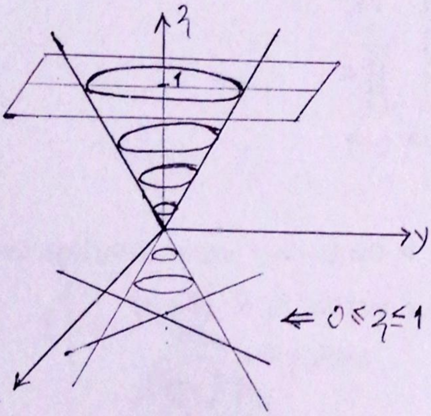
1) Find a parametrization of the cone $z=\sqrt{x^2+y^2}$, $0\leq z\leq 1$.

2) Find a parametrization of the sphere $x^2+y^2+z^2=a^2$.

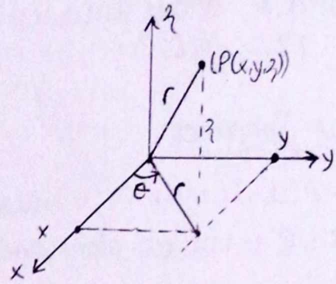
3) Find a parametrization of the cylinder $x^2+(y-3)^2=9$, $0\leq z\leq 5$.

Solutions

1)



Cylindrical Coordinates:



$x=r\cos\theta$

$y=r\sin\theta$

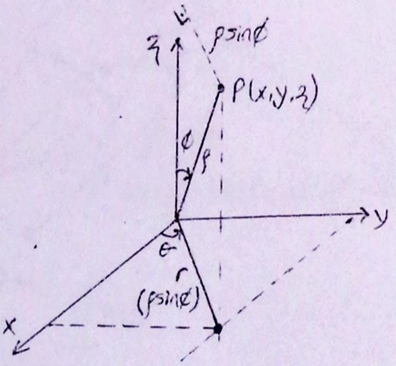
$z=z$

$0\leq\theta\leq 2\pi$

$0\leq r\leq 1$

$$\Rightarrow \left.\begin{aligned} x&=r\cos\theta \\ y&=r\sin\theta \\ z&=r \end{aligned}\right\} \vec{r}(r,\theta)=r\cos\theta\,\vec{i}+r\sin\theta\,\vec{j}+r\vec{k}.$$

2) Spherical Coordinates:



$x=\rho\sin\phi\cos\theta$

$y=\rho\sin\phi\sin\theta$ $\{r=\rho\sin\phi\}$

$z=\rho\cos\phi$

$0\leq\phi\leq\pi$

$0\leq\theta\leq 2\pi$

$-\infty<z<\infty$

$$\Rightarrow \begin{aligned} x&=r\cos\theta \\ y&=r\sin\theta \\ z&=\rho\cos\phi \end{aligned} \quad \underset{(\rho=a)}{\Rightarrow} \quad \begin{aligned} x&=a\sin\phi\cos\theta \\ y&=a\sin\phi\sin\theta \\ z&=a\cos\phi \end{aligned} \Rightarrow \vec{r}(\phi,\theta)=a\sin\phi\cos\theta\,\vec{i}+a\sin\theta\sin\phi\,\vec{j}+a\cos\phi\,\vec{k}.$$

3) Cylindrical Coordinates:

$$x = r\cos\theta \qquad x^2+(y-3)^2=9 \Rightarrow x^2+y^2-6y+9=9$$
$$y = r\sin\theta \Rightarrow \qquad x^2+y^2=6y \qquad \Rightarrow r=0,\ r=6\sin\theta$$
$$z=z \qquad r^2=6r\sin\theta$$

$$\Rightarrow \begin{array}{l} x=6\sin\theta\cos\theta \\ y=6\sin^2\theta \\ z=z \end{array} \Rightarrow \vec{r}(\theta,z)=6\sin\theta\cos\theta\,\vec{i}+6\sin^2\theta\,\vec{j}+z\vec{k}.$$

$0\leq\theta\leq 2\pi$

$0\leq z\leq 5$

Definition: Surface Area

Our goal is to find a double integral for calculating the area of a curved surface S based on the parametrization;

$$\vec{r}(u,v)=f(u,v)\vec{i}+g(u,v)\vec{j}+h(u,v)\vec{k} \quad , a\leq u\leq b,\ c\leq v\leq d.$$

We need S to be smooth for the construction we are about to carry out. The definition of smoothness involves the partial derivatives of $\vec{r}$ with respect to u and v.

$$\vec{r_u}=\frac{d\vec{r}}{du}=\frac{\partial f}{\partial u}\vec{i}+\frac{\partial g}{\partial u}\vec{j}+\frac{\partial h}{\partial u}\vec{k},$$

$$\vec{r_v}=\frac{d\vec{r}}{dv}=\frac{\partial f}{\partial v}\vec{i}+\frac{\partial g}{\partial v}\vec{j}+\frac{\partial h}{\partial v}\vec{k}.$$

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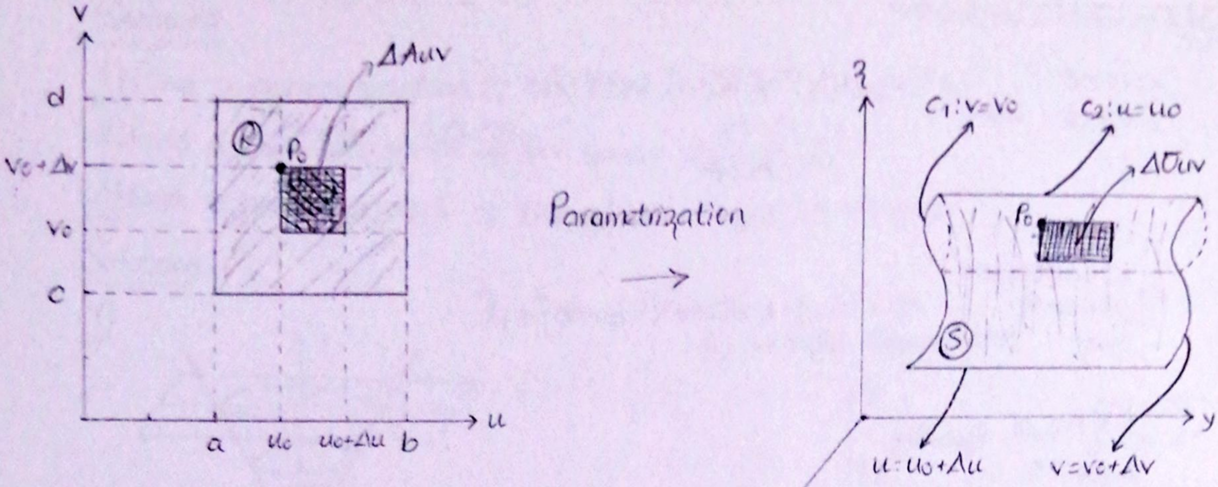
Definition: A parametrized surface $\vec{r}(u,v)=f(u,v)\vec{i}+g(u,v)\vec{j}+h(u,v)\vec{k}$ is smooth if $\vec{r_u}$ and $\vec{r_v}$ are continuous and $\vec{r_u}\times\vec{r_v}$ is never the zero vector in the definition of smoothness mean that the two vectors $\vec{r_u}$ and $\vec{r_v}$ are nonzero and never lie along the same line, so they always determine a plane tangent to the surface.

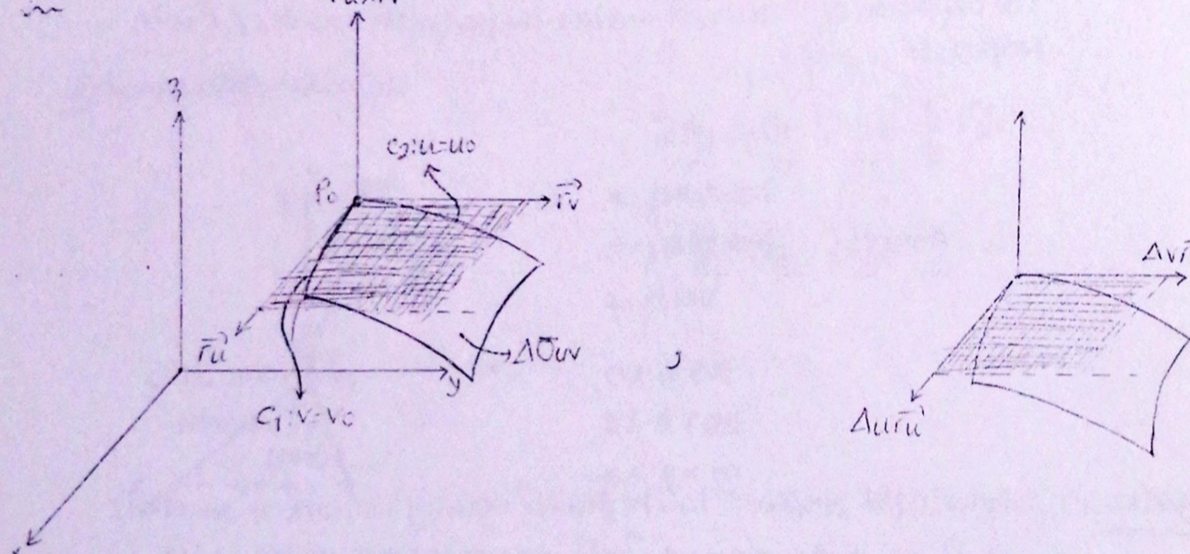
Now, consider a small rectangle ΔA_{uv} in R with sides on the lines $u=u_0$, $u=u_0+\Delta u$, $v=v_0$ and $v=v_0+\Delta v$. (Fig 2). Each side of ΔA_{uv} maps to a curve on the surface S and together these four curves bound a "curved patch element" $\Delta\sigma_{uv}$. In figure, the side $v=v_0$ maps to curve c_1, the side $u=u_0$ maps to c_2 and (u_0,v_0) maps to P_0.



(Fig. 2)

$$\left(\vec{r}(u,v)=f(u,v)\vec{i}+g(u,v)\vec{j}+h(u,v)\vec{k}\right)$$

*



(Magnified view of $\Delta\sigma_{uv}$)

The vecters $\Delta u\vec{r_u}$ and $\Delta v\vec{r_v}$ is defined to be the area of the surface patch element $\Delta\sigma_{uv}$ ($\vec{r_u}\times\vec{r_v}\neq 0$, (Smoothness of S))

* $|\Delta u\vec{r_u}\times\Delta v\vec{r_v}|=|\vec{r_u}\times\vec{r_v}|\Delta u.\Delta v \Rightarrow$ Sum these areas together to obtain an approximation of the surface area of S.

If $n\to\infty$ $(\Delta u,\Delta v\to 0)\Rightarrow$

$$\sum_n|\vec{r_u}\times\vec{r_v}|\Delta u\Delta v=\int_{v=c}^{d}\int_{u=a}^{b}|\vec{r_u}\times\vec{r_v}|\,du\,dv.$$

Definition: The area of the smooth surface,

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}, \quad a \le u \le b,\ c \le v \le d;$$

$$A = \iint_R |\vec{r_u} \times \vec{r_v}|\, dA = \iint_R |\vec{r_u} \times \vec{r_v}|\, du\, dv$$

Surface Area Differential for a Parametrized Surface

$$d\sigma = ds = |\vec{r_u} \times \vec{r_v}|\, du\, dv$$

EX! Find the surface area of the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$

Cylindrical Coordinates;

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{matrix}, \quad \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix} \quad \begin{pmatrix} r = u \\ \theta = v \end{pmatrix}$$

$$\Rightarrow \vec{r}(r,\theta) = r\cos\theta\vec{i} + r\sin\theta\vec{j} + r\vec{k}$$

$$\left.\begin{matrix} \dfrac{\partial\vec{r}}{\partial r} = \vec{r_r} = \cos\theta\vec{i} + \sin\theta\vec{j} + \vec{k} \\ \dfrac{\partial\vec{r}}{\partial\theta} = \vec{r_\theta} = -r\sin\theta\vec{i} + r\cos\theta\vec{j} + 0\vec{k} \end{matrix}\right\} \vec{r_r} \times \vec{r_\theta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\vec{i} - r\sin\theta\vec{j} + r\vec{k}$$

$$|\vec{r_r} \times \vec{r_\theta}| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = r\sqrt{2}$$

$$d\sigma = ds = |\vec{r_r} \times \vec{r_\theta}|\, dr\, d\theta = r\sqrt{2}\, dr\, d\theta$$

$$\Rightarrow A = \iint_R (\vec{r_r} \times \vec{r_\theta})\, dr\, d\theta = \iint_R r\sqrt{2}\, dr\, d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\sqrt{2}\, dr\, d\theta = \sqrt{2}\pi \text{ unitsquared.}$$

EX: Find the surface area of sphere of radius a.

Spherical Coordinates;

$$\begin{cases} \rho = a \\ x = a\sin\phi\cos\theta \\ y = a\sin\phi\sin\theta \\ z = a\cos\phi \end{cases}, \quad \begin{matrix} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\vec{i} + a\sin\phi\sin\theta\,\vec{j} + a\cos\phi\,\vec{k}$$

$$\Rightarrow \quad \vec{r_\theta} = -a\sin\phi\sin\theta\,\vec{i} + a\sin\phi\cos\theta\,\vec{j}$$

$$\vec{r_\phi} = a\cos\phi\cos\theta\,\vec{i} + a\cos\phi\sin\theta\,\vec{j} - a\sin\phi\,\vec{k}$$

$$\vec{r_\theta} \times \vec{r_\phi} = a^2\sin^2\phi\cos\theta\,\vec{i} + a^2\sin^2\phi\sin\theta\,\vec{j} - a^2\sin\phi\cos\phi\,\vec{k}$$

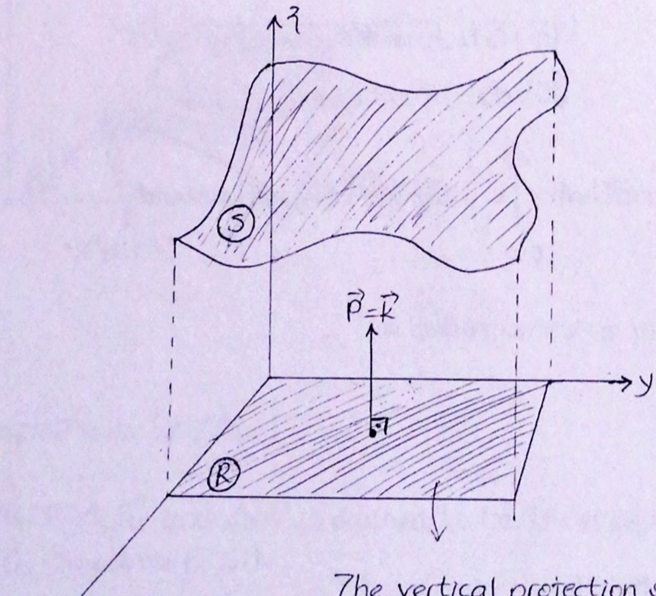
$$|\vec{r_\theta} \times \vec{r_\phi}| = \sqrt{a^4\sin^2\phi} = a^2|\sin\phi| = a^2\sin\phi$$

$$\Rightarrow A(S) = \iint_S dS = \int_{\theta=0}^{2\pi}\left(\int_{\phi=0}^{\pi} a^2\sin\phi\, d\phi\right) d\theta = 4\pi a^2$$

Implicit Surfaces

We now show how to compute the surface area differentiable $d\sigma$ for implicit surfaces. The surface is defined by the equation $F(x,y,z)=c$ and $\vec{p}$ is a unit vector normal to the plane region R. We assume that the surface is smooth (F is differentiable and ∇F is nonzero and continuous on S) and that $\nabla f.\vec{p} \neq 0$, so the surface never folds back over itself. Assume that the normal vector $\vec{p}$ is the unit vector $\vec{k}$, so the region R in figure 1 lies in the xy-plane.

By assumption, we then have $\nabla f.\vec{p} = \nabla f.\vec{k} = F_z \neq 0$ on S. The Implicit Function Theorem implies that S is then the graph of a differentiable function $z=h(x,y)$, although the function $h(x,y)$ is not explicitly known.



The vertical projection or "shadow" of S on a coordinate plane.

Define parameters u and v by $u=x$, $v=y$ and then $z=h(u,v)$;

$$\Rightarrow \vec{r}(u,v) = u\vec{i} + v\vec{j} + h(u,v)\vec{k}$$

$$\vec{r_u} = \vec{i} + \frac{\partial(h(u,v))}{\partial u}\vec{k}$$

$$\vec{r_v} = \vec{j} + \frac{\partial(h(u,v))}{\partial v}\vec{k}$$

Applying the chain rule to $F(x,y,z)=c$ where $u=x$, $v=y$ and $z=h(u,v)$. We obtain the partial derivatives;

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial u}}_{1} + \frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial u}}_{0} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial u} = 0 \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial(h(u,v))}{\partial u} = -\frac{F_x}{F_z}$$

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial v}}_{0} + \frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial v}}_{1} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial v} = 0 \Rightarrow \frac{\partial z}{\partial v} = \frac{\partial(h(u,v))}{\partial v} = -\frac{F_y}{F_z}$$

$$\begin{tikzcd} & z \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ u & & v \end{tikzcd}$$

$$\Rightarrow \left.\begin{aligned} \vec{r_u} &= \vec{i} - \frac{F_x}{F_z}\vec{k} \\ \vec{r_v} &= \vec{j} - \frac{F_y}{F_z}\vec{k} \end{aligned}\right\} \vec{r_u}\times\vec{r_v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{F_x}{F_z} \\ 0 & 1 & -\frac{F_y}{F_z} \end{vmatrix} = \frac{F_x}{F_z}\vec{i} + \frac{F_y}{F_z}\vec{j} + \vec{k} = \frac{1}{F_z}\underbrace{\left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right)}_{\nabla F} = \frac{\nabla F}{\nabla F\cdot\vec{k}}$$

$$\Rightarrow |\vec{r_u}\times\vec{r_v}| = \frac{|\nabla F|}{|\nabla F\cdot\vec{k}|}.$$

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Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x,y,z)=c$ over a closed and bounded plane region R is,

$$\text{Surface Area: } A(S) = \iint_R \frac{|\nabla F|}{|\nabla F\cdot\vec{p}|}\cdot dA$$

where $\vec{p} = \vec{i}, \vec{j}, \vec{k}$ is normal to R and $\nabla F\cdot\vec{p} \neq 0$.

$$\vec{P}=\vec{i} \Rightarrow A(S)=\iint_{R_{yz}} \frac{|\nabla F|}{|\nabla F.\vec{i}|}\,dy\,dz$$

$$\vec{P}=\vec{j} \Rightarrow A(S)=\iint_{R_{xz}} \frac{|\nabla F|}{|\nabla F.\vec{j}|}\,dx\,dz$$

$$\vec{P}=\vec{k} \Rightarrow A(S)=\iint_{R_{xy}} \frac{|\nabla F|}{|\nabla F.\vec{k}|}\,dx\,dy$$

Formula for the Surface Area of a Graph $z=f(x,y)$

For the graph $z=f(x,y)$ over a region R in the xy-plane, the Surface Area's formula's;

$$A(S)=\iint_{R} \sqrt{f_x^2+f_y^2+1}\,dx\,dy$$

* $z-f(x,y)=0 \Rightarrow F(x,y,z)=0$

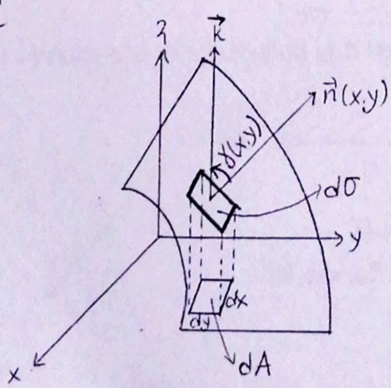
$$\nabla F=\frac{\partial F}{\partial x}\vec{i}+\frac{\partial F}{\partial y}\vec{j}+\frac{\partial F}{\partial z}\vec{k}=-f_x\vec{i}-f_y\vec{j}+\vec{k} \Rightarrow |\nabla F|=\sqrt{f_x^2+f_y^2+1}$$

$$\nabla F.\vec{k}=1 \Rightarrow |\nabla F.\vec{k}|=1$$

$$\Rightarrow A(S)=\iint_{R_{xy}} \frac{\sqrt{f_x^2+f_y^2+1}}{1}\,dx\,dy=\iint_{R_{xy}} \sqrt{f_x^2+f_y^2+1}\,dx\,dy \quad ✓$$

The unit normal vector: $\vec{n}=\dfrac{\nabla F}{|\nabla F|}=\dfrac{-f_x\vec{i}-f_y\vec{j}+k}{\sqrt{f_x^2+f_y^2+1}}$ $\left\{\begin{matrix}\text{upper unit}\\ \text{normal vector}\end{matrix}\right\}$

Remark:



$$\vec{n}.\vec{k}=\underbrace{|\vec{n}|}_{1}.\underbrace{|\vec{k}|}_{1}.\cos\gamma(x,y)=\cos\gamma(x,y)=\frac{1}{\sqrt{f_x^2+f_y^2+1}}$$

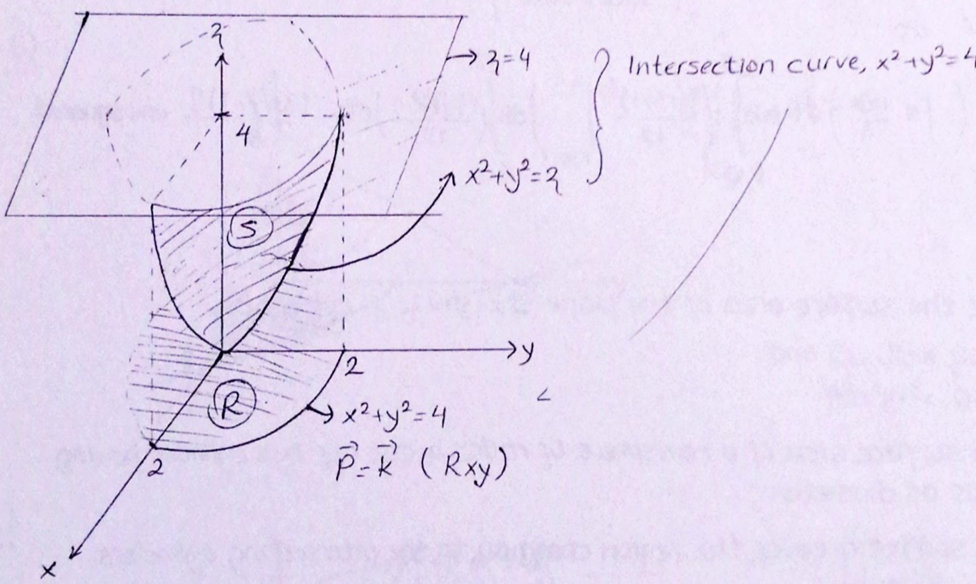
$$\Rightarrow \sec\gamma(x,y)=\sqrt{f_x^2+f_y^2+1} \quad ✓$$

$$A(S)=\iint_{R_{xy}} |\sec\gamma(x,y)|\,dx\,dy=\iint_{R_{xy}} \frac{dA}{|\vec{n}.\vec{k}|}$$

$$A(S)=\iint_{R_{xz}} |\sec\alpha(x,z)|\,dx\,dz=\iint_{R_{xz}} \frac{dA}{|\vec{n}.\vec{j}|} \quad \left\{\sec\alpha(x,z)=\sqrt{f_x^2+f_z^2+1}\right\}$$

$$A(S)=\iint_{R_{yz}} |\sec\beta(y,z)|\,dy\,dz=\iint_{R_{yz}} \frac{dA}{|\vec{n}.\vec{i}|} \quad \left\{\sec\beta(y,z)=\sqrt{f_y^2+f_z^2+1}\right\}$$

EX: Find the area of the surface cut from the bottom of the paraboloid $x^2+y^2-z=0$ by the plane $z=4$.



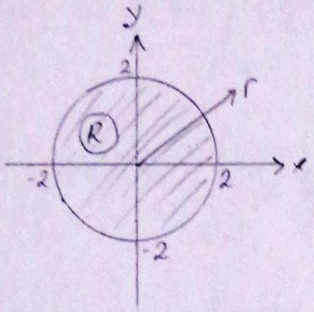
$\vec{P}=\vec{k}$ (R_{xy})

$\nabla F=2x\vec{i}+2y\vec{j}-\vec{k} \Rightarrow |\nabla F|=\sqrt{4x^2+4y^2+1}$

$\nabla F.\vec{k}=-1 \Rightarrow |\nabla F.\vec{k}|=1$

$$\Rightarrow A(S)=\iint_{R_{xy}} \frac{|\nabla F|}{|\nabla F.\vec{k}|}\,dx\,dy$$

$$=\iint_{R_{xy}} \sqrt{4x^2+4y^2+1}\,dx\,dy$$



$x = r\cos\theta$

$y = r\sin\theta$

$dA = dx\,dy$

$\quad = |J(r,\theta)|\,dr\,d\theta$

$\quad = r\,dr\,d\theta$

$0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$

$\Rightarrow$

$$A(S) = \iint_{R_{r\theta}} \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\int_{r=0}^{2} \sqrt{4r^2+1}\; r\,dr \right) d\theta \quad \left\{ \begin{array}{l} 4r^2+1 = u^2 \\ 8r\,dr = 2u\,du \\ 4r\,dr = u\,du \end{array} \right\}$$

$$= \int_{\theta=0}^{2\pi} \left(\int u \cdot \frac{u\,du}{4} \right) d\theta = \int_{\theta=0}^{2\pi} \left(\frac{(4r^2+1)^{3/2}}{12} \Bigg|_{r=0}^{2} \right) d\theta = \int_{\theta=0}^{2\pi} \left(\frac{17\sqrt{17}-1}{12} \right) d\theta = \frac{(17\sqrt{17}-1)\pi}{6} \text{ unit squared.}$$

Examples

1) Determine the surface area of the plane $2x+y+2z=16$ cut off by,

   a) $x=0, y=0, x=2, y=3$ and

   b) $x=0, y=0, x^2+y^2=64$.

2) Find the surface area of a hemisphere of radius a cut off by a cylinder having this radius as diameter.

3) Find the surface area of the region common to the intersecting cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

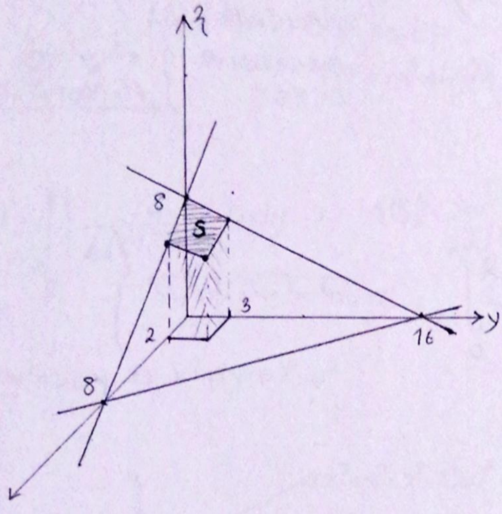
4) Find the area of that part of the cylinder $x^2+y^2=2ay$ that lies inside the sphere $x^2+y^2+z^2=4a^2$.

5) Find the surface area of the paraboloid $x^2+y^2-z=0$ cut off by the planes $z=2$ and $z=6$.

6) Find the surface area of that part of the sphere $x^2+y^2+z^2=4a^2$ that lies inside the cylinder $x^2+y^2=2ay$.

Solutions

1) a)



$\nabla F = 2\vec{i} + \vec{j} + 2\vec{k}$

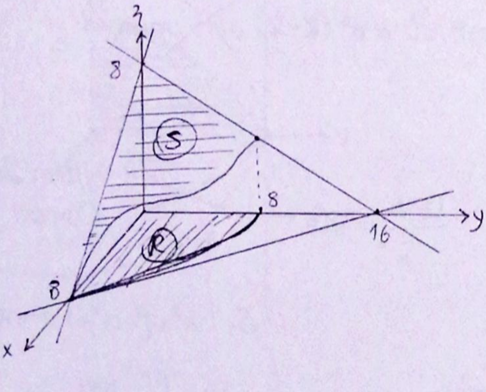
$|\nabla F| = \sqrt{4+1+4} = 3$

$\nabla F \cdot \vec{k} = 2$

$|\nabla F \cdot \vec{k}| = 2$

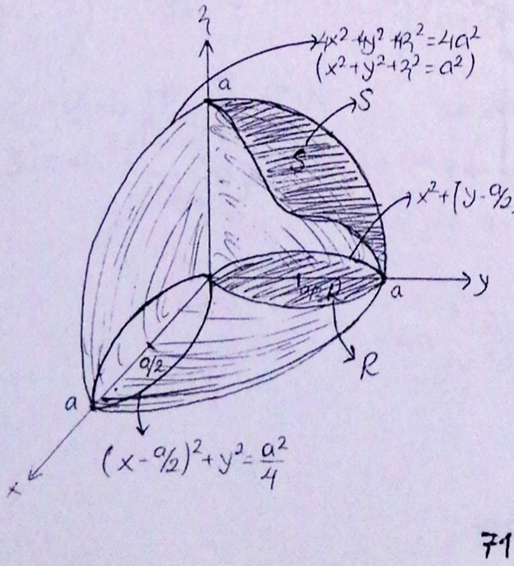
$$\Rightarrow A(S) = \iint_{R_{xy}} \frac{3}{2}\,dx\,dy = \int_{y=0}^{3} \left(\frac{3}{2} \int_{x=0}^{2} dx \right) dy = 9. \checkmark$$

b)



$$\frac{3}{2} \iint_{R_{xy}} dA = 24\pi. \checkmark$$

2)



$$A(S) = \iint_{R_{xy}} \frac{dA}{|\vec{n} \cdot \vec{k}|}$$

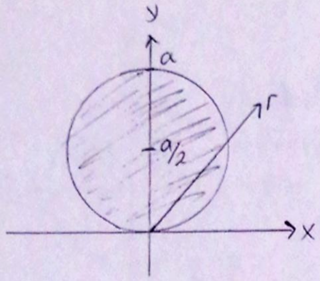
$\nabla F = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$|\nabla F| = \sqrt{4x^2+4y^2+4z^2} = 2a$

$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}$

$\vec{n} \cdot \vec{k} = \frac{z}{a}$

$|\vec{n} \cdot \vec{k}| = \frac{z}{a} \quad (z \geq 0)$



$x^2+(y-a/2)^2=\frac{a^2}{4}$

$\Rightarrow x^2+y^2=ay \Rightarrow$

$x=r\cos\theta$

$y=r\sin\theta$

$dA=r\,dr\,d\theta$

$0\leq r\leq a\sin\theta \quad \left\{ x^2+y^2=ay,\ r^2=ra\sin\theta,\ r=0,\ r=a\sin\theta \right\}$

$0\leq\theta\leq\pi$

$$A(S)=2\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta}\frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta \qquad \left\{ \int_0^{\pi}=2\int_0^{\pi/2},\ z=\sqrt{a^2-x^2-y^2}=\sqrt{a^2-r^2} \right\}$$

$$\left\{ \begin{array}{l} a^2-r^2=u^2 \\ -2r\,dr=2u\,du \\ -r\,dr=u\,du \end{array} \right\}$$

$$A(S)=2a\int_{\theta=0}^{\pi/2}\left(\int-\frac{u\,du}{u}\right)d\theta=2a\int_{\theta=0}^{\pi/2}a(1-\cos\theta)\,d\theta=a^2(\pi-2) \text{ unit squared.}$$

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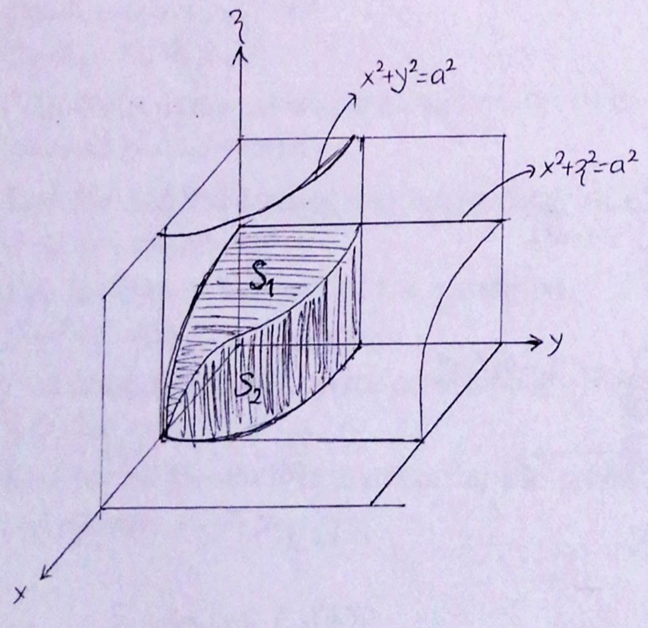
Cuma

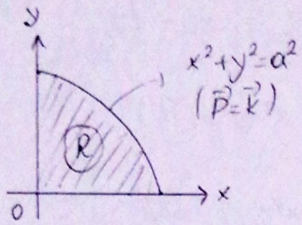
3)



$S_1: x^2+z^2=a^2 \Rightarrow x^2+z^2-a^2=0$

$$A(S)=\iint\frac{|\nabla F|}{|\nabla F\cdot\vec{p}|}\,dA$$

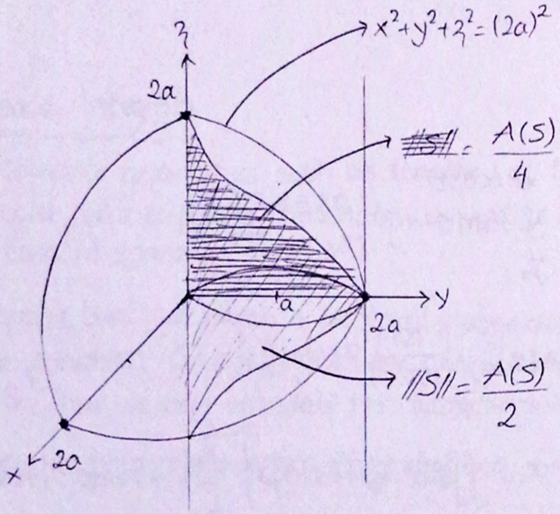
$$\frac{A(S)}{8}=\iint_{R_{xy}}\frac{|\nabla F|}{|\nabla F\cdot\vec{k}|}\,dx\,dy$$



$$\nabla F = 2x\vec{i} + 2z\vec{k}$$
$$|\nabla F| = \sqrt{4x^2+4z^2} = 2a$$
$$\nabla F.\vec{k} = 2z$$
$$|\nabla F.\vec{k}| = |2z| = 2z = 2\sqrt{a^2-x^2} \quad (z \geq 0)$$

$$\Rightarrow \frac{A(S)}{8} = \iint_R \frac{2a}{2\sqrt{a^2-x^2}}\,dxdy \Rightarrow A(S) = 8a\iint_R \frac{dxdy}{\sqrt{a^2-x^2}} = 8a\int_{x=0}^{a}\left(\int_{y=0}^{\sqrt{a^2-x^2}} \frac{dy}{\sqrt{a^2-x^2}}\right)dx = 8a^2. \checkmark$$

4) $x^2+y^2=2ay \Rightarrow x^2+(y-0)^2=a^2$



$$x^2+y^2-2ay=0=F$$
$$\nabla F = 2x\vec{i} + (2y-2a)\vec{j}$$
$$|\nabla F| = 2\sqrt{x^2+(y-a)^2}$$
$$\nabla F.\vec{i} = 2x$$
$$|\nabla F.\vec{i}| = |2x| = 2x \quad (x \geq 0)$$
$$\|S\| = \frac{A(S)}{4} = \iint_R \frac{|\nabla F|}{|\nabla F.\vec{i}|}\,dydz$$

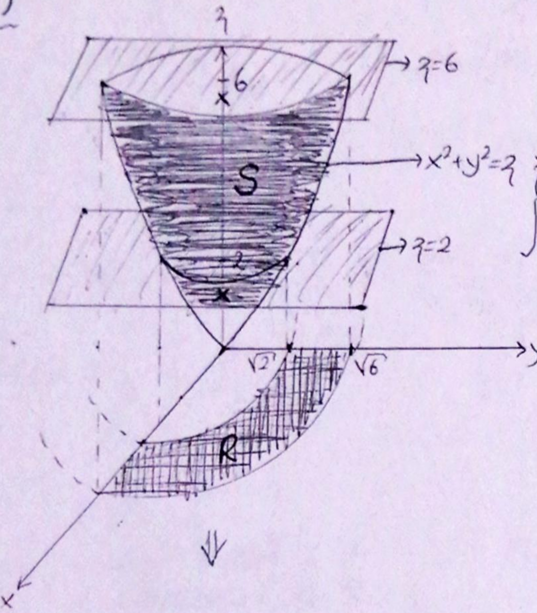
$$\Rightarrow \frac{A(S)}{4} = \iint_R \frac{2\sqrt{x^2+(y-a)^2}}{2x}\,dydz \qquad \left\{\begin{array}{l} x = \sqrt{2ay-y^2} \\ a = \sqrt{x^2+(y-a)^2} \end{array}\right\}$$

$$\Rightarrow A(S) = 4\iint_R \frac{a}{\sqrt{2ay-y^2}}\,dydz = 4a\int_{y=0}^{2a}\left(\int_{z=0}^{\sqrt{4a^2-2ay}} \frac{dz}{\sqrt{2ay-y^2}}\right)dy \qquad \left\{\begin{array}{l} x^2+y^2-2ay=0 \\ x^2+y^2+z^2=(2a^2) \\ \text{Intersection;} \\ z^2=4a^2-2ay \end{array}\right\}$$

$$= 4a\int_0^{2a}\sqrt{\frac{4a^2-2ay}{2ay-y^2}} = 4a\int_0^{2a}\sqrt{\frac{2a(2a-y)}{y(2a-y)}} = 4a\int_0^{2a}\sqrt{\frac{2a}{y}}\,dy$$

$$= 16a^2. \checkmark$$

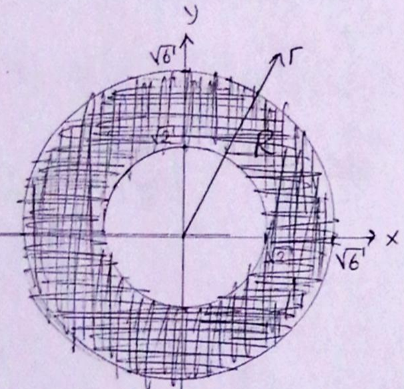
5)



Intersection → $x^2+y^2=(\sqrt{6})^2$

Intersection → $x^2+y^2=(\sqrt{2})^2$

⇓



$x=r\cos\theta$

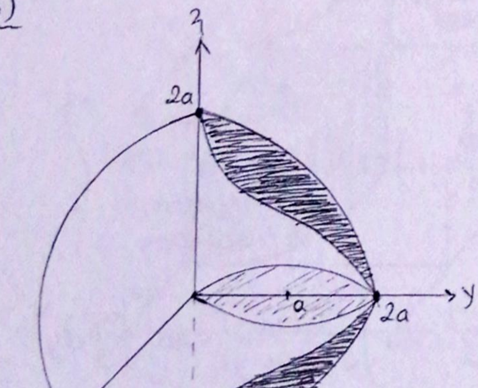
$y=r\sin\theta$, $0\leq\theta\leq 2\pi$

$J=r$, $\sqrt{2}\leq r\leq\sqrt{6}$

$$\Rightarrow A(S)=\iint_R \frac{|\nabla F|}{|\nabla F.\vec{k}|}\,dxdy$$

$$=\iint_R \sqrt{4x^2+4y^2+1}\,dxdy=\int_{\theta=0}^{2\pi}\int_{r=\sqrt{2}}^{\sqrt{6}} \sqrt{4r^2+1}\,r\,dr\,d\theta=\frac{49\pi}{6}.\ ✓$$

6)



$x^2+y^2+z^2-4a^2=0=F$

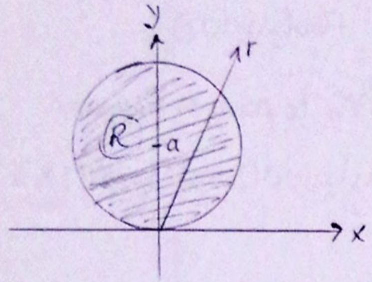
$$\vec{n}=\frac{\nabla F}{|\nabla F|}=\frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2\sqrt{x^2+y^2+z^2}}=\frac{x}{2a}\vec{i}+\frac{y}{2a}\vec{j}+\frac{z}{2a}\vec{k}$$

$$\vec{n}.\vec{k}=\frac{z}{2a}$$

$$|\vec{n}.\vec{k}|=\frac{|z|}{2a}=\frac{z}{2a}\quad (z\geq 0)$$

$$\frac{A(S)}{2}=\iint_R \frac{dA}{|\vec{n}.\vec{k}|}$$

$$\frac{A(S)}{2} = \iint_R \frac{dA}{|\vec{n}\cdot\vec{k}|} = 2a\iint_R \frac{dx\,dy}{\sqrt{4a^2-x^2-y^2}}$$



$x = r\cos\theta$
$y = r\sin\theta$, $0 \le r \le 2a\sin\theta$
$dA = r\,dr\,d\theta$, $0 \le \theta \le \pi$

$$\Rightarrow \frac{A(S)}{2} = 2a\iint_R \frac{dx\,dy}{\sqrt{4a^2-x^2-y^2}} = 2a.2\int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{2a\sin\theta} \frac{r\,dr}{\sqrt{4a^2-r^2}}\right)d\theta \Rightarrow A(S) = 8a^2(\pi-2).$$

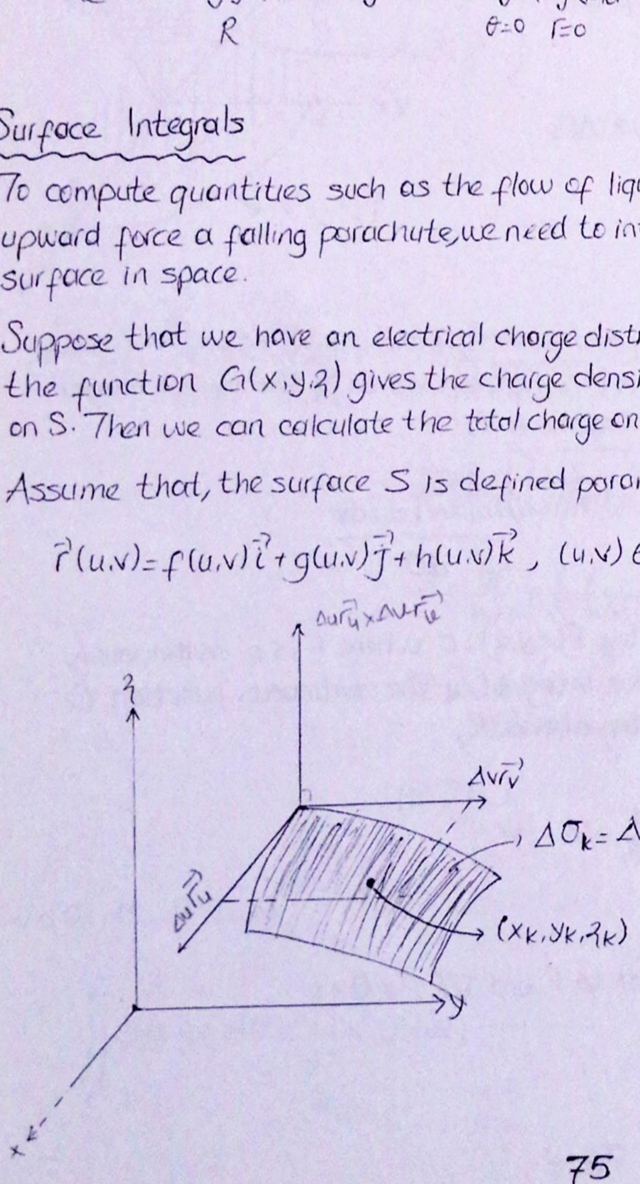
Surface Integrals

To compute quantities such as the flow of liquid across a curved membrane or the upward force a falling parachute, we need to integrate a function over a curved surface in space.

Suppose that we have an electrical charge distributed over a surface S and that the function $G(x,y,z)$ gives the charge density (charge per unit area) at each point on S. Then we can calculate the total charge on S as an integral in the following way.

Assume that, the surface S is defined parametrically on region R in the uv-plane,

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}, \quad (u,v) \in R.$$



The area of the path $\Delta\sigma_k$ is the area of the tangent paralelogram determined by the vectors $\Delta u \vec{r_u}$ and $\Delta v \vec{r_v}$.

Curved surface elements, or patches of area,

$$\Delta\sigma_{uv} \approx |\vec{r_u} \times \vec{r_v}|.\Delta u.\Delta v \approx |\vec{r_u} \times \vec{r_v}| du dv$$

We number the surface element patches $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n$ to form a Riemann sum over S. (the k^{th} patch),

$$\sum_{k=1}^{n} G(x_k, y_k, z_k)\Delta\sigma_k = q$$

Then we take limits as the number of surface patches increases, their areas diminish to zero ($n \to \infty$; $\Delta u \to 0$, $\Delta v \to 0$). The limit defines the surface integral of G over the surface S as,

$$\iint_S G(x,y,z)d\sigma = \lim_{\substack{n \to \infty \\ (\Delta u \to 0 \\ \Delta v \to 0)}} \sum_{k=1}^{n} G(x,y,z)\Delta\sigma_k$$

Formulas for Surface Integrals

1) For a smooth surface S defined parametrically $\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$, $(u,v) \in R$ and a continuous function $G(x,y,z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x,y,z)d\sigma = \iint_R G(f(u,v), g(u,v), h(u,v)) \underbrace{|\vec{r_u} \times \vec{r_v}|.du.dv}_{d\sigma}$$

2) for a surface S given implicitly by $F(x,y,z) = c$ where F is a continuously differentiable function, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,z)\frac{|\nabla F|}{|\nabla F.\vec{P}|} dA$$

where $\vec{P}$ is a unit vector normal to R and $\nabla F.\vec{P} \neq 0$.

3) For a surface S given explicitly as the graph of $z=f(x,y)$, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,z)\sqrt{f_x^2+f_y^2+1}\,dx\,dy$$

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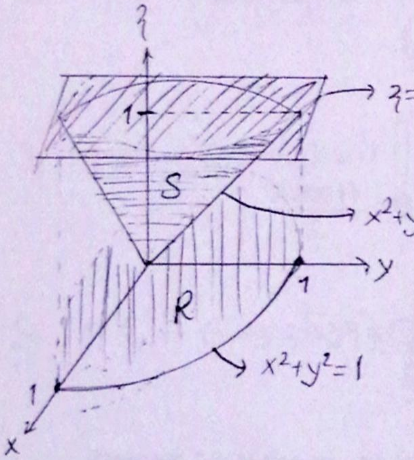
Çarşamba

EX: Integrate $G(x,y,z)=x^2$ over the cone $z=\sqrt{x^2+y^2}$, $0\leq z\leq 1$.



Intersection $\rightarrow x^2+y^2=1$

$$d\sigma = ds = \frac{dA}{|\vec{n}.\vec{p}|} = \frac{dx\,dy}{|\vec{n}.\vec{k}|}, \quad \nabla F = \frac{-2x}{2\sqrt{x^2+y^2}}\vec{i} - \frac{2y}{2\sqrt{x^2+y^2}}\vec{j}+\vec{k}$$

$$|\nabla F| = \sqrt{\left(\frac{-x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{-y}{\sqrt{x^2+y^2}}\right)^2+1^2} = \sqrt{2}$$

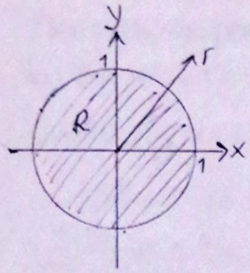
$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{\sqrt{2}}\left(\frac{-x}{\sqrt{x^2+y^2}}\vec{i} - \frac{y}{\sqrt{x^2+y^2}}\vec{j}+\vec{k}\right)$$

$$\vec{n}.\vec{k} = \frac{1}{\sqrt{2}}$$

$$|\vec{n}.\vec{k}| = \frac{1}{\sqrt{2}}$$

$\Rightarrow d\sigma = ds = \sqrt{2}\,dx\,dy$

$$\iint_S G(x,y,z)\,d\sigma = \iint_{R_{xy}} x^2\sqrt{2}\,dx\,dy$$



$$x = r\cos\theta, \quad y = r\sin\theta, \quad dA = r\,dr\,d\theta \qquad , \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

$$\Rightarrow \iint_S G(x,y,z)\,d\sigma = \iint_{R_{xy}} x^2\sqrt{2}\,dx\,dy = \int_{\theta=0}^{2\pi}\Big(\underbrace{\int_{r=0}^{1} (r\cos\theta)^2 \cdot \sqrt{2} \cdot r\,dr}_{\frac{\sqrt{2}}{4}\cos^2\theta}\Big)d\theta$$

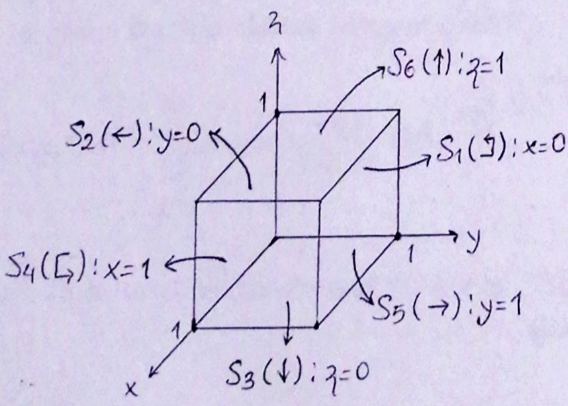
$$= \int_{\theta=0}^{2\pi} \frac{\sqrt{2}}{4} \cdot \cos^2\theta \cdot d\theta \qquad \left\{ \cos^2\theta = \frac{1+\cos 2\theta}{2} \right\}$$

$$= \frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\theta=0}^{2\pi} = \frac{\sqrt{2}}{4}\pi. \checkmark$$

Note: When S is partitioned by smooth curves into a finite number of smooth patches with nonoverlapping interiors then the integral over S is the sum of the the integrals over the patches;

$$\iint_S G(x,y,z)\,d\sigma = \iint_{S_1} G \cdot d\sigma + \iint_{S_2} G \cdot d\sigma + \dots + \iint_{S_n} G \cdot d\sigma$$

EX: Integrate $G(x,y,z) = xyz$ over the surface of the cube cut from the first octant by the planes $x=1$, $y=1$ and $z=1$.



$$\iint_S G(x,y,z)\,d\sigma = \iint_S xyz\,dS = \iint_{S_1} xyz\,dS + \iint_{S_2} xyz\,dS + \dots + \iint_{S_6} xyz\,dS$$

$$\Rightarrow S_1 : x=0 \Rightarrow G(x,y,z)=0 \Rightarrow \iint_{S_1} 0\,d\sigma = \iint_{R_{yz}} 0.\,dy\,dz = 0 \quad \{\vec{n}=\vec{i}\}$$

$$S_2 : y=0 \Rightarrow G(x,y,z)=0 \Rightarrow \iint_{S_2} 0.\,d\sigma = \iint_{R_{xz}} 0.\,dx\,dz = 0 \quad \{\vec{n}=\vec{j}\}$$

$$S_3 : z=0 \Rightarrow G(x,y,z)=0 \Rightarrow \iint_{S_3} 0.\,d\sigma = \iint_{R_{xy}} 0.\,dx\,dy = 0 \quad \{\vec{n}=\vec{k}\}$$

$$S_4 : x=1 \Rightarrow G(x,y,z)=yz \Rightarrow \iint_{S_4} yz\,d\sigma = \iint_{R_{yz}} yz.\,dy\,dz = \int_{z=0}^{1}\left(\int_{y=0}^{1} yz\,dy\right)dz = \frac{1}{4}$$

$$S_5 : y=1 \Rightarrow G(x,y,z)=xz \Rightarrow \iint_{S_5} xz\,d\sigma = \iint_{R_{xz}} xz\,dx\,dz = \int_{x=0}^{1}\left(\int_{z=0}^{1} xz\,dz\right)dx = \frac{1}{4}$$

$$S_6 : z=1 \Rightarrow G(x,y,z)=xy \Rightarrow \iint_{S_6} xy\,d\sigma = \iint_{R_{xy}} xy\,dx\,dy = \int_{y=0}^{1}\left(\int_{x=0}^{1} xy\,dx\right)dy = \frac{1}{4}$$

$$\iint_S G(x,y,z)\,d\sigma = \iint_S xyz\,d\sigma = 0+0+0+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{3}{4}. \checkmark$$

Orientation

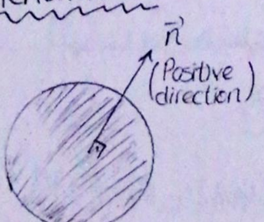
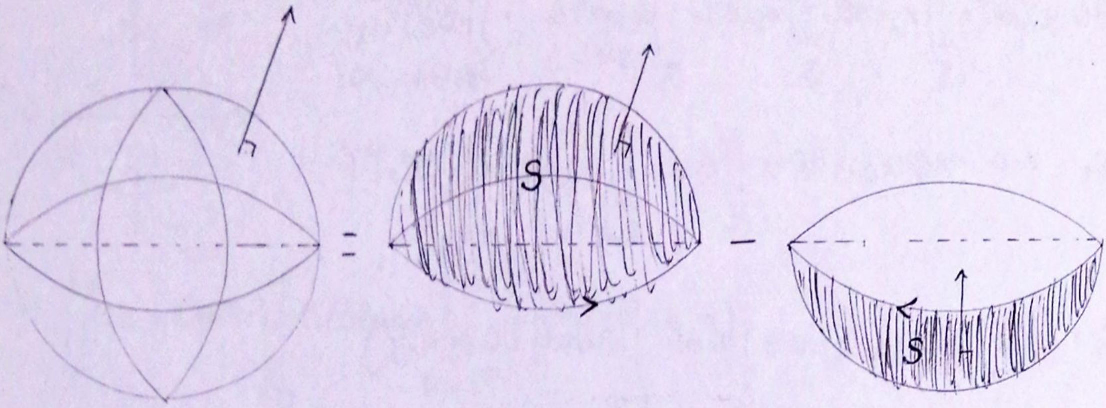


Fig. 1

The outward unit normal vector defines the positive direction at each points and spheres and other smooth closed surfaces in space are orientable. We choose on $\vec{n}$ on a closed surface to point outward. (Fig. 1)



Surface Integral for Flux

Suppose that $\vec{F}$ is continuous vector field defined over an oriented surface S and that $\vec{n}$ is the chosen unit normal field on the surface. We call the integral of $\vec{F}.\vec{n}$ over S the flux of $\vec{F}$ across S in the positive direction. Thus the flux is the integral over S of the scalar component of $\vec{F}$ in the direction of $\vec{n}$.

Definition: The flux of a three-dimensional vector field $\vec{F}$ across an oriented surface S in the direction of $\vec{n}$ is,

$$\text{Flux} = \iint_S \vec{F}.\vec{n}.d\sigma \quad \left\{ \begin{array}{l} \vec{F}: \text{Vector field} \\ \vec{n}: \text{Outward unit normal vector} \end{array} \right\}$$

Let $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ be a vector field and let α, β, γ be the angles between $\vec{n}$ and the coordinate axes x, y, z, respectively. The center unit normal $\vec{n}$ in terms of its direction cosines;

$$\vec{n} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k} \quad \left\{ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,\ |\vec{n}| = 1 \right\}$$

$$\iint_S \vec{F}.\vec{n}.d\sigma = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,d\sigma = \underbrace{\iint_S P\cos\alpha\,d\sigma}_{I_3} + \underbrace{\iint_S Q\cos\beta\,d\sigma}_{I_2} + \underbrace{\iint_S R\cos\gamma\,d\sigma}_{I_1}$$

$$x = \Psi(y,z) \qquad y = \Psi(x,z) \qquad z = f(x,y)$$

$$I_1 \Rightarrow z=f(x,y) = \iint_S \underbrace{R\cos\gamma\, d\sigma}_{dxdy=\cos\gamma d\sigma} \Rightarrow I_1 = \iint_{R_{xy}} R(x,y,f(x,y))\,dxdy$$

$+$

$$I_2 \Rightarrow y=\varphi(x,z) = \iint_S \underbrace{Q\cos\beta\, d\sigma}_{dxdz=\cos\beta d\sigma} \Rightarrow I_2 = \iint_{R_{xz}} Q(x,\varphi(x,z),z)\,dxdz$$

$+$

$$I_3 \Rightarrow x=\psi(y,z) = \iint_S \underbrace{P\cos\alpha\, d\sigma}_{dydz=\cos\alpha d\sigma} \Rightarrow I_3 = \iint_{R_{yz}} P(\psi(y,z),y,z)\,dydz$$

$$= \iint_S \vec{F}\cdot\vec{n}\,d\sigma$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_{R_{yz}} P\,dydz + \iint_{R_{xz}} Q\,dxdz + \iint_{R_{xy}} R\,dxdy.$$

EX: Evaluate;

$$\iint_S x\,dydz + \iint_S dxdz + \iint_S xz^2\,dxdy = \iint_S x\,dydz + dxdz + xz^2\,dxdy$$

where S is the outer part of the sphere $x^2+y^2+z^2=1$ in the first quadrant.

$P=x,\ Q=1,\ R=xz^2$

$$\Rightarrow \iint_{R_{yz}} \sqrt{1-y^2-z^2}\,dydz \quad \left\{\begin{array}{l} y=r\cos\theta \\ z=r\sin\theta \\ dA=r\,dr\,d\theta \end{array},\ \begin{array}{l} 0\le r\le 1 \\ 0\le\theta\le \pi/2 \end{array}\right\} = \frac{\pi}{6}$$

$$\iint_{R_{xz}} (x^2+1+z^2)\,dxdz = \frac{\pi}{4}$$

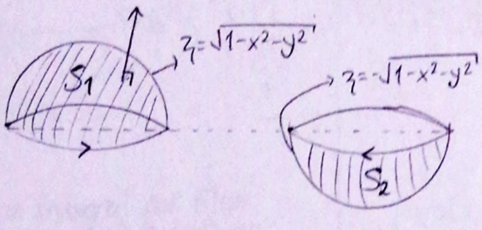
$$\iint_{R_{xy}} x(1-x^2-y^2)\,dxdy = \frac{2}{15}$$

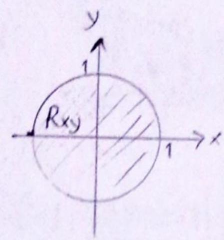
$$\iint_S P\,dydz + Q\,dxdz + R\,dxdy = \frac{25\pi+8}{60}.\ \checkmark$$

Ex! Evaluate,

$$\iint_S z\cos\gamma \, d\sigma$$

where S is the center of the sphere $x^2+y^2+z^2=1$.





$$\left\{ \begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \\ dA=r\,dr\,d\theta \\ 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{array} \right\}$$

$$\iint_S z\cos\gamma\, d\sigma = \iint_{S_1} z\cos\gamma\, d\sigma_1 + \iint_{S_2} z\cos\gamma\, d\sigma_1 = \underbrace{\iint_{Rxy}\sqrt{1-x^2-y^2}\,dx\,dy}_{0\le\theta\le 2\pi} + \underbrace{\iint_{Rxy}-\sqrt{1-x^2-y^2}\,dx\,dy}_{2\pi\ge\theta\ge 0} \quad \{\cos\gamma\, d\sigma = dx\,dy\}$$

$$= 2\iint_{Rxy}\sqrt{1-x^2-y^2}\,dx\,dy$$

$$= 2\int_{\theta=0}^{2\pi}\left(\int_{r=0}^{1}\sqrt{1-r^2}\,r\,dr\right)d\theta = \frac{4\pi}{3}. \checkmark$$

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Çarşamba

If S is part of a level surface $g(x,y,z)=c$ then $\vec{n}$ may be taken to be one of the two fields,

$$\vec{n} = \mp\frac{\nabla g}{|\nabla g|}$$

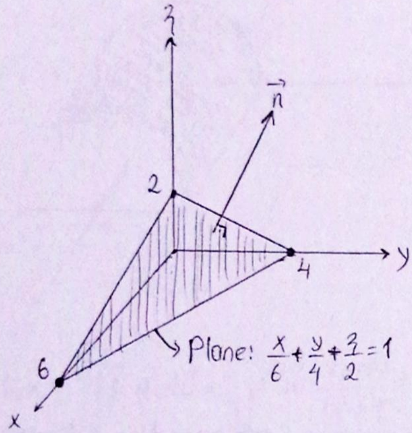
depending on which one gives the preferred direction. The corresponding flux is,

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_R \left(\vec{F}\cdot\mp\frac{\nabla g}{|\nabla g|}\right)\frac{|\nabla g|}{|\nabla g\cdot\vec{p}|}\,dA = \iint_R \vec{F}\cdot\mp\frac{\nabla g}{|\nabla g\cdot\vec{p}|}\,dA\ , \quad \vec{p}=\vec{i},\vec{j},\vec{k}.$$

EX: Evaluate;

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma.$$

Let $\vec{F}(x,y,z) = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ be a vector field where S is the part of the plane $2x+3y+6z=12$ in the first quadrant.



$$2x+3y+6z-12=0=g(x,y,z)$$

$$\vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}$$

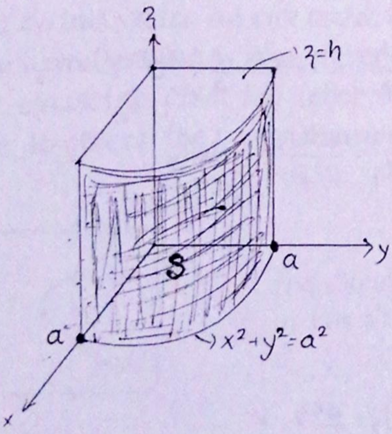
$$\vec{n}\cdot\vec{k} = \frac{6}{7}$$

$$d\sigma = \frac{dA}{|\vec{n}\cdot\vec{p}|} = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \frac{7}{6}dx\,dy.$$

$$= \int_{x=0}^{6}\left[\int_{y=0}^{4-\frac{2x}{3}}(6-2x)\,dy\right]dx = \int_0^6\left(24-12x+\frac{4x^2}{3}\right)dx = 24x-6x^2+\frac{4x^3}{6}\Big|_0^6 = 1024-676 = 6(40-36) = 24$$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_{R_{xy}}\left(\frac{36}{7}z - \frac{36}{7} + \frac{18}{7}y\right)\frac{7}{6}dx\,dy = \iint_{R_{xy}}(6z-6+3y)\,dx\,dy = \iint_{R_{xy}}(\overbrace{12-2x-3y}^{6-2x}-6+3y)\,dx\,dy = 24. \checkmark$$

EX: Find the outward flux of $\vec{F} = \cancel{2x\vec{i}} - \cancel{2x\vec{i}} + z\vec{i} + x\vec{j} + y\vec{k}$ through the piece of the cylinder shown:



$$x^2+y^2-a^2 = g(x,y,z)\ ,\ \vec{P} \neq \vec{k}$$

$$\vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{2x\vec{i}+2y\vec{j}}{\sqrt{4x^2+4y^2}} = \frac{x}{a}\vec{i} + \frac{y}{a}\vec{j}$$

$$\vec{F}\cdot\vec{n} = \frac{xz}{a} + \frac{xy}{a}$$

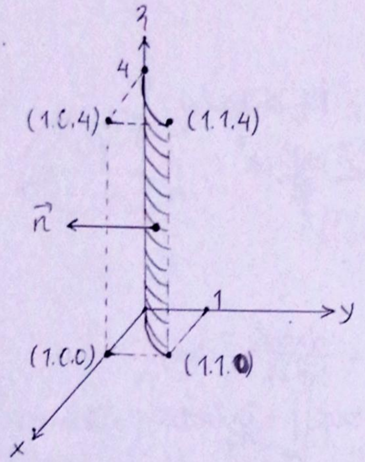
$$d\sigma = \frac{dA}{|\vec{n}\cdot\vec{i}|} = \frac{dy\,dz}{|\frac{x}{a}|} = \frac{a}{x}dy\,dz$$

$$\text{Flux} = \iint_{R_{yz}}\left(\frac{xz}{a} + \frac{xy}{a}\right)\frac{a}{x}dy\,dz = \iint_{R_{yz}}(y+z)\,dy\,dz = \int_{z=0}^{h}\left(\int_{y=0}^{a}(y+z)\,dy\right)dz = \frac{ah^2}{2} + \frac{a^2h}{2}. \checkmark$$

Examples

1) Find the flux (outward) of $\vec{F} = xz\vec{i} + yz\vec{j} + z^2\vec{k}$ through the piece of the sphere of radius a in the first octant.

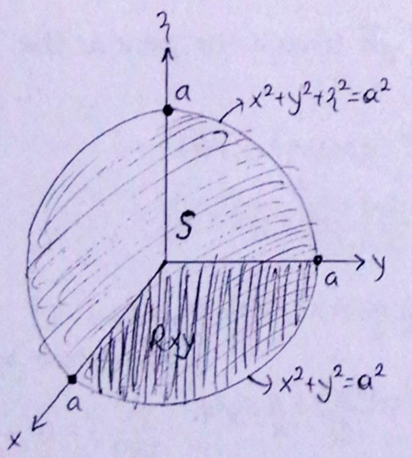
2) Find the flux of $\vec{F} = yz\vec{i} + x\vec{j} - z^2\vec{k}$ through the parabolic cylinder $y = x^2$, $0 \leq x \leq 1$, $0 \leq z \leq 4$ in the direction $\vec{n}$ indrested in figure:



3) Find the flux of $\vec{F} = yz\vec{i} + z^2\vec{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$ by the planes $x = 0$ and $x = 1$.

Solutions

1)



$$x^2 + y^2 + z^2 - a^2 = g(x,y,z) = 0$$

$$\vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x}{a}\vec{i} + \frac{y}{a}\vec{j} + \frac{z}{a}\vec{k}$$

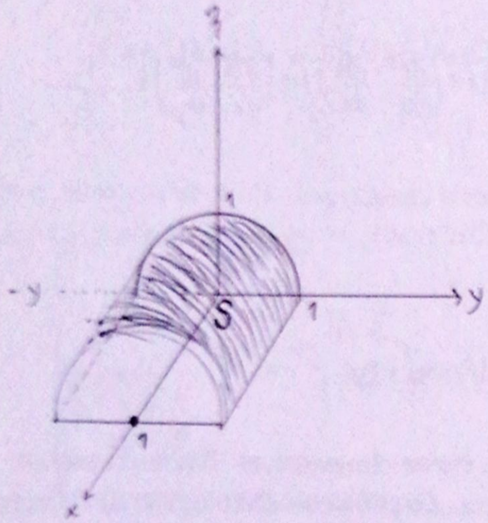
$$\vec{F} \cdot \vec{n} = \frac{x^2 z}{a} + \frac{y^2 z}{a} + \frac{z^2 z}{a} = \frac{z}{a}(x^2 + y^2 + z^2) = az$$

$$d\sigma = \frac{dA}{|\vec{n} \cdot \vec{p}|} = \frac{dx\,dy}{|\vec{n} \cdot \vec{k}|} = \frac{a}{z}\,dx\,dy$$

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{R_{xy}} az \cdot \frac{a}{z}\, dx\,dy = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2 - y^2}} a^2\, dx\,dy = a^2 \underbrace{\iint_{R_{xy}} dx\,dy}_{\text{Area} = \frac{\pi a^2}{4}} = \frac{a^4}{4}\pi.$$

$\underset{\sim}{2)}$ $y = x^2 \Rightarrow x^2 - y = g(x,y,z) \Rightarrow \ldots = 2.$ ✓

$\underset{\sim}{3)}$

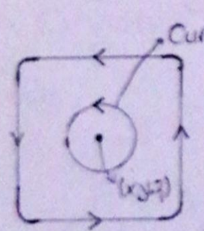


$\Rightarrow \text{Flux} = \iint\limits_{S} \vec{F}.\vec{n}\, d\sigma = \iint\limits_{R_{xy}} \vec{F}.\vec{n}\, \frac{dx\,dy}{|\vec{n}.\vec{k}|} = 2\iint\limits_{R_{xz}} \vec{F}.\vec{n}\, \frac{dx\,dz}{|\vec{n}.\vec{j}|} = \ldots = 2.$ ✓

8 Mayıs 2015
Cuma

STOKE'S THEOREM:

Suppose that $\vec{F}$ is the velocity field of a fluid ~~fl~~ flowing in space. Particles near the point (x,y,z) in the fluid tend to rotate around an axis through (x,y,z) that is parallel to a certain vector we are about to define. This vector points in the direction ~~of~~ of which the rotation is counterclockwise, when viewed looking down onto the plane of the circulation from the tip of the arrow representing the vector.
The length of the vector measures the rate of rotation.



The circulation vector of a point (x,y,z) in a plane in the direction fluid flow.

Figure 1

The vector is called the curl vector and for the vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ it is defined to be,

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\left\{ \nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right\}$$

EX: If $\vec{F} = (x^2 - z)\vec{i} + (-1-y)\vec{j} + xy\vec{k}$, then $\text{Curl}\,\vec{F} = x\vec{i} - (y+1)\vec{j}$. ✓

Stoke's theorem generalized Green's theorem to three dimensions. The circulation curl of from of Green's Theorem relates the counterclockwise circulation of a vector field around a simple closed curve C in the xy-plane to a double integral over the plane region R enclosed by C. Stoke's Theorem relates the circulation of a vector field around the boundary C of an oriented surface S in space Fig 2.

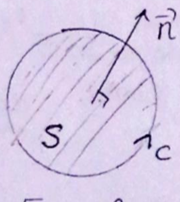


Figure 2

Stoke's Theorem

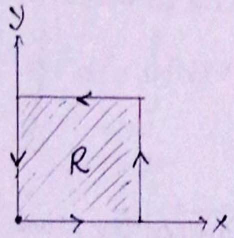
Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose component have continuous first partial derivatives on an open region containing S. Then the circulation of $\vec{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\vec{n}$ equals the integral $(\nabla \times \vec{F}).\vec{n}$ over S,

$$\underbrace{\oint_C \vec{F}.d\vec{r}}_{\substack{\text{Counterclockwise} \\ \text{circulation}}} = \underbrace{\iint_S (\nabla \times \vec{F}).\vec{n}.d\sigma}_{\substack{\text{Curl Integral} \\ (\text{circulation of } \vec{F})}} \qquad (\text{STOKE'S THEOREM})$$

* If two different oriented surfaces S_1 and S_2 have the same boundary c, their curl integrals are equal.

$$\iint_{S_1} (\nabla \times \vec{F}) \vec{n}_1 \, dS_1 = \iint_{S_2} (\nabla \times \vec{F}) \vec{n}_2 \, dS_2$$

* If c is a curve in the xy-plane, oriented counterclockwise and R is the region in the xy-plane bounded by c, then $d\sigma = dxdy$.

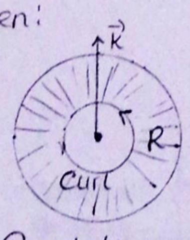


$$\vec{n} = \vec{k}$$

$$(\nabla \times \vec{F}) \cdot \vec{n} = (\nabla \times \vec{F}) \vec{k} = (Q_x - P_y)$$

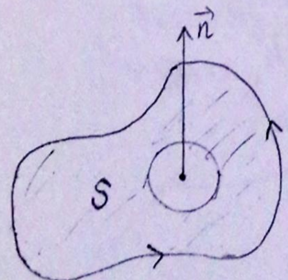
$$\underbrace{\iint_S (\nabla \times \vec{F}) \vec{n} \, d\sigma = \oint_c \vec{F} \cdot dr}_{\text{Stoke's Theorem}} = \underbrace{\iint_R (Q_x - P_y) dx dy}_{\text{Green's Theorem}} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad \left\{ \begin{array}{l} Q_x - P_y = (\nabla \times \vec{F}) \vec{k} \\ dxdy = d\sigma \end{array} \right\}$$

Green:



Circulation

Stoke's:



Circulation

Figure | Comparison of Green's Theorem and Stoke's Theorem

Examples

1) Evaluate;

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\,\vec{n}\,d\sigma$$

for the semisphere S, $x^2+y^2+z^2=9$, $z\geq 0$ is bounding curve $c: x^2+y^2=9$, $z=0$ and the field $\vec{F}=y\vec{i}-x\vec{j}$.

2) Find the circulation of the field $\vec{F}=(x^2-y)\vec{i}+4z\vec{j}+x^2\vec{k}$ around the curve c in which the plane $z=2$ meets the cone $z=\sqrt{x^2+y^2}$ counterclockwise as viewed from above.

HW: Use Stoke's Theorem to evaluate,

$$\oint_C \vec{F}\cdot d\vec{r}$$

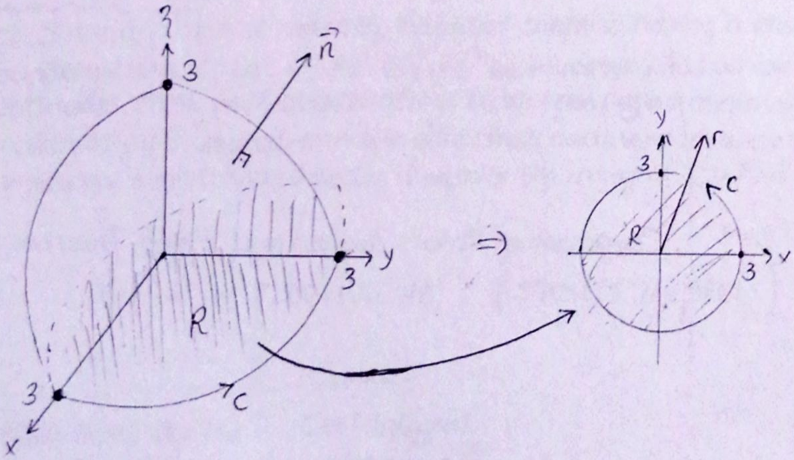
if $\vec{F}=xz\vec{i}+xy\vec{j}+3xz\vec{k}$ and c is the boundary of the partition of the plane $2x+y+z=2$ in the first octant, traversed counterclockwise as viewed from above.

$\{\text{Result}=-1\}$

Solutions

1)

$$\underbrace{\oint_C \vec{F}\,d\vec{r}}_{①} = \underbrace{\iint_S (\nabla\times\vec{f})\,\vec{n}\,d\sigma}_{②}$$



C: $x=3\cos\theta$

$y=3\sin\theta$

$\cancel{dA=3\,dr\,d\theta}$

$\cancel{0\leq r\leq 3}$

$0\leq\theta\leq 2\pi$

$z=0$

$$\nabla \times \vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\vec k$$

$$x^2+y^2+z^2-9 = G(x,y,z) = 0 \Rightarrow \vec n = \frac{\nabla G}{|\nabla G|} = \frac{2x\vec i + 2y\vec j + 2z\vec k}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{3}\vec i + \frac{y}{3}\vec j + \frac{z}{3}\vec k$$

$$d\sigma = \frac{dA}{|\vec n \cdot \vec p|} = \frac{dxdy}{|\vec n \cdot \vec k|} = \frac{3}{z}dxdy = \frac{3}{\sqrt{9-x^2-y^2}}\,dxdy$$

$$\Rightarrow ①: \oint_C \vec F \cdot d\vec r = \oint_C Pdx + Qdy + Rdz \quad \left\{ \begin{array}{ll} P = y & , dx = -3\sin\theta d\theta \\ Q = -x & dy = 3\cos\theta d\theta \\ R = 0 & \end{array} \right\}$$

$$= \int_{\theta=0}^{2\pi} \underbrace{(3\sin\theta)(-3\sin\theta d\theta)}_{-9\sin^2\theta d\theta} + \underbrace{(-3\cos\theta)(3\cos\theta d\theta)}_{-9\cos^2\theta d\theta} + 0 = -18\pi.$$

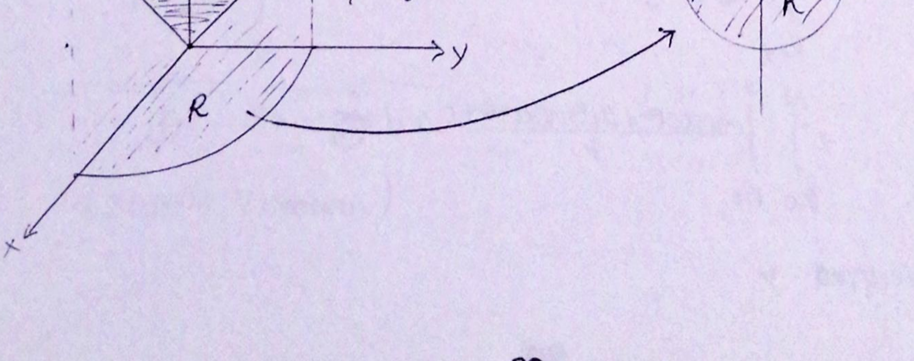
$$②: \iint_S (\nabla\times\vec F)\vec n\, d\sigma = \iint_{R_{xy}} (-2k)\left(\frac{x}{3}\vec i + \frac{y}{3}\vec j + \frac{z}{3}\vec k\right)\left(\frac{3}{z}dxdy\right)$$

$$= \iint_{R_{xy}} \left(-\frac{2}{3}z\right)\left(\frac{3}{z}\right)dxdy = -2\iint_{R_{xy}} dxdy$$

$$= -2(\pi \cdot 3^2) = -18\pi.$$

① = ② Verified. ✓

2)



c: $x = 2\cos\theta$
$y = 2\sin\theta$
$z = 2$

$0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$\underbrace{\oint_C \vec{F}\,d\vec{r}}_{①} = \underbrace{\iint_S (\nabla\times\vec{F})\vec{n}\,d\sigma}_{②}$$

$$\Rightarrow ①: \oint_C \vec{F}\,d\vec{r} = \oint_C P\,dx + Q\,dy + R\,dz \qquad \left\{\begin{array}{l} dx = -2\sin\theta\,d\theta \\ dy = 2\cos\theta\,d\theta \\ dz = 0 \end{array}\right\}$$

$$= \oint_C (x^2-y)dx + (4z)dy + (x^2)dz$$

$$= \int_{\theta=0}^{2\pi} (4\cos^2\theta - 2\sin\theta)(-2\sin\theta\,d\theta) + (4.2)(2\cos\theta\,d\theta) + (4\cos^2\theta)(0)$$

$$= \int_{\theta=0}^{2\pi} (-8\cos^2\theta\sin\theta + 4\sin^2\theta + 16\cos\theta)\,d\theta = \ldots = 4\pi.$$

$$②: \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y & 4z & x^2 \end{vmatrix} = -4\vec{i} - 2x\vec{j} + \vec{k}$$

$$z = \sqrt{x^2+y^2} \Rightarrow z^2 - x^2 - y^2 = G(x,y,z) = 0 \Rightarrow \vec{n} = \frac{\nabla G}{|\nabla G|} = \frac{-2x\vec{i} - 2y\vec{j} + 2z\vec{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{-x\vec{i} - y\vec{j} + z\vec{k}}{\sqrt{2}\cdot\sqrt{x^2+y^2}}$$

$$d\sigma = \frac{dA}{|\vec{n}\cdot\vec{p}|} = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \sqrt{2}\,dx\,dy$$

$$\iint_S (\nabla\times\vec{F})\vec{n}\,d\sigma = \iint_{R_{xy}} \frac{4x + 2xy + z}{\sqrt{2}\sqrt{x^2+y^2}}\sqrt{2}\,dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \frac{4r\cos\theta + 2r^2\sin\theta\cos\theta + r}{r}\cdot r\,dr\,d\theta = 4\pi.$$

① = ② Verified. ✓