Path Independence, Conservative Fields and Potential Function

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Path Independence

Definition: Let $\vec{F}$ be a vector field on an open region D in space and suppose that for any two points A to B in D the line integral $\int_C \vec{F} \cdot d\vec{r}$ along a path C from A to B in D is the same over all points from A to B. Then the integral $\int_C \vec{F} \cdot d\vec{r}$ is path independent in D and the field $\vec{F}$ is conservative on D

$\text{Curl} \vec{F} = \vec{0} \iff \vec{F} = \nabla \phi$

ϕ potential kolumn

We sometimes represent the integral by the symbol $\int_A^B$ rather than the usual line integral symbol $\int_C$

Assumptions on Curves, Vector Fields and Domains:

A curve in the xy-plane is simple if it does not cross itself, when a curve starts and ends at the same point, it is called a closed curve or loop (fig. 1)

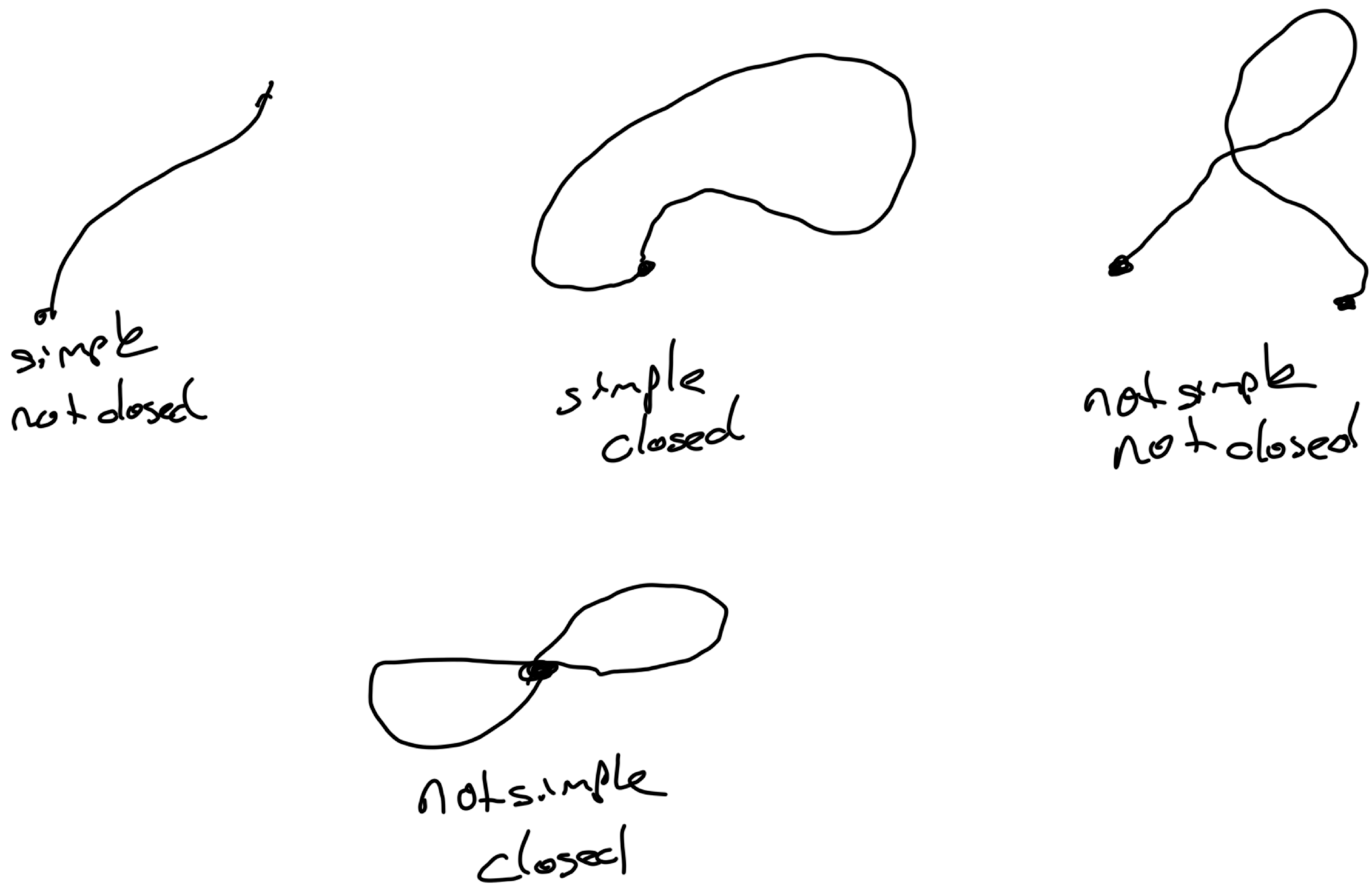

Fig - 1 - Distinguishing Curves

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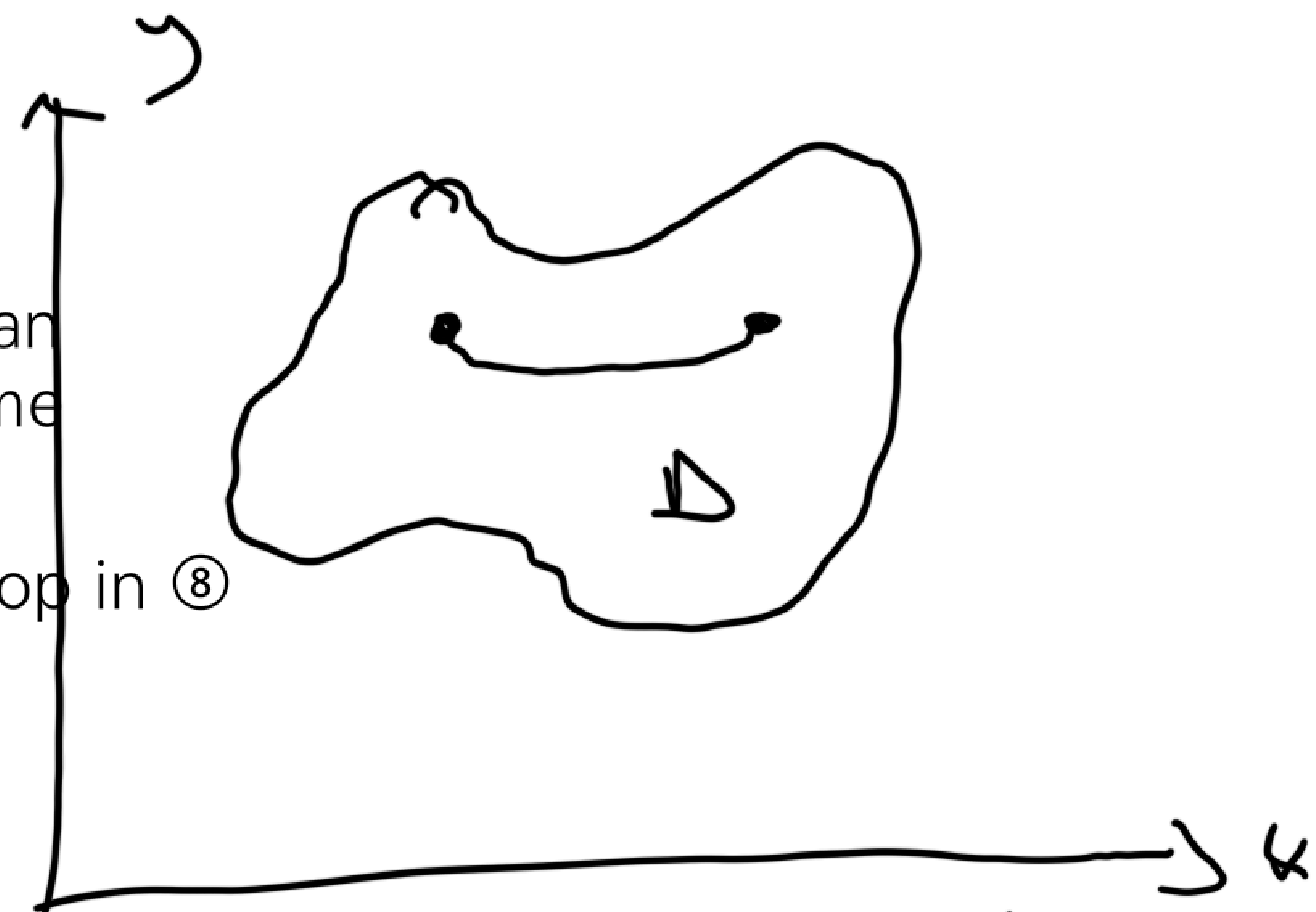
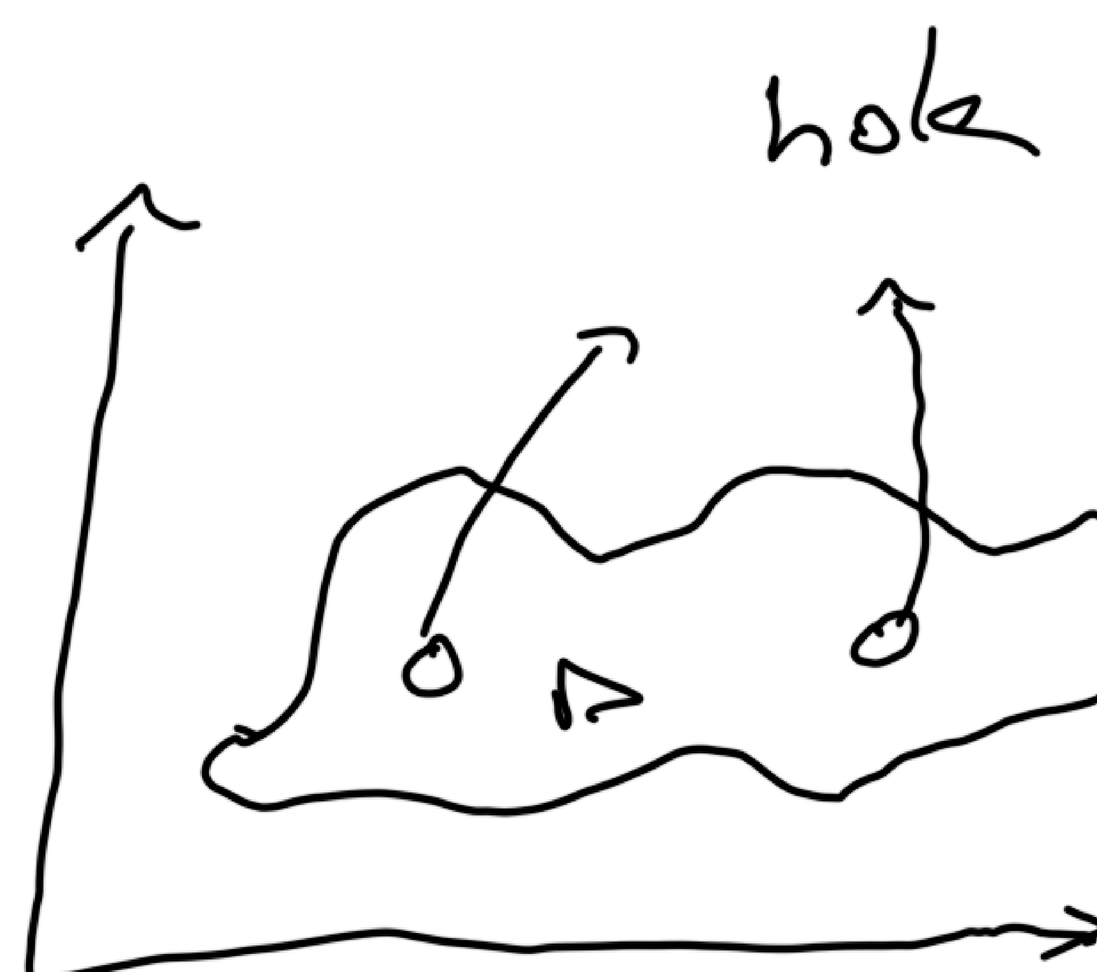
The domains D.ua consider are open regions in space, so every point in D in the center of an open ball that lies entirely in@. we also assume ⑤ to be connected. Finally, we assume ⑧ is simply connected, which means that every loop in ⑧

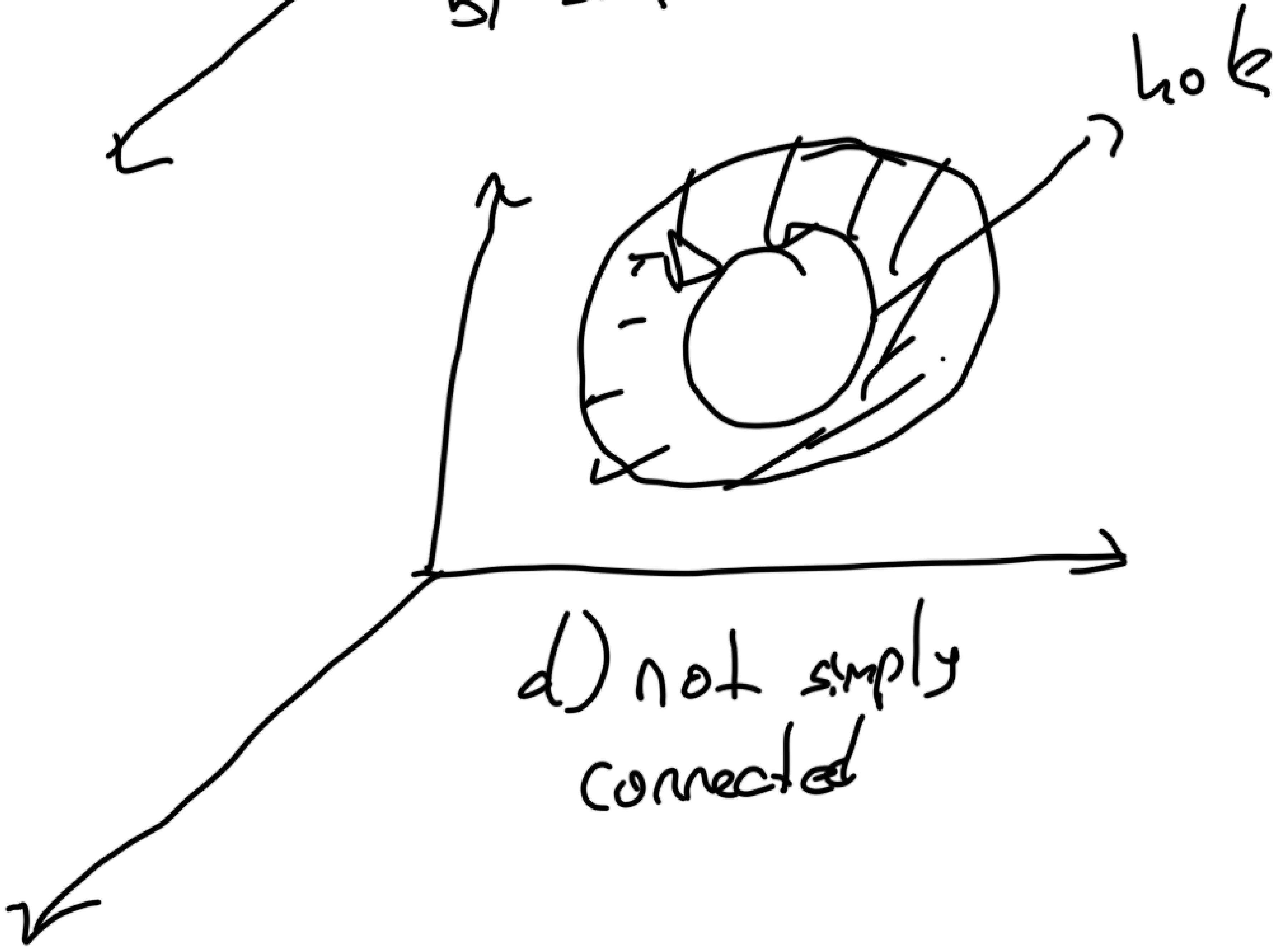


a) Simply connected

b) Simply connected



c) not simply connected



d) not simply connected

Line Integrals in Vector field
Theorem: Fundamentals of Line Integrals

Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\vec{r}(t)$. Let ϕ be a differentiable function with a continuous gradient vector $\vec{F} = \nabla\phi$ on a domain D containing C. Then

$$\int_C \vec{F}\, d\vec{r} = \phi(B) - \phi(A)$$

Proof of theorem 1: Suppose that A and B are two points in region D.

$$C: \{ r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \} \quad a \le t \le b$$

ϕ is a differentiable function of t.

$$\frac{d\phi}{dt} = \frac{d\phi}{dx}\cdot\frac{dx}{dt} + \frac{d\phi}{dy}\cdot\frac{dy}{dt} + \frac{d\phi}{dz}\frac{dz}{dt}$$

$$\Rightarrow \frac{d\phi}{dt} = \underbrace{\left(\frac{d\phi}{dx}\vec{i} + \frac{d\phi}{dy}\vec{j} + \frac{d\phi}{dz}\vec{k}\right)}_{\vec{F} = \nabla\phi} \cdot \underbrace{\left(\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}\right)}_{\frac{d\vec{r}}{dt}}$$

$$\int_C \vec{F}\, d\vec{r} = \int_{t=a}^{b} \underbrace{\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}}_{\frac{d\phi}{dt}}\, dt = \phi\Big|_a^b = \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a))$$

$$= \phi(B) - \phi(A)$$

Example:

Suppose the force field $\vec{F} = \nabla\phi$ is the gradient of the function

$$\phi(x, y, z) = \frac{1}{x^2+y^2+z^2}.$$

find the work done by $\vec{F}$ in moving an object along a smooth curve a joining the $(1,0,0)$ to $(0,0,2)$ that does not pass through the origin.

$$\text{work} = w = \int_C \vec{F}\,d\vec{r} = \phi\Big|_{(1,0,0)}^{(0,0,2)} = \phi(0,0,2) - \phi(1,0,0) = \frac{3}{4}$$

2nd method

Theorem 2: Conservative fields are Gradient fields

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space, then $\vec{F}$ is conservative iff $\vec{F}$ is gradient field $\nabla\phi$, for a differentiable function ϕ. $(\vec{F} = \nabla\phi)$

Theorem 3: Loop Property of Conservative fields

The following statements are equivalent.

1) $\oint_C \vec{F} \, d\vec{r} = 0$ around every loop.

2) The field $\vec{F}$ is conservative on D

Example 3 : Evaluate the integral

$$I = \int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over any path from $(1,1,1)$ to $(2,3,-1)$

$\mathrm{curl}\,\vec{F} = \vec{0}$?

$$\int_C \vec{F}\,d\vec{r} = \int_{t=a}^{b} \left(\vec{F}(r(t)) \cdot \frac{d\vec{r}}{dt} \right) dt$$

$$= \int_C P\,dx + Q\,dy + R\,dz$$

$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$\vec{F}_{(x,y,z)} = y\vec{i} + x\vec{j} + z\vec{k}$

vector field

$$\mathrm{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y & x & 4 \end{vmatrix}$$

$$= \vec{i}\left(\overbrace{\frac{d(4)}{dy}}^{=0} - \overbrace{\frac{d(x)}{dz}}^{0}\right) + \vec{j}\left(\overbrace{\frac{d(y)}{dz}}^{0} - \overbrace{\frac{d(4)}{dx}}^{0}\right) + \vec{k}\left(\frac{d(x)}{d(x)} - \frac{d(y)}{dy}\right)$$

$$= \vec{0}$$

- $\vec{F}$ is conservative (or irrational) so $\vec{F} = \nabla\phi$ $\quad \phi$: Potencial function

- The line integral is path integrant

$$I = \phi(2,3,-1) - \phi(1,1,1)$$ / veya doğru parçası ile yapılırsa aynısı olur

1) $\frac{d\phi}{dx} = y$

2) $\frac{d\phi}{dy} = x$

3) $\frac{d\phi}{dz} = 4$

$$\int d\phi = \int 4\,dz \Rightarrow \phi(x,y,z) = \int 4\,dz + \int h(x,y) = 4z + h(x,y)$$

$$= 4z + xy + C$$

Potential function

diffrent both given w.r.t y

$$\frac{d\phi}{dy} = 0 + \frac{d\,h(x,y)}{dy}$$

$$x = \frac{d\,h(x,y)}{dy} \Rightarrow d\,h(x,y) = x\,dy \Rightarrow h(x,y) = xy + g(x)$$

diffrent both ispect word respect to x

$$\frac{d\phi}{dx} = 0 + y + \frac{d\,g(x)}{dx}$$

$$\cancel{y} = 0 + \cancel{y} + \frac{d\,g(x)}{dx} \Rightarrow \frac{d\,g(x)}{dx} = 0 \Rightarrow \int d\,g(x) = \int 0\,dx$$

$$\Rightarrow g(x) = C \text{ potantiel}$$

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz =$$

$$\phi(2,3,-1) - \phi(1,1,1)$$

$$= (-4 + 6 + c) - (5 + c) = -3$$

Green's Theorem in the Plane:

Theorem: Let C be a piecewise smooth, simple closed curve enclosing a region R in xy-plane.

Let $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ be a vector field with P and Q having continuous first partial derivatives in an open region containing R, then the counterclockwise circulation of $\vec{F}$ around C equals the double integral over R:

$$\boxed{\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C \vec{F}\cdot d\vec{r} = \underbrace{\oint_C P\,dx + Q\,dy}_{1} = \underbrace{\iint_R \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy}_{2}}$$

(Left Hand Side) $\longleftarrow$ LHS

RHS (Right Hand Side)

Green's Theorem in the plane

<u>note:</u> C eğrisi düzlem eğrisi olmalı

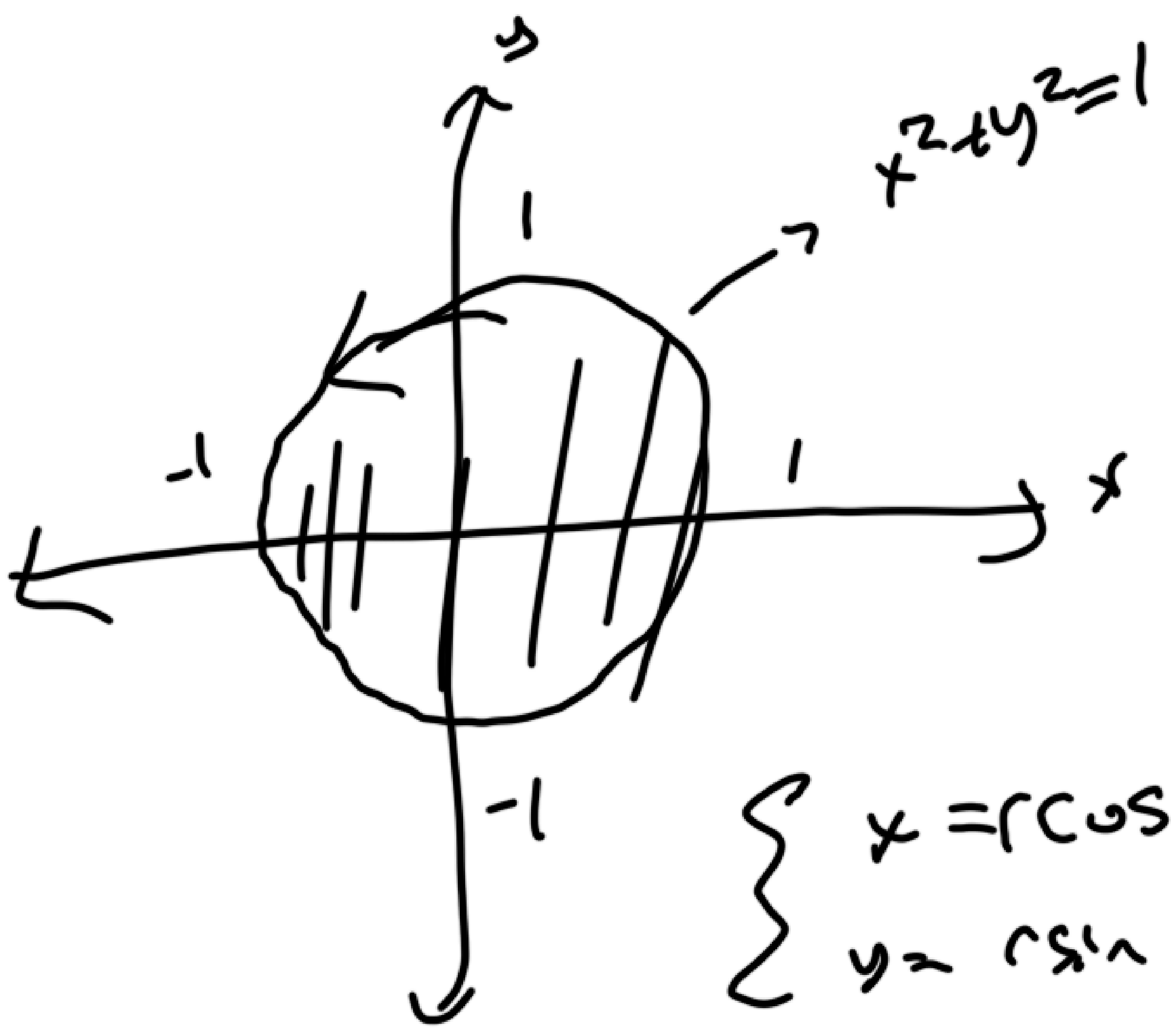
İspati var 49. sayfa

Example Verify Green's Theorem for the vector field $\vec{F}(x,y) = (x-y)\vec{i} + x\vec{j}$ and the region R bounded by the unit circle.

$$C: \left\{ \vec{F}(t) = \cos t\,\vec{i} + \sin \vec{j}, \quad 0 \leq t < 2\pi. \right.$$

(saatin dönme tersi yönü

LHS i1 = $\oint_C$



$$\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right. \qquad r=1 \qquad \begin{array}{l} x = \cos\theta \\ y = \sin\theta \end{array}$$

$t = \theta$

$$\oint_C \vec{F}\,d\vec{r} = \oint_{t=a}^{b} \left[\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \right] dt$$

$$= \int_{t=0}^{2\pi} \Big[\underbrace{\big((\cos t - \sin t)\vec{i} + \cos t\,\vec{j} \big) \cdot \big(-\sin t\,\vec{j} + \cos t\,\vec{j} \big)}_{\left(\frac{1}{2}\right)\frac{-2\sin t\cos t}{\sin 2t} + \underbrace{\sin^2 t + \cos^2 t}_{1}} \Big] dt = t = \frac{1}{4}\cos 2t \Big|_{t=0}^{2\pi}$$

$$= (2\pi + 1/4) - (0 + 1/4)$$

$$= \cos 2t = 2\pi$$

RHS: 2

$$\iint_R (Q_x - P_y)\,dx\,dy = \iint_R (2)\,dx\,dy = 2\iint_R dx\,dy = 2\pi$$

$$Q_x = 1 \qquad P_y = -1$$

$$Q_x - P_y = 1-(-1) = 2$$

$$\boxed{= 2\int_{\theta=0}^{2\pi}\left(\int_{r=0}^{1} r\,dr\right)d\theta = 2\pi}$$