Multiple Integration

Double Integrals

(S): three differansial region

(D): domain of integration.

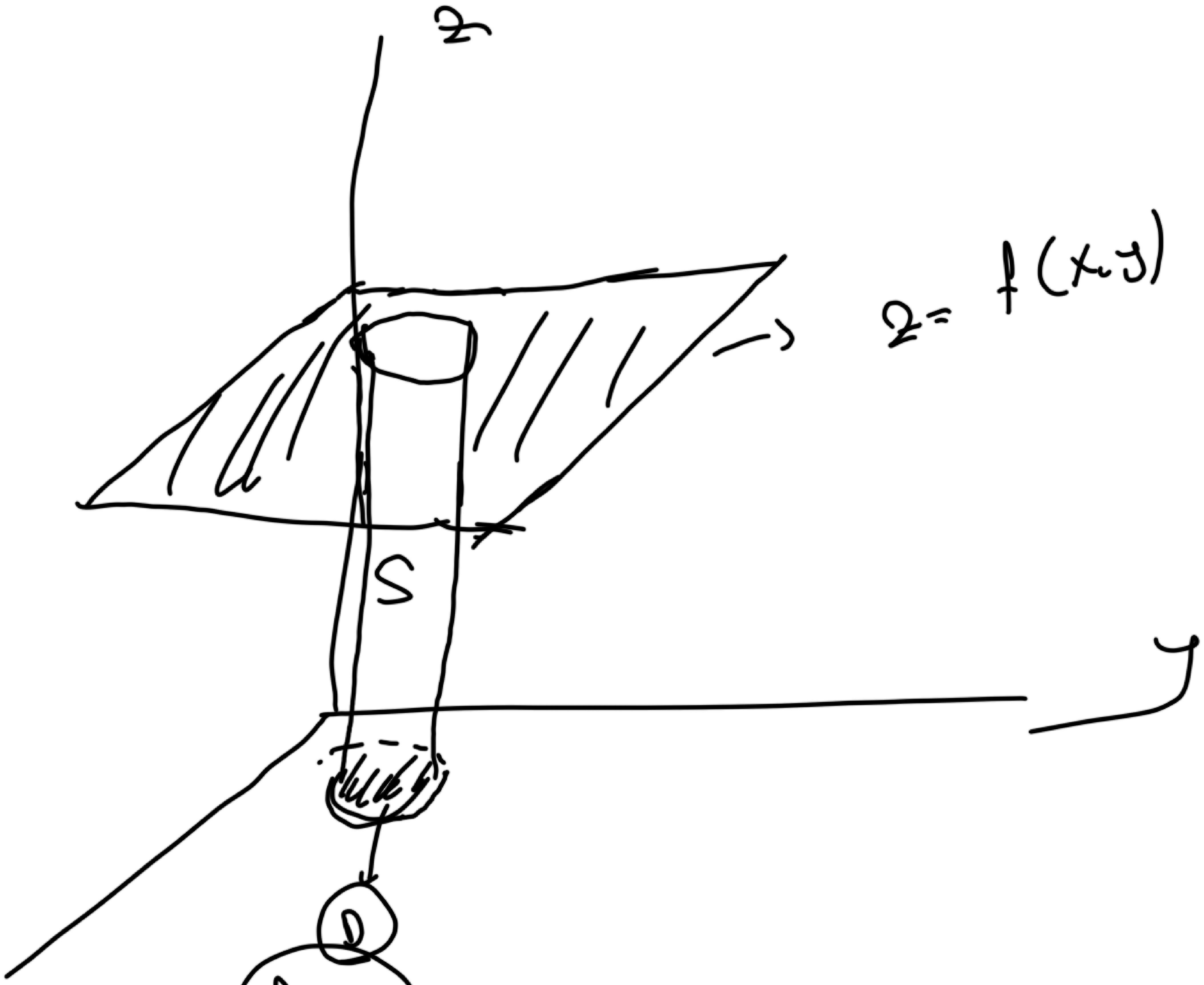
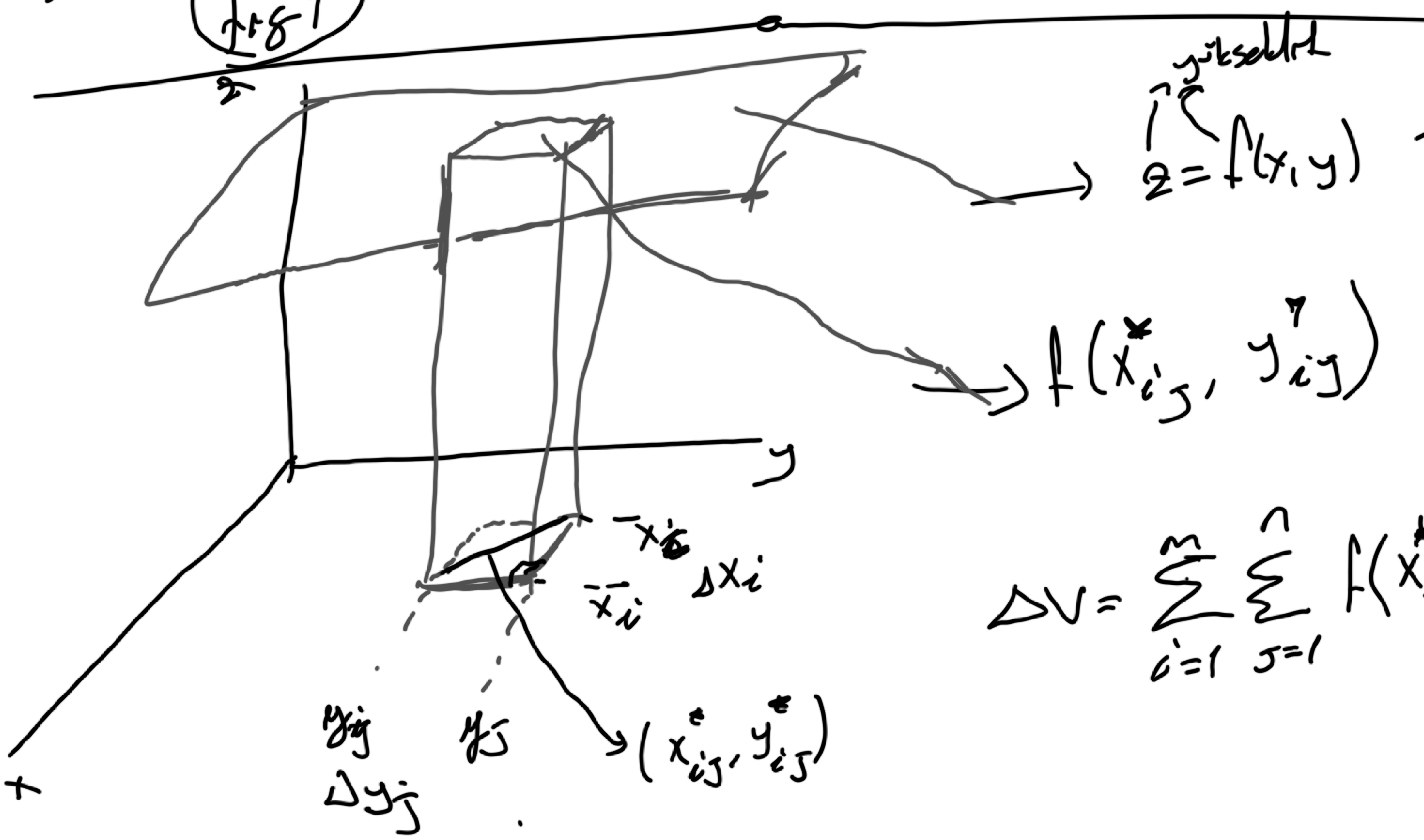


fig 1



R_{ij} = Rectangular box

$$\Delta V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta x_i \cdot \Delta y_j$$

fig 2 The Riemann Sum is a sum of values of such boxes.

There, for positive functions (f), the Riemann sum $R(f,P)$ approximates the volume above (D) and under the graph of (f)

$$V = \iint_D f(x,y)\, dA \qquad \left(\lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \cdot \Delta A_{ij} = \iint_D f(x,y)\, dA = V \right)$$

Remark

$dA = dx\,dy$
$= dy\,dx$

(integrasyon sınırı sabitse dydx ya da dxdy fark etmez, ama farklıysa değişken sınırı içte yazılır. sonuç fonksiyon olmasın diye)

Ex If $0 \le x \le 1$, $0 \le y \le 1$ the evaluate $\iint_D (x^2+y)\, dA$

$$\int_{x=0}^{1} \int_{y=0}^{1} (x^2+y)\, dy\, dx = \int_{x=0}^{1} \left(x^2 y + \frac{y^2}{2} \right) \Big|_{y=0}^{1} dx$$

$x^2 + \frac{1}{2}$

$$= \int_{x=0}^{1} \left(x^2 + \frac{1}{2} \right) dx = \frac{x^3}{3} + \frac{x}{2} \Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

Double Integrals over more General Domains

fig: Bounded domain D is a subset of rectangle R

Definitions: If $f(x,y)$ is defined and bounded on domain D, let $\hat{f}$ be the extension of f that D zero everywhere outside D:

$$\hat{f}(x,y) = \begin{cases} f(x,y), & \text{if } f(x,y) \text{ belongs to } D \\ 0, & \text{if } f(x,y) \text{ does not belong to } D \end{cases}$$

If $\hat{f}$ is integrable over R, we say f is integrable over D and define the double integral of f over D to be

$$\boxed{\iint_R \hat{f}(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_{\substack{(x,y)\in R \\ (x,y)\notin D}} 0\,dA = \iint_D f(x,y)\,dA}$$

Iteration of Double Integrals in Cartesian Coordinates

The existence of the double integral $\iint_D f(x,y)\,dA$ depends on f and the domain D. As we shall see evaluation of double integrals easiest when the domain of integration is of simple type

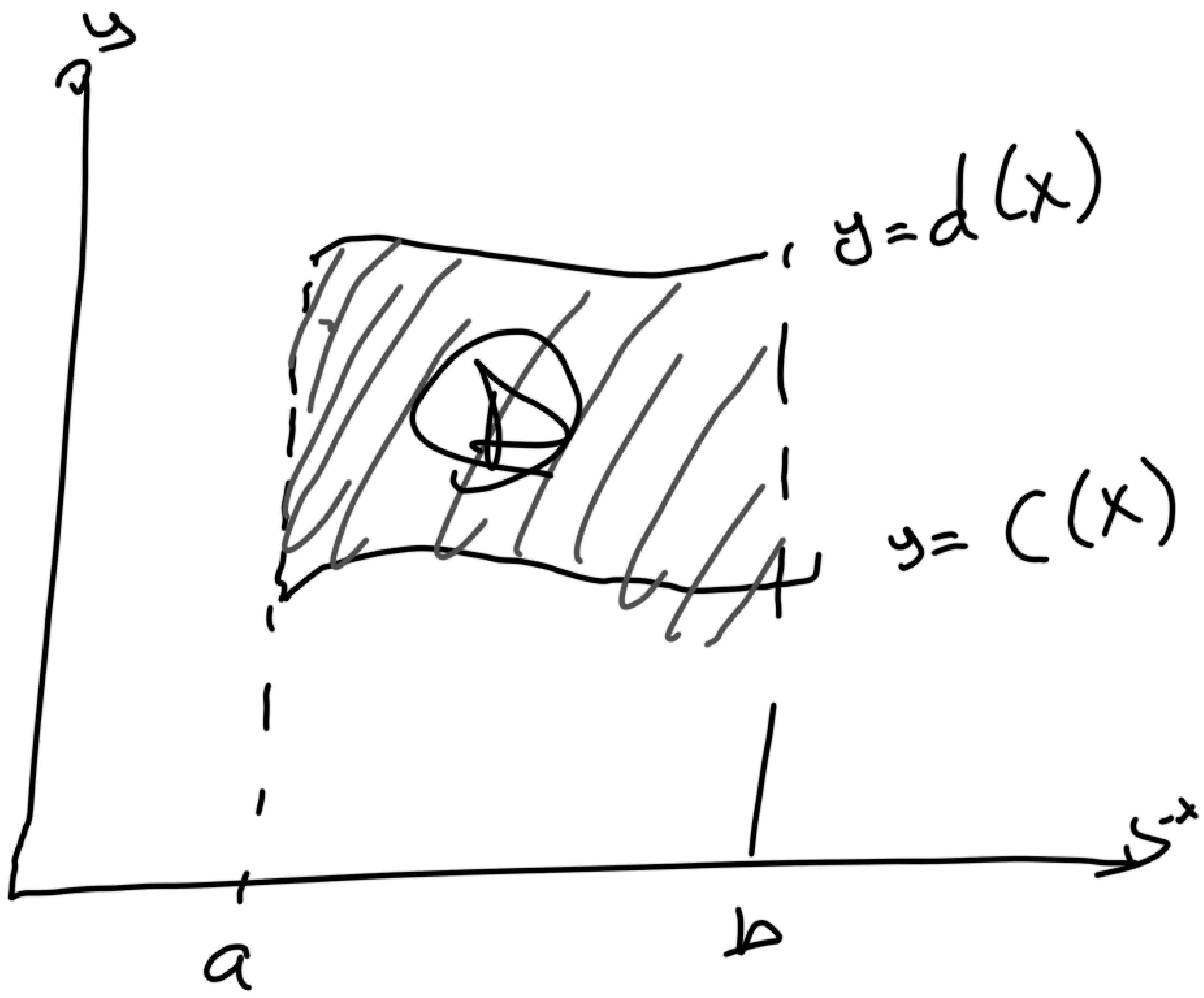


figure 1: A y-simple domain

$D:\ a \le x \le b\ ;\ c(x) < y < d(x)$

$f(x,y)$ continous in this domain

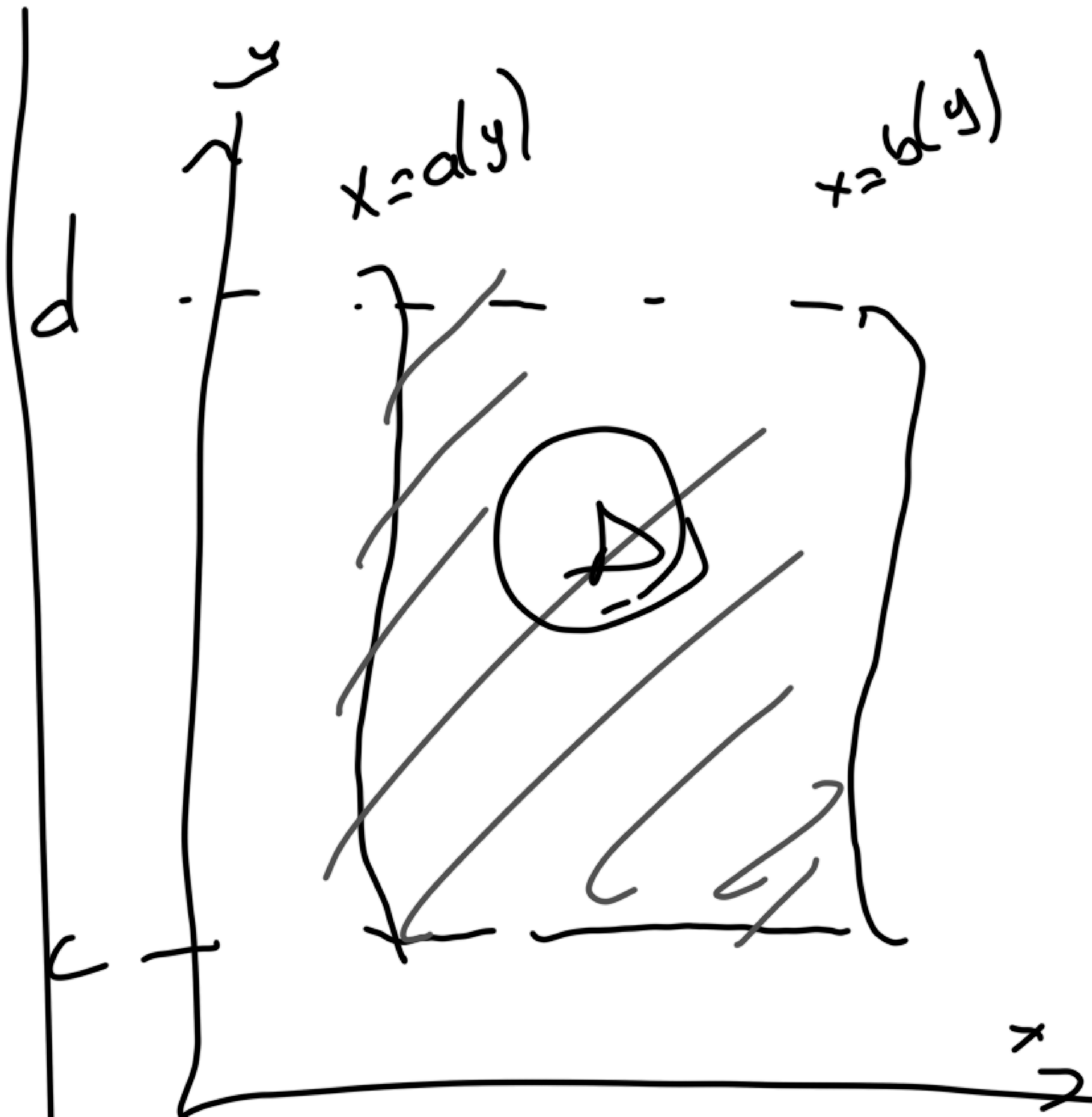


fig 2: A x-simple domain

$D:\ c \le y \le d\ ;\ a(y) \le x \le b(y)$

$x = f(y)$ continous in this domain

$$\Rightarrow \text{Fig 1}: \iint_D f(x,y)\,dA = \int_{x=a}^{b}\left(\int_{y=c(x)}^{d(x)} f(x,y)\,dy\right)dx$$

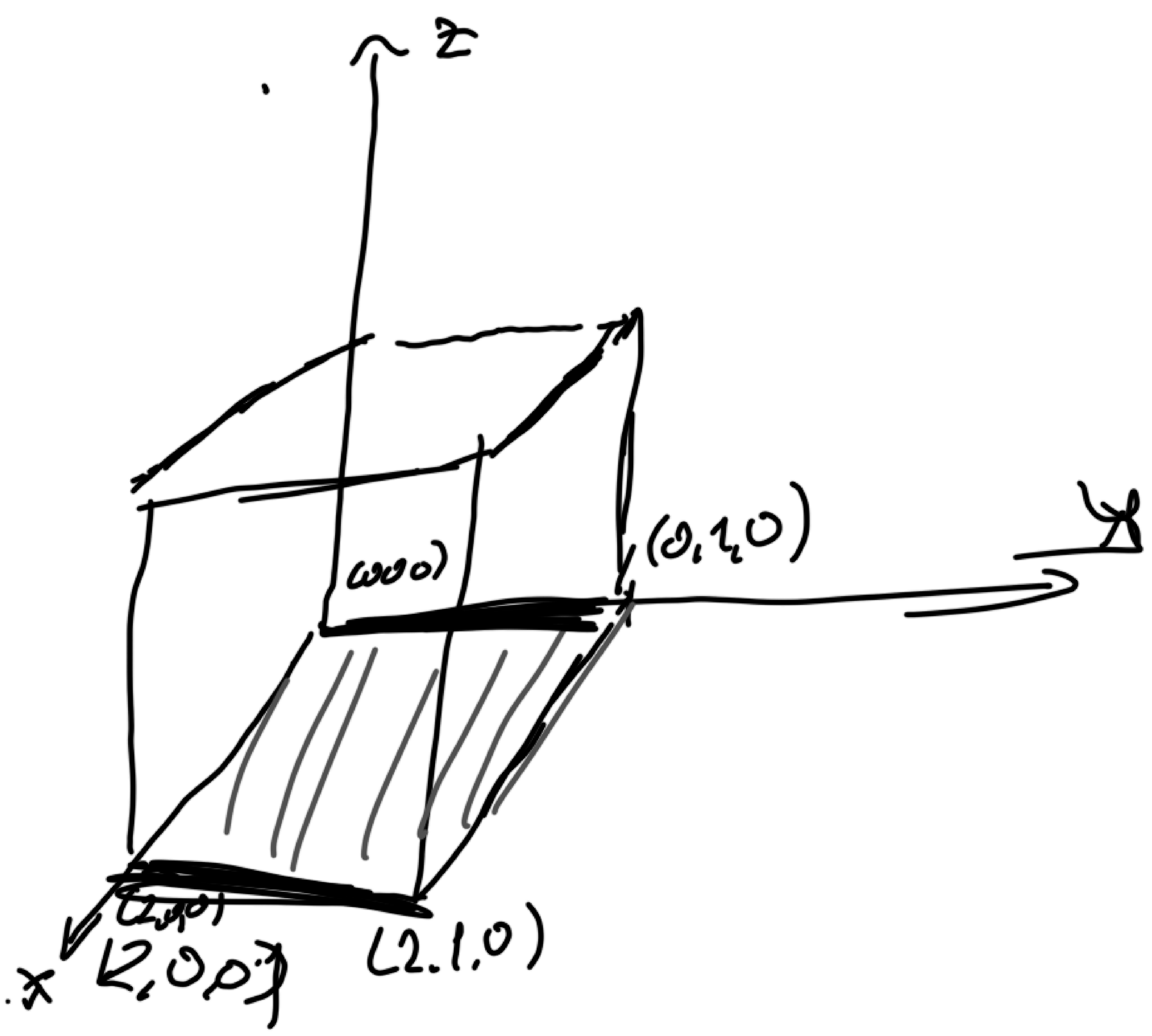
$$\Rightarrow \text{fig 2}: \iint_D f(x,y)\,dA = \int_{y=c}^{d}\left(\int_{x=a(y)}^{b(y)} f(x,y)\,dx\right)dy$$

Ex $z = 4 - x - y$

$x + y + z = 4$

$\frac{x}{4} + \frac{y}{4} + \frac{z}{4} = 1$, Calculate the value under the plane over the rectangular region $R: 0 \le x \le 2,\ 0 \le y \le 1$ in the xy-plane

$x = 0 \quad x = 2 \qquad y = 0 \quad y = 1$



$$\text{Volume} = \iint_D f(x,y)\,dA = \iint_D (4-x-y)\,dy\,dx$$

$$= \int_{y=0}^{1} \int_{x=0}^{2} (4-x-y)\,dx\,dy$$

$$= 4x - \frac{x^2}{2} - yx \Big|_0^2$$

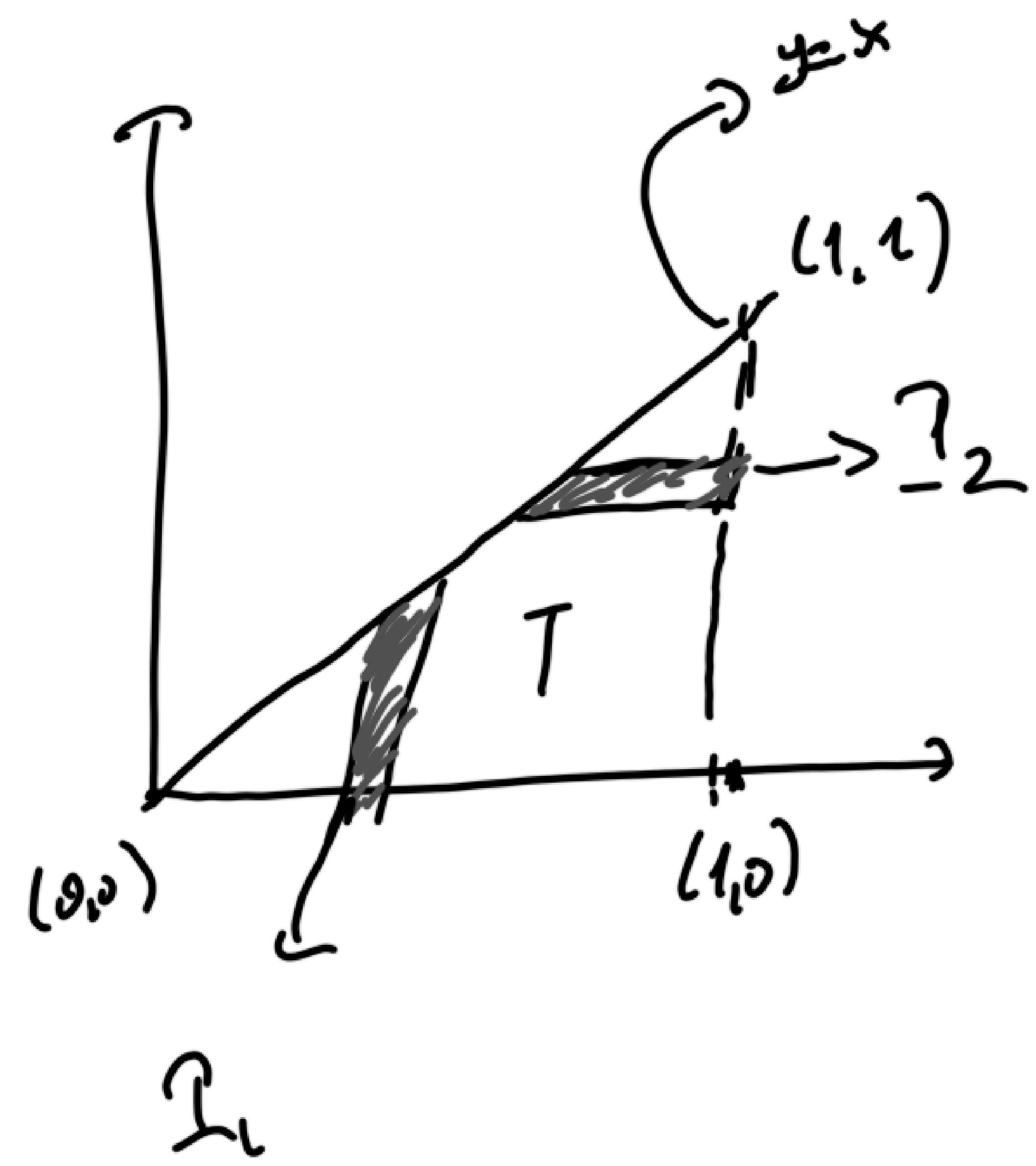
$$= 8 - 2 - 2y = 6 - 2y$$

$$\int_{y=0}^{1} (6-2y)\,dy = 6y - y^2 \Big|_0^1$$

$$= 6 - 1 = 5$$

Example 8

Evaluate $\iint xy\,dA$ over the triangle T with vertices $(0,0)$, $(1,0)$ and $(1,1)$



$$I_1 = \iint_D xy\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{1} xy\,dx\right)dy$$

$$\int_{x=0}^{1} \frac{\cancel{x}}{2}\,dx = \frac{1}{4}$$

Ex: Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 100 - 6x^2y$

and $R = \{(x,y)\ \ 0 \le x \le 2$ and $-1 \le y \le 1\}$

$x=0$ $x=2$ $y=-1$ $y=1$

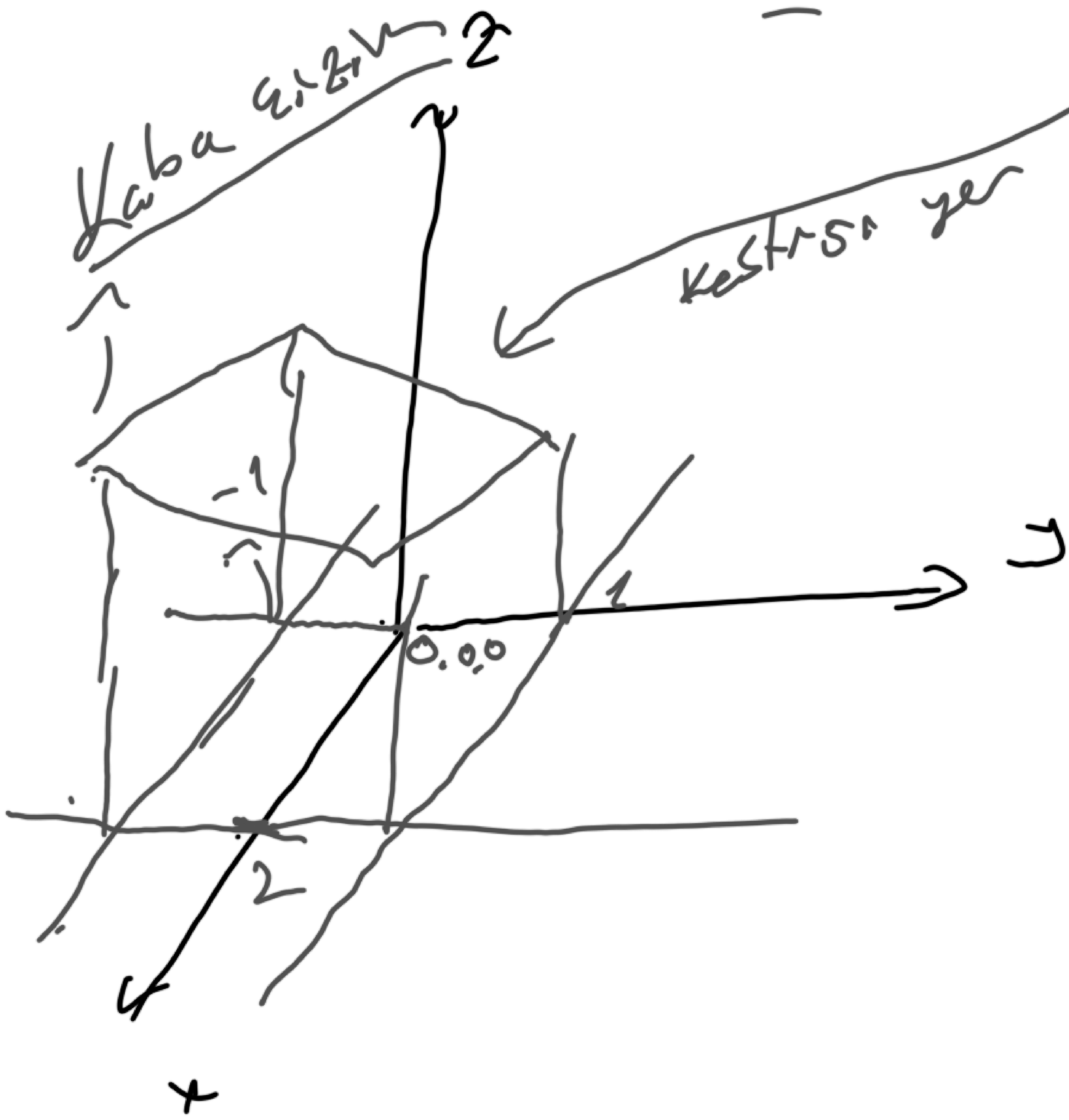
$$\int_{x=0}^{2}\int_{y=1}^{1}(100-6x^2y)\,dy\,dx$$

$$100y - 3x^2y^2\Big|_{-1}^{1}$$

$$100 - 3\cancel{x^2} + 100 + 3\cancel{x}$$

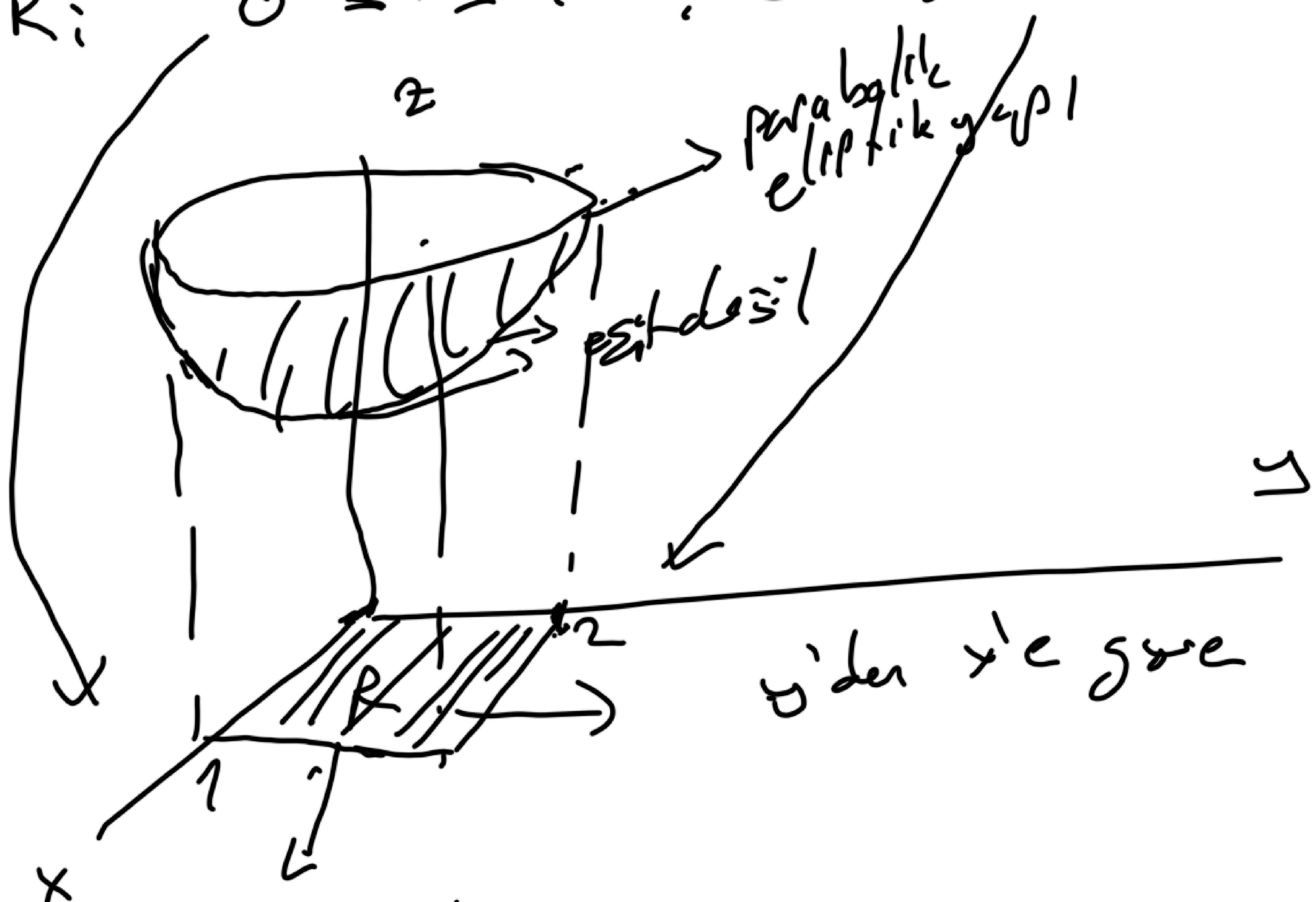
$$\int_0^2 200\,dx$$

$$200x\Big|_0^2 = 400$$



Ex Find the volume of the region bounded above the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle

$R: \quad 0 \le x \le 1 \ , \ 0 \le y \le 2$



doğan bölge

$$V = \iint_R f(x,y)\, dA = \int_{x=0}^{1} \left(\int_{y=0}^{2} (10 + x^2 + 3y^2)\, dy \right) dx$$

$$= \int_{x=0}^{1} 10y + x^2 y + y^3 \Big|_0^2 \ (y)$$

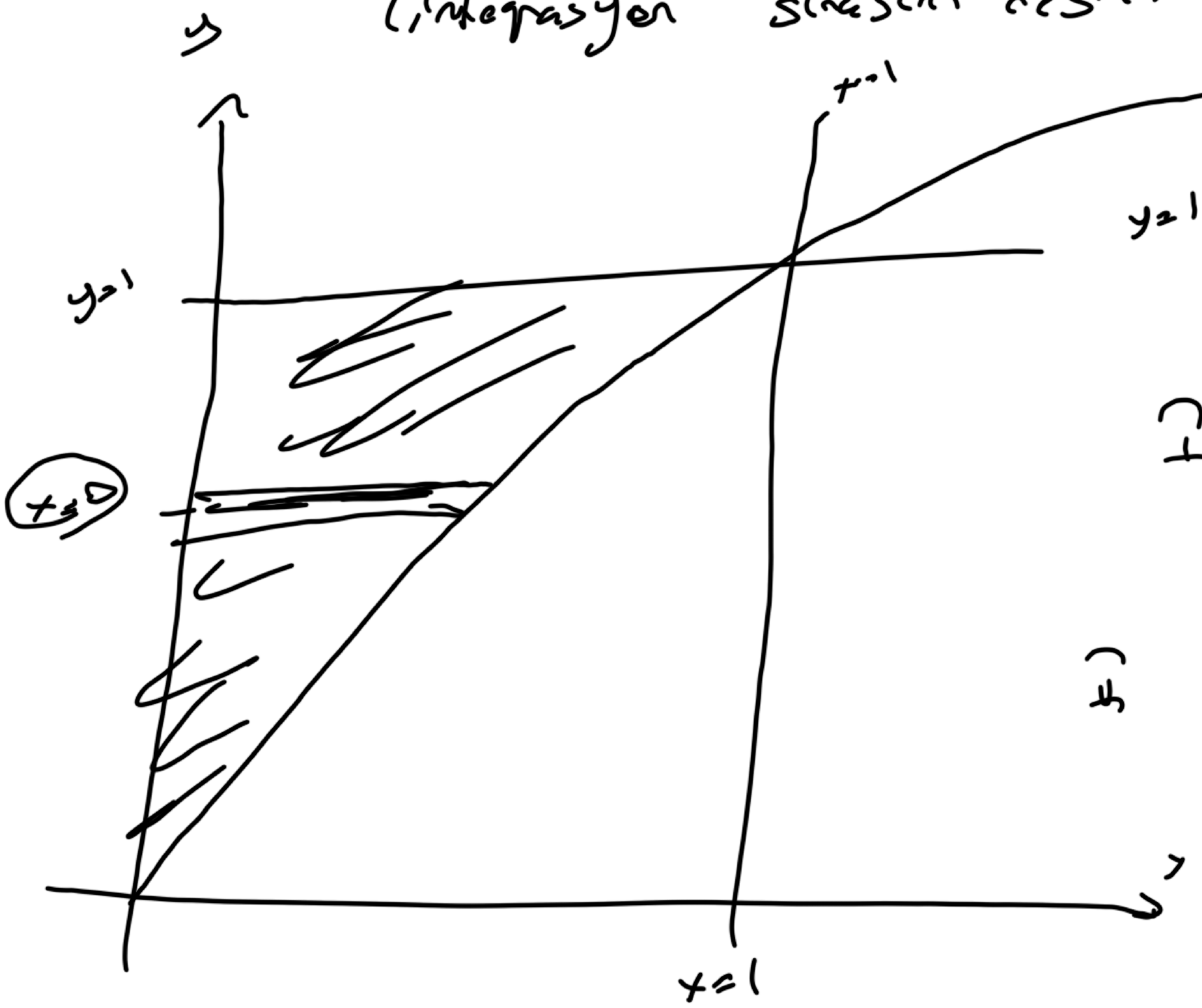
$$= \int_{x=0}^{1} (20 + 2x^2 + 8)\, dx$$

$$= 20x + \frac{2x^2}{3} + 8x \Big|_0^1$$

$$= 20 + \frac{2}{3} + 8 = \frac{86}{3}$$

Order Of Integration Reversal

(integrasyon sırasını değiştirme)



$$I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3} dy = \int_{x=0}^1 \left(\int_{y=\sqrt{x}}^1 e^{y^3} dy \right) = ?$$

$$I = \int_{y=0}^1 \left(\int_{x=0}^{y^2} e^{y^3} dx \right) dy$$

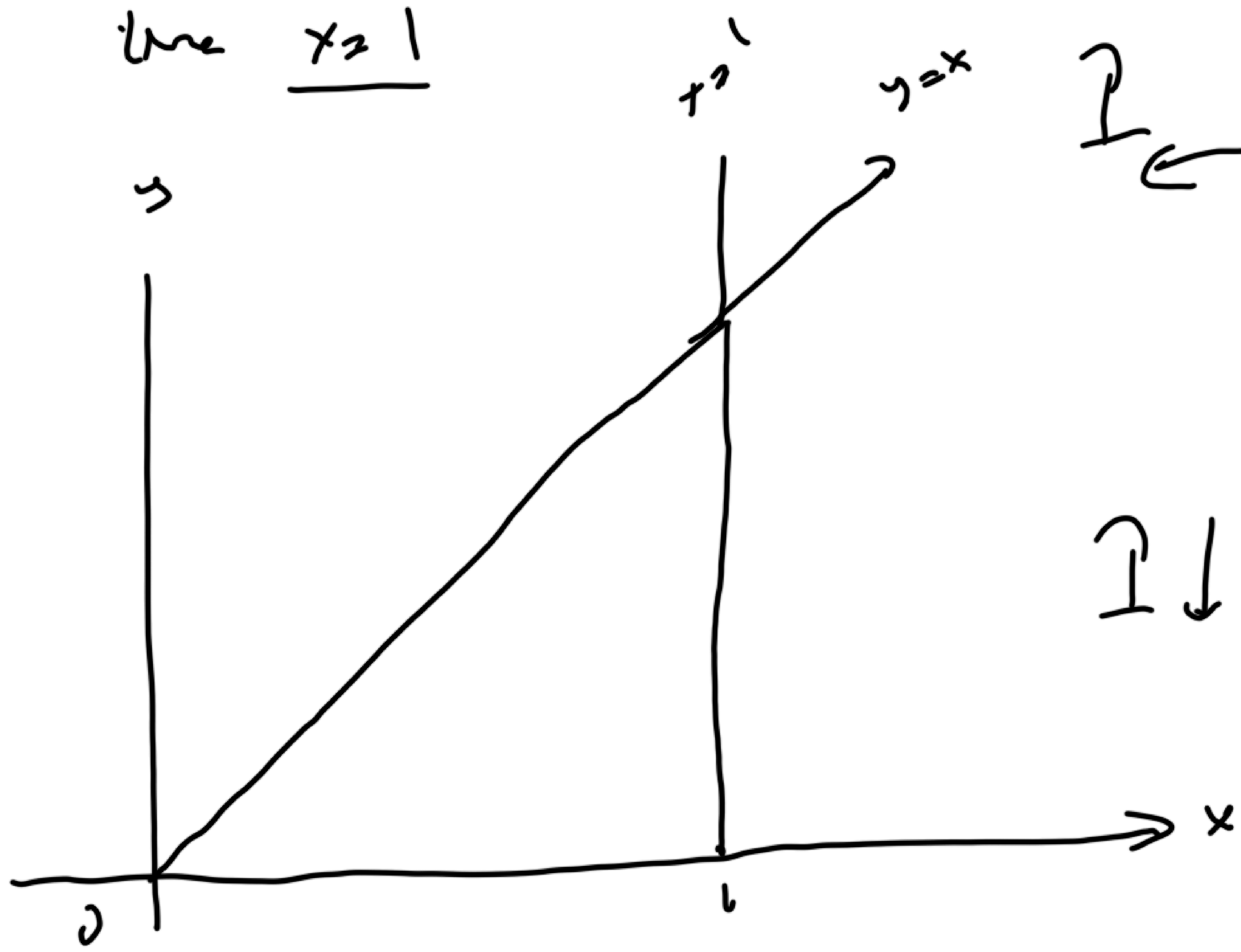
$$= \int_{y=0}^1 \left(x \cdot e^{y^3} \right) \Big|_{x=0}^{y^2}$$

$$= \int_{y=0}^1 y^2 \cdot e^{y^3} = \frac{e^{y^3}}{3} \Big|_0^1$$

$$= \frac{e-1}{3}$$

Examp

Calculate $\iint_R \frac{\sin x}{x}\, dA$. where R is the triangle in the xy-plane bounded by the x-axis, the line $y=x$ and the line $x=1$



I $\leftarrow$ $\iint_R \frac{\sin x}{x}\, dA = \int_{y=0}^{1}\left(\int_{x=y}^{1}\left(\frac{\sin x}{x}\right)dx\right)dy = ?$

II $\downarrow$ $\iint_R \frac{\sin x}{x}\, dA = \int_{x=0}^{1}\left(\int_{y=0}^{x}\frac{\sin x}{x}\, dy\right)dx$

$$y\cdot\frac{\sin x}{x}\Big|_{y=0}^{x} = \cancel{x}\cdot\frac{\sin x}{\cancel{x}} - \cancel{0\cdot\frac{\sin x}{x}}$$

$$= \sin x$$

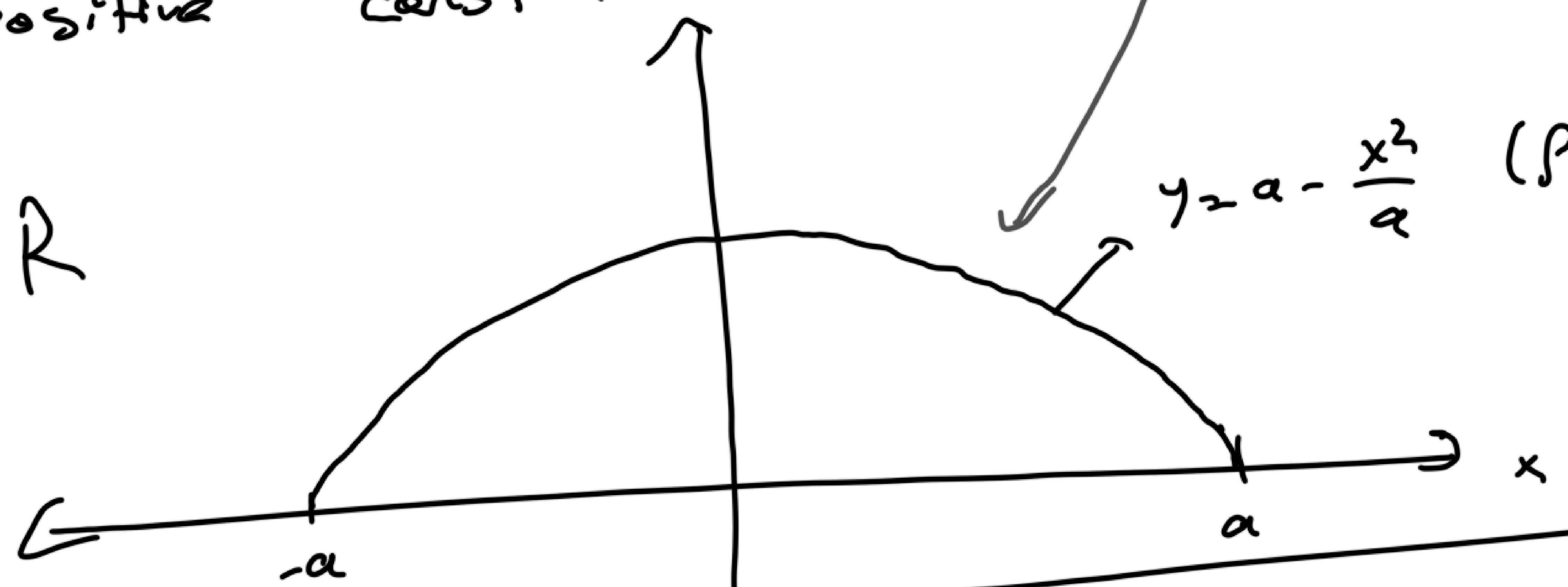
$$\int_{x=0}^{1}\sin x\, dx = -\cos x\Big|_{x=0}^{1}$$

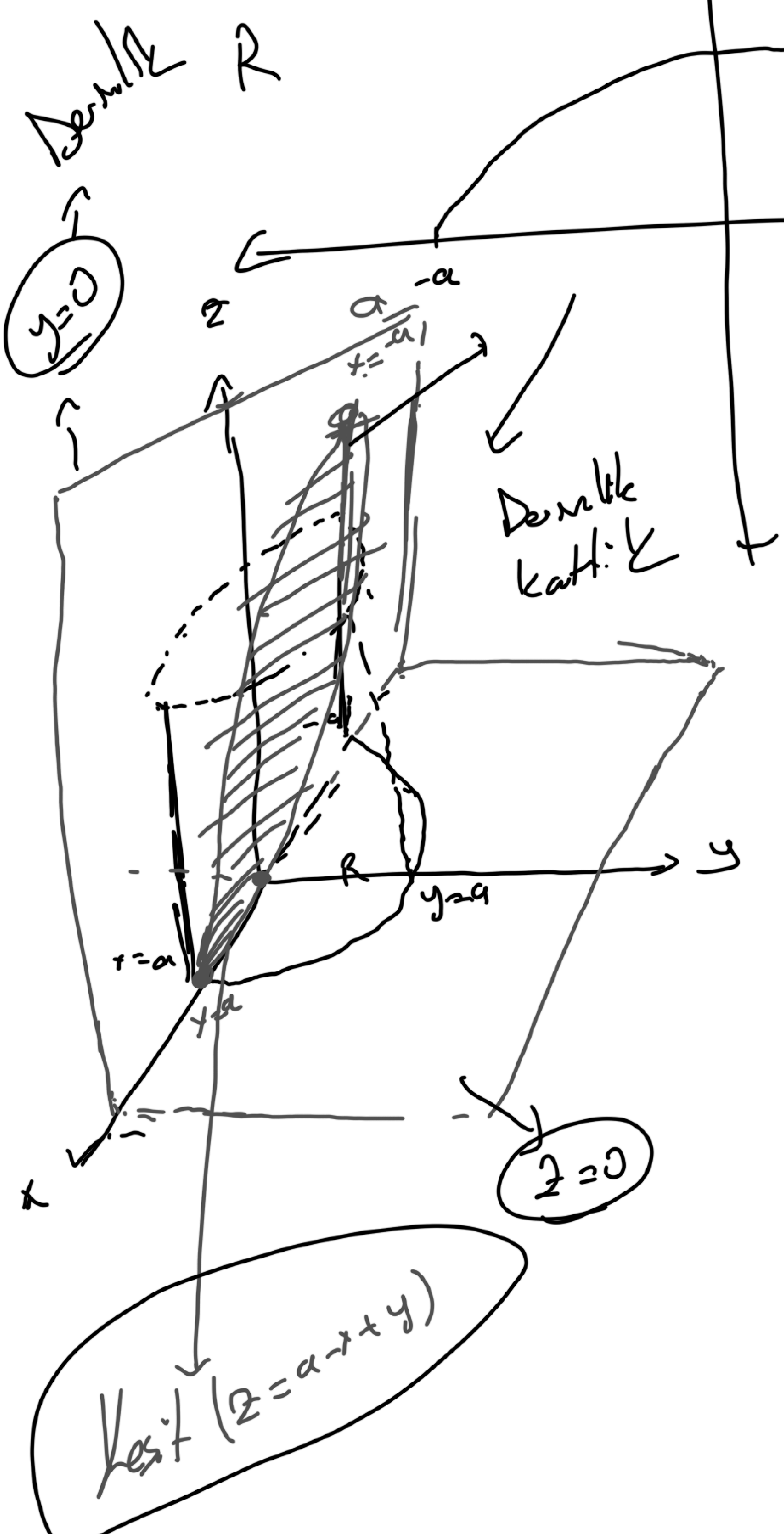
$$= -\cos(1)+\cos(0) =$$

$$= 1-\cos(1)$$

* ifade ilk tarama şekline göre çözülemezdir, o yüzden dikey tarama yapılmıştır.

Example: Sketch and find the volume of the solid bounded by the planes $y=0$, $z=0$ and $z=a-x+y$ and the the parabolic cylinder $y=a-\frac{x^2}{a}$, where a is a positive constant.





$x-y+z=a \to$ kesit

$\frac{x}{a}-\frac{y}{a}+\frac{z}{a}=1$

$z=a-x+y$ $\quad \to z=0$

height $= z_2-z_1 = a-x+y-(0) = a-x+y$

$\underline{z}$

(1) $V=\iint_R f(x,y)\,dA = \iint_R (a-x+y)\,da$

base area

$$\int_{x=-a}^{0}\left(\int_{y=0}^{a-\frac{x^2}{a}}(a-x+y)\,dy\right)dx$$