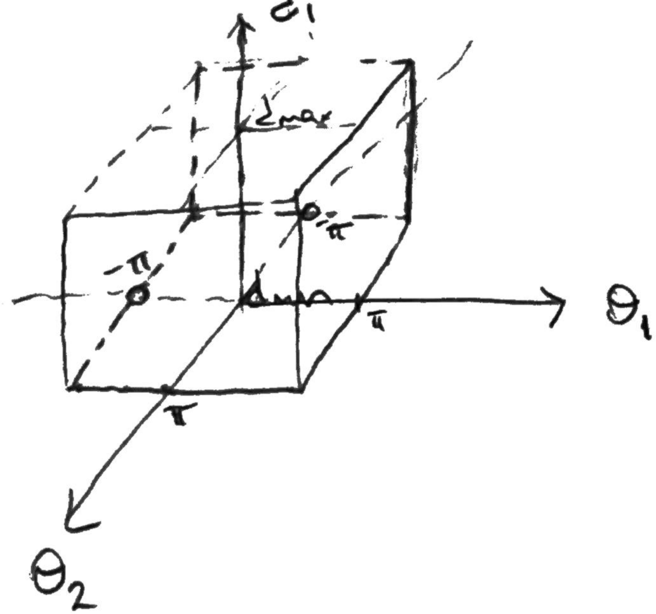
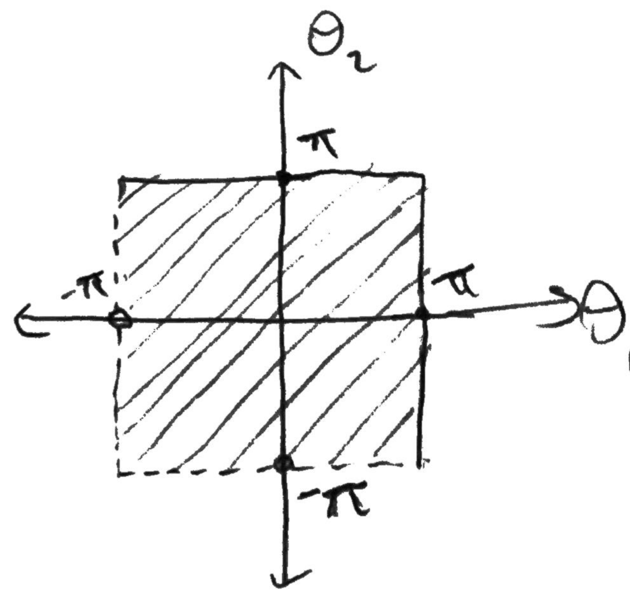
1)

a)



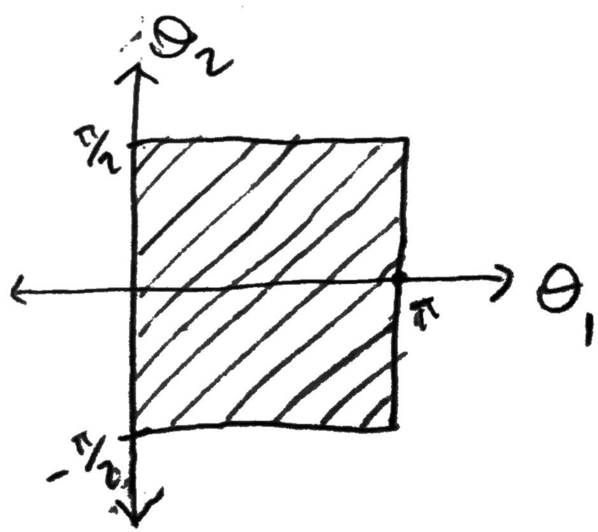
d_1
d_{max}
$-\pi$
π
d_{min}
π
θ_1
π
θ_2

b)



θ_2
π
$-\pi$
π
θ_1
$-\pi$

c)



θ_2
$\pi/2$
θ_1
π
$-\pi/2$

2) There are 4 DH parameters

- Joint angle, θ_j, is the angle between x_{j-1} and x_j axes about z_{j-1} axis
- Link offset, d_j, distance between origin of $j-1$'s frame and x_j axis along z_{j-1} axis
- Link length, a_j, distance between z_{j-1} and z_j axes along the x_j axis
- Link twist, α_j, angle between z_{j-1} and z_j axes about the x_j axis

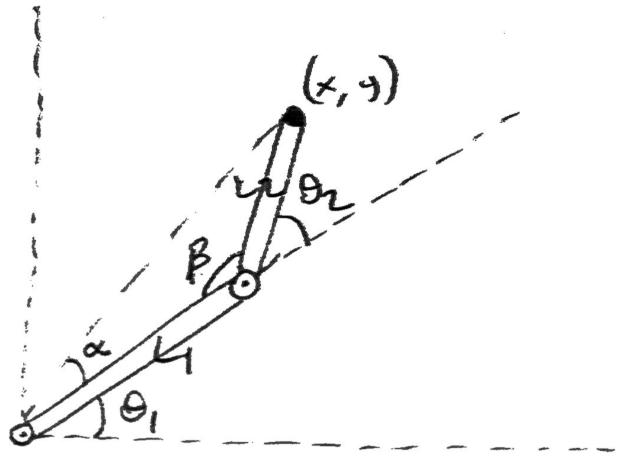
In our case,

- θ_1 and θ_2 joint angles are given as θ_1 and θ_2 respectively.
- d_1 and d_2 parameters are 0 since origin of the joint 1 is on the x axis of joint 2.
- a_1 and a_2 link lengths are given as d_1 and d_2 respectively.
- α_1 and α_2 link twists are 0 since z axes of both joints are parallel.

Hence, forward kinematics for the end effector becomes

$$\boxed{R(\theta_1)\,T(d_1)\,R(\theta_2)\,T(d_2)}$$

3)



Assume $\gamma = \theta_1 + \alpha$.

$\gamma = \text{atan2}(y, x)$

From the cosine theorem we can calculate α and β

$$x^2 + y^2 + L_1^2 = L_2^2 + 2 L_1 \sqrt{x^2+y^2} \cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{x^2+y^2+L_1^2-L_2^2}{2L_1\sqrt{x^2+y^2}}\right)$$

$$L_1^2 + L^2 = x^2 + y^2 + 2L_1L_2\cos\beta$$

$$\beta = \cos^{-1}\left(\frac{L_1^2+L_2^2-x^2-y^2}{2L_1L_2}\right)$$

Hence, we have two solutions

$$\boxed{\theta_1 = \gamma - \alpha, \quad \theta_2 = \pi - \beta}$$

and

$$\boxed{\theta_1 = \gamma + \alpha, \quad \theta_2 = \beta - \pi}$$

a) If $\sqrt{x_0^2+y_0^2} = d_1+d_2$, it has 1 solution.

$\alpha = 0, \beta = \pi \Rightarrow \theta_1 = \gamma = atan2(y_0, x_0), \theta_2 = 0.$

If $0 < \sqrt{x_0^2+y_0^2} < d_1+d_2$, it has 2 solutions just like the given equations.

If x_0 and y_0 are 0, it has infinite solutions for θ_1. ($\theta_2 = \pi$).

If $\sqrt{x_0^2+y_0^2} > d_1+d_2$, it has no solution.

b) and c)

In these cases it has infinite solutions since θ_2 can take any value between ~~0 and~~ $\pm\pi$.

d) If $d_{max}-d_2 < \sqrt{x_0^2+y_0^2} < d_{max}+d_2$, it has infinite solutions since there are infinite triangle whose corners are joints and end effector.

If $\sqrt{x_0^2+y_0^2} = d_{max}+d_2$, it has 1 solution.

$\alpha = 0, \beta = \pi \Rightarrow \theta_1 = \gamma = atan2(y_0, x_0), \theta_2 = 0$

If $\sqrt{x_0^2+y_0^2} > d_{max}+d_2$, it has no solution.

4) We know

$$S = \frac{d}{d\Theta} R(\Theta)\, R^T(\Theta)$$

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$\omega = [\omega_x, \omega_y, \omega_z]$$

Hence

$$\frac{d\theta}{dt} S = \frac{d\theta}{dt} \cdot \frac{d}{d\theta} R(\Theta)\, R^T(\Theta)$$

$$S(\omega) = \frac{d}{dt} R(\Theta)\, R^T(\Theta)$$

a) $$S_1(\omega_1) = \frac{d}{dt} A\, A^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a\sin(at) & a\cos(at) \\ 0 & -a\cos(at) & -a\sin(at) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(at) & -\sin(at) \\ 0 & \sin(at) & \cos(at) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}$$

$$\Rightarrow \omega_1 = [-a, 0, 0]$$

b)

$$S_2(\omega_2) = \frac{d}{dt} B B^T = \begin{bmatrix} -2bt \sin(bt^2) & 0 & 2bt\cos(bt^2) \\ 0 & 0 & 0 \\ -2bt\cos(bt^2) & 0 & -2bt \sin(bt^2) \end{bmatrix} \begin{bmatrix} \cos(bt^2) & 0 & -\sin(bt^2) \\ 0 & 1 & 0 \\ \sin(bt^2) & 0 & \cos(bt^2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2bt \\ 0 & 0 & 0 \\ -2bt & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \omega_2 = [0, 2bt, 0]$$

c)

$$S(\omega_3) = \frac{d}{dt}(AB)(AB)^T = \left(\frac{d}{dt} A B + A \frac{d}{dt} B\right) B^T A^T$$

$$= \frac{d}{dt} A A^T + A \frac{d}{dt} B B^T A^T$$

$$= S_1(\omega_1) + A S_2(\omega_2) A^T$$

$$= \begin{bmatrix} 0 & 2bt\sin(at) & 2bt\cos(at) \\ -2bt\sin(at) & 0 & a \\ -2bt\cos(at) & -a & 0 \end{bmatrix}$$

$$\Rightarrow \omega_3 = [-a, 2bt\cos(at), 2bt\sin(at)]$$