CENG460 HW1

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1 Q1

In order to find the transformation matrices, first we need to find the coordinates of the 4 points. For the points A, B and D it is easy and can be seen just by looking the three perspective.

$P_A = (-5, 2, 0)$, $P_B = (1, -1, 0)$, $P_D = (0, 0, 3)$

We can calculate the x and y coordinates of the point C with trigonometry from top view. Moreover, we know the height of the breaking points of the rectangle is 2. Hence height of the C is $2 + 2 * \cos(30)$.

$P_C = (-4 + \frac{2}{\sqrt{5}}, 4 - \frac{1}{\sqrt{5}}, 2 + \sqrt{3})$

Know, since we know the translations, we need to calculate rotations. We can see the angle between x axis of the point A and y direction is $\theta = \arctan(2)$. Therefore transformation matrices are the following:

$$T_A = T(-5, 2, 0) \cdot R_x(\frac{\pi}{2}) \cdot R_y(\arctan(2))$$

$$T_B = T(1, -1, 0) \cdot R_y(\pi)$$

$$T_C = T(-4 + \frac{2}{\sqrt{5}}, 4 - \frac{1}{\sqrt{5}}, 2 + \sqrt{3})) \cdot R_x(\pi) \cdot R_z(\arctan(\frac{1}{2})) \cdot R_y(\frac{-2\pi}{3})$$

$$T_D = T(0, 0, 3) \cdot R_x(\frac{-\pi}{2}) \cdot R_z(\frac{\pi}{4})$$

Finally, we can calculate the transformation matrices in the question.

$$\begin{aligned} {}^{A}T_C &= T_A^{-1} \cdot T_C \\ &= R_y(-\arctan(2)) \cdot R_x(\frac{-\pi}{2}) \cdot T(5 + \frac{2}{\sqrt{5}}, 2 - \frac{1}{\sqrt{5}}, 2 + \sqrt{3}) \cdot R_x(\pi) \cdot R_z(\arctan(\frac{1}{2})) \cdot R_y(\frac{-2\pi}{3}) \end{aligned}$$

$$
\begin{aligned}
{}^{B}T_D &= T_B^{-1} \cdot T_D \\
&= R_y(-\pi) \cdot T(-1,1,3) \cdot R_x(\frac{-\pi}{2}) \cdot R_z(\frac{\pi}{4})
\end{aligned}
$$

$$
\begin{aligned}
{}^{C}T_D &= T_C^{-1} \cdot T_D \\
&= R_y(\frac{2\pi}{3}) \cdot R_z(-\arctan(\frac{1}{2})) \cdot R_x(-\pi) \cdot T(4-\frac{2}{\sqrt{5}}, -4+\frac{1}{\sqrt{5}}, 1-\sqrt{3})) \cdot R_x(\frac{-\pi}{2}) \cdot R_z(\frac{\pi}{4})
\end{aligned}
$$

2 Q2

a) First, we need to translate the line so that it passes from the origin. Then we need to rotate it around x axis by some angle α so that it lies on x-z plane. Then, rotate around y axis with some angle $-\beta$ so that it lies on global z axis. Therefore, the transformation matrix is:

$$
\begin{aligned}
\tan(\alpha) &= \frac{b}{c} \\
\tan(\beta) &= \frac{a}{\sqrt{b^2+c^2}} \\
T &= R_y(-\arctan(\frac{a}{\sqrt{b^2+c^2}})) \cdot R_x(\arctan(\frac{b}{c})) \cdot T(-d,-e,-f)
\end{aligned}
$$

b) If we transform the point with the matrix in part a, rotation becomes just rotation around axis z. Then, we need transform back so that the point can be represented global coordinates again.Hence,

$$Q = T^{-1} \cdot R_z(\theta) \cdot T$$

c) All the vectors passing from point p hold the equation. Therefore, we can generalize the vectors as,

$$v = [x, y, z]^T t + [p_x, p_y, p_z]$$

with some arbitrary x,y and z.

d) İf the line is same, then it means alignment transformations are same for T_1 and T_2. Let's say alignment transformation is A and rotations are $R_z(\alpha)$,$R_z(\beta)$. Then, we can easily show the equality with the following

$$
\begin{aligned}
T_1 \cdot T_2 &= A^{-1} \cdot R_z(\alpha) \cdot A \cdot A^{-1} \cdot R_z(\beta) \cdot A \\
&= A^{-1} \cdot R_z(\alpha) \cdot R_z(\beta) \cdot A \\
&= A^{-1} \cdot R_z(\alpha+\beta) \cdot A
\end{aligned}
$$

$$
\begin{aligned}
T_2 \cdot T_1 &= A^{-1} \cdot R_z(\beta) \cdot A \cdot A^{-1} \cdot R_z(\alpha) \cdot A \\
&= A^{-1} \cdot R_z(\beta) \cdot R_z(\alpha) \cdot A \\
&= A^{-1} \cdot R_z(\beta + \alpha) \cdot A
\end{aligned}
$$

As it can be seen from equation (10) and (11), as long as the rotation line stays the same, the order of the rotations angles does not affect the result.

3 Q3

a) Let's assume that,

$\theta_3 = |\theta_2 - \theta_1|$

$\theta_4 = 2\pi - \theta_3$

where one of these angles is the smaller angle while the other one is the greater between θ_1 and θ_2. Then, the shortest angle α between θ_1 and θ_2 becomes,

$$
\begin{aligned}
\alpha &= \frac{(\theta_4 + \theta_3) - |\theta_4 - \theta_3|}{2} \\
&= \frac{(2\pi - \theta_3 + \theta_3) - |2\pi - \theta_3 - \theta_3|}{2} \\
&= \pi - |\pi - \theta_3| \\
&= \pi - |\pi - (\theta_2 - \theta_1)|
\end{aligned}
$$

b) We know that for a given quaternion q,

$q = \cos\frac{\theta}{2} < v \sin\frac{\theta}{2} >$

where v is the line rotate about and θ is the rotation angle. Therefore, we can get the rotation angles of 2 quatrains from formula above and translation this situation into the situation that we handled in part a so that we can find minimum angle between them.

4 Q4

a) For $n = 0$, we have,

$$R^0 = R^{-1}R = I$$

$$\begin{bmatrix} \cos 0 & -\sin 0 & 0 \\ \sin 0 & \cos 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Now, assume that the equation holds for n. Moreover, for $(n+1)$,

$$
\begin{aligned}
R^{n+1} = R \cdot R^n = R(\theta)R(n\theta) &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos n\theta & -\sin n\theta & 0 \\ \sin n\theta & \cos n\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta\cos n\theta - \sin\theta\sin n\theta & -\cos\theta\sin n\theta - \sin\theta\cos n\theta & 0 \\ \sin\theta\cos n\theta + \cos\theta\sin n\theta & -\sin\theta\sin n\theta + \cos\theta\cos n\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta + n\theta) & -\sin(\theta + n\theta) & 0 \\ \sin(\theta + n\theta) & \cos(\theta + n\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos((n+1)\theta) & -\sin((n+1)\theta) & 0 \\ \sin((n+1)\theta) & \cos((n+1)\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= R((n+1)\theta)
\end{aligned}
$$

b) Let's say that $R^n(\theta)$ is R^n, and we know that any invertible matrix R can be represented as

$R = PDP^{-1}$

where P is eigenvalue matrix and and D is diagonal eigenvector matrix. Therefore,

$R^n = e^{n\log(R)} = e^{n\log(PDP^{-1})} = e^{n(P\log(D)P^{-1})} = e^{(P\log(D^n)P^{-1})}$

Since the eigenvalues of the rotation matrix is the rotation angles, taking nth power of matrix D is just multiplying angles with n. Assume D' is the diagonal matrix having angles $n\theta$. Hence,

$e^{(P\log(D^n)P^{-1})} = e^{(P\log D'P^{-1})} = e^{\log(PD'P^{-1})} = PD'P^{-1} = R'$

where R' is $R(n\theta)$.

5 Q5

a) We know that inverse of a rotation matrix exists since it is equal to rotation with the same angle but with the minus sign. Moreover, let V and D be the eigenvector and eigenvalue matrices of rotation matrix R respectively.

$R = VDV^{-1}$

If we look at the diagonalized vector D, we can see two eigenvalues are in complex form like $a + bi$, $a - bi$ which are conjugate, and the other one is simply 1. Therefore, matrix D becomes like a rotation matrix such that,

$a + bi = \cos\theta + i\sin\theta$

Finally with the euler formula we can calculate the angle θ.

6 Q6

We know that for two lines being linearly dependent, they must be collinear, i.e. one of them is scalar multiplied version of the other one. We also know the quaternion multiplication formula,

$q_1 \circ q_2 = s_1s_2 - v_1 \cdot v_2 < s_1v_2 + s_2v_1 + v_1 \times v_2 >$

s_1s_2 multiplication is nothing but a scalar multiplication.

Since dot product is commutative the $v_1 \cdot v_2$ part is not effected by the order.

$s_1v_2 + s_2v_1$ part is not

However, the $v_1 \times v_2$ part changes if the order changes since $i \times j = k$ and $j \times i = -k$, so we need to keep the cross product part as 0 in order not to be effected from the order of inputs. For two lines in 3D, their cross product is 0 if and only if they are collinear.