

# Quantum Many-Body Simulations of Double Dot System

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# Introduction

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## 1. Small semiconductor nanostructures

# Quantum-Dot Model

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- Schrödinger equation

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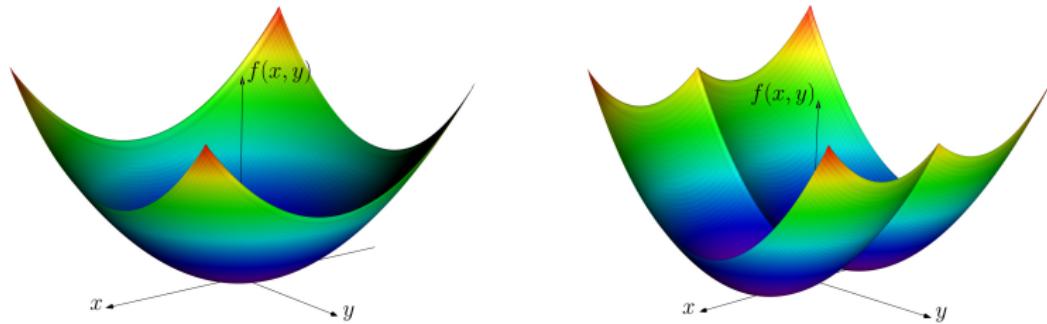
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<sup>1</sup>S. Kvaal. "Harmonic Oscillator Eigenfunction Expansions, Quantum dots, and Effective Interactions". In: *Phys. Rev. B* (2009).

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 $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$        $V(\mathbf{r}) = \frac{1}{2}m\omega^2(r^2 - \delta R|x| + R^2)$



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## Methods

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**Hartree-Fock  
Variational Monte-Carlo**

## Methods: Variational Principle

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$$E_0 \leq \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

# **Slater Determinant and Energy Functional**

## Methods: Slater Determinant and Energy Functional

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- $E[\Psi] = \frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \sum_p \langle p | \mathcal{H}_0 | p \rangle + \frac{1}{2} \sum_{p,q} [ \langle pq | f_{12} | pq \rangle \pm \langle pq | f_{12} | qp \rangle ]$

# Hartree-Fock

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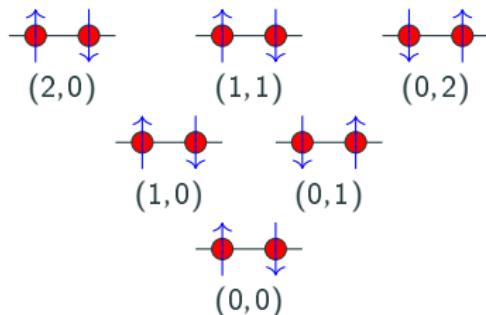
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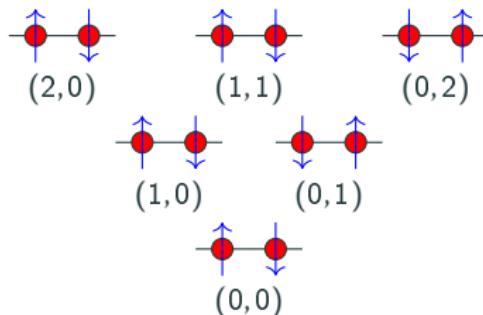
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    - $\mathcal{J} \equiv \langle \psi_k^* | f_{12} | \psi_k \rangle = \int \psi_k^*(\mathbf{r}) f_{12} \psi_k(\mathbf{r}) d\mathbf{r}$
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    - $\mathcal{F} |\psi\rangle = \boldsymbol{\varepsilon} |\psi\rangle, \boldsymbol{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_N)$

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- Roothan-Hall:  $\mathbf{FC}_i = \epsilon S \mathbf{C}_i$ 
  - $F_{pq} = h_{pq} + \sum_{pq} \rho_{pq} (2D_{prqs} \pm D_{prsq})$
  - $h_{pq} \equiv \langle p | h | q \rangle$
  - $\rho_{pq} \equiv \sum_{i=1}^{\frac{N}{2}} C_{pi} C_{qi}^*$
  - $D_{pqrs} \equiv \langle pq | f_{12} | rs \rangle$
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- Poople-Nesbet:  $\mathbf{F}^+ \mathbf{C}^+ = \boldsymbol{\epsilon} \mathbf{S} \mathbf{C}^+, \mathbf{F}^- \mathbf{C}^- = \boldsymbol{\epsilon}^- \mathbf{S} \mathbf{C}^-$ 
  - $F_{pq}^\pm = h_{pq} + \sum_{k_\pm} \sum_{rs} C_{rk_\pm}^{\pm\dagger} C_{sk_\pm}^{\pm\dagger} [D_{prqs} - D_{prsq}] + \sum_{k_\mp} \sum_{rs} C_{rk_\mp}^{\mp\dagger} C_{sk_\mp}^{\mp\dagger} D_{prqs}$

# Variational Monte-Carlo

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- Rewrite expectation value:  $\frac{\langle \Psi | \mathcal{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \int \frac{\Psi^* \mathcal{H} \Psi}{\int \Psi^* \Psi dr} dr = \int \frac{|\Psi|^2}{\int \Psi^* \Psi dr} E_L dr$

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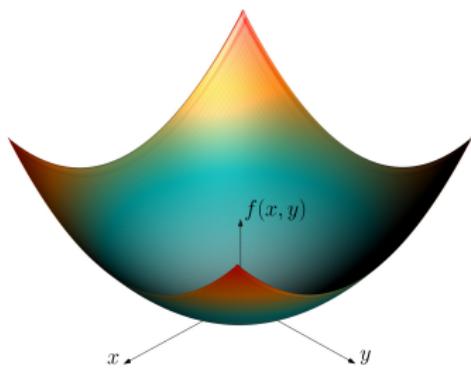
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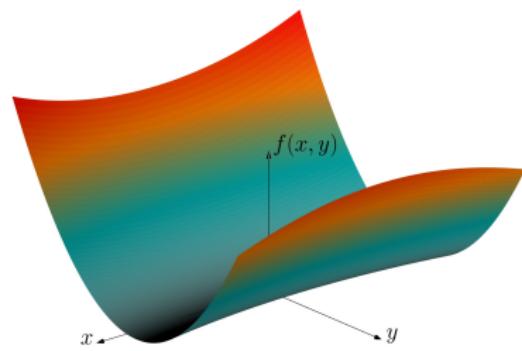
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  - Importance Sampling
    - $r^{\text{new}} = r^{\text{old}} + D \Delta t F^{\text{old}} + \sqrt{\Delta t} \xi$
    - $F = \frac{2}{\Psi} \nabla \Psi$
    - $\frac{T(b, a, \Delta t)}{T(a, b, \Delta t)} = \sum_i \exp\left(-\frac{(r_i^{(b)} - r_i^{(a)} - D \Delta t F_i^{(a)})^2}{4 D \Delta t} + \frac{(r_i^{(a)} - r_i^{(b)} - D \Delta t F_i^{(b)})^2}{4 D \Delta t}\right)$

# Minimization

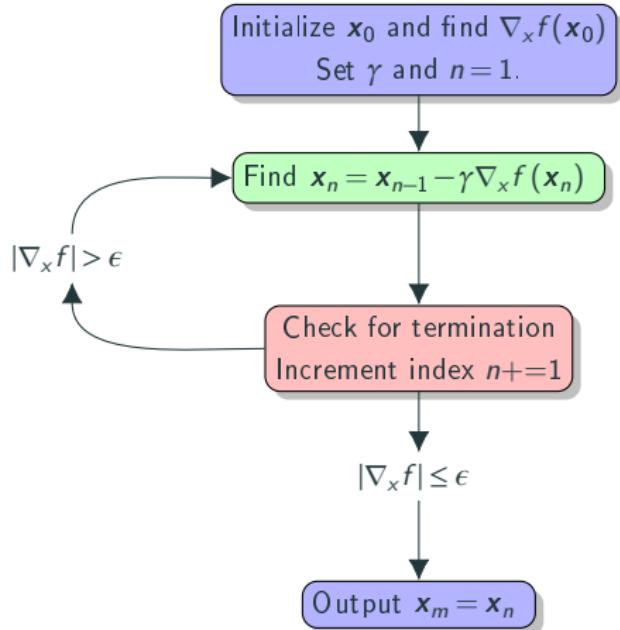
## Single-Well



## Rosenbrock



# Minimization: Gradient Descent



# Minimization: Gradient Descent

$x_0$	$\gamma$	Iterations	$x_m$	$f(x_m)$
(5, 5)	0.9	20	(-0.072, -0.072)	0.010
(5, 5)	0.9	50	( $-8.920 \times 10^{-5}$ , $-8.920 \times 10^{-5}$ )	$1.591 \times 10^{-8}$
(5, 5)	0.9	100	( $-1.273 \times 10^{-9}$ , $-1.273 \times 10^{-9}$ )	$3.242 \times 10^{-18}$
(5, 5)	0.5	20	(0.0, 0.0)	0.0
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(0, 0.5)	0.001	100	(0.181, 0.030)	0.034
(0, 0.5)	0.001	500	(0.512, 0.258)	0.327
(0, 0.5)	0.001	1000	(0.675, 0.454)	0.106
(0, 0.5)	0.001	100000	(1.000, 1.000)	0.0
(0, 0.5)	0.0001	100	(0.027, 0.068)	1.399
(0, 0.5)	0.0001	500	(0.105, 0.009)	0.801
(0, 0.5)	0.0001	1000	(0.184, 0.031)	0.666
(0, 0.5)	0.0001	100000	(0.994, 0.989)	$3.131 \times 10^{-5}$

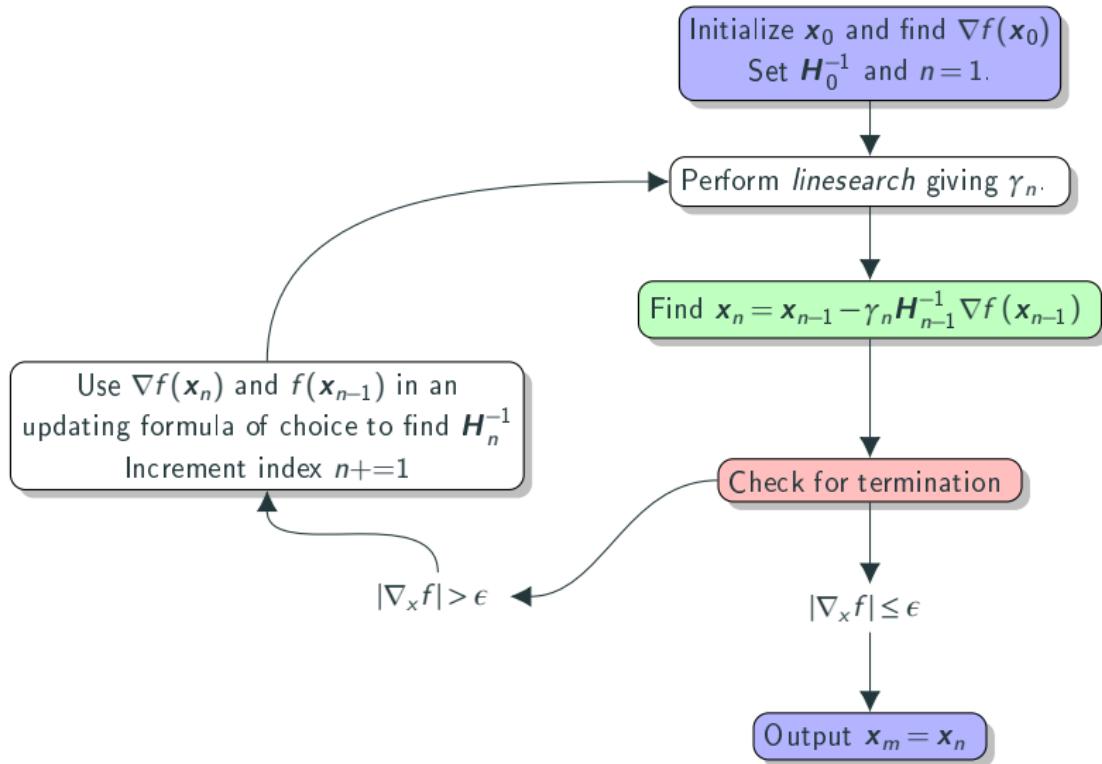
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(-1,2)	1	(0.447,-0.894)	1.000
(1,1)	2	(0.000,0.000)	0.000
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(-0.5,2.0)	1	(-0.706,0.708)	7.280
(-0.5,2.0)	2	(-0.780,0.649)	3.342
(-0.5,2.0)	10	(0.238,0.051)	0.584
(-0.5,2.0)	30	(1.000,1,000)	0.000
(5.5,-10.0)	1	(-0.996,0.091)	85.214
(5.5,-10.0)	2	(-0.908,1.087)	10.549
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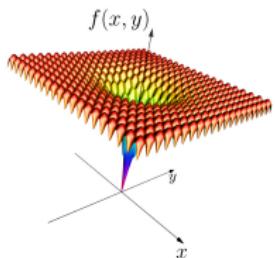
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(5.5,-10.0)	1	(-0.996,0.091)	85.214
(5.5,-10.0)	2	(-0.908,1.087)	10.549
(5.5,-10.0)	10	(0.027,0.012)	0.9613
(5.5,-10.0)	30	(1.000,1,000)	0.000

## Minimization: Simulated Annealing

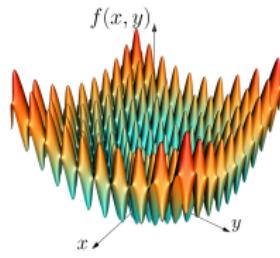
---

# Minimization: Simulated Annealing

Ackley



Rastrigin



# Wavefunction

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# Wavefunction

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$$\phi_i(\mathbf{r}) = \sum_{p=1}^L C_{pi} \chi_p(\mathbf{r})$$

# Wavefunction

---

$$\phi_i(\mathbf{r}) = \sum_{p=1}^L C_{pi} \chi_p(\mathbf{r})$$

## Wavefunction: Integral Elements

---

$$\langle \phi_i(\mathbf{r}) | \phi_j(\mathbf{r}) \rangle$$

$$\langle \phi_i(\mathbf{r}) | x_d^k | \phi_j(\mathbf{r}) \rangle$$

$$\langle \phi_i(\mathbf{r}) | \nabla^2 | \phi_j(\mathbf{r}) \rangle$$

$$\langle \phi_i(\mathbf{r}_1) \phi_j(\mathbf{r}_2) | f_{12} | \phi_k(\mathbf{r}_1) \phi_l(\mathbf{r}_2) \rangle$$

## Wavefunction: Single-Well

---

- Hermite Function:  $\psi_n(\mathbf{r}) \equiv \prod_d N_d H_{n_d}(\sqrt{\omega}x_d) \exp\left(-\frac{\omega}{2}x_d^2\right)$

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- Solution in polar<sup>3</sup>

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<sup>3</sup>E. Anisimovas and A. Matulis. “Energy spectra of few-electron quantum dots”. In: *Journal of Physics: Condensed Matter* (1998), p. 601.

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- Solution in polar<sup>3</sup>
- $\psi_n(\mathbf{r}) = \prod_d N_d \sum_{l=1}^{n_d} C_{n_d l}^{\text{Hermite}} g_l\left(\frac{\omega}{2}, \mathbf{r}, \mathbf{0}\right)$

---

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# Wavefunction: Single-Well

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- Solution in Cartesian<sup>4</sup>

$$\langle g_i(\mathbf{r}) | g_j(\mathbf{r}) \rangle$$

$$\langle g_i(\mathbf{r}) | x_d^k | g_j(\mathbf{r}) \rangle$$

$$\langle g_i(\mathbf{r}) | \nabla^2 | g_j(\mathbf{r}) \rangle$$

$$\langle g_i(\mathbf{r}_1) g_j(\mathbf{r}_2) | f_{12} | g_k(\mathbf{r}_1) g_l(\mathbf{r}_2) \rangle$$

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<sup>4</sup>J. Olsen T. Helgaker P. Jørgensen. *Molecular Electronic-Structure Theory*. Wiley, 2014. isbn: 978-0-47-196755-2. doi: 10.1002/9781119019572.

# Wavefunction: Single-Well Integral Elements

$$\begin{aligned}\langle \psi_i^{\text{HO}} | \psi_j^{\text{HO}} \rangle &= N_i \delta_{ij} \\ \langle \psi_i^{\text{HO}} | h^{\text{HO}} | \psi_j^{\text{HO}} \rangle &= N_i \epsilon_i^{\text{HO}} \delta_{ij}\end{aligned}$$

$$\langle \psi_i^{\text{HO}} \psi_j^{\text{HO}} | \frac{1}{r_{12}} | \psi_k^{\text{HO}} \psi_l^{\text{HO}} \rangle = \frac{aN_{ijkl}}{\sqrt{2\omega}} \sum_{tuvw}^{ijkl} H_{tuvw}^{ijkl} \sum_{pq}^{t+v,u+w} E_p^{tv} E_q^{uw} (-1)^q \xi_{p+q} \left( \frac{\omega}{2}, \mathbf{0} \right)$$

$$E_t^{i+1,j} = \frac{1}{2(\alpha + \beta)} E_{t-1}^{ij} - \frac{\beta}{\alpha + \beta} (A_x - B_x) E_t^{ij} + (t+1) E_{t+1}^{ij}$$

$$E_t^{i,j+1} = \frac{1}{2(\alpha + \beta)} E_{t-1}^{ij} - \frac{\alpha}{\alpha + \beta} (A_y - B_y) E_t^{ij} + (t+1) E_{t+1}^{ij}$$

$$E_0^{00} = K_{AB}$$

$$\xi_{t+1,u}^n = t \xi_{t-1,u}^{n+1} + X_{AB} \xi_{t,u}^{n+1}$$

$$\xi_{t,u+1}^n = u \xi_{t,u-1}^{n+1} + Y_{AB} \xi_{t,u}^{n+1}$$

$$\xi_{00}^n = \left( \frac{-2\alpha\beta}{\alpha + \beta} \right)^n \zeta_n \left( \frac{\alpha\beta}{\alpha + \beta} R_{AB}^2 \right)$$

$$\zeta_n(x) = \int_{-1}^1 \frac{u^{2n}}{\sqrt{1-u^2}} e^{-u^2 x} du$$

$$\xi_{t+1,u,v}^n = t \xi_{t-1,u,v}^{n+1} + X_{AB} \xi_{t,u,v}^{n+1}$$

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## Wavefunction: Double-Well

---

- Perturbation of harmonic oscillator:  $U^{\text{DW}}(r) = V^{\text{HO}}(r) + V_n^{\text{DW}}(r)$

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---

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- Eigenvalue equation:  $H^{\text{DW}} C^{\text{DW}} = \epsilon^{\text{DW}} C^{\text{DW}}$ 
  - $H_{ij}^{\text{DW}} = \epsilon_i^{\text{HO}} \delta_{ij} + \langle \psi_i^{\text{HO}} \left| V_n^{\text{DW}} \right| \psi_j^{\text{HO}} \rangle$

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- Integral-Elements

$$\langle \psi_p^{\text{DW}} \left| \psi_q^{\text{DW}} \right\rangle = \delta_{pq}$$

$$\langle \psi_p^{\text{DW}} \left| h^{\text{DW}} \right| \psi_q^{\text{DW}} \rangle = \epsilon_p^{\text{DW}} \delta_{pq}$$

$$\left\langle \psi_p^{\text{DW}} \psi_q^{\text{DW}} \left| \frac{1}{r_{12}} \right| \psi_r^{\text{DW}} \psi_s^{\text{DW}} \right\rangle = \sum_{tuvw}^{ijkl} C_{tp}^{\text{DW}} C_{uq}^{\text{DW}} C_{vr}^{\text{DW}} C_{ws}^{\text{DW}} \left\langle \psi_t^{\text{HO}} \psi_u^{\text{HO}} \left| \frac{1}{r_{12}} \right| \psi_v^{\text{HO}} \psi_w^{\text{HO}} \right\rangle$$

## Wavefunction: Slater-Jastrow

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- NQS:  $J_{\text{NQS}} = e^{-\sum_{i=1}^N \frac{(r_i - a_i)^2}{2\sigma^2}} \prod_j^M \left( 1 + e^{b_j + \sum_{i=1}^N \sum_{d=1}^D \frac{x_i^{(d)} w_{i+d,j}}{\sigma^2}} \right)$

## Wavefunction: Slater-Jastrow

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- Padé-NQS:  $J = J_{\text{Padé}} J_{\text{NQS}}$

# Implementation

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# Implementation

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- C++ and Eigen

# Implementation

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- C++ and Eigen
  - Performance

# Implementation

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- C++ and Eigen
  - Performance
  - Generalization

# Implementation

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- C++ and Eigen
  - Performance
  - Generalization
- Python

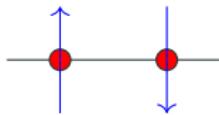
# Implementation

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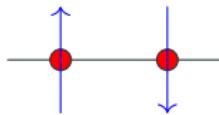
- C++ and Eigen
  - Performance
  - Generalization
- Python
  - Generate C++ code

## Implementation: Cartesian

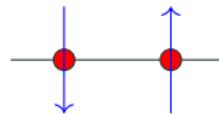
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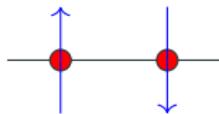
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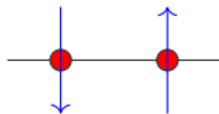
(1,1)



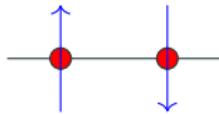
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(1,0)

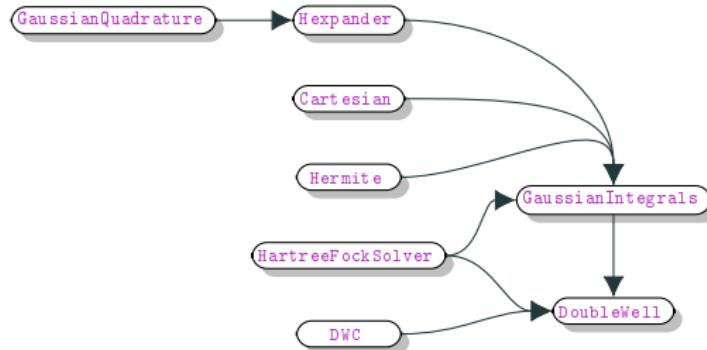


(0,1)

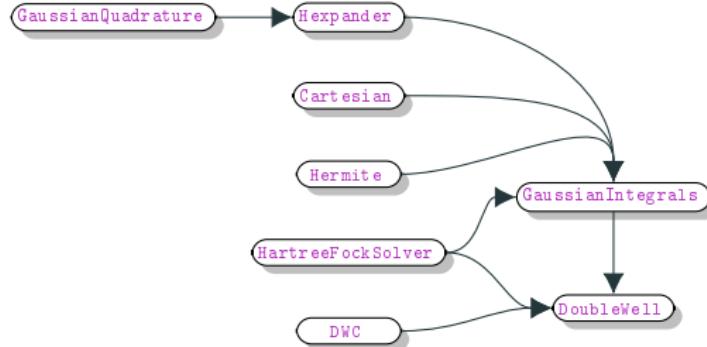


(0,0)

# Implementation: Hartree-Fock

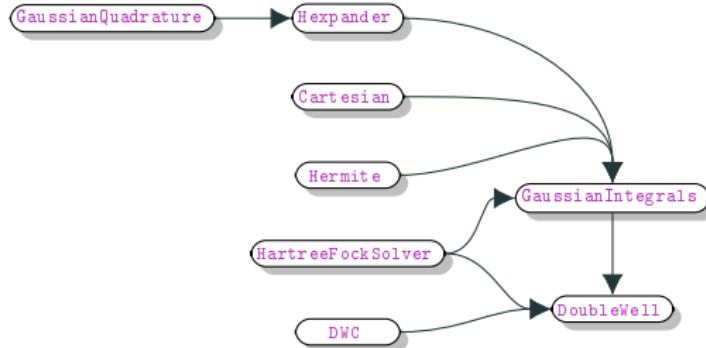


# Implementation: Hartree-Fock



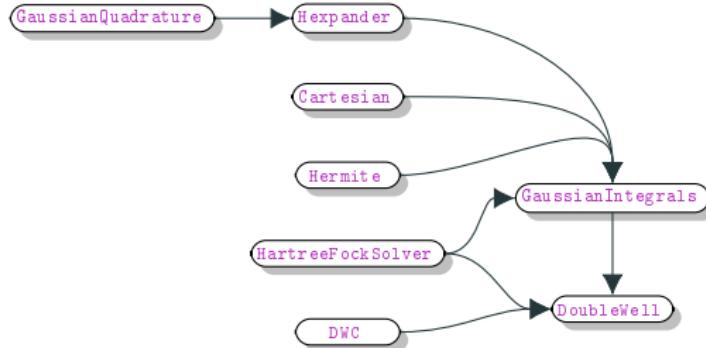
- Parallelization

# Implementation: Hartree-Fock



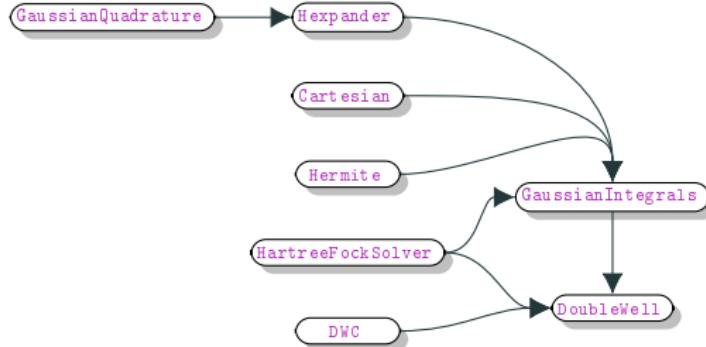
- Parallelization
  - Two-body element is computationally expensive
  - $S_i = \sum_{j=0}^{P_i} \prod_d (n_{j_d} + 1)$

# Implementation: Hartree-Fock



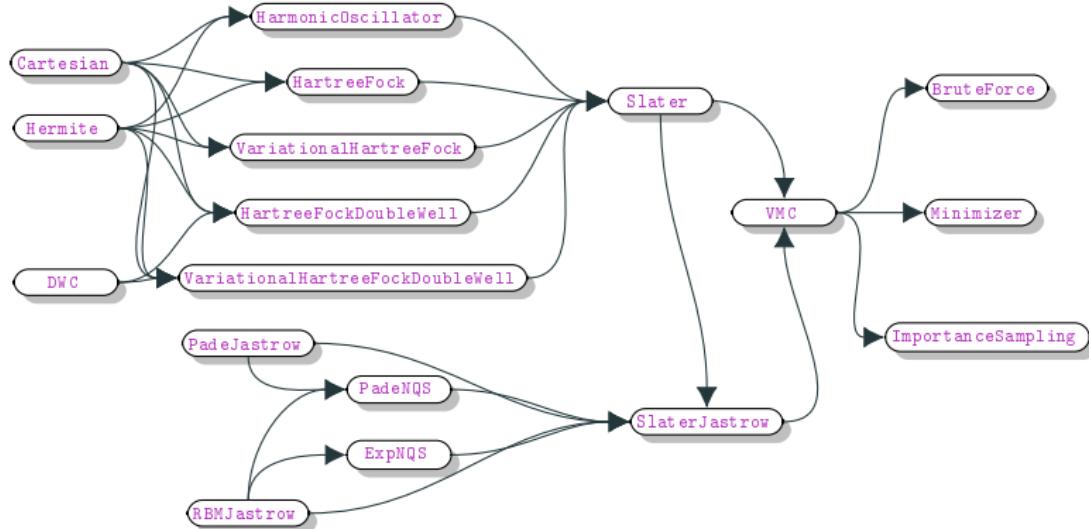
- Parallelization
  - Two-body element is computationally expensive
  - $S_i = \sum_{j=0}^{P_i} \prod_d (n_{jd} + 1)$
- Hartree-Fock algorithm only run on one process

# Implementation: Hartree-Fock

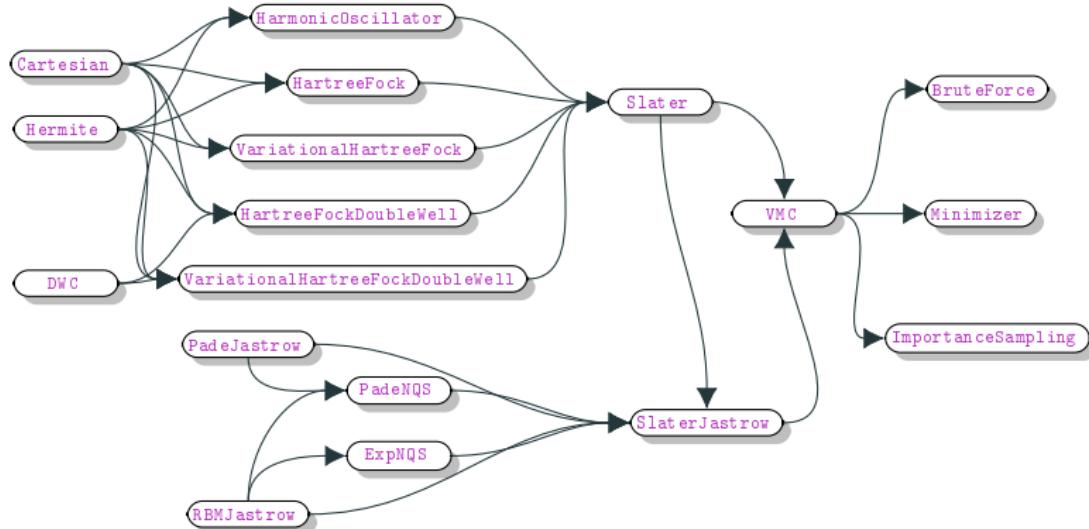


- Parallelization
  - Two-body element is computationally expensive
  - $S_i = \sum_{j=0}^{P_i} \prod_d (n_{jd} + 1)$
- Hartree-Fock algorithm only run on one process
- Tabulation of Two-Body matrix

# Implementation: Variational Monte-Carlo



# Implementation: Variational Monte-Carlo



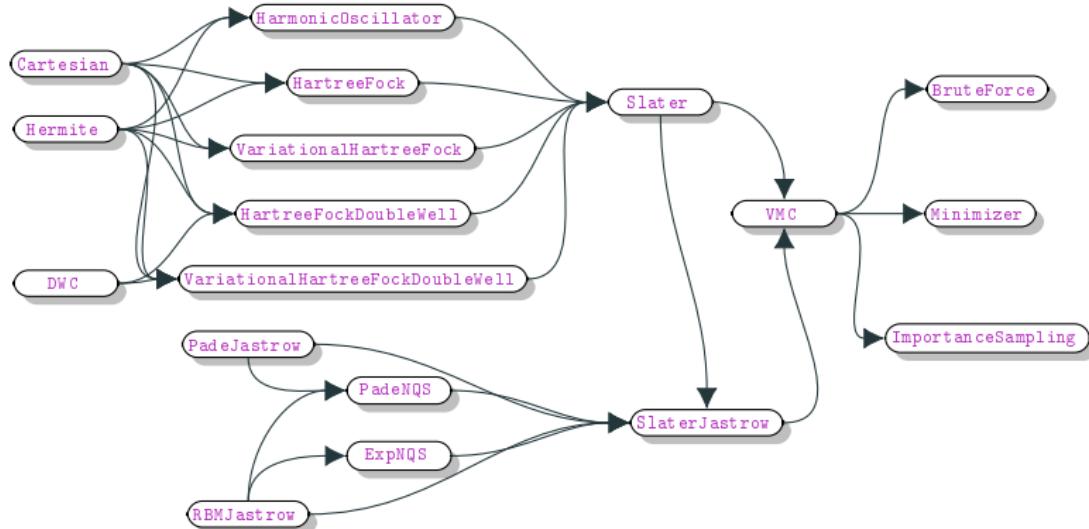
- Hermite generated with Python and SymPy

# Implementation: Variational Monte-Carlo

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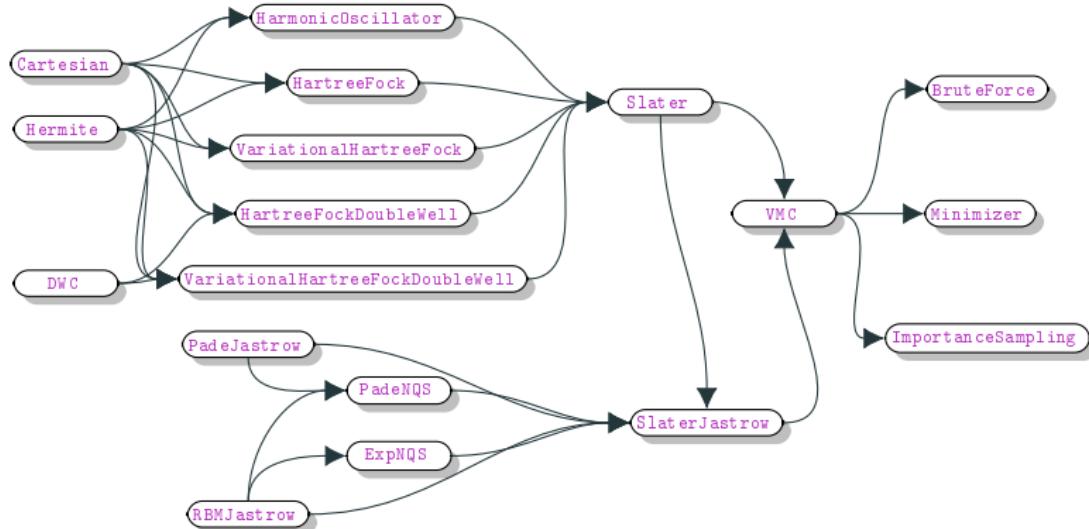
- `set`: Called during initialization (before each sampling)
- `reSetAll`: Sets all matrices to zero (used in testing)
- `initializeMatrices`: Allocate memory
- `update`: Update positions and wavefunction
- `reset`: Revert to previous positions and wavefunction
- `resetGradient`: Revert to previous gradient
- `acceptState`: Update previous positions and wavefunction to current
- `acceptGradient`: Update previous gradient to current one

# Implementation: Variational Monte-Carlo



- **Hermite** generated with Python and SymPy

# Implementation: Variational Monte-Carlo



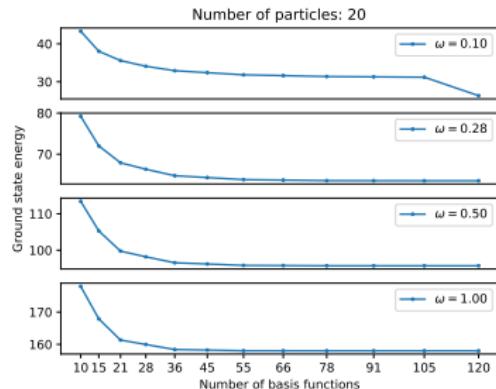
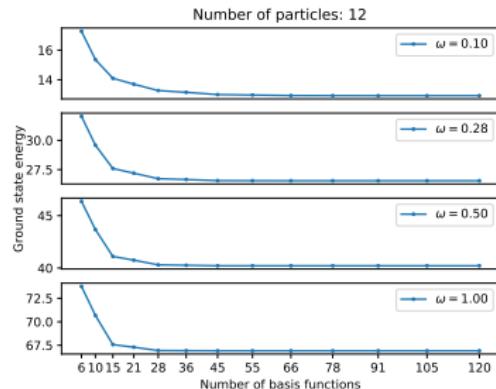
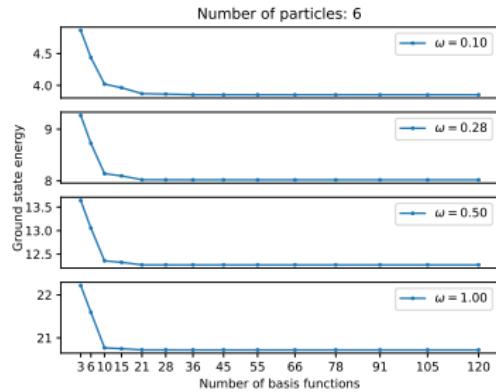
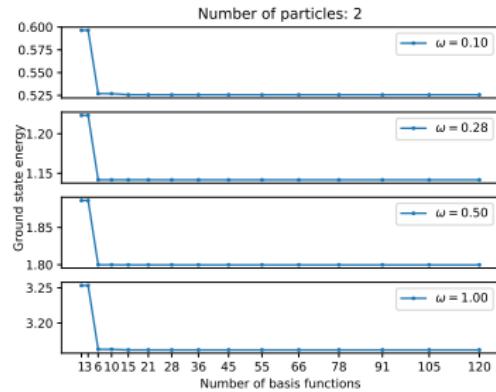
- Hermite generated with Python and SymPy
- Wavefunction class can be created with Python

# Results

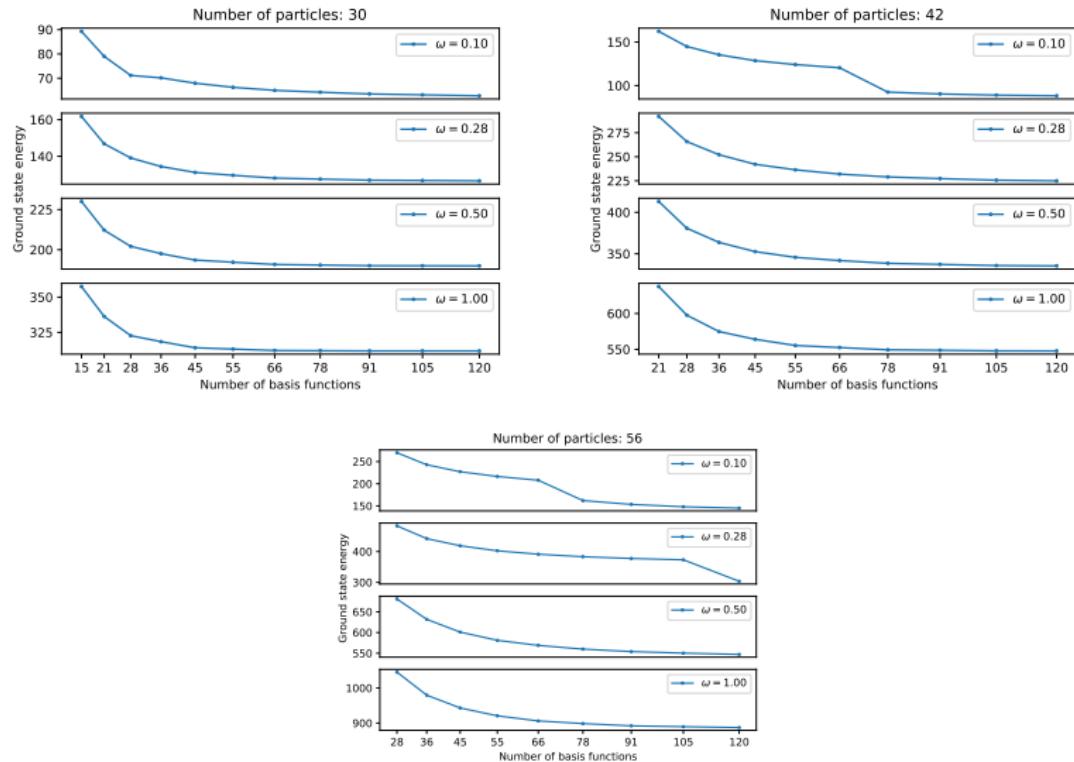
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# Benchmark

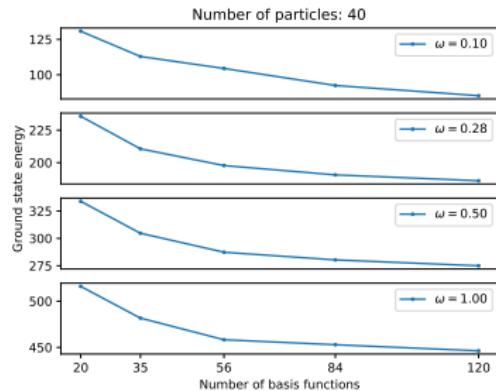
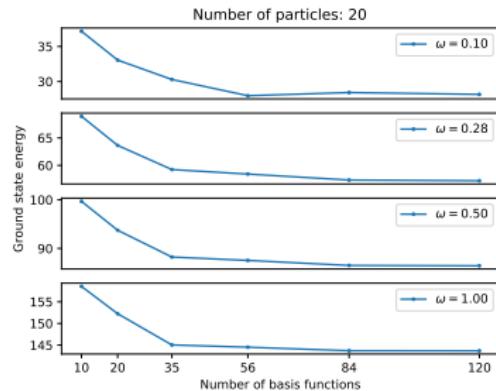
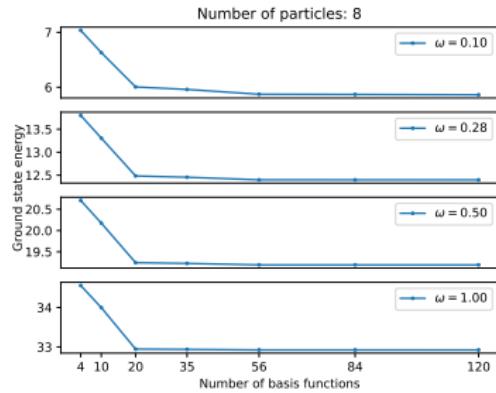
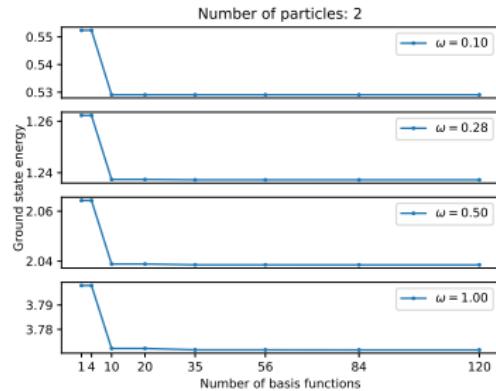
# Results: Benchmark



# Results: Benchmark



# Results: Benchmark



## Results: Benchmark

$\omega$ [a.u]	$N$			
	2	6	12	20
0.1	0.4407(4)	3.5650(4)	12.3164(4)	30.0480(4)
0.28	1.0020(4)	7.6198(4)	25.5948(3)	61.8090(3)
0.5	1.6650(4)	11.8017(4)	39.3166(3)	93.9240(2)
1.0	3.0000(5)	20.2863(3)	68.1465(3)	156.2778(2)

$\omega$ [a.u]	$N$	
	2	8
0.1	0.50006(5)	5.80479(4)
0.28	1.20156(5)	12.48178(4)
0.5	2.00027(5)	19.33356(4)
1.0	3.72985(5)	33.30958(4)

$$\psi = \psi^{\text{HO}}(\sqrt{\alpha\omega}) J_{\text{Pad\'e}}$$

## Results: Benchmark

$\omega$ [a.u]	$N$			
	2	6	12	20
0.1	0.46552(5){15}	3.70137(4){36}	12.64342(4){91}	-
0.28	1.04939(4){6}	7.89627(4){36}	26.21301(4){66}	62.93503(5){120}
0.5	1.70130(4){6}	12.02776(4){21}	39.76442(3){45}	95.21976(3){91}
1.0	3.05625(4){6}	20.45876(3){36}	66.37115(3){45}	157.41119(3){78}

$\omega$ [a.u]	$N$			
	2	6	12	20
0.10	0.44473(5){15}	3.63897(4){36}	12.46408(4){91}	-
0.28	1.04978(4){6}	7.72929(4){36}	25.96595(4){66}	62.65652(3){120}
0.50	1.66418(4){6}	11.97781(4){21}	39.57182(3){45}	94.76303(3){91}
1.00	3.00624(4){6}	20.38811(3){36}	66.28996(3){45}	157.46167(3){78}

$$\psi_p = \sum_l C_{lp} \psi_l^{\text{HO}}(\sqrt{\omega} r) J_{\text{Pad\'e}}, \quad \psi_p = \sum_l C_{lp} \psi_l^{\text{HO}}(\sqrt{\alpha \omega} r) J_{\text{Pad\'e}}$$

## Results: Benchmark

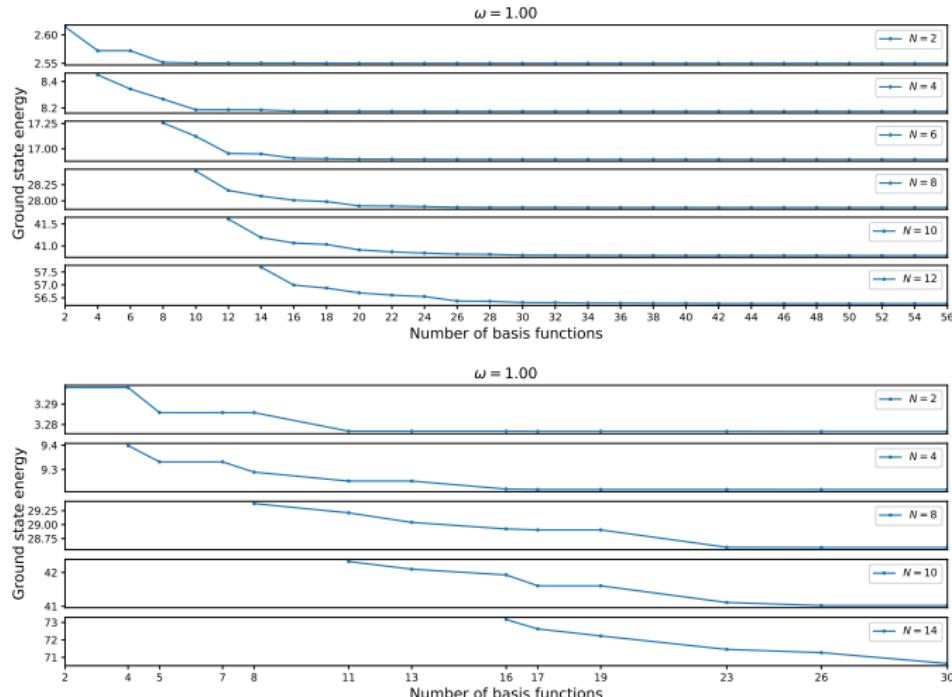
$\omega$	$N$	
	2	8
0.1	0.51122(5){70}	5.87372(4){120}
0.28	1.21844(5){70}	12.36177(4){168}
0.5	2.02030(4){20}	19.15006(4){112}
1.0	3.72918(5){20}	33.58046(4){168}

$\omega$	$N$	
	2	8
0.1	0.50751(5){70}	5.84082(4){240}
0.28	1.20320(5){20}	12.37435(4){168}
0.5	2.01439(4){20}	19.09917(4){112}
1.0	3.72959(5){70}	33.04162(4){168}

$$\psi_p = \sum_l C_{lp} \psi_l^{\text{HO}}(\sqrt{\omega} r) J_{\text{Pad\'e}}, \quad \psi_p = \sum_l C_{lp} \psi_l^{\text{HO}}(\sqrt{\alpha\omega} r) J_{\text{Pad\'e}}$$

# Results: Double-Well Hartree-Fock



## Results: Double-Well Variational Monte-Carlo

$\omega$	$N$			
	2	4	6	8
1.0	2.42238(4){10}	7.95247(4){42}	16.61419(4){44}	27.54453(3){50}

$$\psi_p = \sum_l C_{lp}^{\text{HF}} \sum_k C_{kl}^{\text{DW}} \psi_k^{\text{HO}} (\sqrt{\omega} r) J_{\text{Pad\'e}}$$

$\omega$	$N$			
	2	4	6	8
1.0	2.36618(4){10}	7.90232(4){42}	16.55609(4){44}	27.58524(4){50}

$$\psi_p = \sum_l C_{lp}^{\text{HF}} \sum_k C_{kl}^{\text{DW}} \psi_k^{\text{HO}} (\sqrt{\alpha \omega} r) J_{\text{Pad\'e}}$$

## Results: Double-Well Variational Monte-Carlo

$\omega$	$N$		
	2	4	8
1.0	3.25118(4){11}	9.17489(4){17}	28.49671(4){26}

$$\psi_p = \sum_l C_{lp}^{\text{HF}} \sum_k C_{kl}^{\text{DW}} \psi_k^{\text{HO}} (\sqrt{\omega} r) J_{\text{Pad\'e}}$$

$\omega$	$N$		
	2	4	8
1.0	3.22226(4){11}	9.17013(4){17}	28.62826(4){26}

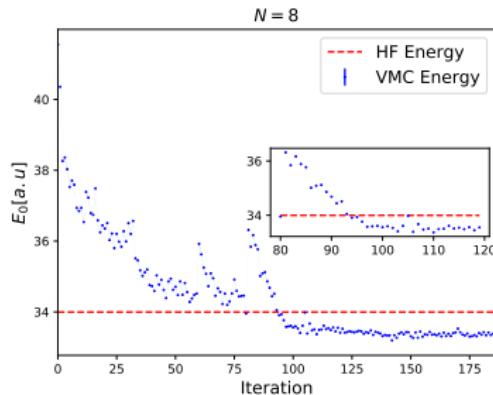
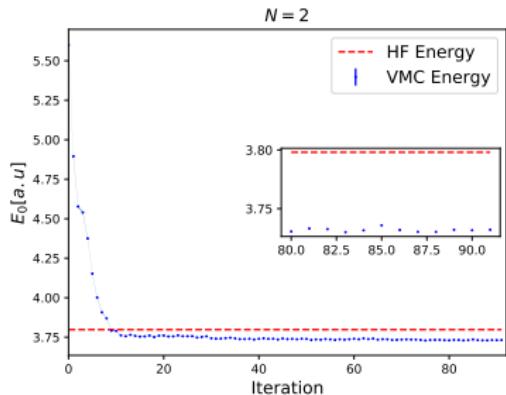
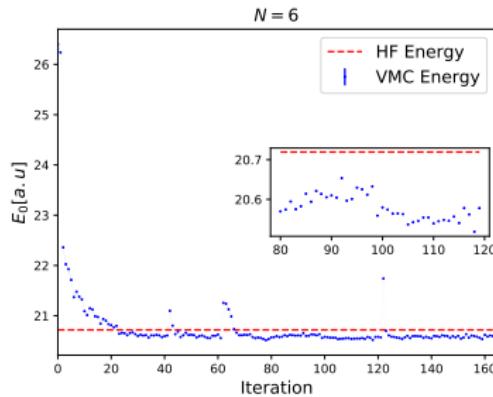
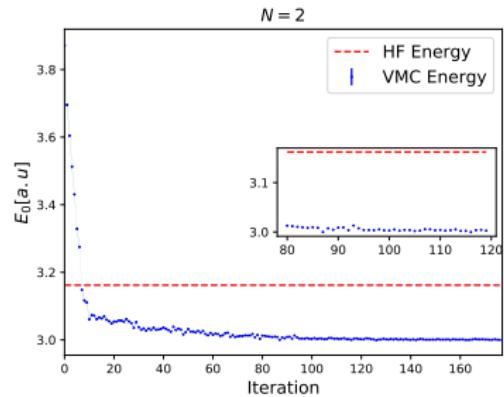
$$\psi_p = \sum_l C_{lp}^{\text{HF}} \sum_k C_{kl}^{\text{DW}} \psi_k^{\text{HO}} (\sqrt{\alpha\omega} r) J_{\text{Pad\'e}}$$

## Results: NQS-Jastrow

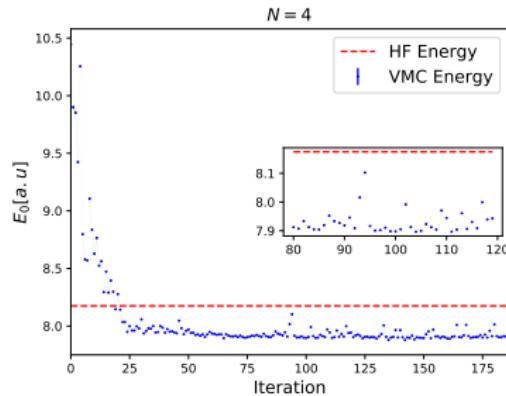
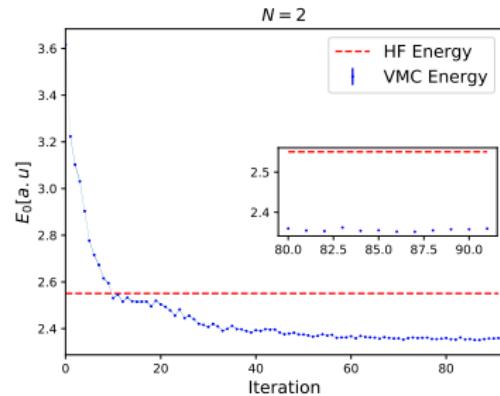
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$$J_{\text{NQS}} = e^{-\sum_{i=1}^N \frac{(r_i - a_i)^2}{2\sigma^2}} \prod_j^M \left( 1 + e^{b_j + \sum_{i=1}^N \sum_{d=1}^D \frac{x_i^{(d)} w_{i+d,j}}{\sigma^2}} \right)$$

# Results: NQS-Jastrow Harmonic Oscillator



# Results: NQS-Jastrow Double-Well



## Summary and Conclusion

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# Summary

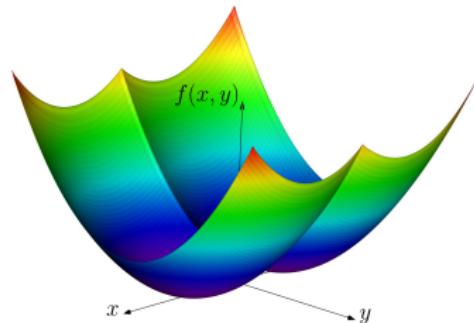
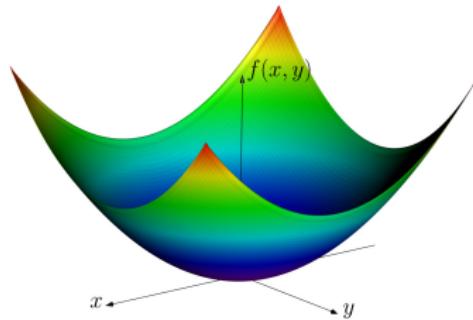
- Interaction:

- $f(\mathbf{r}_i, \mathbf{r}_j) = \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$ , Coulomb Repulsion

- Confinement: Harmonic Oscillator, Double-Well

$$V(\mathbf{r}) = \frac{1}{2} m \omega^2 r^2$$

$$V(\mathbf{r}) = \frac{1}{2} m \omega^2 (r^2 - \delta R|x| + R^2)$$



**Hartree-Fock  
Variational Monte-Carlo**

# Summary

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$$\begin{aligned}\left\langle \psi_i^{\text{HO}} \middle| \psi_j^{\text{HO}} \right\rangle &= N_i \delta_{ij} \\ \left\langle \psi_i^{\text{HO}} \middle| h^{\text{HO}} \right\rangle &= N_i \epsilon_i^{\text{HO}} \delta_{ij} \\ \left\langle \psi_i^{\text{HO}} \psi_j^{\text{HO}} \middle| \frac{1}{r_{12}} \right\rangle \left\langle \psi_k^{\text{HO}} \psi_l^{\text{HO}} \right\rangle &= \frac{aN_{ijkl}}{\sqrt{2\omega}} \sum_{tuvw}^{ijkl} H_{tuvw}^{ijkl} \sum_{pq}^{t+v,u+w} E_p^{tv} E_q^{uw} (-1)^q \xi_{p+q} \left( \frac{\omega}{2}, \mathbf{0} \right)\end{aligned}$$

$$E_t^{i+1,j} = \frac{1}{2(\alpha + \beta)} E_{t-1}^{ij} - \frac{\beta}{\alpha + \beta} (A_x - B_x) E_t^{ij} + (t+1) E_{t+1}^{ij}$$

$$E_t^{i,j+1} = \frac{1}{2(\alpha + \beta)} E_{t-1}^{ij} - \frac{\alpha}{\alpha + \beta} (A_y - B_y) E_t^{ij} + (t+1) E_{t+1}^{ij}$$

$$E_0^{00} = K_{AB}$$

$$\xi_{t+1,u}^n = t \xi_{t-1,u}^{n+1} + X_{AB} \xi_{t,u}^{n+1}$$

$$\xi_{t,u+1}^n = u \xi_{t,u-1}^{n+1} + Y_{AB} \xi_{t,u}^{n+1}$$

$$\xi_{00}^n = \left( \frac{-2\alpha\beta}{\alpha + \beta} \right)^n \zeta_n \left( \frac{\alpha\beta}{\alpha + \beta} R_{AB}^2 \right)$$

$$\zeta_n(x) = \int_{-1}^1 \frac{u^{2n}}{\sqrt{1-u^2}} e^{-u^2 x} du$$

$$\xi_{t+1,u,v}^n = t \xi_{t-1,u,v}^{n+1} + X_{AB} \xi_{t,u,v}^{n+1}$$

$$\xi_{t,u+1,v}^n = u \xi_{t,u-1,v}^{n+1} + Y_{AB} \xi_{t,u,v}^{n+1}$$

$$\xi_{t,u,v+1}^n = u \xi_{t,u,v-1}^{n+1} + Y_{AB} \xi_{t,u,v}^{n+1}$$

$$\xi_{000}^n = (-2\alpha\beta)^n \zeta_n \left( \frac{\alpha\beta}{\alpha + \beta} R_{AB}^2 \right)$$

$$\zeta_n(x) = \int_{-1}^1 u^{2n} e^{-u^2 x} du$$

## Summary

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- Perturbation of harmonic oscillator:  $U^{\text{DW}}(r) = V^{\text{HO}}(r) + V_n^{\text{DW}}(r)$
- Expand in HO-functions:  $\left| \psi_p^{\text{DW}} \right\rangle = \sum_l C_{lp}^{\text{DW}} \left| \psi_l^{\text{HO}} \right\rangle$
- Eigenvalue equation:  $H^{\text{DW}} C^{\text{DW}} = \epsilon^{\text{DW}} C^{\text{DW}}$ 
  - $H_{ij}^{\text{DW}} = \epsilon_i^{\text{HO}} \delta_{ij} + \langle \psi_i^{\text{HO}} \left| V_n^{\text{DW}} \right| \psi_j^{\text{HO}} \rangle$
- Integral-Elements

$$\langle \psi_p^{\text{DW}} \left| \psi_q^{\text{DW}} \right\rangle = \delta_{pq}$$

$$\langle \psi_p^{\text{DW}} \left| h^{\text{DW}} \right| \psi_q^{\text{DW}} \rangle = \epsilon_p^{\text{DW}} \delta_{pq}$$

$$\left\langle \psi_p^{\text{DW}} \psi_q^{\text{DW}} \left| \frac{1}{r_{12}} \right| \psi_r^{\text{DW}} \psi_s^{\text{DW}} \right\rangle = \sum_{tuvw}^{ijkl} C_{tp}^{\text{DW}} C_{uq}^{\text{DW}} C_{vr}^{\text{DW}} C_{ws}^{\text{DW}} \left\langle \psi_t^{\text{HO}} \psi_u^{\text{HO}} \left| \frac{1}{r_{12}} \right| \psi_v^{\text{HO}} \psi_w^{\text{HO}} \right\rangle$$

# Conclusion

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- Single-Well and Double-well

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- Single-Well and Double-well
- General C++ code implemented
- Hartree-Fock limit reached
- Hartree-Fock energies improved with Variational Monte-Carlo
- Stable minimization with NQS-Jastrow

## Further Work

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- Binding with Python

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- Optimize and clean-up code

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- Rewrite with sparse matrices in Hartree-Fock

## Further Work

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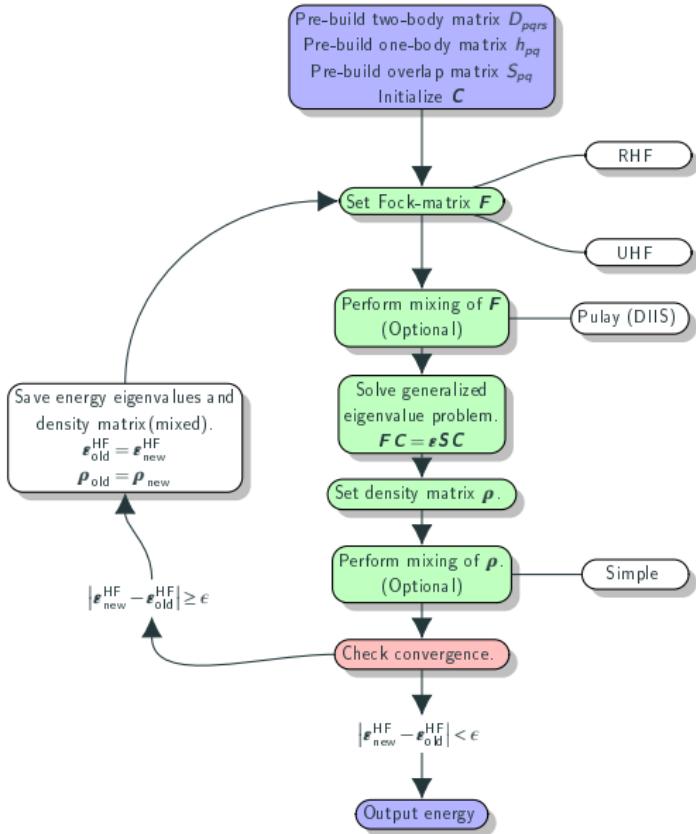
- Binding with Python
- Optimize and clean-up code
- Rewrite with sparse matrices in Hartree-Fock
- Extend to non-isotropic gaussians

**Questions?**

# Questions

Questions?

# Methods: Hartree-Fock



## Methods: Variational Monte-Carlo

