6. Implicit method for nonlinearities; Neumann and Periodic B. C.s

How should we adapt our implicit methods for nonlinear PDEs? While it would be possible to solve nonlinear equations each timestep, that is not usually necessary, or advisable. Instead, we can seek linear approximations whose errors are the same order as our original truncation. For example,consider the Fisher equation $u_t = u_{xx} + f(u)$, for some function f. The simplest thing to do would be to treat the nonlinear term $f(u)$ explicitly. Alternatively, if using a Crank-Nicolson formulation, we might expect best accuracy if we centre $f(u)$ about $j + \frac{1}{2}$, writing

$$U_n^{j+1} - U_n^j = \tfrac{1}{2} r \left(\delta^2 U_n^j + \delta^2 U_n^{j+1}\right) + \tfrac{1}{2} k \left(f(U_n^j) + f(U_n^{j+1})\right). \tag{6.1}$$

If $f(u)$ is differentiable, we could then approximate the nonlinear unknown term by

$$f(U_n^{j+1}) = f(U_n^j) + f'(U_n^j)\left(U_n^{j+1} - U_n^j\right) + O\left(|U_n^{j+1} - U_n^j|^2\right). \tag{6.2}$$

Combining (6.2) and (6.1), U_n^{j+1} now appears linearly on the RHS. We can then modift the matrices A and B in (5.6): $AU^{j+1} = BU^j + c^j$ accordingly. Note that in general, the stability of the method depends on the eigenvalues of $A^{-1}B$, which may change if we alter A and B

Neumann Boundary Conditions:

So far we have been assuming that u is known on the boundaries. The condition $u = g$ on a boundary is called a **Dirichlet** condition. An important alternative is the **Neumann** boundary condition, $u_x = f$ at $x = 0$, say. Physically, $f = 0$ corresponds to permitting no flux across the boundary, so that if u corresponds to temperature, this would be an insulating boundary. Slight differences occur to our system in this case. Firstly, if $u_x = f$ is known on $x = 0$, then the value of u is unknown there. One could use the value of f to relate the boundary value to that one point in, and set $U_0^j = U_1^j - fh$, which keeps the same number of unknowns as in the Dirichlet case. This would change the (1,1) element of A in (5.7) to $1 + \frac{1}{2}r$. A better way of dealing with this condition is to introduce a fictitious point at $n = -1$, and then require $U_{-1} = U_1$. Then $(\delta^2 u)_0 = 2u_1 - 2u_0$, which we can use in the FD scheme evaluated on the boundary. However, in that case we will have an additional row and column in A as we have more unknowns. If both boundaries are Neumann, then we have $(N + 1)$ unknowns rather than $(N - 1)$.

Periodic Boundary Conditions:

Another important case to consider are **Periodic** boundary conditions, where we identify $x = 0$ and $x = 1$. The new boundary value $U_N = U_0$, but also the derivatives at each end must match. So we also idenitfy the fictitious points U_{-1} and U_{N+1} as U_{N-1} and U_1 respectively. When we evaluate the FDM on the boundary, we get an extra equation, and the matrix A takes the form (5.7) but with the top right and bottom left corners changed to $-r/2$. As a result, the matrix is no longer strictly tri-diagonal, and we may have to modify our solving routine, as we'll see next time. In this case we have N unknowns.

M345N10 First Project – Hyperdiffusion

This project counts for 20% of the entire module. It is due in by 23:59 on Monday 4th February. It should be submitted electronically on Blackboard – instructions will follow.

The function $u(x,t)$ is 2π-periodic in x. As $t \to \infty$ it **may** approach a limit, $u \to u_\infty(x)$

(1) Using a uniform grid of size h, find a centred finite difference formula for the 4th derivative

$$u_{xxxx} \simeq a\,(u_{n+2} + u_{n-2}) + b\,(u_{n+1} + u_{n-1}) + cu_n.$$

(2) Write a program using an explicit method to solve the hyperdiffusion equation

$$u_t = -u_{xxxx} + Q(x,\,t) \qquad \text{in} \quad 0 < x < 2\pi \qquad \text{with periodic boundary conditions.} \quad ,$$

together with the initial condition $u(x,0) = f(x)$ for (periodic) functions f and Q of your choice. Devise a suitable test of your program and determine the largest possible time-step k for given space-step h for which your method converges.

(3) Now solve the same problem using an implicit method. Compare the computation time taken for the two methods to obtain the same overall accuracy for $u(x,t)$.

(4) A PDE which is known to give rise to many different solutions is

$$u_t = -u_{xxxx} - 2u_{xx} + Gu + qu^2 - u^3 \qquad \text{in} \quad 0 < x < 2\pi,$$

with periodic boundary conditions and an initial condition $u(x,0) = f(x)$. For fixed values of the constants G and q, your task is to investigate the steady states which are approached for large times for different starting conditions. Before you start can you spot any possible steady solutions?

It is suggested you use between 100 and 500 points in the x-direction, and an initial condition $f(x) = A\cos px + B\cos qx$ for constants A and B and integers p and q but you may investigate different periodic functions. You may if you wish fix $q = 0.5$, and consider a large value of G, $G > 100$, but again you may try any values.

Be warned that if your time-step k is too big you may miss some possible solutions, but of course you don't want your program to be too slow. Likewise, if your convergence criterion is too lax, you may think u has settled down whereas in fact it is still evolving.

Practical considerations: You should submit your program files and a pdf file containing a write-up of your project. You have a fair bit of choice. Your write-up should explain what choices you made, suitable output from your programs, and make appropriate comments and observations about your tests, expectations and results. The pdf title should include your name – I don't want to have to process 20 files all called Project1.pdf! I shall return a marked version of your work with comments.