PERMUTATIONS OF MINIMUM DEPTH

A **permutation** w is a sequence $w = w_1 w_2 \cdots w_n$ of n elements where each positive integer in $\{1, \ldots, n\}$ appears exactly once. (Note: One can think of a permutation as a bijection from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. This point of view is not so important for this project, but it may explain some notation at some point.)

Denote the set of all permutations of n elements by S_n, and the set of all permutations of any number of elements by S (so $S = \bigcup_{n=1}^{\infty} S_n$.

Given two positive integers i, j with $1 \leq i, j \leq n$, define $t_{i,j} : S_n \to S_n$ to be the **transposition** (or swap) switching the entries in the i-th and j-th positions. Formally, given $w = w_1 \cdots w_n \in S_n$, $t_{i,j}(w)$ is the permutation with $[t_{i,j}(w)]_k = w_k$ if $k \neq i, j$, $[t_{i,j}(w)]_i = w_j$, and $[t_{i,j}(w)]_j = i$.

Given a positive integer i with $1 \leq i \leq n-1$, define $s_i = t_{i,i+1}$; these functions are called **adjacent** or **simple** transpositions.

For each positive integer n, the **identity permutation** $id \in S_n$ is the permutation $12 \cdots n$ (so $id_i = i$ for all i with $1 \leq i \leq n$).

Given a permutation $w \in S_n$ the **length** of w (denoted $\ell(w)$) is the smallest positive integer ℓ such that there exist ℓ simple transpositions $s_{i_1}, \ldots, s_{i_\ell}$ with $s_{i_1}(s_{i_2}(\cdots(s_{i_\ell}(w))\cdots)) = id$. (From now on, to make things easier on the eyes, I won't type all those parentheses.) By convention $\ell(id) = 0$.

Given a permutation $w \in S_n$, the **inversion set** of w is the set

$$\mathrm{Inv}(w) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i < j \leq n, w_i > w_j\}$$

of pairs of positions that are "out of order."

Exercise:

Prove that $\ell(w) = |\mathrm{Inv}(w)|$. (Hint: You will need to prove separately that $\ell(w) \leq |\mathrm{Inv}(w)|$ and $\ell(w) \geq |\mathrm{Inv}(w)|$. One will require induction on $\ell(w)$ and the other induction on $|\mathrm{Inv}(w)|$. The idea is that any simple transposition decrease the number of inversions by at most 1, and there always exists (except for $w = id$, which is the base case) a simple transposition that decreases the number of inversions by 1.)

Given a permutation $w \in S_n$ the **reflection length** of w (denoted $r(w)$) is the smallest positive integer r such that there exist r (not necessarily simple) transpositions $t_{i_1,j_1}, \ldots, t_{i_r,j_r}$ with $t_{i_1,j_1} \cdots t_{i_r,j_r} w = id$.

Given a permutation $w \in S_n$, define the equivalence relation $\equiv_w$ on $\{1, \ldots, n\}$ by declaring that $i \equiv_w j$ if $w_i = j$ and taking the transitive closure. Let $\mathrm{Cyc}(w)$ be the set of equivalence classes of $\equiv_w$.

Exercise: Prove that $r(w) = n - |\mathrm{Cyc}(w)|$. (Hint: First show that $|\mathrm{Cyc}(t_{i,j}(w))| = |\mathrm{Cyc}(w)| + 1$ if and only if $i \equiv_w j$. (Otherwise, $|\mathrm{Cyc}(t_{i,j}(w))| = |\mathrm{Cyc}(w)| - 1$.) Then use the hint for the previous exercise.)

Given a transposition $t_{i,j}$, define the **depth** of $t_{i,j}$, denoted $\mathrm{dp}(t_{i,j})$, to be $|i-j|$. Given an arbitrary permutation $w \in S_n$ define $\mathrm{dp}(w)$ to be the smallest positive integer d such that there exists an integer k and transpositions $t_{i_1,j_1}, \ldots, t_{i_k,j_k}$ with $t_{i_1,j_1} \cdots t_{i_k,j_k} w = id$ and $\sum_{m=1}^{k} \mathrm{dp}(t_{i_m,j_m}) = d$.

Given a permutation $w \in S_n$, define the **total exceedences** $e(w)$ to be $\sum_{i=1}^{n} \max(0, w_i - i)$.

Exercise:

Prove that $\mathrm{dp}(w) = e(w)$. (Hint: Use induction as before. To show $\mathrm{dp}(w) \leq e(w)$, for $w \neq id$, define t_{i_k,j_k} (the first transposition applied to w) by letting i_k be the largest integer with $w_{i_k} > i_k$, and let j_k be the smallest integer $j_k > i_k$ with $w_{j_k} < j_k$. (Of course you have to explain why such integers exist.))

Exercise:

Prove that, for any permutation w, $\mathrm{dp}(w) \geq (\ell(w) + r(w))/2$. (Hint: You want to use the definitions, not the results of any of the previous exercises.)

Exercise:

Prove that $\mathrm{dp}(w) = (\ell(w) + r(w))/2$ if and only if:

- $w = id$, or
- There exists $t_{i,j}$ such that, if we let $v = t_{i,j}(w)$, then $\mathrm{dp}(v) = (\ell(v) + r(v))/2$, $r(v) = r(w) - 1$, and $\ell(v) = \ell(w) - (2|i-j| - 1)$.

(Actually, I haven't thought through this exercise, so it's possible that it's a good deal harder than I thought.)

Now on to the actual problem:

Let $m \leq n$. If $v \in S_m$ and $w \in S_n$, we say w **contains** v if there exist integers $i_1, \ldots, i_m$, with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, such that $v_j \leq v_k$ if and only if $w_{i_j} \leq w_{i_k}$.

If w does not contain v, then we say w **avoids** v.

We say w is **shallow** if $\mathrm{dp}(w) = (r(w) + \ell(w))/2$.

Problem:

Prove the following are equivalent:

- w avoids 4231 and is shallow.
- w avoids 4231, 3412, 34521, 54123, 365214, 541632, 7652143, and 5476321

$r(w)$ = least # of swaps to identity

$\ell(w)$ = # of pairs are out of order

$2 \cdot e(w) \geq r($