Assignment # 1: MATH1051

Jamie Chen
Student Number: 48093189
Semester 2, 2023

August 17, 2023

1. **(1 mark each)** Determine the domains (as a subset of $\mathbb{R}$) of the functions:

   (a) $f_1(x) = \frac{1}{e^x - e^{-x}}$

   > The domain of this function $f_1(x)$ is $\mathbb{R} \setminus \{0\}$. As the denominator of the function $f_1(x)$ is $e^x - e^{-x}$, we can see that the function is undefined when $e^x - e^{-x} = 0$. This occurs when $e^x = e^{-x}$, which is when $x = 0$. Therefore, the domain of this function is $\mathbb{R} \setminus \{0\}$.

   (b) $f_2(x) = \frac{1}{\sqrt{4-x^2}}$

   > The domain of this function $f_2(x)$ is $(-2, 2)$ (non-inclusive). As the denominator of the function $f_2(x)$ is $\sqrt{4 - x^2}$, we can see that the function is undefined when $\sqrt{4 - x^2} = 0$. This occurs when $4 - x^2 = 0$, which is when $x = \pm 2$. Therefore, the domain of this function is $(-2, 2)$ (non-inclusive).

   (c) $f_3(x) = \log \arccos x$

   > The domain of this function $f_3(x)$ is $[-1, 1)$ (inclusive from the left, exclusive from the right). As the function $f_3(x)$ is a composition of two functions, we must consider the domain of both functions. The domain of the function $\arccos x$ is $[-1, 1]$, and the domain of the function $\log x$ is $(0, \infty)$. Therefore, the domain of the function $f_3(x)$ is $[-1, 1)$ (inclusive from the left, exclusive from the right).

2. **(3 marks)** Given is the function $g(x) = x^2 + 3x$. For a second function f with $f(3) = 0$ we find $(g \circ f)(x) = x^2 - 3x$. What is the function f? Is f unique?

The function f can be a simple linear function, such as $f(x) = x - 3$.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= (x-3)^2 + 3(x-3) \\ &= x^2 - 6x + 9 + 3x - 9 \\ &= x^2 - 3x\end{aligned}$$

We can prove that this function is unique or not by substituting $f(x) = x - 3$ into the function $g(x)$ and seeing if it is equal to the function $(g \circ f)(x)$.

$$\begin{aligned}g(x) &= x^2 + 3x \\ g(f(x)) &= (x-3)^2 + 3(x-3) \\ &= x^2 - 6x + 9 + 3x - 9 \\ &= x^2 - 3x\end{aligned}$$

3. **(1 mark each)** Determine which of the following functions are 1-1? Prove your answer.

   (a) $f_1(x) = e^{-x^2}$

Defined in the workbook:

> A function $f : X \to Y$ is called **one-to-one** (or **injective**) if $\forall x_1, x_2 \in \mathbb{R} \cap X,\ f(x_1) = f(x_2) \implies x_1 = x_2$.

The function $f_1(x) = e^{-x^2}$ is one-to-one as it is a strictly decreasing function. Additionally, if we follow the definition of a one-to-one function:

$$\begin{aligned} f_1(x_1) &= f_1(x_2) \\ e^{-x_1^2} &= e^{-x_2^2} \\ \ln(e^{-x_1^2}) &= \ln(e^{-x_2^2}) \\ -x_1^2 &= -x_2^2 \\ x_1^2 &= x_2^2 \\ \implies x_1 &= x_2 \end{aligned}$$

Therefore, the function $f_1(x) = e^{-x^2}$ is one-to-one.

(b) $f_2(x) = 2x^2 - 3x + 1$

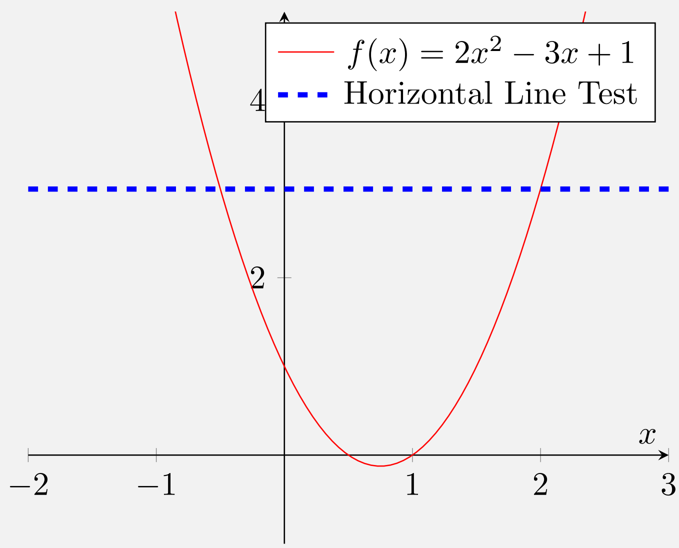
The function $f_2(x) = 2x^2 - 3x + 1$ is not one-to-one. There are several reasons for this:

1. The function $f_2(x)$ is not strictly increasing or decreasing.
2. The function $f_2(x)$ is a quadratic function, and therefore can have two values of x that correspond to the same value of y.
3. Using the definition of a one-to-one function:

$$f_1(x_1) = f_1(x_2) \implies x_1 = x_2$$

$$2x_1^2 - 3x_1 + 1 = 2x_2^2 - 3x_2 + 1 \implies x_1 = x_2$$

We can see that the function $f_2(x)$ is not one-to-one as there are multiple values of x that correspond to the same value of y.



(c) $f_3(x) = |x| + 2 \cdot x$

The function $f_3(x) = |x| + 2 \cdot x$ is one-to-one. There are several reasons for this (using the definition of a one-to-one function):

$$\begin{aligned} f_3(x_1) &= f_3(x_2) \\ |x_1| + 2 \cdot x_1 &= |x_2| + 2 \cdot x_2 \end{aligned}$$

In this case, lets consider that there are two cases:

The first case is when $x_1, x_2 \geq 0$:

In this case, we can assume that $|x_1| = x_1$ and $|x_2| = x_2$. We can then determine the equation for this case:

$$|x_1| + 2 \cdot x_1 = |x_2| + 2 \cdot x_2$$

Simplifying this equation gives us:

$$3x_1 = 3x_2$$

Dividing both sides by 3 gives us:

$$x_1 = x_2$$

We can now determine the second case.

The second case is when $x_1, x_2 < 0$:

In this case, we can assume that $|x_1| = -x_1$ and $|x_2| = -x_2$. We can then determine the equation for this case:

$$-x_1 + 2 \cdot x_1 = -x_2 + 2 \cdot x_2$$

Simplifying this equation gives us:

$$x_1 = x_2$$

After determining both cases for the absolute value function, we can see that the function $f_3(x) = |x| + 2 \cdot x$ is one-to-one.

4. **(1 mark each)** Determine what the following limits are or show that they do not exist.

(a) $\lim_{n\to\infty} \frac{(n^2+4n-27)(n^3-1)}{(n(n-1))^2}$

We can determine the limit of this function by using the following steps:

We can first expand the numerator and denominator of the function and then divide each term by the highest power of n in the denominator:

$$\begin{aligned}\lim_{n\to\infty} \frac{(n^2+4n-27)(n^3-1)}{(n(n-1))^2} &= \lim_{n\to\infty} \frac{n^5+4n^4-27n^3-n^3+1}{n^4-2n^3+n^2} \\ &= \lim_{n\to\infty} \frac{n^5+4n^4-28n^3+1}{n^4-2n^3+n^2} \\ &= \lim_{n\to\infty} \frac{n+4-\frac{28}{n}+\frac{1}{n^3}}{1-\frac{2}{n}+\frac{1}{n^2}} \\ &= \frac{\infty}{1} \\ &= \infty\end{aligned}$$

Therefore the limit does not exist when $n \to \infty$ as the limit is ∞.

(b) $\lim_{n\to\infty} \frac{3n^2-9n+48}{4n^3}$

We can determine the limit for this function by using the same steps as the previous question.

We can divide each term by n^2 in the denominator.

As $n \to \infty$, $\frac{1}{n^2} \to 0$. Therefore the term will approach 0.

$$\begin{aligned}\lim_{n\to\infty} \frac{3n^2-9n+48}{4n^3} &= \lim_{n\to\infty} \frac{3-\frac{9}{n}+\frac{48}{n^2}}{4n} \\ &= \lim_{n\to\infty} \frac{3}{4n} - \frac{9}{4n^2} + \frac{12}{n^3} \\ &= 0 \quad \text{Since the limit of } \frac{1}{n} = 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore the limit of the function is 0.

(c) $\lim_{n\to\infty} \frac{(3n+1)^3-27n^3}{n^2}$

Similar to part (a), we can determine the limit of this function by using the following steps:

We can first expand the numerator and denominator of the function and then divide each term by the highest power of n in the denominator.

As $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$ as $n \to \infty$, we can ignore these terms as they will eventually approach 0.

$$\begin{aligned}\lim_{n\to\infty} \frac{(3n+1)^3-27n^3}{n^2} &= \lim_{n\to\infty} \frac{27n^3+27n^2+9n+1-27n^3}{n^2} \\ &= \lim_{n\to\infty} \frac{27n^2+9n+1}{n^2} \\ &= \lim_{n\to\infty} (27 + \frac{9}{n} + \frac{1}{n^2}) \\ &= 27\end{aligned}$$

$\therefore$ The limit is 27 as $n \to \infty$.

(d) $\lim\limits_{n\to\infty} \frac{2n^2}{2n-1} - n$

To determine this limit, we can use similar steps as the previous question in part (a).

We can divide each term in the numerator by the highest power of n in the denominator.

As $\frac{1}{n} \to 0$ as $n \to \infty$, we can ignore this term as it will eventually approach 0.

$$\begin{aligned}
\lim_{n\to\infty} \frac{2n^2}{2n-1} - n &= \lim_{n\to\infty} \frac{2n^2}{2n-1} - \frac{n(2n-1)}{2n-1} \\
&= \lim_{n\to\infty} \frac{2n^2-n(2n-1)}{2n-1} \\
&= \lim_{n\to\infty} \frac{2n^2-2n^2+n}{2n-1} \\
&= \lim_{n\to\infty} \frac{n}{2n-1} \\
&= \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} \\
&= \frac{1}{2} \quad \text{Since the limit of } \tfrac{1}{n} = 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore, the limit is $\frac{1}{2}$ as $n \to \infty$.

(e) $\lim_{n\to\infty} \sqrt{n(n+1)} - n$

In this limit, we can find the limit by rationalising the numerator.

To do this we can multiply the numerator and denominator by the conjugate of the numerator:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt{n(n+1)} - n &= \lim_{n\to\infty} \sqrt{n(n+1)} - n \times \frac{\sqrt{n(n+1)}+n}{\sqrt{n(n+1)}+n} \\
&= \lim_{n\to\infty} \frac{n(n+1)-n^2}{\sqrt{n(n+1)}+n}
\end{aligned}$$

We can then simplify the equation to find the limit:

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n(n+1)}+n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n(n+1)}+n} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}+1}} \\
&= \frac{1}{2} \quad \text{Since the limit of } \frac{1}{n} = 0 \text{ as } n \to \infty.
\end{aligned}$$

5. **(1 mark each)** Consider the sequence a_n defined by the recursion

$$a_n = a_{n-1} - \frac{1}{4}a_{n-2} \tag{1}$$

for $n = 3, 4, 5, \ldots$.

(a) Calculate h such that $a_n = h^{n-1}$ fulfils the recursive definition.

To calculate h we can substitute $a_n = h^{n-1}$ into the recursive definition:

$$h^{n-1} = h^{n-2} - \frac{1}{4}h^{n-3}$$

We can now divide both sides by h^{n-3}:

$$h^2 = h - \frac{1}{4}$$

Solving h in terms of the quadratic formula:

$$h = \frac{-(-1) \pm \sqrt{1 - 4 \cdot 1 \cdot \frac{1}{4}}}{2 \cdot 1}$$

$$h = \frac{1 \pm \sqrt{1 - 1}}{2}$$

$$h = \frac{1 \pm 0}{2}$$

$$\therefore h = \frac{1}{2}$$

(b) What is the limit of the sequence a_n (if it exists)?

To find the limit of the sequence a_n we can start by looking at the recursive definition.

Since $a_n = a_{n-1} - \frac{1}{4}a_{n-2}$, we can tell that the sequence is dependent on two previous terms; a_{n-1} and a_{n-2}.

Because of this, the limit when n approaches infinity will be dependent on the previous two terms.

Since the previous terms will always be smaller than the current term and that the terms will get infinitely smaller, the limit of the sequence a_n will be 0.

We can also prove this by using the limit definition:

We can assume that there is a limit L such that:

$$\lim_{n\to\infty} a_n = L$$

Since the sequence is defined recursively, we can substitute L into the recursive definition:

$$L = L - \frac{1}{4}L$$

We can then simplify the equation to find the limit:

$$L = 0$$

Therefore, if the limit exists for the sequence, the limit of the sequence a_n is 0.

6. **(1 mark each)** Given is the sequence b_n defined in recursive form

$$b_n = \frac{1}{2}\left(b_{n-1} + \frac{A}{b_{n-1}}\right)$$

for a given $A > 0$. You can assume that all values of b_n are non-zero.

(a) For $A = 2$ use your calculator (or MATLAB) to calculate the first four values of the sequence b_n starting from $b_1 = A$ (this is for $n = 1, 2, 3, 4$). Inspecting these values do you expect the sequence to be convergent or to be divergent?

Given that A > 0, I expect the sequence to be convergent. This is because as n increases, the value of b_n will approach the limit of the sequence.

Substituting $n = 1, 2, 3, 4$ into the recursive definition:

$$\begin{aligned} b_1 &= A \\ b_2 &= \tfrac{1}{2}\left(b_1 + \tfrac{A}{b_1}\right) \\ &= \tfrac{1}{2}\left(A + \tfrac{A}{A}\right) \\ &= \tfrac{1}{2}(A+1) \\ &= \tfrac{A+1}{2} \\ &= \tfrac{A}{2} + \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} b_3 &= \tfrac{1}{2}\left(b_2 + \tfrac{A}{b_2}\right) \\ &= \tfrac{1}{2}\left(\tfrac{A}{2} + \tfrac{1}{2} + \tfrac{A}{\frac{A}{2}+\frac{1}{2}}\right) \\ &= \tfrac{A^2+6A+1}{4A+4} \\ b_4 &= \tfrac{1}{2}\left(b_3 + \tfrac{A}{b_3}\right) \\ &= \tfrac{1}{2}\left(\tfrac{A^2+6A+1}{4A+4} + \tfrac{A}{\frac{A^2+6A+1}{4A+4}}\right) \end{aligned}$$

We can then substitute $A = 2$ into the recursive definition:

$$\begin{aligned} b_1 &= 2 \\ b_2 &= \tfrac{2}{2} + \tfrac{1}{2} \\ b_2 &= \tfrac{3}{2} \end{aligned}$$

$$\begin{aligned} b_3 &= \tfrac{1}{2}\left(\tfrac{2}{2} + \tfrac{1}{2} + \tfrac{2}{\frac{2}{2}+\frac{1}{2}}\right) \\ b_3 &= \tfrac{17}{12} \\ b_4 &= \tfrac{1}{2}\left(\tfrac{17}{12} + \tfrac{2}{\frac{17}{12}}\right) \\ b_4 &= \tfrac{577}{408} \end{aligned}$$

(b) Assume you know the sequence b_n is converging, what would be its limit (or its limits)? Justify your answer. Is it consistent with your result of part (a)?

We can show this by using the recursive definition:

Because the sequence is increasing and bounded above, we can find the limit and equality of the sequence by solving the equation:

Let L be the limit of the sequence b_n:

$$L = \frac{1}{2}\left(L + \frac{A}{L}\right)$$

Multiplying both sides by 2:

$$2L = L + \frac{A}{L}$$

Subtracting L from both sides:

$$L = \frac{A}{L}$$

Multiplying both sides by L:

$$L^2 = A$$

Taking the square root of both sides:

$$L = \pm\sqrt{A}$$

Since $A > 0$, we know that the Limit will be at $L = \sqrt{A}$.

Since we know that $A = 2$, we can substitute this into the equation:

$$L = \sqrt{2}$$

Therefore, the limit of the sequence b_n is $\sqrt{2}$.

7. **(1 mark each)** Assume you have given a sequence c_n with non-zero values ($c_n \neq 0$ for $n = 1, 2, \ldots$) that fulfils the condition

$$\left| \frac{c_n}{c_{n-1}} \right| \leq q \tag{2}$$

for all $n = 1, 2, 3, \ldots$ for some fixed constant q with $0 < q < 1$.

   (a) Show that $c_n \to 0$ for $n \to \infty$. (Hint: use the squeeze theorem)

To show that the sequence c_n converges to 0, we can use the squeeze theorem. We can start by rearranging the inequality:

$$|c_n| \leq q|c_n - 1|$$

Since we know that $0 < q < 1$, the absolute value of c_n will always be less than the absolute value of c_{n-1}. This means that the sequence c_n will always be squeezed between 0 and $q|c_n - 1|$. Since $q|c_n - 1|$ converges to 0, the sequence c_n must also converge to 0. We can show this by using the squeeze theorem:

$$|c_n| \leq q^n |c_{n-1}|$$

Since $q^n |c_{n-1}|$ converges to 0, the sequence c_n must also converge to 0.

Therefore the sequnce c_n converges to 0 when $n \to \infty$.

(b) Use the result from part(a) to show that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

Using the result from Part A, we can show that the $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

We can start by defining a new sequence a_n:

$$a_n = \frac{2^n}{n!} \quad \text{for } n = 1, 2, 3, \ldots$$

We then divide the sequence a_n by a_{n-1}:

$$\frac{a_n}{a_{n-1}} = \frac{\frac{2^n}{n!}}{\frac{2^{n-1}}{(n-1)!}}$$

We can then simplify the equation:

$$= \frac{2^n}{n!} \times \frac{(n-1)!}{2^{n-1}}$$

We can then cancel out the 2^{n-1} and $(n-1)!$:

$$= \frac{2^n}{n \times (n-1)!} \times \frac{(n-1)!}{2^{n-1}}$$

$$= \frac{2^n}{n} \times \frac{1}{2^{n-1}}$$

We can then simplify the equation:

$$= \frac{2 \times 2^{n-1}}{n} \times \frac{1}{2^{n-1}}$$

We can now cancel out the 2^{n-1}:

$$= \frac{2}{n}$$

Since $\frac{2}{n} \leq 1$ for all $n \geq 2$, we can apply the result from part A to show that the sequence a_n converges to 0 as n approaches infinity.

(c) Again: use the result from part(a) to show that $\lim_{n\to\infty} \dfrac{1}{3^n n^3} = 0$.

Similar to Part B, we can use the result from Part A to show that the $\lim_{n\to\infty} \dfrac{1}{3^n n^3} = 0$.

We can start by defining a new sequence b_n:

$$b_n = \frac{1}{3^n n^3} \quad \text{for } n = 1, 2, 3, \ldots$$

We then divide the sequence b_n by b_{n-1}:

$$\frac{b_n}{b_{n-1}} = \frac{\frac{1}{3^n n^3}}{\frac{1}{3^{n-1}(n-1)^3}}$$

Simplify by reciprocating the denominator and multiplying by the numerator:

$$= \frac{1}{3^n n^3} \times \frac{3^{n-1}(n-1)^3}{1}$$

Simplify further:

$$\frac{1}{n^3} \times \frac{(n-1)^3}{1} = \frac{(n-1)^3}{3n^3} = \frac{n^3 - 3n^2 + 3n - 1}{3n^3}$$

We can then divide the numerator and denominator by n^3:

$$= \frac{1}{3} - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3}$$

We now determine this limit:

$$\lim_{n\to\infty} \frac{1}{3} - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} = \frac{1}{3} - 0 + 0 - 0 = \frac{1}{3}$$

Since $\frac{1}{3} \leq 1$ for all $n \geq 2$, we can apply the result from part A to show that the sequence b_n converges to $\frac{1}{3}$ as n approaches infinity.