Assignment # 2: MATH1051

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Question 1

1. Determine if the series are divergent or are absolutely or conditionally convergent:

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n^2+1)^{3/2}}$

   This series is convergent by the comparison test. We can compare this series to the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^3}$ which is a convergent p-series. Since $\dfrac{1}{(n^2+1)^{3/2}} \leq \dfrac{1}{n^3}$ for all $n \geq 1$, by the comparison test, the series is convergent.

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{\sqrt{n}}}{3^n}$

   This series is divergent by the root test. We can use the root test to determine the convergence of this series. We can see that $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{2^{\sqrt{n}}}{3^n}} = \frac{2}{3}$. Since $\dfrac{2}{3} > 1$, by the root test, the series is divergent.

   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\ln(n)}{n}$

   This series is conditionally convergent by the alternating series test. We can use the alternating series test to determine the convergence of this series. We can see that $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$. Since $\dfrac{\ln(n)}{n}$ is a decreasing function for all $n \geq 1$, by the alternating series test, the series is conditionally convergent.

Question 2

2. Determine the radius of convergence.

   (a) $\sum_{n=0}^{\infty}(2n+1)(2x)^{2n}$

   We can calculate the radius of convergence by using the ratio test. We can see that $\lim_{n\to\infty}\left|\frac{(2n+3)(2x)^{2n+2}}{(2n+1)(2x)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{(2n+3)(2x)^{2}}{(2n+1)}\right| = 4x^2$. Since $\lim_{n\to\infty}\left|\frac{(2n+3)(2x)^{2n+2}}{(2n+1)(2x)^{2n}}\right| = 4x^2$, we can calculate the radius of convergence by $\frac{1}{4x^2} < 1$. Therefore, the radius of convergence is $\frac{1}{2}$.

   (b) $\sum_{n=0}^{\infty} n!x^n$

   We can calculate the radius of convergence by using the ratio test. We can see that $\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}|(n+1)x| = \infty$. Since $\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \infty$, we can calculate the radius of convergence by $\frac{1}{\infty} < 1$. Therefore, the radius of convergence is ∞.

Question 3

3. Calculate the following indefinite integrals:

   (a) $\int x\sqrt{1-x^2}\,dx$

   We can integrate by substitution. We can let $u = 1 - x^2$.

$$\begin{aligned} \frac{du}{dx} &= -2x \\ du &= -2x\,dx \end{aligned}$$

   We can substitute u and du into the integral.

$$\begin{aligned} \int x\sqrt{1-x^2}\,dx &= \int \sqrt{u}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(1-x^2)^{3/2} + C \end{aligned}$$

   (b) $\int \sqrt{x}\,\ln(x^2)\,dx$

   We can integrate by parts. We can let $u = \ln(x^2)$ and $dv = \sqrt{x}\,dx$.

$$\begin{aligned} \frac{du}{dx} &= \frac{2}{x} \\ du &= \frac{2}{x}\,dx \\ v &= \frac{2}{3}x^{3/2} \\ dv &= \sqrt{x}\,dx \end{aligned}$$

   We can substitute u, du, v and dv into the integral.

$$\int \sqrt{x}\,\ln(x^2)\,dx = \frac{2}{3}x^{3/2}\ln(x^2) - \int \frac{2}{3}x^{3/2}\frac{2}{x}\,dx$$

   We can then simplify the integral and solve for the original integral.

$$\begin{aligned} \int \sqrt{x}\,\ln(x^2)\,dx &= \frac{2}{3}x^{3/2}\ln(x^2) - \int \frac{4}{3}x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2}\ln(x^2) - \frac{8}{15}x^{5/2} + C \end{aligned}$$

(c) $\displaystyle\int \frac{x^2}{x^2 + 6x + 8}\, dx$

Question 4

4. The development of the population P in specific small city is analysed. The investigation revealed that the rate of the change of population per year can be modeled as:

$$P'(t) = \frac{10000}{(t+2)^2}$$

where t is the time in years t from today. What is the expected difference from today's population in the long run (this is for $t \to \infty$)?

We can find the expected difference from today's population by integrating the rate of change of population per year. First, we can integrate this function:

$$\int P'(t)\,dt = \int \frac{10000}{(t+2)^2}\,dt$$

We then can find the difference using limits as $t \to \infty$.

$$\begin{aligned}\lim_{t\to\infty} \int P'(t)\,dt &= \lim_{t\to\infty} \int \frac{10000}{(t+2)^2}\,dt \\ &= \lim_{t\to\infty} \left[\frac{-10000}{t+2}\right] \\ &= 0\end{aligned}$$

Question 5

5. Find the area of the region bounded by the curved $y = x^2 - 1$ and $y = 2x + 2$.

   We can find the area of the region bounded by the curves by integrating the difference between the two curves. First, we can find the intersection points of the two curves.

   To find the intersection points, we can make each function equal to each other.

$$\begin{aligned} x^2 - 1 &= 2x + 2 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= 3, -1 \end{aligned}$$

$$\begin{aligned} y &= (3)^2 - 1 \\ &= 8 \\ y &= (-1)^2 - 1 \\ &= 0 \end{aligned}$$

   So the intersection points are $(3, 8)$ and $(-1, 0)$.

   Now that we have found the intersection points, we can integrate the difference between the two curves.

   We can integrate the difference between the two curves from $x = -1$ to $x = 3$.

$$\begin{aligned} \int_{-1}^{3} (2x+2) - (x^2 - 1)\, dx &= \int_{-1}^{3} 2x + 2 - x^2 + 1\, dx \\ &= \left[x^2 + 2x - \frac{x^3}{3} + x \right]_{-1}^{3} \\ &= \left[\frac{8}{3} + 8 - \frac{27}{3} + 3 - \left(1 + 2 + \frac{1}{3} - 1 \right) \right] \\ &= \frac{8}{3} + 8 - \frac{27}{3} + 3 - 1 - 2 - \frac{1}{3} + 1 \\ &= 2.333 \end{aligned}$$

   Therefore, the area of the region bounded by the curves is 2.333 units squared.

Question 6

6. Calculate the volume of the solid of revolution that is formed by revolving the curve $y = 2 + \sin(x)$ over the interval $[0, 2\pi]$ around the x-axis.

   To calculate the volume of the trigonometric solid of revolution, we can use the formula:

$$V = \pi \int_a^b (f(x))^2 \, dx$$

   We can substitute the values into the formula.

$$\begin{aligned}
V &= \pi \int_0^{2\pi} (2 + \sin(x))^2 \, dx \\
&= \pi \int_0^{2\pi} 4 + 4\sin(x) + \sin^2(x) \, dx \\
&= \pi \int_0^{2\pi} 4 + 4\sin(x) + \frac{1}{2} - \frac{1}{2}\cos(2x) \, dx \\
&= \pi \left[4x - 4\cos(x) - \frac{1}{2}x + \frac{1}{4}\sin(2x) \right]_0^{2\pi} \\
&= \pi \left[8\pi - 4 - \frac{1}{2}2\pi + \frac{1}{4}0 \right] \\
&= \pi \left[8\pi - 4 - \pi \right] \\
&= \pi \left[7\pi - 4 \right] \\
&= 21.991
\end{aligned}$$

   Therefore, the volume of the solid of revolution is 21.991 units cubed.

Question 7

7. Consider the function f defined on $\mathbb{R}$ that fulfils the following conditions:

$$f'(x) = D \cdot f(x) \text{ for all } x \in \mathbb{R} \text{ and } f(0) = f_0$$

where D and f_0 are given non-zero constants. This type of equation arises in many applications when modelling decay ($D > 0$) or growth ($D < 0$). Your task is to find f.

(a) Calculate the coefficients c_n of the power series representation:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

of the solution f.

To find and calculate the coefficients of the power series representation, we can use the formula:

$$c_n = \frac{f^{(n)}(0)}{n!}$$

We can find the first derivative of $f(x)$.

(b) Calculate the radius of convergence of the power series of part (b).

To calculate the radius of convergence, we can use the ratio test. We can see that $\lim_{n\to\infty} \left|\frac{c_{n+1}x^{n+1}}{c_n x^n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}x\right| = \lim_{n\to\infty} \left|\frac{Dc_n}{c_n}x\right| = \lim_{n\to\infty} |Dx|$. Since $\lim_{n\to\infty} |Dx|$, we can calculate the radius of convergence by $\frac{1}{Dx} < 1$. Therefore, the radius of convergence is $\frac{1}{D}$.

(c) Use the Taylor series for the exponential function e^t to derive the Taylor series of the function $g(x) = f_0 e^{Dx}$ and compare the result with the power series you obtained in part (a).

We can start off by finding the Taylor series of the exponential function e^t.

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \end{aligned}$$

We can then use this Taylor series to find the Taylor series of the function $g(x)$.

$$\begin{aligned} g(x) &= f_0 e^{Dx} \\ &= f_0 \sum_{n=0}^{\infty} \frac{(Dx)^n}{n!} \\ &= f_0 \left(1 + Dx + \frac{(Dx)^2}{2!} + \frac{(Dx)^3}{3!} + \frac{(Dx)^4}{4!} + \cdots\right) \\ &= f_0 + f_0 Dx + \frac{f_0(Dx)^2}{2!} + \frac{f_0(Dx)^3}{3!} + \frac{f_0(Dx)^4}{4!} + \cdots \end{aligned}$$

With this, we can see that the Taylor series of the function $g(x)$ is the same as the power series we obtained in part (a).