Assignment # 4: MATH1051

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1. **(1 mark each)** Determine the domains (as a subset of $\mathbb{R}$) of the functions:

   (a) $f_1(x) = \frac{1}{e^x - e^{-x}}$

   The domain of this function $f_1(x)$ is $\mathbb{R} \setminus \{0\}$.

   (b) $f_2(x) = \frac{1}{\sqrt{4-x^2}}$

   The domain of this function $f_2(x)$ is $(-2, 2)$ (non-inclusive).

   (c) $f_3(x) = \log \arccos x$

   The domain of this function $f_3(x)$ is $[-1, 1]$ (inclusive).

2. **(3 marks)** Given is the function $g(x) = x^2 + 3x$. For a second function f with $f(3) = 0$ we find $(g \circ f)(x) = x^2 - 3x$. What is the function f? Is f unique?

The function f can be a simple linear function, such as $f(x) = x - 3$.

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= (x-3)^2 + 3(x-3) \\
&= x^2 - 6x + 9 + 3x - 9 \\
&= x^2 - 3x
\end{aligned}
$$

However, there are other functions that can be used to achieve the same result. (NOT PROVEN YET)

3. **(1 mark each)** Determine which of the following functions are 1-1? Prove your answer.

   (a) $f_1(x) = e^{-x^2}$

Defined in the workbook:

> A function $f : X \to Y$ is called **one-to-one** (or **injective**) if $\forall x_1, x_2 \in \mathbb{R} \cap X$, $f(x_1) = f(x_2) \implies x_1 = x_2$.

The function $f_1(x) = e^{-x^2}$ is one-to-one as it is a strictly decreasing function. Additionally, if we follow the definition of a one-to-one function:

$$\begin{aligned} f_1(x_1) &= f_1(x_2) \\ e^{-x_1^2} &= e^{-x_2^2} \\ \ln(e^{-x_1^2}) &= \ln(e^{-x_2^2}) \\ -x_1^2 &= -x_2^2 \\ x_1^2 &= x_2^2 \\ \implies x_1 &= x_2 \end{aligned}$$

Therefore, the function $f_1(x) = e^{-x^2}$ is one-to-one.

You can see that there are other things:

(b) $f_2(x) = 2x^2 - 3x + 1$

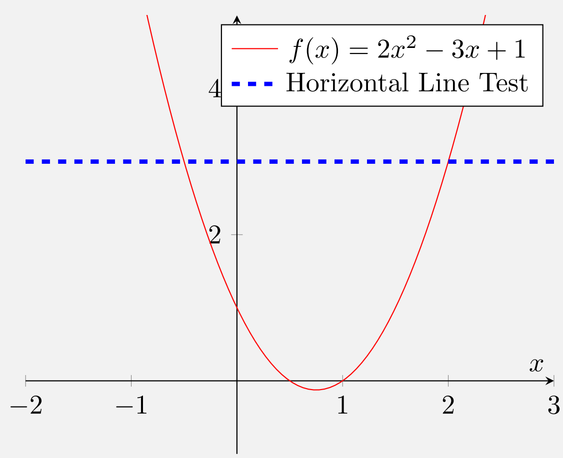
The function $f_2(x) = 2x^2 - 3x + 1$ is not one-to-one. There are several reasons for this:

1. The function $f_2(x)$ is not strictly increasing or decreasing.
2. The function $f_2(x)$ is a quadratic function, and therefore can have two values of x that correspond to the same value of y.
3. Using the definition of a one-to-one function:

$$f_1(x_1) = f_1(x_2) \implies x_1 = x_2$$

$$2x_1^2 - 3x_1 + 1 = 2x_2^2 - 3x_2 + 1 \implies x_1 = x_2$$

We can see that the function $f_2(x)$ is not one-to-one as there are multiple values of x that correspond to the same value of y.



(c) $f_3(x) = |x| + 2 \cdot x$

The function $f_3(x) = |x| + 2 \cdot x$ is not one-to-one. There are several reasons for this:

1. The function $f_3(x)$ is not strictly increasing or decreasing.
2. The function $f_3(x)$ is a piecewise function, and therefore can have two values of x that correspond to the same value of y.
3. Using the definition of a one-to-one function:

4. **(1 mark each)** Determine what the following limits are or show that they do not exist.

   (a) $\lim_{n\to\infty} \frac{(n^2+4n-27)(n^3-1)}{(n(n-1))^2}$

   $$\begin{aligned}
   \lim_{n\to\infty} \frac{(n^2+4n-27)(n^3-1)}{(n(n-1))^2} &= \lim_{n\to\infty} \frac{n^5+4n^4-27n^3-n^3+1}{n^4-2n^3+n^2} \\
   &= \lim_{n\to\infty} \frac{n^5+4n^4-28n^3+1}{n^4-2n^3+n^2} \\
   &= \lim_{n\to\infty} \frac{n+4-\frac{28}{n}+\frac{1}{n^3}}{1-\frac{2}{n}+\frac{1}{n^2}}
   \end{aligned}$$

   (b) $\lim_{n\to\infty} \frac{3n^2-9n+48}{4n^3}$

   (c) $\lim_{n\to\infty} \frac{(3n+1)^3-27n^3}{n^2}$

   (d) $\lim_{n\to\infty} \frac{2n^2}{2n-1} - n$

   (e) $\lim_{n\to\infty} \sqrt{n(n+1)} - n$

5. **(1 mark each)** Consider the sequence a_n defined by the recursion

$$a_n = a_{n-1} - \frac{1}{4}a_{n-2} \tag{1}$$

for $n = 3, 4, 5, \ldots$.

(a) Calculate h such that $a_n = h^{n-1}$ fulfils the recursive definition.

> To calculate h we can substitute $a_n = h^{n-1}$ into the recursive definition:
>
> $$h^{n-1} = h^{n-2} - \frac{1}{4}h^{n-3}$$
>
> We can now divide both sides by h^{n-3}:
>
> $$h^2 = h - \frac{1}{4}$$
>
> Solving h in terms of the quadratic formula:
>
> $$h = \frac{-(-1) \pm \sqrt{1 - 4 \cdot 1 \cdot \frac{1}{4}}}{2 \cdot 1}$$
>
> $$h = \frac{1 \pm \sqrt{1-1}}{2}$$
>
> $$h = \frac{1 \pm 0}{2}$$
>
> $$\therefore h = \frac{1}{2}$$

(b) What is the limit of the sequence a_n (if it exists)?

> To find the limit of the sequence a_n we can use the formula.

6. **(1 mark each)** Given is the sequence b_n defined in recursive form

$$b_n = \frac{1}{2}\left(b_{n-1} + \frac{A}{b_{n-1}}\right)$$

for a given $A > 0$. You can assume that all values of b_n are non-zero.

(a) Use your calculator (or MATLAB) to calculate the first four values of the sequence b_n starting from $b_1 = A$ (this is for $n = 1, 2, 3, 4$). Inspecting these values do you expect the sequence to be convergent or to be divergent?

(b) Assume you know the sequence b_n is converging, what would be its limit (or its limits)? Justify your answer. Is it consistent with your result of part (a)?

7. **(1 mark each)** Assume you have given a sequence c_n with non-zero values ($c_n \neq 0$ for $n = 1, 2, \ldots$) that fulfils the condition

$$\left|\frac{c_n}{c_n - 1}\right| \leq q \tag{2}$$

for all $n = 1, 2, 3, \ldots$ for some fixed constant q with $0 < q < 1$.

(a) Show that $c_n \to 0$ for $n \to \infty$. (Hint: use the squeeze theorem)

(b) Use the result from part(a) to show that $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

(c) Again: use the result from part(a) to show that $\lim_{n\to\infty} \frac{1}{3^n n^3} = 0$.