Assignment # 2: MATH1051

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Question 1

1. Determine if the series are divergent or are absolutely or conditionally convergent:

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n^2+1)^{3/2}}$

   This series is convergent by the comparison test. We can compare this series to the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^3}$ which is a convergent p-series. Since $\displaystyle\frac{1}{(n^2+1)^{3/2}} \leq \frac{1}{n^3}$ for all $n \geq 1$, by the comparison test, the series is convergent.

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{\sqrt{n}}}{3^n}$

   This series is absolutely convergent by the ratio test. We can use the ratio test to determine the convergence of this series.
   We can see that $\displaystyle\lim_{n\to\infty} \left| \frac{2^{\sqrt{n+1}}}{3^{n+1}} \cdot \frac{3^n}{2^{\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{2^{\sqrt{n+1}}}{2^{\sqrt{n}}} \cdot \frac{3^n}{3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2^{\sqrt{n+1}-\sqrt{n}}}{3} \right|$.
   Since $\displaystyle\lim_{n\to\infty} \left| \frac{2^{\sqrt{n+1}-\sqrt{n}}}{3} \right| = 0$, by the ratio test, the series is absolutely convergent.

   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\ln(n)}{n}$

   This series is conditionally convergent by the alternating series test. We can use the alternating series test to determine the convergence of this series. We can see that $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$. Since $\displaystyle\frac{\ln(n)}{n}$ is a decreasing function for all $n \geq 1$, by the alternating series test, the series is conditionally convergent.

Question 2

2. Determine the radius of convergence.

   (a) $\sum_{n=0}^{\infty}(2n+1)(2x)^{2n}$

   We can calculate this using the ratio test first. The ratio test is:

$$\lim_{n\to\infty}\left|\frac{(2(n+1)+1)(2x)^{2(n+1)}}{(2n+1)(2x)^{2n}}\right|$$

   We can simplify the ratio test.

$$\begin{aligned}\tfrac{a_{n+1}}{a_n} &= \frac{(2(n+1)+1)(2x)^{2(n+1)}}{(2n+1)(2x)^{2n}}\\ &= \frac{2(n+1)}{2n+1}\cdot(2x)^2\end{aligned}$$

   Taking trhe limit of this simplified ratio test, we can see that

$$L = \lim_{n\to\infty}\left|\frac{2(n+1)}{2n+1}\cdot(2x)^2\right| = 4x^2$$

   Setting $L < 1$ for convergence, we can see that

$$\begin{aligned}4x^2 &< 1\\ x^2 &< \frac{1}{4}\\ x &< \frac{1}{2}\end{aligned}$$

   Therefore, the radius of convergence is $\frac{1}{2}$.

   (b) $\sum_{n=0}^{\infty} n!x^n$

   For finding the radius of convergence, we use the ratio test. The ratio test is:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

   We can simplify the ratio test by cancelling out the $n!$ terms.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)\cdot n!\cdot x^{n+1}}{n!\cdot x^n}\right|\\ &= \lim_{n\to\infty}|(n+1)x|\end{aligned}$$

   Taking the limit of the simplified ratio test, we can see that

$$\lim_{n\to\infty}|(n+1)x| = \infty$$

   Since the limit is infinity, it means that the series is divergent for all $x \neq 0$. Therefore, the radius of convergence is 0.

Question 3

3. Calculate the following indefinite integrals:

   (a) $\int x\sqrt{1-x^2}\,dx$

   We can start by substituting $u = 1 - x^2$.

$$\int xu^2\,dx$$

   We can rearrange the equation to solve for dx.

$$du = -2x\,dx$$

   We can substitute u and dx into the integral.

$$= \int -\frac{1}{2}u^{1/2}\,du$$

   We can then do the integration for the whole equation.

$$= -\frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{2}\cdot\frac{2u^{3/2}}{3} + C$$

   We can then substitute u back into the equation.

$$= -\frac{1}{2}\cdot\frac{2(1-x^2)^{3/2}}{3} + C$$

(b) $\int \sqrt{x}\ \ln(x^2)\, dx$

We can start by simplifying the integral:

$$= 2\int x^{1/2}\ \ln(x)\, dx$$

We can then start by integrating by parts, in which the formula is:

$$\int uv' = uv - \int u'v$$

We let $u = ln(x)$ and $v = x^{1/2}$.

$$u' = 1/x \quad \text{and} \quad v' = \frac{2}{3}x^{3/2}$$

We can then substitute u, u', v, and v' into the formula.

$$\begin{aligned} \int uv' &= uv - \int u'v \\ &= \frac{2x^{3/2}\ln(x)}{3} - \int \frac{2x^{1/2}}{3}\, dx \end{aligned}$$

We can then integrate the second part of the equation.

$$= \frac{2x^{3/2}\ln(x)}{3} - \frac{4x^{3/2}}{9} + C$$

We times the integral by 2 to get the final answer.

$$= \frac{2x^{3/2}\ln(x)}{3} - \frac{8x^{3/2}}{9} + C$$

Therefore, the final answer is $\frac{2x^{3/2}\ln(x)}{3} - \frac{8x^{3/2}}{9} + C$.

(c) $\displaystyle\int \frac{x^2}{x^2+6x+8}\,dx$

We can start by simplifying the integral and bring the integral of 1 to the front.

This can be achieved through polynomial long division.

$$\begin{array}{r|l} & 1 \\ \hline x^2+6x+8 & x^2 \\ & -x^2-6x-8 \\ \hline & -6x-8 \end{array}$$

We can then split the integral into two parts by using the remainder of the polynomial long division.

$$\begin{aligned}\int \frac{x^2}{x^2+6x+8}\,dx &= \int \frac{x^2+6x+8-6x-8}{x^2+6x+8}\,dx \\ &= \int 1\,dx - \int \frac{6x+8}{x^2+6x+8}\,dx \\ &= \int 1\,dx - 2\int \frac{3x+4}{x^2+6x+8}\,dx\end{aligned}$$

We can further simplify the second integral.

$$\begin{aligned}\int \frac{3x+4}{x^2+6x+8}\,dx &= \int \left(\frac{3(2x+6)}{2(x^2+6x+8)} - \frac{5}{x^2+6x+8}\right)dx \\ &= 3\int \frac{x+3}{x^2+6x+8}\,dx - 5\int \frac{1}{x^2+6x+8}\,dx\end{aligned}$$

We can then use the substitution method to solve the first integral by letting $u = x^2+6x+8$.

$$du = (2x+6)dx$$

We can then substitute u and du into the first integral.

$$\frac{1}{2}\int \frac{1}{u}\,du$$

We can then integrate the first integral and simplify.

$$\begin{aligned}\frac{1}{2}\int \frac{1}{u}\,du &= \frac{1}{2}\ln(u) + C \\ &= \frac{1}{2}\ln(x^2+6x+8) + C \quad \text{since } u = x^2+6x+8\end{aligned}$$

We now solve the second integral by partial fractions.

$$= \int \left(\frac{1}{2(x+2)} - \frac{1}{2(x+4)}\right)dx$$

We can then integrate the second integral and simplify.

$$= \frac{1}{2}\ln(x+2) - \frac{1}{2}\ln(x+4) + C$$

We can then bring this back to the original integral.

$$\begin{aligned}\int 1\,dx - 2\int \frac{3x+4}{x^2+6x+8}\,dx &= x - 3\ln(|x^2+6x+8|) - 5\ln(|x+4|) + 5\ln(|x+2|) + C \\ &= x - 3\ln(|x+2|\,|x+4|) + 5\ln(|x+2|) - 5\ln(|x+4|) + C\end{aligned}$$

Question 4

4. The development of the population P in specific small city is analysed. The investigation revealed that the rate of the change of population per year can be modeled as:

$$P'(t) = \frac{10000}{(t+2)^2}$$

where t is the time in years t from today. What is the expected difference from today's population in the long run (this is for $t \to \infty$)?

We can start by integrating this function between the time now ($t = 0$) and the time in the long run ($t \to \infty$).

$$\int_0^\infty \frac{10000}{(t+2)^2}\,dt$$

We can then integrate the function.

$$\begin{aligned}
\int_0^\infty \frac{10000}{(t+2)^2}\,dt &= \int_0^\infty 10000(t+2)^{-2}\,dt \\
&= \left[-10000(t+2)^{-1}\right]_0^\infty \\
&= \left[-\frac{10000}{t+2}\right]_0^\infty
\end{aligned}$$

We let $k = t$ and $k = 0$ as $t \to \infty$.

$$\begin{aligned}
&= \left(\lim_{k\to\infty} -\frac{10000}{k+2}\right) - \left(-\frac{10000}{0+2}\right) \\
&= 0 - \left(-\frac{10000}{2}\right) \\
&= \frac{10000}{2} \\
&= 5000
\end{aligned}$$

Therefore, the expected difference from today's population in the long run is 5000 people.

Question 5

5. Find the area of the region bounded by the curved $y = x^2 - 1$ and $y = 2x + 2$.

   We can find the area of the region bounded by the curves by integrating the difference between the two curves. First, we can find the intersection points of the two curves.

   To find the intersection points, we can make each function equal to each other.

$$\begin{aligned} x^2 - 1 &= 2x + 2 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= 3, -1 \end{aligned}$$

$$\begin{aligned} y &= (3)^2 - 1 \\ &= 8 \\ y &= (-1)^2 - 1 \\ &= 0 \end{aligned}$$

   So the intersection points are $(3, 8)$ and $(-1, 0)$.

   Now that we have found the intersection points, we can integrate the difference between the two curves.

   We can integrate the difference between the two curves from $x = -1$ to $x = 3$.

$$\begin{aligned} \int_{-1}^{3} (2x+2) - (x^2 - 1)\, dx &= \int_{-1}^{3} 2x + 2 - x^2 + 1\, dx \\ &= \left[x^2 + 2x - \frac{x^3}{3} + x\right]_{-1}^{3} \\ &= \left[\frac{9}{3} + 6 - \frac{27}{3} + 3 - \left(1 - 2 + \frac{1}{3} - 1\right)\right] \\ &= \frac{9}{3} + 6 - \frac{27}{3} + 3 - 1 + 2 - \frac{1}{3} + 1 \\ &= 10.333 \end{aligned}$$

   Therefore, the area of the region bounded by the curves is 10.333 units squared.

Question 6

6. Calculate the volume of the solid of revolution that is formed by revolving the curve $y = 2 + \sin(x)$ over the interval $[0, 2\pi]$ around the x-axis.

   To calculate the volume of the trigonometric solid of revolution, we can use the formula:

$$V = \pi \int_a^b (f(x))^2 \, dx$$

   We can substitute the values into the formula.

$$\begin{aligned}
V &= \pi \int_0^{2\pi} (2 + \sin(x))^2 \, dx \\
&= \pi \int_0^{2\pi} 4 + 4\sin(x) + \sin^2(x) \, dx \\
&= \pi \int_0^{2\pi} 4 + 4\sin(x) + \frac{1}{2} - \frac{1}{2}\cos(2x) \, dx \\
&= \pi \left[4x - 4\cos(x) + \frac{1}{2}x + \frac{1}{4}\sin(2x) \right]_0^{2\pi} \\
&= \pi \left(\left(8\pi - 4 + \frac{1}{2}2\pi + \frac{1}{4} \right) - (-4) \right) \\
&= \pi \, [9\pi - 4 + 4] \\
&= \pi \, [9\pi] \\
&= 9\pi^2
\end{aligned}$$

   Therefore, the volume of the solid of revolution is $9\pi^2$ units cubed.

Question 7

7. Consider the function f defined on $\mathbb{R}$ that fulfils the following conditions:

$$f'(x) = D \cdot f(x) \text{ for all } x \in \mathbb{R} \text{ and } f(0) = f_0$$

where D and f_0 are given non-zero constants. This type of equation arises in many applications when modelling decay ($D > 0$) or growth ($D < 0$). Your task is to find f.

(a) Calculate the coefficients c_n of the power series representation:

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

of the solution f.

$$f(a) = \sum_{n=0}^{\infty} c_n a^n$$

Now, take the derivative of both sides of the power series representation of $f(x)$:

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

Substituting $x = a$ into this expression, we get:

$$f'(a) = \sum_{n=1}^{\infty} n c_n a^{n-1}$$

We can continue taking derivatives of both sides of the power series representation of $f(x)$ to obtain expressions for higher-order derivatives of $f(x)$ evaluated at $x = a$. Specifically, we have:

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$
$$f''(a) = \sum_{n=2}^{\infty} n(n-1) c_n a^{n-2}$$
$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) c_n x^{n-3}$$
$$f'''(a) = \sum_{n=3}^{\infty} n(n-1)(n-2) c_n a^{n-3}$$
$$\vdots$$

We can use these expressions to solve for the coefficients c_n using a system of equations. Specifically, we have:

$$\begin{aligned} f(a) &= c_0 \\ f'(a) &= c_1 \\ f''(a) &= 2c_2 \\ f'''(a) &= 3 \cdot 2 \cdot c_3 \\ &\vdots \\ f^{(n)}(a) &= n! \cdot c_n \\ &\vdots \end{aligned}$$

Solving this system of equations for the coefficients c_n, we get:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

where $f^{(n)}(a)$ denotes the nth derivative of $f(x)$ evaluated at $x = a$.

(b) Calculate the radius of convergence of the power series of part (a).

To calculate the radius of convergence, we can use the ratio test.

We can see that

$$\lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} x \right| = \lim_{n\to\infty} \left| \frac{Dc_n}{c_n} x \right| = \lim_{n\to\infty} |Dx| .$$

Since $\lim_{n\to\infty} |Dx|$, we can calculate the radius of convergence by $\frac{1}{Dx} < 1$.

Therefore, the radius of convergence is $\frac{1}{D}$.

(c) Use the Taylor series for the exponential function e^t to derive the Taylor series of the function $g(x) = f_0 e^{Dx}$ and compare the result with the power series you obtained in part (a).

We can start off by finding the Taylor series of the exponential function e^t.

$$\begin{aligned} e^t &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \end{aligned}$$

We can then use this Taylor series to find the Taylor series of the function $g(x)$.

$$\begin{aligned} g(x) &= f_0 e^{Dx} \\ &= f_0 \sum_{n=0}^{\infty} \frac{(Dx)^n}{n!} \\ &= f_0 \left(1 + Dx + \frac{(Dx)^2}{2!} + \frac{(Dx)^3}{3!} + \frac{(Dx)^4}{4!} + \cdots\right) \\ &= f_0 + f_0 Dx + \frac{f_0 (Dx)^2}{2!} + \frac{f_0 (Dx)^3}{3!} + \frac{f_0 (Dx)^4}{4!} + \cdots \end{aligned}$$

With this, we can see that the Taylor series of the function $g(x)$ is the same as the power series we obtained in part (a).