Assignment # 1: MATH1051

Jamie Chen
Student Number: 48093189
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1. **(1 mark each)** Determine the domains (as a subset of $\mathbb{R}$) of the functions:

   (a) $f_1(x) = \frac{1}{e^x - e^{-x}}$

   > The domain of this function $f_1(x)$ is $\mathbb{R} \setminus \{0\}$. As the denominator of the function $f_1(x)$ is $e^x - e^{-x}$, we can see that the function is undefined when $e^x - e^{-x} = 0$. This occurs when $e^x = e^{-x}$, which is when $x = 0$. Therefore, the domain of this function is $\mathbb{R} \setminus \{0\}$.

   (b) $f_2(x) = \frac{1}{\sqrt{4-x^2}}$

   > The domain of this function $f_2(x)$ is $(-2, 2)$ (non-inclusive). As the denominator of the function $f_2(x)$ is $\sqrt{4 - x^2}$, we can see that the function is undefined when $\sqrt{4 - x^2} = 0$. This occurs when $4 - x^2 = 0$, which is when $x = \pm 2$. Therefore, the domain of this function is $(-2, 2)$ (non-inclusive).

   (c) $f_3(x) = \log \arccos x$

   > The domain of this function $f_3(x)$ is $[-1, 1)$ (inclusive from the left, exclusive from the right). As the function $f_3(x)$ is a composition of two functions, we must consider the domain of both functions. The domain of the function $\arccos x$ is $[-1, 1]$, and the domain of the function $\log x$ is $(0, \infty)$. Therefore, the domain of the function $f_3(x)$ is $[-1, 1)$ (inclusive from the left, exclusive from the right).

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2. **(3 marks)** Given is the function $g(x) = x^2 + 3x$. For a second function f with $f(3) = 0$ we find $(g \circ f)(x) = x^2 - 3x$. What is the function f? Is f unique?

The function f can be a simple linear function, such as $f(x) = x - 3$.

$$(g \circ f)(x) = g(f(x))$$
$$= (x - 3)^2 + 3(x - 3)$$
$$= x^2 - 6x + 9 + 3x - 9$$
$$= x^2 - 3x$$

We can prove that this function is unique or not by substituting $f(x) = x - 3$ into the function $g(x)$ and seeing if it is equal to the function $(g \circ f)(x)$.

$$g(x) = x^2 + 3x$$
$$g(f(x)) = (x - 3)^2 + 3(x - 3)$$
$$= x^2 - 6x + 9 + 3x - 9$$
$$= x^2 - 3x$$

3. **(1 mark each)** Determine which of the following functions are 1-1? Prove your answer.

(a) $f_1(x) = e^{-x^2}$

Defined in the workbook:

> A function $f : X \to Y$ is called **one-to-one** (or **injective**) if
> $\forall x_1, x_2 \in \mathbb{R} \cap X, \ f(x_1) = f(x_2) \implies x_1 = x_2.$

The function $f_1(x) = e^{-x^2}$ is one-to-one as it is a strictly decreasing function. Additionally, if we follow the definition of a one-to-one function:

$$f_1(x_1) = f_1(x_2)$$
$$e^{-x_1^2} = e^{-x_2^2}$$
$$\ln(e^{-x_1^2}) = \ln(e^{-x_2^2})$$
$$-x_1^2 = -x_2^2$$
$$x_1^2 = x_2^2$$
$$\implies x_1 = x_2$$

Therefore, the function $f_1(x) = e^{-x^2}$ is one-to-one.

You can see that there are other things:

(b) $f_2(x) = 2x^2 - 3x + 1$

The function $f_2(x) = 2x^2 - 3x + 1$ is not one-to-one. There are several reasons for this:

1. The function $f_2(x)$ is not strictly increasing or decreasing.

2. The function $f_2(x)$ is a quadratic function, and therefore can have two values of x that correspond to the same value of y.

3. Using the definition of a one-to-one function:

$$f_1(x_1) = f_1(x_2) \implies x_1 = x_2$$

$$2x_1^2 - 3x_1 + 1 = 2x_2^2 - 3x_2 + 1 \implies x_1 = x_2$$

We can see that the function $f_2(x)$ is not one-to-one as there are multiple values of x that correspond to the same value of y.
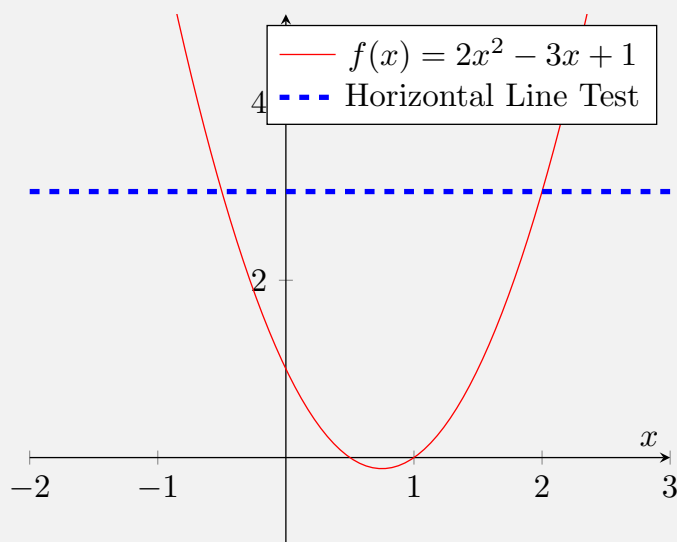
(c) $f_3(x) = |x| + 2 \cdot x$

The function $f_3(x) = |x| + 2 \cdot x$ is one-to-one. There are several reasons for this (using the definition of a one-to-one function):

$$f_3(x_1) = f_3(x_2)$$

$$|x_1| + 2 \cdot x_1 = |x_2| + 2 \cdot x_2$$

In this case, lets consider that there are two cases:

The first case is when $x_1, x_2 \geq 0$:

In this case, we can assume that $|x_1| = x_1$ and $|x_2| = x_2$. We can then determine the equation for this case:

$$|x_1| + 2 \cdot x_1 = |x_2| + 2 \cdot x_2$$

Simplifying this equation gives us:

$$3x_1 = 3x_2$$

Dividing both sides by 3 gives us:

$$x_1 = x_2$$

We can now determine the second case.

The second case is when $x_1, x_2 < 0$:

In this case, we can assume that $|x_1| = -x_1$ and $|x_2| = -x_2$. We can then determine the equation for this case:

$$-x_1 + 2 \cdot x_1 = -x_2 + 2 \cdot x_2$$

Simplifying this equation gives us:

$$x_1 = x_2$$

After determining both cases for the absolute value function, we can see that the function $f_3(x) = |x| + 2 \cdot x$ is one-to-one.

4. **(1 mark each)** Determine what the following limits are or show that they do not exist.

(a) $\displaystyle\lim_{n\to\infty} \frac{(n^2+4n-27)(n^3-1)}{(n(n-1))^2}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{(n^2+4n-27)(n^3-1)}{(n(n-1))^2} &= \lim_{n\to\infty} \frac{n^5+4n^4-27n^3-n^3+1}{n^4-2n^3+n^2} \\
&= \lim_{n\to\infty} \frac{n^5+4n^4-28n^3+1}{n^4-2n^3+n^2} \\
&= \lim_{n\to\infty} \frac{n+4-\frac{28}{n}+\frac{1}{n^3}}{1-\frac{2}{n}+\frac{1}{n^2}} \\
&= \frac{\infty}{1} \\
&= \infty
\end{aligned}
$$

(b) $\displaystyle\lim_{n\to\infty} \frac{3n^2-9n+48}{4n^3}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{3n^2-9n+48}{4n^3} &= \lim_{n\to\infty} \frac{3-\frac{9}{n}+\frac{48}{n^2}}{4n} \\
&= \lim_{n\to\infty} \frac{3}{4n} - \frac{9}{4n^2} + \frac{12}{n^3} \\
&= 0 \quad \text{Since the limit of } \tfrac{1}{n} = 0 \text{ as } n\to\infty.
\end{aligned}
$$

(c) $\displaystyle\lim_{n\to\infty} \frac{(3n+1)^3-27n^3}{n^2}$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{(3n+1)^3-27n^3}{n^2} &= \lim_{n\to\infty} \frac{27n^3+27n^2+9n+1-27n^3}{n^2} \\
&= \lim_{n\to\infty} \frac{27n^2+9n+1}{n^2} \\
&= \lim_{n\to\infty} \left(27 + \frac{9}{n} + \frac{1}{n^2}\right) \\
&= 27
\end{aligned}
$$

$\therefore$ The limit is 27 as $n\to\infty$.

(d) $\lim\limits_{n\to\infty} \frac{2n^2}{2n-1} - n$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{2n^2}{2n-1} - n &= \lim_{n\to\infty} \frac{2n^2}{2n-1} - \frac{n(2n-1)}{2n-1} \\
&= \lim_{n\to\infty} \frac{2n^2 - n(2n-1)}{2n-1} \\
&= \lim_{n\to\infty} \frac{2n^2 - 2n^2 + n}{2n-1} \\
&= \lim_{n\to\infty} \frac{n}{2n-1} \\
&= \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} \\
&= \frac{1}{2} \quad \text{Since the limit of } \tfrac{1}{n} = 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore, the limit is $\frac{1}{2}$ as $n \to \infty$.

(e) $\lim\limits_{n\to\infty} \sqrt{n(n+1)} - n$

In this limit, we can find the limit by rationalising the numerator.

To do this we can multiply the numerator and denominator by the conjugate of the numerator:

$$
\begin{aligned}
\lim_{n\to\infty} \sqrt{n(n+1)} - n &= \lim_{n\to\infty} \sqrt{n(n+1)} - n \times \frac{\sqrt{n(n+1)}+n}{\sqrt{n(n+1)}+n} \\
&= \lim_{n\to\infty} \frac{n(n+1)-n^2}{\sqrt{n(n+1)}+n}
\end{aligned}
$$

We can then simplify the equation to find the limit:

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n(n+1)}+n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n(n+1)}+n} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\
&= \frac{1}{2} \quad \text{Since the limit of } \tfrac{1}{n} = 0 \text{ as } n \to \infty.
\end{aligned}
$$

5. **(1 mark each)** Consider the sequence a_n defined by the recursion

$$a_n = a_{n-1} - \frac{1}{4}a_{n-2} \tag{1}$$

for $n = 3, 4, 5, \ldots$.

(a) Calculate h such that $a_n = h^{n-1}$ fulfils the recursive definition.

To calculate h we can substitute $a_n = h^{n-1}$ into the recursive definition:

$$h^{n-1} = h^{n-2} - \frac{1}{4}h^{n-3}$$

We can now divide both sides by h^{n-3}:

$$h^2 = h - \frac{1}{4}$$

Solving h in terms of the quadratic formula:

$$h = \frac{-(-1) \pm \sqrt{1 - 4 \cdot 1 \cdot \frac{1}{4}}}{2 \cdot 1}$$

$$h = \frac{1 \pm \sqrt{1 - 1}}{2}$$

$$h = \frac{1 \pm 0}{2}$$

$$\therefore h = \frac{1}{2}$$

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(b) What is the limit of the sequence a_n (if it exists)?

> To find the limit of the sequence a_n we can start by looking at the recursive definition.
>
> Since $a_n = a_{n-1} - \frac{1}{4}a_{n-2}$, we can tell that the sequence is dependent on two previous terms; a_{n-1} and a_{n-2}.
>
> Because of this, the limit when n approaches infinity will be dependent on the previous two terms.
>
> Since the previous terms will always be smaller than the current term and that the terms will get infinitely smaller, the limit of the sequence a_n will be 0.

6. **(1 mark each)** Given is the sequence b_n defined in recursive form

$$b_n = \frac{1}{2}\left(b_{n-1} + \frac{A}{b_{n-1}}\right)$$

for a given $A > 0$. You can assume that all values of b_n are non-zero.

(a) Use your calculator (or MATLAB) to calculate the first four values of the sequence b_n starting from $b_1 = A$ (this is for $n = 1, 2, 3, 4$). Inspecting these values do you expect the sequence to be convergent or to be divergent?

> Given that A ¿ 0, I expect the sequence to be convergent. This is because as n increases, the value of b_n will approach the limit of the sequence.
>
> Substituting $n = 1, 2, 3, 4$ into the recursive definition:
>
> $$b_1 = \frac{1}{2}\left(b_{1-1} + \frac{A}{b_{1-1}}\right)$$
> $$= \frac{1}{2}\left(b_0 + \frac{A}{b_0}\right)$$
> $$= \frac{1}{2}\left(A + \frac{A}{A}\right)$$
> $$= \frac{1}{2}(A + 1)$$
> $$= \frac{A+1}{2}$$
> $$= \frac{A}{2} + \frac{1}{2}$$
>
> $$b_2 = \frac{1}{2}\left(b_{2-1} + \frac{A}{b_{2-1}}\right)$$
> $$= \frac{1}{2}\left(b_1 + \frac{A}{b_1}\right)$$
> $$= \frac{1}{2}\left(\frac{A}{2} + \frac{1}{2} + \frac{A}{\frac{A}{2}+\frac{1}{2}}\right)$$
>
> $$b_3 = \frac{1}{2}\left(b_{3-1} + \frac{A}{b_{3-1}}\right)$$
> $$= \frac{1}{2}\left(b_2 + \frac{A}{b_2}\right)$$
> $$= \frac{1}{2}\left(\frac{A}{2} + \frac{1}{2} + \frac{A}{\frac{A}{2}+\frac{1}{2}}\right)$$
>
> $$b_4 = \frac{1}{2}\left(b_{4-1} + \frac{A}{b_{4-1}}\right)$$
> $$= \frac{1}{2}\left(b_3 + \frac{A}{b_3}\right)$$
> $$= \frac{1}{2}\left(\frac{A}{2} + \frac{1}{2} + \frac{A}{\frac{A}{2}+\frac{1}{2}}\right)$$

(b) Assume you know the sequence b_n is converging, what would be its limit (or its limits)? Justify your answer. Is it consistent with your result of part (a)?

> The limit is 1. Please show working.

7. **(1 mark each)** Assume you have given a sequence c_n with non-zero values ($c_n \neq 0$ for $n = 1, 2, \dots$) that fulfils the condition

$$\left| \frac{c_n}{c_n - 1} \right| \leq q \tag{2}$$

for all $n = 1, 2, 3, \dots$ for some fixed constant q with $0 < q < 1$.

(a) Show that $c_n \to 0$ for $n \to \infty$. (Hint: use the squeeze theorem)

> To show that the sequence c_n converges to 0, we can use the squeeze theorem. We can start by rearranging the inequality:
>
> $$|c_n| \leq q|c_n - 1|$$
>
> Since we know that $0 < q < 1$, the absolute value of c_n will always be less than the absolute value of $c_n - 1$. This means that the sequence c_n will always be squeezed between 0 and $q|c_n - 1|$. Since $q|c_n - 1|$ converges to 0, the sequence c_n must also converge to 0. We can show this by using the squeeze theorem:
>
> $$|c_n| \leq q^n |c_{n-1}|$$
>
> Since $q^n |c_{n-1}|$ converges to 0, the sequence c_n must also converge to 0.
>
> Therefore the sequnce c_n converges to 0 when $n \to \infty$.

(b) Use the result from part(a) to show that $\lim\limits_{n \to \infty} \dfrac{2^n}{n!} = 0$.

> Using the result from Part A, we can show that the $\lim\limits_{n \to \infty} \dfrac{2^n}{n!} = 0$.
>
> We can start by defining a new sequence a_n:
>
> $$a_n = \frac{2^n}{n!} \quad \text{for } n = 1, 2, 3, \dots$$
>
> We then divide the sequence a_n by a_{n-1}:
>
> $$\frac{a_n}{a_{n-1}} = \frac{\frac{2n}{n!}}{\frac{2n-1}{(n-1)!}} = \frac{2n(n-1)!}{n!2n-1} = \frac{2}{n}$$
>
> Since $\frac{2}{n} \leq 1$ for all $n \geq 2$, we can apply the result from part A to show that the sequence a_n converges to 0 as n approaches infinity.

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(c) Again: use the result from part(a) to show that $\displaystyle\lim_{n\to\infty} \frac{1}{3^n n^3} = 0$.

Similar to Part B, we can use the result from Part A to show that the $\displaystyle\lim_{n\to\infty} \frac{1}{3^n n^3} = 0$.

We can start by defining a new sequence b_n:

$$b_n = \frac{1}{3^n n^3} \quad \text{for } n = 1, 2, 3, \ldots$$

We then divide the sequence b_n by b_{n-1}:

$$\frac{b_n}{b_{n-1}} = \frac{\frac{1}{3^n n^3}}{\frac{1}{3^{n-1}(n-1)3}} = \frac{(n-1)^3}{3n^3} = \frac{n^3 - 3n^2 + 3n - 1}{3n^3} = \frac{1}{3} - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3}$$

We now determine this limit:

$$\lim_{n\to\infty} \frac{1}{3} - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} = \frac{1}{3} - 0 + 0 - 0 = \frac{1}{3}$$

Since $\frac{1}{3} \leq 1$ for all $n \geq 2$, we can apply the result from part A to show that the sequence b_n converges to 0 as n approaches infinity.