ELEC 421

Digital Signal and Image Processing

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Course Roadmap for DSP

Lecture	Title
Lecture 0	Introduction to DSP and DIP
Lecture 1	Signals
Lecture 2	Linear Time-Invariant System
Lecture 3	Convolution and its Properties
Lecture 4	The Fourier Series
Lecture 5	The Fourier Transform
Lecture 6	Frequency Response
Lecture 7	Discrete-Time Fourier Transform
Lecture 8	Introduction to the z-Transform
Lecture 9	Inverse z-Transform; Poles and Zeros
Lecture 10	The Discrete Fourier Transform
Lecture 11	Radix-2 Fast Fourier Transforms
Lecture 12	The Cooley-Tukey and Good-Thomas FFTs
Lecture 13	The Sampling Theorem
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Lecture 15	MATLAB Implementation of Filter Design

Lecture 7:
Discrete-Time Fourier Transform

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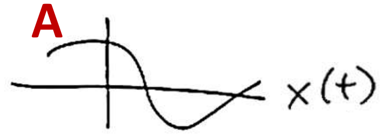
The Discrete-Time Fourier Transform (DTFT); How are the CTFT and DTFT different?

CONTINUOUS TIME FT (CTFT)

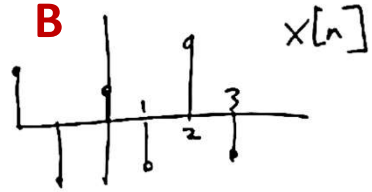
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \quad \textbf{(1)}$$

A



DEFINE THE DISCRETE-TIME FT (DTFT)

B



$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}} \quad \textbf{(2)}$$

(Forward FT)

- So far, for Fourier transform, we have been mostly staying in the continuous time, although we have tackled the discrete time on some occasions as well. Now, we want to discuss the **discrete time Fourier transform**.
- For the **continuous time FT**, which sometimes may be abbreviated as **CTFT**, to obtain **X(ω)** from **x(t)**, shown in Graph **A**, we used **(1)**.
- **How can we deal with a signal that is not continuous but instead is represented like the sticks of different heights?** This discrete signal is represented by **x[n]** and is shown in Graph **B**.
- We assume that the sticks in Graph **B** are ***indexed by integers***. So, our intuition is that what we want to do is to **turn the integral into a sum**. Instead of integrating over all the continuous values of **x** and **t**, we sum over all the discrete values.
- Now, we can define the **discrete time FT** or **DTFT**. To obtain **X(ω)** from **x[n]**, we can use **(2)**.

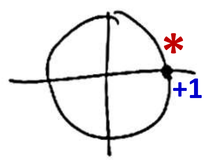
DTFTs are 2π-periodic

CTFT's HAVE A FREQUENCY RANGE $\omega \in (-\infty, \infty)$.

DTFT's HAVE A FREQUENCY RANGE OF WIDTH 2π.

$$X(\omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \underbrace{e^{-j2\pi \overbrace{n}^{\text{INTEGER}}}}_{1}$$



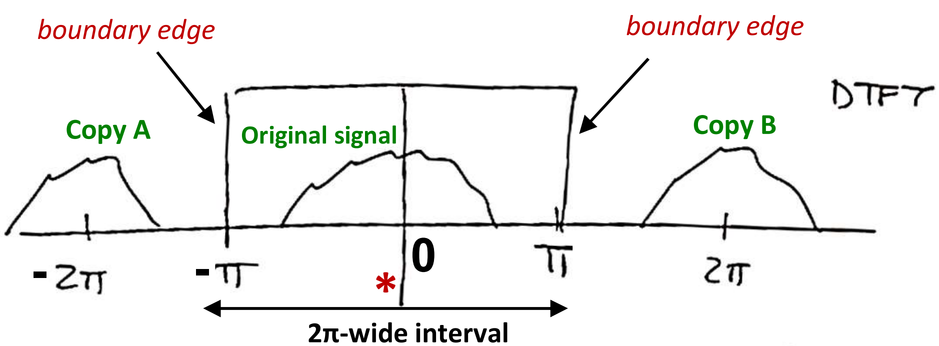
$$= X(\omega)$$

$$\rightarrow \boxed{X(\omega + 2\pi) = X(\omega)}$$

- The difference between the two **X(ω)**'s is that for the continuous time FT, we can have various frequencies (e.g., 5,000 Hertz or 10,000 Hertz). But in discrete time, it turns out that we only have a fixed set of frequencies. Why is that?
- We see that **exp(-j2πn)** is always at the point shown by * on the unit circle (which is **+1**). No matter what the value of the integer **n** is, we always end up at the same point. If **n = 0**, we are at point *. If we go around once, that is still *, go around twice, that is **4π**, and so on. Hence, no matter what **n** is, **exp(-j2πn)** always reduces to **1**. So, **X(ω+2π) is the same as X(ω)**.
- In summary, for continuous time FT's, we have a frequency range that is basically unlimited. But for the **DTFT**'s, we have a frequency range of width **2π**.

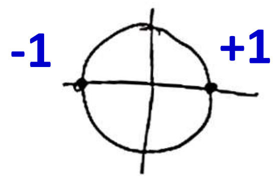
DTFTs are 2π-periodic

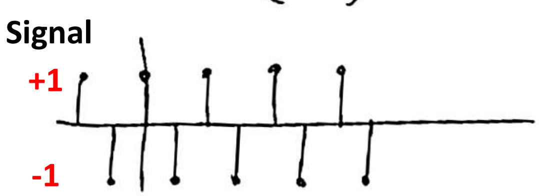
(1)



(2) $e^{j\pi n} = \cos \pi n + j \sin \pi n$

$= \pm 1 = (-1)^n$ (3)





- **Note:** In graph **(1)**, there are some **symmetries** that when everything is real, we often just draw from **0** to **+π**. This is because we know that we can predict what the left hand side looks like given the right hand side.

- It would be more accurate to say that the DTFT can be evaluated for any omega, but it is **2π-periodic**. What it would look like would generally be something that looks like **(1)**, where we have repeated copies of whatever is happening in the middle, every **2π**-units. What happens in practice is that we usually restrict our attention to the range between **–π** and **+π** and we only draw the DTFT inside the **2π**-wide interval (shown by *) with the understanding that the DTFT actually has other copies out there (such as **Copy A** or **Copy B**).

- One of the reasons why we choose **–π** and **+π** as the boundaries is the behavior of **exp(jπn)**. This term is presented in **(2)**. When **n = 0**, we are at **+1**, and when **n = 1**, we are at **-1**. For **n = 2**, we are at **+1** again. So, basically, we are bouncing back and forth between **+1** and **-1**.

- If we draw that signal, it looks like something that alternates between **+1** and **-1** as fast as it can go. In some sense, this is the highest frequency discrete time signal that we can create. So, it stands to reason that we draw our *boundary edges* at **–π** and **+π**.

Deriving the inverse DTFT

CT IFT: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$

DT IFT: $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$

BY ANALOGY:

PROOF: CONSIDER $\int_{-\pi}^{\pi} \underbrace{X(\omega)}_{\text{forward FT}} e^{j\omega n} \, d\omega =$

$$= \int_{-\pi}^{\pi} \underbrace{\sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m}}_{\text{forward FT}} e^{j\omega n} \, d\omega \quad \textbf{(1)}$$

- So far, we talked about the **forward FT** of a discrete time signal. Now, **how would we go back and get the discrete time signal, x[n], from the FT, X(ω)?**
- Here, **CT IFT** (or **CTIFT**) refers to continuous time inverse Fourier transform and **DT IFT** (or **DTIFT** or **IDTFT**) refers to discrete time inverse Fourier transform.
- In **(1)**, we are going to use **m** here instead of **n** because we have got an **n** here already, and so for the purposes of this expression, **n** is already some fixed number. So, we need to use a different variable here as our **dummy summation variable**. Now, what we would like to be able to do is to switch the integral and the sum.

Deriving the inverse DTFT

A SUFFICIENT CONDITION TO SWITCH THE ORDER:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \textbf{(2)}$$

(*)

$$\int \sum = \sum_{m=-\infty}^{\infty} x[m] \boxed{\int_{-\pi}^{\pi} e^{j\omega(n-m)}\, d\omega}$$

$$\rightarrow \int_{-\pi}^{\pi} [\cos \omega(n-m) + j \sin \omega(n-m)]\, d\omega \quad \textbf{(3)}$$

$n-m=0$: $\displaystyle\int_{-\pi}^{\pi} 1\, d\omega = 2\pi$

$n-m \neq 0$: $\displaystyle\int_{-\pi}^{\pi}$ 

or 

(4)

- A sufficient condition to switch the integral and the sum is that if we sum up all of the values of the absolute value of **x[n]**, we get something that converges, shown by **(2)**.
- In **(3)**, when **n # m** (or **n – m # 0**), we have an integral from **–π** to **+π** of a signal that is wiggling between this region, shown by (4). This oscillation happens an integer number of times inside this interval. When we are inside this interval, and when **n # m**, we always have as much above the line as we have below the line (the shaded areas). So, all these integrals turn to **0** when **n # m**.
- **Summary for Integral (*):** When **n = m**, it is equal to **2π**, and when **n # m**, it turns into **0**.

For n = m:

$$\sum_{m=-\infty}^{\infty} x[m] \int_{-\pi}^{\pi} e^{j\omega(n-m)}\, d\omega = 2\pi \cdot x[n] \longrightarrow \textbf{(1)} = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega = 2\pi \cdot x[n]$$

Sub for the integral term in $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \longrightarrow x[n] = \frac{1}{\cancel{2\pi}} \cdot \cancel{2\pi} \cdot x[n] \longrightarrow$ **LHS = x[n]** and **RHS = x[n]**.

So, we proved the formula for DT IFT (or **IDTFT**) in the previous slide.

The DTFT and IDTFT formulas

DTFT: $$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

IDTFT: $$\underbrace{x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega}_{}$$ **(1)**

LOOKS LIKE FOURIER SERIES EXPANSION FOR A SIGNAL WITH PERIOD 2π

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j\overset{1}{\overset{\shortparallel}{\frac{2\pi}{T}}} kt} \, dt$$ **(2)**

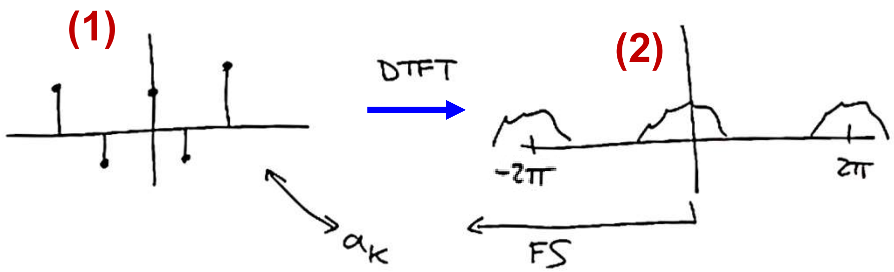
- Let us it put it all together. We see that **(1)** looks a lot like **(2)**. Here, instead of **ω**, we have **t**, and instead of **dω**, we have **dt**. To get **(1)** from **(2)**, we also need to let **T = 2π** in **(2)**.

Reminder for Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \, dt$$

The DTFT and IDTFT formulas



$$x[n] \underset{\text{FS}}{\overset{\text{DTFT}}{\rightleftarrows}} X(\omega)$$

- One way to think about **the relationship between the DTFT and the FS** is saying if we were to take signal **(1)**, which is our original discrete time signal, it will have some DTFT that is **2π-periodic**, shown in **(2)**. The idea would be that if we take this **2π-periodic** signal in **(2)** and we go backwards and compute the Fourier series of **(2)**, we will get the $\mathbf{a_k}$'s in **(1)**.
- **Conclusion:** The DTFT can be viewed as a ***Fourier series in reverse***. If you take the Fourier series of the DTFT **X(ω)**, the Fourier coefficients $\mathbf{a_k}$ will give you the discrete-time samples **x[n]**.

The 4 kinds of Fourier Transforms

Table:

	CONTINUOUS	DISCRETE
PERIODIC	FS	? (DFT) ↓ AFTER Z-TRANSFORM
NONPERIODIC	FT	DTFT

- Let us make a table, where in the columns, we have either a ***continuous*** or ***discrete*** signal and in the rows, we have either a ***periodic*** or a ***non-periodic*** signal.
- We have already showed that when we have a continuous periodic signal, we take the **FS** and when we have a continuous non-periodic signal, we take the **FT**.
- What we just showed here was if we have a discrete non-periodic signal, we take the discrete time FT (**DTFT**). What we have yet to fill in is the entry for discrete time periodic signal, shown by question mark. This is going to be the discrete FT (**DFT**). We are going to do this after we do the **z**-transform.

How does the DTFT converge?

IF $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ **(1)**

(2) $\lim_{N\to\infty} |X(\omega) - X_N(\omega)| = 0$

AS N GETS BIG, $X_N(\omega) = \sum_{n=-N}^{n=N} x[n]e^{-j\omega n}$ **(3)**

CONVERGES AT EACH POINT TO $X(\omega)$.

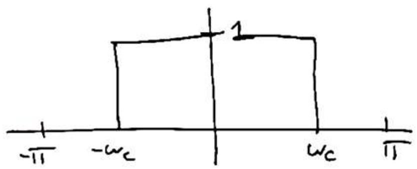
IF $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$ **(4)** (LOOSER CONDITION)

(5) $\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 d\omega = 0$

- If we have the convergence property **(1)**, then we have what we would call a ***really good convergence***, shown by **(2)**.
- $\mathbf{X_N(\omega)}$ is the partial sum where we only add up a finite amount of FS terms. The limit **(2)** says that, in the limit and as **N** gets big, $\mathbf{X_N(\omega)}$ converges at each point to the actual value, i.e., to $\mathbf{X(\omega)}$. So, the actual value and the approximate value get closer to each other. It is like saying that if we want to have the approximation of our FT to be as close as possible to the true thing, we will get there if we ***add up enough terms*** at every point.
- There is a ***looser*** **condition** that says we can have **(4)** instead. This is called **mean square convergence**. It means there are signals that satisfy **(4)** but do not satisfy **(1)**. In that case, what we have instead is **(5)**. So, **(2)** may not be true at every point exactly, but the integral of difference between the true thing and the approximation gets small smaller and smaller. This is related to the **Gibbs phenomenon**.

Inverse DTFT of a pulse (i.e., an ideal lowpass filter); DTFT examples

Example:

If $X(\omega) =$  **then** $x[n] = ?$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega = \frac{1}{2\pi j n} e^{j\omega n}\Big]_{-\omega_c}^{\omega_c}$$

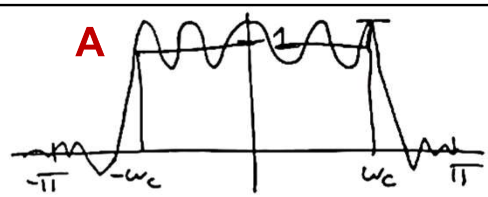
$$= \frac{1}{2\pi j n}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right) = \frac{1}{\pi n}\sin(\omega_c n) \longrightarrow$$

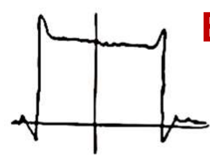
$$x[n] = \frac{\omega_c}{\pi}\,\mathrm{sinc}(\omega_c n)$$

SATISFIES $\sum |\,|^2 < \infty$

NOT $\sum |\,| < \infty$

$X(\omega) =$
(using partial sums)

A 

 **B**

$$\longrightarrow \boxed{x[n] = \frac{\omega_c}{\pi}\,\mathrm{sinc}(\omega_c n)}$$

(1) $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ **(2)** $X_N(\omega) = \sum_{n=-N}^{n=N} x[n] e^{-j\omega n}$

- The outcome of our computation is **x[n]** equation, which is what we expected. That is, a pulse in one domain corresponds to a sinc in other domain.
- This sinc function satisfies that "the sum of the square is less than infinity" ***but not*** that "the sum of the absolute value is less than infinity".
- If we start from **x[n]** and we try to go back to **X(ω)**, we need to use **(1)** or, in practice, its approximation, **(2)**, the **partial sums**, $\mathbf{X_N(\omega)}$. What do we get if we consider the partial approximations in the time domain, and adding **x[n]** terms? In the frequency domain, we will get something that looks like the **Gibbs phenomenon** shown in **A**. This looks like an ***approximate filter***. Again, there is always some little oscillations at the shoulder. This means that we do not converge at every point but we ***do*** get something that gets closer and closer, and eventually we will get something that looks like **B**. Mathematically, it means that the **integral difference** between the approximation and the square wave gets smaller and smaller, and hence, we get convergence in that sense.

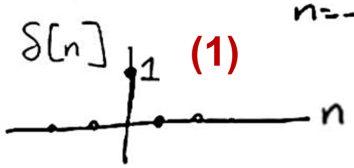
Delta function

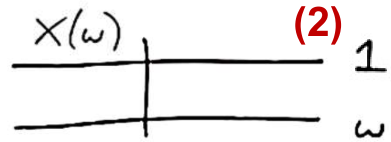
Example:

$x[n] = \delta[n]$ **(in the time domain)** **(A): Given x[n] find X(ω).**

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$ **(in the frequency domain)**

(1)





(2)

DTFT

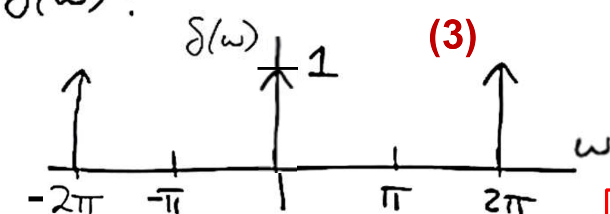
$$\boxed{X(\omega) = 1}$$

WHAT IF $X(\omega) = \delta(\omega)$? **(B): Given X(ω) find x[n].**

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}$$

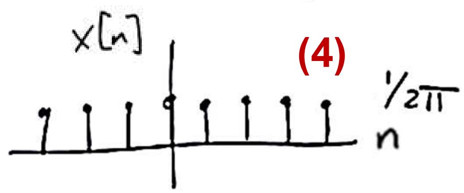
(3)



IDTFT

$$\boxed{x[n] = \frac{1}{2\pi}}$$

(4)



x[n] = The Inverse FT of X(ω) or IFT of δ(ω) = 1/(2π)

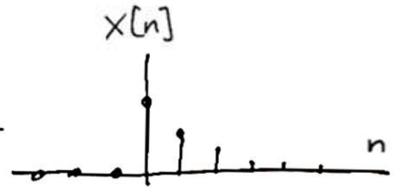
- In **(A)**, **x[n]** is given and we are looking for **X(ω)**. What if we go the other way around, i.e., case **(B)**? What if **X(ω) = δ(ω)** and we are looking for **x[n]**? To be clear though, we cannot actually have a delta function sitting alone at zero in the same way that we had for the continuous time FT. This is because when we have a discrete-time signal, everything in the frequency domain world has to be periodic. So, what we are saying is that we have a delta function that repeats every **2π** units. Because it is periodic, we can ask what is the inverse Fourier transform of this repeating signal.

Conclusion: It is still true that a delta function in one domain corresponds to a constant in the other domain, but we just have to be a little bit careful with what we mean by a delta function in the frequency domain world, i.e., about the graph of **(3)**.

One-sided exponential

Example:

$$x[n] = a^n u[n] \qquad |a| < 1$$



$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n \quad \textbf{(1)} \rightarrow \boxed{X(\omega) = \frac{1}{1 - a e^{-j\omega}}}$$

DTFT

MAGNITUDE RESPONSE $|X(\omega)|$

PHASE RESPONSE $\angle X(\omega)$

- The DTFT of a **finite-length sequence** is always periodic with period **2π**. The DTFT of an **infinite-length sequence** is **not necessarily periodic** and may or may not exhibit repetitive behavior, depending on the signal.

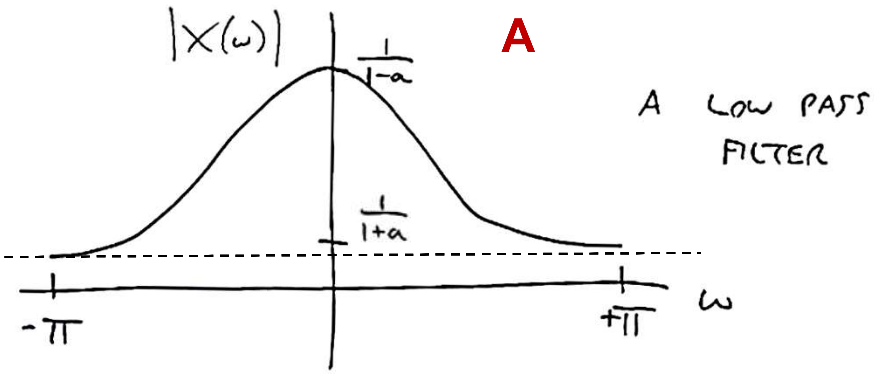
- In this example, we have an exponential (i.e., an infinite-length sequence) that is decaying. This is a valid signal to sum up. We already know what the infinite sum formula for this signal would look like and that it satisfies the conditions or our rules. That means that we could take the FT of it.
- In **(1)**, we can use the ***infinite sum formula*** since each one of the terms of **a.exp(-jω)** has a magnitude of less than **1**. This was the requirement to use this formula.
- **X(ω)** is a **complex function**. So, even though we may have a real signal, **x[n]**, we end up with complex FT. That being said, we can still look at the **magnitude** and the **phase** of this complex function to get a sense of what the filter is doing.
- The **magnitude response** is defined as the absolute value or the magnitude of **X(ω)** and the **phase response** is the angle. These two are complimentary ways of looking at what the filter is doing. The magnitude response is the one that really defines the character of the filter, i.e., “Is it a lowpass, is it a highpass, etc.?”.

One-sided exponential

$$|X(\omega)| = \left|\frac{1}{1-ae^{-j\omega}}\right| = \frac{1}{|(1-a\cos\omega)+ja\sin\omega|}$$

$$= \frac{1}{\sqrt{(1-a\cos\omega)^2+(a\sin\omega)^2}}$$

A

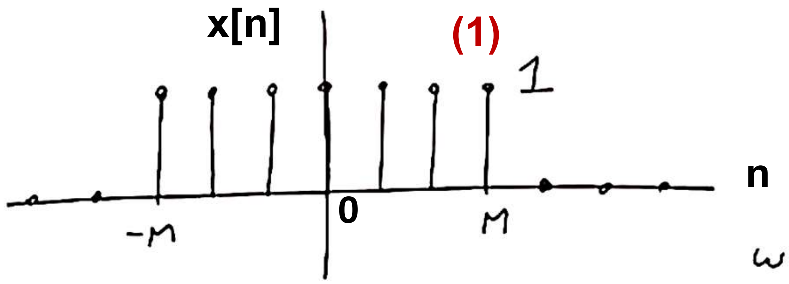


- Both the magnitude of this complex function, $|\mathbf{X}(\boldsymbol{\omega})|$, and "**a**" are going to be some real number. Since **a < 1**, as $\boldsymbol{\omega}$ gets bigger, $|\mathbf{X}(\boldsymbol{\omega})|$ is going to get smaller, as shown in **A**. So, the function starts out high and dips down, not to zero, but to **1/(1+a)**.
- **Conclusion:** $|\mathbf{X}(\boldsymbol{\omega})|$ looks like a **crude lowpass filter** because frequencies in the middle is getting passed through and frequencies at the edges is getting attenuated.

Time-domain pulse

Example:

PULSE IN THE TIME DOMAIN



$$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}$$

$$=\sum_{n=-M}^{M} e^{-j\omega n} \overset{(2)}{=} e^{j\omega M}\overbrace{\sum_{n=0}^{2M} e^{-j\omega n}}^{(3)} \longrightarrow$$

$$= e^{j\omega M}\left(\frac{1-e^{-j\omega(2M+1)}}{1-e^{-j\omega}}\right) \quad (4)$$

FINITE SUM FORMULA.

* $\omega=0$

$X(\omega)=2M+1$

- **What does a pulse (i.e., a discrete pulse) in the time domain look like?** A pulse in the time domain that is centered at **0** is shown in **(1)**. It is just going to be between **–M** and **+M**, for some integer **M**, and **0** everywhere else.
- The FT of **x[n]** is going to be the sum of the values of the signal **x[n]** times **exp(-jωn)**. Equation **(2)** can be turned into **exp(jωM)** times the sigma shown by **(3)**. So, based on **(3)**, we can instead start the sum at **0** and go up to **2M** (this is just a little bit of a mathematical trick!).
- For the new sigma **(3)**, we can use the **finite sum formula**. As a special case, when **ω = 0**, instead of **(4)**, which gives **0/0**, we will use **(2)**, which will give us **X(ω) = 2M+1** (as shown in the side note of *****).

- We had the following before (the sum of a finite geometric series):

$$\sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad 0<\alpha<1$$

Time-domain pulse

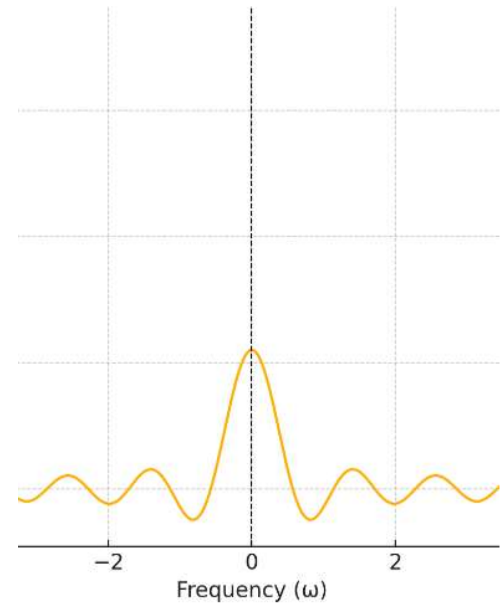
$$X(\omega) = e^{j\omega M}\left(\frac{1-e^{-j\omega(2M+1)}}{1-e^{-j\omega}}\right)$$

$$= \underbrace{e^{j\omega M}\,\frac{e^{-j\omega\frac{2M+1}{2}}}{e^{-j\omega/2}}}_{=1}\;\frac{\left(e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}\right)}{\left(e^{j\omega/2} - e^{-j\omega/2}\right)} \longrightarrow$$

$$\boxed{X(\omega) = \frac{\sin\left(\omega\frac{2M+1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}}$$

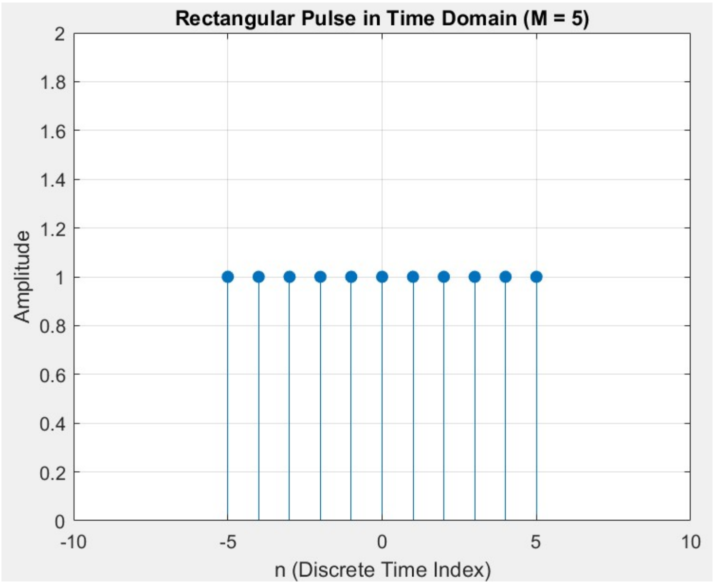
DTFT

Time-domain pulse



MATLAB code for graphing discrete time rectangular pulse:

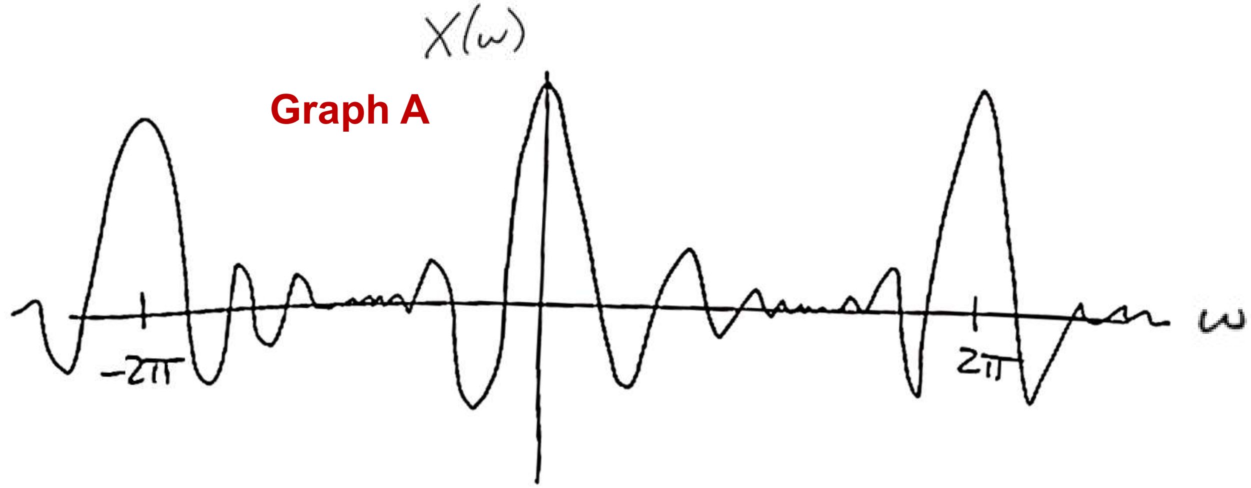
```
>> % Parameters
M = 5;  % Change M to modify the pulse length

% Time indices from -M to M
n = -M:M;

% Pulse of height 1 between -M and M
x_n = ones(1, 2*M + 1);

% Plot the rectangular pulse
figure;
stem(n, x_n, 'filled');
title(['Rectangular Pulse in Time Domain (M = ', num2str(M), ')']);
xlabel('n (Discrete Time Index)');
ylabel('Amplitude');
grid on;

% Set axis limits
xlim([-10 10]);  % X-axis from -10 to 10
ylim([0 2]);     % Y-axis from 0 to 2
```



Time-domain pulse

Sketch of X(ω):

$$X(\omega) = \frac{\sin\left(\omega \frac{2M+1}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \quad \textbf{(1)}$$



- **Why this is not a sinc function? X(ω)** in **(1)** does not look like a sinc function. That is, **Graph A** does not look like a ***conventional sinc function***. First, we know that whenever we have a discrete time signal in the time domain, it has to be periodic in the frequency domain. Equation **(1)** ***is*** period but not quite a sinc function. Between **-2π** and **-2π**, this is the closest we can get to making a periodic version of the sinc. This graph looks like what we would get if we took sinc functions and shifted them to be centered at multiples of **2π** and then added all that stuff up. The outcome would be **(1)**, which is the closed form summation of this process.
- So, the intuition is still good. It is still a pulse in the discrete time domain that has converted to a ***sinc-like function*** (a ratio of sines), in the frequency domain, but the kind of sinc that we get in the frequency domain has to necessarily be a little bit modified to make it periodic.
- **Conclusion:** The functional form for the DTFT is a little bit different but if we were to look at the big picture it still kind of gives us that sinc-like function!

DTFT convolution property

WE WANT TO USE THE DTFT TO STUDY LTI SYSTEMS.

OUR CONVOLUTION PROPERTY STILL HOLDS:

$$y[n] = x[n] * h[n] \quad \textbf{(1)}$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega n} \qquad \begin{matrix} m=n-k \\ n=m+k \end{matrix} \quad \textbf{(2)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega(m+k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}\right)\left(\sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m}\right) \quad \textbf{(3)}$$

$$Y(\omega) = X(\omega) H(\omega) \quad \textbf{(4)}$$

- Our main goal is to analyze how a system operates on a signal. That is, we want to use the DTFT to study LTI systems. So, let us prove that our convolution property still holds.
- In **(1)**, let us say that we have **y[n]**, which is going to be **x[n]** convolved with **h[n]**. This is the kind of convolution that we need to do for discrete time signals. Next, we will find the DTFT of **y[n]**, i.e., **Y(ω)**.
- In **(2)**, we are going to move the sums around. Here, we also do a quick change of variables.
- The terms in the parentheses in **(3)** are exactly the definition of the FT of **x[n]** or **X(ω)** and the FT of **h[n]** or **H(ω)**. After re-writing the two sums as the product of these two sums, we arrive at **(4)**.
- **Conclusion:** We showed that starting with a discrete signal of **x[n]** and an impulse response of **h[n]**, the output Fourier transform, **Y(ω)**, is the product of the input Fourier transform, **X(ω)**, and **H(ω)**.

LTI systems and sinusoidal inputs: new frequencies cannot be introduced

$$x[n] = Ae^{j\omega_0 n}$$

$$y[n] = x[n] * h[n] \quad \textbf{(1)}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, Ae^{j\omega_0 (n-k)}$$

$$= Ae^{j\omega_0 n} \overbrace{\sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}}^{**}$$

$$y[n] = \underbrace{H(\omega_0)}_{*} \cdot Ae^{j\omega_0 n} \quad \textbf{(2)}$$

- **What is the intuition for the frequency response?** Let us suppose that we have an input signal that is some amplitude, **A**, times a complex exponential at some frequency, **exp(jω_0n)**. The intuition for the **frequency response** is that it takes every subcomponent frequency of the input, **x[n] = A.exp(jω_0n)** and modulates it by some amplification or attenuation and shift it by some phase. Put differently, the intuition is that the value of the frequency response at some fixed frequency, **H(ω_0)**, can tell us how cosines and sines are at that frequency at the output, **y[n] = H(ω_0).A.exp(jω_0n)**.
- **What happens when we put x[n] through our system?** This is the convolution, **(1)**, when we have our arbitrary impulse response. The sigma shown by ** is exactly what we defined as the frequency response evaluated at **ω_0**. So, this is like saying we take **H(ω_0)** and we multiply it by **A.exp(jω_0n)** to get **y[n]**, as shown by **(2)**.
- **Conclusion:** If we put **A.exp(jω_0n)** in, then what comes out is the very same thing multiplied by the complex number, **H(ω_0)**, which is the frequency response evaluated at that frequency.

Response of a real LTI system to a real cosine

(1)

$$A e^{j\omega_0 n} \rightarrow \underbrace{A|H(\omega_0)|}_{\text{AMPLITUDE SCALING}} e^{j(\overbrace{\angle H(\omega_0)}^{\text{PHASE SHIFTING}} + \omega_0 n)} \quad \textbf{(2)}$$

IN THE SAME WAY, IF $h[n]$ IS REAL,

(3)

$$\cos(\omega_0 n + \phi) \longrightarrow |H(\omega_0)| \cos(\omega_0 n + \phi + \angle H(\omega_0)) \quad \textbf{(4)}$$

$$\longrightarrow \boxed{y[n] = A|H(\omega_0)| e^{j(\angle H(\omega_0) + \omega_0 n)}}$$

- To make it even more explicit, let us see what happens if we put signal **(1)** through the system. Here, what comes out is ***an amplitude scaling*** and ***a phase shifting***, shown by **(2)**.
- If we have a real-valued cosine, **(3)**, that has some frequency and phase, what comes out of the system is the same kind of amplitude scaling times the cosine at the ***same frequency*** plus a phase shift, shown by **(4)**. This is always true whether or not **h[n]** is real-valued or complex-valued.
- As shown in **(4)**, what we get when we put the cosine in is the same cosine coming out with the amplitude scaling of $|\mathbf{H}(\boldsymbol{\omega_0})|$ and the phase shifting of $\angle\mathbf{H}(\boldsymbol{\omega_0})$.
- **Conclusion:** We cannot introduce new frequencies into an LTI system. The only thing we can do is to take the magnitude of the frequency components that we have and amplify or attenuate them and move them around.

Example: system responses involving pure cosines

Example:

$$y[n] = A|H(\omega_0)|\, e^{j(\angle H(\omega_0) + \omega_0 n)}$$

$$h[n] = \left(\tfrac{1}{3}\right)^n u[n] \longrightarrow H(\omega) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}} \quad \text{(1)}$$

$$x[n] = 2e^{j\pi/3\, n} \quad \text{(2)}$$

$$H\left(\tfrac{\pi}{3}\right) = \frac{1}{1 - \frac{1}{3}e^{-j\frac{\pi}{3}}} \quad \text{(3)}$$

$$= \frac{1}{1 - \frac{1}{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}j\right)} = \frac{1}{\frac{5}{6} + \frac{\sqrt{3}}{6}j} \quad \text{(4)}$$

$$|\text{(4)}|, \ \angle|\text{(4)}| \longrightarrow$$

$$\boxed{y[n] = 2\left|H\left(\tfrac{\pi}{3}\right)\right| e^{j\left(\frac{\pi}{3}n + \angle H\left(\frac{\pi}{3}\right)\right)}}$$

- **What is y[n] for the given x[n] and h[n]?** We are given the impulse response, **h[n]**, and also an input signal which is a standard looking exponential. The first thing that we should recognize is that in **(2)** (the input signal), we only have a single sinusoid at the frequency of **π/3**. So, we just need to plug in **π/3** into **(1)** to get **H(π/3)**.
- So, all we need to know is what is happening in the frequency response at **π/3**. In other words, all we need to do in order to solve this problem is to figure out the magnitude and phase of this single complex number, **(4)**. There is no need to take **x[n]** into the frequency domain. All that is needed is ***observe*** that when we have a pure combination of cosines and sines in the input, we can only get attenuated or amplified versions of those pure cosines and sines in the output.

Example: system responses involving pure cosines

MATLAB code for $|H(\omega_0)|$ and $\angle H(\omega_0)$, for ω_0 = π/3:

First method:

```
>> % Define the complex number H(pi/3)
H_pi_over_3 = 1 / (5/6 + (sqrt(3)/6)*1i);

% Calculate the magnitude and angle
magnitude = abs(H_pi_over_3);
angle_in_radians = angle(H_pi_over_3);  % Angle in radians
angle_in_degrees = rad2deg(angle_in_radians);  % Convert to degrees

% Display the results
disp(['Magnitude of H(pi/3): ', num2str(magnitude)]);
disp(['Angle of H(pi/3) in radians: ', num2str(angle_in_radians)]);
disp(['Angle of H(pi/3) in degrees: ', num2str(angle_in_degrees)]);
Magnitude of H(pi/3): 1.1339
Angle of H(pi/3) in radians: -0.33347
Angle of H(pi/3) in degrees: -19.1066
```

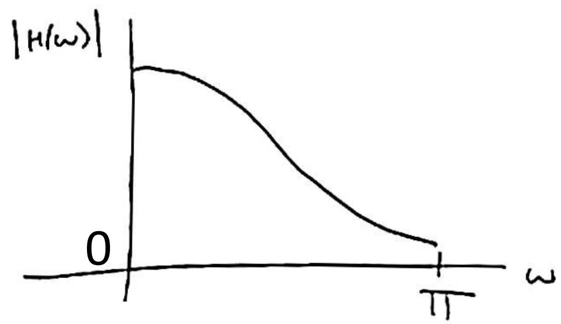
Second method:

```
>> z=1/(1 - (1/3)*exp(-i*pi/3)); M = abs(z)
M =
 1.1339
>> Ph2 = atan2(imag(z), real(z))
Ph2 =
 -0.3335
```

Magnitude and phase response

Magnitude Response

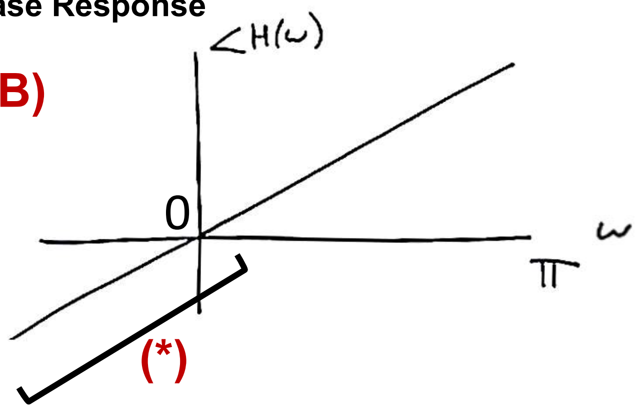
(A)



WHEN h[n] IS REAL, |H(ω)| IS EVEN.

Phase Response

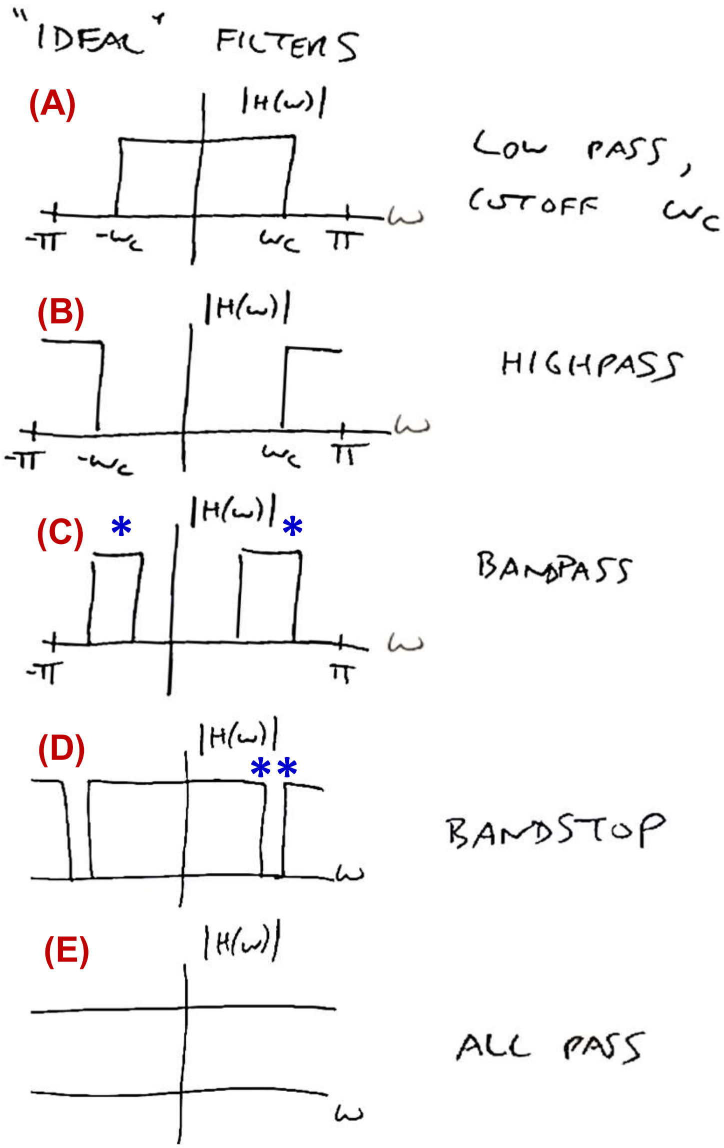
(B)



WHEN h[n] IS REAL ∠H(ω) IS ODD

- Next, we will discuss the magnitude and phase response plots. A lot of times, we tend to pay more attention to the magnitude and less attention to the phase. All the asymmetry and shifting properties we discussed before can be carried over to magnitude and phase response plots.
- **Magnitude Response:** When **h[n]** is **real**, which is often the case, the magnitude response is **even**, shown in **(A)**. What that means is that often times, we only plot the magnitude response that is between **ω = 0** and **π** with the understanding that **0** represents the lowest frequency, which is DC, and **π** represents the highest frequency, which is the cosine that is alternating up and down all the time.
- **Phase Response:** When **h[n]** is **real**, the angle or the phase response is **odd**. The phase response plot often looks like **(B)**. It continues off to the left side, but often times, we will only plot the positive value, i.e., between **ω = 0** and **π** and ignore **(*)**. Here, we drew this as a straight line because that is a desirable thing. We will discus why this is true at a later time.

Ideal filters in the frequency domain

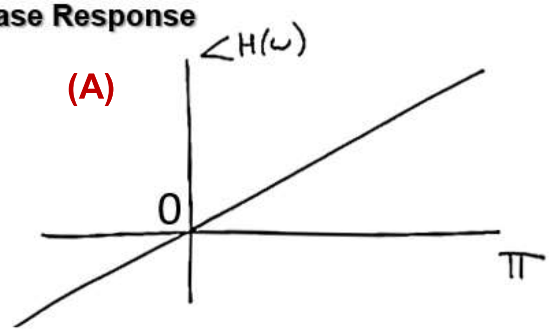


- **Different Types of Digital Filters:** Usually the digital filters are classified according to what their magnitude response looks like (sometimes called **ideal filters**).
- **Lowpass filter (A):** We should keep in mind that the world of frequency is really bounded in the **2π**-wide interval. Here, there is an upper bound on how high the frequency could get. However, in continuous time, there is no such upper bound. Now, we can put lower bound and upper bound on the frequency band. **(A)** is a lowpass filter, where we have a **cutoff frequency of ω_c**.
- **Highpass filter (B):** This is the opposite of **(A)**. Here, the shape of the filter is so that it is chopping out lower frequencies and retaining higher frequencies.
- **Bandpass filter (C):** We can just keep a little section of the frequency band (shown by *).
- **Bandstop filter (D):** We can also do a bandstop filter, where we basically keep everything except for little narrow area (shown by **).
- **All pass filter (E):** Here, the frequency content is not getting changed but the phases of the individual cosines and sines are going to change. One may think that this is like a useless filter but what is happening under the hood is that the phases of things are getting changed and that can be important.

Why is linear phase response desirable?

Phase Response

(A)



PHASE RESPONSE.

CONSTANT

WHY IS $\angle H(\omega) = -c\omega$ DESIRABLE? **(1)**

SAY $|H(\omega)|$ IS PIECEWISE CONSTANT.

IN THE PASS REGION, $(|H(\omega)| = 1)$

$$Y(\omega) = X(\omega)H(\omega) \quad \textbf{(2)}$$

$$= X(\omega)\,|H(\omega)|\,e^{j\angle H(\omega)} \quad \textbf{(3)} \longrightarrow$$

$$Y(\omega) = X(\omega)\,e^{-jc\omega} \xrightarrow{\text{IDTFT}}$$

FOR LINEAR PHASE FILTER. $\angle H(\omega) = -c\omega$.

$$\boxed{y[n] = x[n-c]}$$

(4)

PURE DELAY OF THE INPUT — NOT CONSIDERED DISTORTION.

- **Why is a linear phase response, shown in (A), desirable?** In **(1)**, **c** (also shown by **τ**) is just some constant (either positive or negative).
- Let us say that the magnitude of **H(ω)** is ***piecewise constant***. In all the pictures we drew for various filters, we noticed that all are piecewise constant. For instance, the magnitude is 1 in the ***pass band***, or 0 in the ***stop band***. FT of the output is going to be FT of the input times the frequency response, shown by **(2)**. In **(3)**, we are just decomposing $\mathbf{H(\omega)}$ into magnitude and phase.
- In the pass region, since $|\mathbf{H(\omega)}| = \mathbf{constant} = \mathbf{1}$, then the magnitude is unchanged and what we have is just exp(j.$\angle\mathbf{H(\omega)}$) for a linear phase filter.

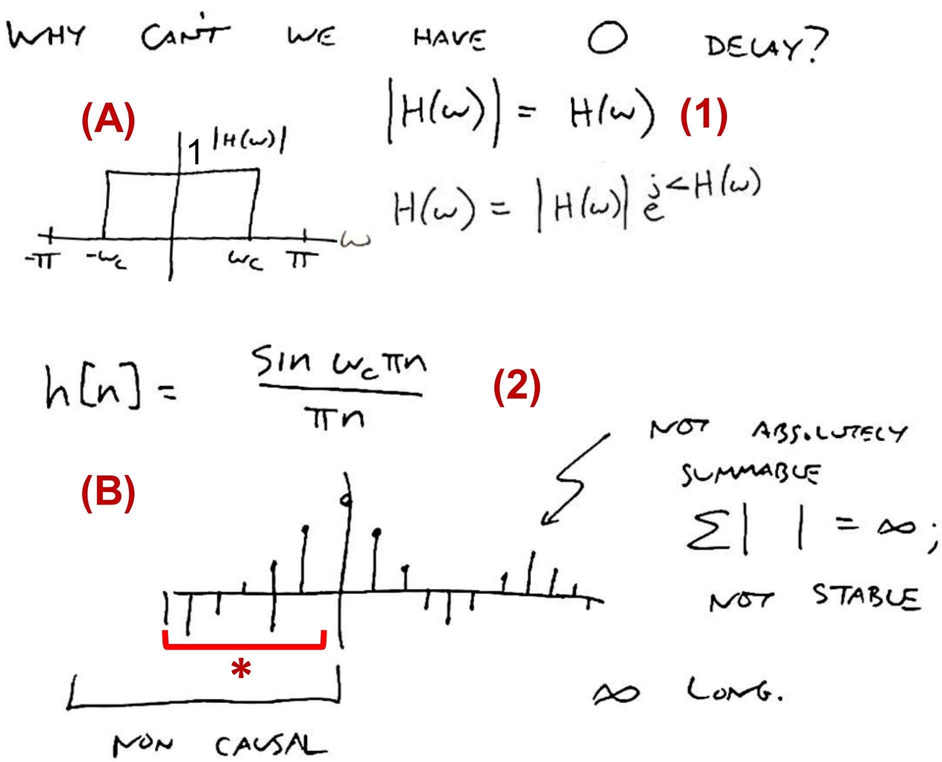
Why is linear phase response desirable?

(4) $y[n] = x[n-c]$

PURE DELAY OF THE INPUT – NOT CONSIDERED DISTORTION.

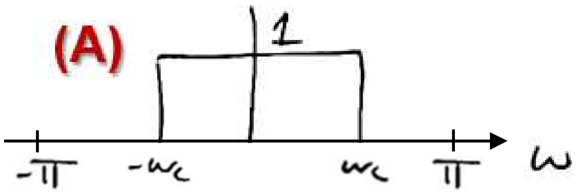
- In **(4)**, we see that this is basically just a phase shift in the frequency domain. We can prove that this is corresponding to a **delay** in the time domain (for positive **c**).
- This is good because it just means that after we apply this linear phase filter to a signal, all we are really doing is delaying the output.
- So, in some sense, what is happening is that all of the cosines and sines are getting delayed ***by the same amount***. Hence, it is not considered to be **distortion** of the signal.
- If all the cosines and sines that were making up the signal were being delayed at different rates then our output signal would sound very weird. But if everything is shifting at the same rate, all we are doing is delaying the whole thing.
- **Conclusion:** We have a pure delay of the input which is not considered as distortion.

Why cannot we have zero delay in the real world?



- **If we are going to be delaying the signal why not just have zero delay all the time?** This is like saying we want to basically not have any shifting at all. We already showed that if we have, for example, a lowpass filter **(A)**, and when we have zero delay, that means the magnitude, $|\mathbf{H}(\boldsymbol{\omega})|$, is equal to the actual filter, $\mathbf{H}(\boldsymbol{\omega})$, shown in **(1)**.
- We showed that the corresponding time domain signal was basically a sinc function, **(2)**.
- The graph of **(2)** looks like **(B)**. Here, the first observation is that it goes on infinitely, but more importantly, it has these negative values on the left hand side, shown by (*). So, all this means is that **the filter is non-causal**.
- Another factor is that this filter **is not absolutely summable**. That means that the sum of the absolute value of the terms equals infinity. This, in turn, means that **the filter is not stable**. We have not not talk about stability in great details yet, but we will later. For now, just suffice to say that it is not desirable!
- Another factor is that it is infinitely long.

Why cannot we have zero delay in the real world?



- **Conclusion:** There are lots of reasons why we cannot take this lowpass filter ideally and get it to actually work in a physical system, i.e., our impulse response is not very desirable. However, as it turns out, we ***can*** make a nice impulse response that comes pretty close to **(A)** and has linear phase. This means that all we are doing is really conducting a nice filtering of the signal and only delaying the output. So, <u>if we want to filter a signal, we have to tolerate a little bit of delay in our output</u> and that is fine.
- As an example, if we put a CD into a stereo, there is no guarantee that the bits that are getting read off the CD are getting to us in real time. We do not know how much delay is built into that system, and in most cases, it does not really matter to us.

End of Lecture 7