# ELEC 421

# Digital Signal and Image Processing

**Siamak Najarian, Ph.D., P.Eng.,**

Professor of Biomedical Engineering (retired),

Electrical and Computer Engineering Department,

University of British Columbia

# Course Roadmap for DSP

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|---|---|
| Lecture 0 | Introduction to DSP and DIP |
| Lecture 1 | Signals |
| Lecture 2 | Linear Time-Invariant System |
| Lecture 3 | Convolution and its Properties |
| Lecture 4 | The Fourier Series |
| Lecture 5 | The Fourier Transform |
| Lecture 6 | Frequency Response |
| Lecture 7 | Discrete-Time Fourier Transform |
| Lecture 8 | Introduction to the z-Transform |
| Lecture 9 | Inverse z-Transform; Poles and Zeros |
| Lecture 10 | The Discrete Fourier Transform |
| Lecture 11 | Radix-2 Fast Fourier Transforms |
| Lecture 12 | The Cooley-Tukey and Good-Thomas FFTs |
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| Lecture 15 | MATLAB Implementation of Filter Design |

# Lecture 6:
# Frequency Response

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# Proving the convolution property of the Fourier Transform

$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega)H(\omega) \quad \textbf{(1)}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau\right)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\overbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)\,e^{-j\omega t}\,dt\right]}^{(*)}d\tau \quad \textbf{(2)}$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\mathcal{F}\big(h(t-\tau)\big)\,d\tau \quad \textbf{(3)}$$

$$\mathcal{F}\big(h(t)\big) = H(\omega) \longrightarrow \mathcal{F}\big(h(t-\tau)\big) = H(\omega)e^{-j\omega\tau} \xrightarrow{\text{sub in (3)}}$$

$$= \int_{-\infty}^{\infty} x(\tau)\,H(\omega)e^{-j\omega\tau}\,d\tau$$

$$= H(\omega)\int_{-\infty}^{\infty} x(\tau)\,e^{-j\omega\tau}\,d\tau = H(\omega)X(\omega) \longrightarrow \boxed{Y(\omega) = X(\omega)H(\omega)}$$

- If we have a signal, **y(t)**, that is the convolution of two other signals, **x(t)** and **h(t)**, then there is an FT property that tells us that the FT of **y(t)**, **Y(ω)**, is simply the product of the two FTs, as shown in **(1)**. Let us prove this.
- In **(2)**, we are going to do the standard trick of moving around the two integrals. So, we are going to interchange. This means that we can take out the terms that do not depend on **t**. Now, we see that **(*)** looks like an FT itself, i.e., the FT of **h(t - τ)**. This FT is abbreviated as ***curly F*** of the time-domain signal of **h(t - τ)**.

# The frequency response: the Fourier Transform of the impulse response

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad \textbf{(1)}$$

$$\delta(t) \rightarrow \boxed{\text{LTI System}} \rightarrow h(t)$$

IMPULSE RESPONSE

$$\mathcal{F}\big(h(t)\big) = H(\omega)$$

FREQUENCY RESPONSE

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) \qquad \textbf{(2)}$$

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow Y(\omega) \qquad \textbf{(3)}$$

- The reason that we care about the convolutions like this in the first place, i.e., **(1)**, is that we will be dealing with systems in which **x(t)** is the input, **h(t)** is the impulse response of the system, and **y(t)** is the output. In that case, we give this **H(ω)** a special name.
- If we put the delta function into the system (and again this only works for an LTI system), what comes out is **h(t)**, which we call the **impulse response**. Now, we define this new thing, which is the FT of **h(t)** or **H(ω)**, and we call that the **frequency response**. This is a very important concept. Because that is what lets us take things that were complicated in the time domain and make them easier in the frequency domain. This is why sometimes we will interchangeably use either of the block diagrams shown in **(2)** or **(3)**.
- In diagram **(3)**, in some references, the input and output are still shown in time-domain, i.e., **x(t)** and **y(t)**, instead of **X(ω)** and **Y(ω)**. We will use both methods of representation. Here, using either notation, it should be clear that **h(t)** is the impulse response of the system and **H(ω)** is the frequency response of the system.

# Series of systems in the frequency domain

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t) \qquad \textbf{(1)}$$

$$x(t) \rightarrow \boxed{H_1(\omega)} \rightarrow \boxed{H_2(\omega)} \rightarrow y(t) \qquad \textbf{(2)}$$

$$Y(\omega) = X(\omega)H_1(\omega)H_2(\omega) = X(\omega)H_2(\omega)H_1(\omega) \qquad \textbf{(3)}$$

$$\left.\begin{array}{l} x(t) \rightarrow \boxed{H_2(\omega)} \rightarrow \boxed{H_1(\omega)} \rightarrow y(t) \\ x(t) \rightarrow \boxed{h_2(t)} \rightarrow \boxed{h_1(t)} \rightarrow y(t) \end{array}\right] \qquad \textbf{(4)}$$

- We can now put it all together. Let us suppose we put the signal, **x(t)**, through two impulse responses, **(1)**. We can equivalently talk about putting that signal through two frequency responses, **(2)**. In the frequency domain, the output, **Y(ω)**, is going to be the input **X(ω)** times one frequency response, **$H_1$(ω)**, times the other frequency response, **$H_2$(ω)**, as represented by **(3)**.
- In **(3)**, we know that the order does not matter, since this is just a ***multiplication***. There is no problem interchanging these two things. This is another way of saying that if we interchange the two impulse responses, we get the same result.
- In **(4)**, the two block diagrams are equivalent.

# Interpreting the frequency response: the action of the system on each complex sinusoid

WHAT DOES FREQUENCY RESPONSE MEAN?

$$x(t) = e^{j\omega_0 t}$$

$$X(\omega) = 2\pi\,\delta(\omega - \omega_0) \quad \textbf{(1)}$$

- We used the following from earlier:

$$X(\omega) = \delta(\omega - k\omega_0)$$

**IFT** $$x(t) = \frac{1}{2\pi}\int \delta(\omega - k\omega_0)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\, e^{jk\omega_0 t}$$

- When **k = 1**, we get **(1)**.

- **What does the frequency response mean?** Let us put a complex exponential into our system. This exponential has a fixed frequency $\boldsymbol{\omega_0}$. One way to think about this **x(t)** is like a constant amplitude signal, **1**, times the exponential, or **x(t) = 1.exp(j$\omega_0$t) = exp(j$\omega_0$t)**.
- **What is X(ω)?** That is, what would be the FT of **x(t)**? It is just going to be a delta function. We can find that the FT of **x(t)** or **X(ω)** is just a shifted delta function, shown in **(1)**.

# Interpreting the frequency response: the action of the system on each complex sinusoid

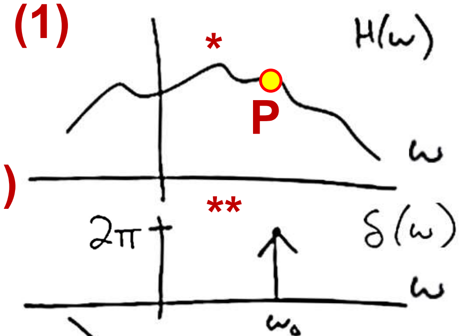
WHAT DOES FREQUENCY RESPONSE MEAN?

$$x(t) = e^{j\omega_0 t}$$

$$X(\omega) = 2\pi\,\delta(\omega-\omega_0) \quad \textbf{(1)}$$

$$Y(\omega) = \underbrace{H(\omega)}_{\text{FREQ. RESP}}\,X(\omega) \quad \textbf{(2)}$$

$$= H(\omega)\,2\pi\,\delta(\omega-\omega_0) \rightarrow$$




$$Y(\omega) = H(\omega_0)\,2\pi\,\delta(\omega-\omega_0) \quad \textbf{(3)} \quad \xrightarrow{\text{inverse FT}}$$

$$\boxed{y(t) = H(\omega_0)\,e^{j\omega_0 t}} \quad \textbf{(4)}$$

- Let us find out how the system responds to this shifted delta function, i.e., **how does the system respond to X(ω), and how to get Y(ω), shown in (2)?**
- In **(3)**, we are multiplying the delta function, **$2\pi.\delta(\omega-\omega_0)$**, shown in Graph ******, by whatever the complex number of **$H(\omega_0)$** is (shown by point **P** in Graph *****).
- As shown in Graphs ***** and ******, this is like we have some kind of arbitrary frequency response **$H(\omega_0)$** and we are multiplying it by a delta function that is shifted to fire at **$\omega_0$**. This leads us to the equation of **$Y(\omega) = H(\omega_0).2\pi.\delta(\omega - \omega_0)$**, **(3)**.
- In **(4)**, we undo the frequency domain to get back into time domain and to find the corresponding output of the system, **y(t)**. Here, we did an **inverse FT** on **(3)**.

# Interpreting the frequency response: the action of the system on each complex sinusoid

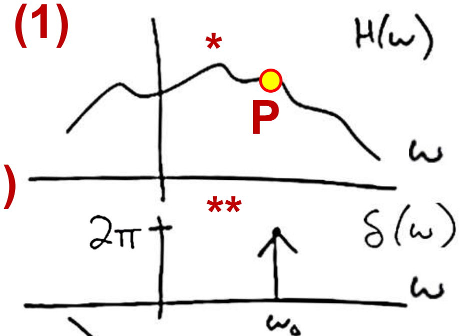
WHAT DOES FREQUENCY RESPONSE MEAN?

$$x(t) = e^{j\omega_0 t}$$

$$X(\omega) = 2\pi\,\delta(\omega-\omega_0) \quad \textbf{(1)}$$

$$Y(\omega) = \underbrace{H(\omega)}_{\text{FREQ. RESP}}\,X(\omega) \quad \textbf{(2)}$$

$$= H(\omega)\,2\pi\,\delta(\omega-\omega_0) \longrightarrow$$




$$Y(\omega) = H(\omega_0)\,2\pi\,\delta(\omega-\omega_0) \quad \textbf{(3)} \quad \xrightarrow{\text{inverse FT}}$$

$$\boxed{y(t) = H(\omega_0)\,e^{j\omega_0 t}} \quad \textbf{(4)}$$

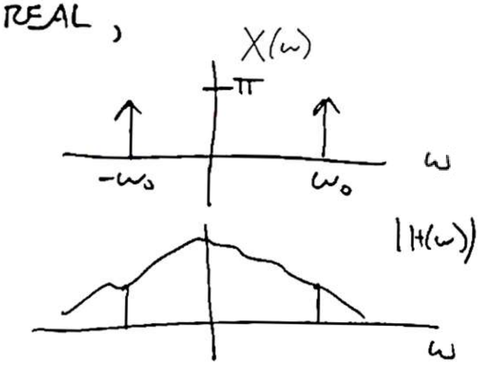
- **What is the meaning of equation (4)?** This mean that if we put a complex exponential in, with a certain frequency $\boldsymbol{\omega_0}$, then what comes out is ***the same complex exponential just multiplied by some complex number***. So, we are not changing anything about the frequencies in this signal. The only thing that is happening is we are taking that complex exponential, and then multiplying it by some **scaler**. That scaler, $\mathbf{H(\omega_0)}$, has a magnitude and an angle. Here, we are not changing the fundamental frequency content of the signal.
- **Conclusion:** The key idea is that the system cannot introduce new frequencies into the output that were not present in the input.

# A real LTI system only changes the magnitude and phase of a real cosine input

SUPPOSE $h(t)$ IS REAL,

$x(t) = \cos \omega_0 t$ **(1)**

$H(\omega) = (H(-\omega))^*$ **(2)**




$Y(\omega) = X(\omega) H(\omega)$

$= \pi \left( \delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right) H(\omega)$

$= \pi \left( H(\omega_0)\delta(\omega-\omega_0) + H(-\omega_0)\delta(\omega+\omega_0) \right)$

$= \pi \left( H(\omega_0)\delta(\omega-\omega_0) + (H(\omega_0))^*\delta(\omega+\omega_0) \right)$

$= \pi \left( |H(\omega_0)| e^{j\angle H(\omega_0)}\delta(\omega-\omega_0) + |H(\omega_0)| e^{-j\angle H(\omega_0)}\delta(\omega+\omega_0) \right) \xrightarrow{\text{IFT}}$

$\boxed{y(t) = |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))}$ **(3)**

**magnitude of the frequency response**

- For a real-valued signal **h(t)**, the Fourier transform **H(ω)** satisfies the conjugate symmetry property:

$$H(\omega_0) = (H(-\omega_0))^*$$

$$H(-\omega_0) = (H(\omega_0))^*$$

- In the complex world, concepts are a little bit abstract. So, let us make things little more concrete by assuming that **h(t)**, the impulse response, is real. What happens now is that since the impulse response is real, we can use the symmetry properties of the Fourier transform that we disused before.
- Here, if we put a cosine, shown by **(1)**, into this real-valued impulse response system, what comes out is the very same cosine multiplied by the magnitude of the frequency response and phase-shifted by the angle, shown by **(3)**. Here, we used equation **(2)**, which is complex conjugate property. So, the same cosine comes out perhaps, amplified or attenuated and shifted around.
- So, this is a different way of thinking about if the signal is made up of a **real decomposition of cosines and sines**, then we cannot get any other cosines and sines out that were not present originally.

# An LTI system cannot introduce new frequencies

**Example:**

$\cos 3t \rightarrow \boxed{\phantom{xxxxx}} \rightarrow \cos 5t$

CAN'T BE LTI.

- As an example, if **cos(3t)** goes in and **cos(5t)** comes out, we can immediately conclude this system is not an LTI system. Because, we have introduced a different frequency that was not present in the original signal. Here, there was no cosine or sine of **5t** in the original part, so this cannot be LTI.

# Introduction to filters

WE USUALLY INTERPRET $H(\omega)$ AS WHAT THE SYSTEM DOES TO FREQUENCIES OF THE INPUT SIGNAL. "FILTER"

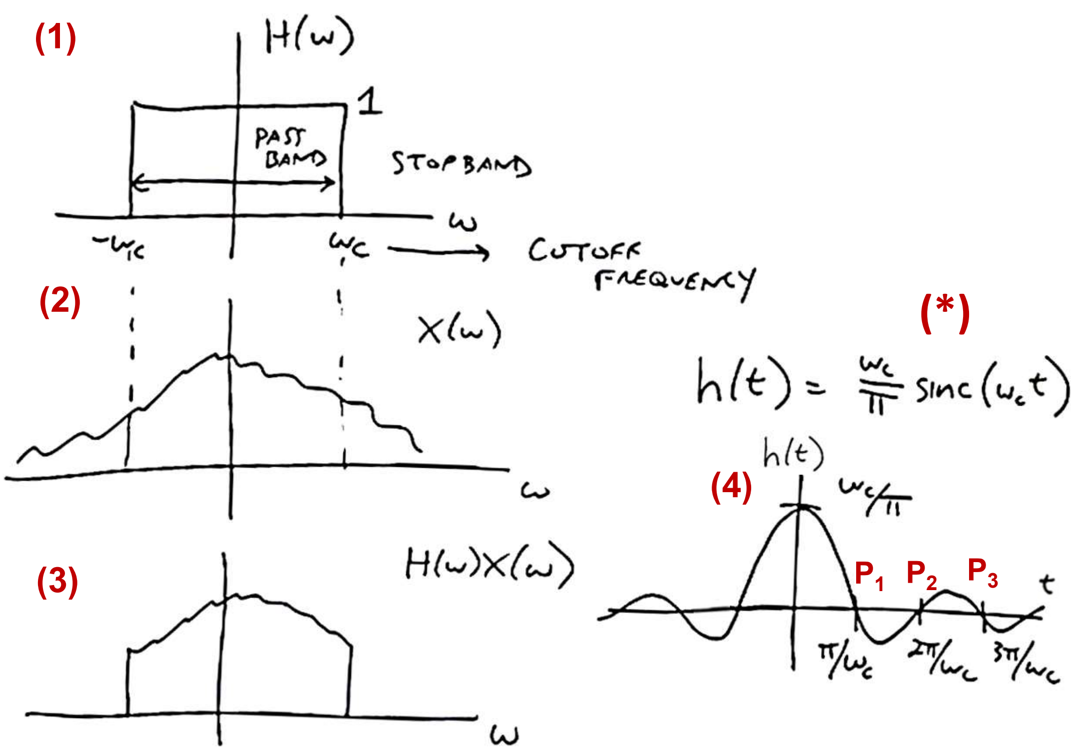
THE MOST IMPORTANT OF THESE IS THE LOW PASS FILTER.

- The equation describing a lowpass filter in the frequency domain is ***the frequency response of the filter.*** It tells us how different frequencies are modified by the filter, either being passed (retained) or attenuated. In this context, the frequency response of the filter provides the relationship between the input and output for different frequency components.

- What is usually happening is that we are taking frequencies of the input and then we are damping down or fully removing certain frequencies. That is why we call it a **filter** because only certain things are passing through. The most important of these filter is the **lowpass filter**.

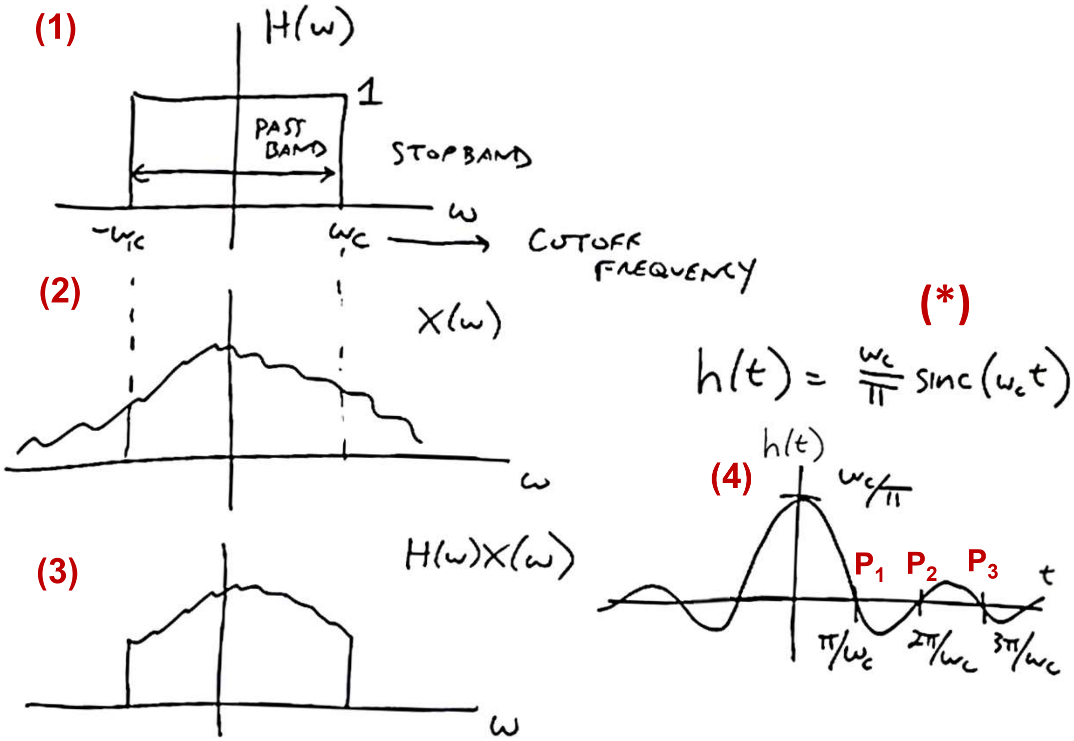
- **Frequency Response:** As pointed out before, the frequency response **H(ω)** of a system is a function that describes how the system processes different frequency components of an input signal. It provides information about how the system modifies both the amplitude and phase of each frequency component of the input. Specifically, the frequency response represents the Fourier transform of the system's impulse response **h(t)**. In other words, if **h(t)** is the impulse response of a system, then its Fourier transform **H(ω)** is the frequency response.

# Introduction to filters



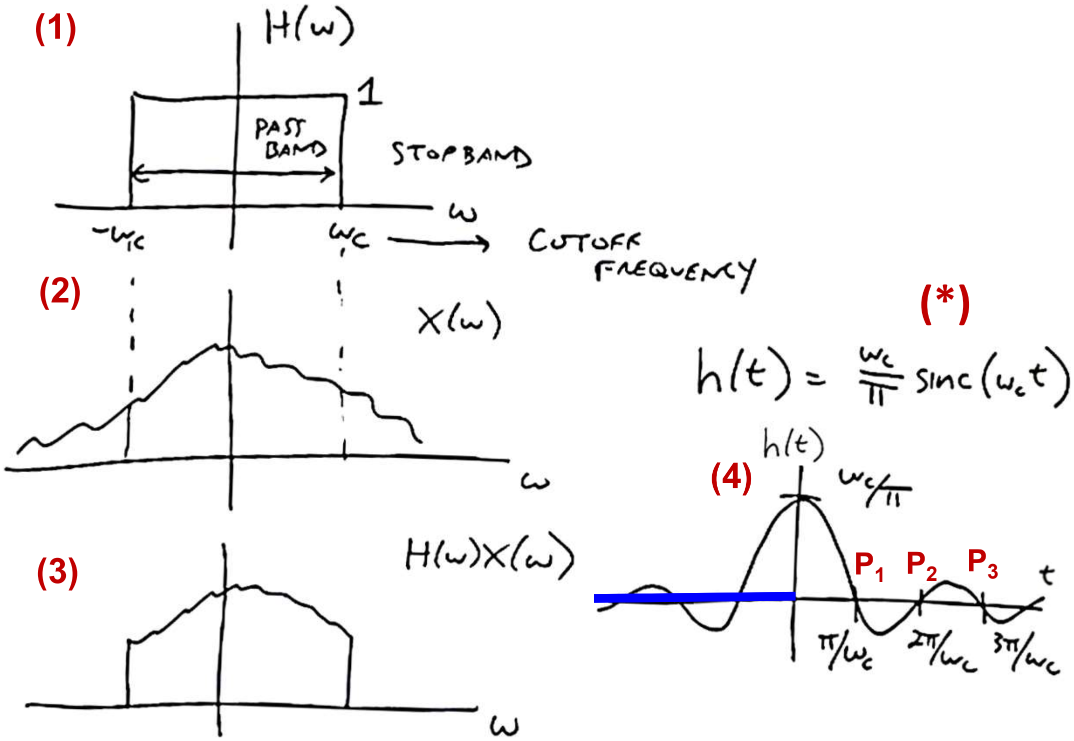

- Let us sketch a lowpass filter in the frequency domain **(1)**. We have a frequency response that looks like **H(ω)**. The height is **1** in the range between **$-\omega_c$** and **$+\omega_c$** and **0** elsewhere.
- The result of applying **H(ω)** to some input signal that has some arbitrary FT of **X(ω)** (shown in **(2)**) is that it cuts out all the frequencies that are higher than **$\omega_c$**. So, what we get at the end is something like **(3)**.
- In terms of terminology, the **passband** and the **stopband** are shown in **(1)**. What is under the filter is the passband and what is outside of it is a stopband.
- **What would be the corresponding impulse response of the system? That is, how H(ω) and h(t) are related?** We know that a pulse in one world corresponds to a sinc function in the other world. So, we would expect that the corresponding impulse response is a sinc function **(*)** related to the cut-off frequency, shown in **(4)**. In **h(t)** graph, we have an evenly-spaced set of ***zero crossings*** and these zero-crossings occur at multiples of **$\pi/\omega_c$**, shown by **$P_1$**, **$P_2$**, and **$P_3$**.

# Introduction to filters




- **More on h(t) graph:** Graph **(4)** illustrates an important ***tradeoff in filter design***. Graph **(1)** is often exactly what we want in the frequency domain, something that cuts out only what we want and throws away ***exactly*** all the rest. But if we think about how we are going to actually implement the filter in Graph **(1)** in the time domain, then Graph **(4)** is not so great. Why?

# Introduction to filters



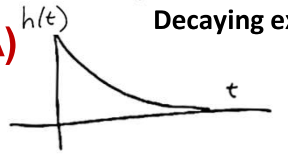

- **Reasons why (4) is not great: (a)** It is not causal. It goes in both directions. That means we are looking into the future in order to filter the signal corresponding to the pieces of the impulse response that are to the left of the **t**-axis (shown by **blue** line). **(b)** The other thing is the ***wiggliness*** (oscillations). Even if we were to somehow truncate things, Graph **(4)** goes out infinitely far in both directions and so we would have to eventually stop at some point. We do not want to be using a ***huge delay*** looking back into time for thousands of samples to be able to do our filtering. Also, it turns out that this wiggliness in the time domain causes some problems in the output. That is, if we were to use an approximation of this in order to filter a signal, this wiggliness in the time domain will make some ***rippliness*** in the output signal that we may not want.
- **Note:** Everything we discussed here is still in the continuous world, whereas in real life, we have to ***design digital filters*** that approximate **H(ω)**.

# Example: frequency response for a one-sided exponential impulse response

**Example:**

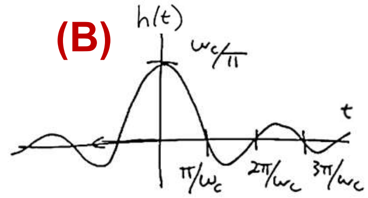
CONSIDER $h(t) = e^{-at} u(t) \quad a>0$

(A) Decaying exponential




$h(t) = \frac{\omega_c}{\pi} \operatorname{sinc}(\omega_c t)$

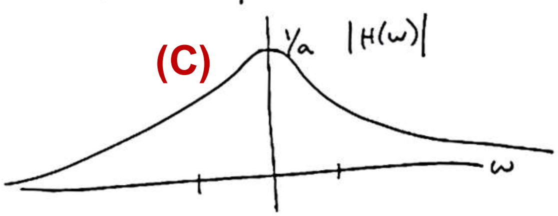
(B)




$$H(\omega) = \frac{1}{a + j\omega}$$

$$|H(\omega)| = \left| \frac{a - j\omega}{a^2 + \omega^2} \right| = \frac{1}{a^2 + \omega^2} \sqrt{a^2 + \omega^2} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

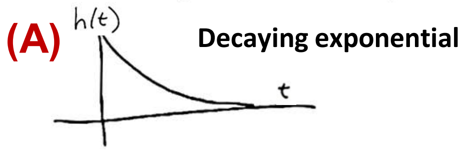
(C)



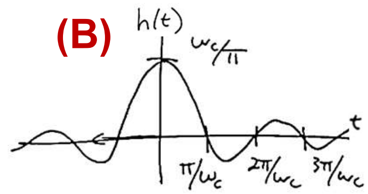

- **What would be a good approximation to an ideal lowpass filter?** If we compare **(A)** with **(B)**, there is a similarity. There is no negative part in **(A)**, but there is this trend that the filter values get smaller as we go out in time.
- The FT graph of **(A)** is shown in **(C)**. Let us see whether or not **(C)** would make a good lowpass filter. For the moment, let us not worry about the phase. Typically, for filtering, the first thing we care about is the magnitude of the frequency response. We usually look at the ***absolute value*** of things, i.e., $|\mathbf{H}(\boldsymbol{\omega})|$. Also, we care about the phase. We are going to talk about phase when we discuss filter design.

# Example: frequency response for a one-sided exponential impulse response

CONSIDER $h(t) = e^{-at}u(t) \quad a>0$

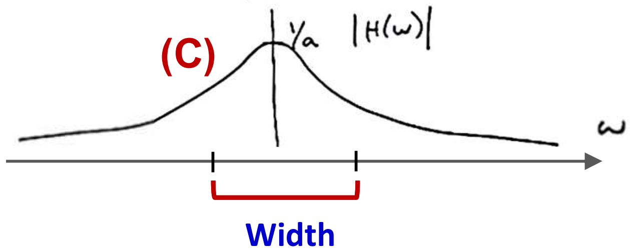



$h(t) = \frac{\omega_c}{\pi}\,\mathrm{sinc}(\omega_c t)$




$$H(\omega) = \frac{1}{a + j\omega}$$

$$|H(\omega)| = \left|\frac{a - j\omega}{a^2+\omega^2}\right| = \frac{1}{a^2+\omega^2}\sqrt{a^2+\omega^2} = \frac{1}{\sqrt{a^2+\omega^2}}$$




- In **(C)**, as **ω** increases, we are falling off. So, the magnitude of this filter goes to zero as we go further up. Hence, this is a fairly **crude lowpass filter**.
- Now, how could we make the fall off of this filter wider or thinner? That depends on the value of "**a**" that we choose. The value of "**a**" is related to the **width** of the filter in **(C)**.
- **Larger a:** The decay is slower, causing the curve to stretch horizontally (making it wider).
- **Smaller a:** The decay is faster, resulting in a narrower curve.
- **Effect on Width:** The width of the graph refers to how far along the **ω**-axis the significant part of the curve extends. When **a** is larger, the response decays more gradually, making the graph broader and extending further along the **ω**-axis. When **a** is smaller, the response decays more sharply, leading to **a** narrower width.

# Computing outputs for arbitrary inputs using the frequency response

- **What is the general setup to solving LTI systems?**

1) COMPUTE $X(\omega)$ FROM $x(t)$
2) MULTIPLY $X(\omega) \cdot H(\omega) = Y(\omega)$
3) TAKE INVERSE FT TO GET $y(t)$

- Usually, step 3 is the hardest part where we have got to take the **inverse FT**.

# Partial fractions

**Example 1:**

$$x(t) = e^{-5t}u(t) \quad , \quad h(t) = e^{-3t}u(t)$$

$$X(\omega) = \frac{1}{5+j\omega} \qquad H(\omega) = \frac{1}{3+j\omega}$$

$$X(\omega) \cdot H(\omega) = Y(\omega)$$

$$Y(\omega) = \frac{1}{(5+j\omega)(3+j\omega)} = \frac{A}{5+j\omega} + \frac{B}{3+j\omega} \quad \textbf{(1)}$$

$$= \frac{A(3+j\omega) + B(5+j\omega)}{(5+j\omega)(3+j\omega)} = \frac{(3A+5B) + (A+B)j\omega}{(5+j\omega)(3+j\omega)}$$

$$\rightarrow \frac{1+0(j\omega)}{(5+j\omega)(3+j\omega)} = \frac{(3A+5B)+(A+B)j\omega}{(5+j\omega)(3+j\omega)}$$

$$\begin{bmatrix} 3A + 5B = 1 \\ A + B = 0 \end{bmatrix} \rightarrow \begin{matrix} 2B = 1 \\ A = -B \end{matrix} \qquad B = 1/2 , \quad A = -1/2$$

$$Y(\omega) = \frac{-1/2}{5+j\omega} + \frac{1/2}{3+j\omega} \xrightarrow{\text{IFT}}$$

$$\boxed{y(t) = -\frac{1}{2}e^{-5t}u(t) + \frac{1}{2}e^{-3t}u(t)}$$

- Let us go over some examples on how we can actually solve LTI systems using the setup we just discussed.

- Here, we use the **method of partial fractions**, where we hypothesize that we could decompose **Y(ω)**, **(1)**, in terms of the two simple fractions.

# A more complicated example

## Example 2:

$$x(t) = e^{-t}u(t) \qquad H(\omega) = \frac{2+j\omega}{(1+j\omega)(3+j\omega)}$$

$$\downarrow$$

$$X(\omega) = \frac{1}{1+j\omega} \rightarrow Y(\omega) = \frac{2+j\omega}{(1+j\omega)^2(3+j\omega)} \rightarrow$$

$$Y(\omega) = \frac{A}{(1+j\omega)^2} + \frac{B}{(1+j\omega)} + \frac{C}{3+j\omega}$$

$$= \frac{A(3+j\omega) + B(1+j\omega)(3+j\omega) + C(1+j\omega)^2}{(1+j\omega)^2(3+j\omega)}$$

$$= \frac{\overset{*}{(3A+3B+C)} + \overset{***}{(A+4B+2C)}j\omega + (B+C)j\omega^2}{(1+j\omega)^2(3+j\omega)} \quad \textbf{(2)}$$

$$= \frac{\overset{**}{2} + j\omega + (0)j\omega^2}{(1+j\omega)^2(3+j\omega)} \quad \textbf{(1)} \rightarrow$$

**(3)**

$$\left[\begin{array}{l} 3A + 3B + C = 2 \\ A + 4B + 2C = 1 \\ B + C = 0 \end{array}\right] \begin{array}{l} \rightarrow \; 3A + 2B = 2 \\ \rightarrow \; A + 2B = 1 \\ C = -B \qquad 2A = 1 \end{array}$$

$$\left[\begin{array}{l} A = 1/2 \\ B = 1/4 \\ C = -1/4 \end{array}\right. \rightarrow$$

$$Y(\omega) = \frac{1/2}{(1+j\omega)^2} + \frac{1/4}{(1+j\omega)} + \frac{-1/4}{(3+j\omega)}$$

$$\boxed{y(t) = \frac{1}{2}te^{-t}u(t) + \frac{1}{4}e^{-t}u(t) - \frac{1}{4}e^{-3t}u(t)}$$

- **Note:** When we compare **(1)** with **(2)** to obtain the set of equation shown by **(3)**, we perform the process of **term matching**. Term * corresponds to the constant term, which is **2** (shown by **), and *** corresponds to **1**, since we have **(1)jω** in **(1)**. The term (**B + C**) in **(2)** is equal to **0**, because there is no **jω²** term up there in the numerator of **(1)**.

# Using the Fourier Transform to solve differential equations

- **Techniques to make solving differential equations easier:**

**1)** $y(t) = \frac{d}{dt} x(t) \qquad Y(\omega) = j\omega X(\omega)$

**Example 3:**

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$(j\omega)^2 Y(\omega) + 4j\omega Y(\omega) + 3Y(\omega) = j\omega X(\omega) + 2X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2+j\omega}{(1+j\omega)(3+j\omega)} \longrightarrow$$

$$\boxed{H(\omega) = \frac{2+j\omega}{(1+j\omega)(3+j\omega)}}$$

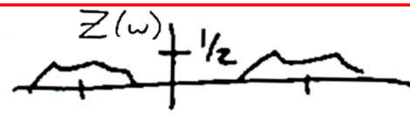
# Convolution in the frequency domain is multiplication in the time domain

(*)

**2)** $z(t) = x(t)y(t) \rightarrow Z(\omega) = \frac{1}{2\pi} X(\omega) * Y(\omega)$

**Example 4:**

$Z(\omega) = \frac{1}{2\pi}$ 


$z(t) = x(t)\cos(\omega_0 t) \rightarrow$ $Z(\omega) =$ 


$$\rightarrow \boxed{Z(\omega) = \frac{1}{2}\left[X(\omega - w_0) + X(\omega + w_0)\right]}$$

$$f(\omega) * \delta(\omega - \omega_0) = f(\omega - \omega_0)$$
$$X(\omega) * \pi\delta(\omega - w_0) = \pi X(\omega - w_0)$$

- We showed that convolution in the time domain is the same thing as multiplication in the frequency domain. The same thing applies if we flip it around because of the **Principle of Duality**. So, we could say that multiplication in the time domain is the same as convolution in the frequency domain.
- Following this and as shown in **z(t) = x(t)y(t)**, there is a property that says that if we have a product in the time domain, then in the frequency domain, we have a convolution. The place where this comes up the most is when we are doing operations like **amplitude modulation**.
- In this example, let us say we have a cosine function in time domain that represents **y(t)**. Then, in the frequency domain, where we have the **Z(ω)**, what we get is the convolution of the original signal, **X(ω)**, (Graph **A**) convolved with a cosine (Graph **B**). So, the result of this operation is two half-height copies of **X(ω)** that are centered at the modulation frequency of $\omega_0$. Note that equation (*) is a general equation and is valid for any two signals **x(t)** and **y(t)**.

# End of Lecture 6