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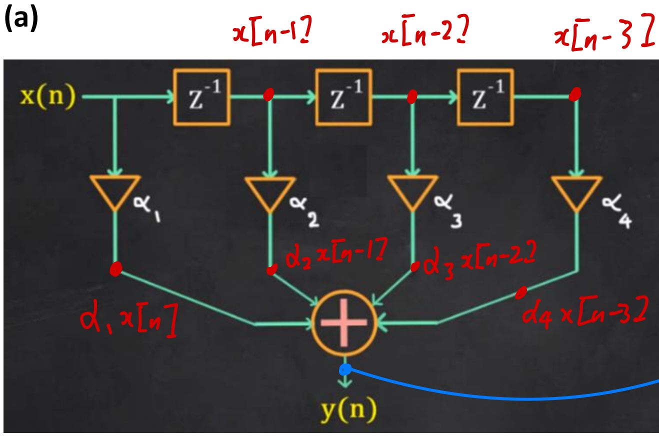
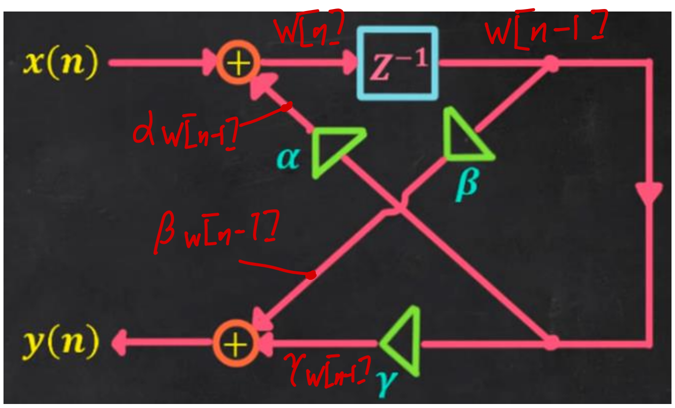
Q1: Find y(n) expression in the following block diagrams, **(a)** and **(b)**. In **(a)**, assume that α's are scalar values. In **(b)**, assume α, β, γ are all scalar values.

(a)



$$\boxed{y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]}$$

(b)



$$w[n] = x[n] + \alpha w[n-1]$$

$$y[n] = \beta w[n-1] + \gamma w[n-1]$$

$$w[n-1] = \frac{1}{\alpha}\left(w[n] - x[n]\right)$$

Through LTI property

$$y[n+1] = \beta w[n] + \gamma w[n]$$

$$\Rightarrow w[n] = \frac{y[n+1]}{\beta+\gamma}$$

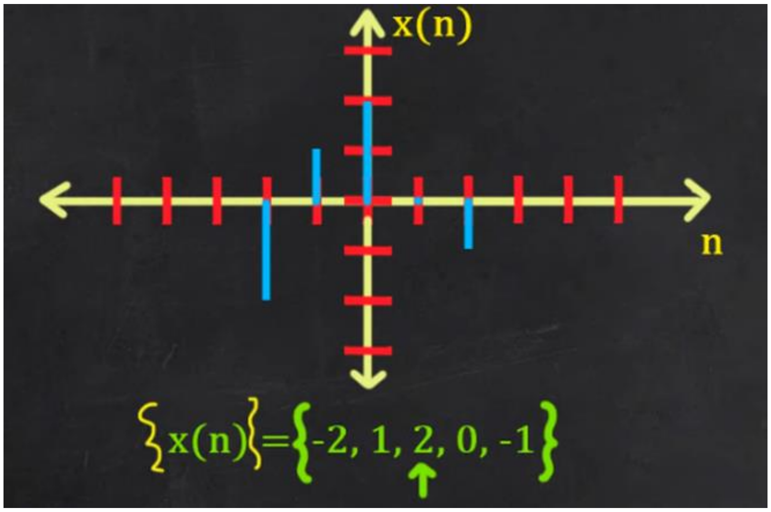
$$\Rightarrow w[n-1] = \frac{1}{\alpha}\left(\frac{y[n+1]}{\beta+\gamma} - x[n]\right)$$

$$\Rightarrow y[n] = \frac{\beta+\gamma}{\alpha}\left(\frac{y[n+1]}{\beta+\gamma} - x[n]\right)$$

$$y[n-1] = \frac{\beta+\gamma}{\alpha}\left(\frac{y[n]}{\beta+\gamma} - x[n-1]\right)$$

$$\Rightarrow \boxed{y[n] = \alpha y[n-1] + (\beta+\gamma)\, x[n-1]}$$

Q2: Find the expressions for the even digital sequence/signal, $x_e(n)$, and the odd digital sequence/signal, $x_o(n)$, for the given digital sequence/signal, $x(n)$. Plot each sequence/signal.



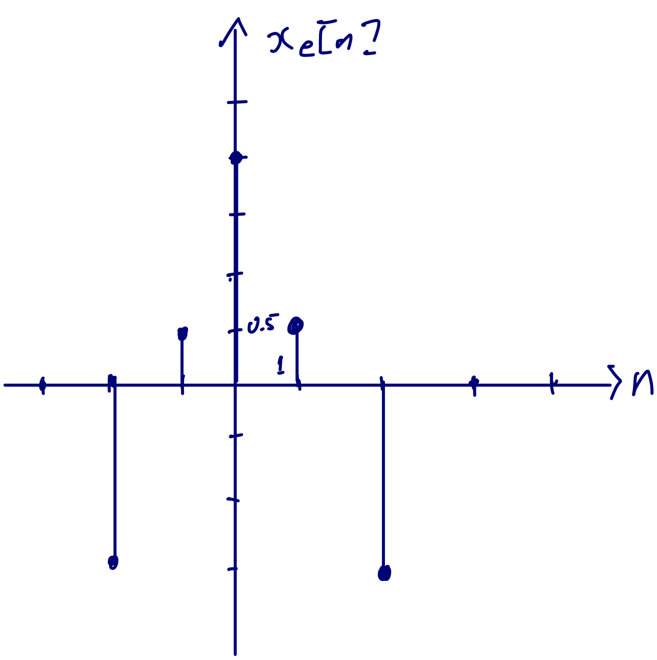
$$x_e[n] = \frac{x[n] + x[-n]}{2} \quad , \quad x[-n] = \{-1, 0, \underline{2}, 1, -2\}$$

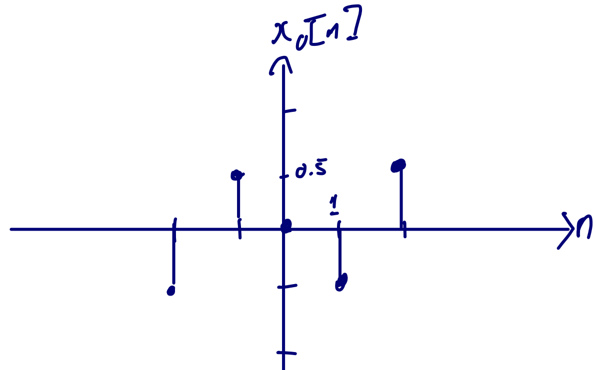
$$\Rightarrow x_e[n] = \frac{\{-2, 1, \underline{2}, 0, -1\} + \{-1, 0, \underline{2}, 1, -2\}}{2} = \frac{\{-3, 1, \underline{4}, 1, -3\}}{2}$$

$$x_e[n] = \left\{-\frac{3}{2}, \frac{1}{2}, \underline{2}, \frac{1}{2}, -\frac{3}{2}\right\}$$

$$x_o[n] = \frac{x[n] - x[-n]}{2} = \frac{\{-2, 1, \underline{2}, 0, -1\} - \{-1, 0, \underline{2}, 1, -2\}}{2} = \frac{\{-1, 1, \underline{0}, -1, 1\}}{2}$$

$$x_o[n] = \left\{-\frac{1}{2}, \frac{1}{2}, \underline{0}, -\frac{1}{2}, \frac{1}{2}\right\}$$





Q3: Determine which of the following signals are energy signals or power signals.

$x(n) = 4\ ,\ n > 0$	$x(n) = \left(\frac{1}{4}\right)^n u(n)$	$x(n) = \begin{cases} 3(-1)^n, n \geq 0 \\ 0 \quad ,n < 0 \end{cases}$

Use the following hints:

	CT Signals	DT Signals
Instantaneous Power	$P(t) = \lvert x(t)\rvert^2$	$P(n) = \lvert x(n)\rvert^2$
Total Energy	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt$	$E = \sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2$
Average Power	$P = \lim_{T\to\infty}\left[\frac{\int_{-T}^{T}\lvert x(t)\rvert^2\, dt}{2T}\right]$	$P = \lim_{N\to\infty}\left[\frac{\sum_{n=-N}^{N}\lvert x(n)\rvert^2}{2N+1}\right]$

Energy Signal & Power Signal

Energy Signal : $0 < E < \infty$ (Finite Total Energy) and $P = 0$ (Zero Average Power)

Power Signal : $0 < P < \infty$ (Finite Average Power) and $E = \infty$ (Infinite Total Energy)

A signal cannot be both an energy signal and a power signal

There can be signals that are neither an energy signal nor a power signal

No physical signal can have infinite energy or infinite average power

For DT Signals:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P = \lim_{N\to\infty}\left[\frac{\sum_{n=-N}^{N} |x[n]|^2}{2N+1}\right]$$

A.

$x[n] = 4\ ,\ n > 0$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=0}^{\infty} 16 = \infty$$

$$P = \lim_{N\to\infty} \frac{\left[\sum_{n=-N}^{N} |x[n]|^2\right]}{2N+1} = \lim_{N\to\infty} \frac{\sum_{0}^{N} 16}{2N+1} = \lim_{N\to\infty} \frac{16N}{2N+1} = 8$$

power signal

$\boxed{x[n] = 4 \text{ is a power signal with a power of } 8}$

B. $x[n] = \left(\frac{1}{4}\right)^n u[n]$

$$E[n] = \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n = \frac{1}{1-\frac{1}{16}} = \frac{16}{15}$$

$$P = \lim_{N\to\infty} \frac{16/15}{2N+1} = 0$$

$x[n]$ is an energy signal with energy 16/15.

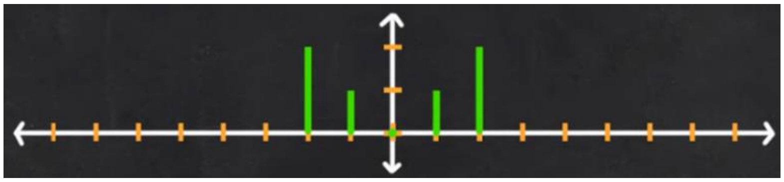
C. $x[n] = \begin{cases} (-1)^n \cdot 3 & , n \geq 0 \\ 0 & , n < 0 \end{cases}$

$$E = \sum_{n=0}^{\infty} 9 = \infty$$

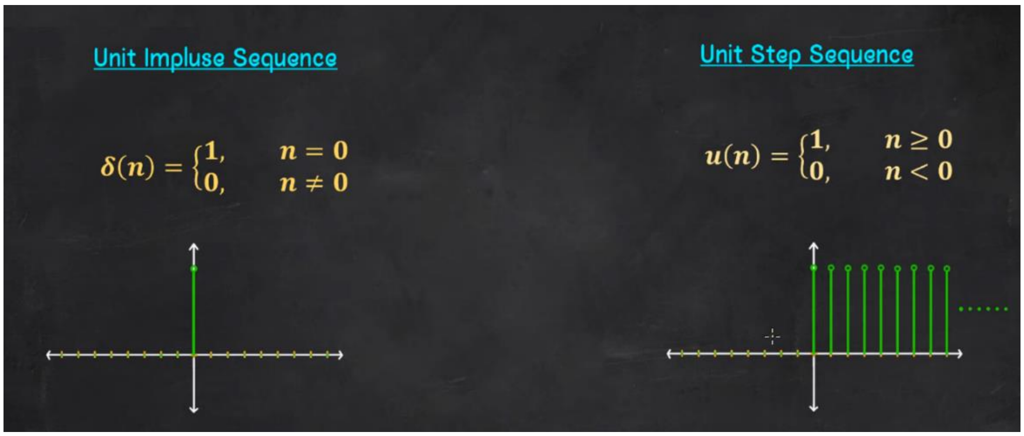
$$P = \frac{\lim_{N\to\infty} \sum_{n=0}^{N} 9}{2N+1} = \lim_{N\to\infty} \frac{9N}{2N+1} = 4.5$$

$x[n] = (-1)^n 3$ is a power signal

Q4: Express the following signal in terms of **(a)** unit impulse signal, and **(b)** unit step signal.



Use the mathematical definitions of unit impulse sequence and unit step sequence given below.



a. $x[n] = 2\delta[n+2] + \delta[n+1] + \delta[n-1] + 2\delta[n-2]$

b. $x[n] = 2u[n+2] - u[n+1] - u[n] + u[n-1] + u[n-2] - 2u[n-3]$

Q5: Find the fundamental period for the following digital signals:

$$\bar{x}_1(n) = 4\cos\left(\frac{2\pi n}{5}\right) \qquad\qquad \bar{x}_2(n) = \sin(0.6\pi n + 0.6\pi)$$

Note that a sinusoidal discrete signal is periodic if 2π/ω equals a rational number, as shown below:

$$\omega N = 2\pi r \quad , \quad r \rightarrow \text{+ve integer}$$

$$\frac{2\pi}{\omega} = \frac{N}{r}$$

In the above, both N and r are positive integers, where N is the period of the digital signal.

$$\omega N = 2\pi r$$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{N}{r} \quad , \text{ where } N, r \in \mathbb{Z}^+$$

a) $T = \dfrac{2\pi}{\left(\frac{2\pi}{5}\right)} = 5 = \dfrac{5}{1}$

$\therefore N = 5$

b) $T = \dfrac{2\cancel{\pi}}{0.6\cancel{\pi}} = \dfrac{10}{3}$

$\therefore N = 10$

Q6: Find the fundamental period for the following digital signal:

$$\bar{x}(n) = \cos(1.2\pi n + 0.65\pi) - 4\sin(0.9\pi n) + 5\cos(0.5\pi n)$$

Use the following hint, where LCM is the Least Common Multiple. Note that the LCM is the smallest positive integer that is evenly divisible by two or more given numbers.

$$\bar{x}_1(n) \qquad \bar{x}_2(n) \qquad \bar{x}_3(n)$$

$$N_1 \qquad N_2 \qquad N_3$$

$$\bar{x}_4(n) = \alpha\bar{x}_1(n) + \beta\bar{x}_2(n) + \gamma\bar{x}_3(n)$$

$$N_4 = LCM(N_1, N_2, N_3)$$

You need to show the graphs of each digital signal using MATLAB. Also, plot the graph of the overall signal of $x_4(n)$.

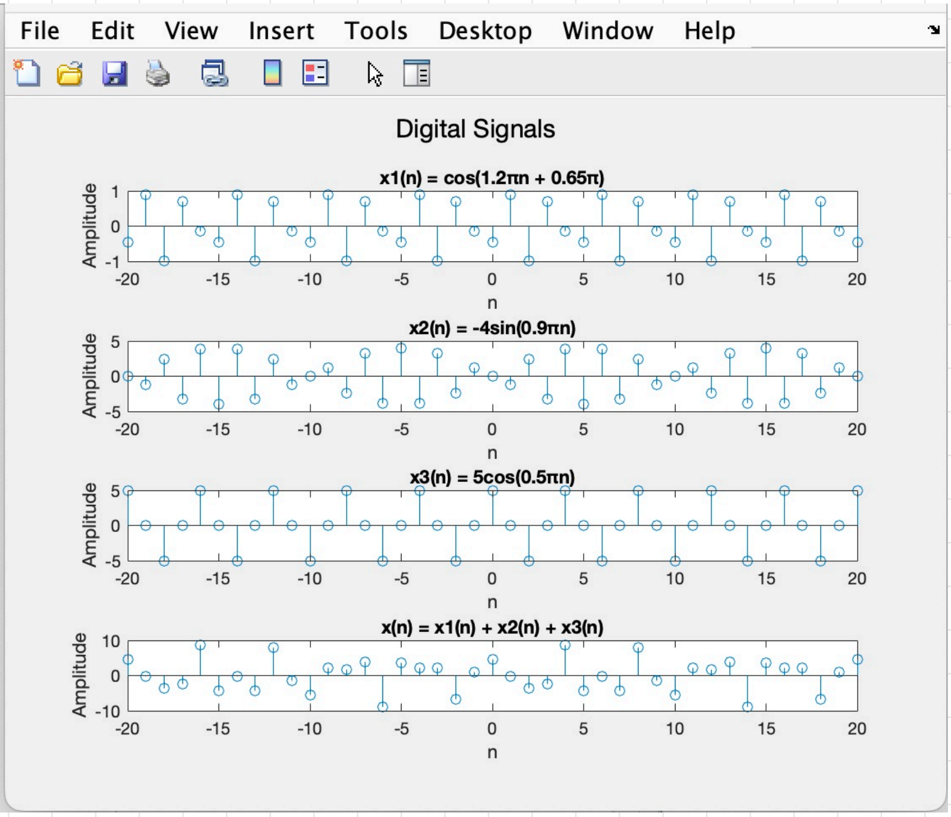
$\bar{x}_1[n]$: $\frac{2\pi}{1.2\pi} = \frac{5}{3} \Rightarrow N_1 = 5$

$\bar{x}_2[n]$: $\frac{2\pi}{0.9\pi} = \frac{20}{9} \Rightarrow N_2 = 20$

$\bar{x}_3[n]$: $\frac{2\pi}{0.5\pi} = \frac{4}{1} \Rightarrow N_3 = 4$

$$\boxed{N_4 = LCM(5, 20, 4) = 20}$$

```matlab
% A1 Q6
% Define the range of n
n = -20:20;  % two periods of the fundamental period (N = 20)

% Define each digital signal
x1 = cos(1.2*pi*n + 0.65*pi);
x2 = -4*sin(0.9*pi*n);
x3 = 5*cos(0.5*pi*n);

% Define the overall signal
x = x1 + x2 + x3;

% Plot each digital signal
figure;
subplot(4,1,1);
stem(n, x1);
title('x1(n) = cos(1.2πn + 0.65π)');
xlabel('n');
ylabel('Amplitude');

subplot(4,1,2);
stem(n, x2);
title('x2(n) = -4sin(0.9πn)');
xlabel('n');
ylabel('Amplitude');

subplot(4,1,3);
stem(n, x3);
title('x3(n) = 5cos(0.5πn)');
xlabel('n');
ylabel('Amplitude');

% Plot overall signal
subplot(4,1,4);
stem(n, x);
title('x(n) = x1(n) + x2(n) + x3(n)');
xlabel('n');
ylabel('Amplitude');

% Adjust the layout
sgtitle('Digital Signals');
```



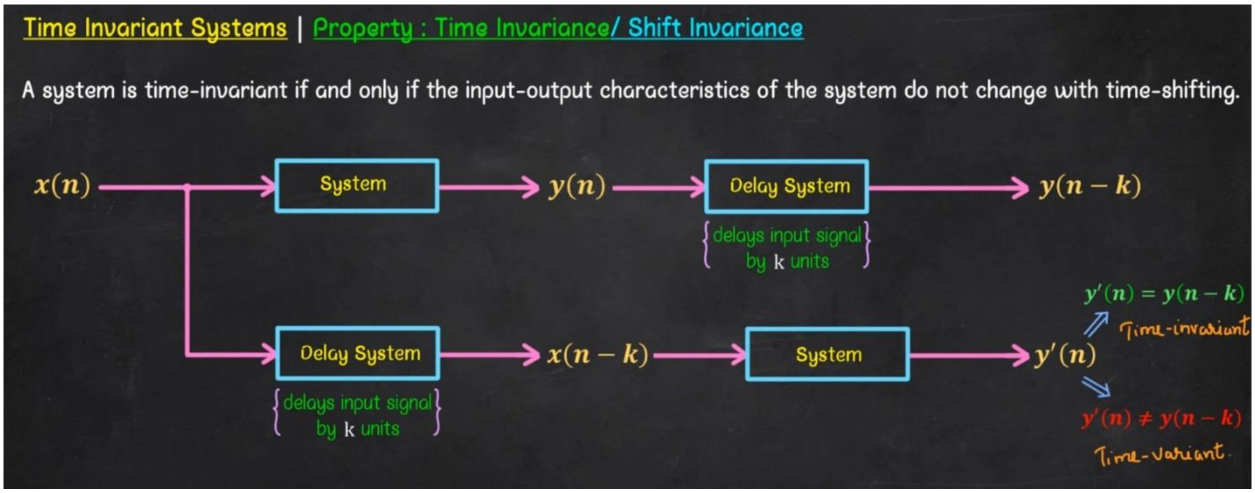
Q7: Show whether the given digital signals are time-variant or time-invariant.

Part 1: y(n) = x(2n).

Part 2: y(n) = 2n + x(n).

Part 3: y(n) = n.x(n).

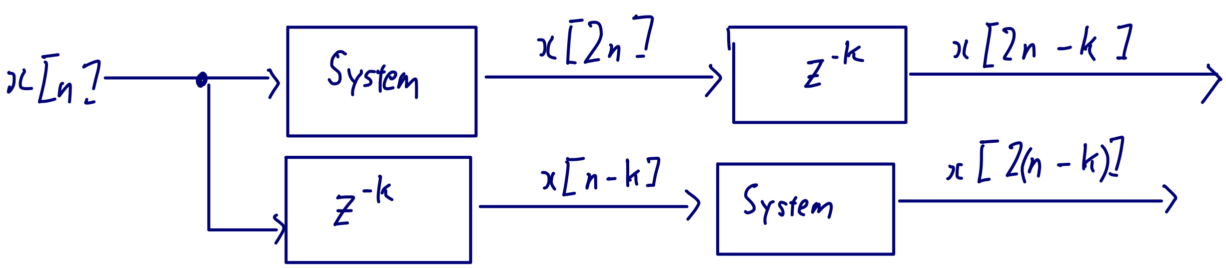
Use the given hints.



For a system to be time-invariant :

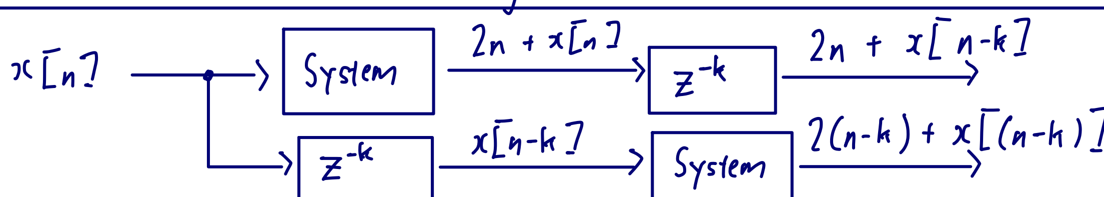
1. It should not have time scaling property
2. Added/subtracted term in the system equation should be constant or zero
3. Coefficient of terms in the system equation should be constant

part 1



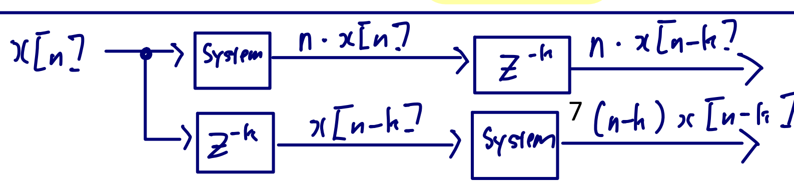
$y[n] = x[2n]$ has time scaling property, hence is a time-Variant system

part 2



$y[n] = 2n + x[n]$ is time-Variant because the system adds a non-constant value

part 3



$y[n] = n\,x[n]$ is time-Variant because the coefficient is not constant

Q8: Show whether the given digital signals are causal or non-casual.

Part 1: y(n) = x(n) – x(n-1). ⟶ No $x[n+k]$ terms, $k \geq 1$, $k \in \mathbb{Z}^+$ therefore the signal is causal

Part 2: y(n) = a.x(n). ⟶ $y[n]$ is scaled version of present input $x[n]$, therefore it is causal.

Part 3: y(n) = $\sum_{k=-\infty}^{n} x(k)$. ⟶ $y[n] = \ldots + x[0] + x[1] + \ldots + x[n-1] + x[n]$

$y[n]$ depends only on values before n and is therefore causal

Part 4: y(n) = x(n) + 3x(n + 4). ⟶ $3x[n+4]$ depends on a value 4 steps ahead what we currently have & is therefore non-causal

Part 5: y(n) = x(n^2). ⟶ $y[-n] = x[n^2]$. Negative values of n depend on future values, hence, non-causal

Part 6: y(n) = x(-n). ⟶ $y[-1] = x[1]$, output of system at negative values depends on future values. Therefore non-causal.

Use the given hints.

Causal Systems | Property : Causality

A system is said to be causal if the output of the system at any time n depends only on the present and past inputs, but does not depend on future inputs.

For a causal system,

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

The output of a non-causal system depends not only on the present and past inputs, but also on the future inputs.

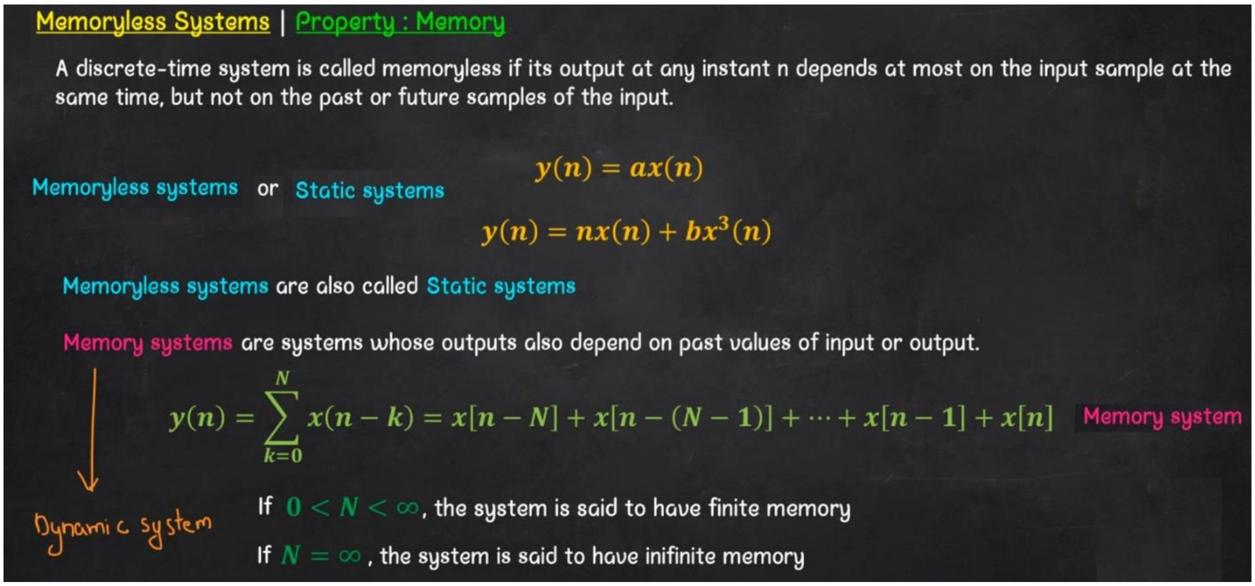
Q9: Show whether the given digital systems are static system (memoryless) or dynamic (memory) system.

Part 1: y(n) = = [2 + sin (n - 1)].x(n)

Part 2: y(n) = max[x(n), x(n - 1), ..., x(−∞)]

Part 3: y(n) = x(-n).

Use the given hints.



Part 1: → Memoryless system because $y[n]$ depends only on a present value of $x[n]$ and is static.

Part 2: → Suppose $x[n] = \delta[n]$, then $y[n] = 1 \ \forall\, n \geq 0$, i.e. $y[1] = x[0]$, $y[2] = x[0]$, ..., $y[k] = x[0] \ \forall\, k \in \mathbb{Z}^+$. Through counterexample, this system has memory and is a dynamic system.

Part 3: → given $n = -k$, $y[-k] = x[k]$ which depends on a non-present value, and has memory and is a dynamic system.

Q10: Consider the following LTI system, y[k] – 0.8y[k – 1] = x[k].

Part 1:

Find the frequency response of the system, i.e., both the amplitude response and the phase response. Only find the equations and no need to plot them.

Part 2:

Let x[k] = 1 be the input with the frequency response obtained in **Part 1**. Find the system response.

Part 3:

Let $x[k] = \cos\left(\frac{\pi}{6}k - 0.4\right)$ be the input with the frequency response obtained in **Part 1**. Find the system response.

Part 4:

Let $x(t) = \cos(1500t)$ be the ***continuous*** input with the frequency response obtained in **Part 1**. Find the system response if we sample the continuous signal every 0.001 seconds.

Part 1

$$Y(\omega) - 0.8e^{-j\omega}Y(\omega) = X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = \frac{1}{1-0.8e^{-j\omega}}$$

$$H(\omega) = \frac{1}{1-0.8e^{-j\omega}} = \frac{1}{1-0.8(\cos(\omega) - j\sin(\omega))} = \frac{1}{[1-0.8\cos(\omega)] + j(0.8)\sin(\omega)}$$

$$|H(\omega)| = \frac{1}{\sqrt{(1-0.8\cos(\omega))^2 + (0.8\sin(\omega))^2}}$$

$$\measuredangle H(\omega) = -\arctan\left(\frac{0.8\sin(\omega)}{1-0.8\cos(\omega)}\right)$$

Part 2

$x[k] = 1$ is a DC signal with $\omega = 0$

$$|H(0)| = \frac{1}{\sqrt{(1-0.8)^2+0^2}} = \frac{1}{0.2} = 5$$

$$\measuredangle H(0) = -\arctan(0) = 0$$

$$\Rightarrow y[k] = 5$$

Part 3

$x[k] = \cos\left(\frac{\pi}{6}k - 0.4\right)$ has $\omega = \frac{\pi}{6}$

$$\left|H\left(\tfrac{\pi}{6}\right)\right| = \frac{1}{\sqrt{\left(1-\frac{4}{5}\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{4}{10}\right)^2}} = \frac{1}{\sqrt{\left(\frac{10-4\sqrt{3}}{10}\right)^2 + \left(\frac{4}{10}\right)^2}}$$

$$\measuredangle H\left(\tfrac{\pi}{6}\right) = -\arctan\left(\frac{4/10}{1-\frac{4}{5}\frac{\sqrt{3}}{2}}\right) \simeq -0.9159$$

$$\Rightarrow y[k] \simeq 1.9828\cos\left(\tfrac{\pi}{6}k - 1.3159\right)$$

Part 4

$x(t) = \cos(1500t)$ sampled every 0.001 s, i.e. $f_s = 1000$

$t = n \cdot T_s$

$$x[n] = \cos\left(\frac{1500}{1000} \cdot n\right) = \cos(1.5n)$$

$$\Rightarrow \omega = 1.5$$

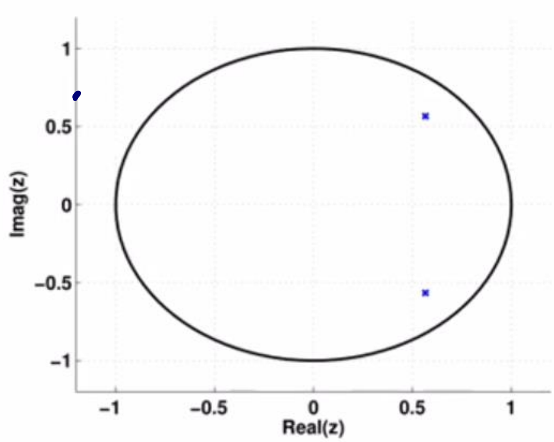
$$|H(1.5)| = \frac{1}{\sqrt{(1-0.8\cos(1.5))^2 + (0.8\sin(1.5))^2}} \simeq 0.8093$$

$$\measuredangle H(1.5) = -\arctan\left(\frac{0.8\sin(1.5)}{1-0.8\cos(1.5)}\right) \simeq -0.7021$$

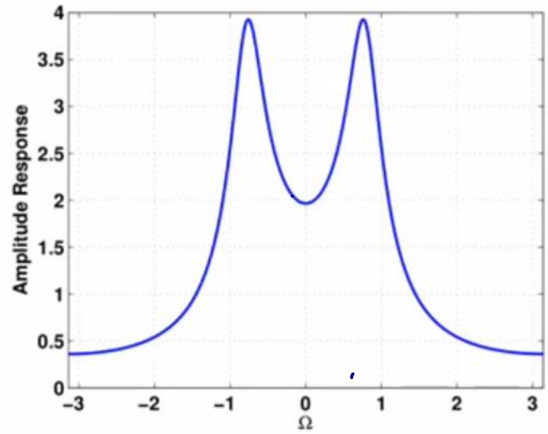
$$y[n] \simeq 0.8093\cos(1.5n - 0.7021)$$

Q11: Explain the general shapes of the amplitude response using the given pole/zero plots.

Part 1:

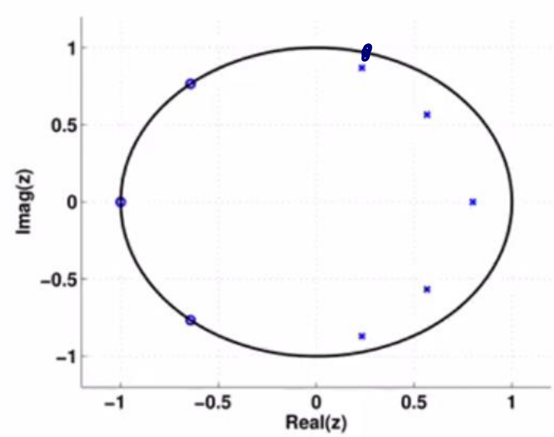


(a) Pole/Zero Plot

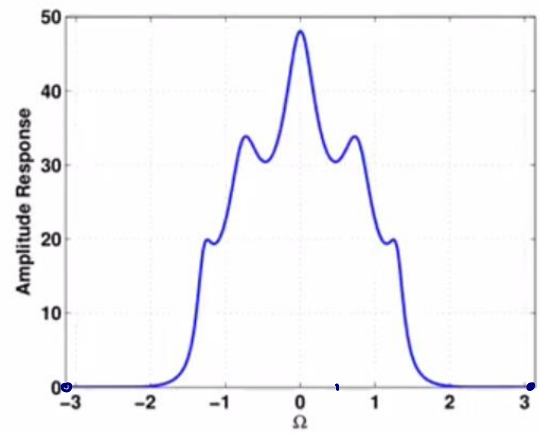


(b) Frequency Response

Part 2:



(c) Pole/Zero Plot



(d) Frequency Response

Use the following hints:

Part 1: Amplitude response is given by: $\left| K \frac{\text{Product of distances of zeros to } e^{j\Omega}}{\text{Product of distances of poles to } e^{j\Omega}} \right|$. Peaks are approximately at $\Omega = \pm 0.75$ and a trough at $\Omega = 0$. $e^{j0} = 1$. We want to be as far as possible to a zero & as close as possible to a pole. We get close to a pole near $\Omega = \frac{\pi}{4}$ which happens to be close to the poles. At $\Omega = \pm\pi$ we see the lowest on the amplitude plot because the distance is at a max. Also, at $\Omega = 0$ we get to the maximum seperation before approaching a new pole, hence the trough.

Part 2: We are close to a pole at approx $\Omega = 0, \pm\frac{2\pi}{5}, \pm\frac{\pi}{3}$ and a zero at $\Omega = \frac{3\pi}{4}, \pi$. We see great attenuation at the edges where the zeros are located and peaks around $\Omega = \pm 1.2, \pm 0.75$, and 0 all corresponding to the respective values of Ω as we expect. Again, being close to the poles amplifies, while being close to zeros attenuates when we go around the unit circle.

- The Transfer Function Of A DT LTI System Can Be Written As

$$H(z) = K\frac{(z - z_1)(z - z_2)\cdots(z - z_n)}{(z - \gamma_1)(z - \gamma_2)\cdots(z - \gamma_n)}$$

where z_i and γ_i are Zeros and Poles of the System.

- Let $z = e^{j\Omega}$ (i.e evaluate $H(z)$ on the unit circle). This Gives The System Frequency Response

$$H(e^{j\Omega}) = H(\Omega) = K\frac{(e^{j\Omega} - z_1)(e^{j\Omega} - z_2)\cdots(e^{j\Omega} - z_n)}{(e^{j\Omega} - \gamma_1)(e^{j\Omega} - \gamma_2)\cdots(e^{j\Omega} - \gamma_n)}$$

- The Quantity $e^{j\Omega}$ Is A Point On the Unit Circle
- The Quantity $e^{j\Omega} - z_i$ Is A Vector From z_i to $e^{j\Omega}$
- The Quantity $e^{j\Omega} - \gamma_i$ Is A Vector From γ_i to $e^{j\Omega}$
- Use Polar Coordinates To Define

$$r_i e^{j\phi_i} = e^{j\Omega} - z_i$$

and

$$d_i e^{j\theta_i} = e^{j\Omega} - \gamma_i$$

- The Frequency Response Can Now Be Written As

$$\begin{aligned} H(\Omega) &= K\frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_n e^{j\phi_n})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_n e^{j\theta_n})} \\ &= K\frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} e^{j[(\phi_1+\phi_2+\cdots+\phi_n)-(\theta_1+\theta_2+\cdots+\theta_n)]} \end{aligned}$$

- The Amplitude Response Is

$$\begin{aligned} |H(\Omega)| &= \left| K\frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_n e^{j\phi_n})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_n e^{j\theta_n})} \right| \\ &= K\frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} \\ &= K\frac{\text{product of distances of zeros to } e^{j\Omega}}{\text{product of distances of poles to } e^{j\Omega}} \end{aligned}$$

- The Phase Response Is

$$\begin{aligned} \angle H(\Omega) &= \angle\left(K\frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_n e^{j\phi_n})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_n e^{j\theta_n})} \right) \\ &= (\phi_1 + \phi_2 + \cdots + \phi_n) \\ &- (\theta_1 + \theta_2 + \cdots + \theta_n) \\ &= \text{sum of zero angles to } e^{j\Omega} \\ &- \text{sum of pole angles to } e^{j\Omega} \end{aligned}$$

- The Amplitude and Phase Response Of A Filter Is Controlled By Its Pole and Zero Locations