# ELEC 421

# Digital Signal and Image Processing

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# Course Roadmap for DSP

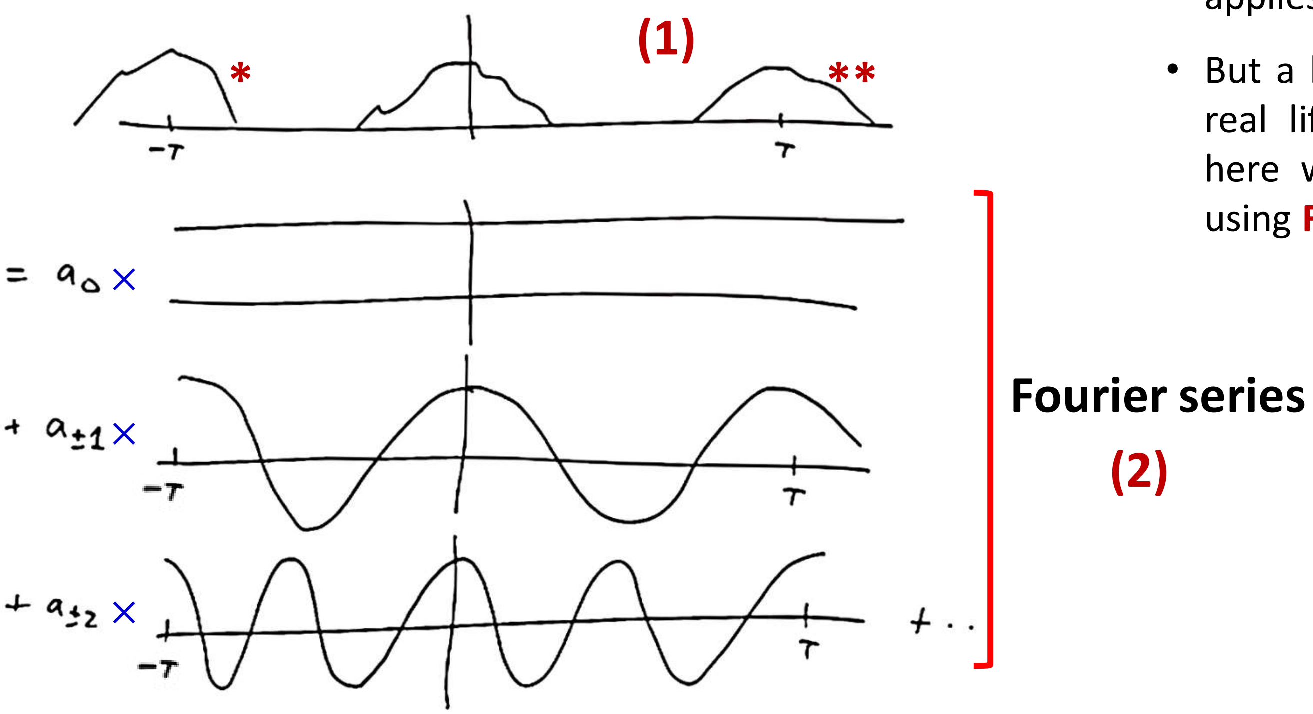
| Lecture | Title |
|---|---|
| Lecture 0 | Introduction to DSP and DIP |
| Lecture 1 | Signals |
| Lecture 2 | Linear Time-Invariant System |
| Lecture 3 | Convolution and its Properties |
| Lecture 4 | The Fourier Series |
| Lecture 5 | The Fourier Transform |
| Lecture 6 | Frequency Response |
| Lecture 7 | Discrete-Time Fourier Transform |
| Lecture 8 | Introduction to the z-Transform |
| Lecture 9 | Inverse z-Transform; Poles and Zeros |
| Lecture 10 | The Discrete Fourier Transform |
| Lecture 11 | Radix-2 Fast Fourier Transforms |
| Lecture 12 | The Cooley-Tukey and Good-Thomas FFTs |
| Lecture 13 | The Sampling Theorem |
| Lecture 14 | Continuous-Time Filtering with Digital Systems; Upsampling and Downsampling |
| Lecture 15 | MATLAB Implementation of Filter Design |

# Lecture 5:
# The Fourier Transform

# Table of Contents

# The Fourier Transform

## Review of Fourier series:



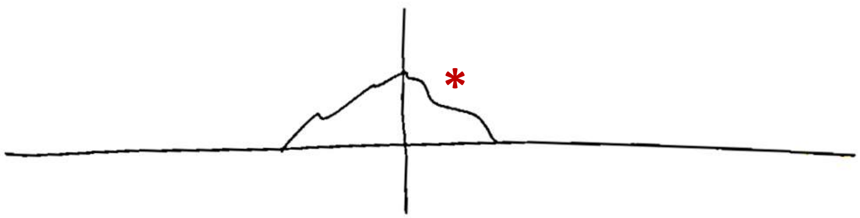

- We talked about the Fourier series, which only applies to periodic signals.
- But a lot of times, the signals we care about in real life are not periodic (i.e., **aperiodic**) and here we will investigate how to handle them using **Fourier transform**.

- Just as a refresher, the idea of the ***Fourier series*** was that we have some of ***periodic signal*** with period **T**. The definition of periodic means it repeats every **T** units as shown in **(1)**.
- What we did in Fourie series was to write the signal as the combination of a bunch of other simpler periodic signals, which turned out to be sinusoids as shown in **(2)**.
- Now, **what do we do when our signal is not periodic?** An aperiodic signal would look like **(1)** just without any copies (such as * or **).

# The Fourier Transform

FOURIER TRANSFORM

WHAT ABOUT AN APERIODIC SIGNAL?




**(1)** $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$

$\downarrow$

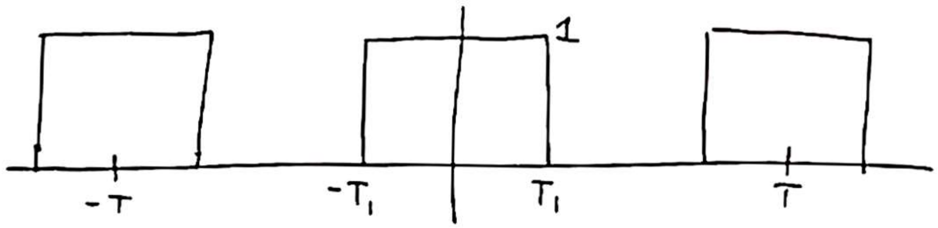
**(2)** $x(t) = \int_{\omega=-\infty}^{\infty} \boxed{\text{FT}}\, e^{j\omega t}\, d\omega$

- Let us look at an ***aperiodic*** or a ***non-periodic*** signal. This is just like we have one lonely thing in the middle (see *) and no copies. For the Fourier transform, we can imagine that **we have copies that are infinitely far away**. The idea is that in the limit, the Fourier series coefficients somehow become what is called the **Fourier transform**.

- For the Fourier series representation, we used **(1)**. **What happens when we have a signal where T is very large?** When **T** is very large, that means $\boldsymbol{\omega_0}$ is very small and that, in turn, means that the sinusoids that we are adding together are actually very close together in frequency. As **T** goes to infinity and $\boldsymbol{\omega_0}$ goes to **0**, the sum will turn into an **integral**. Eventually, the terms are so finely spaced that we will have a continuous function of frequency shown by the integral **(2)**. What goes in the **red** box is going to be the Fourier transform. We will derive how that works.

# Thinking of the Fourier series coefficients as samples of a function

SQUARE WAVE




$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$$\text{(1)} \qquad T a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0}$$

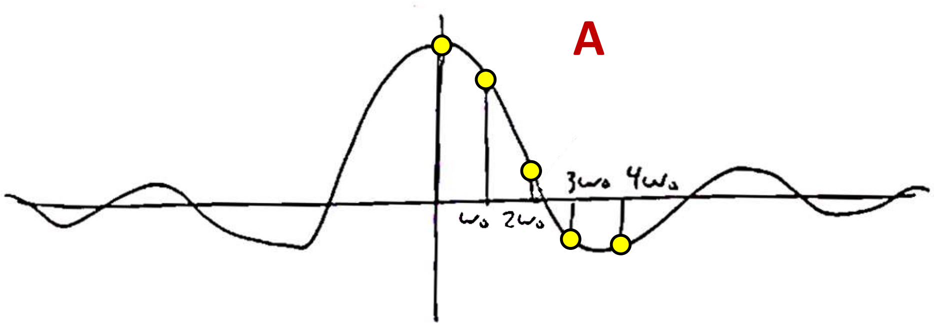
- We have a pulse train (square wave) the width of which is $\pm$**$T_1$**.
- The Fourier series coefficients turned out to be the sinc functions. The right-hand side of **(1)** says that we are evaluating **$2\sin(\omega T_1)/\omega$** at **$\omega = k\omega_0$**. It means that in order to get the **$a_k$**'s, we need to take this ***continuous function of omega***, i.e., **$2\sin(\omega T_1)/\omega$** and ***sample it*** every **$\omega_0$** units.

# Thinking of the Fourier series coefficients as samples of a function

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

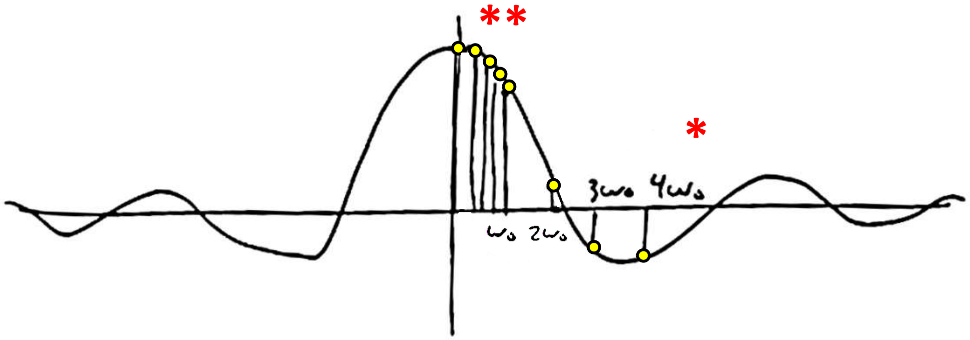
$$T a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0} = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0} \qquad \textbf{(1)}$$

**A**




- We just showed that in order to get the $\mathbf{a_k}$'s, we would be sampling $\mathbf{2sin(\omega T_1)/\omega}$ every $\boldsymbol{\omega_0}$ units, as shown in graph **A**.
- That is a nice way to think about how the Fourier series and the Fourier transform are related.
- Equation **(1)** is basically saying that ***the underlying concept is the Fourier transform*** and to get the Fourier series, i.e., $\mathbf{a_k}$'s, we sample the Fourier transform at these equally spaced values.

# Deriving the Fourier Transform from a Fourier Series whose period goes to infinity

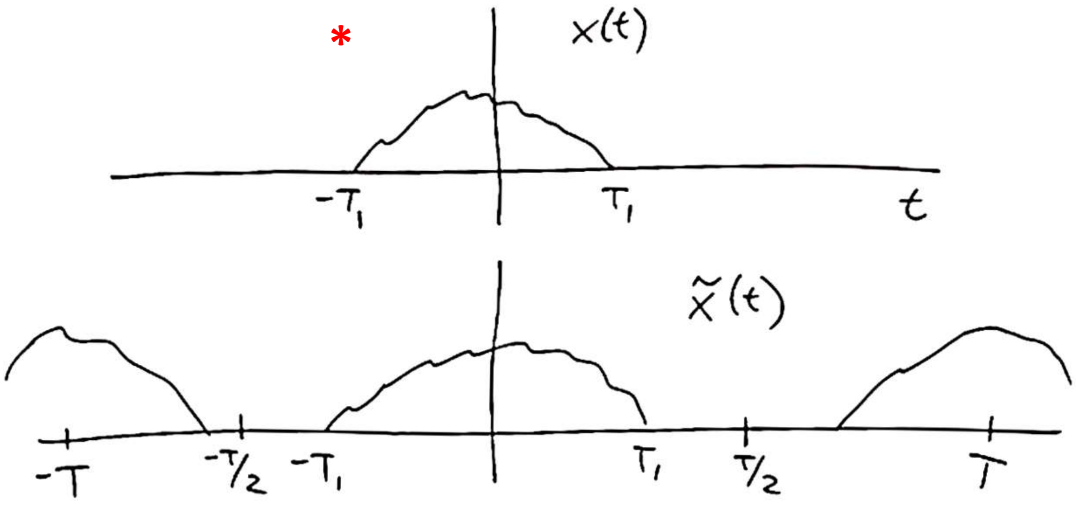



INTUITION: $\{a_k\}$ ARE EVENLY SPACED VALUES OF THIS CONTINUOUS FUNCTION OF $\omega$

AS $T \to \infty$, $\omega_0 \to 0$, SAMPLES GET INFINITELY CLOSE.

- As **T** goes to infinity, $\boldsymbol{\omega_0}$ goes to zero and samples get infinitely close. So, if we wanted to push the square wave off to infinity, eventually, what we would have would be the samples shown by vertical lines in region **. That is, instead of looking like *, they would look more like **. In the limit, we will be getting every point along this continuous function.

# Deriving the Fourier Transform from a Fourier Series whose period goes to infinity



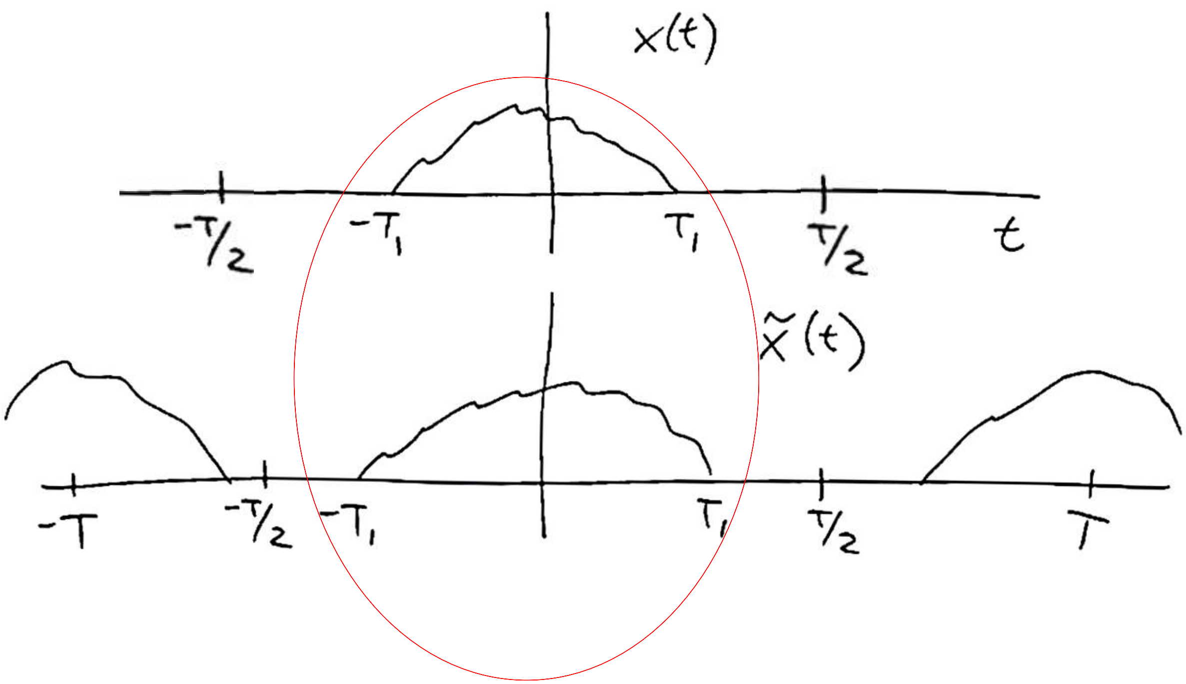

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \textbf{(1)}$$

- Let us mathematically derive how this works. Assume that we begin with a non-periodic signal *. We want the Fourier transform of this signal. What we can do is make a **related signal**. Let us assume that the signal has finite extent for the moment. This means that there is some **T**, large enough, so that we can make copies of the signal way out on the **t**-axis. Now, we can take this **x(t)** and create periodic copies of it for some **T**. Let us call this signal $\tilde{\mathbf{x}}(\mathbf{t})$, **x** tilde of **t**. It is related to **x(t)** but not the same as **x(t)**. The copies do not overlap. Now, we have a signal that we can take the Fourier series of. This is a periodic signal that can be shown by equation **(1)**. Put differently, the signal $\tilde{\mathbf{x}}(\mathbf{t})$ is going to have some Fourier series.
- **What are the $a_k$'s?** We need to use the other formula, the **analysis formula**, to figure out what the $a_k$'s are.

# Deriving the Fourier Transform from a Fourier Series whose period goes to infinity

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$




- We can just replace $\tilde{\mathbf{x}}(\mathbf{t})$ with the actual **x(t)** in the range between **–T/2** and **+T/2** since these two signals are exactly the same in this range.
- Now, we can say if we are going to integrate from **–T/2** and **+T/2**, we are assuming the signal is zero outside this range (shown by **red** oval). So, we do not lose anything by turning the integral from $-\infty$ to $+\infty$. Nothing is lost because all we are doing is just adding zero.

# Deriving the Fourier Transform from a Fourier Series whose period goes to infinity

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} X(k\omega_0) \quad \textbf{(1)}$$

DEFINE $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

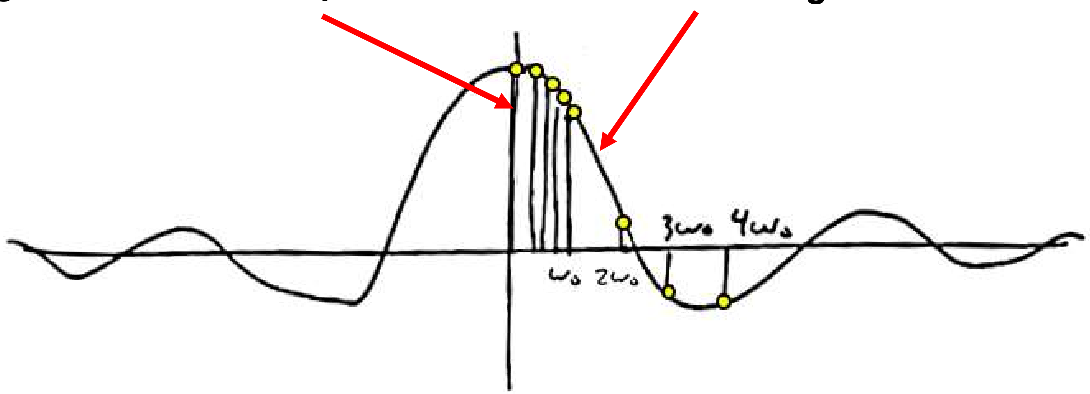
"THE ENVELOPE"

- As pointed out before, we do not lose anything by turning the integral from $-\infty$ to $+\infty$. This is shown in **(1)**.
- The continuous function of $\mathbf{X(k\omega_0)}$ is, in fact, $\mathbf{X(\omega)}$ that is being evaluated at $\mathbf{k\omega_0}$ .
- So, we see now that the $\mathbf{a_k}$'s are related to $\mathbf{X(\omega)}$. This continuous function acts like an **envelope**. What we get for the $\mathbf{a_k}$'s are ***samples of this envelope***. This is shown in **Graph A**.

**Graph A**

**The $a_k$'s are sampling inside this envelope.**

**The sinc signal is like the envelope.**



# Deriving the Fourier Transform from a Fourier Series whose period goes to infinity

$$a_k = \frac{1}{T}X(k\omega_0) \quad \textbf{(1)}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \textbf{(2)}$$

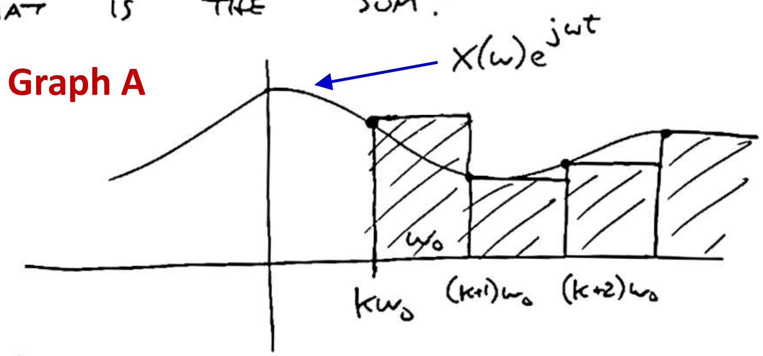
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \overbrace{\frac{1}{T}X(k\omega_0)}^{a_k} e^{jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}, \quad \frac{\omega_0}{2\pi} = \frac{1}{T}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \overbrace{\left[X(k\omega_0)e^{jk\omega_0 t}\right]\omega_0}^{*}$$

WHAT IS THE SUM?

Graph A




$\lim_{\omega \to 0}$ OF BOTH SIDES

- Now that we have $a_k$'s, **(1)**, we are going to plug them into $\tilde{\mathbf{x}}(\mathbf{t})$ equation, **(2)**, and reconstruct our signal.
- Let us think about a function that is defined as **X(ω)** (shown in **Graph A**). Let us see what ***** is (i.e., the terms inside the box of sigma). Here, we are summing up the area of the shaded rectangles. The width of the rectangle is $\boldsymbol{\omega_0}$ and the height of the rectangle is the continuous function, $\mathbf{X(\omega).e^{j\omega t}}$, sampled at $\mathbf{k\omega_0}$ (i.e., the term in the square bracket in *****).
- As we make the shaded boxes narrower and narrower, eventually, we will get things that add up to exactly the integral. So, the sum converges to the integral. The sum of these boxes (or rectangles) is called the **Riemann sum**.
- We want to make $\boldsymbol{\omega_0}$ very small (or **T** very large). So, we take the limit of both sides. That is, we find the limit of $\tilde{\mathbf{x}}(\mathbf{t})$ and the limit of the sigma and we let $\boldsymbol{\omega_0}$ approach **0**. So, the left hand side, the limit of $\tilde{\mathbf{x}}(\mathbf{t})$, is going to become **x(t)**, which corresponds to pushing the copies infinitely far away, and the right hand side is going to become the integral.

# The result: formulas for the forward and inverse Fourier Transform

$$\lim_{\omega \to 0} \tilde{x}(t) = \lim_{\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[ X(k\omega_0) e^{jk\omega_0 t} \right] \omega_0$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \qquad \text{INVERSE FOURIER TRANSFORM} \qquad \textbf{(1)}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \qquad \text{FOURIER TRANSFORM} \qquad \textbf{(2)}$$

$$\begin{array}{l} \nearrow \ \text{PERIODIC} \rightarrow \text{FS} \ (\text{DISCRETE } a_k) \\ x(t) \ \text{CONTINUOUS} \\ \searrow \ \text{APERIODIC} \rightarrow \text{FT} \ (\text{CONTINUOUS } X(\omega)) \end{array}$$

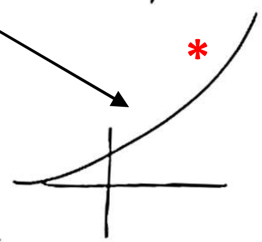
- **Fourier transform (FT)** tells us how to go from the time domain to the omega domain and the **inverse Fourier transform (IFT)** tells us how to go backwards.
- Now, we can go both backwards and forwards for any ***finite-length aperiodic*** signal using these two equations, **(1)** and **(2)**. The Fourier Transform is not limited to finite-length signals; it applies to infinite-length aperiodic signals as well, provided the signal is absolutely integrable. This concludes the derivation of the Fourier transform.
- In summary, let us say we have **x(t)** being a continuous function. If it is ***periodic***, what we get is the ***Fourier series***, which has a ***discrete*** set of $\mathbf{a_k}$. And if it is ***not periodic*** (***aperiodic***), we get the ***Fourier transform***, which has a ***continuous*** **X(ω)**.

- Later on, we are going to generalize the above formulation.
- We will discuss about what happens when **x(t)** is discrete or digital, i.e., when we have **x[n]** instead of **x(t)**. In DSP context, it turns out that there are also **digital equivalents** to both of the above equations. We will later learn about what the other two formulations are.

# When can we compute the Fourier Transform?

WHEN IS THIS "LEGAL"?

1) $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$ FINITE ENERGY




2) FINITE # OF EXTREMA

3) FINITE # OF DISCONTINUITIES

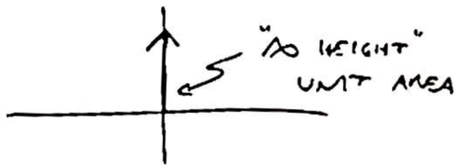
WE DO ALLOW THE FT OF PERIODIC SIGNALS.

- **Property #1:** We should have an integral where the energy converges. We have to have finite energy signal. For example, we cannot have a signal like * that is aperiodic and infinite area.
- **Property #2:** It is about having a finite number of extrema (maxima and minima). This is the equivalent of saying not wiggling infinitely much.
- **Property #3:** It has to have a finite number of discontinuities.
- These are basically the same things we discussed when dealing with Fourier series. The Fourier series and the Fourier transform are really the same thing. ***The Fourier series is like a special case of the Fourier transform in some sense.***
- We ***do*** allow the Fourier transform of periodic signals.

# Fourier Transform examples; Delta function; Shifted delta function

## Examples for FT:

1) $x(t) = \delta(t)$




$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega(0)} = 1 \quad \text{FOR ALL } \omega$$

2) $x(t) = \delta(t - t_0)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t_0} \quad \text{FOR ALL } \omega$$

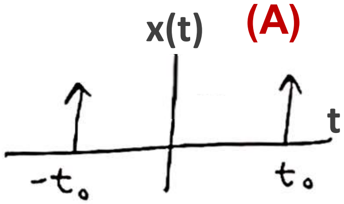
**Note:** $$\int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-j\omega t}\, dt = e^{-j\omega t_0}$$

- **Example 1:** It is just saying what happens when we integrate with the delta function. The integral basically corresponds to looking at the places where the delta function is firing. So, the FT of the delta function is a constant.
- **Example 2:** What happens if we shift the delta function? It is like shifting where the delta function fires. Now, all that is going to happen is that the delta function, instead of firing at $\mathbf{t = 0}$, is firing at $\mathbf{t = t_0}$.
- **Conclusion:** Both in **Examples 1** and **2**, the functions have the same magnitude. The magnitudes are just some point on the unit circle. But, they have different phase. When we time shift the signal, we phase shift the FT, the same way we phase shift the FS coefficients. We are not really changing the contribution of things. All we are doing is changing ***when*** these cosines and sines ***start***.

# Delta functions at ± $t_0$; One-sided exponential

(A)

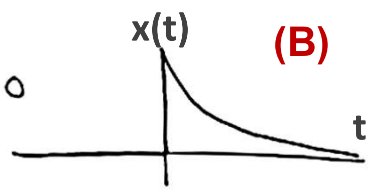
3)




$$X(\omega) = e^{j\omega t_0} + e^{-j\omega t_0} \rightarrow$$

$$\boxed{X(\omega) = 2\cos(\omega t_0)}$$

(B)




$$4) \quad x(t) = e^{-at}u(t) \qquad a>0$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

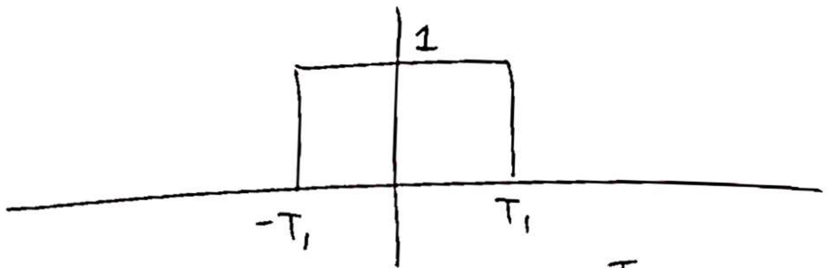
$$= \frac{-1}{a+j\omega}\, e^{-(a+j\omega)t}\Big]_{t=0}^{t=\infty} = \frac{1}{a+j\omega} \rightarrow$$

$$\boxed{X(\omega) = \frac{1}{a+j\omega}}$$

- **Example 3:** In **(A)**, we want to see what happens if we have sums of delta functions. So, when we have two impulses like the ones shown, we get a cosine.
- **Example 4:** In **(B)**, we have an exponential function that turns on at zero and is decreasing. The equation for **X(ω)** is similar to that of the Laplace transform of the signal. Just a note that **as the Fourier transform is more general than the Fourier series, the Laplace transform is more general than the Fourier transform**.

# Pulse

5)




$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

$$= \frac{-1}{j\omega} e^{-j\omega t} \Big]_{t=-T_1}^{t=T_1} = \frac{-1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right)$$

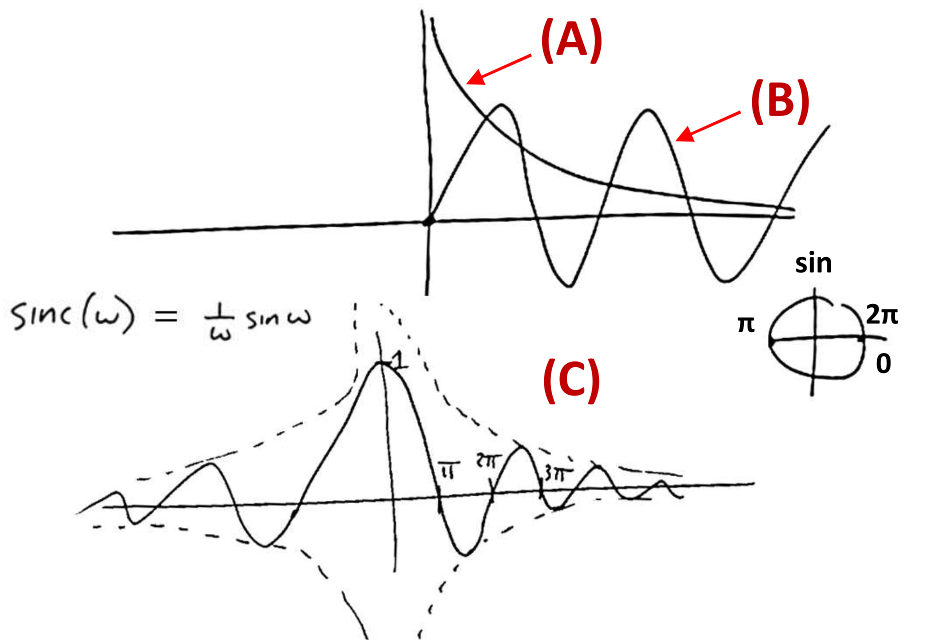
$$= \frac{2\sin(\omega T_1)}{\omega} = \frac{2T_1 \sin \omega T_1}{\omega T_1} = 2T_1 \operatorname{sinc}(\omega T_1)$$

$$\rightarrow \boxed{X(\omega) = 2T_1 \operatorname{sinc}(\omega T_1)}$$

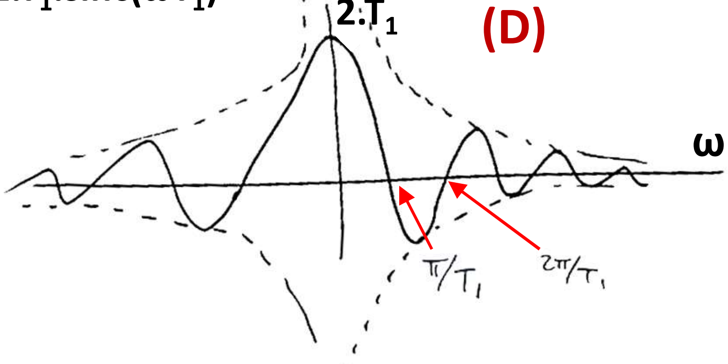
- **Example 5:** One signal that is really important is the pulse (not repeated), just a **regular pulse**. The result tells us that a pulse in the time-domain corresponds to a **sinc function** in the frequency-domain and vice versa.

# Comments on the sinc function

- **What does sinc(ω) look like?**




X(ω) = 2.$T_1$.sinc(ω$T_1$)




- The function **sinc(ω)** is like **sin(ω)/ω**. Here, if we draw them separately, **1/ω** is looks like **(A)** and **sin(ω)** looks like **(B)**. When we multiply these two things together, **1/ω** will act like an **envelope** that modulates (or multiplies) the **sin(ω)** that is oscillating inside it. The sinc function has a central peak at **ω = 0** and ***sidelobes*** that decay on either side.
- The function **sin(ω)/ω** crosses zero (i.e., the **x**-axis) in the same places that **sin(ω)** crosses the **x**-axis. This occurs when we have **π**, **2π**, **3π** and so on. Basically, these are multiples of **π**. As we approach **ω = 0**, the function **sin(ω)/ω** approaches **1**. The result is shown in **(C)**.
- Finally, because our target is to sketch **sinc(ω$T_1$)**, we need to scale the **x**-axis by **1/$T_1$**, as shown in **(D)**. Here, the zeros are function of **$T_1$**.

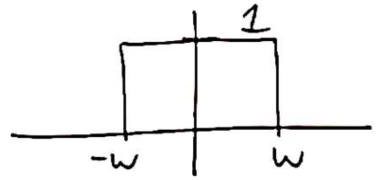
# A pulse in the frequency domain

- An example of ***inverse Fourier transform***:

**Example:**

WHAT IF

$X(\omega) =$




$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{jt\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} e^{jt\omega}\, d\omega = \frac{1}{2\pi j t}\, e^{jt\omega}\Big]_{\omega=-W}^{\omega=W}$$

$$= \frac{1}{\pi t}\cdot\frac{1}{2j}\left(e^{jtW} - e^{-jtW}\right) = \frac{\sin tW}{\pi t} = \frac{W}{\pi}\,\mathrm{sinc}\,(tW)$$

| T | F |
|---|---|
| $\delta(t)$ | CONST. |
| CONST. | $\delta(\omega)$ |

**(A)**

**T: Time domain**

**F: Frequency domain**

| T | F |
|---|---|
| PULSE | SINC |
| SINC | PULSE |

**(B)**

**This is the case shown above.**

$$\longrightarrow \boxed{x(t) = \frac{W}{\pi}\,\mathrm{sinc}\,(tW)}$$

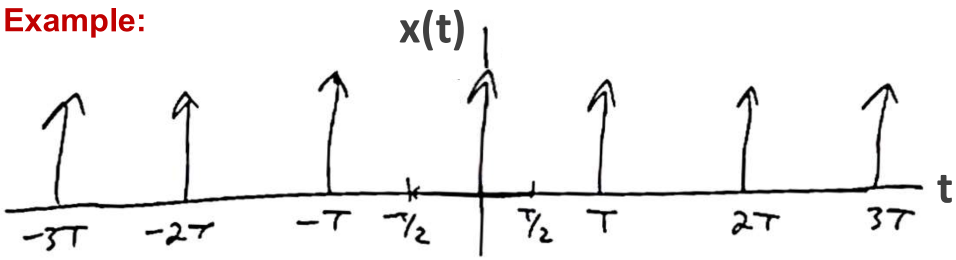
- **Duality Property:** We showed that if we have a pulse in the time domain, we get a sinc in the frequency domain. In this example, we have a pulse in the frequency domain, and as shown, we get a sinc in the time domain. This is not a coincidence and is called **Duality**.
- In table **(A)**, if we have a delta function in the time domain, then in the frequency domain, we get a constant. In a similar way, if we have a delta function in the frequency domain, we get a constant in the time domain.
- A summary of two examples are shown in tables **(A)** and **(B)**.
- Another example, that we had before, is that a pair of impulses in the time domain corresponds to a cosine in the frequency domain. And, in the same way, a cosine in the time domain corresponds to a pair of impulses in the frequency domain.

# Periodic signals

WHAT IF $x(t)$ IS PERIODIC?

WE CAN TAKE EITHER THE FS OR FT

**Example:**




$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$ IMPULSE TRAIN.

$$FS \Rightarrow \quad a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi}{T}t} \, dt$$

$$= \frac{1}{T}$$ FOR ALL $k$.

$\rightarrow$ $\boxed{a_k = \frac{1}{T}}$ FOR ALL $k$.

- So far, none of the functions for which we did the FT has been periodic. But, we mentioned that we are also allowed to do the FT of signals that are periodic. So, let us see what happens if we have a **periodic signal**. Here, we can take either the FS or the FT. FT and FS are closely related.
- The **impulse train** is an important signal used in the **sampling theorem**.
- When we do the FS, in the integral, in the range of **–T/2** to **+T/2**, the only place that delta function fires is at **0**, where its value is **1**. So, we just plug in **0** for **t** in **exp(-jk2πt/T)**. Eventually, for all the $a_k$'s, the impulse train gives us a constant for these FS coefficients, i.e., **1/T**.
- Later on, we will discuss how to find the FT of this impulse train. That is, we will finish this example at a later time and after we fully cover some additional concepts.

## The relationship between the Fourier Series coefficients and Fourier Transform for a periodic signal

LET'S START MORE GENERALLY;

SUPPOSE $x(t)$ IS A PERIODIC SIGNAL,

SO IT CAN BE REPRESENTED AS

(1) $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ FS

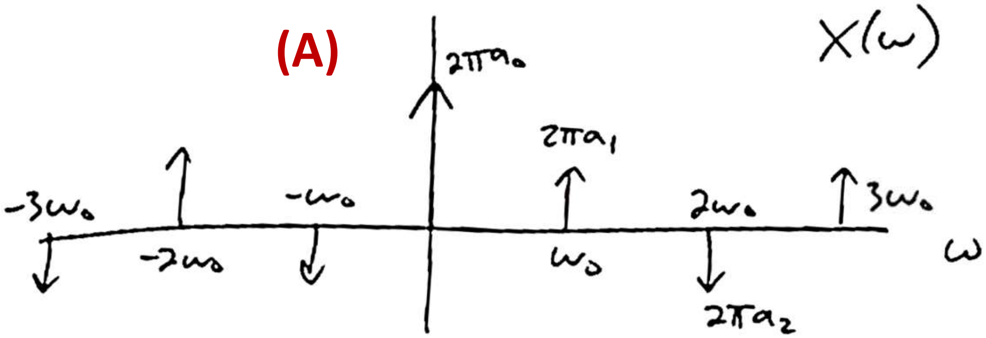
(2) $$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$ FT

- **What is the general equation for the computation of the Fourier transform of a periodic signal?** That is, if we have **x(t)**, **(1)**, in time-domain, how can we get **X(ω)**, which is in the frequency-domain? To do this, we can use equation **(2)**.
- **Conclusion:** The coefficients in FS, i.e., $\mathbf{a_k}$, are related to the coefficients in FT, i.e., $\mathbf{2\pi a_k}$. That is, to get the coefficients in FT, we just multiply the coefficients in FS by **2π**. So, the Fourier transform of a periodic function results in a discrete spectrum of delta functions at harmonics of the fundamental frequency $\boldsymbol{\omega_0}$. In other words, the Fourier transform of a periodic function does not yield a continuous function but rather a sum of delta functions (spikes) at these harmonic frequencies.

## The relationship between the Fourier Series coefficients and Fourier Transform for a periodic signal

LET'S SUPPOSE WE HAVE A SIGNAL WHOSE FOURIER TRANSFORM LOOKS LIKE:

**(A)**




$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot a_k \cdot \overbrace{\delta(\omega - k\omega_0)}^{*}$$

WHAT IS THE CORRESPONDING $x(t)$?

- Let us suppose we have a signal whose FT looks like **(A)**. **What is the corresponding x(t)?**
- The impulses are spaced apart by $\boldsymbol{\omega_0}$. So, $\mathbf{X(\omega)}$ is the sum of whatever the height of the impulse is, i.e., $\mathbf{2\pi . a_k}$, times the delta function spaced out at multiples of $\boldsymbol{\omega_0}$.
- The key thing is understanding what the inverse FT of the shifted delta function is, shown by *. After that, we can find **x(t)**.

## The relationship between the Fourier Series coefficients and Fourier Transform for a periodic signal

SUPPOSE $X(\omega) = \delta(\omega - k\omega_0)$ **(1)**

**(2)** $x(t) = \frac{1}{2\pi} \int \delta(\omega - k\omega_0) e^{j\omega t} \, d\omega$ **IFT**

$$= \frac{1}{2\pi} e^{jk\omega_0 t}$$

SO FOR $X(\omega) = \sum 2\pi a_k \delta(\omega - k\omega_0)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ *

= FOURIER SERIES OF $x(t)$.

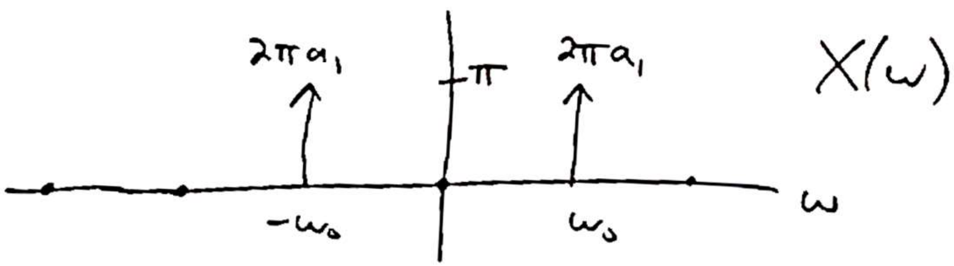
FOR A PERIODIC SIGNAL WITH $T = \frac{2\pi}{\omega_0}$

Graph A


- Let us just take one of the delta functions, i.e., just $\boldsymbol{\delta(\omega - k\omega_0)}$, shown in **(1)** and find the corresponding **x(t)**.
- Equation **(2)** is the **IFT** or the inverse Fourier transform of the shifted delta function.
- As shown in *, if we have a series of impulses (which may or may not be the same height in the frequency domain), that are also equally spaced, when we take that back in the time domain, what we get is a periodic signal.
- We can interpret the **heights of these impulses** (**$2\pi a_k$**) as being read off from our FS coefficients. That is, the FS coefficients are the heights of those impulses that have been divided by **$2\pi$**. An example is the height **$2\pi a_0$** in **Graph A**. The corresponding coefficient in **x(t)** will be **$a_0$**, i.e., **$a_0 = (2\pi a_0)/(2\pi)$**.

# Reading off time domain signals from delta functions in the frequency domain

**Example:** Given **X(ω)** find **x(t)**.




- **Sample calculation for $a_1$:** From **X(ω)** graph, at $\omega_0$, the height of the impulse is $2\pi a_1$. Solving for $a_1$, $2\pi a_1 = \pi$, we get $a_1 = 1/2$.

$$\left[\begin{array}{llll} \omega = 0 & \Rightarrow & X(\omega) = 0 & \\ \omega = \pm\omega_0 & & X(\omega): \text{DELTA} & \text{HEIGHT } \pi \\ \omega = \pm k\omega_0 & & X(\omega) = 0 & \\ k>1 & & & \end{array}\right]$$

$$a_0 = 0, \quad a_{\pm 1} = \frac{1}{2}, \quad a_{\pm k} = 0 \quad k>1.$$

$$\left[\begin{array}{l} X(\omega) = \sum 2\pi a_k \, \delta(\omega - k\omega_0) \\ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \end{array}\right]$$

$$x(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} = \cos \omega_0 t \rightarrow \boxed{x(t) = \cos \omega_0 t}$$

- The above reconfirms what we already knew. We knew that when we had a cosine in one domain (here, $\mathbf{\cos(\omega_0 t)}$), we got the two impulses in the other domain. So, moving backwards, if we start from $\mathbf{x(t) = \cos(\omega_0 t)}$, this is like saying we can take the cosine, compute the Fourier series of it, put those Fourier coefficients on top of the impulses in **X(ω)**, and eventually, we can get the Fourier transform, **X(ω)**.

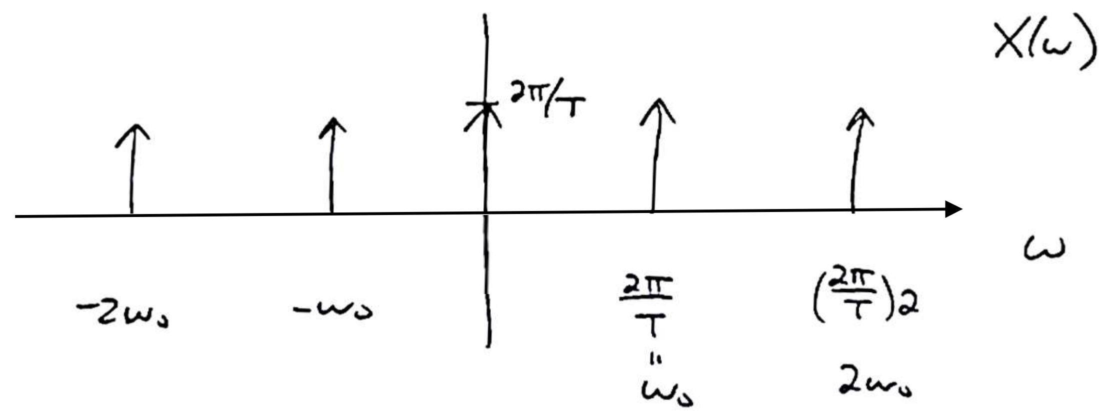
# Fourier Transform of an impulse train

**Example:** Here, we revisit the impulse train and will find **X(ω)**.

FOR PERIODIC IMPULSE TRAIN, PERIOD T

$a_k = \frac{1}{T}$ FOR ALL $k$. We showed this before.

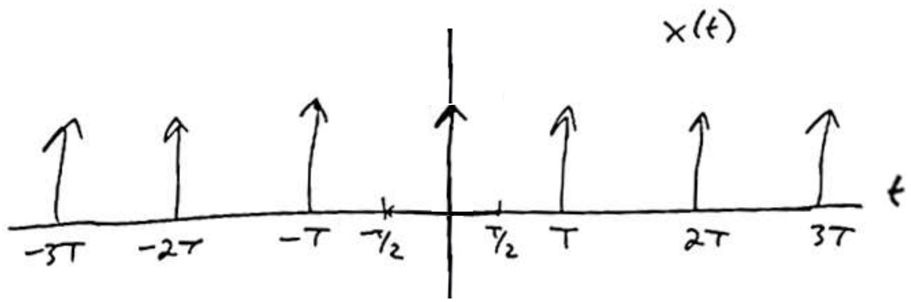
SO THE FOURIER TRANSFORM OF IMPULSE TRAIN



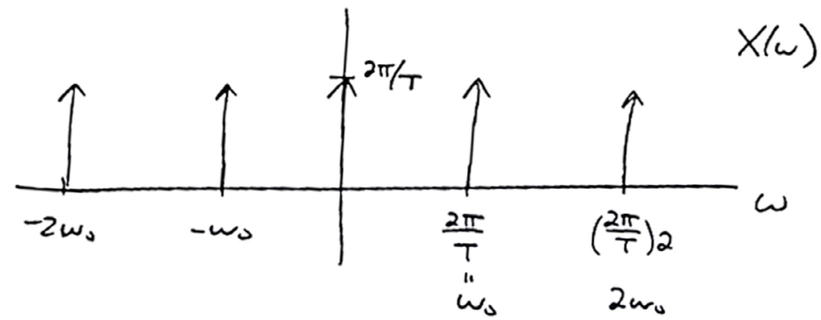

- Given that $a_k = 1/T$, the heights of the impulses in **X(ω)** are going to be $2\pi . a_k$ or $2\pi/T$.

# Fourier Transform of an impulse train

**Graph A**




**Graph B**




- In summary, if we have an impulse train in the time domain (**Graph A**), where the impulses are spaced apart by **T**, what we have in the frequency domain is also an impulse train where the impulses are spaced apart by **2π/T** (**Graph B**).
- The wider the impulses are in **Graph A**, the closer they are together in **Graph B**. This makes sense because the wider the impulses in **Graph A** are (i.e., if **T** is getting really large), the more it is pushing it out. So, it is not periodic at all anymore. So, eventually, we just have one impulse in the middle and nothing else in **Graph A**.
- The corresponding thing that would happen in **Graph B** would be that the impulses would all push together so closely that they become a constant. We already knew that the delta function in the time domain (i.e., a single delta function and not an impulse train) corresponds to a constant in the frequency domain.

# Fourier Transform properties

FOURIER TRANSFORM PROPERTIES

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$ **Inverse Fourier Transform**

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$ **Forward Fourier Transform**

$X(j\omega)$ $X(s)$ $s = j\omega$

1) LINEARITY

$x(t) \leftrightarrow X(\omega)$ $y(t) \leftrightarrow Y(\omega)$

$ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$

- In some references instead of **X(ω)**, they use **X(jω)**. We prefer **X(ω)** since we usually prefer to plot **X(ω)** vs. **ω**.

- **Property #1 (Linearity):** The FT exhibits linearity. This means that if we take the FT of a sum of signals, **ax(t) + by(t)**, it is equivalent to taking the FT of each signal individually and then summing the results: **FT [ax(t) + by(t)] = FT [ax(t)] + FT [by(t)] = aX(ω) + bY(ω)**. So, the FT is a linear operator.

# Symmetry properties

2) TIME SHIFT:

$$x(t-t_0) \leftrightarrow X(\omega)e^{-j\omega t_0} \quad \textbf{(1)}$$

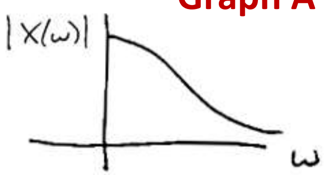
$$|X(\omega)e^{-j\omega t_0}| = |X(\omega)| \quad \textbf{(2)}$$

3) SYMMETRY PROPERTIES:

$$X(\omega) = R(\omega) + jI(\omega)$$

IF $x(t)$ IS REAL,
- $R(\omega) = R(-\omega)$ REAL PART EVEN
- $I(\omega) = -I(-\omega)$ IMAG PART ODD
- $|X(\omega)|$ IS EVEN, $\angle X(\omega)$ IS ODD

**Graph A**




- **Property #2 (Time Shift):** If we take a signal and we shift it, the thing that happens to its FT is a ***phase shift***. The magnitude of **(1)** is the same as whatever the magnitude was before, as shown in **(2)**. So, all we are doing is shifting the phase of the sinusoids in the integral that we add up. We are not really shifting any of the important content.
- **Property #3 (Symmetry Properties):** Let us say we have **X(ω)** and it is equal to a real part plus an imaginary part. If **x(t)** is real, then we have some useful properties that tell us the ***real part*** is ***even*** and the ***imaginary part*** is ***odd***. Additionally, the magnitude of **X(ω)** is even and the angle of **X(ω)** is odd. So, for real signals, given that **|X(ω)| = |X(-ω)|** for all frequencies **ω**, we do not have to plot the FT for all values of **ω**. Since the magnitude is even, we usually do not bother plotting the stuff on the left hand side because we know it is just the mirror image across the **y**-axis, as shown in **Graph A**.

# Symmetry properties

4) CAN SHOW

$$\begin{cases} Ev(x(t)) \longleftrightarrow Re(X(\omega)) \\ Od(x(t)) \longleftrightarrow j\,Im(X(\omega)) \end{cases}$$

$$\begin{cases} x(t) \text{ REAL, EVEN,} & X(\omega) \text{ REAL, EVEN} \\ x(t) \text{ REAL, ODD,} & X(\omega) \text{ IMAGINARY, ODD} \end{cases}$$

- **Property #4 (Even and Odd Parts):** We can show that the even part of **x(t)**, or **Ev(x(t))**, corresponds to the real part of **X(ω)**, or **Re(X(ω))**.
- Also, the odd part of **x(t)**, or **Od((x)t))**, corresponds to the imaginary part of **X(ω)**, **jIm(X(ω))**.
- Additionally, if **x(t)** is real and even, then **X(ω)** is also real and even. And, if **x(t)** is real and odd, then **X(ω)** is purely imaginary and odd.
- The applications of the above is that we can ***predict*** some simple things about a signal without actually having to compute the FT if we recognize the relationship between the even parts and the odd parts.

# Differentiation/integration

5) DIFFERENTIATION / INTEGRATION

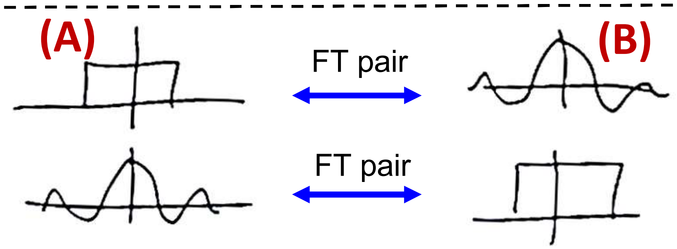
$$x'(t) \leftrightarrow j\omega X(\omega) \quad \text{similar to } s.X(s)$$

(1) $$\int_{-\infty}^{t} x(\tau)\,d\tau \leftrightarrow \frac{1}{j\omega} X(\omega) + \underbrace{\pi X(0)\,\delta(\omega)}_{(*)}$$

6) TIME SCALING

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

7) DUALITY




- **Property #5 (Differentiation and Integration):** In **(1)**, there is just a little extra bit (*****) that could happen at **ω = 0**. This could be a DC offset that we get when we do the integration.
- **Property #6 (Time Scaling):** If we scale the signal, **x(at)**, what we get is a scaled version of the FT, going the other way, i.e., **1/a**. So, if we shrink the time domain signal, we spread out the frequency domain signal, and vice versa.
- **Property #7 (Duality):** Duality says when we have a ***pulse*** in one world, **(A)**, it corresponds to a ***sinc*** in the other world, **(B)**. Then if something happens in one domain, we can relate it to what happens in the other domain. That is, we can switch them and be sure that it will work out. It is like ***a simple change of variable***. Duality implies that all the properties we just wrote down also applies to the other variables (relating **t** to **ω**). The duality property highlights the inherent connection between a function's behavior in the time and frequency domains. It allows us to predict the frequency content of a signal based on its time-domain shape (and vice versa) for certain functions.

# Convolution and preview of frequency response

8) PARSEVAL

$$\overbrace{\int_{-\infty}^{\infty} |x(t)|^2 \, dt}^{(1)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

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9) KEY PRINCIPLE: CONVOLUTION:

IF $y(t) = x(t) * h(t)$

$$Y(\omega) = X(\omega)H(\omega)$$

IMPULSE RESPONSE

$$x(t) \to \boxed{h(t)} \to y(t)$$

$$X(\omega) \to \boxed{H(\omega)} \to Y(\omega) \qquad (2)$$

FREQUENCY RESPONSE

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- **Property #8 (Parseval Theorem):** This says that the integral **(1)** in the time domain (that gives us the total energy) is the same as the total energy in the frequency domain (except for this **2π** factor). So, basically, it is saying that we have the same total energy either way and that we do not lose anything.
- **Property #9 (Convolution):** Convolution says that if **y(t)** is **x(t)** convolve with **h(t)**, then **Y(ω)** is **X(ω)** ***times*** **H(ω)**. This basically says that if we have an LTI system and we put a signal through it that has a certain impulse response, we can have an equivalent way of representing this system with the **frequency response**, **H(ω)**, shown in block diagram **(2)**. Multiplying things together in the frequency domain and taking the inverse for a transform is generally a lot easier than manually doing the time domain convolution. Given that our system is assumed to be LTI, this suddenly makes our life much easier!

# End of Lecture 5