ELEC 421
Digital Signal and Image Processing

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Course Roadmap for DSP

Lecture	Title
Lecture 0	Introduction to DSP and DIP
Lecture 1	Signals
Lecture 2	Linear Time-Invariant System
Lecture 3	Convolution and its Properties
Lecture 4	The Fourier Series
Lecture 5	The Fourier Transform
Lecture 6	Frequency Response
Lecture 7	Discrete-Time Fourier Transform
Lecture 8	Introduction to the z-Transform
Lecture 9	Inverse z-Transform; Poles and Zeros
Lecture 10	The Discrete Fourier Transform
Lecture 11	Radix-2 Fast Fourier Transforms
Lecture 12	The Cooley-Tukey and Good-Thomas FFTs
Lecture 13	The Sampling Theorem
Lecture 14	Continuous-Time Filtering with Digital Systems; Upsampling and Downsampling
Lecture 15	MATLAB Implementation of Filter Design

Lecture 8:
Introduction to the z-Transform

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Review of CTFT/DTFT; what is DT version of the Laplace transform

Z-TRANSFORM

CONTINUOUS	DISCRETE
$X(s) = \int_{t=-\infty}^{\infty} x(t) e^{-st} dt$ LAPLACE or FS	$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ Z-TRANSFORM
$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ CTFT	$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ DTFT
(A)	**(B)**

From before:

	CONTINUOUS	DISCRETE
PERIODIC	FS	? (DFT) ↓ AFTER Z-TRANSFORM
NONPERIODIC	FT	DTFT

- When we deal with signals and systems and differential equations, in general, we use the **Laplace transform (LPT)**. In the continuous world, the Laplace transform is a generalization of the Fourier transform, **(A)**. We just replace **jω** with the more generic complex variable, **s**. The reason that we care about the Laplace transform is that it allows us to solve problems that we could not solve with the FT alone. The advantage of the FT is that it has this very intuitive and concrete interpretation as "***frequency***". A lot of real world engineering systems involve frequency.
- **What is the discrete time version (DT version) of the Laplace transform?** This is called the **z-transform, (B)**. And again, the idea is the same. We turn an infinite integral into an infinite sum. Also, by comparing **X(z)** and **X(ω)** in column **(B)**, we are replacing **exp(-jωn)** with the term $\mathbf{z^{-n}}$. In **z-transform (ZT)**, **z** is a more generic idea. Before, the **ω** was only a real-valued frequency variable, now, **z is some complex number**.

Why is z^n a special signal for DT LTI systems?; Introduction to the transfer function

RECALL:

$$e^{j\omega_0 n} \rightarrow \boxed{H} \rightarrow H(\omega_0)e^{j\omega_0 n} \quad (1)$$

z^n IS "SPECIAL" FOR DT LTI SYSTEMS

$x[n] = z^n$ FOR SOME COMPLEX z.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k] = \sum_{k=-\infty}^{\infty} h[k]\,z^{n-k}$$

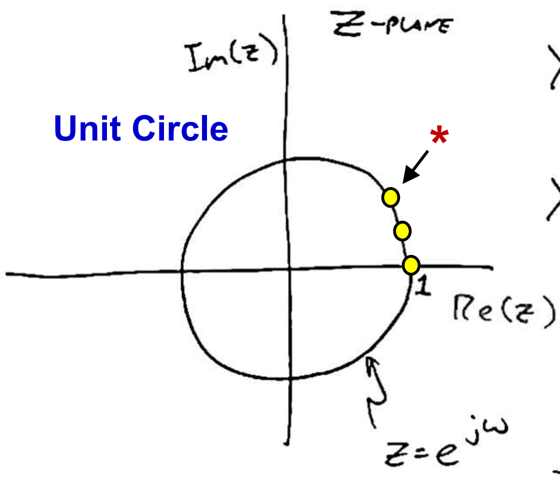
$$= z^n \underbrace{\sum_{k=-\infty}^{\infty} h[k] z^{-k}} = \underbrace{H(z)}_{\text{TRANSFER FUNCTION}} z^n$$

$$\longrightarrow \boxed{y[n] = H(z) z^n} \quad (2)$$

- **Why does the z-transform make sense as an operator for a digital signal?** We showed that for both the continuous and discrete time FT's, the ***exponentials*** (**e** to the power of something) were important for LTI systems in the sense that when **exp(jω_0n)** went in, the same thing came out, just modulated by the frequency response, **H(ω_0)**. This is shown again in **(1)**. The same thing is actually true for the **z**-transform.
- **H(z)** is called **z-transform impulse response** or just **transfer function**. The result, **(2)**, shows that if we put **x[n] = z^n** in, what comes out, **y[n]**, is the same **z^n** just modulated or multiplied by some complex number, **H(z)**, which is evaluated at some complex number **z**.
- Before, we showed that this was true for some specific set of complex numbers, the ones that are on the unit circle. But here, ***this is actually true for any complex number***. That is why we can talk about ***more generic transformations*** than just the frequency response.

How are the DTFT and z-transform related?

RELATIONSHIP BETWEEN Z-TRANSFORM ($X(z)$) AND DTFT ($X(\omega)$):



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad (1)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad (2)$$

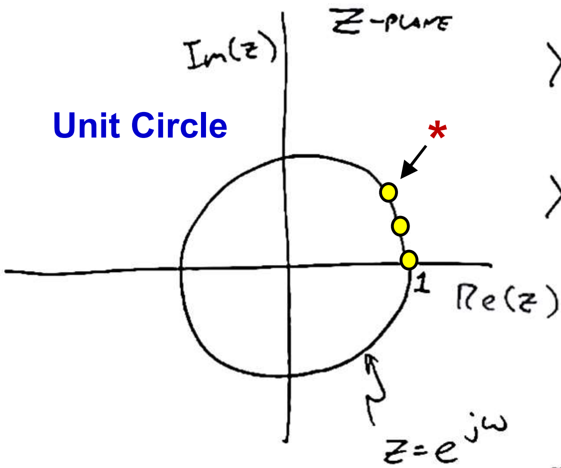
$$= X(z)\Big|_{z=e^{j\omega}} \quad (3)$$

DTFT IS 2π-PERIODIC.

- **What is the relationship between z-transform and DTFT?** The formulas, **(1)** and **(2)**, look pretty similar. In fact, **X(ω)** is exactly what we would get if we took **X(z)** and evaluated it at the points **z = e^{jω}**. This is represented by **(3)**. What that means is like saying that if the DTFT exists, the FT is exactly at those places where that complex number happens to be on the **unit circle**, shown by some representative points such as *.
- **ROC in z-transform:** The **z**-transform has a region, which is called the **region of convergence (ROC)**. **X(z)** formula is not automatically defined for any **z** that we choose. There are only certain **z**'s that make this formula ***converge***. When both the ZT and FT exist on the unit circle, then they are related by **z = e^{jω}**. Since we are evaluating omegas as they go from **0** on the unit circle to **π**, this makes it clear that the ***DTFT has to be periodic***, i.e., because we have this **circular relationship**. So, DTFT is **2π**–periodic.

The unit circle plays a critical role for the z-transform

RELATIONSHIP BETWEEN Z-TRANSFORM AND DTFT:



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad (1)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad (2)$$

$$= X(z)\Big|_{z=e^{j\omega}} \quad (3)$$

DTFT IS 2π-PERIODIC.

UNIT CIRCLE IS KEY FOR DT SYSTEMS (LIKE jω-AXIS WAS FOR CT SYSTEMS). **(A)**

For Laplace Transform

- We are not going to talk about the **Laplace transform** too much (see statement **(A)**), other than to say that just in the same way that the **jω-axis**, the **imaginary axis**, plays a big role in analyzing continuous time systems, the **unit circle** plays a big role in analyzing discrete time systems.
- In continuous time analysis, we spend a lot of time worrying about, when did the poles, for example, crossed over the **jω**-axis that affected the **system stability**, and so on. We are going to talk about exactly the same concepts here, except now ***we care about when do poles stay inside the unit circle***.
- **Conclusion:** There is a **symmetry** or a **relationship** between **z**-transform and DTFT, and in this relationship, the key is the **unit circle**.

Why do we need the z-transform?

WHY DO WE NEED THE Z-TRANSFORM?

(A) • THE DTFT DOESN'T ALWAYS CONVERGE/EXIST.

$$\left(\sum_{n=-\infty}^{\infty} |x[n]| < \infty \right)$$

(B) • THE Z-TRANSFORM MAY CONVERGE IN PLACES WHERE THE DTFT DOESN'T EXIST

(C) • NOTATION IS EASIER – POLYNOMIALS IN Z RATIONAL FUNCTIONS OF Z

(D) • HELPS A LOT WHEN DESIGNING FILTERS.

- **Why do we need the z-transform?**
- **Point A:** One reason is that the DTFT does not always converge or exist. In **(A)**, we need to be able to sum the values of the signal for that sum to converge, i.e., the sum should be finite value. We talked about how we could relax this condition slightly, but for the moment, let us just assume this is the condition that we need for the DTFT to exist. Not every signal satisfies this constraint, **(A)**. If we are staying in Fourier transform world, there are some signals that we cannot analyze with the FT, because the FT does not exist. So, the **z**-transform allows us to talk about more general class of signals.

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- **Point (B):** The **z**-transform may converge in places where the Fourier transform does not exist.
- **Point (C):** The notation is generally easier. We get a lot of **polynomials** or **rational** functions in **z**. Here, we can lose all those $e^{j\omega}$ things that were a little bit tedious to keep in mind and keep track of. Working with **z**'s are much cleaner, i.e., working with polynomials in **z**-world.
- **Point (D):** The **z**-transform helps a lot when ***designing digital filters in DSP***. The idea is that when we look at what are called the poles and the zeros of the **z**-transform, understanding what those poles and zeros are doing can help us extremely in terms of designing digital filters. By examining the poles and zeros, we can make the filters do what we want them to do. The key point is that **it is tough to design filters directly in the frequency domain, but it is a lot easier to do it in the z-world.**

The region of convergence (ROC)

THE REGION OF CONVERGENCE (ROC)

WRITE $z = re^{j\omega}$ **(1)**

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \overbrace{\left(x[n] r^{-n}\right)}^{(2)} e^{-j\omega n}$$

$$= \text{DTFT}\left(x[n] r^{-n}\right) \quad \textbf{(3)}$$

$\Leftrightarrow$ Z-TRANSFORM CONVERGES IF

$$\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty \quad \textbf{(4)}$$

From before:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

DTFT

- Let us drill down a little bit into **Point (B)**. This is related to the critical concept of **the region of convergence (ROC)**. First, we write down **z** in polar form, **(1)**, and then we plug it into **X(z)**. Now we can see that **(2)** is some sort of ***modified signal***. In some sense, what we are doing in **(3)** is taking the DTFT of the modified signal. We have the same convergence condition that we had before, **(4)**.
- Based on **(4)**, it is clear that the **z**-transform may converge in places where the FT does not. This is because now we have an extra part, r^{-n}. For **r = 1** and if the sum converges and since **(4)** is true, then the DTFT exists and the **z**-transform also has a region of convergence that includes **r = 1**. But for ***changing values of r***, if we were to crank up or damp down the original signal by multiplying it by some decreasing exponential, maybe we could get the sum to converge. So, the **z**-transform exists for that certain value of **r**. In other words, **what we are doing is figuring out *what the range of r is* for which this sum converges.**

Example: the step function

Example:

$u[n]$: $\sum_{n=-\infty}^{\infty} u[n] = \infty$ **(1)**

(2)

WHAT ABOUT $u_m[n] = u[n]\, r^{-n}$ FOR SOME REAL r.

(3)

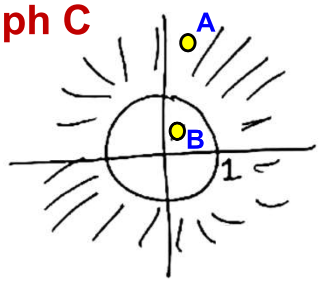
$$\sum_{n=-\infty}^{\infty} u[n]\, r^{-n} = \sum_{n=0}^{\infty} r^{-n} = \frac{1}{1-r^{-1}} = \frac{1}{1-\frac{1}{r}} \quad \textbf{(4)}$$

IF $|r| > 1$. **(5)**

$$= \sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^{n} \rightarrow \boxed{\sum_{n=-\infty}^{\infty} u[n]\, r^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^{n}} \quad \textbf{(6)}$$

THE ROC OF THE Z-TRANSFORM OF $u_m[n]$ IS $\boxed{|r| > 1}$

Graph C

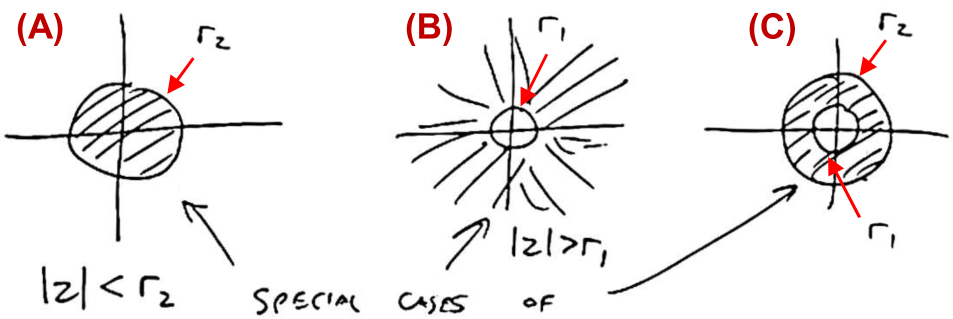


- Let us consider the step function, **u[n]**. If we were to look at this sum, **(1)**, it will get infinity. So, we cannot take the Fourier transform of the step function. It is simply not possible to do that. However, if we were to look at **(2)**, for some real **r**, in that case, the sum would look like **(4)**, if $|\mathbf{r}| > \mathbf{1}$ (shown by **(5)**). Here, $\mathbf{u_m[n]}$ is the modified version of **u[n]**, i.e., a modified discrete step signal.
- A different way of looking at **(3)** is that we need to make sure that each of these terms in **(6)** is less than **1**. In that case, the sum will converge (in the region of $|\mathbf{r}| > \mathbf{1}$). So, we can say that the ROC of the **z**-transform of $\mathbf{u_m[n]}$ is $|\mathbf{r}| > \mathbf{1}$.
- The way we draw the ROC, **Graph C**, is to sketch a unit circle and then draw the **sunburst**. This says that any **z** we pick out in the sunburst region (such as point **A**) will satisfy this constraint and therefore the sum, **(3)**, will converge. We ***can*** talk about the transform only in this region. But, anywhere inside the circle (such as point **B**), we cannot. This is simply because the radius is less than **1** and this sum will just blow up.
- **Conclusion:** This is a signal, i.e., **(1)**, that we can analyze with the **z**-transform in certain regions of the complex plane, but we cannot analyze it by Fourier transform.

What do ROCs look like?; If the ROC includes the unit circle, the system is stable

CONVERGENCE OF THE Z-TRANSFORM DEPENDS ONLY ON $|z| = r$. **(1)**

ROC'S LOOK LIKE:



$$r_1 < |z| < r_2 \quad \textbf{(2)}$$

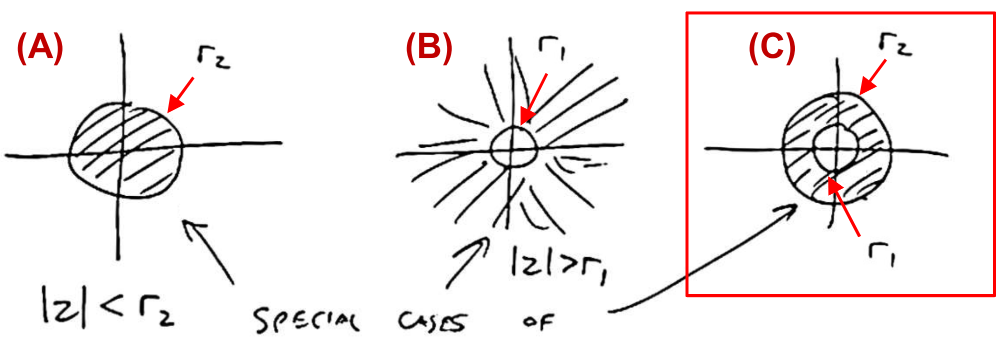
IF ROC INCLUDES $|z| = 1$ $(r = 1)$, THEN THE DTFT EXISTS. (CONVERGES)

- Convergence of the **z**-transform only depends on the magnitude of **z**, which we were calling **r**, shown by **(1)**. So, this means that regions of convergence look like things that only depend on the radius, **r**.
- Here, we could have a region of convergence that looks like **(A)**, or the one we discussed in the previous example, that looks like **(B)**, i.e., everything outside of a circle. We could even have something like **(C)**, where we have things that are between two circles.
- In some sense, **(A)** and **(B)** are actually like special cases of **(C)**. So, if we were to take **(C)** and let the outer circle go to infinity, we will get a picture that looks like the sunburst, i.e., **(B)**. If in **(C)**, we take the inner circle and bring it down to **0**, then we get **(A)**.

What do ROCs look like?; If the ROC includes the unit circle, the system is stable

CONVERGENCE OF THE Z-TRANSFORM DEPENDS ONLY ON $|z| = r$. **(1)**

ROC'S LOOK LIKE:



$$\boxed{r_1 < |z| < r_2}$$ **(2)**

IF ROC INCLUDES $|z| = 1$ $(r = 1)$, THEN THE DTFT EXISTS. (CONVERGES)

- So, generally, ROC looks like **(C)** graphically and mathematically can be presented by **(2)**.
- **Conclusion:** When the ROC includes the unit circle, i.e., if it includes $|z| = 1$ (or **r = 1**), that means we can talk about both the **z**-transform ***and the Fourier transform***. In other words, if the ROC includes the unit circle, then the DTFT exists. This is another way of saying that the DTFT sum converges. This convergence of the DTFT has a special name and that is called **stability**.

From before:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

DTFT

If the ROC includes the unit circle, the system is stable; Poles and zeros

IF THE Z-TRANSFORM OF AN IMPULSE RESPONSE $h[n]$ FOR AN LTI SYSTEM CONVERGES ON THE UNIT CIRCLE (DTFT EXISTS), THEN THE SYSTEM IS <u>STABLE</u>.

- **Stability:** If the z-transform of an impulse response, **h[n]**, for an LTI system, converges on the unit circle, (i.e., if DTFT exists), then the system is **stable**.
- We are going to talk a lot more about stability and what it means in future lectures. Right now, it is more about definition than anything else.
- A system is stable if the region of convergence of its **z**-transform includes the unit circle, meaning its Fourier transform exists. This is because the ***impulse response is absolutely summable***, satisfying the BIBO stability criterion. Recall that BIBO is short for Bounded Input Bounded Output.
- In some sense, the stability of the system means the system is doing good things and would not blow up. It means that if we put some arbitrary signal into the system, it cannot do crazy things that produce some output that explodes out to infinity. In other words, it means the system, in general, is **well-behaved**.

Poles and zeros

IF THE Z-TRANSFORM OF AN IMPULSE RESPONSE $h[n]$ FOR AN LTI SYSTEM CONVERGES ON THE UNIT CIRCLE (DTFT EXISTS), THEN THE SYSTEM IS STABLE.

$$X(z) = \frac{N(z)}{D(z)} \quad \text{(1)}$$

POLYNOMIALS IN Z

$N(z) = 0 \Rightarrow X(z) = 0$ "ZEROS"

$D(z) = 0 \Rightarrow X(z) = \infty$ "POLES"

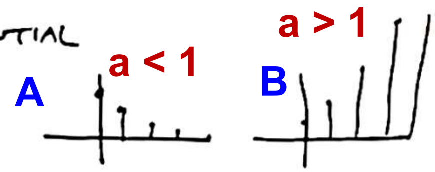
- We can write the **z**-transform as some numerator over some denominator, **(1)**, where both are polynomials in **z**.
- When the numerator is **0**, **N(z) = 0**, the **z**-transform or **X(z)** is **0**, and these are naturally called **zeros** of the transfer function.
- When the denominator is **0**, **D(z) = 0**, that means that **X(z)** blows up to infinity, and we call these **poles** (also called **singularities**).
- There are important connections between the zeros and the poles of the polynomials and the ROC of the **z**-transform. We will see, especially when it comes to ***filter design***, it is all about understanding where the zeros and the poles of transfer function are in order to make the filter do what we want it to do.

Z-transform examples; Right-sided exponential

Example:

RIGHT-SIDED EXPONENTIAL

$$x[n] = a^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \quad (1)$$

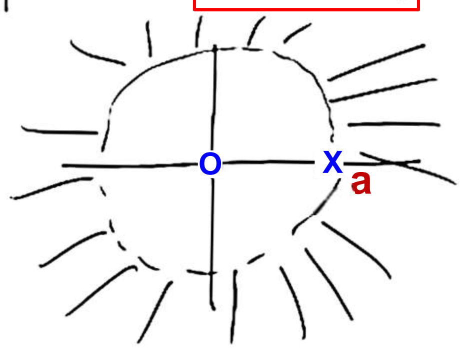
X(z) CONVERGES IF $\left|\frac{a}{z}\right| < 1$ $\boxed{|z| > |a|}$

THEN $X(z) = \dfrac{1}{1 - \frac{a}{z}}$

$= \dfrac{z}{z-a}$

O = ZERO

X = POLE



Pole-zero Diagram

$$\rightarrow \boxed{X(z) = \frac{z}{z-a}} \quad (2)$$

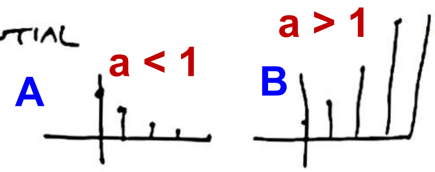
- Let us do an example of an exponential sequence. This is a right-sided exponential digital signal, because it only starts at **0** and it goes on to infinity off to the right hand side.
- When the magnitude of **a** is less than **1**, assuming **a** is a positive number, we get a decreasing exponential, **A**. When **a** is greater than **1**, we get an increasing exponential, **B**. We can really predict that the ROC has to do with when **a < 1**. This is because otherwise we are never going to get sums that converge with pictures like **B**, when **a > 1**.
- So, let us compute the **z**-transform, **X(z)**. This leads to **(1)**. Here, **(1)** will then converge to **(2)**.
- Now, we can draw what is called the **pole-zero diagram**. There is only one place where the **z**-transfer is **0**, and that is when **z = 0**. We draw a **circle** (**O**) at this point to indicate zero. We draw a **cross** (**X**) at **z = a** to indicate a pole.

Right-sided exponential

Example:

RIGHT-SIDED EXPONENTIAL

$$x[n] = a^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \quad (1)$$

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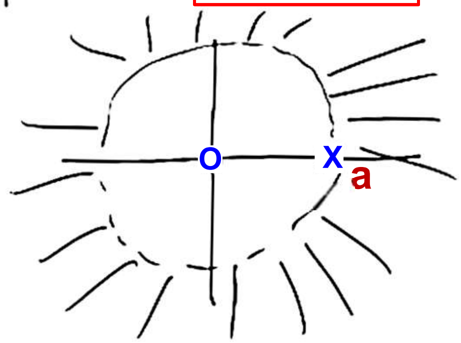


Pole-zero Diagram

$$\rightarrow \boxed{X(z) = \frac{z}{z-a}} \quad (2)$$

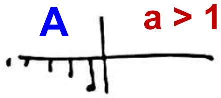
- In general, **a** could be some complex number, but here, we are assuming that it is a real positive number. By plotting the pole-zero diagram, we see that the ROC is going to be ***everything outside the circle***. We can also see that if **a** is strictly less than **1**, the Fourier transform exists. Otherwise, the FT does not converge.
- We saw that when we were doing the DTFT of this signal, then **a** had to be a number such as **1/4** or **1/3** or **1/2**. We could not do this with **a** being a number such as **2**. Otherwise, the Fourier transform would not exist.
- Note that in causal systems, the impulse response **h[n]** is typically right-sided. This means that **h[n]** is non-zero only for **n ≥ 0**. If the original signal **x[n]** is also right-sided (causal), the output **y[n]** of the system, calculated as the convolution **y[n] = x[n]∗h[n]**, will also be right-sided. This is because convolution of two right-sided signals yields a right-sided result.

Left-sided exponential

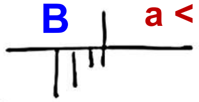
Example:

LEFT-SIDED EXPONENTIAL:

A a > 1



B a < 1



$$x[n] = -a^n u[-n-1] \quad (1)$$

"ON" FROM $-\infty$ TO -1

- Now, let us talk about the **left-sided exponential**. This is kind of the opposite of the right-sided exponential.

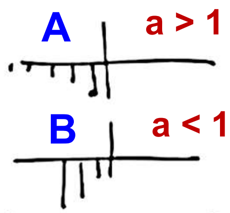
- First, let us take a look at **u[-n-1]** and see how it behaves. When **n = 0**, we get **u[-1]**, which is step function of **-1**, and that is **0**. When **n = 1**, we get step function at **-2**, which is also **0**. So, basically, for any **n** $\geq$ **0**, the number inside the brackets (i.e., "**-n-1**") is going to be negative and we are not going to get any contribution. Whereas when **n = -1**, we are going to get **u[0]**, and that is **1**. This is the first time that the function turns on. For example, for **n = -4**, when we have **u[-(-4)-1]** or **u[+3] = 1**, the step function will also be **ON**. So, this is a function that is basically **ON** from **-∞** to **-1** and turns off at **0**.

- **Graph of** $\mathbf{x[n] = -a^n.u[-n-1]}$**:** For equation **(1)**, we are going to get functions that look like **A** or **B**, depending on the size of **a**. That is, it could look like something that is decreasing like **A** or increasing like **B** (when we move towards the left side of the **n**-axis)

Two functions can have the same algebraic z-transform but different ROCs- specifying both is important

Example:

LEFT-SIDED EXPONENTIAL:



$$x[n] = -a^n u[-n-1] \quad \text{(1)}$$

"ON" FROM $-\infty$ TO -1

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1]\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty} -\left(\frac{z}{a}\right)^n \quad \text{(2)}$$

CONVERGES IF $\left|\frac{z}{a}\right| < 1$

$$\boxed{|z| < |a|}$$

$$= \frac{-1}{1-\frac{z}{a}} + 1$$

$$= \frac{-a}{a-z} + \frac{a-z}{a-z} = \frac{-z}{a-z} = \frac{z}{z-a}$$

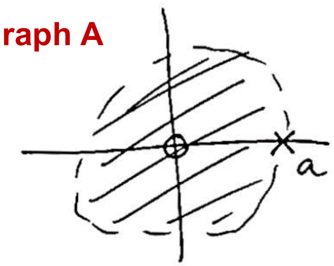
$$\rightarrow \boxed{X(z) = \frac{z}{z-a}} \quad \text{(3)}$$

- Let us compute **z**-transform, **X(z)**. We will arrive at **(2)**. Now, **when does this sum converge?** It converges if the absolute value of **z/a** is less than **1**. We need to have each of the terms of $(z/a)^n$ getting smaller and smaller.
- **What is the actual value of the sum?** We can show that it will be **(3)**. It seems that **(3)** looks exactly like the **z**-transform that we got last time for the right-sided exponential and that seems wrong! It seems like we should not be able to get two discrete time signals that have the same **z**-transform. But the key idea here is that they do not have the same **z**-transform because ***z-transform comes in two parts, both X(z) and the region of convergence (ROC).*** And, here, the region of convergence is actually opposite of what we got for the right-sided exponential. Last time, we got the region of convergence being **z** ***outside*** of the circle, or $|z| > |a|$. Here, the region convergence is the ***inside*** of the circle, or $|z| < |a|$.

Two functions can have the same algebraic z-transform but different ROCs- specifying both is important

ROC HERE:

Graph A

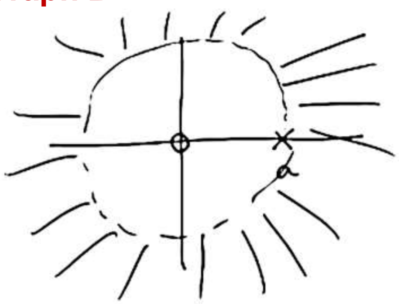


* Z-TRANSFORM CONSISTS OF BOTH AN ALGEBRAIC FORM OF X(Z) AND AN ROC THAT SAYS WHERE THIS IS VALID.

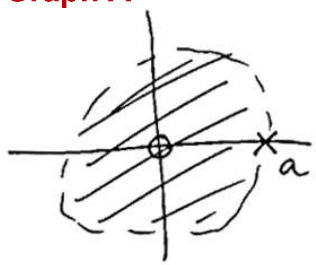
- In **Graph A**, the region of convergence is the inside of the circle, or $|z| < |a|$. Like before, we have a zero and a pole, but this time, the ROC is ***inside the circle*** as shown by the shaded area.
- **Conclusion:** The key idea is that **z**-transform is both an algebraic form of **X(z)** and an ROC that says where this is valid. So, we cannot just use the **z**-transform of a signal alone, and then compute the inverse transform. We also need to be given the ROC in order to disambiguate which of the possible time domain signals we might mean. That is the reason why we always have to make sure that we note the ROC when we are working on the design of a digital filter. Otherwise, we can get confused!

Two functions can have the same algebraic z-transform but different ROCs- specifying both is important

Graph B



Graph A



- Now it is fair to say that in a lot of the design problems that we do, ***we are going to deal with right-sided signals***.
- When we say it is a right-sided signal, that pins down what the ROC is. Put differently, this means we do not have to be told exactly what the ROC is. We are just needed to be told whether we are looking for a right-side signal, or a left-sided signal. That distinguishes between the pattern in **Graph B** (the sunburst pattern) and the pattern in **Graph A**.

The sum of two right-sided signals

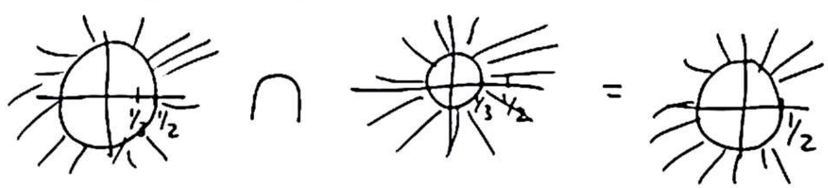
Example:

Right-sided   Right-sided

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{-1}{3}\right)^n u[n]$$

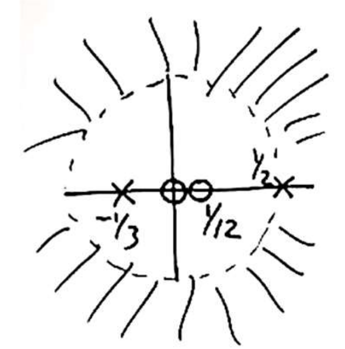
$$X(z) = \underbrace{\frac{z}{z-\frac{1}{2}}}_{(1)} + \underbrace{\frac{z}{z+\frac{1}{3}}}_{(2)} = \frac{2z^2 - \frac{1}{6}z}{(z-\frac{1}{2})(z+\frac{1}{3})} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

ROC $|z| > \frac{1}{2}$ $\cap$ ROC $|z| > \frac{1}{3}$ $=$ $|z| > \frac{1}{2}$



Graph A   Graph B   Graph C

ROC and pole-zero diagram



Graph D

z-transform

$$\boxed{X(z) = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}}$$

$\boxed{|z| > \frac{1}{2}}$ ROC equation

- We are going to use ***the linearity of the z-transform***. Here, we can immediately find the **z**-transform of each components in **x[n]** along with their ROC's. For the final ROC, we need to look at the **intersection** of the two ROC's. The ROC in **Graph A** is more restrictive. So, the final ROC looks like a sunburst extending from **1/2**, as shown in **Graph C**.
- The pole-zero diagram is shown in **Graph D**. We have a zero at **0**, and another zero at **1/12**. We also have a pole at **1/2** and another at **-1/3**.
- **Graph D** shows both the ROC, that looks like a circle that extends out from **1/2**, together with the poles and zeros.
- **Conclusion:** We can see that the ROC is related to the poles. The zeros do not really play any role in what the ROC could be, but the poles play a huge role. This is mainly because in **(1)** and **(2)**, we only need to look at the terms that involve the denominator (i.e., the poles).

Right-sided plus left-sided

Example:

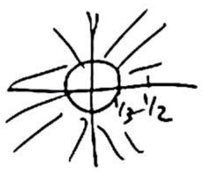
CONSIDER

Right-sided **(1)** Left-sided **(2)**

$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$\text{ROC: } |z| > \tfrac{1}{3} \;\cap\; \text{ROC: } |z| < \tfrac{1}{2}$$



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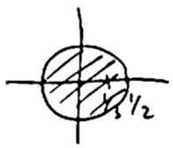

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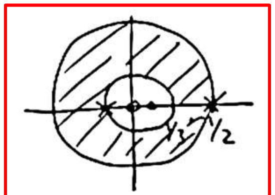


Graph A

Graph B

Graph C

ROC and pole-zero diagram

z-transform

$$\boxed{X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}}$$

$$\boxed{\tfrac{1}{3} < |z| < \tfrac{1}{2}}$$

ROC equation

RIGHT-SIDED EXPONENTIAL

$$x[n] = a^n u[n] \qquad X(z) = \frac{z}{z-a} \qquad |z| > |a|$$

LEFT-SIDED EXPONENTIAL

$$x[n] = a^n u[-n-1] \qquad X(z) = \frac{z}{z-a} \qquad |z| < |a|$$

- In this example, we have a signal which has a part that is right-sided and a part that is left-sided. So, it is like saying that **(1)** covers what is happening for $\mathbf{n} \geq \mathbf{0}$ and **(2)** covers what is happening when $\mathbf{n} < \mathbf{0}$.
- It turns out that we can work this out to the very same algebraic equation we had in the previous example, but the ROC is different now. We see that the intersection of **Graphs A** and **B** is all the shaded area that is inside the ring-shaped picture shown in **Graph C**.
- Even though we have pretty much the same pole-zero diagram, here, we get a very different ROC. In most cases, we are assuming that things start at zero (i.e., right-sided). One of the reasons is that, most of the time, the signals that we care about are impulse responses and for impulse responses to correspond to causal systems, we should have signals that are right-sided. We do not want to have responses for which **h[n]** is non-zero for values of $\mathbf{n} < \mathbf{0}$. So, we want our impulse response to be right-sided. Because of this, we will mostly get ROC's that look like sunbursts, i.e., going outward.

Finite-length exponential

Example:

FINITE LENGTH EXPONENTIAL

$$x[n] = \begin{cases} a^n & n \in [0, N-1] \\ 0 & \text{ELSE} \end{cases}$$

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(\frac{a}{z}\right)^n$$

FINITE # OF TERMS; FORMULA IS ALWAYS VALID

$$; \quad \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \qquad 0<\alpha<1$$

$$= \frac{1-\left(\frac{a}{z}\right)^N}{1-\frac{a}{z}} = \frac{z^N - a^N}{z^N - a z^{N-1}}$$

$$= \frac{z^N - a^N}{z^{N-1}(z-a)}$$

N ZEROS AT ROOTS OF $z^N = a^N$

N-1 POLES AT $z=0$

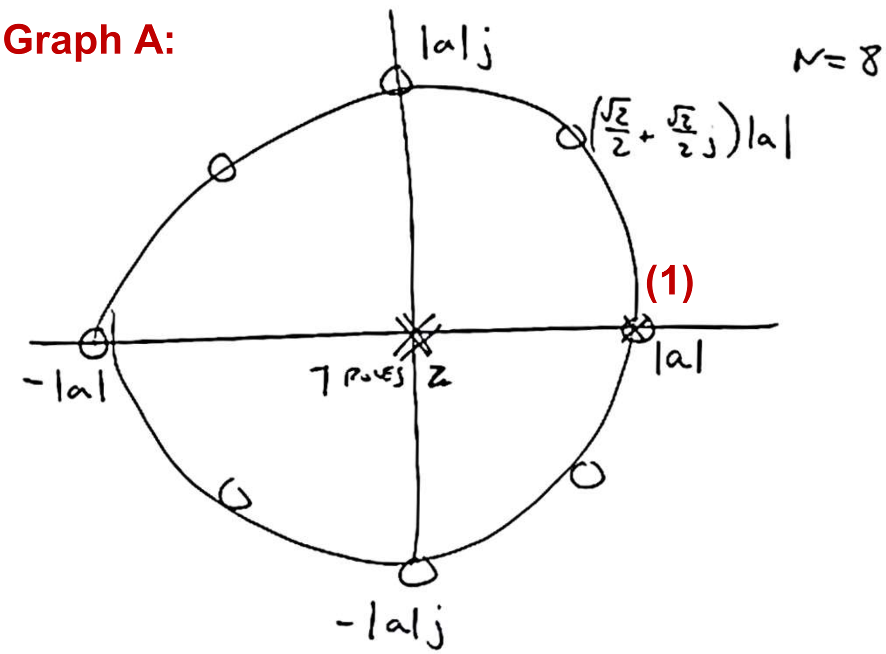
POLE AT $z=a$

$$\rightarrow \boxed{X(z) = \frac{z^N - a^N}{z^{N-1}(z-a)}} \quad \textbf{(1)}$$

- According to **(1)**, we have **N** number of zeros at the roots of $\mathbf{Z^N = a^N}$, which are the zeros of the top polynomial, and **N-1** numbers of poles at $\mathbf{z = 0}$, which are the zeros of the bottom polynomial. We also have a pole at $\mathbf{z = a}$.

Finite-length exponential

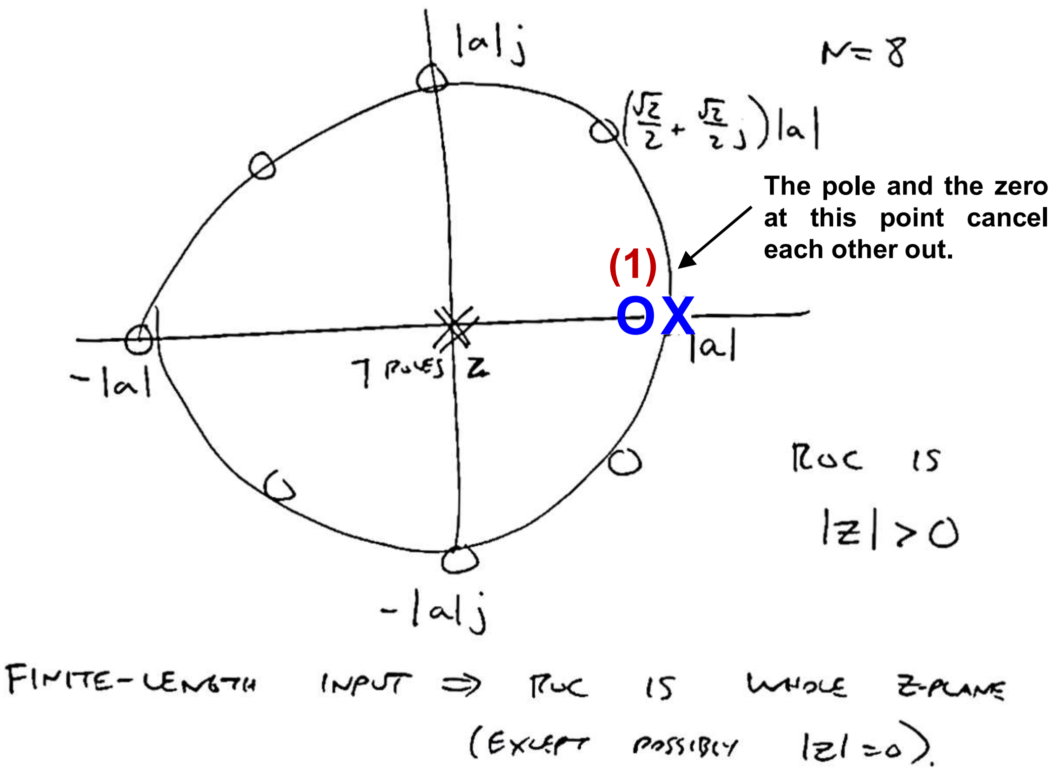
Graph A:



- **What are the Nth root of some number a or Z^N = a^N?** In **Graph A**, let us say we have **N = 8**. That means we are going to have a pattern of things that look like evenly-spaced around the circle. The zeros of the top polynomial will be **8** complex numbers, shown by "**O**". We also have a whole bunch of poles at the origin, i.e., **7** poles at the origin, and also **1** pole at point **(1)**, shown by "**X**".

Finite-length exponential

Graph A:



- In **Graph A**, and at point **(1)**, we have both a pole and a zero. They will cancel each other out. In fact, there is not going to be a zero or a pole at **(1)**, since they cancel one another at the numerator and the denominator.
- One thing to observe here is that the ROC is any value of **z**, that is not **0** or ROC: $|\mathbf{z}| > \mathbf{0}$. That means we can take any value of **z** for the **z**-transform for this signal. **This is generally true when the input signal is only finite length**.
- If we have only a finite number of terms to add up, nothing can ever go wrong. So, for a finite-length input, the ROC is the whole **z**-plane, except possibly at $|\mathbf{z}| = \mathbf{0}$.
- **Conclusion:** When we have a finite-length input or a finite-length input impulse response, life is always pretty good and there is no need to worry about poles because all the poles are clustered together.

Exponential times a cosine

Example:

$$x[n] = 2^n \cos(3n)\, u[n]$$

$$= 2^n \left(\frac{e^{j3n} + e^{-j3n}}{2}\right) u[n]$$

$$= \frac{1}{2}\left((2e^{3j})^n + (2e^{-3j})^n\right) u[n]$$

$$X(z) = \frac{1}{2}\left(\frac{1}{1 - \frac{2e^{3j}}{z}}\right) + \frac{1}{2}\left(\frac{1}{1 - \frac{2e^{-3j}}{z}}\right)$$

$$= \frac{1}{2}\left(\frac{z}{z - 2e^{3j}} + \frac{z}{z - 2e^{-3j}}\right)$$

$$\left.\begin{aligned} x[n] &= a^n u[n] \\ X(z) &= \frac{1}{1 - \frac{a}{z}} \end{aligned}\right\} \quad \textbf{(1)}$$

- We used the pair of **(1)** that we had before to find the **z**-transform of, e.g., $\mathbf{(2e^{3j})^n}$. Note that **a** can be any number, including a complex number. In this case, we have $\mathbf{a = 2e^{3j}}$.

Exponential times a cosine

$$X(z) = \frac{1}{2}\left(\frac{z}{z-2e^{3j}} + \frac{z}{z-2e^{-3j}}\right)$$

$$= \frac{z(z-2e^{-3j}) + z(z-2e^{3j})}{2(z-2e^{3j})(z-2e^{-3j})} \rightarrow$$

POLES AT

$z = 2e^{\pm 3j}$

$$= \frac{2z^2 - 2z(e^{-3j} + e^{+3j})}{2(z^2 - 2z(e^{-3j}+e^{3j}) + 4)}$$

$$= \frac{2z^2 - (4\cos 3)z}{2(z^2 - (4\cos 3)z + 4)} = \frac{z^2 - (2\cos 3)z}{z^2 - (4\cos 3)z + 4}$$

ZEROS AT

$z = 0$

$z = 2\cos 3$

Exponential times a cosine

$$\boxed{X(z) = \frac{1}{2}\left(\frac{z}{z-2e^{3j}} + \frac{z}{z-2e^{-3j}}\right)}$$

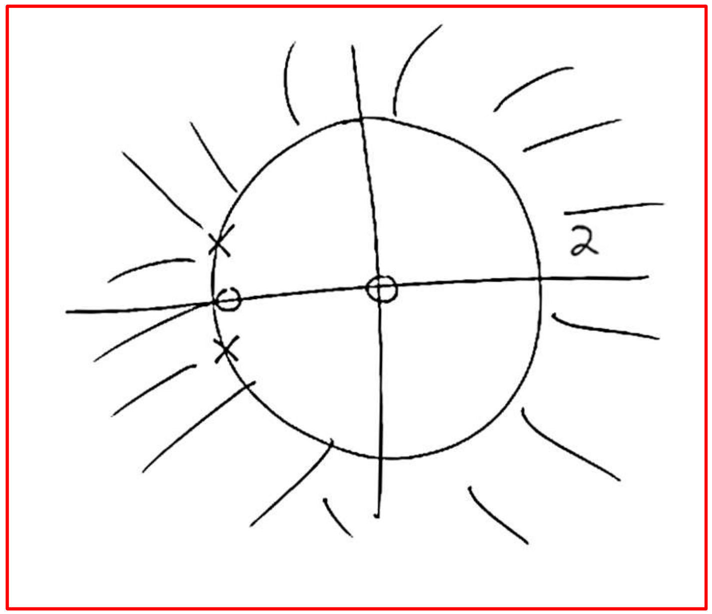
$$\boxed{|z| > 2}$$

ROC:

FOR $x[n] = a^n u[n]$

WAS $|z| > |a|$.

$$a = 2e^{\pm 3j} \longrightarrow |a| = 2 \longrightarrow |z| > 2$$



POLES AT

$z = 2e^{\pm 3j}$

ZEROS AT

$z = 0$

$z = 2\cos 3$

ROC rules

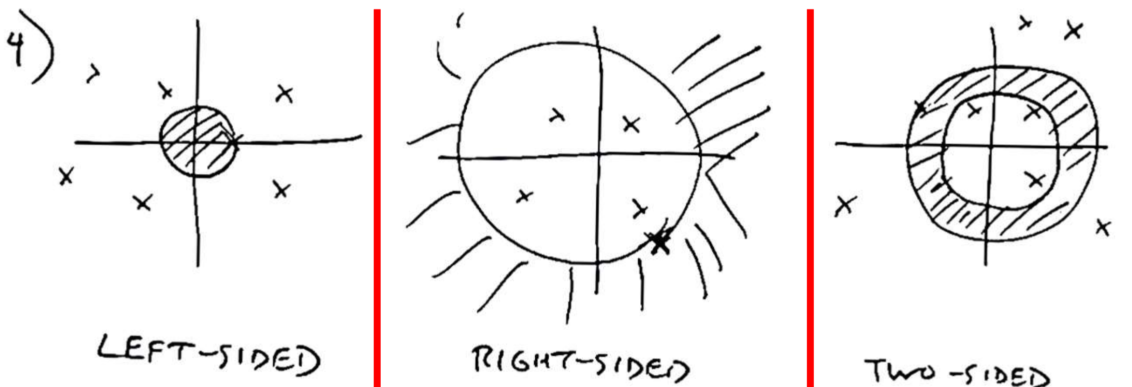
RULES ABOUT THE ROC:

1) ROC IS A RING OR DISC CENTERED AT 0.

2) ROC CONTAINS NO POLES

3) IF $x[n]$ IS FINITE-LENGTH, ROC IS ENTIRE Z-PLANE (EXCEPT POSSIBLY $z=0$, $z=\infty$)

4)



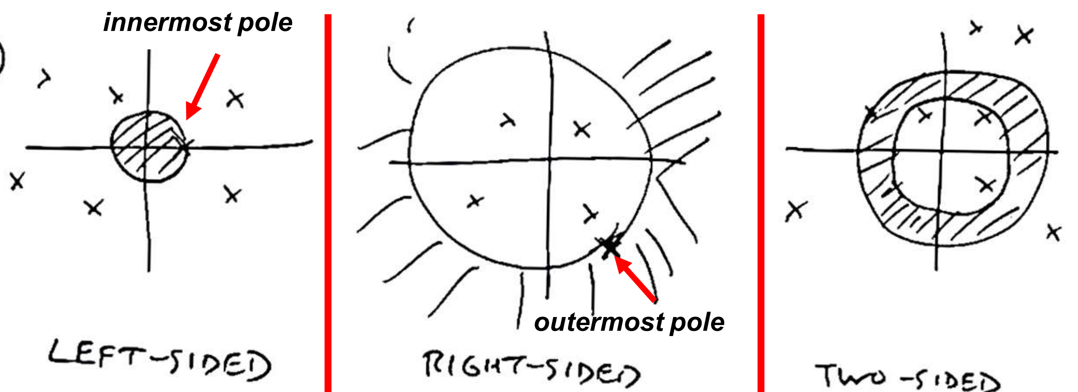
Graph A | Graph B | Graph C

- Let us discuss **the rules about the ROC**.
- **Rule #1:** The ROC is a ***ring*** or a ***disc*** centered at the origin. We have already shown why that is true, because it depends only on the radius of the complex variable. So, that means it has got to be something that looks like a **circle**.
- **Rule #2:** The ROC contains no poles. And, that is by definition. Because if the **z**-transform is infinity, that is what a pole means, then certainly it does not converge there. So, the ROC never contains any poles by definition.
- **Rule #3:** We showed earlier that if **x[n]** is finite length, the ROC is the entire **z**-plane, except possibly, we may run into problems at exactly **0** or exactly infinity.

ROC rules

RULES ABOUT THE ROC:

1) ROC IS A RING OR DISC CENTERED AT 0.
2) ROC CONTAINS NO POLES
3) IF $x[n]$ IS FINITE-LENGTH, ROC IS ENTIRE Z-PLANE (EXCEPT POSSIBLY $z=0$, $z=\infty$)
4)



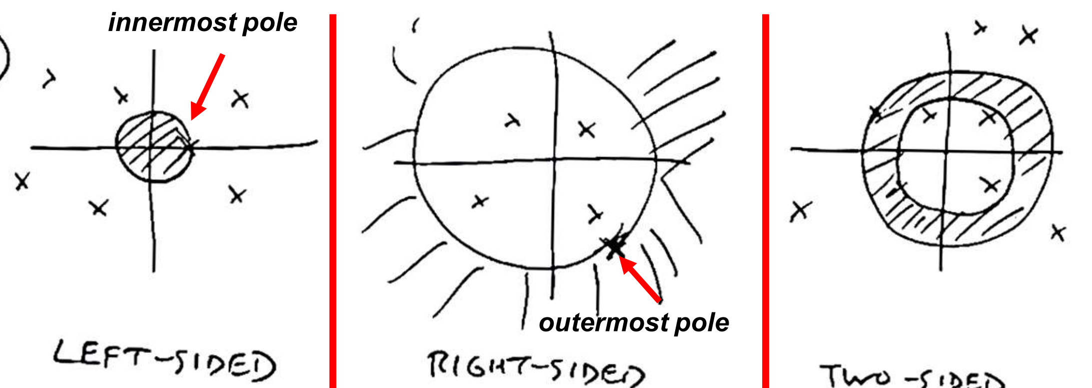
Graph A   Graph B   Graph C

- **Rule #4:** Let us look at three different cases, shown in **Graphs A**, **B**, and **C**.
- In **Graph A**, **x[n]** is a left-sided signal, and hence, the ROC is going to be the region ***inside the innermost pole***. So, if we have a bunch of poles out here, the ROC is going to be the shaded region (i.e., inside the circle).
- In **Graph B**, **x[n]** is a right-sided signal, and hence, the ROC is going to be the reverse. It is going to be the region that is ***outside the outermost pole***.

ROC rules

RULES ABOUT THE ROC:

1) ROC IS A RING OR DISC CENTERED AT 0.
2) ROC CONTAINS NO POLES
3) IF $x[n]$ IS FINITE-LENGTH, ROC IS ENTIRE Z-PLANE (EXCEPT POSSIBLY $z=0$, $z=\infty$)
4)



Graph A | Graph B | Graph C

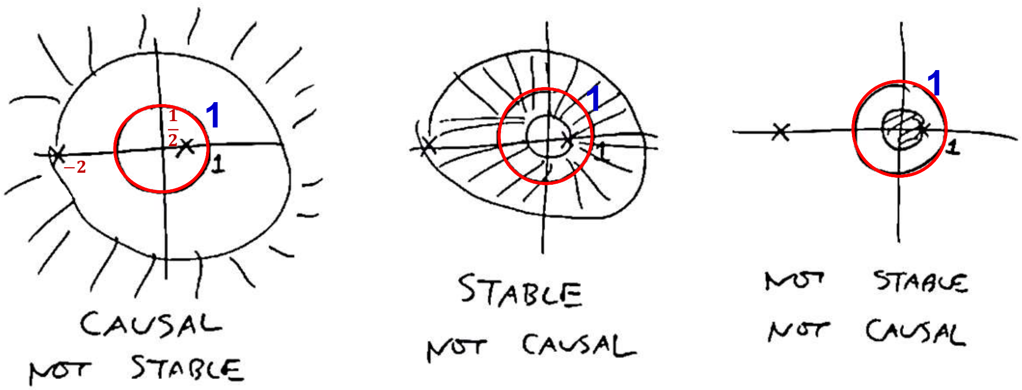
- In **Graph C**, **x[n]** is a two-sided signal, and hence, the ROC look like some sort of a ***doughnut***, that is defined by the ring between two poles. So, we could have more poles either inside or outside the ring (not on the shaded area), but in general, the ROC is bounded by the poles. So, this is a way of saying that the ROC is always bounded by the poles, and that in special cases, we know that a left-sided signal looks like **Graph A**, and a right-sided signal looks like **Graph B**. In the case of **Graph C**, there are many possible doughnut-shaped ROC's that correspond to different, two-sided signals, and hence, we have to disambiguate within there which one we mean.
- **Summary of Graphs:** One thing that we cannot have is that we cannot have a pole in the ROC (that is **Rule #2**).

The ROC, stability, and causality

5) ROC TELLS US: IS THE SYSTEM STABLE?
↳ FOR A TRANSFER FUNCTION $H(z)$ IS THE SYSTEM CAUSAL?

STABLE ⟺ ROC INCLUDES UNIT CIRCLE

CAUSAL ⟺ IMPULSE RESPONSE IS RIGHT-SIDED



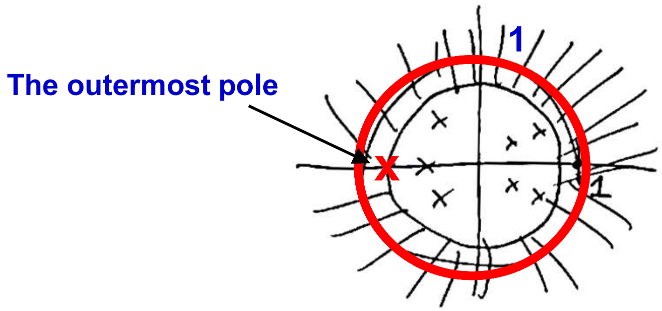
Graph A   Graph B   Graph C

- **Rule #5:** The final thing to point out here is that the ROC basically tells us whether the system is stable or not. And, is the system causal?
- **Stable** implies that the ROC includes the unit circle. **Causal** implies that the impulse response is right-sided. Let us look at some possible ROC's (shown in **Graphs A**, **B**, and **C**). First, we draw the unit circle for each case. Second, let us suppose we have poles at **1/2** and **-2**. We can have three different kinds of ROC's.
- **Graph A:** This is a right-sided signal. So, it is causal, but since it does not include the unit circle, it is not stable.
- **Graph B:** This includes the unit circle. So, it is a stable system. But because the ROC does not go out to infinity, this is not causal.
- **Graph C:** This one neither includes the unit circle nor goes out to infinity. So, this one is not stable, and it is not causal either.

Desirable ROCs: all poles are inside the unit circle

BEST-CASE (MOST DESIRABLE) SCENARIO:

– ROC LOOKS LIKE



Graph D

i.e., ROC EXTENDS OUTWARDS FROM LARGEST-MAGNITUDE POLE, AND ALL POLES ARE INSIDE THE UNIT CIRCLE.

- The case that we desire the most is a system where all the poles are inside the unit circle and the ROC looks like a **sunburst**, going outside that ***outermost pole*** (**Graph D**).
- **Conclusion:** When we design systems, especially for control systems, we want to make sure that all the poles are inside the unit circle and that the ROC extends beyond the outermost pole. This is exactly analogous to what we did in signals for the continuous time systems where we wanted, for example, to push all the poles into the left-half plane.

- All we talked about up to this point, was the ***forward definition of the z-transform***. What we are going to do next time is to see how we can deal with ***inverse z-transforms***, and what are some **z**-transform properties that can make our life easier!

End of Lecture 8