ELEC 421

Digital Signal and Image Processing

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Course Roadmap for DSP

Lecture	Title
Lecture 0	Introduction to DSP and DIP
Lecture 1	Signals
Lecture 2	Linear Time-Invariant System
Lecture 3	Convolution and its Properties
Lecture 4	The Fourier Series
Lecture 5	The Fourier Transform
Lecture 6	Frequency Response
Lecture 7	Discrete-Time Fourier Transform
Lecture 8	Introduction to the z-Transform
Lecture 9	Inverse z-Transform; Poles and Zeros
Lecture 10	The Discrete Fourier Transform
Lecture 11	Radix-2 Fast Fourier Transforms
Lecture 12	The Cooley-Tukey and Good-Thomas FFTs
Lecture 13	The Sampling Theorem
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Lecture 15	MATLAB Implementation of Filter Design

Lecture 9:
Inverse z-Transform; Poles and Zeros

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Review of known z-transform pairs

$$\alpha^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha| \qquad (1)$$

$$\cos \omega_0 n \, u[n] \longleftrightarrow \frac{1-(\cos\omega_0) z^{-1}}{1-(2\cos\omega_0) z^{-1} + z^{-2}} \qquad |z| > 1 \qquad (2)$$

$$r^n \sin \omega_0 n \, u[n] \longleftrightarrow \frac{r \sin\omega_0 \, z^{-1}}{1-(2r\cos\omega_0) z^{-1} + r^2 z^{-2}} \qquad |z| > r \qquad (3)$$

- **How do we do the inverse z-transform? How do we connect up the z-transform with the discrete time Fourier transform?** We often care about frequency when we are talking about LTI systems, but the **z**-transform is more abstract. How does it help us design frequency domain filters, for example?
- In **(1)**, that we discussed before, unlike the Fourier transform where we had to put constraints on α for the Fourier transform to exist, here we can take the **z**-transform of this signal no matter what α is. But the ROC of that may or may not include the unit circle. So, if $\alpha < 1$, then the Fourier transform exists and everything is good. If $\alpha > 1$, we would normally say that it would not converge, but we can show that for certain **z**'s, it will still work.
- In **(2)**, we are going to get two poles that are complex conjugates, and a couple of zeros.
- In **(3)**, we are kind of combining **(1)** and **(2)**.

Why do all the formulas have z^{-1} instead of z?

$$\alpha^n u[n] \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha| \qquad (1)$$

$$\cos\omega_0 n \, u[n] \longleftrightarrow \frac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}} \qquad |z|>1 \qquad (2)$$

$$r^n \sin\omega_0 n \, u[n] \longleftrightarrow \frac{r\sin\omega_0 \, z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2 z^{-2}} \qquad |z|>r \qquad (3)$$

SAY $x[n]=0 \quad n<0$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n} \longrightarrow$$

$$\longrightarrow \quad X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots \qquad (4)$$

- One important thing to notice is that all of these **z**-transforms, **(1)** to **(3)**, are formulas that depend on only ***negative*** powers of **z**. **Why is that?**
- The first thing to say is that, suppose we have **x[n] = 0**, for **n < 0**. If that is true for all these three functions, then the **z**-transform is as shown in **(4)**.
- **Conclusion:** We can see that for a causal signal, the only powers of **z** that we get are negative. That means that for many of these types of signals we care about, i.e., ***the right-sided signals***, we only see negative powers of **z**.

Why do all the formulas have z^{-1} instead of z?

$$\cos\omega_0 n\, u[n] \longleftrightarrow \frac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}} \quad |z|>1 \qquad (1)$$

$$\frac{1-(\cos\omega_0)z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}}\cdot\frac{z^2}{z^2} = \frac{z^2-(\cos\omega_0)z}{z^2-2\cos\omega_0 z+1} \qquad (2)$$

- Suppose that we are a little bit shaky about what **(1)** looks like. What we could do is just multiply **(1)** by $\mathbf{z^2/z^2}$, then we would get $\mathbf{z^{+2}}$ and $\mathbf{z^{+1}}$, which leads us to **(2)**. So, we ***can*** make everything look like positive powers of **z** if we want to. And, in some sense, this is a little bit more comfortable when we are trying to find the poles and the zeros,.
- For example, if **(1)** is given, and we are tasked with finding the poles and zeros, **(1)** is a little weird mathematically, whereas the terms in **(2)** are just **polynomials** that we can factor out and easily find the poles and zeros.
- On the downside, whenever we look at the **z-transform tables**, and by convention, all the inverse transforms are always in terms of negative powers of **z** and so this will not help us with going backwards, unless we multiple everything through by negative powers of **z**.

What is the inverse z-transform?

WHAT IS THE INVERSE z-TRANSFORM?

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz \quad \textbf{(1)}$$

COMPLEX CONTOUR INTEGRAL FOR SOME $|z| = r$ IN THE ROC.

- **How do we undo the z-transform?** Just like in the Fourier domain, the whole point of the **z**-transform method is that we take the original input signals, we multiply them together in the transform domain, and we take the ***inverse transform***. So, we have to know how to go both forwards and backwards.
- The downside is that the inverse **z**-transform is something that involves "**Complex Analysis**", and hence, the computation of the integral **(1)**. The integral sign, that has a circle symbol with an arrow, is used in **complex contour integrals** for some circle, $|\mathbf{z}| = \mathbf{r}$, in the ROC. Here, instead of doing the complex contour integral, we will be looking at the patterns and will be using a combination of **tables** and the general understanding of what the **z**-transform does.
- The table is called **Table of z-Transform Pairs**.

Table of z-transform pairs

Table of z-Transform Pairs

$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$		$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$	ROC
transform	$x[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X(z)$	R_x
time reversal	$x[-n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X(\frac{1}{z})$	$\frac{1}{R_x}$
complex conjugation	$x^*[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X^*(z^*)$	R_x
reversed conjugation	$x^*[-n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X^*(\frac{1}{z^*})$	$\frac{1}{R_x}$
real part	$\Re\mathfrak{e}\{x[n]\}$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	R_x
imaginary part	$\Im\mathfrak{m}\{x[n]\}$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	R_x
time shifting	$x[n-n_0]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$z^{-n_0}X(z)$	R_x
scaling in $\mathcal{Z}$	$a^n x[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X\left(\frac{z}{a}\right)$	$\lvert a\rvert R_x$
downsampling by N	$x[Nn], N \in \mathbb{N}_0$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{N}\sum_{k=0}^{N-1} X\left(W_N^k z^{\frac{1}{N}}\right) \quad W_N = e^{-\frac{j2\omega}{N}}$	R_x
linearity	$ax_1[n] + bx_2[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$aX_1(z) + bX_2(z)$	$R_x \cap R_y$
time multiplication	$x_1[n]x_2[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{2\pi j}\oint X_1(u)X_2\left(\frac{z}{u}\right)u^{-1}du$	$R_x \cap R_y$
frequency convolution	$x_1[n] * x_2[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$X_1(z)X_2(z)$	$R_x \cap R_y$
delta function	$\delta[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	1	$\forall\ z$
shifted delta function	$\delta[n-n_0]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	z^{-n_0}	$\forall\ z$

Table of z-transform pairs

step	$u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| > 1$
	$-u[-n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-1}$	$\|z\| < 1$
ramp	$nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
	$n^2u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
	$-n^2u[-n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| < 1$
	$n^3u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| > 1$
	$-n^3u[-n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\|z\| < 1$
	$(-1)^n$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z+1}$	$\|z\| < 1$
exponential	$a^nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
	$-a^nu[-n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-a}$	$\|z\| < \|a\|$
	$a^{n-1}u[n-1]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1}{z-a}$	$\|z\| > \|a\|$
	$na^nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
	$n^2a^nu[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{az(z+a)}{(z-a)^3}$	$\|z\| > \|a\|$
	$e^{-an}u[n]$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{z-e^{-a}}$	$\|z\| > \|e^{-a}\|$
exp. interval	$\begin{cases} a^n & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$	$\overset{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Table of z-transform pairs

sine	$\sin(\omega_0 n)\, u[n]$	$\stackrel{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z\sin(\omega_0)}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
cosine	$\cos(\omega_0 n)\, u[n]$	$\stackrel{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z-\cos(\omega_0))}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > 1$
	$a^n \sin(\omega_0 n)\, u[n]$	$\stackrel{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{za\sin(\omega_0)}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
	$a^n \cos(\omega_0 n)\, u[n]$	$\stackrel{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z(z-a\cos(\omega_0))}{z^2-2a\cos(\omega_0)z+a^2}$	$\|z\| > a$
differentiation in $\mathcal{Z}$	$nx[n]$	$\stackrel{\mathcal{Z}}{\Longleftrightarrow}$	$-z\frac{dX(z)}{dz}$	R_x
integration in $\mathcal{Z}$	$\frac{x[n]}{n}$	$\stackrel{\mathcal{Z}}{\Longleftrightarrow}$	$-\int_0^z \frac{X(z)}{z}dz$	R_x
	$\frac{\prod_{i=1}^{m}(n-i+1)}{a^m m!} a^m u[n]$	$\stackrel{\mathcal{Z}}{\Longleftrightarrow}$	$\frac{z}{(z-a)^{m+1}}$	

Methods for computing inverse z-transforms; D(z) has real roots: partial fractions

Example:

$$X(z) = \frac{7 - 13z^{-1}}{1 - 2z^{-1} - 3z^{-2}} \qquad |z| > 3$$

Let z^{-1} = B, so 1 - 2B - $3B^2$ = (1-3B)(1+B)

$$(1 - 3z^{-1})(1 + z^{-1})$$

$$= \frac{A}{1 - 3z^{-1}} + \frac{B}{1 + z^{-1}}$$

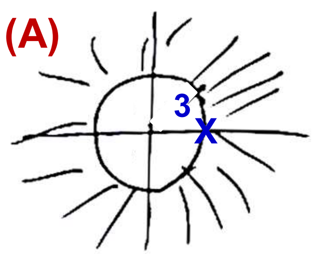
$$= \frac{(A+B) + (A-3B)z^{-1}}{(1-3z^{-1})(1+z^{-1})}$$

$$A + B = 7$$
$$A - 3B = -13$$
$$4B = 20$$
$$B = 5,\ A = 2$$

$$= \frac{2}{1-3z^{-1}} + \frac{5}{1+z^{-1}} \quad \textbf{(1)}$$

$$\boxed{x[n] = 2(3)^n u[n] + 5(-1)^n u[n]} \quad \textbf{(2)}$$

(A)



- Remember that every **z**-transform comes with an additional region of convergence (ROC). When we take things back into the time domain, there is an ambiguity about what that time domain signal could be depending on what the ROC was. We cannot just assume that everything is always right-sided.
- When going from **(1)** to **(2)**, we know from the start that we have got a right-sided signal because of the fact that the ROC looks like the starburst (or sunburst) shown in **(A)**. That means that, at this stage, when we are doing the inverse transform, we choose right-sided signals to make it work.

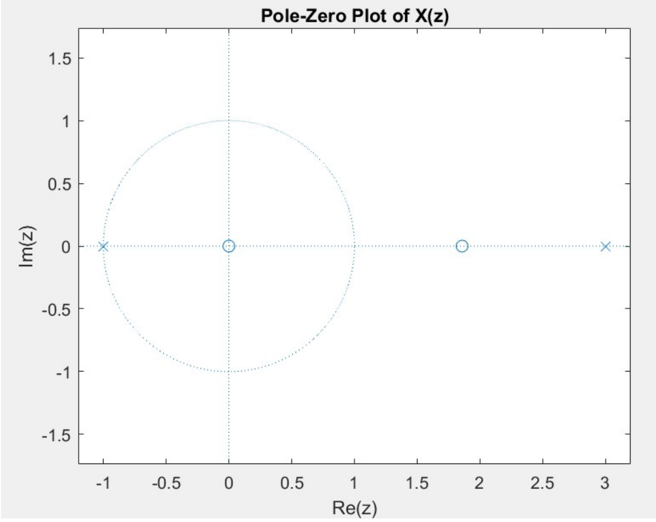
From the table:

$$a^n u[n] \quad \overset{\mathcal{Z}}{\Longleftrightarrow} \quad \frac{z}{z-a} \qquad |z| > |a|$$

Methods for computing inverse z-transforms; D(z) has real roots: partial fractions

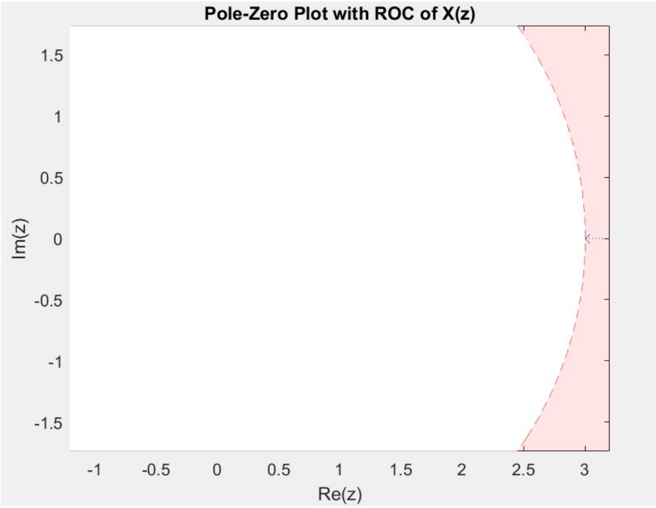
- Here is a MATLAB code to sketch the pole-zero plot:

```
>> % Define the numerator and denominator coefficients of X(z)
num = [7 -13 0];  % Coefficients of 7z^2 - 13z
den = [1 -2 -3];  % Coefficients of z^2 - 2z - 3

% Use zplane to plot the poles and zeros
zplane(num, den);

% Add title and labels
title('Pole-Zero Plot of X(z)');
xlabel('Re(z)');
ylabel('Im(z)');
```



- Here is a MATLAB code to sketch the ROC:

```
% Hold the current plot to add the ROC shading
hold on;

% Define points for drawing the ROC (outside the circle with radius 3)
theta = linspace(0, 2*pi, 100);  % Create angle points for a circle
x = 3 * cos(theta);  % X points (real axis)
y = 3 * sin(theta);  % Y points (imaginary axis)

% Plot the ROC boundary (circle with radius 3)
plot(x, y, 'r--');  % Dashed red circle at |z| = 3

% Fill the area outside the circle to represent the ROC
fill([-10 -10 10 10], [-10 10 10 -10], 'r', 'FaceAlpha', 0.1, 'EdgeColor', 'none');  % Fill entire area
fill(x, y, 'w', 'EdgeColor', 'none');  % Erase the inside of the circle

% Finish the plot
hold off;
```



D(z) has complex roots: matching to cosine/exponentials from table

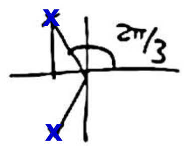
Example:

RIGHT-SIDED SIGNAL

$$X(z) = \frac{3z}{z^2+2z+4} = \frac{3z^{-1}}{1+2z^{-1}+4z^{-2}}$$

$$\frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}j = 2e^{\pm \frac{2\pi}{3}j}$$

$$\underbrace{2}_{r}\, e^{\pm \overbrace{\tfrac{2\pi}{3}}^{\omega_0} j}$$



$$X(z) = \frac{3z^{-1}}{1+\underbrace{2}_{2r\cos\omega_0}z^{-1}+4z^{-2}}$$

$$2 \cdot 2 \cdot \frac{-1}{2} = -2$$

$$r\sin\omega_0 = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$X(z) = \frac{3z^{-1}}{1+2z^{-1}+4z^{-2}} \quad \textbf{(1)}$$

$$r^n \sin\omega_0 n\, u[n] \longleftrightarrow \frac{r\sin\omega_0\, z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} \quad |z|>r \quad \textbf{(2)}$$

- We are told that **X(z)** is a right-sided signal. This is another way of saying that the ROC is all those **z**'s outside of the circle of radius $\mathbf{2}$ or $|\mathbf{z}| > \mathbf{2}$.
- We use **Pattern Matching** between **(1)** and **(2)** in order to find the inverse **z**-transform or **x[n]**.

$$X(z) = \frac{3}{\sqrt{3}} \frac{\sqrt{3}z^{-1}}{1+2z^{-1}+4z^{-2}} = \sqrt{3}\frac{\sqrt{3}z^{-1}}{1+2z^{-1}+4z^{-2}}$$

$$= \sqrt{3} \cdot \left(\frac{r\sin\omega_0\, z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} \right)\Bigg|_{\substack{r=2 \\ \omega_0=\frac{2\pi}{3}}} \longrightarrow \boxed{x[n] = \sqrt{3}\left(2^n \sin\left(\tfrac{2\pi}{3}n\right) u[n]\right)}$$

From the table:

$$a^n \sin(\omega_0 n)\, u[n] \overset{Z}{\Longleftrightarrow} \frac{za\sin(\omega_0)}{z^2-2a\cos(\omega_0)z+a^2} \quad |z|>a$$

D(z) has complex roots: sine and cosine parts

Example:

$$X(z) = \frac{3z+5}{z^2+2z+4} \quad |z| > 2 \longrightarrow$$

$$x[n] = A r^n \sin\omega_0 n \; u[n] + B r^n \cos\omega_0 n \; u[n] \longrightarrow$$

$$\boxed{x[n] = \frac{1}{3} \cdot 2^{n-2} \left[7\sqrt{3}\sin\left(\frac{2\pi n}{3}\right) - 15\cos\left(\frac{2\pi n}{3}\right)\right]}$$

Region of Convergence (ROC) for a causal system: $|z| > 2.$

- This example is similar to the previous example, except now we have added a term (here, **5**) to the numerator.
- In this case, we can show that the inverse **z**-transform will be a combination of sine and cosine.

X(z) is not a rational function: power series

Example:

$$X(z) = 3z^{-2} + 5z^{-1} - \frac{1}{2} + 3z^{3} \qquad \begin{array}{c}\text{ROC} \\ 0<|z|<\infty\end{array} \qquad \textbf{(1)}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad \textbf{(2)}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots + x[-1]z + x[-2]z^{2} + x[-3]z^{3} \cdots \longrightarrow$$

$$\boxed{x[n] = 3\delta[n-2] + 5\delta[n-1] - \frac{1}{2}\delta[n] + 3\delta[n+3]}$$

From the table:

$$\delta[n-n_0] \quad \overset{z}{\Longleftrightarrow} \quad z^{-n_0} \qquad \forall z$$

- If we have a simple **X(z)** in the form of a polynomial of powers of **z**, we can immediately figure out what the **x[n]** would have to be just **by inspection**.
- In this example, for **X(z)** given in **(1)**, the ROC is almost the entire **z**-plane. Here, we can just look at the definition of the **z**-transom, shown in **(2)**. It is a series in the powers of **z**'s. **So, X(z) is just like a series where the coefficients of z^{-n} are the values of x[n]**. In this case, for the equation of **x[n]**, we just write down the coefficients in the **X(z)** equation and multiply each term by the corresponding delta function.
- **Conclusion:** For simple **X(z)** that are in the form of polynomials (not rational functions), we can find **x[n]** just by inspection, without having to do any sort of manipulation.

X(z) is not a rational function: power series

Example:

$$X(z) = e^{z} \qquad |z| < \infty$$

POWER SERIES

$$= 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$x[0] \quad x[-1] \quad x[-2] \quad \ldots$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^{n}$$

$$\boxed{x[n] = \begin{cases} 0 & n \geq 1 \\ \frac{1}{(-n)!} & n \leq 0 \end{cases}}$$

- In this example, we use **Taylor's series expansion,** which is a form of power series. Here, we can see that **x[n]** is a left-sided signal, because we have **x[0]**, **x[-1]**, **x[-2]**, etc.

Long division

Example:

- In this example, we use **Long Division** and try to make **X(z)** look like a **power series**.

LONG DIVISION

$$X(z) = \frac{1 + 2z^{-1}}{1 + z^{-1}}$$

$$= \begin{array}{r|l} & 1 + z^{-1} - z^{-2} + z^{3} - z^{4} \dots \\ \hline 1 + z^{-1} & 1 + 2z^{-1} \\ & \underline{1 + z^{-1}} \\ & \qquad z^{-1} \\ & \qquad \underline{z^{-1} + z^{-2}} \\ & \qquad\qquad -z^{-2} \end{array}$$

$$\longrightarrow \boxed{X(z) = 1 + z^{-1} - z^{-2} + z^{-3} - z^{-4} + z^{-5} - \dots} \qquad |z| > 1$$

Z-transform properties; Time shift

1) LINEARITY $\quad a x_1[n] + b x_2[n] \leftrightarrow a X_1(z) + b X_2(z)$

2) TIME SHIFT: $\quad x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$

- **Property #1 (Linearity):** If we have a combination of inputs, when we do the **z**-transform, we get the expected combination of outputs.
- **Property #2 (Time Shift):** If we have a shift in time, the corresponding thing we get is a multiplication by some power of **z**, or z^{-n_0}. This is like the time shift in one domain corresponding to a phase shift in the other domain for frequency domain methods. But in this case, z is playing the role of a phase shift. In the Fourier world, this was like $e^{j\omega}$. But, in theory, here this z could be any magnitude. So, we have to be a little more careful.
- The term z^{-n_0} may also have the effect of introducing some extra poles and zeros into the ROC. Because if n_0 number is positive, by multiply z^{-n_0}, we are putting extra poles of **0**. And, if n_0 number is negative, we are putting extra zeros in. So, we just have to be a little careful when doing the ROC.

Time shift

Example:

$$X(z) = \frac{1 + 2z^{-1}}{1 + z^{-1}}$$

$$= \frac{1}{1+z^{-1}} + \overbrace{2z^{-1}\frac{1}{1+z^{-1}}}^{*}$$

$$\downarrow \qquad \downarrow$$

$$\boxed{x[n] = (-1)^n u[n] + 2(-1)^{n-1} u[n-1]}$$

- This is the problem that we just did by long division. An easier way to do this is to use the **Time Shift Property**.
- The term z^{-1} in * represents a time shift of whatever the transfer of the term it is multiply into is. Here, we originally have **$(-1)^n.u[n]$**, and when we delay it by **-1**, we get the inverse **z**-transform of the term * as **$(-1)^{n-1}.u[n-1]$**.
- **Conclusion:** Usually, for the **z**-transforms in the tables to work, we want the degree of the polynomial of **z** in the denominator to be higher than the degree of the polynomial of **z** in the numerator. If that is not true, and there is big powers of **z** up in the numerator, we need to take them out and use them as ***delays***.

Scaling, time reversal, convolution, differentiation, initial value theorem

3) SCALING $a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right)$

4) TIME-REVERSAL $x[-n] \leftrightarrow X\left(\frac{1}{z}\right)$

5) CONVOLUTION $x[n] * h[n] \leftrightarrow X(z)H(z)$

6) DIFFERENTIATION $n\,x[n] \leftrightarrow -z\frac{dX(z)}{dz}$

7) INITIAL VALUE THEOREM:

$$x[0] = \lim_{z\to\infty} X(z)$$

- **Property #3 (Scaling):** If we multiply **x[n]** by some exponential, it will have the form of **X(z/a)** in the **z**-domain. This is related to how the ROC works. If we were to multiply the input signal by some exponential, that is going to have an effect on the location of the poles and zeros, and hence, the ROC.
- **Property #4 (Time Reversal):** If we flip the signal (change "**n**" to "**–n**" in **x[n]**), we will have an inversion of the poles and zeros across the unit circle.
- **Property #5 (Convolution):** If we have **x[n]** convolved with **x[n]** in the time domain, then in the frequency domain, we have the product of the corresponding **z**-transforms.
- **Property #6 (Differentiation):** If we want to know what the derivative of **X(z)** is in the **z**-domain, it corresponds to multiplying **x[n]** by an "**n**" in the time domain.
- **Property #7 (Initial Value Theorem):** It says that in the time domain, the DC value of the signal (i.e., **x[n = 0]**) is the limit of **X(z)** in the **z**-domain when **z** approaches infinity.

The poles and zeros of rational H(z)

$$\underset{\text{TRANSFER FUNCTION}}{H(z)} = \frac{N(z)}{D(z)} \quad \leftarrow \text{POLYNOMIALS} \quad \textbf{(1)}$$

$$N(z) = 0 \Rightarrow H(z) = 0 \Rightarrow \text{ZEROS}$$

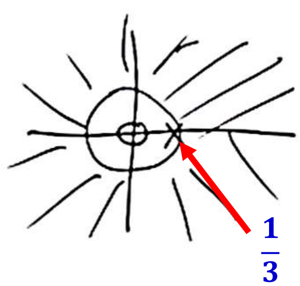
$$D(z) = 0 \Rightarrow H(z) = \infty \Rightarrow \text{POLES.}$$

Example:

$$h[n] = \left(\tfrac{1}{3}\right)^n u[n] \quad \textbf{(2)} \qquad H(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \ \textbf{(3)} \qquad |z| > \tfrac{1}{3} \ \textbf{(4)}$$

$$= \frac{z}{z - \frac{1}{3}}$$

(A)



- **Why do we care about the z-transform in the first place? How does it help us? What does it tell us?** In most of the situations in the real world systems, we are usually going to talk about systems where the transfer functions are some polynomial over some other polynomials **(1)**.
- When **N(z) = 0**, we have the numerator equal to **0**. That means that the whole transfer function is equal to **0**. We call those **zeros**.
- When **D(z) = 0**, we have the denominator equal to **0**. That means that the transfer function blows up. We call those **poles**.
- For example, if we have **(2)** as the impulse response of our system, then our transfer function **H(z)** and the ROC are **(3)** and **(4)**, respectively. So, the ROC is going to be everything outside of the circle with a radius of **1/3**, as shown in **(A)**. Here, **h[n]** is a right-sided signal.

The poles and zeros of rational H(z)

$$H(z) = \frac{N(z)}{D(z)} \quad \begin{matrix}\leftarrow \\ \leftarrow\end{matrix} \quad \text{POLYNOMIALS} \qquad \textbf{(1)}$$

TRANSFER FUNCTION

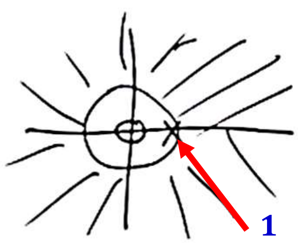
$$N(z) = 0 \Rightarrow H(z) = 0 \Rightarrow \text{ZEROS}$$
$$D(z) = 0 \Rightarrow H(z) = \infty \Rightarrow \text{POLES.}$$

Example:

$$h[n] = \left(\tfrac{1}{3}\right)^n u[n] \ \textbf{(2)} \qquad H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \ \textbf{(3)} \qquad |z| > \tfrac{1}{3} \ \textbf{(4)}$$

$$= \frac{z}{z-\frac{1}{3}}$$

(A)



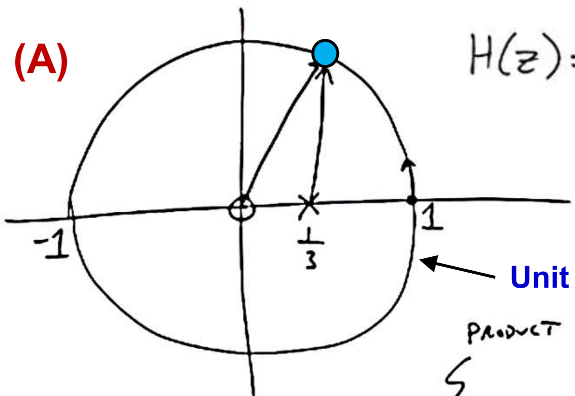
$\frac{1}{3}$

- By examining the poles and zeros in the Laplace transform, we can make some inferences as about what is happening in the frequency domain.
- The **z**-transform agrees with the Fourier transform on the unit circle when the ROC includes the unit circle. So, the unit circle is a critical object for the discrete time analysis in the same way that the **jω**-axis was a critical place for continuous time analysis.
- What we care about is what is happening on the unit circle. This is because if we evaluate the **z**-transform going around the unit circle and that if we unwrap it and plot it as if it was going from **-π** to **+π** we get the picture of frequency response.

Sketching the magnitude response using geometric intuition about poles and zeros

Example:

(A)



Unit Circle

$$H(z) = \frac{z}{z-\frac{1}{3}}$$

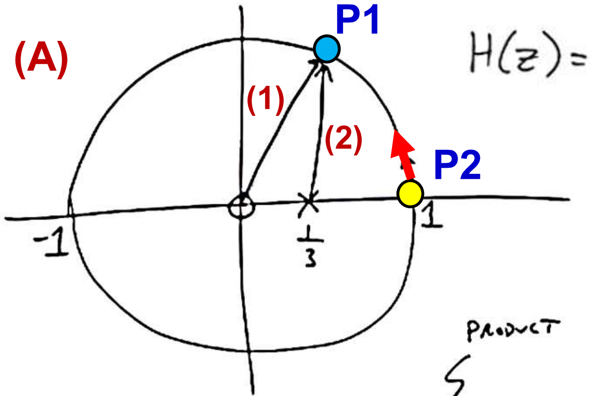
(1)

$$\left|H(e^{j\omega_0})\right| = \left|\frac{N(e^{j\omega_0})}{D(e^{j\omega_0})}\right| = \frac{\overset{\text{PRODUCT}}{\prod}\left(\text{LENGTH OF VECTOR FROM EACH O TO } e^{j\omega_0}\right)}{\prod\left(\text{LENGTH OF VECTOR FROM EACH X TO } e^{j\omega_0}\right)}$$

- **Geometric Intuition: How do we infer what is going on in the frequency domain using pole-zero plot?** Let us draw the pole and zero plot along with the unit circle. How can we look at this plot and make some judgments about what is happening in the frequency domain?
- What we want to do is to evaluate the **z**-transform as we move around the unit circle. Let us start off by the magnitude of **H** evaluated at some point on the unit circle, i.e., $|H(e^{j\omega_0})|$. If we are able to write this magnitude in terms of a numerator polynomial over a denominator polynomial, fundamentally, what we get in the numerator is the ***product*** of the length of the vector from each zero ("**O**") to $e^{j\omega_0}$ over the product of the length of the vector from each pole ("**X**") to $e^{j\omega_0}$. This is shown in equation **(1)**. So, **what do we mean by that?**

Sketching the magnitude response using geometric intuition about poles and zeros

(A)

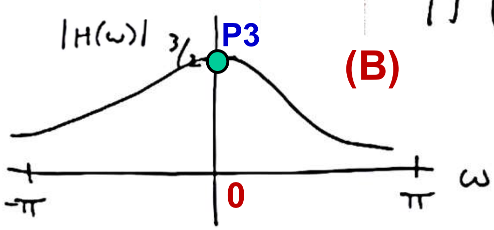


$$H(z) = \frac{z}{z - \frac{1}{3}}$$

(*)

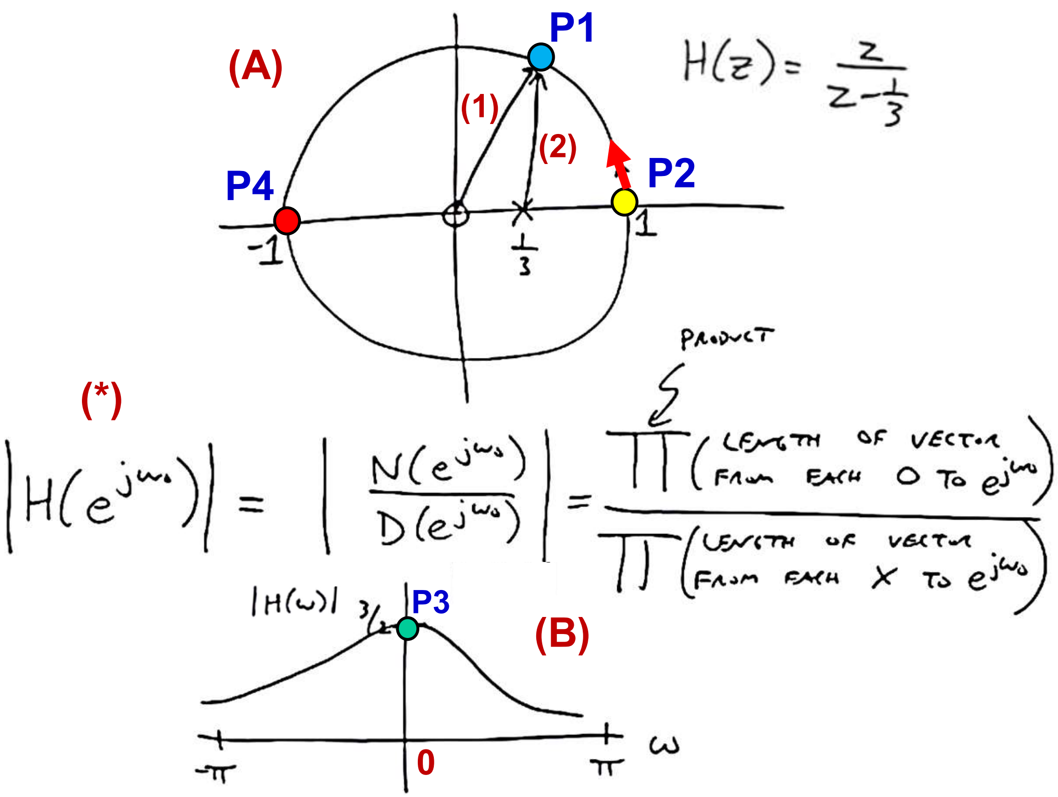
$$\left|H(e^{j\omega_0})\right| = \left|\frac{N(e^{j\omega_0})}{D(e^{j\omega_0})}\right| = \frac{\overbrace{\prod}^{\text{PRODUCT}}\left(\text{LENGTH OF VECTOR FROM EACH O TO } e^{j\omega_0}\right)}{\prod\left(\text{LENGTH OF VECTOR FROM EACH X TO } e^{j\omega_0}\right)}$$

(B)



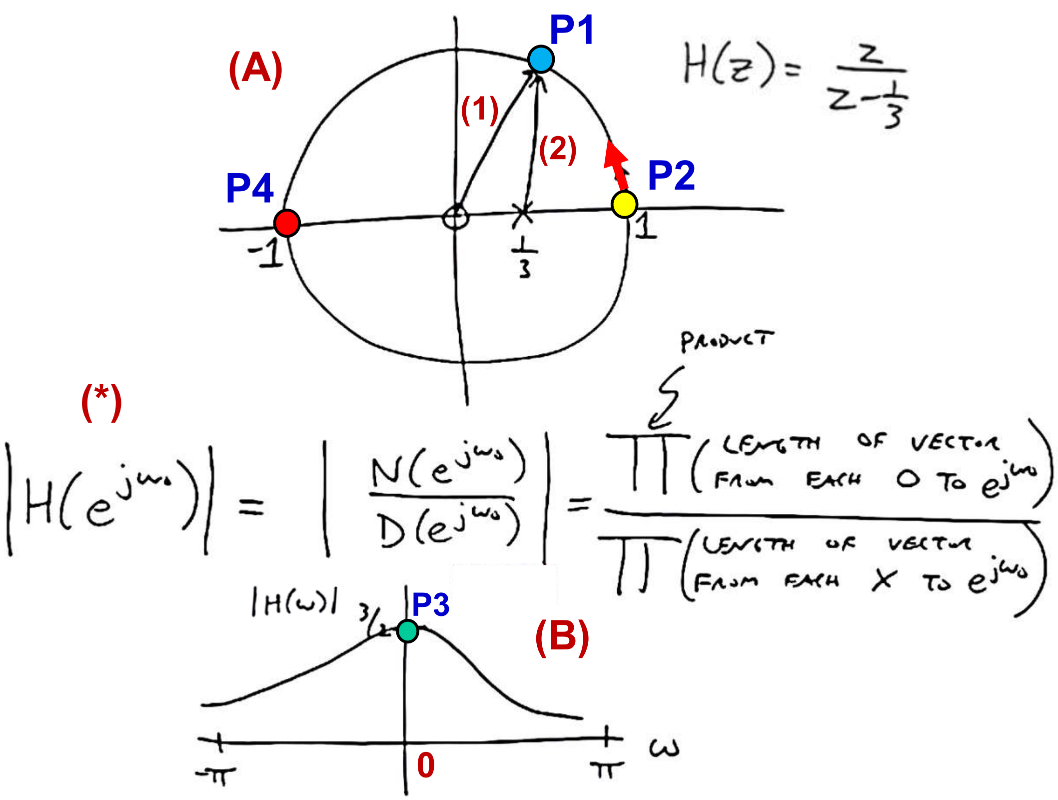
- By referring to **(A)**, the interpretation of **(*)** is like saying if we want to know what is going on with the Fourier transform at point **P1**, we look at the length of vector **(1)**, over the length of vector **(2)**.
- Based on the above, we can make a **crude plot** of the magnitude of the frequency response $|\mathbf{H}(\boldsymbol{\omega})|$ vs. frequency, $\boldsymbol{\omega}$. This is shown in **(B)**. When **ω = 0**, i.e., point **P2** in **(A)**, the length of the vector from zero ("**O**") to **P2** is **1** and the length from pole ("**X**") to **P2** is **2/3**. This corresponds to **P3** in Graph **(B)**, and is equal to **3/2**.
- Let us see what happens when we move our point around the unit circle. Moving counter-clockwise means that the length of vector **(2)** is getting longer, but their ratio is getting smaller. For example, at point **P1**, the vectors are a lot closer in size, than when they were at **P2**. So, as we move this point around the unit circle, the ratio of the vectors is going to get smaller and smaller. Now, we can roughly sketch out graph **(B)**. We see that it is going to dip off as we move away from the point where **ω = 0**. The final graph is shown in **(B)**.

The fabric analogy



- The way we would like to think about the **z**-transform and the pole-zero plot is as follows:
- **Analogy:** Imagine that there is a ***billowy fabric*** that is blowing up from the plane of this page, and the fabric has an inclination to push up. When we have a **zero**, that is like we **nailed** the fabric down to the ground. And, when we have a **pole**, it is like we have a **tent pole** that is pushing the fabric up. So, as we travel around the unit circle, we want to think about what is the fabric inclined to do at a certain point on the unit circle. Point **P2** is the closest point to the pole. And, so this is the place where that tent pole is going to have the most effect.

The fabric analogy



- As we move away counter-clockwise from the tent point starting from **P2**, vector **(2)** is getting larger. In this case, the distance from zero ("**O**") for any point on the circle is always the same and is equal to **1**. So, in this case, we only need to consider how far away we are from the pole.
- At **P2**, we are the closest to pole that we are ever going to get. And now, as we move around, our distance to the pole is increasing. ***The influence of the pole*** at point **P4** should be as low as we can get. Because, number one, we are far from the pole, and number two, the zero has more of an effect.
- **Conclusion:** The idea is that we draw the pole-zero plot and then we travel around the unit circle. Then, we think, how these poles and zeros are affecting the position we are on the unit circle. Following that, we can roughly sketch the magnitude of the frequency response $|\mathbf{H}(\boldsymbol{\omega})|$ vs. frequency, $\boldsymbol{\omega}$.

Complex-pole example

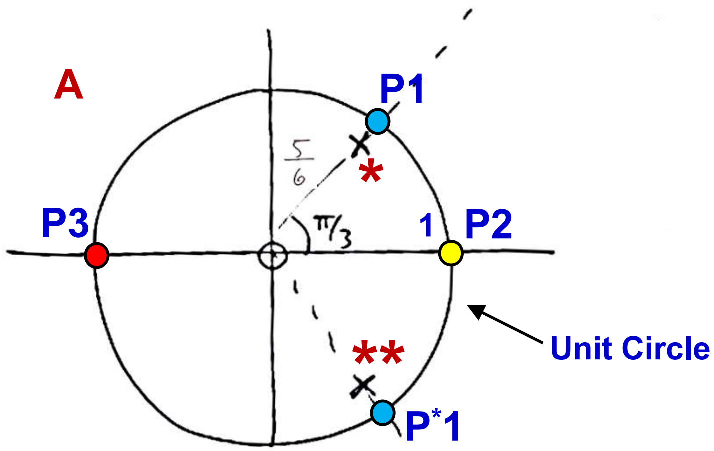
Example:

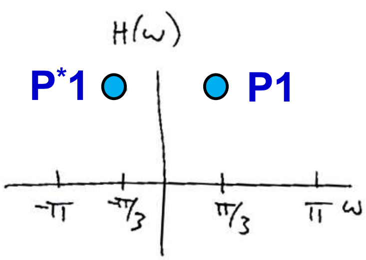
$$h[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3}n\right) u[n]$$

$$a^n \sin(\omega_0 n)\, u[n] \quad \overset{Z}{\Longleftrightarrow} \quad \frac{za\sin(\omega_0)}{z^2 - 2a\cos(\omega_0)z + a^2} \quad |z| > a$$

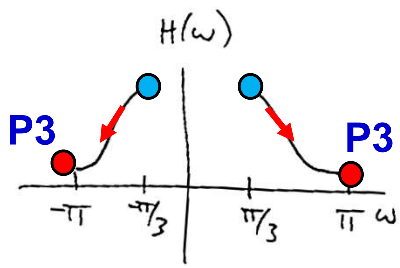
$$H(z) = \frac{\frac{5}{6}\sin\frac{\pi}{3}\, z^{-1}}{1 - \frac{10}{6}\cos\frac{\pi}{3}\, z^{-1} + \left(\frac{5}{6}\right)^2 z^{-2}} \qquad \textbf{(1)}$$

A





B



C

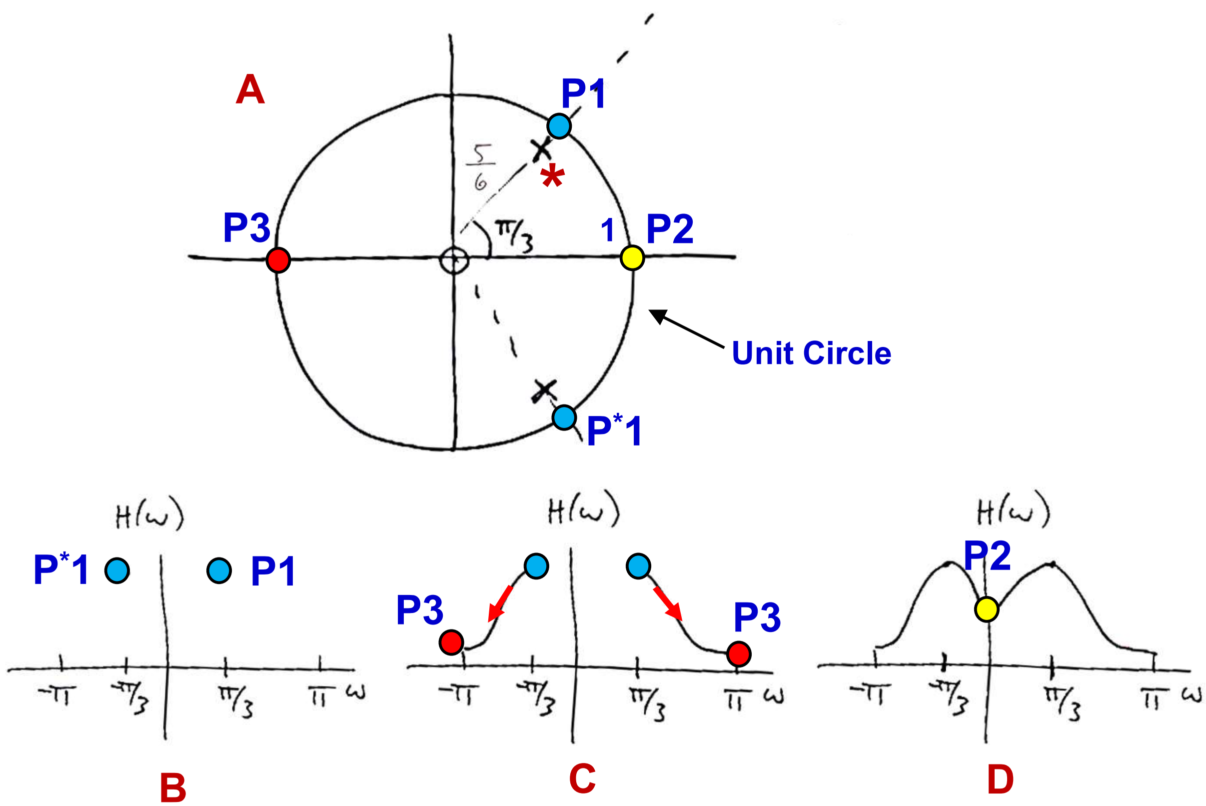
- In **(1)**, **H(z)** is the frequency response, **5/6** is the radius, and **π/3** is the angle. We could already read off where the poles are going to be here. The pole-zero plot looks like **A**. It turns out, we will have a zero ("**O**") at **0** (the origin) and poles with an angle of **π/3** (on the dashed-line) with a magnitude of **5/6**.
- **If we think about this as a filter, what does the frequency response look like?** So, we are thinking about where the influence of these poles is felt. When we are at points **P1** or **P*1**, the **tent poles** (shown at points * and **) are really sticking up. And, that is going to be a ***local maximum*** of the frequency response, **H(ω)**, shown in **B**. So, the frequency response at **+π/3** or **-π/3** is going to be highest. As we travel away counter-clockwise from these poles over towards the high frequencies, i.e., towards **P3**, the influence of both these poles is going to get a lot smaller. Eventually, we are getting a dip down in the direction of **red** arrows in **C**.

Complex-pole example

Example:

$$h[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3}n\right) u[n]$$

$$H(z) = \frac{\frac{5}{6}\sin\frac{\pi}{3}\, z^{-1}}{1 - \frac{10}{6}\cos\frac{\pi}{3}\, z^{-1} + \left(\frac{5}{6}\right)^2 z^{-2}} \quad \textbf{(1)}$$

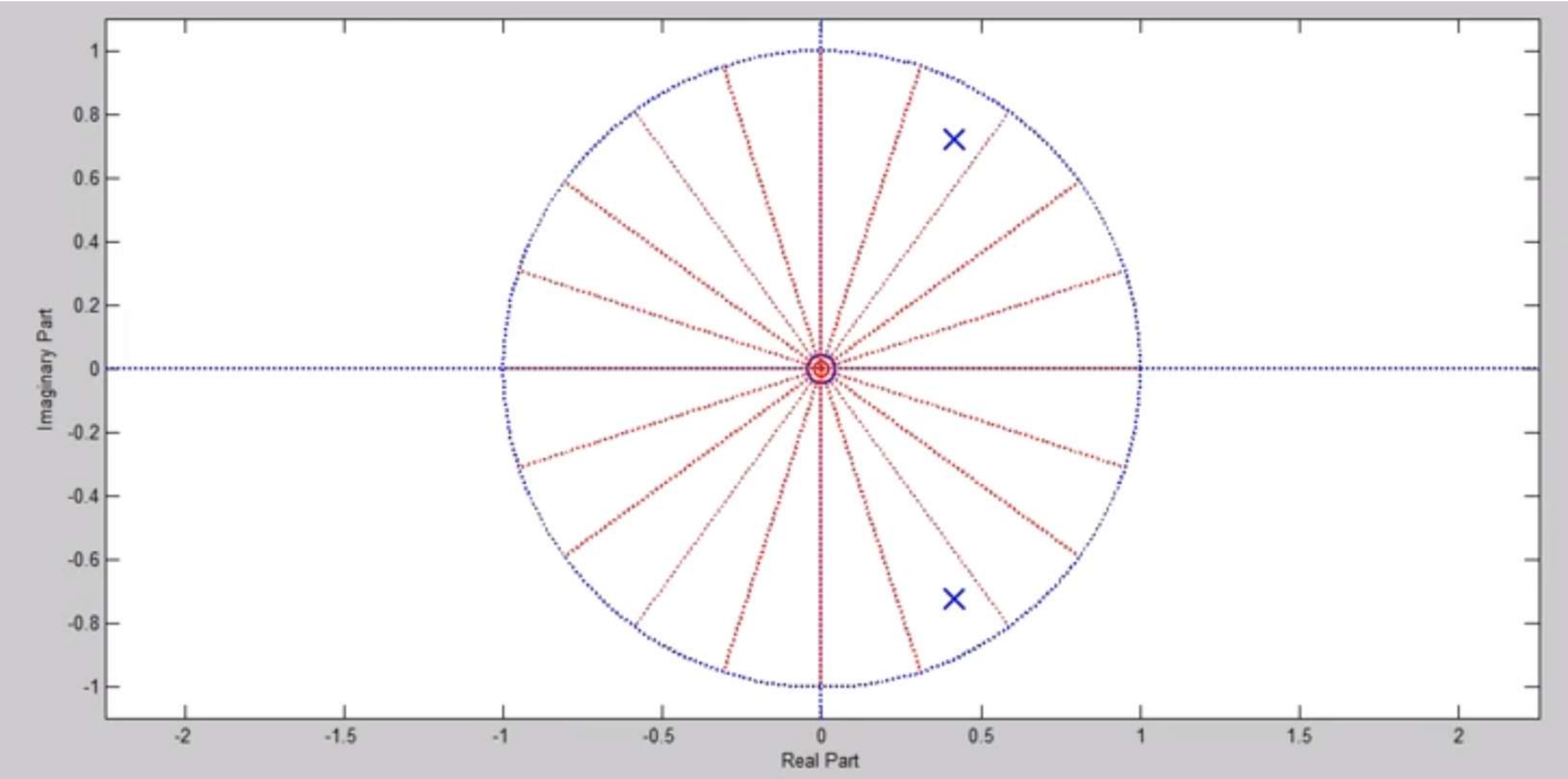


- Now, let us move on the unit circle again but this time clockwise. If we start at **P1** and go towards **P2**, which corresponds to **ω = 0**, the pole's influence (the pole shown by *) is going to get smaller. But it is not going to be as small at **P2** as it is at **P3**. This is because at **P2**, we are still kind of pretty close to the pole located at *. The final **H(ω)** is shown in **D**.
- **Conclusion:** The final **H(ω)** looks like a ***crude bandpass filter*** where the region that we are passing through is closest to the poles.

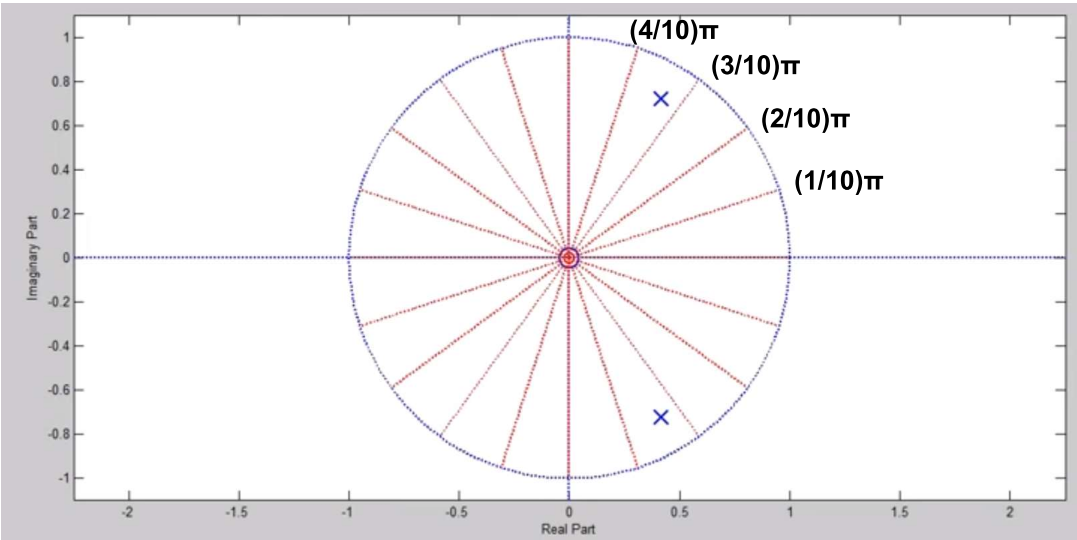
MATLAB demonstrations of moving poles and zeros

- Let us see the same example in MATLAB:

```
>> r = 5/6; b = [0 sin(pi/3)*r]; a = [1 -2*r*cos(pi/3) r^2];
>> figure(1); zplane(b,a)
```



MATLAB demonstrations of moving poles and zeros

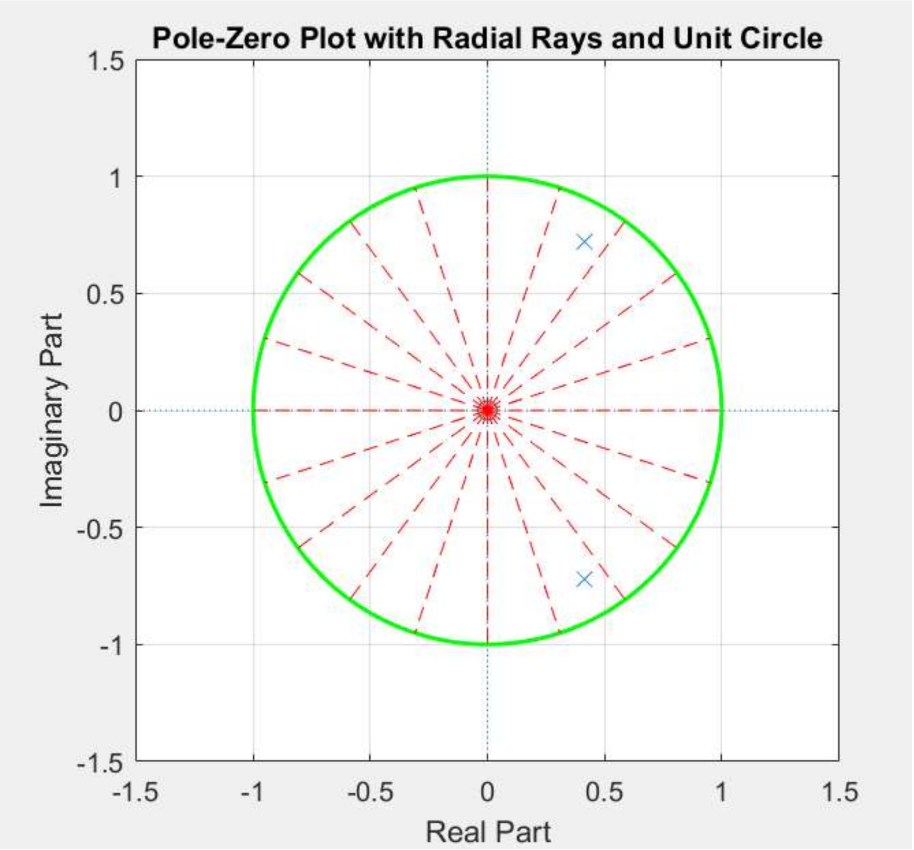


A

- This is exactly the same as the picture we showed before, just looking better! We have marked off the red dotted lines in Graph **A**, which are basically, **(1/10)π**, **(2/10)π**, **(3/10)π**, **(4/10)π**, etc. So, we can see that these poles occur just over **(3/10)π** because we know that "**X**" is at **π/3**.
- By default, the **unit circle** is also drawn on the plot for reference. The unit circle represents the boundary between the stable and unstable regions of the **z**-plane.

MATLAB demonstrations of moving poles and zeros

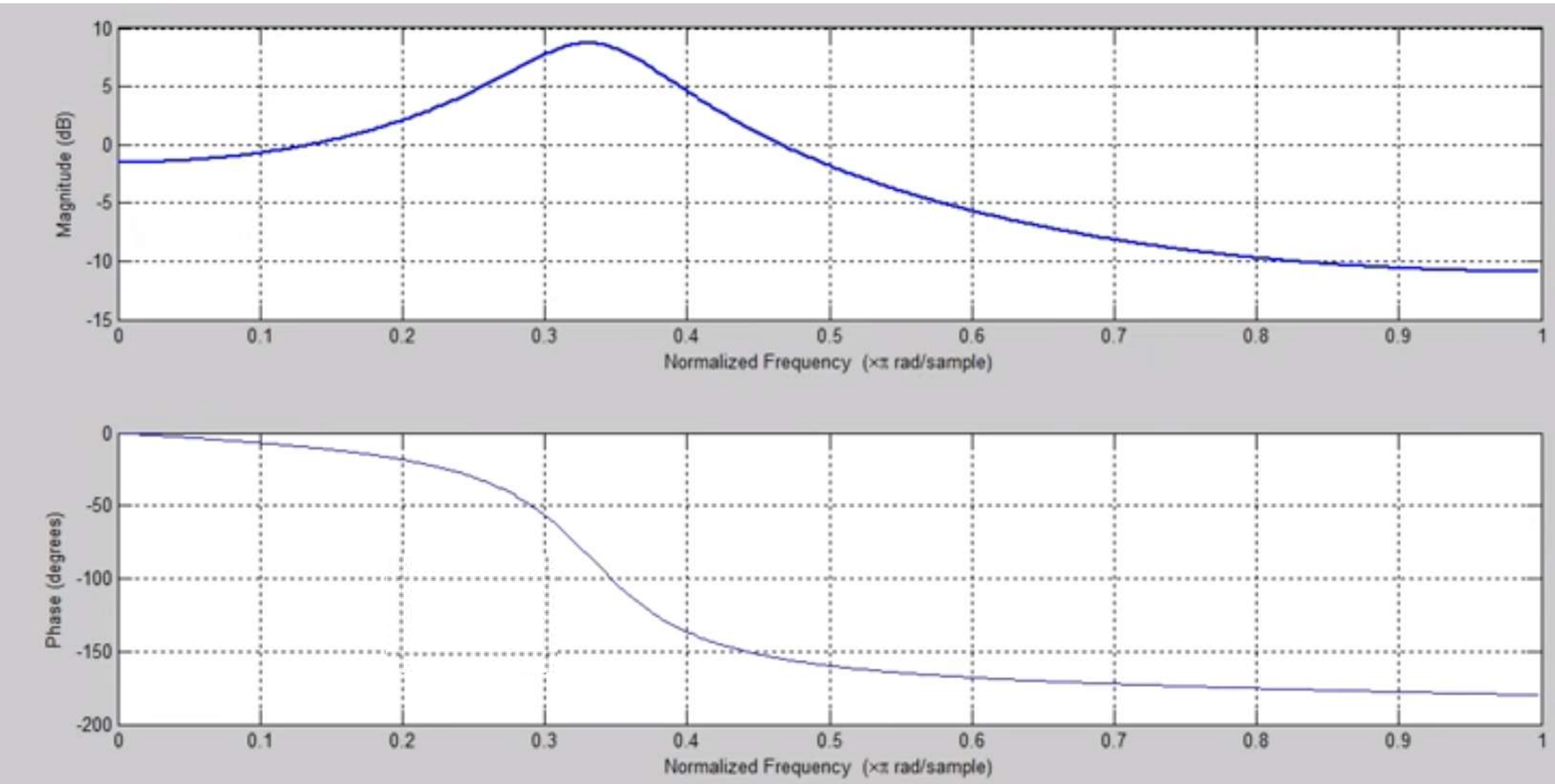
```
>> % Define the parameters
r = 5/6; % Given radius
b = [0 sin(pi/3)*r]; % Zero coefficients
a = [1 -2*r*cos(pi/3) r^2]; % Pole coefficients

% Create a figure for the pole-zero plot
figure;

% Generate the pole-zero plot
zplane(b, a);
hold on;

% Draw the unit circle as a dark green solid line
theta_unit_circle = linspace(0, 2*pi, 100);
x_unit_circle = cos(theta_unit_circle);
y_unit_circle = sin(theta_unit_circle);
plot(x_unit_circle, y_unit_circle, 'g-', 'LineWidth', 1.5); % Dark green solid line

% Define the angles for the rays (20 rays around the circle)
theta_rays = (0:1:20) * (pi/10); % Angles from 0 to 2*pi in increments of (1/10)π

% Calculate the x and y coordinates for the rays ending at the unit circle
x_rays = cos(theta_rays);
y_rays = sin(theta_rays);

% Plot the rays as red dashed lines from the origin to the unit circle
for k = 1:length(theta_rays)
    plot([0, x_rays(k)], [0, y_rays(k)], 'r--'); % Draw red dashed lines
end

% Add title and labels
title('Pole-Zero Plot with Radial Rays and Unit Circle');
xlabel('Real Part');
ylabel('Imaginary Part');

% Set axis limits
axis equal; % Keep aspect ratio equal
xlim([-1.5 1.5]);
ylim([-1.5 1.5]);
grid on; % Add grid for better visualization
hold off;
```



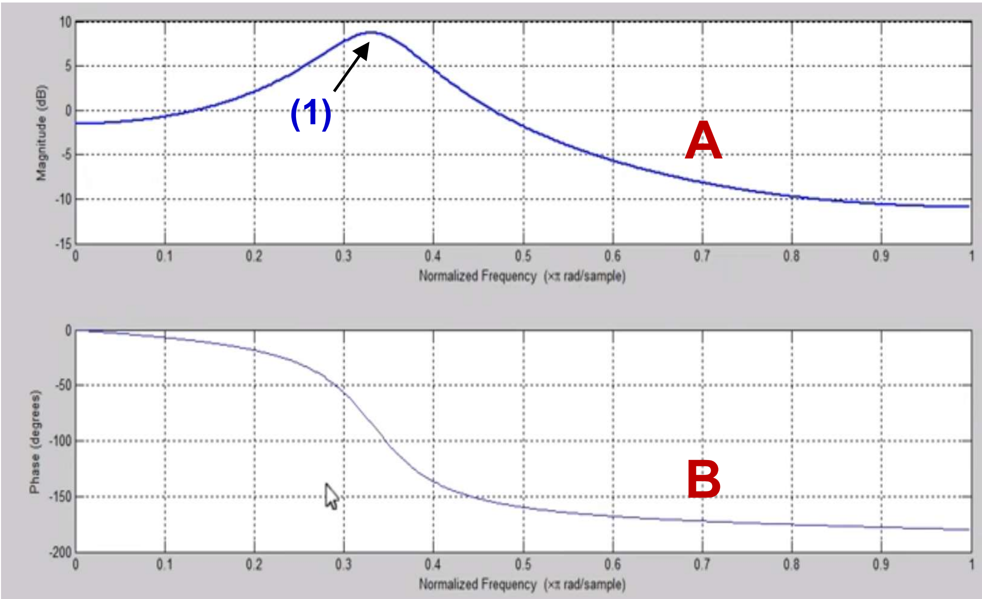
Reading off magnitude response from more complicated examples

- Let us look at the frequency response in MATLAB:

```
>> freqz(b,a)
```



Reading off magnitude response from more complicated examples



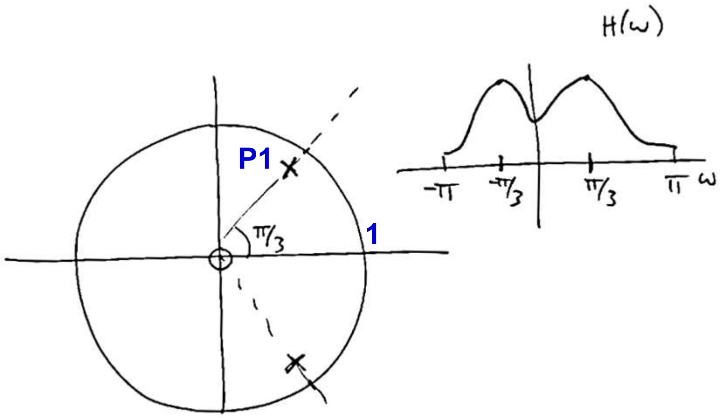
- In MATLAB, there is a command called **freqz (b,a)** where we pass in the coefficients and it will sketch the frequency response.
- Graph **A** (the **magnitude plot**) is exactly what we just drew manually. This is also called **DB-plot** (or decibel plot). Here, we see the **peak** that is just around **π/3** at **(1)**. Note that the magnitude is in decibels (dB).
- We can also look at the **phase plot** (Graph **B**). The phase plot is related to the angle between the poles and the zeros. But, here, we want to talk mostly about the magnitude for the moment.

- The **zplane(b,a)** command is a valuable tool for visualizing the poles and zeros of a digital filter on the complex plane (**z**-plane). It helps analyze the filter's behavior by revealing the locations of its poles and zeros, which influence stability, frequency response, and other characteristics. The **zplane** command takes two mandatory arguments:
  - **b:** This is a vector representing the numerator coefficients of the filter's transfer function in polynomial form. The order of the coefficients corresponds to the powers of **z** in the numerator. For example, **b = [2 1]** represents a numerator of **2z + 1**.
  - **a:** This is a vector representing the denominator coefficients of the transfer function in polynomial form. Similar to **b**, the order of the coefficients corresponds to the powers of **z** in the denominator. For example, **a = [1 -0.5]** represents a denominator of **z - 0.5**.

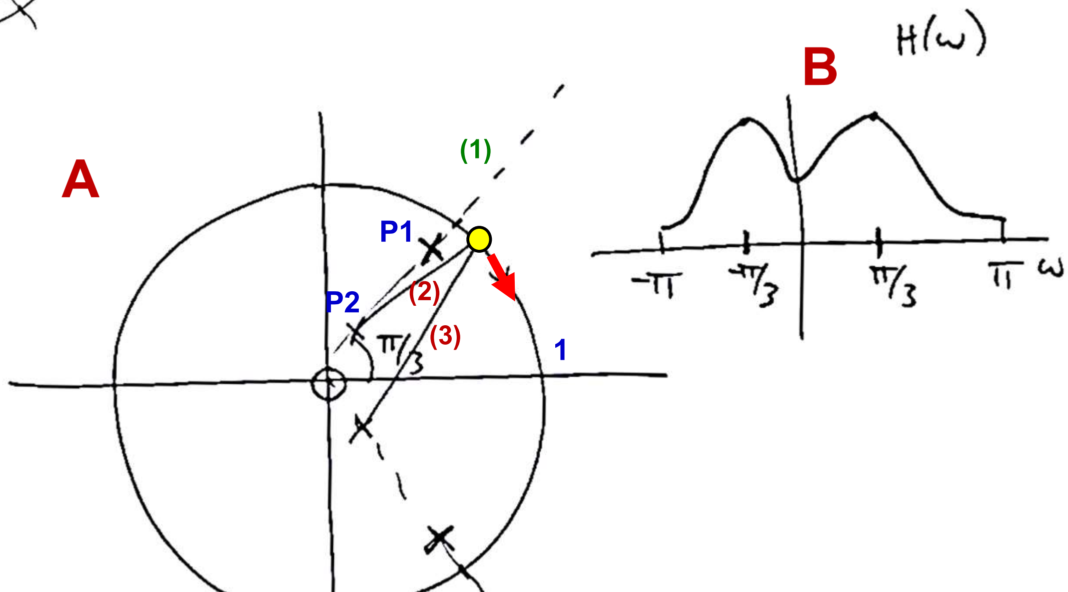
Reading off magnitude response from more complicated examples

$$h[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3}n\right) u[n]$$

$$H(z) = \frac{\frac{5}{6}\sin\frac{\pi}{3}\, z^{-1}}{1 - \frac{10}{6}\cos\frac{\pi}{3}\, z^{-1} + \left(\frac{5}{6}\right)^2 z^{-2}}$$



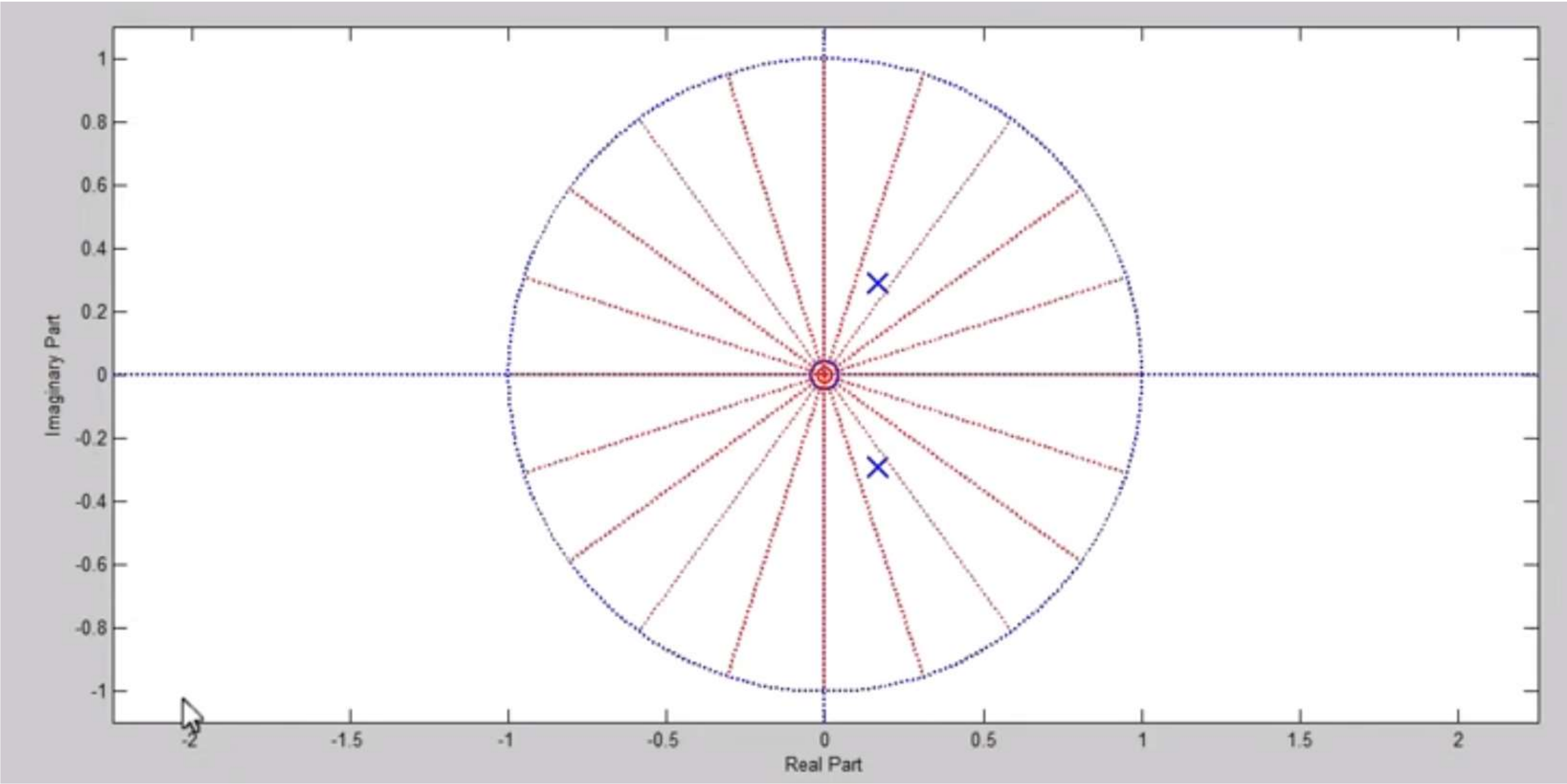
- **What happens if we were to change the positions of these poles?** Let us keep the same angle but move the poles. Here, let us move the pole that is initially at point **P1** (on the dashed-line, **(1)**) towards the origin. So, we are keeping the same angle. What we would expect would be, number one, the overall influence of these **tent poles** would be less because we are moving them ***further interior to the unit circle***. And, number two, as it turns out, the local maxima positions in graph **B** also will change slightly. Because if we move the poles really far inwards, then maybe it is not true anymore that the product of vector **(2)** and vector **(3)** in graph **A** is the local maximum. It could be that in graph **B**, the local maximum moves a little bit closer to **ω = 0**. In graph **A**, it means we are moving in the direction of the **red** arrow. Let us just check this out in MATLAB.



Reading off magnitude response from more complicated examples

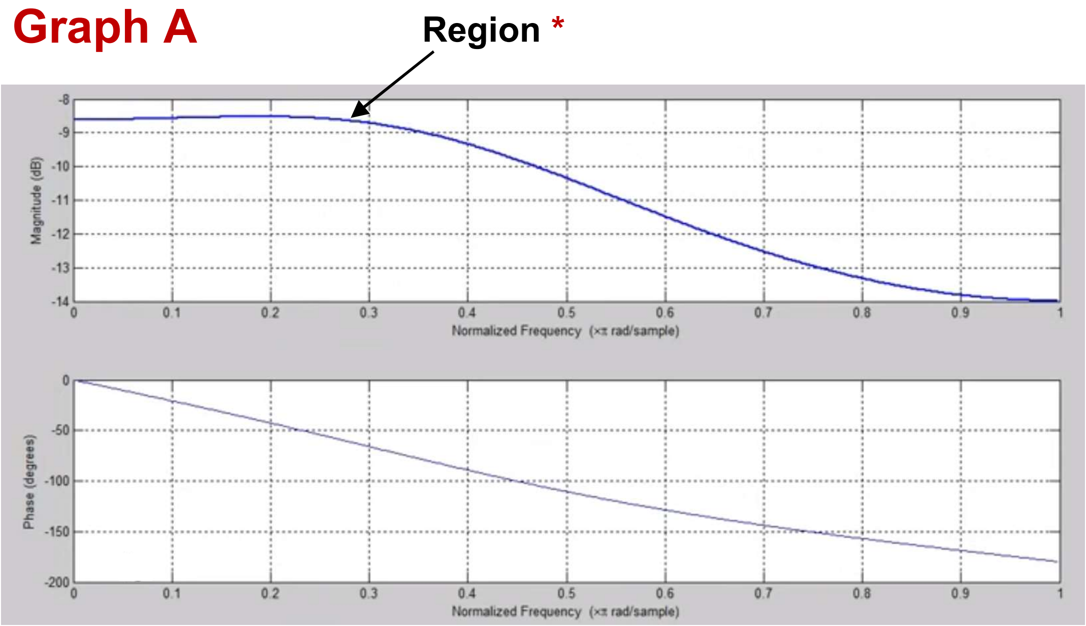
- Let us suppose that instead of using **r = 5/6**, we use **r = 1/3**. Now the poles are closer to the origin.

```
>> r = 1/3; b = [0 sin(pi/3)*r]; a = [1 -2*r*cos(pi/3) r^2]; zplane(b,a)
```



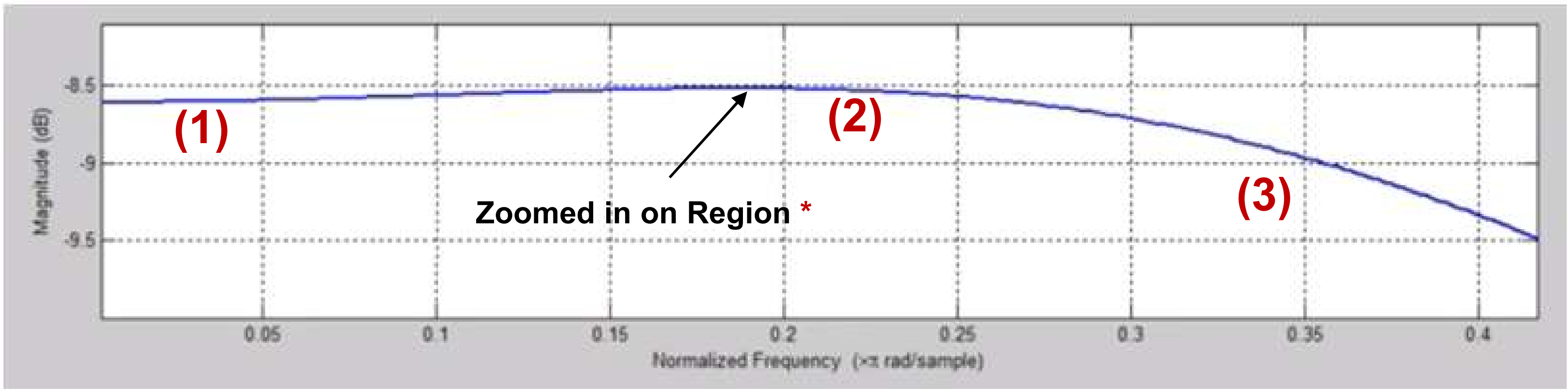
Reading off magnitude response from more complicated examples

Graph A



- At the frequency domain (**Graph A**), number one, this is not a very dramatic filter. We see that there is still some bandpass characteristic. If we were to zoom in on **Region** *, we would see that it does start low **(1)**, gets a little bit higher **(2)**, and then dips **(3)**. So there is a very, very weak bandpass effect here, but the position of where that bandpass is happening (in **H(ω)** vs. **ω** plot) is no longer at angle **π/3**. It has moved a little bit closer to angle **0**, because those poles are so far away from the unit circle.

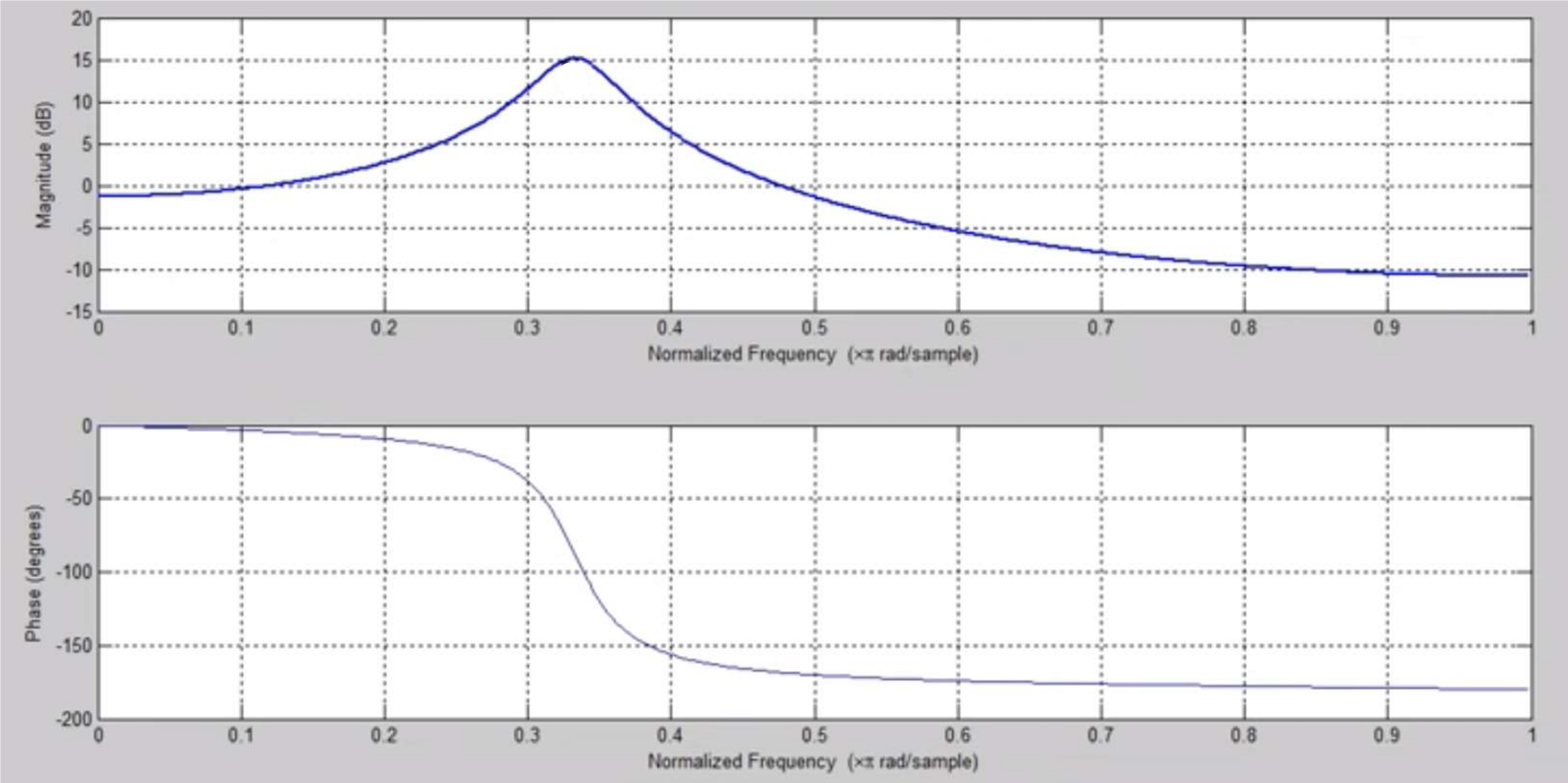
Graph B



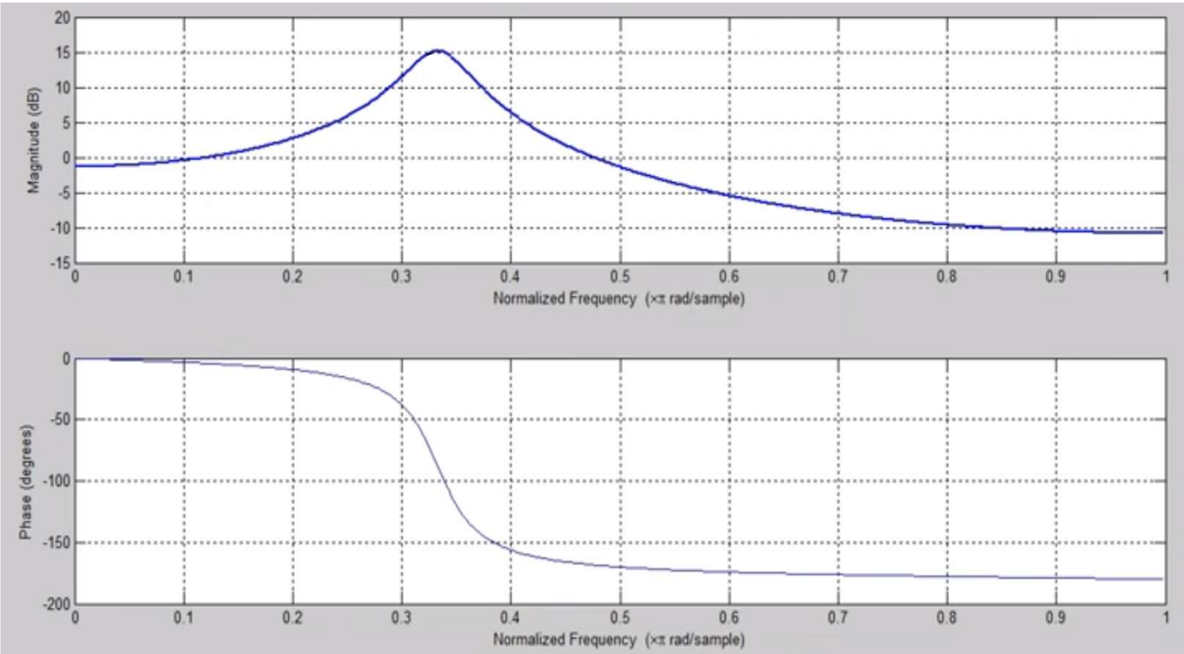
Reading off magnitude response from more complicated examples

- If we wanted to really crank up the influence of the poles, what we could do would be to make the radius a lot larger, say, we make it **r = 11/12**.

```
>> r = 11/12; b = [0 sin(pi/3)*r]; a = [1 -2*r*cos(pi/3) r^2];
>> freqz(b,a)
```



Reading off magnitude response from more complicated examples

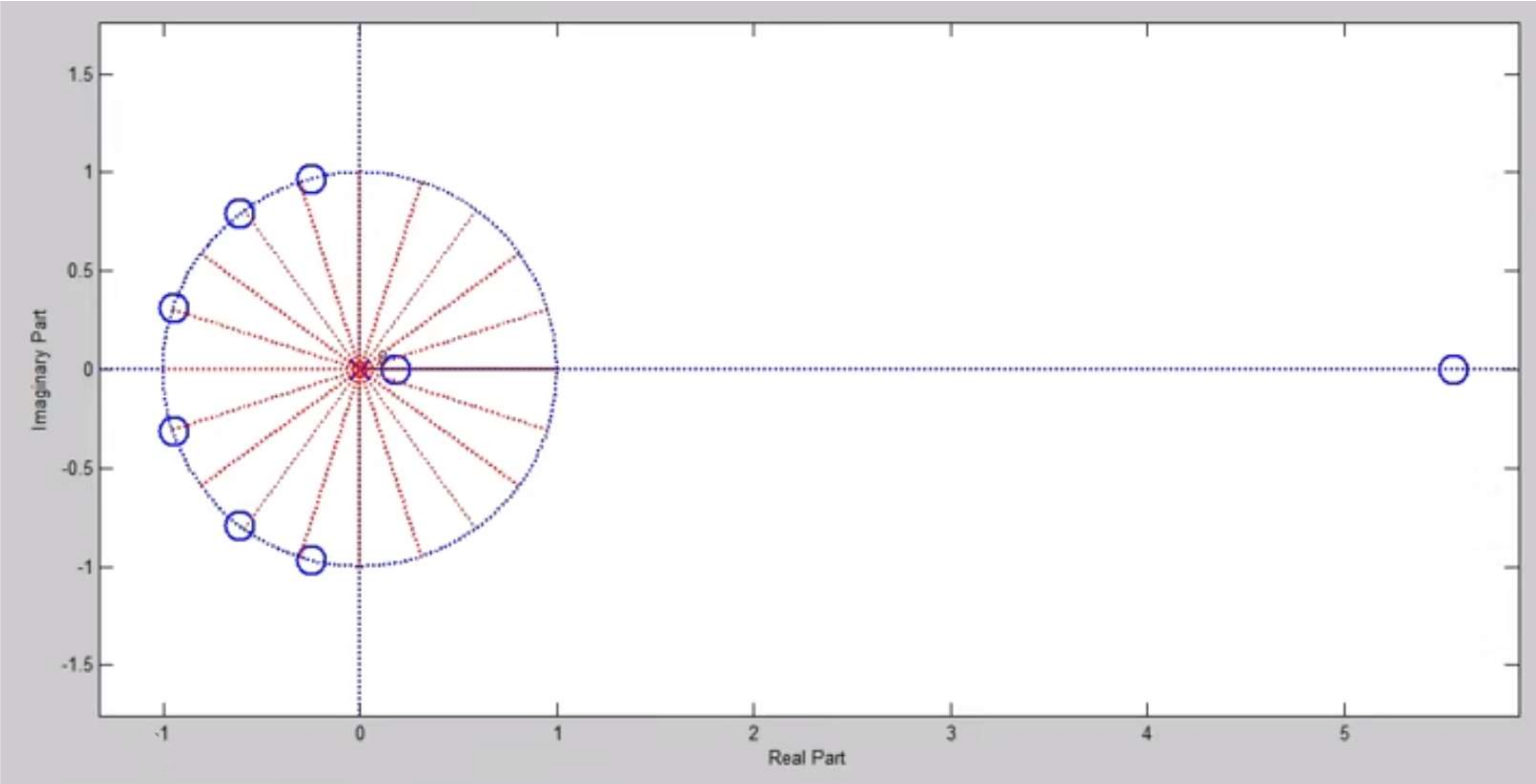


- When we look at the frequency response again, we get this ***very sharp peak***. If were were to compare this to the previous plot, we would say that the magnitude of the frequency response right across the pole is much, much higher because that pole is sticking up very close to the position on the unit circle,
- **Conclusion:** We can look at a frequency response diagram or we can look at the pole-zero diagram and read off what the filter is doing.

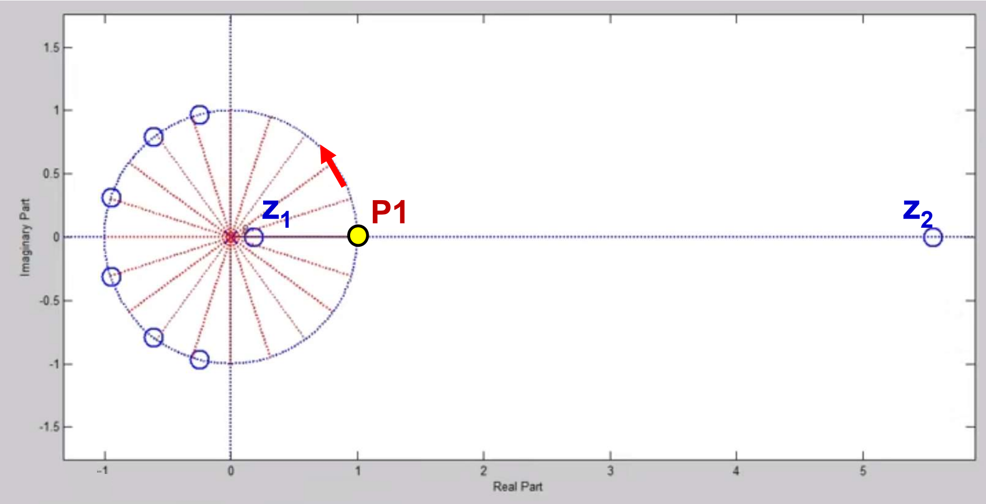
Reading off magnitude response from more complicated examples

Example:

- Is this pole-zero diagram showing a lowpass filter, a highpass filter, or a bandpass filter?



Reading off magnitude response from more complicated examples

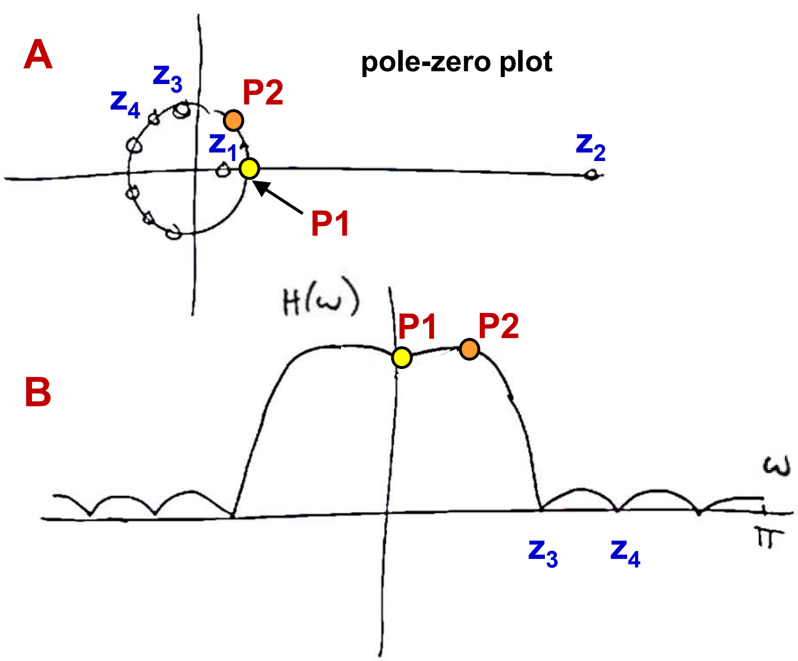


- **Note:** **P1** is just our starting point. It is neither a pole nor a zero.

- Let us first use our **Geometric Intuition**. All the poles are at the origin, i.e., at **0**. That means when we travel around a unit circle, all these poles are going to have ***the same effect***. The poles are not really part of the process. Instead, we have a lot of zeros. Recall that a zero is like nailing that fabric down to the page. So, the fabric is kind of inclined to billow up a little bit between zeros. For example, here we can tell that when we plot the frequency response, we will actually get places where the response is exactly **0**. And, in between those places, i.e., between the zeros, the fabric is inclined to push up a little bit.

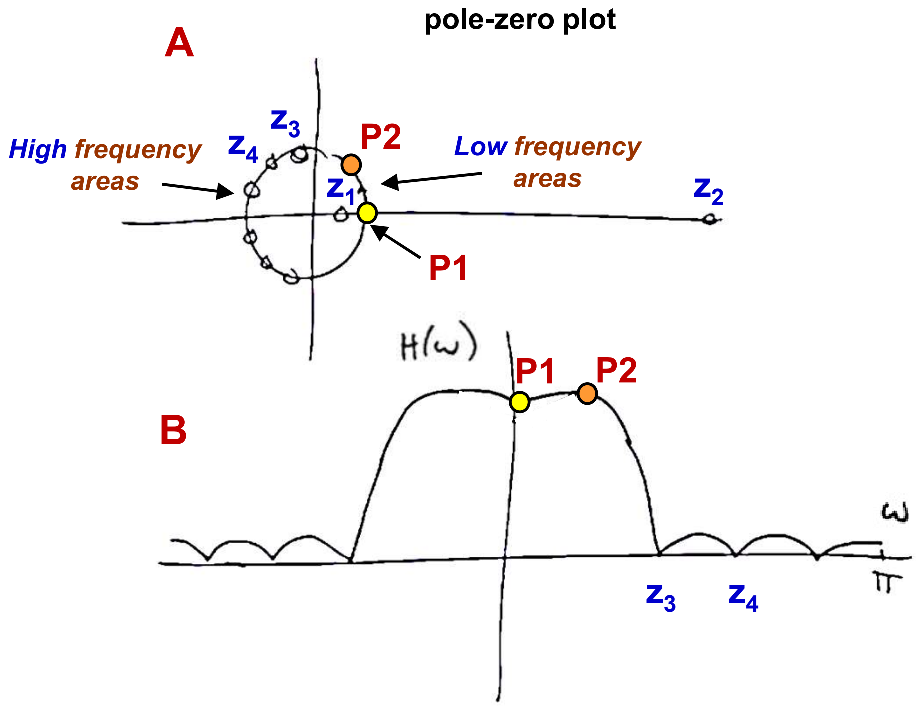
- At point **P1**, this is like saying, there are some zeros (**z_1** and **z_2**), that are ***constraining the fabric*** in that region, and in some sense, the influence of those zeros is going to be strongest around region of point **P1**. As we move counter-clockwise in the direction of **red** arrow, the influence of the zeros on the left will get a little bit higher.

Reading off magnitude response from more complicated examples



- Let us predict what is going to happen to **H(ω)** if we were to sketch the **pole-zero plot** and then move around the unit circle. The pole-zero plot could look like **A**, where we have a bunch of zeros on the left and a couple of zeros on the right. What we predict we might see would be something like **B**. There is going to be a **local minimum** at **ω = 0**, shown by **P1** in both **A** and **B**, where zeros z_1 and z_2 are affecting the position on the unit circle the most, i.e., affecting position **P1** in **B**. Then we predict that it is going to **go up a little bit** because we are going to travel away from these two zeros (moving towards position **P2**) and the influence of the left zeros is going to be felt more.
- As we go further in counter-clockwise direction towards the bunch of zeros on the left, we are going to start to get drawn in by these zeros. So, graph **B** is going to **go down**. At some point, we are going to hit the first zero on the left, shown by z_3 and we are going to nail that frequency response to the floor.
- Between z_3 and z_4 , we are going to let the fabric billow up a little bit and then we are going to nail it down again when we get to z_3, and so on and so forth.
- This is our ***expected frequency response***, but the actual frequency response might not be as dramatic as we did here.

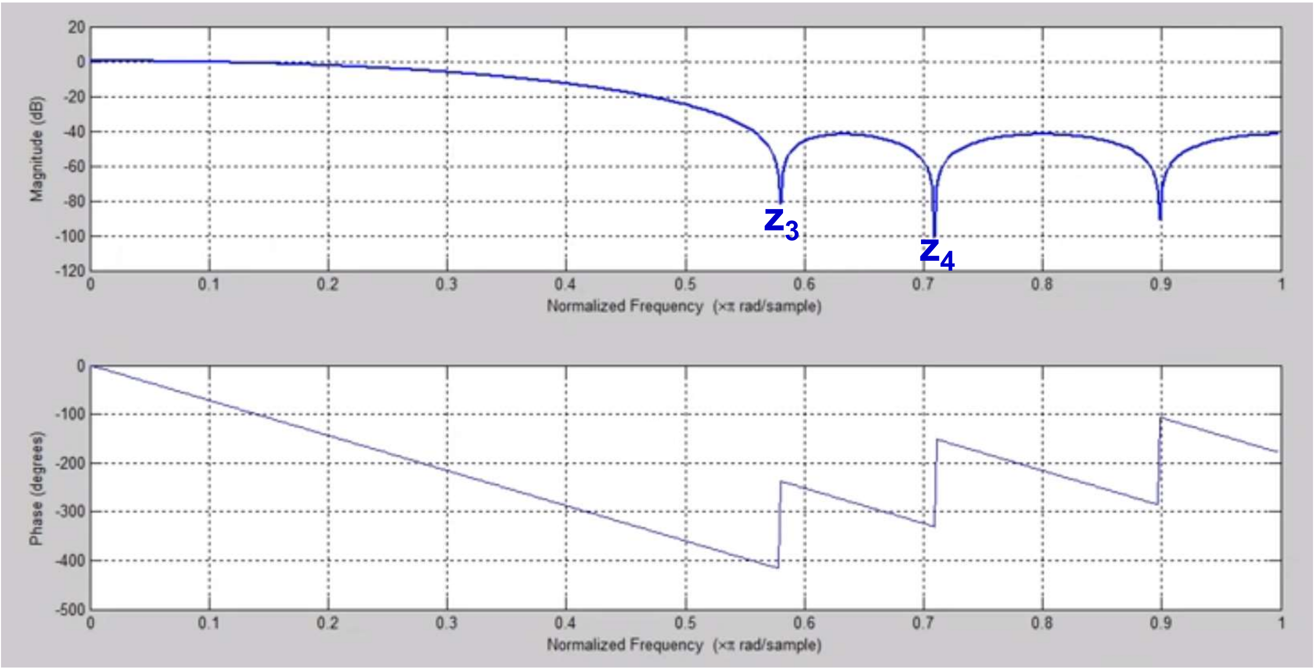
Reading off magnitude response from more complicated examples



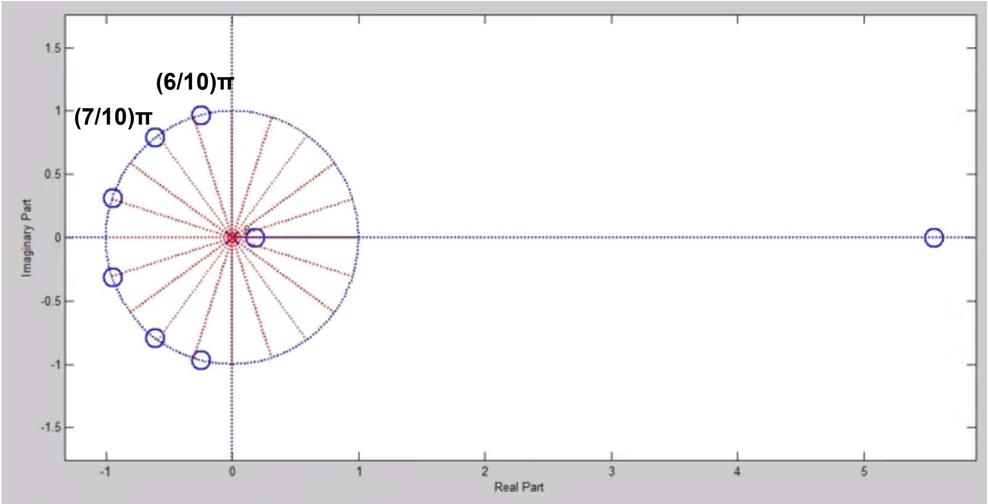
- By looking at graph **B**, the filter looks like a **lowpass filter**. The reason is that there is nothing constraining the fabric in the ***low frequency areas***. So, the fabric is free to push up as it likes around the position of **P1**. However, there is lots of stuff constraining the fabric at the ***high frequency areas***, around the positions where the bunch of zeros are. In this area, there are all these zeros that are pushing that fabric down.

Reading off magnitude response from more complicated examples

A

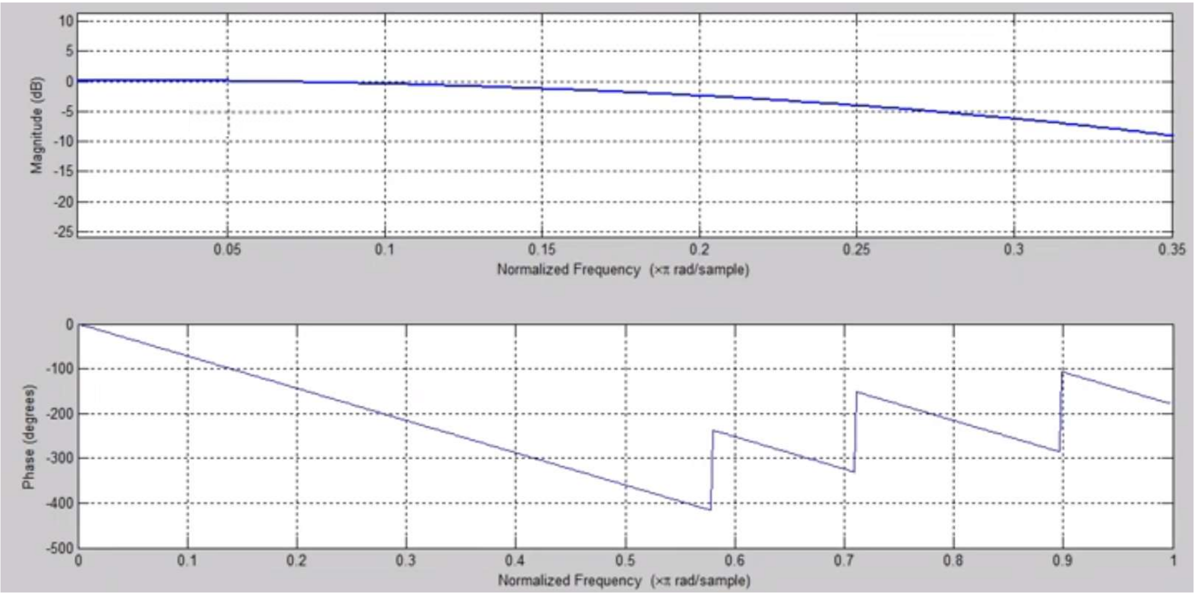


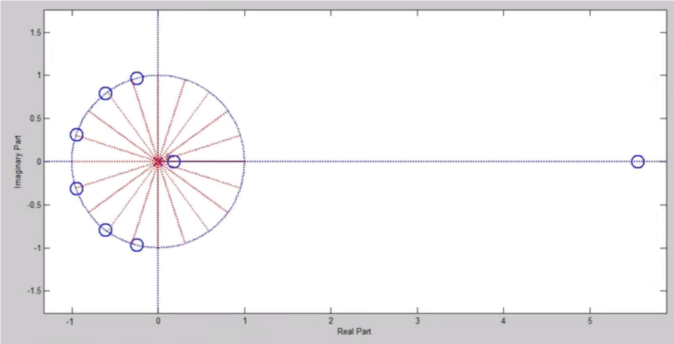
B



- Let us see what we actually get in MATLAB? The **dips** here (such as z_3) are the zeros. So, if we go back and take a look at the pole-zero plot in **B**, we can see that, as we expected, we get one zero, z_3, that is just before **(6/10)π**, another zero, z_4, at about **(7/10)π**, etc. We can see that the frequency response is billowing up between those areas (i.e., between the zeros).
- We do not see the dipping effect of z_1 and z_2, maybe because they are not close enough to the unit circle to really pull down the magnitude response. Next, we can investigate this further by zooming in on the low frequency area.

Reading off magnitude response from more complicated examples





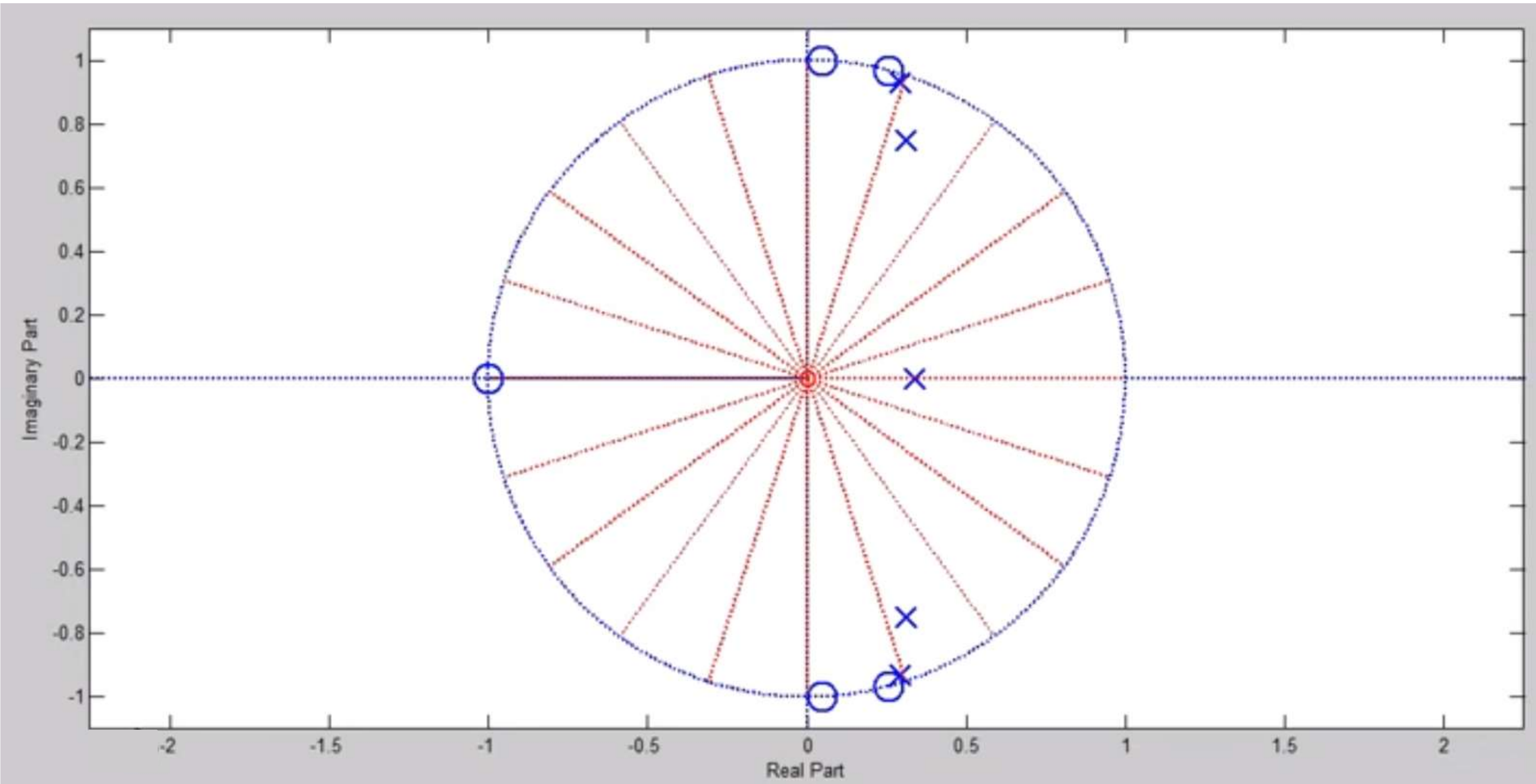
- If we zoom in on the left hand-side of the plot, it looks like the zeros are not close enough to the unit circle to really pull down the magnitude response right at **0**. But the general character of what we got is some sort of a **lowpass filter**.

Reading off magnitude response from more complicated examples

Example:

- Is this pole-zero diagram showing a lowpass filter, a highpass filter, or a bandpass filter?

```
>> [b,a] = ellip(5,0.5,20,0.4); zplane(b,a)
```



Reading off magnitude response from more complicated examples

Code [b,a] = ellip(5, 0.5, 20, 0.4) in MATLAB:

- The code **[b,a] = ellip(5, 0.5, 20, 0.4)** in MATLAB designs an elliptic filter and returns its coefficients. Here is a breakdown of what it does:

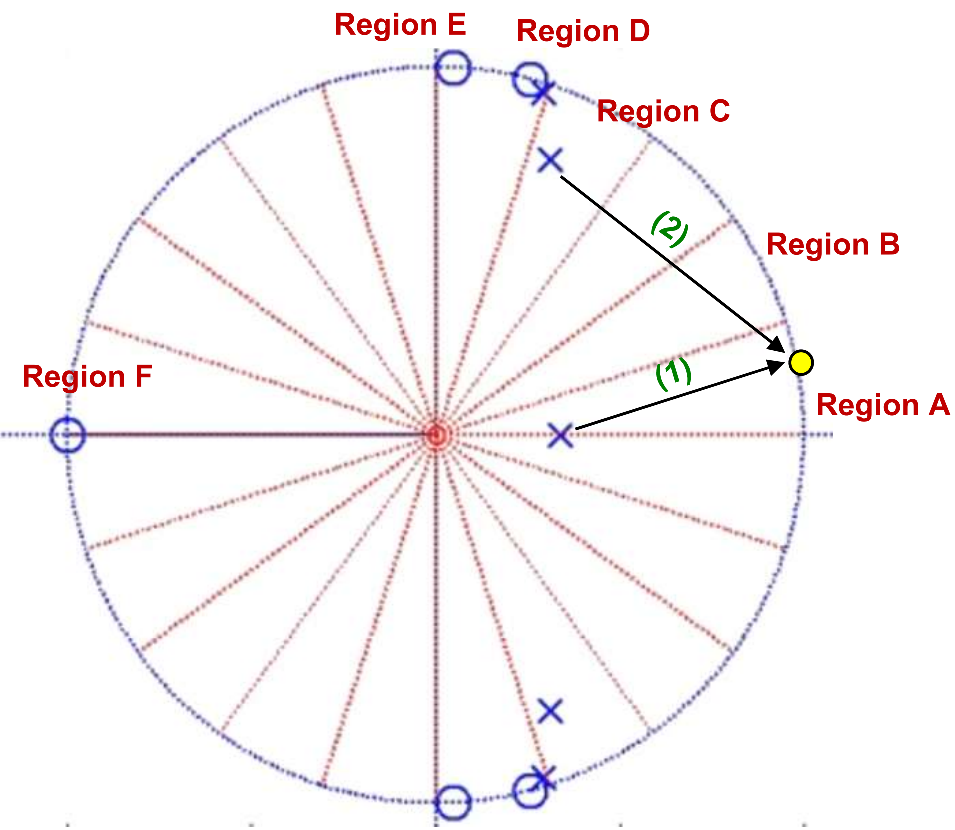
Elliptic Filter Design:

- The **ellip function** is used to design an elliptic filter based on the provided specifications. The arguments define:
  - **5:** This is the desired order of the filter (number of poles).
  - **0.5:** This specifies the passband ripple in decibels (dB). Lower values result in a flatter passband but require a higher filter order.
  - **20:** This defines the stopband attenuation in dB. Higher values provide more attenuation in the stopband but may also affect the passband width.
  - **0.4:** This specifies the normalized passband edge frequency. The frequency range from **0** to this value is the passband. Frequencies above this value are attenuated in the stopband. The normalization is with respect to the sampling frequency (which is assumed to be **1** here).

Output:

- The function returns two vectors:
  - **b:** This vector contains the numerator coefficients of the designed elliptic filter's transfer function.
  - **a:** This vector contains the denominator coefficients of the transfer function.

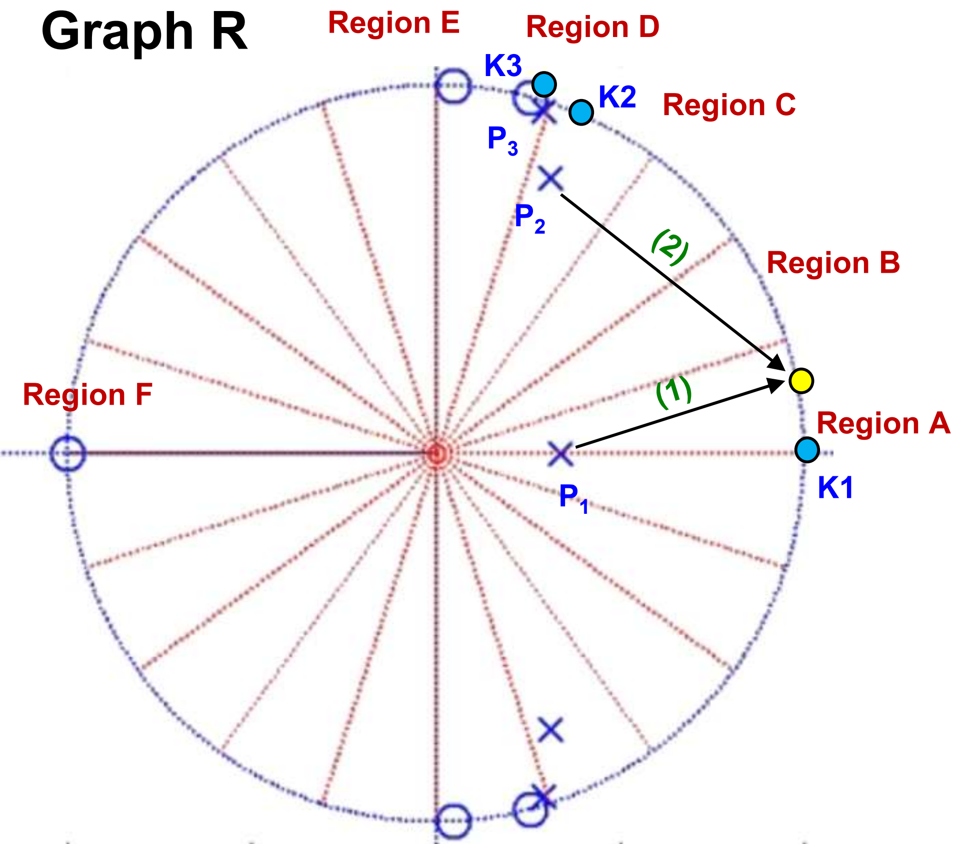
Reading off magnitude response from more complicated examples



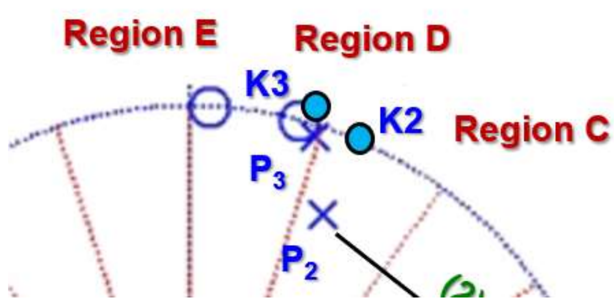
- The **X**'s are going to push the frequency response up. The **O**'s are going to pull the frequency response down. So, definitely, the **O**'s are going to nail down the frequency response at a few points around **Region E**, and things are definitely nailed down at the highest frequency around **Region F**.
- We are definitely going to have some extremely high values in **Regions A** to **C**, in the low frequency bands. This is still going to be a lowpass filter of some sort, because in **Region A**, what is happening is that the length of the vectors to the poles (such vectors **(1)** or **(2)**), are all going to be smaller than the length of the vectors to zeros. So, in this region, we expect to have some high action. There is really nothing that is pulling the frequency response down at these points.
- As we move around from **Region A** towards **Region D**, number one, the effect of the poles is getting less, and number two, the effect of the zeros is getting stronger.

Reading off magnitude response from more complicated examples

Graph R

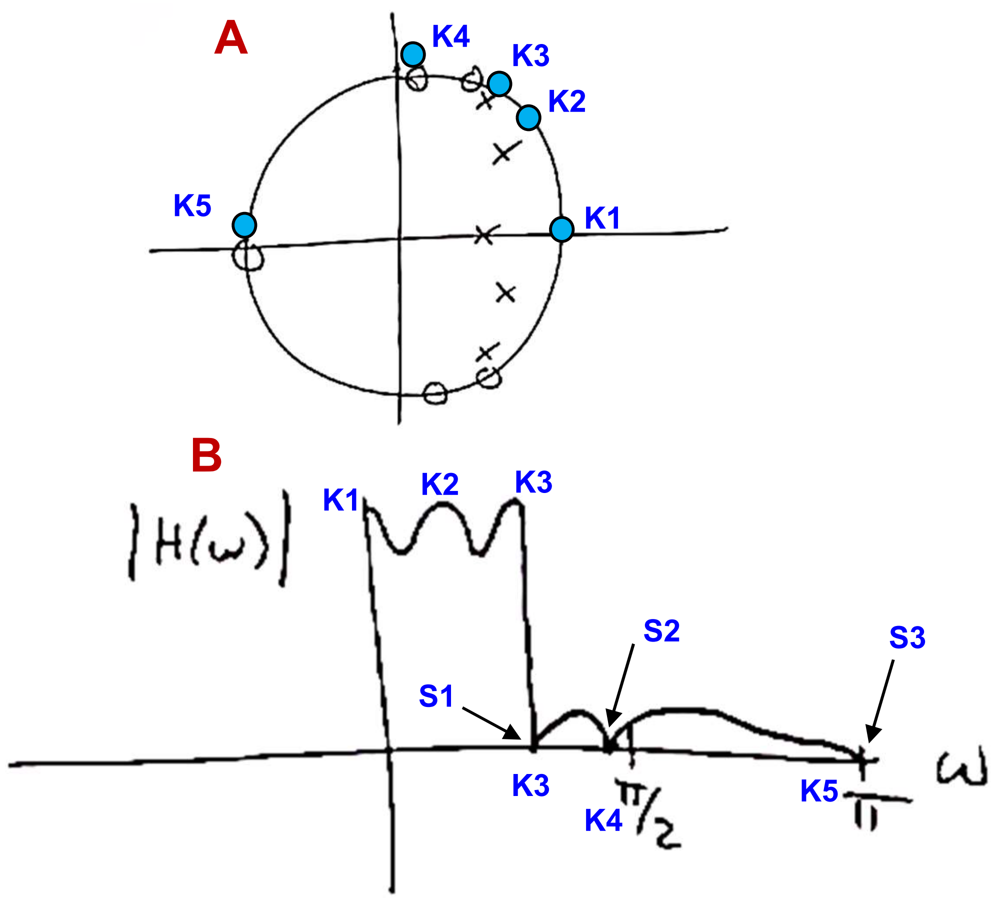


Graph Q



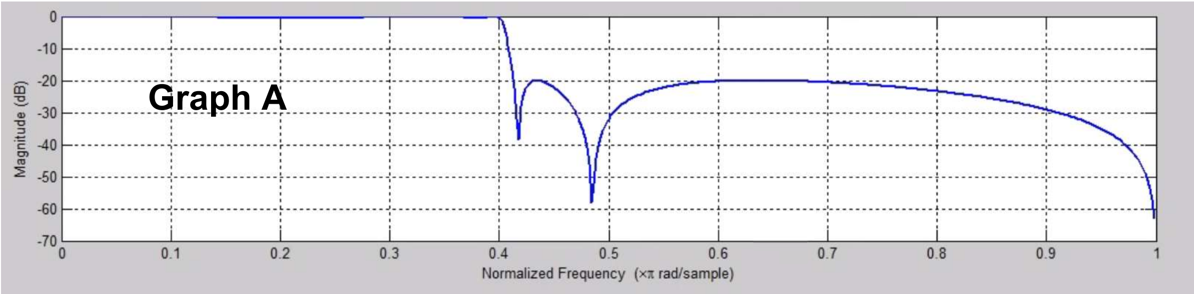
- Once the frequency response has been nailed down to **0** in **Region E**, there is nothing that the poles on the right is going to do to pull it very much up again. It is like saying, how much we could have that fabric be affected if we had nailed it down, even if there is a tent pole in the **Regions A** to **D**.
- If we were to make the sketch, we would expect to see a local maximum in **Region A** (or at $\omega = 0$) or point **K1**, due to the influence of pole P_1. Then we would expect to see another local maximum at around **K2**, due to the influence of pole P_2 (see **Graph Q** for the enlarged part of **Graph R**). Then we expect to see a very sharp local maximum in **Region D** at point **K3**, because this pole, P_3, is so close to the unit circle, and then there is a zero that is right after that pole. So, we expect that there is going to be a very sharp drop here. That is, in **Region D**, something is pulling the frequency response up right there and pulling it back down again. And, finally, we are going to see a couple of ripples in **Regions E** to **F**.

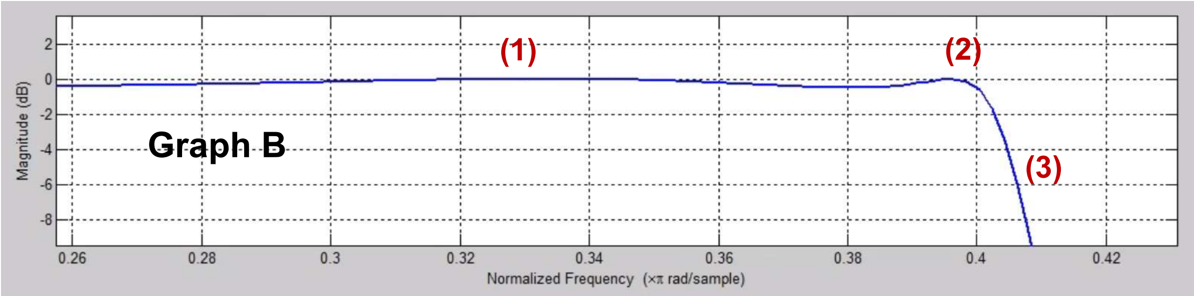
Reading off magnitude response from more complicated examples

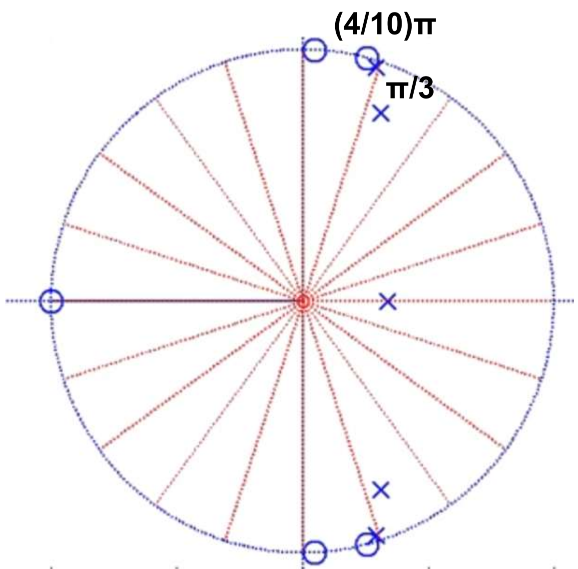


- Let us move from point **K1** towards point **K5**, and sketch the frequency response manually.
- At **K1**, we expect a local maximum. Then we are going to have another local maximum at **K2**. Here, the height of this point may be a little bit greater because it is closer to the unit circle. Then there is going to be a very sharp local maximum at **K3**. Because of the zero that is close to point **K3**, (i.e., to the left of it in graph **A**), we are going to nail that down to the ground at somewhere like point **S1**, shown in graph **B**. And, then because of the zero at point **K4**, we are going to nail it down to the ground at point **S2**. Finally, because of the zero at point **K5**, we are going to nail it down to the ground at point **S3**.

Reading off magnitude response from more complicated examples





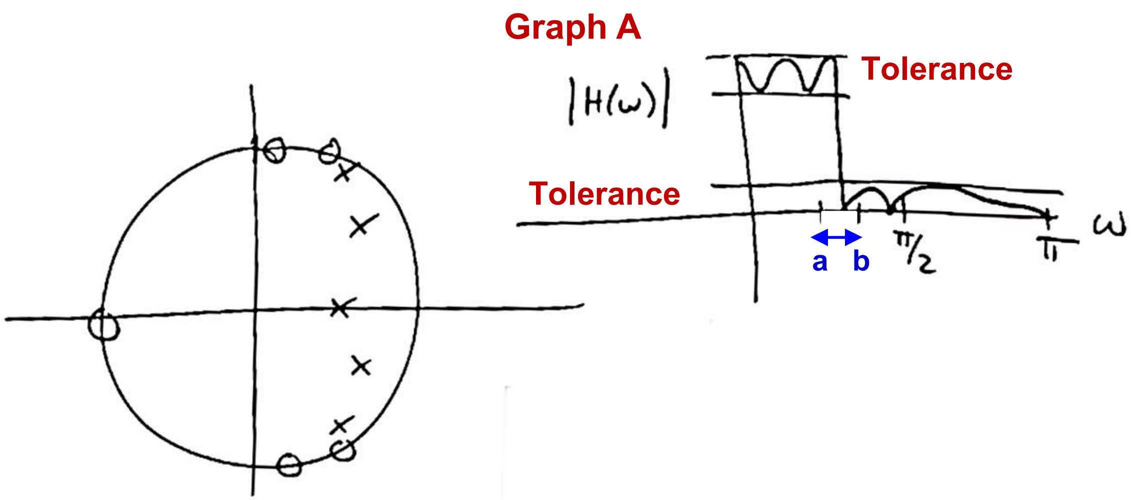


- **Let us see what we actually get in MATLAB**. Here, **Graph B** is the same as **Graph A**, but we zoomed in on the left part of the graph.
- Around **(4/10)π** is where the action is. We can see in **(2)**, this is where that pole is really pushing up right around **(4/10)π** before we have this precipitous drop, i.e., **(3)**. Then just before that in **(1)**, we have a maximum here, that is a little bit flatter due to the influence of that other pole that is closer to about **π/3**. Here, **π/3** is weaker, so the influence on magnitude response is not as dramatic. However, the influence of **(4/10)π** is really strong.
- In MATLAB, whenever in the magnitude plot, we see that the dB dips down to like **-40 dB** or **-60 dB**, those are actually the zeros. These are the places that are nailed down.

Reading off magnitude response from more complicated examples

- Last time we were talking about, what good is the **z**-transfer? Why do we care about these poles and zeros that are way off in places and they are not being on the unit circle?
- So, what we discussed in this lecture is the kind of the intuition for why we care. It is true that in most cases we do not really care about the value of the **z**-transform for **z**'s that are far away from the unit circle. ***What we care about is what is happening on and around the unit circle for the purposes of understanding how filters work***.
- Our analysis suggests that the way that we can design filters is by pushing these poles and zeros around, trying to get them to be in good places. And, one thing that is true, and we already saw in these pictures, is that we are mostly concerned with **real-valued filters** and so the poles and zeros are always going to come in ***complex conjugate pairs***. As we will see, for certain classes of filters, it is not like we can just arbitrarily put these poles and zeros wherever we want. There are some constraints on the way that they can appear.

Reading off magnitude response from more complicated examples



- Usually, the way it works in filter design is that, e.g., we want to have a passband that goes for approximately **0** to **π/3**. We want a stopband that is really close to **0**, e.g., between **2π/3** and **π**. And, then we put **tolerances** on how much **oscillations** we are willing to tolerate in the passband, i.e., how much our picture can actually wiggle around (as shown in **Graph A**). We can also talk about how wide **a** to **b** distance on the omega axis should be.

- In **Graph A**, we have shown two different tolerances, one for the top part of the graph and one for the bottom part. We are going to talk about how we can take all those kinds of constraints and figure out how we can design filters that work. If we were given as many zeros and poles as we wanted, then we could imagine that we could push these **thumbtacks** (i.e., the **zeros**) into the fabric and to push up the **tent poles** to get a very fine control over the frequency response. In practice, we are constrained by how many poles and zeros we are given. In the language of digital signal processing, those are called **taps** or **coefficients**.

- For example, your design team may be tasked with the design of a lowpass filter that needs **15**-tap, i.e., a **15**-tap filter, and has a certain property in the passband and another property in the stopband. Obviously, we cannot guarantee that easily. We cannot get everything that we want at the same time. There are always **trade-offs**.

Reading off magnitude response from more complicated examples

- **Conclusion:** The intuition of reading off the frequency response from the pole-zero plot, after we do the **z**-transform, is actually very handy to do. That is a good kind of rule of thumb for working professional engineers to be able to look at something and say, e.g., this is a highpass filter and we can see why the designed passband response is not so good because these poles are too far away from the unit circle. So, maybe we should move the poles closer. This is the start of getting intuition for how we should manipulate things to make the filter preform the task that we expect it to do.

Solving difference equations with z-transforms

Example:

$$y[n] + \frac{1}{4} y[n-1] = x[n] + \frac{1}{5} x[n-1] \quad \textbf{(1)}$$

WHAT IS $H(z)$?

$$Y(z) + \frac{1}{4} z^{-1} Y(z) = X(z) + \frac{1}{5} z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{5} z^{-1}}{1 + \frac{1}{4} z^{-1}} \quad \textbf{(2)} \quad |z| > \frac{1}{4}$$

WHAT IS RESPONSE TO A UNIT STEP?

$$X(z) = \frac{1}{1 - z^{-1}} \quad \textbf{(3)} \quad Y(z) = \frac{1 + \frac{1}{5} z^{-1}}{(1 + \frac{1}{4} z^{-1})(1 - z^{-1})} \rightarrow$$

$$Y(z) = \frac{A}{1 + \frac{1}{4} z^{-1}} + \frac{B}{1 - z^{-1}}$$

$$\rightarrow \boxed{y[n] = \frac{24}{25} u[n] + \frac{1}{25} \left(-\frac{1}{4} \right)^n u[n]}$$

From the table:

$$a^n u[n] \overset{\mathcal{Z}}{\Longleftrightarrow} \frac{z}{z-a} \qquad |z| > |a| \quad \textbf{(4)}$$

- The other thing that we often use **z**-transforms for is **solving difference equations**. A system is often specified in terms of something like equation **(1)**. It has something involving the inputs, and something involving the outputs. We want to know what the transfer function of the system is, i.e., **H(z)**.
- We would take the **z**-transfer on both sides. We know the transfer function is always the output over the input, **(2)**. If we wanted to have a right-sided impulse response, we would say that $|\mathbf{z}| > \frac{1}{4}$.
- **What is the response to a digital unit step signal?** For this particular input, **X(z)** is **(3)**. We used equation **(4)** for **a = 1**. So, **Y(z)** is going to be the product of **X(z)** and **H(z)**. If we wanted to continue down this line, we would have a partial fractions kind of problem. As seen here, **z**-transform makes solving difference equations much easier.

End of Lecture 9