ELEC 421

Digital Signal and Image Processing

Siamak Najarian, Ph.D., P.Eng.,
Professor of Biomedical Engineering (retired),
Electrical and Computer Engineering Department,
University of British Columbia

Course Roadmap for DSP

Lecture	Title
Lecture 0	Introduction to DSP and DIP
Lecture 1	Signals
Lecture 2	Linear Time-Invariant System
Lecture 3	Convolution and its Properties
Lecture 4	The Fourier Series
Lecture 5	The Fourier Transform
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Lecture 7	Discrete-Time Fourier Transform
Lecture 8	Introduction to the z-Transform
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Lecture 4:
The Fourier Series

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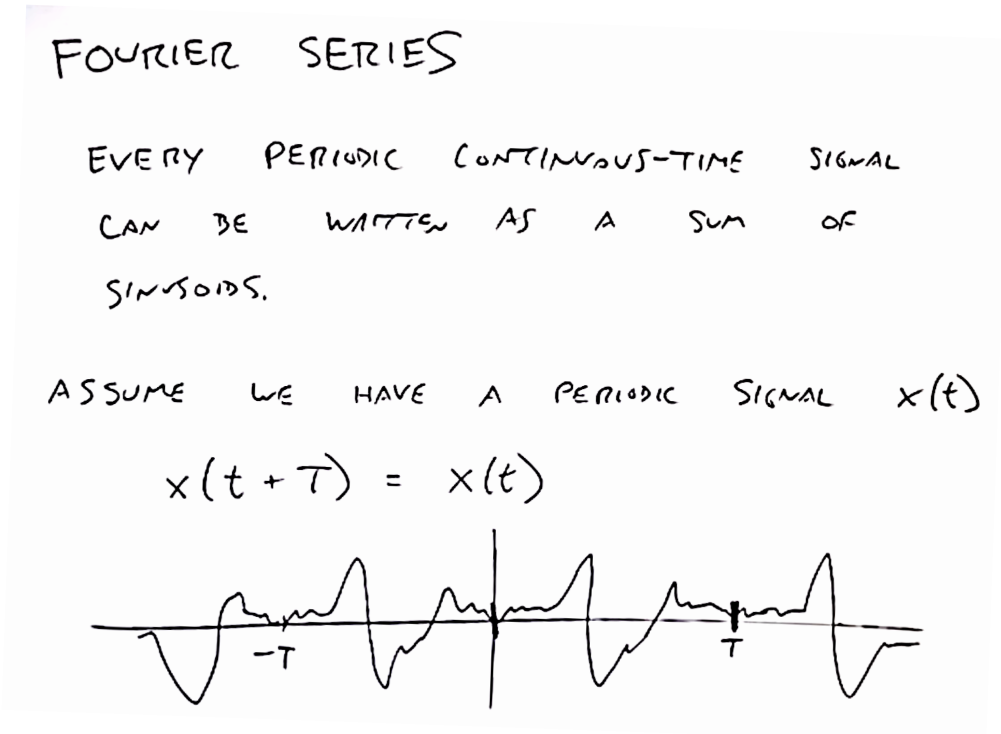
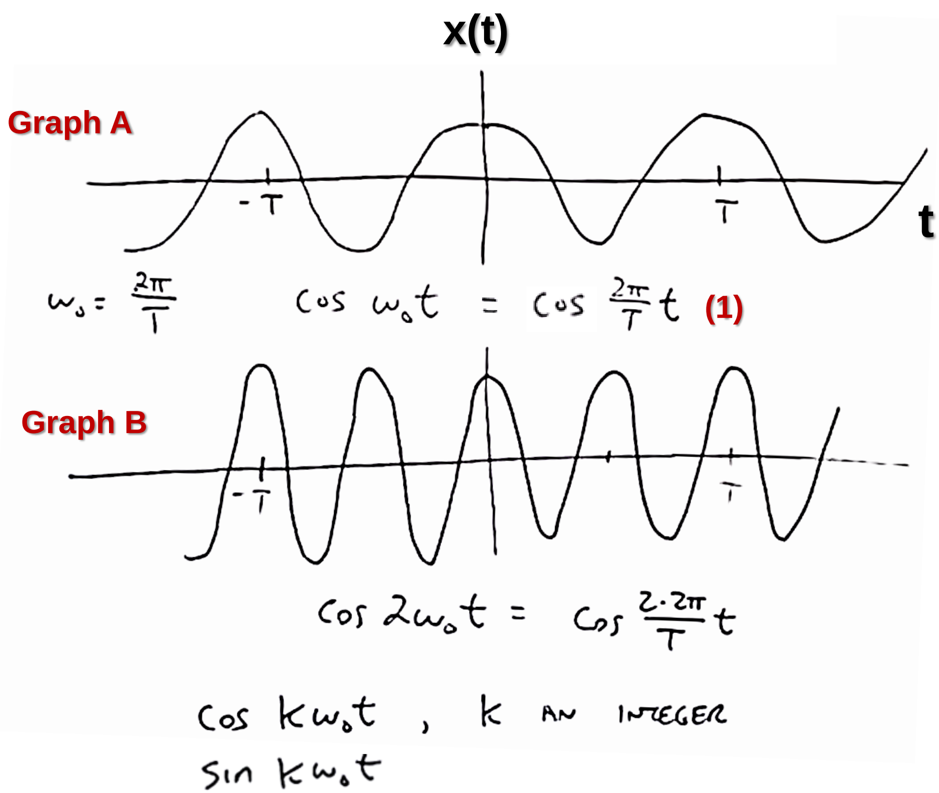
The Fourier Series



- It is very important to know what the Fourier Series is and how it works. This is because the **DFT (Digital Fourier Transform)** is like a discrete version of the Fourier Series.

- We noticed that convolution process can be tedious.
- The great strength of LTI signals is the ability to use what are called **transform methods** to make solving LTI systems easier.
- That means we bring the input, the system, and the output into a new domain which is going to be either the **Fourier domain** or the **z-transform domain**.
- In **Fourier analysis** we do the decomposition of a signal into sines and cosines.
- Sinusoids naturally occur in a lot of situations, such as swinging pendulums, spinning wheels, and communications theory. Cellphones, AM/FM radio are built on things like carrier waves that are high-frequency sinusoids. LTI systems respond to sinusoidal inputs in a particularly special way. That is why we use Fourier Transforms in the first place.
- Every periodic continuous-time signal can be written as a sum of sinusoids. We want to see how this works.

Assumption: x(t) is periodic with period T



- The first question is what other signals are periodic. The most natural signal that has the period **T** is a sinusoid (Graph A).
- **Fundamental Frequency:** Here, ω_0 is called the fundamental frequency. This is because if we plug in **t = 1.T** in (1), we get **cos(2π)**, which is **1**. If we plug it **t = 2.T**, we get **cos(4π)**, which is also **1**, and so on. Put differently, the fundamental frequency of a sinusoid is ***the lowest frequency component*** present in the signal.
- In Graph B, the cosine wiggles twice as fast.
- We can also have **sin(ω_0t)**, since it is basically a cosine function that has shifted over to the right. It also has a period of **T**.
- Both **cos(kω_0t)** and **sin(kω_0t)** are periodic and have the same period of **T**.
- In **cos(kω_0t) = cos[k(2π/T)t]**, the angular frequency is scaled by a factor of **k** (**k** being an integer greater than **1**). This effectively increases the frequency, causing the function to oscillate faster.

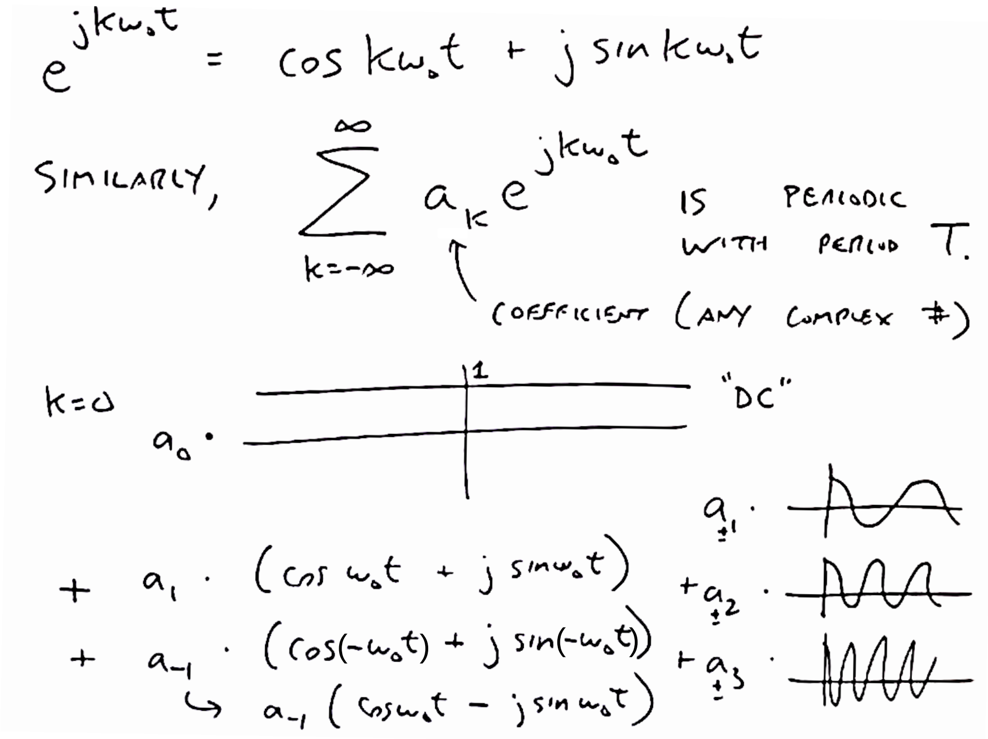
Complex exponentials with period T

$$e^{jk\omega_0 t} = \cos k\omega_0 t + j \sin k\omega_0 t$$

SIMILARLY, $\displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ IS PERIODIC WITH PERIOD T.

COEFFICIENT (ANY COMPLEX #)

Interpreting the Fourier Series sum



- **Interpretation of a_k's:** Coefficients (a_k's) could be any complex number. Multiplying the signal by these coefficients, such as by **2** or **3** or by **1 + j**, does not fundamentally change its period; it just changes its amplitude and phase.
- The interpretation is that as we increase **k**, the signal wiggles more and more.

Deriving the formula for the $\{a_k\}$

OUR GOAL: REPRESENT

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}$$

"SYNTHESIS"

FOURIER SERIES

(Synthesis equation)

HOW TO COMPUTE $\{a_k\}$ FOR A GIVEN $x(t)$?

n IS A FIXED INTEGER.

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \quad \textbf{(1)}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

- Our goal is to represent **x(t)** in the form of sum of terms. This is called the **Fourier series**.
- **Synthesis equation:** It means how we synthesize the signal from the $\mathbf{a_k}$'s. So, if we are given the $\mathbf{a_k}$'s, the synthesis equation shows us how we add up the $\mathbf{a_k}$'s to get back to the **x(t)**. Next, we need formulas for $\mathbf{a_k}$'s to get **x(t)**. The synthesis equation in Fourier series is used to ***reconstruct*** a periodic function from its Fourier coefficients. It essentially shows how to combine these coefficients with sinusoidal functions (sines and cosines) to ***recreate*** the original signal.
- In **(1)**, we multiply both sides by $\mathbf{exp(-jn\omega_0 t)}$.

Deriving the formula for the $\{a_k\}$

INTEGRATE BOTH SIDES FROM 0 TO T:

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\, dt \quad \textbf{(2)}$$

$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_0^T e^{j(k-n)\omega_0 t}\, dt}_{\textbf{(3)}}$$

$$\textbf{(3)} \int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos((k-n)\omega_0 t)\, dt + j \int_0^T \sin((k-n)\omega_0 t)\, dt$$

SAY $k=n$. $\int_0^T 1\, dt = T$. $\qquad \int_0^T 0\, dt = 0$.

FOR $k=n$, THE INTEGRAL IS $= T$.

- In **(2)**, we are going to assume that everything is well-behaved enough so that we can switch the order of the sum and the integral.

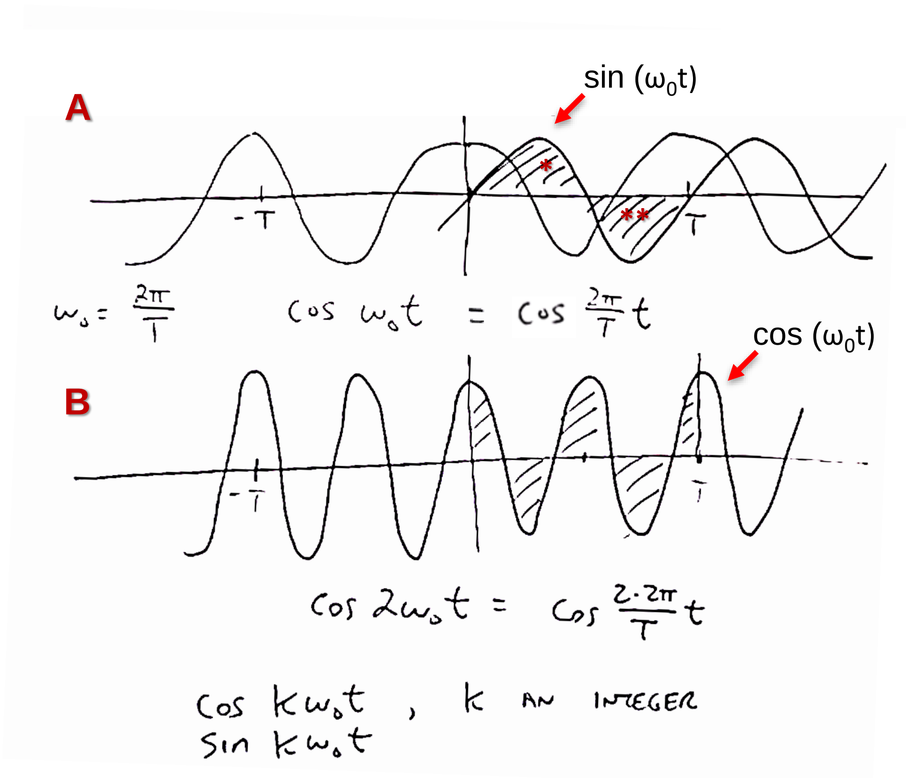
Deriving the formula for the $\{a_k\}$

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos((k-n)\omega_0 t)\, dt + j\int_0^T \sin((k-n)\omega_0 t)\, dt$$

$$k \neq n \qquad \int_0^T \cos((\text{INTEGER})\omega_0 t)\, dt + j\int_0^T \sin((\text{INTEGER})\omega_0 t)\, dt$$

- Let us focus on these integral.

Deriving the formula for the $\{a_k\}$



- When we integrate the sine, as shown in **A**, we get part ***** and part ****** that cancels out entirely. We would have just as much above the line as we would below the line. So, no matter what the integer is, as long as we have an ***integer number of ripples*** inside the **0** to **T** interval, the integral turns out to be zero. The same is true for the integration of cosine, as shown in **B**.

The result of the derivation

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt$$

$$\sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt \quad *$$

$$k \neq n \qquad \int_0^T \underbrace{\cos((\text{INTEGER})\omega_0 t)}_{\downarrow}\, dt + j \sin \ldots$$

OSCILLATES AN EQUAL # OF TIMES INSIDE $[0,T] \Rightarrow$ INTEGRAL IS 0.

$$\text{THUS} \quad ** \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = a_n T$$

(when **k = n**) **(Analysis equation)**

$$\boxed{a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt}$$

$\{a_k\}$ ARE THE FOURIER SERIES COEFFICIENTS OF $x(t)$.

- In *, even though we have a complicated sum of infinite number of integrals, only one of these integrals is actually not equal to zero and that is when this **k** matches whatever the **n** we had there. So, that means the whole integral equals only a nonzero term when **k = n**.
- In **, when **k = n**, we will have $\mathbf{a_n \times T}$.
- The $\mathbf{a_k}$'s equations are called the **analysis equations**. This is because we are taking the original signal **x(t)** and we are analyzing it down into the $\mathbf{a_k}$'s.
- The $\mathbf{a_k}$'s are called the **Fourier series coefficients of x(t)** or sometimes called the **spectral coefficients**.

Symmetries in $\{a_k\}$ for real x(t)

WHAT IF x(t) IS REAL? $\{a_k\}$ ARE COMPLEX, BUT THERE ARE PATTERNS.

Assumption:

x(t) is real

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\| \qquad\qquad \|$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$(x + jy)^*$
$= x - jy$

IF z IS REAL, $z = z^*$

$a_1 = 1+2j$
$a_{-1} = 1-2j$

Example

$$\boxed{a_k = a_{-k}^*}$$

(Complex Conjugate Symmetry)

- Even if we have a real input, we are going to get complex Fourier series coefficients.
- **x*(t)** is the complex conjugate of **x(t)**.
- If **x(t)** is real, then the equation of **x(t)** is equal to the equation of **x*(t)**.
- **Complex Conjugate Symmetry:** It can be shown that $\mathbf{a_k = a_{-k}^*}$. This means we only need to keep track of positive indices, i.e., positive **k**'s in $\mathbf{a_k}$'s. The negative ones are just the complex conjugates of the positive ones.

Different forms of the Fourier Series for real signals

FOR $x(t)$ REAL, WE CAN ALSO WRITE

$$x(t) = a_0 + \sum_{k=1}^{\infty} \underset{\substack{\uparrow \\ \text{AMPLITUDE}}}{2A_k} \cos(\underset{\substack{\uparrow \\ \text{PHASE}}}{\theta_k} + k\omega_0 t) \quad \textbf{(1)}$$

THESE ARE COMPUTABLE FROM THE $\{a_k\}$

ALTERNATELY

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} (B_k \cos k\omega_0 t - C_k \sin k\omega_0 t) \quad \textbf{(2)}$$

- For each of the sinusoids in **(1)**, there is an amplitude and there is a phase plus some shift. The $\mathbf{A_k}$'s and $\boldsymbol{\theta_k}$'s are computable from the $\{\mathbf{a_k}\}$.
- If we do not like to deal with the phase, we can use the alternative form shown by **(2)**. Later on, we need to compute $\mathbf{B_k}$'s and $\mathbf{C_k}$'s, which are also related to the $\{\mathbf{a_k}\}$.

Fourier Series examples

Example:

$$x(t) = 5 + 2\cos(\omega_0 t)$$

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \textbf{(1)}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \quad \textbf{(2)}$$

$$x(t) = 5 + 2\left(\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right)$$

$$= 5 + e^{j\omega_0 t} + e^{-j\omega_0 t}$$

$$\downarrow$$

$$a_0 \cdot e^{0 j\omega_0 t} + a_1 e^{1 j\omega_0 t} + a_{-1} e^{-1 j\omega_0 t} + \dots$$

We use Pattern Matching:

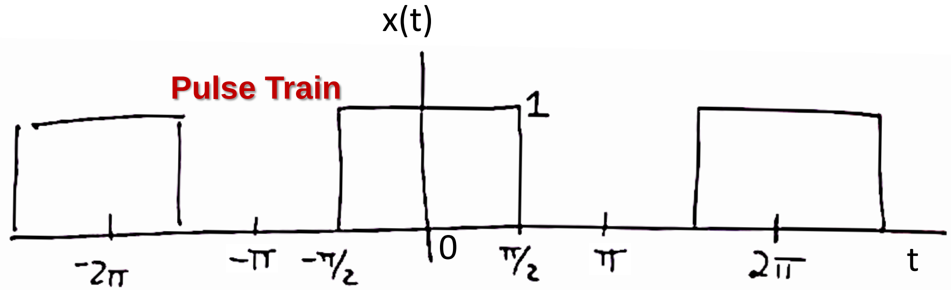
$$\boxed{a_0 = 5 \qquad a_1 = 1 \qquad a_{-1} = 1 \qquad \text{ALL OTHER } a_k = 0.}$$

- Whenever we have something that is strictly visibly a sum of cosines, we can immediately figure out what the Fourier series is. Here, instead of using **(1)**, we can use **(2)** directly. That is, we are using **Fourier series decomposition** concept as a shortcut. Basically, we are ***reading off*** the Fourie series. This method is called **pattern matching**.

Fourier Series for a pulse train

Example:

$x(t) =$



$$T = 2\pi, \qquad \omega_0 = \frac{2\pi}{T} = 1$$

DC term: $$a_0 = \frac{1}{T}\int_0^T x(t)\,e^{-j\cdot 0\cdot \omega_0 \cdot t}\,dt = \frac{1}{T}\int_0^T x(t)\,dt$$

= AVERAGE VALUE OF SIGNAL OVER ONE PERIOD.

(1) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\,dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\,dt = \frac{\pi}{2\pi} = \frac{1}{2}$$

or **(2)** $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt$$

$$\longrightarrow \boxed{a_0 = \frac{1}{2}}$$

- Because the signal is periodic with period **T**, we can choose any interval of length **T** that makes sense for us to integrate on. Both **(1)** and **(2)** will lead to the same final result.
- From **0** to **2π** in **(2)**, we have to integrate two pieces of **x(t)**. Whereas if we chose the interval of **–π** to **+π** (as shown in **(1)**), we will have a nice symmetric pulse with only one piece of **x(t)**.

Fourier Series for a pulse train; The sinc function

General a_k terms:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)\, e^{-jkt}\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-jkt}\, dt = \frac{1}{2\pi}\,\frac{-1}{jk}\, e^{-jkt}\Big]_{t=-\pi/2}^{t=\pi/2}$$

$$= \frac{-1}{2\pi jk}\left(e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right)$$

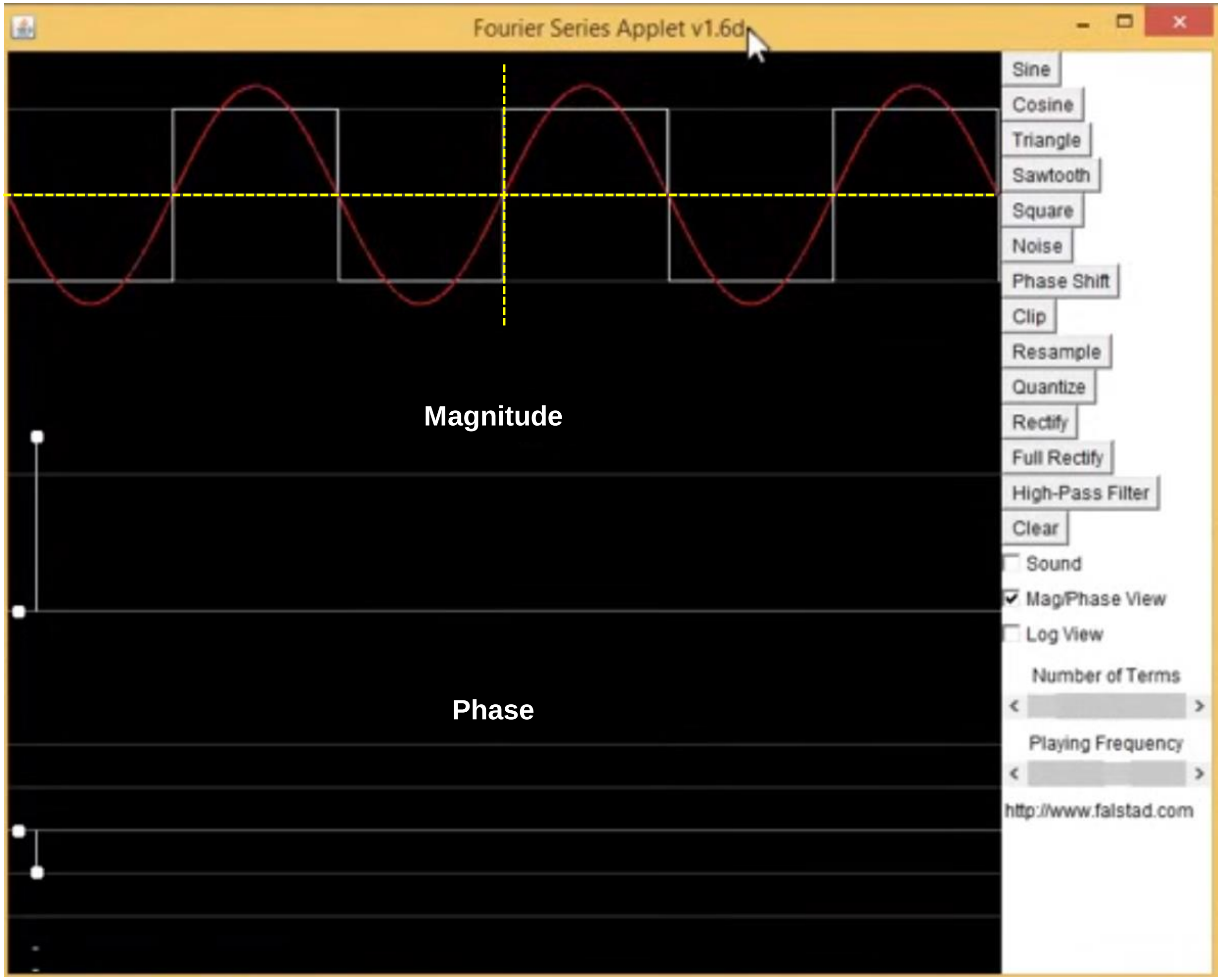
$$= \frac{1}{\pi k}\cdot\frac{1}{2j}\left(e^{jk\pi/2} - e^{-jk\pi/2}\right) = \frac{\sin k\frac{\pi}{2}}{\pi k} \xrightarrow{\text{sinc } x = \frac{\sin x}{x}} = \frac{1}{2}\,\text{sinc}\,\frac{k\pi}{2} \rightarrow \boxed{a_k = \frac{1}{2}\,\text{sinc}\,\frac{k\pi}{2}}$$

$$a_1 = \sin\frac{\pi}{2}\Big/\pi = \frac{1}{\pi}, \quad a_{-1} = a_1^* = \frac{1}{\pi}, \quad a_2 = \sin\pi/2\pi = 0 \qquad \textbf{(1)}$$

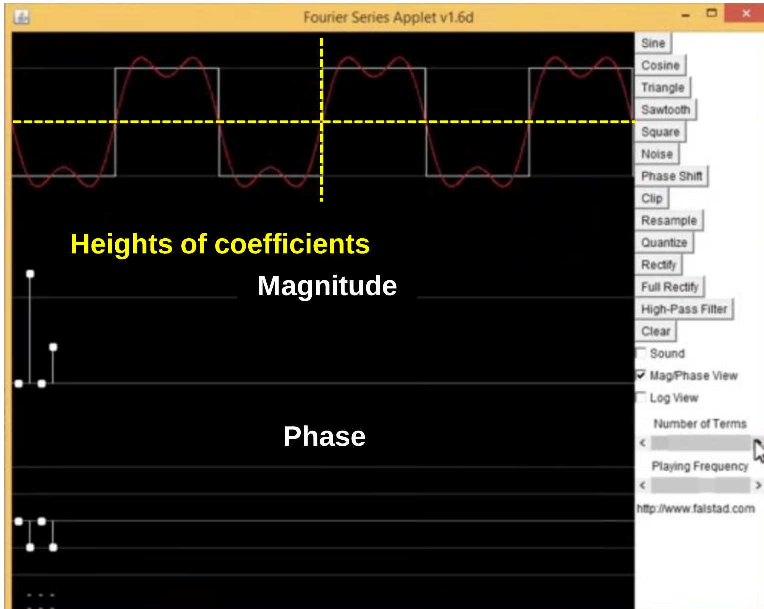
$$a_k = a_{-k}^*$$

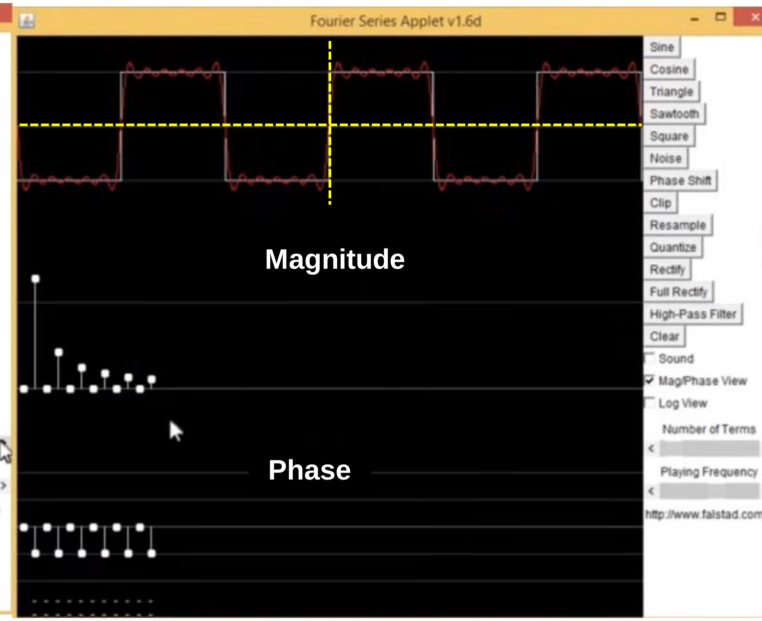
- We are going to use the fact that we can change the limits of the integral to be any interval of length **T**.
- For $\mathbf{a_k}$'s, **(1)**, we see an alternating pattern of zeros and non-zeros.
- Here, **Sinc Function** is defined as **sinc(X) = sin(1.X)/X = sin(X)/X**. Sometimes, it is defined as **sinc(X) = sin(πX)/X**. We are using the first definition, i.e., **sinc(X) = sin(X)/X**.

Fourier Series applet

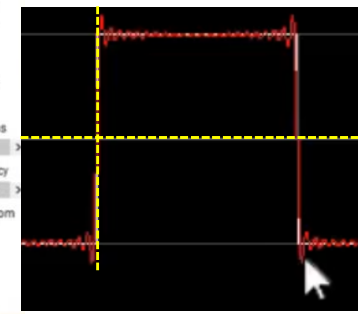


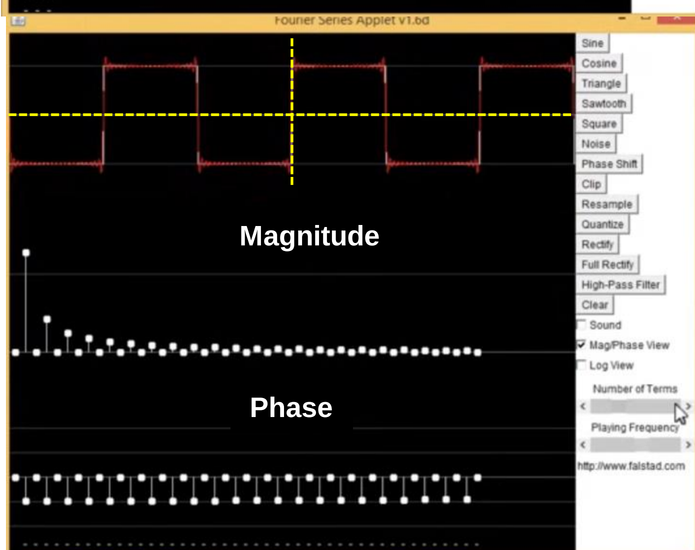
Fourier Series applet





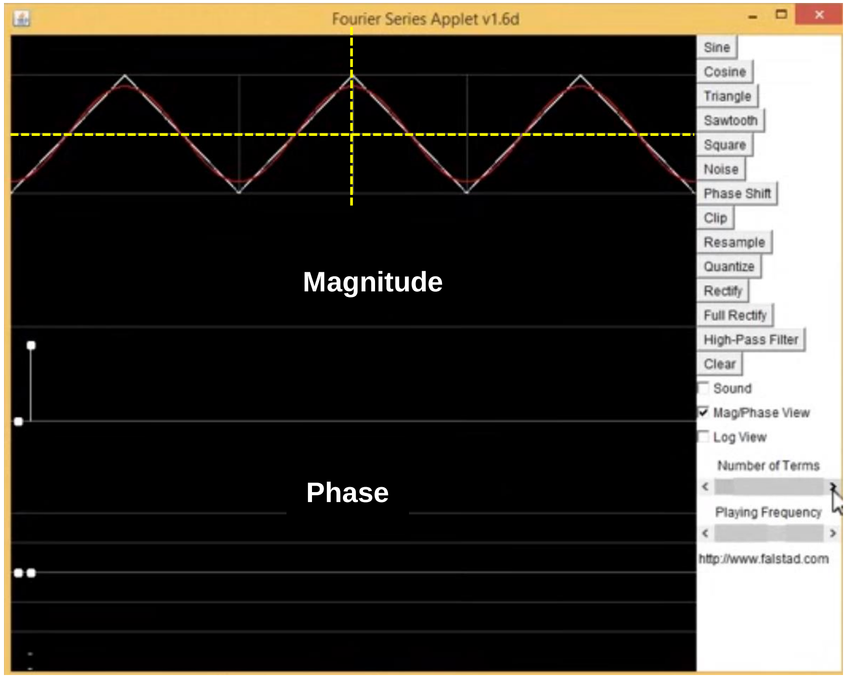
(1)

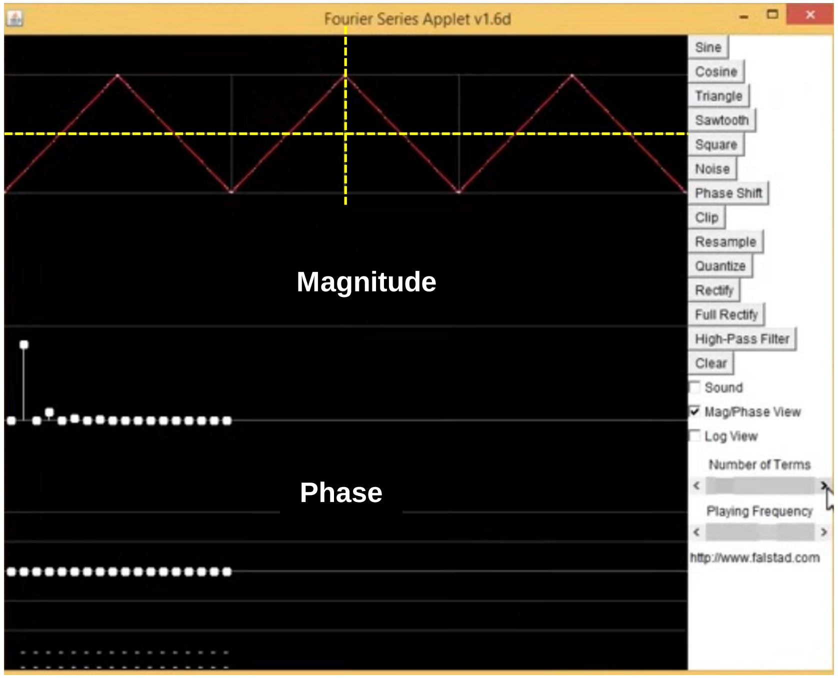




- As we add more terms, we get a better and better approximation to the signal. Also, we see that as we add more terms, the **heights of the coefficients** are decreasing.
- We will get better approximation (the **red** signal) to the **square wave** by increasing the number of terms.
- As shown in **(1)**, there is a little bit of **slop** or **overshoot/undershoot** or **shoulders** around the edges. This has to do with the fact that the signal is **not continuous**. It actually has a **jump**.

Fourier Series applet

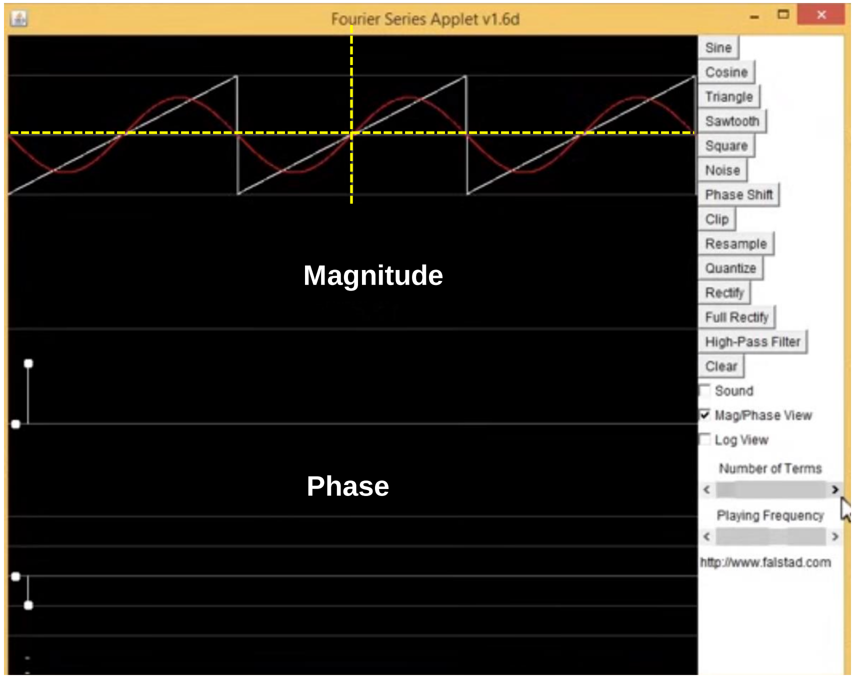


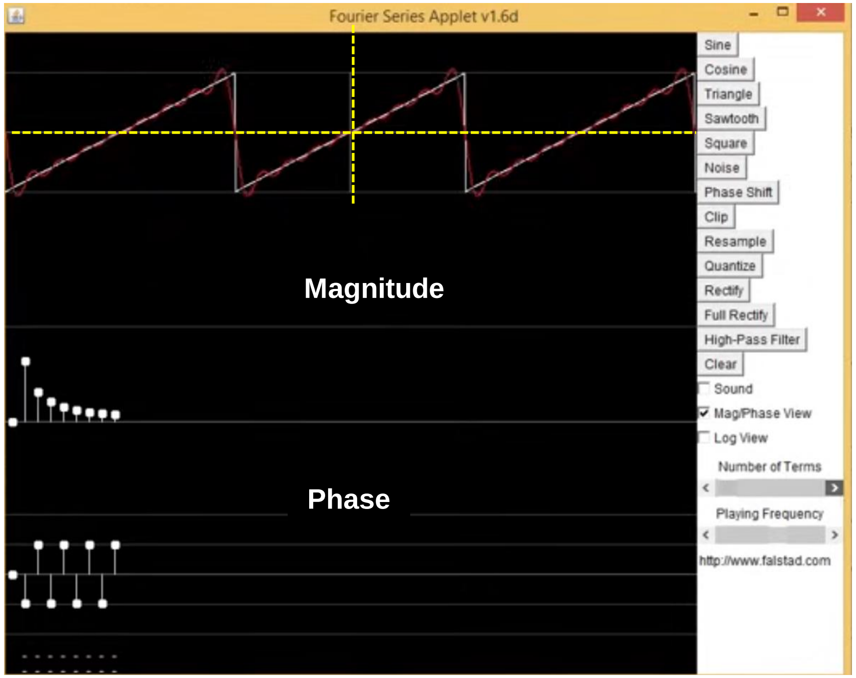


- Let us do the same kind of ***decomposition*** for a different signal, such as for the given **triangular pulse**. A sinusoid is already a pretty good approximation to this triangle and as we add one more term to the DC of the triangle, we only make it better and better. We start to force the sinusoid up into the corners at the top and the bottom to make the approximation better and better. Eventually, after not that many terms, the approximation looks about as good as the original thing.
- The approximation is much better than for the same number of coefficients of the square wave and that again has to do with the fact that the triangle is ***continuous*** whereas a square wave is not.

Fourier Series applet

- The same process is shown below for the **sawtooth signal**:

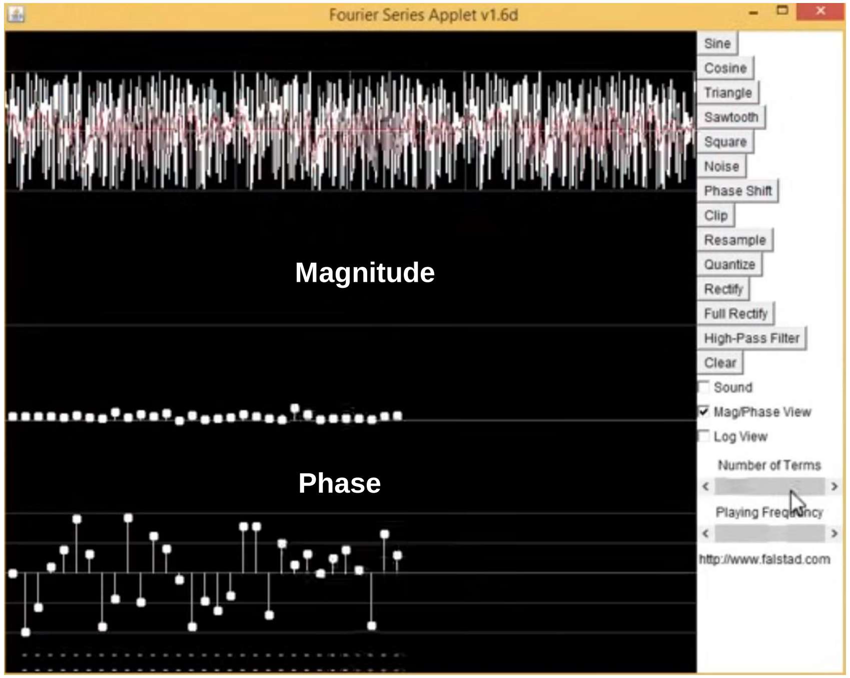


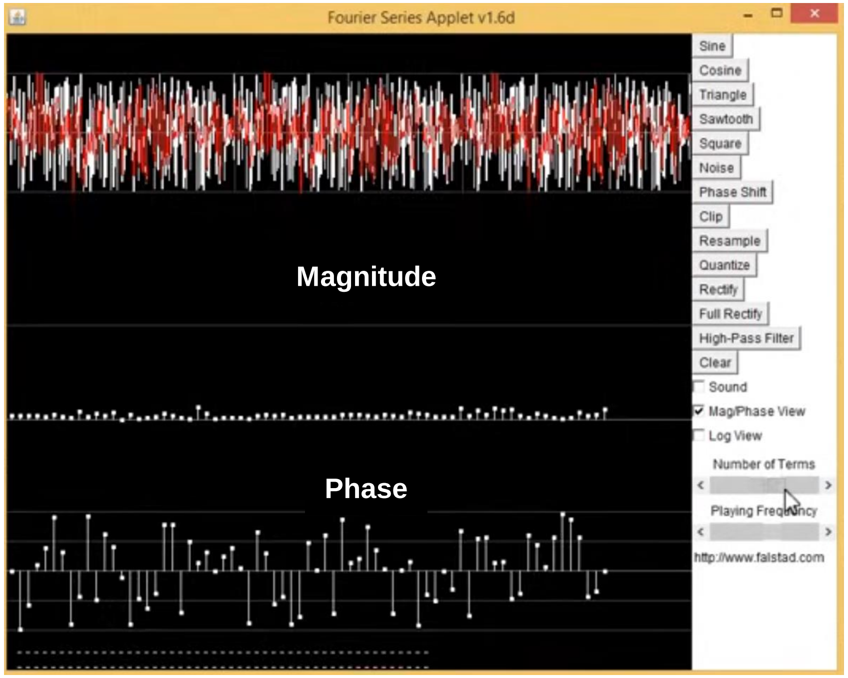


- The first approximation is not so good. The second one is a little better and then slowly things start to improve. These sine waves are increasingly trying to bring up the slope of that line, where the ***discontinuity*** is, to being positive fitting. So, as we increase the number of terms, the slope of that line gets steeper and steeper although we still never really lose the little overshoots at the corners.

Fourier Series applet

- The same process is shown below for a **noise signal**:

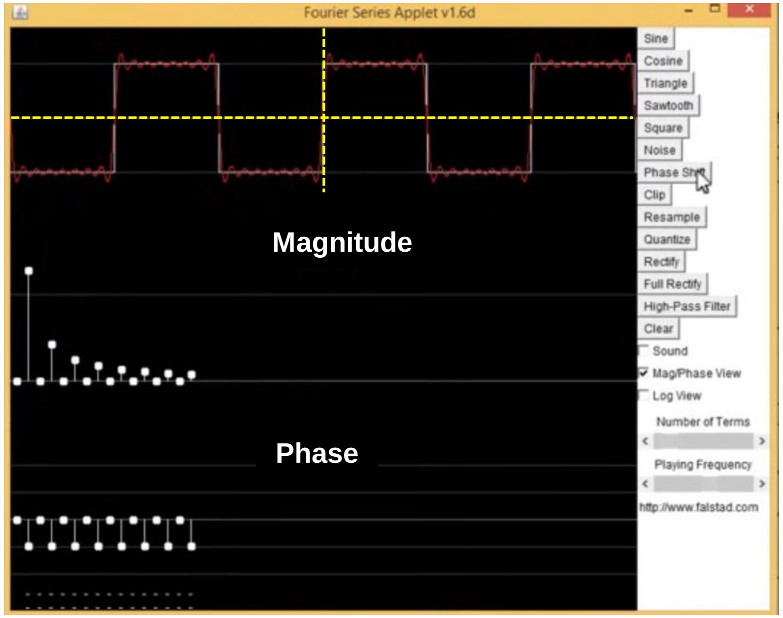


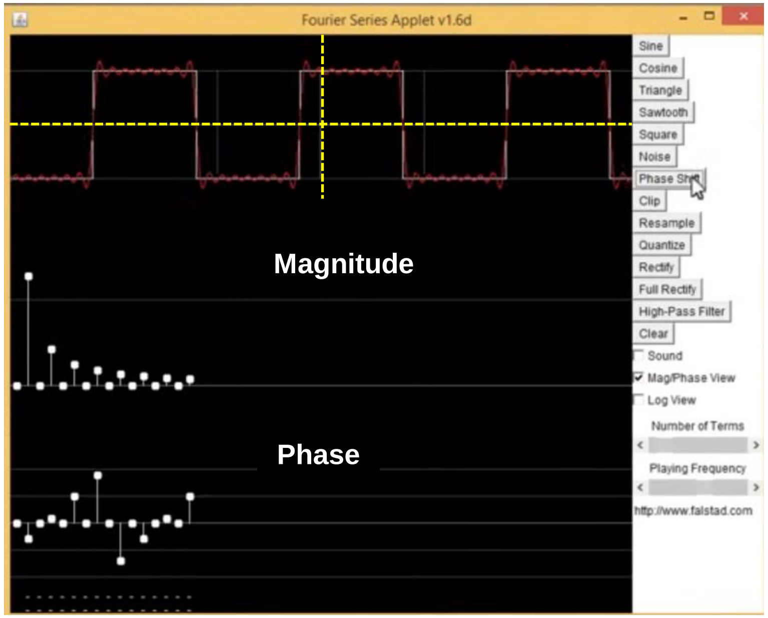


- For a typical periodic noise, if we crank up the number of terms, eventually we can even get a noisy signal to match up with all these Fourier series coefficients. But, here, we clearly do not have enough Fourier series coefficients to make the noise. We would need a lot more than what we would need for a triangle or a sawtooth or a square wave signal. **So, the simpler the signal, the fewer Fourier series coefficients we would need to represent the signal accurately**.

Fourier Series applet

- What happens when we **change the phase**?





- Again let us take a look at the square wave. What happens as we shift the wave left and right? The intuition is that really we should not be changing the amount of every cosine or sine that we need to make up that signal. All we are doing is changing the phases of the underlying sinusoids. So, here, as we move the phase shift around what we are going to see is that the magnitudes of the Fourier series coefficients do not change, but the phases are changing. This is like saying we are multiplying the $\mathbf{a_k}$ by some complex number who has a magnitude of **1**. We are just shifting that phase around.

Fourier Series applet

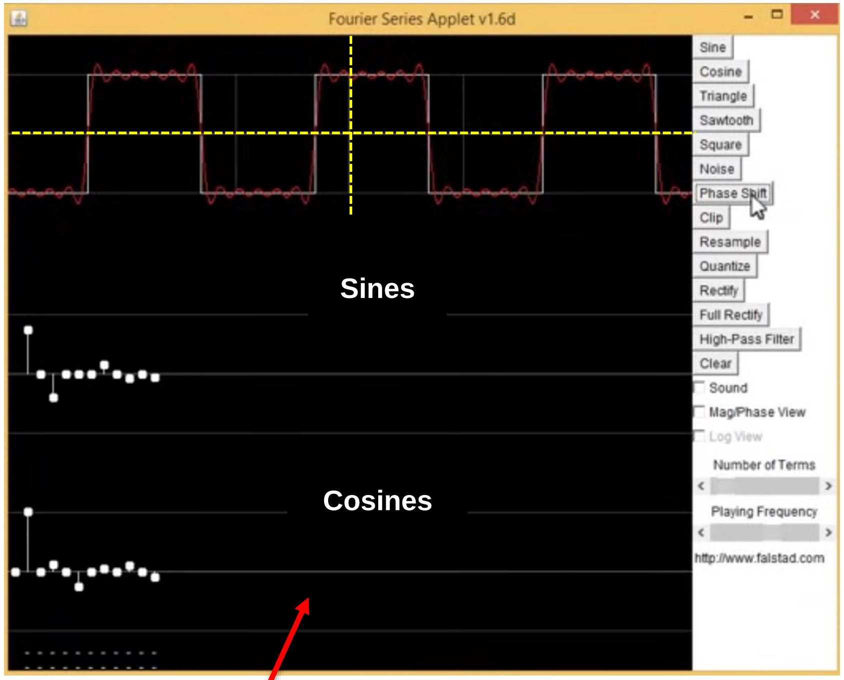
Alternate Form of Signal Decomposition:

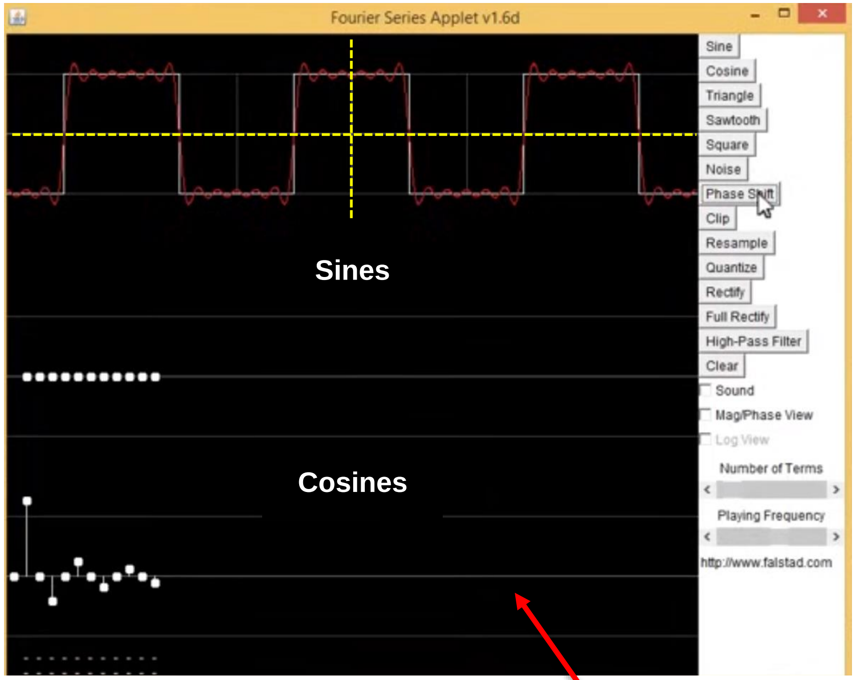
- We could also view the **alternate form of signal decomposition** of cosines and sines, shown in equation **(1)**.
- Let us go back to square wave. This process says how much of a cosine and how much of a sine we need (see also **(1)**). Here, this shows that we can make up this signal fully out of sines (no cosines).
- Based on the above, in our graph, all the cosines coefficients are zero.

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left(B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right) \quad \textbf{(1)}$$

Fourier Series applet

Alternate Form of Signal Decomposition:

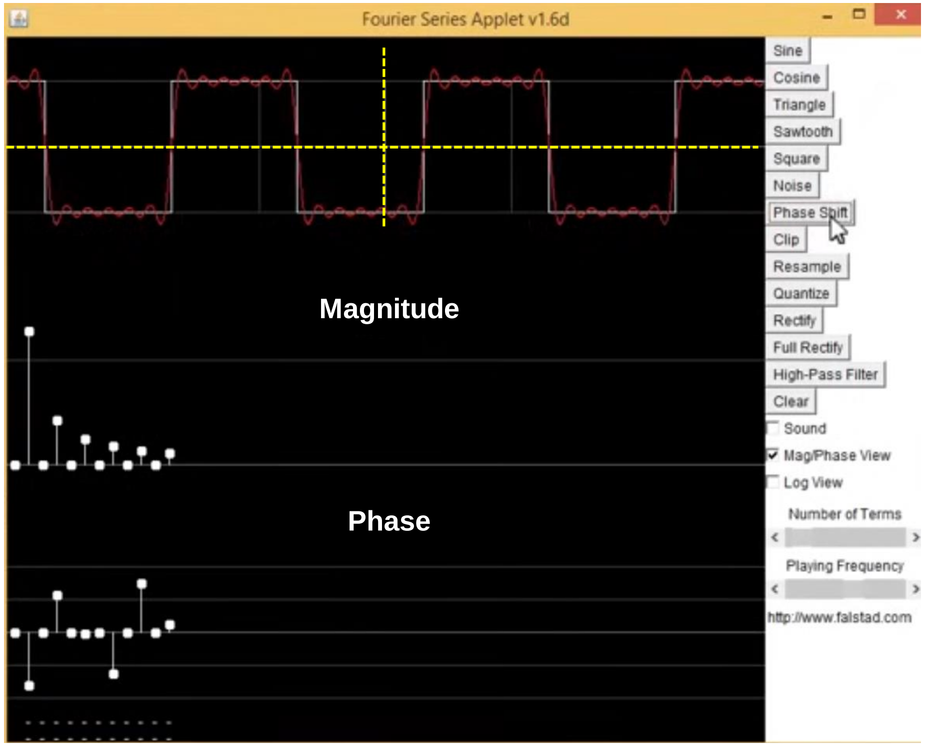


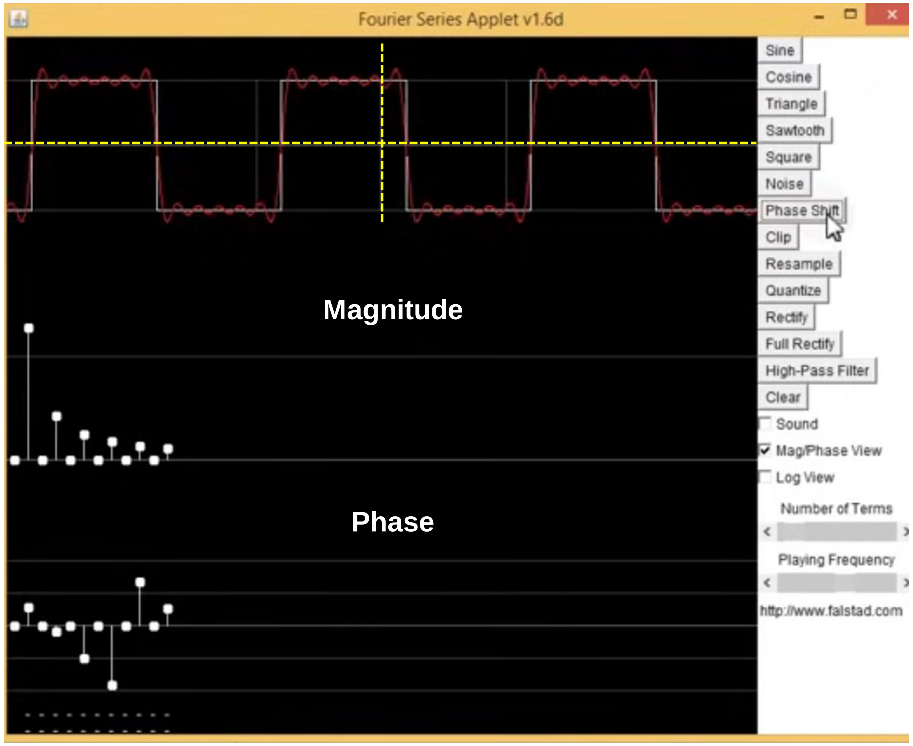


- If we start to **change the phase**, then we are going to need some sines and some cosines.

- This goes on until eventually, if we align the signal, it is purely even, and then we can make the signal purely out of cosines. So, as the signal approaches being even, we will have no sine input and all cosines. This makes sense because we should be able to get even signal out of other even signals.

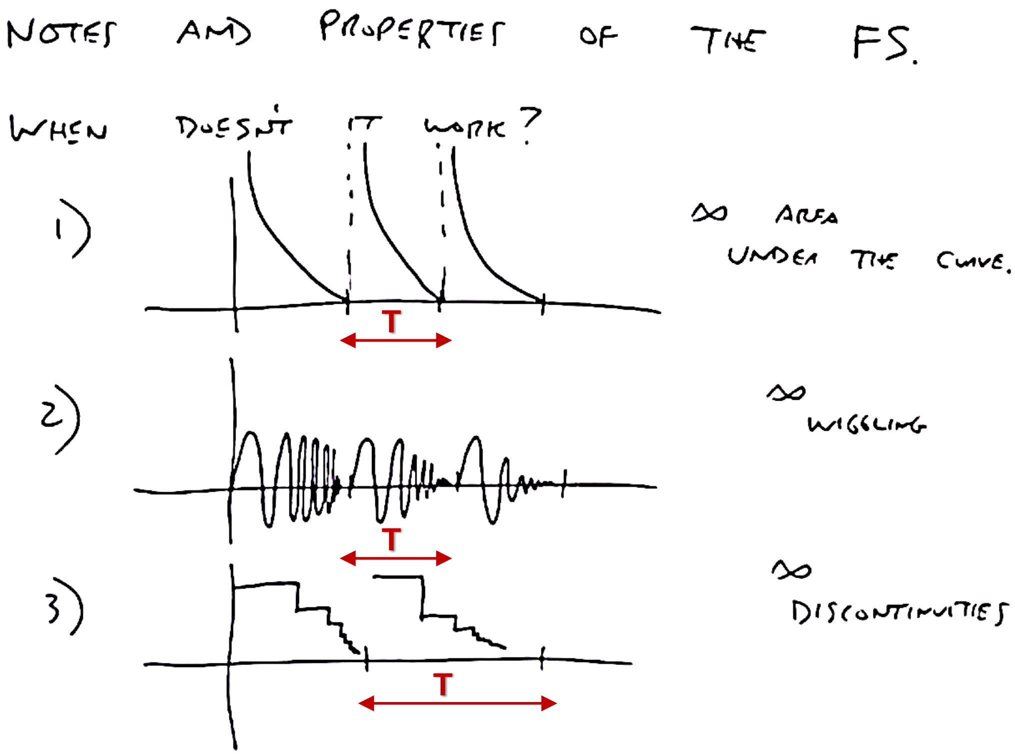
Fourier Series applet





- Just to refresh, changing the phase does not change the magnitude of the $\mathbf{a}_k$'s.

When can we not compute the Fourier Series?



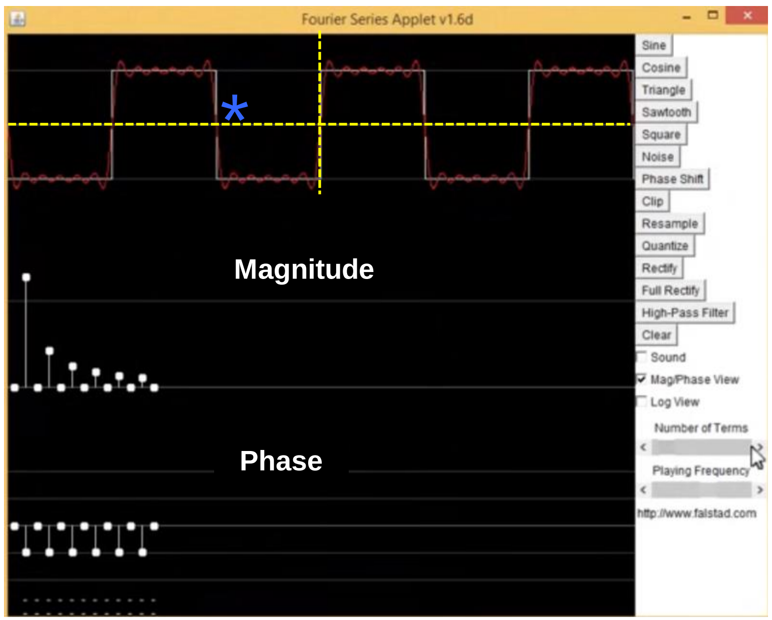
- **1)** One reason that **FS (Fourier series)** might not work is when we have some ***strange asymptote***. Here, the signal approaches the asymptote. The signal is periodic but we cannot make the Fourier series out of it. So, this is like a problem with ***infinite area under the curve***.
- **2)** The next one that is equally pathological is called ***infinite wiggling***. Here, we have a strange signal that as we approach the period, it wiggles faster and faster. So, we kind of never get there.
- **3)** The next one that is also pathologic is when we have ***infinite discontinuities*** in the signal.

Discontinuities and the Gibbs phenomenon

WHAT HAPPENS AT A DISCONTINUITY?

a) FS CONVERGES AT EVERY CONTINUOUS POINT

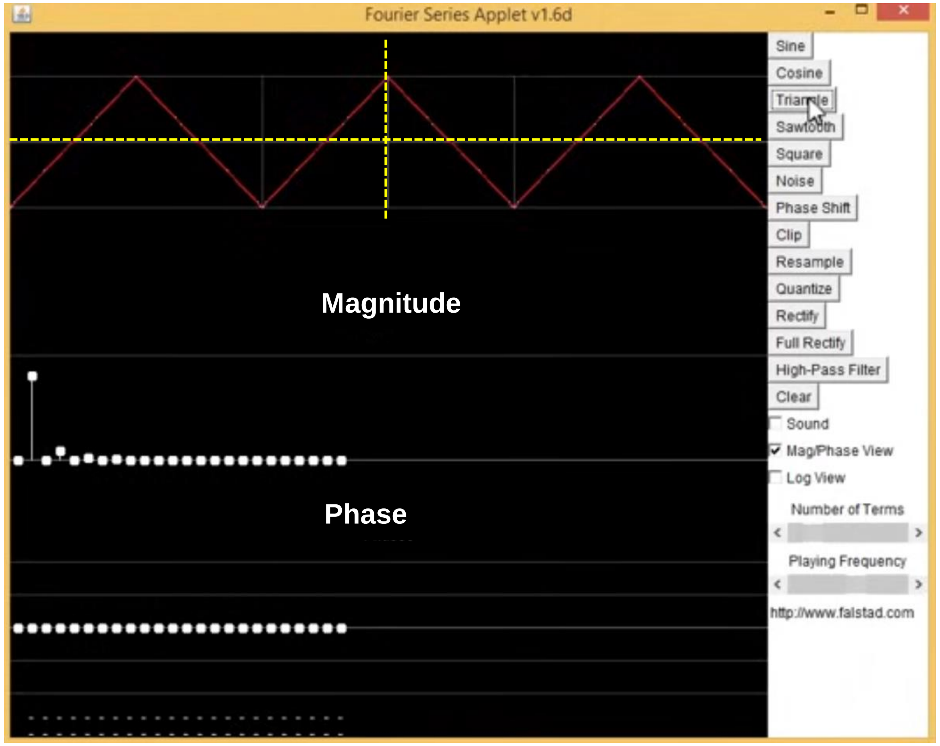
b) CONVERGES TO THE AVERAGE VALUE AT EVERY DISCONTINUITY.



- We can see that no matter how many terms we add, the value at *, where the signal is discontinuous, is **0**. The value is **+1** on one side of discontinuity and **-1** at the other side. So, no matter how many terms we add, the **red** signal is always going to be splitting the difference. We are never going to be able to do better than that.
- One thing to notice is that as the slope at the discontinuity approaches infinity we get a steeper and steeper curve, but we are always going to split the difference as to where that actually hits the discontinuity.

Discontinuities and the Gibbs phenomenon

- We do not have the problem of discontinuity with **triangular pulse**.



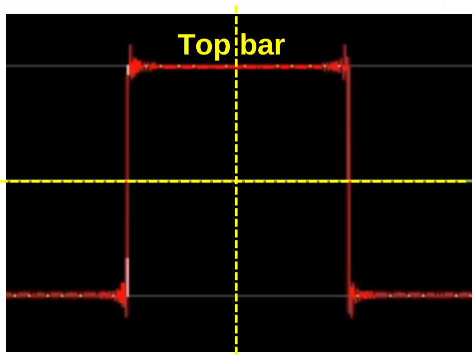
- Unlike in the case of a square pulse, a triangle has no discontinuity. Here, there is no problem with convergence. At every point, the Fourier series will converge and that is why even after adding only a few terms, everything looks so good.

Discontinuities and the Gibbs phenomenon

WHAT HAPPENS AT A DISCONTINUITY?

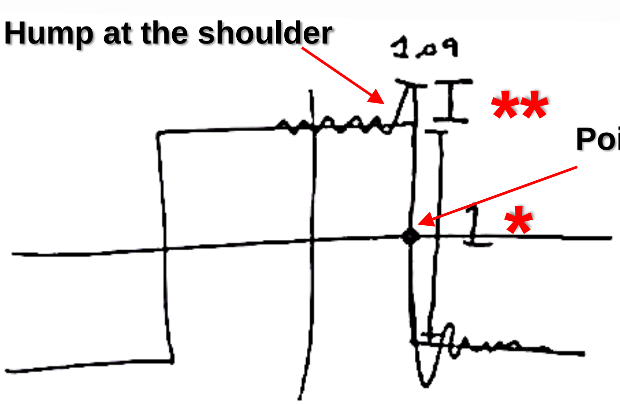
a) FS CONVERGES AT EVERY CONTINUOUS POINT

b) CONVERGES TO THE AVERAGE VALUE AT EVERY DISCONTINUITY.



GIBBS PHENOMENON

NEVER GET THE OVERSHOOT BELOW ≈ 9% OF THE HEIGHT OF THE DISCONTINUITY



- **Gibbs Phenomenon:** What happens at the **shoulder**, where there is a **hump**, is called the **Gibbs phenomenon**. It is the name for the behavior that happens at a discontinuity.
- The Gibbs phenomenon tells us that we can never get the overshoot below approximately **9%** of the height of discontinuity (shown by *, which is assumed to be equal to **1**). Here, given that the height is **1**, the height shown by ** is never going to be less than **9%** of the height of **1** or less than **1.09**.
- If we keep on adding more terms (i.e., amp up the number of terms), eventually the Fourier series will be continuous all the way across the **top bar** and it becomes a decreasing problem. But we can never get rid of the hump.

Properties of the Fourier Series (time shift, differentiation, Parseval, convolution)

PROPERTIES THAT MAKE FS EASIER.

WE HAVE $x(t)$ PERIODIC w/ PERIOD T,

FOURIER SERIES $\{a_k\}$ $\quad x(t) \longleftrightarrow \{a_k\}$

Time domain side — Fourier domain side

1) LINEARITY

$$x(t) \longleftrightarrow \{a_k\}$$
$$y(t) \longleftrightarrow \{b_k\}$$

$$\alpha x(t) + \beta y(t) \longleftrightarrow \{\alpha a_k + \beta b_k\} \quad *$$

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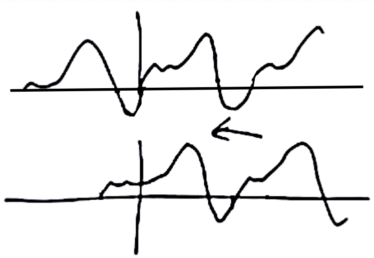
2) TIME SHIFTING

Original signal:

Shifted signal:



- Let us look at the properties that make Fourier series easier to work with.
- **Property #1 (Linearity):** If we have two signals that have given Fourier series and they both have the same period, then if we were to combine these two signals with a **linear combination**, the corresponding Fourier series is exactly what we expect, i.e., a linear combination of the Fourier series coefficients (shown by *).
- **Property #2 (Time Shifting):** If we have a periodic signal and if we were to shift the signal over by some constant phase shift, the idea is that the contents of the shifted signal, in terms of the frequency domain, should not change. The only thing that should change is the **phase** of the corresponding Fourier series coefficients.

Properties of the Fourier Series (time shift, differentiation, Parseval, convolution)

$$x(t) \longleftrightarrow \{a_k\}$$

LET $y(t) = x(t - t_0)$ *

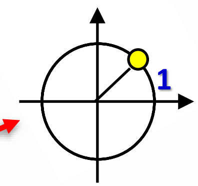
$$y(t) \longleftrightarrow a_k e^{-jk\omega_0 t_0}$$

<u>PHASE</u> SHIFT OF FS COEFFICIENTS.

MAGNITUDE DOESN'T CHANGE.

$$|b_k| = |a_k|$$

Unit Circle



- **More on Time Shifting Property:** Let us say we have the Fourier series coefficients of the original signal. That is, we have the $\mathbf{a_k}$'s set for **x(t)**. Then we let **y(t)** equal a time shifted version of the original signal (a delayed version). So, $\mathbf{y(t) = x(t - t_0)}$. Using time shifting property, we can conclude that the Fourier series coefficients of **y(t)** are simply shifted by $\mathbf{t_0}$ but the magnitude of the $\mathbf{a_k}$'s does not change. That is $|\mathbf{b_k}| = |\mathbf{a_k}|$.
- Note that when we see a number like *, it is just ***a point*** somewhere on the **unit circle** in the complex plane. As long as we are on the unit circle, the magnitude is always going to be equal to **1**.

Properties of the Fourier Series (time shift, differentiation, Parseval, convolution)

3) DIFFERENTIATION

$$x(t) \longleftrightarrow \{a_k\}$$

$$x'(t) \longleftrightarrow \{jk\omega_0 a_k\}$$

Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ **(Synthesis equation)**

$$x'(t) = \sum_{k=-\infty}^{\infty} (a_k jk\omega_0) e^{jk\omega_0 t}$$

- **Property #3 (Differentiation):** Using this property, if we have the Fourier series of **x(t)**, we can easily find **x´(t)**. All we need to do is to multiply the a_k's by $jk\omega_0$.
- So, the differentiation property of Fourier series allows us to differentiate the terms of a Fourier series representation of a function and obtain **the Fourier series representation of its derivative**.

Properties of the Fourier Series (time shift, differentiation, Parseval, convolution)

4) PARSEVAL

$$\frac{1}{T}\int_0^T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

AVERAGE POWER OF SIGNAL

POWER OF FS COEFFICIENTS

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5) CONVOLUTION

$$x(t) \leftrightarrow \{a_k\}$$
$$y(t) \leftrightarrow \{b_k\}$$

$$x(t)\,y(t) \longleftrightarrow \sum_{\ell=-\infty}^{\infty} a_\ell\, b_{k-\ell} = a * b \quad \textbf{(1)}$$

$$\int_0^T x(\tau)\,y(t-\tau)\,d\tau \longleftrightarrow T a_k b_k \quad \textbf{(2)}$$

- **Property #4 (Parseval's Theorem):** This theorem basically says that when we look at the ***average power of a signal***, that is the same thing as looking at the power of the Fourier series coefficients. So, it is another way of saying that there is the same amount of power in the time domain as there is in the frequency domain. It means that there is no information lost by converting between the two domains.
- **Property #5 (Convolution):** If we have two time domain signals and we multiply them in the time domain, then what we get is the equivalent of convolving the two sets of numbers ($\mathbf{a}_k$'s and $\mathbf{b}_k$'s sets). In **(1)**, this is like invoking the convolution of the two vectors of Fourier series coefficients.
- Another form of this theorem (which is more useful than **(1)**), is when we are doing the integral **(2)**. This is the **convolution integral**, which is really the property that we care about for LTI systems. So, convolution in the time domain becomes multiplication in frequency domain. This makes life much easier!

End of Lecture 4