

# Asymptotic Behavior of the Number of Interval Sums

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## Abstract

Interval sum is sum of consecutive terms in a sequence. An asymptotic formula is found for the number of interval sums of a class of sequences. Namely  $(f(i))_{i=1}^{\infty}$ , where  $f$  is measurable, locally bounded away from 0 and  $\infty$  and eventually monotonic with index  $\alpha > 1$ . This extends results of O'Sullivan et al. on interval sums of prime powers.

The main theorem provides an asymptotic formula for the interval sum counting function

$$S(x) = \# \left\{ (i, j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x \right\}. \quad (1)$$

That is

$$S(x) \sim n^2 C(\alpha) \quad n = n(x) \rightarrow \infty \quad (2)$$

where  $x \sim \frac{n f(n)}{\alpha+1}$ ,  $x \rightarrow \infty$  and

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1}) \Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \quad (3)$$

The proof relies on classical results from the theory of regular variation, in particular the uniform convergence theorem and Potter's bound. The function  $S(x)$  can be split into a sum of  $s_i$  that count the number of intervals of length  $i$ . The main idea in the proof is that  $s_i$  can be related to the inverse of

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \quad (4)$$

which allows us to relate  $S(x)$  asymptotically to

$$n^2 \int_0^1 \phi^{-1}(t) dt. \quad (5)$$

The integral can be calculated using standard substitution and using analytically extended beta-function, which completes the proof.

# 1 Introduction

Interval sum is sum of consecutive terms in a sequence. Determining the asymptotic behavior of the count of interval sums less than  $x$  has been investigated for primes and prime powers [3, 4, 5]. Moser [3] determined asymptotic behavior of interval sums of primes. Tongsomporn [5] et al. derived strict bounds for the number of interval sums of prime squares. O’Sullivan et al. [4] found asymptotic for the number of interval sums of prime powers.

To the best of our knowledge, no asymptotic formula is known for more general sequences. An asymptotic formula is found for the number of interval sums for a class of sequences. Namely  $(f(i))_{i=1}^{\infty}$ , where  $f$  is measurable, locally bounded away from 0 and  $\infty$  and eventually monotonic with index  $\alpha > 1$ .

Our main theorem provides an asymptotic for the interval sum counting function

$$S(x) = \# \left\{ (i, j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x \right\}. \quad (6)$$

That is

$$S(x) \sim n^2 C(\alpha) \quad n = n(x) \rightarrow \infty \quad (7)$$

where  $x \sim \frac{nf(n)}{\alpha+1}, x \rightarrow \infty$  and

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \quad (8)$$

The proof relies on classical results from the theory of regular variation, in particular the uniform convergence theorem and Potter’s bound [1]. The function  $S(x)$  can be split into a sum of  $s_i$  that count the number of intervals of length  $i$ . The main idea in the proof is that  $s_i$  can be related to the inverse of

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \quad (9)$$

which allows us to calculate the exact coefficient in the asymptotic.

This paper is organized as follows. Section 2 defines our notation. Section 3 first proves basic lemmas about asymptotics, then a lemma that calculates the integral of the inverse of  $\phi$ , and finally a lemma that relates  $\phi$  and  $s_i$ . Section 4 proves the main theorem 1 and one technical lemma bounding the sum of  $s_i$  for small  $i$ .

## 2 Notation

- $RV_\alpha^+$  – measurable, eventually monotonic regularly varying (see definition 1 or [1]) functions locally bounded away from 0 and  $\infty$  with index  $\alpha$  (see [1]).
- $S(x) = \# \{(i, j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \dots + f(j) \leq x\}$  – the number of interval sums below  $x$ .
- $s_k(x) = \# \{(i, j) \in \mathbb{N}^2 \mid j - i = k - 1, f(i) + f(i+1) + \dots + f(j) \leq x\}$  – the number of interval sums of length  $k$  below  $x$ .
- $f(x) \sim g(x), x \rightarrow a$  – meaning  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$ . The part " $x \rightarrow a$ " is often omitted, when it is clear from context.
- $f(x) \lesssim g(x)$  – meaning  $\limsup \frac{f(x)}{g(x)} \leq 1$
- $f(x) \gtrsim g(x)$  – meaning  $\liminf \frac{f(x)}{g(x)} \geq 1$ .
- sums of the form

$$\sum_{i=x}^y F(i). \tag{10}$$

are denoted with

$$\sum_{x \leq i < y} F(i) \tag{11}$$

- $f(x) = O(g(x))$  – there exists  $C > 0$  such that  $|f(x)| \leq Cg(x)$  for all sufficiently large  $x$ .
- $f(x) = o(g(x))$  – shorthand for

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$$

## 3 Preliminaries

**Definition 1.** Function  $f: [0, \infty) \rightarrow [0, \infty)$  is **regularly varying** with **index**  $\alpha$  if for all  $\lambda > 0$

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha. \tag{12}$$

**Definition 2.** Function  $L: [0, \infty) \rightarrow [0, \infty)$  is *slowly varying* if for all  $\lambda > 0$

$$\lim_{x \rightarrow \infty} \frac{L(\lambda x)}{L(x)} = 1. \quad (13)$$

It is assumed throughout this paper that  $n = n(x)$  is the unique integer satisfying

$$\sum_{i=1}^n f(i) \leq x < \sum_{i=1}^{n+1} f(i) \quad (14)$$

where  $f \in RV_\alpha^+$ , that is  $f(x) = x^\alpha L(x)$  (see section 1.5 in [1]), for some slowly varying function  $L$ . That is,  $n$  is the length of the largest interval.

**Lemma 1.** *We have*

$$x \sim \frac{n^{\alpha+1} L(n)}{\alpha + 1} = \frac{n f(n)}{\alpha + 1} \quad (15)$$

for  $\alpha > -1$ .

*Proof.* This is a direct consequence of Karamata's theorem (see proposition 1.5.8 in [1] or Proposition 1 in the appendix).  $\square$

**Lemma 2.** *Let  $\varepsilon > 0$  and let  $k, l > 0$  such that  $k - l > \varepsilon$ . If  $\alpha > 1$  and*

$$\sum_{i=nl}^{nk} f(i) < x \quad (16)$$

*then uniformly in  $k$  and  $l$*

$$\sum_{i=nl}^{nk} f(i) \sim \frac{1}{\alpha + 1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}). \quad (17)$$

*Proof.* From the definition of  $n$  it follows that  $k \leq l + 1$ . Since  $f$  is eventually increasing

$$\sum_{i=nl}^{nk} f(i) > n(k - l)f(nl) > n\varepsilon f(nl). \quad (18)$$

Using Potter's bound (see Theorem 1.5.6 [1] or Theorem 2 in the Appendix) and Lemma 1

$$\frac{1}{\alpha + 1} n^{\alpha+1} L(n) \gtrsim n\varepsilon f(nl) \quad (19)$$

$$= n^{\alpha+1} \varepsilon l^\alpha L(nl) \quad (20)$$

$$\gtrsim \varepsilon n^{\alpha+1} L(n) l^\alpha \frac{1}{2^{\max(l, l^{-1})}}. \quad (21)$$

Thus

$$l \lesssim \max \left( \left( \frac{2}{\varepsilon(\alpha+1)} \right)^{\frac{1}{\alpha+1}}, \left( \frac{2}{\varepsilon(\alpha+1)} \right)^{\frac{1}{\alpha-1}} \right). \quad (22)$$

Now  $l$  and  $k$  are bounded above and below, so we can use the uniform convergence theorem (see Theorem 1.2.1 in [1] or Theorem 1 in the Appendix)

$$\sum_{i=nl}^{nk} f(i) \sim \sum_{i=nl}^{nk} i^\alpha L(n) \quad (23)$$

$$= L(n) \sum_{i=nl}^{nk} i^\alpha \quad (24)$$

$$\sim L(n) \int_{nl}^{nk} t^\alpha dt \quad (25)$$

$$= \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}). \quad (26)$$

Since the uniform convergence theorem applies uniformly this convergence is also uniform.  $\square$

**Lemma 3.** *The function  $\phi : (0, \infty) \rightarrow (0, 1)$ ,  $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$  is a strictly decreasing bijection, has an inverse and*

$$\int_0^1 \phi^{-1}(t) dt = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \quad (27)$$

for  $\alpha > 1$ .

*Proof.* Taking the derivative of  $\phi$

$$\frac{d\phi}{dx} = (1 + x^{\alpha+1})^{-\frac{\alpha}{\alpha+1}} x^\alpha - 1 \quad (28)$$

$$= \frac{x^\alpha}{\left( (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} \right)^\alpha} - 1 \quad (29)$$

$$< 1 - 1 = 0 \quad (30)$$

where the last inequality follows from  $(1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} > x$ . Because the derivative of  $\phi$  is negative everywhere,  $\phi$  is strictly decreasing. Now  $\phi$  is a bijection, because  $\phi(0) = 1$  and  $\lim_{x \rightarrow \infty} \phi(x) = 0$ , hence has an inverse.

Since  $\phi$  is a strictly decreasing bijection

$$\int_0^1 \phi^{-1}(x) dx = \int_0^\infty \phi(x) dx. \quad (31)$$

Next we show that the integral converges. Since  $x^{\frac{1}{\alpha+1}}$  is increasing with a decreasing derivative, we have  $(1+y)^{\frac{1}{\alpha+1}} < y^{\frac{1}{\alpha+1}} + \frac{1}{\alpha+1}y^{-\frac{\alpha}{\alpha+1}}$ . Thus

$$\phi(x) < x^{-\alpha}, \quad (32)$$

so the integral converges.

Doing a change of variables  $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$ ,  $x = (\frac{t}{1-t})^{\frac{1}{\alpha+1}}$ , we get

$$dx = \frac{1}{\alpha+1} \left( \frac{t}{1-t} \right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \quad (33)$$

and

$$\phi(x) = (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \quad (34)$$

$$= \left( \frac{1}{1-t} \right)^{\frac{1}{\alpha+1}} - \left( \frac{t}{1-t} \right)^{\frac{1}{\alpha+1}} \quad (35)$$

$$= (1-t)^{-\frac{1}{\alpha+1}} (1-t^{\frac{1}{\alpha+1}}). \quad (36)$$

Putting it together

$$\int_0^\infty \phi(x) dx = \int_0^1 (1-t)^{-\frac{1}{\alpha+1}} (1-t^{\frac{1}{\alpha+1}}) \frac{1}{\alpha+1} \left( \frac{t}{1-t} \right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \quad (37)$$

$$= \frac{1}{\alpha+1} \int_0^1 (1-t)^{\frac{\alpha}{\alpha+1} - \frac{1}{\alpha+1} - 2} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1} - \frac{\alpha}{\alpha+1}}) dt \quad (38)$$

$$= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{2}{\alpha+1} - 1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt. \quad (39)$$

Let  $g_r(t) = (1-t)^{r - \frac{2}{\alpha+1} - 1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$  and  $g(t) = (1-t)^{-\frac{2}{\alpha+1} - 1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$ . Since for each  $r > 0$ ,  $g_r(t) \rightarrow g(t)$ ,  $r \rightarrow 0$  pointwise and  $|g_r(t)| < |g(t)|$ , we can use the dominated convergence theorem. Thus

$$\int_0^1 (1-t)^{-\frac{2}{\alpha+1} - 1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt \quad (40)$$

$$= \lim_{r \rightarrow 0^+} \int_0^1 (1-t)^{r - \frac{2}{\alpha+1} - 1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt. \quad (41)$$

Let  $B$  be the beta function (see NIST DLMF § 5.12.1 [2])

$$B(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} \quad (42)$$

The beta function can be analytically extended using Pochhammer's integral (see NIST DLMF § 5.12.12 [2]).

$$= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt \quad (43)$$

$$= \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \int_0^1 (1-t)^{r-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt \quad (44)$$

$$= \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \left( B\left(\frac{1}{\alpha+1}, r - \frac{2}{\alpha+1}\right) - B\left(\frac{2}{\alpha+1}, r - \frac{2}{\alpha+1}\right) \right). \quad (45)$$

Using the gamma expression for beta functions we get

$$\int_0^\infty \phi(x) dx = \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \left( \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(r - \frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1} + r - \frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(r - \frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1} + r - \frac{2}{\alpha+1})} \right) \quad (46)$$

$$= \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \left( \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)} \right). \quad (47)$$

Since  $\Gamma(t) = \frac{\Gamma(1+t)}{t}$  we have  $\Gamma(t) \sim \frac{1}{t}, t \rightarrow 0$ . Thus

$$\lim_{r \rightarrow 0} \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)} = 0 \quad (48)$$

so the expression simplifies to

$$\int_0^\infty \phi(x) dx = \frac{1}{\alpha+1} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \quad (49)$$

$$= \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \quad (50)$$

where the last equality follows from  $z\Gamma(z) = \Gamma(z+1)$ . □

**Lemma 4.** *Let  $\varepsilon > 0$  and  $i > \varepsilon n$ , then  $s_i(x)$  has the following asymptotic*

$$s_i(x) \sim n\phi^{-1}\left(\frac{i}{n}\right) \quad (51)$$

where  $\phi(x) = (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ .

*Proof.* Let  $l, k > 0$  be such that  $n(k-l) = i$  and

$$\sum_{j=nl}^{nk} f(j) \leq x < \sum_{j=nl+1}^{nk+1} f(j) \quad (52)$$

Clearly it follows that  $l$  and  $k$  are unique up to producing the same range of integer indices in the sum.

Using Lemma 2 and Lemma 1

$$\frac{1}{\alpha+1} L(n)n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}) \sim \sum_{j=nl}^{nk} f(j) \sim x \sim \sum_{j=1}^n f(j) \sim \frac{n^{\alpha+1}L(n)}{\alpha+1}. \quad (53)$$

Thus

$$\frac{n^{\alpha+1}L(n)}{\alpha+1}(k^{\alpha+1}-l^{\alpha+1}) \sim \frac{n^{\alpha+1}L(n)}{\alpha+1} \quad (54)$$

$$k^{\alpha+1}-l^{\alpha+1} \sim 1 \quad (55)$$

$$k-l \sim (1+l^{\alpha+1})^{\frac{1}{\alpha+1}}-l \quad (56)$$

$$i \sim n((1+l^{\alpha+1})^{\frac{1}{\alpha+1}}-l) \quad (57)$$

$$ln \sim n\phi^{-1}\left(\frac{i}{n}\right). \quad (58)$$

Lemma 3 proves that  $\phi^{-1}$  is well defined. Since  $s_i(x)$  counts the number of intervals of length  $i$ , we get

$$s_i(x) = nl \sim n\phi^{-1}\left(\frac{i}{n}\right). \quad (59)$$

□

## 4 Proof of the Main Theorem

**Lemma 5.** *Let  $\varepsilon > 0$  and  $\alpha > 1$ , then*

$$\sum_{i=1}^{n\varepsilon} s_i(x) = O(n^2)\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})}, \quad (60)$$

where the  $O(n^2)$  term is independent of  $\varepsilon$ .

*Proof.* Since

$$x \geq f(i) + \dots + f(i+k-1) \geq kf(i) \quad (61)$$

$$f^{-1}\left(\frac{x}{k}\right) \geq i, \quad (62)$$

it follows that  $s_k(x) \leq f^{-1}\left(\frac{x}{k}\right)$ . The inverse of a regularly varying function is regularly varying with index

$\frac{1}{\alpha}$  (see Theorem 1.5.12 of [1] or Theorem 3 in the Appendix). Let  $f^{-1}(x) = x^{\frac{1}{\alpha}}\tilde{L}(x)$ . We get

$$\sum_{i=1}^{n\varepsilon} f^{-1}\left(\frac{x}{i}\right) \sim \int_1^{n\varepsilon} f^{-1}\left(\frac{x}{t}\right) dt = x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt. \quad (63)$$

By Potter's bound

$$\tilde{L}\left(\frac{x}{t}\right) \leq M\tilde{L}\left(\frac{x}{n}\right)\left(\frac{n}{t}\right)^{\delta}, \quad (64)$$



where  $M$  is the constant from Potter's bound and  $\delta = \frac{1}{2}(1 - \frac{1}{\alpha})$ . Now we can bound the integral

$$x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt \leq x^{\frac{1}{\alpha}} M \tilde{L}\left(\frac{x}{n}\right) n^\delta \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta}. \quad (65)$$

By definition

$$f^{-1}(f(n)) \sim n \quad (66)$$

$$n \tilde{L}(n^\alpha L(n)) L(n)^{1/\alpha} \sim n \quad (67)$$

$$\tilde{L}(n^\alpha L(n)) L(n)^{1/\alpha} \sim 1. \quad (68)$$

Using the relation between  $x$  and  $n$  obtained in Lemma 1

$$\tilde{L}\left(\frac{x}{n}\right) \sim \tilde{L}(n^\alpha L(n)). \quad (69)$$

Thus from equations 69 and 68

$$x^{\frac{1}{\alpha}} M \tilde{L}\left(\frac{x}{n}\right) n^\delta \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \sim M n^2 L(n)^{\frac{1}{\alpha}} \tilde{L}(n^\alpha L(n)) \frac{2\alpha}{\alpha-1} \varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})} \quad (70)$$

$$= O(n^2) \varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})} \quad (71)$$

□

**Theorem 1.** *The function  $S(x)$  has the asymptotic  $n^2 C(\alpha)$  for  $\alpha > 1$ , where*

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1}) \Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \quad (72)$$

*Proof.* Let  $\varepsilon > 0$ . From the definition of  $S$  and  $s_i$  it follows that

$$S(x) = \sum_{i=1}^n s_i(x). \quad (73)$$

Splitting the sum

$$S(x) = \sum_{i=1}^{n\varepsilon} s_i(x) + \sum_{i=n\varepsilon}^n s_i(x). \quad (74)$$

Using Lemma 4 we get

$$\sum_{i=n\varepsilon}^n s_i(x) \sim \sum_{i=n\varepsilon}^n n\phi^{-1}\left(\frac{i}{n}\right) \quad (75)$$

$$= n \sum_{i=n\varepsilon}^n \phi^{-1}\left(\frac{i}{n}\right) \quad (76)$$

$$\sim n \int_{n\varepsilon}^n \phi^{-1}\left(\frac{t}{n}\right) dt \quad (77)$$

$$= n^2 \int_{\varepsilon}^1 \phi^{-1}(t) dt. \quad (78)$$

Since the integral converges

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \phi^{-1}(t) dt = \int_0^1 \phi^{-1}(t) dt \quad (79)$$

From Lemma 5 it follows that the first sum is negligible in the asymptotics, so taking  $\varepsilon \rightarrow 0$  finishes the proof.  $\square$

**Corollary 1.** *The set of distinct interval sums*

$$S'(x) = \left\{ y \mid \sum_{n=i}^j f(n) = y < x \right\} \quad (80)$$

has inequality  $|S'(x)| \lesssim n^2 C(\alpha)$ .

*Proof.* Since  $|S'(x)| \leq S(x)$ , this is direct consequence of theorem 1.  $\square$

**Corollary 2.** *If furthermore  $n \in \mathbb{N} \implies f(n) \in \mathbb{N}$ , then natural density of  $S' = \bigcup_{x=1}^{\infty} S'(x)$  is 0.*

*Proof.* This is equivalent to

$$\lim_{x \rightarrow \infty} \frac{S'(x)}{x} = 0 \quad (81)$$

Using corollary 1 and lemma 1 we get

$$\frac{S'(x)}{x} \lesssim \frac{n^2 C(\alpha)}{n^{1+\alpha}/(\alpha+1)} = O(n^{1-\alpha}). \quad (82)$$

Since  $1 - \alpha < 0$  the claim follows.  $\square$

## 5 Convergence of the Asymptotic Formula

Figures 1 and 2 show the impact of the slowly varying part and the index on convergence, respectively. We can observe that increasing the index, slows down the convergence, and a larger growth rate in the slowly varying part corresponds to slower convergence.

See the table 1 below for relative error of the asymptotic at  $x = 10^{20}$  for varying functions.

$f(x)$	relative error at $x = 10^{20}$
$x^2$	0.037%
$x^3$	0.045%
$x^4$	0.194%
$x^3 \log(x)$	1.433%
$x^3 \log(x)^2$	2.836%

Table 1: Relative error of the asymptotic at  $x = 10^{20}$  for various functions.

These findings align with our analysis: the proofs rely on the uniform convergence theorem which gives slower convergence for larger growth rates and the convergence of the sum to integral is slower for larger indices.

## 6 Conclusion

The main theorem gives an asymptotic for the number of interval sums below  $x$ , namely  $S(x) \sim n^2 C(\alpha)$ . It remains an open question what happens in the case  $\alpha \leq 1$ ; our methods can't handle that case since the integral in Lemma 3 diverges. Future work could therefore investigate the case  $\alpha \leq 1$ , which requires a different approach as the integral for  $C(\alpha)$  diverges.

## 7 Use of AI

AI has been used in making of this article. AI was used to finalize writing, proof checking and in finding u-substitution ideas to solve the integral in lemma 3. The AI models used were GPT-5 mini, GPT-5.1,

Gemini 3 Flash and Gemini 3 Pro. Gemini was only used for writing assistance and GPT was also used for proof checking and u-substitution ideas. Everything AI produced was thoroughly checked.

## References

- [1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.
- [2] *NIST Digital Library of Mathematical Functions*. <https://dlmf.nist.gov/>, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [3] L. Moser. On the sum of consecutive primes. *Canadian Mathematical Bulletin*, 6(2):159–161, 1963.
- [4] C. O’Sullivan, J. P. Sorenson, and A. Stahl. Algorithms and bounds on the sums of powers of consecutive primes. *Integers*, 24, 2024.
- [5] A. Tongsonporn, S. Wananiyakul, and J. Steuding. Sums of consecutive prime squares. *Integers*, 22, 2022.