Asymptotic Behavior of the Number of Interval Sums

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Abstract

In this paper we extend the results of O'Sullivan et al. on interval sums of prime powers to the broader class of regularly varying functions that are measurable, eventually monotonic and locally bounded away from 0 and ∞. We find an asymptotic formula for the number of interval sums below x. Our approach relies on approximating the number of fixed length interval sums using the function $\phi(x) = (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ that comes naturally from inverting the asymptotic relation. The contribution from the number of interval sums of small length is negligible, allowing us to obtain the full asymptotic.

1 Introduction

Determining the asymptotic behavior of the count of interval sums less than x has been investigated for primes and prime powers [3][4][5]. To the best of our knowledge, no asymptotic formula is known for more general sequences. In this paper we find an asymptotic formula for a class of sequences, namely those $\big(f(i)\big)_{i=1}^{\infty}$, where f is measurable, locally bounded away from 0 and ∞ and eventually monotonic with index $\alpha > 1$.

Our main theorem provides an asymptotic for the interval sum counting function

$$S(x) = \#\left\{(i,j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x\right\}. \tag{1}$$

That is, we have

$$S(x) \sim n^2 C(\alpha) \qquad n = n(x) \to \infty \tag{2}$$

where $x \sim \frac{nf(n)}{\alpha+1}, x \to \infty$ and

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \tag{3}$$

The proof relies on classical results from the theory of regular variation, in particular the uniform convergence theorem and Potter's bound [1]. We can split the $S(x)$ into a sum of s_i that count the number of intervals of length i. The main idea in the proof is that we can relate s_i to the inverse of

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \tag{4}$$

which allows us to calculate the exact coefficient in the asymptotic.

This paper is organized as follows. Section 2 defines our notation. Section 3 first proves basic lemmas about asymptotics, then a lemma that calculates the integral of the inverse of ϕ, and finally a lemma that relates ϕ and s_i. Section 4 proves the main theorem 1 and one technical lemma bounding the sum of s_i for small i.

2 Notation

- RV_α^+ – measurable, eventually monotonic regularly varying (see definition 1 or [1]) functions locally bounded away from 0 and ∞ with index α (see [1]). We assume the domain and range are $\mathbb{R}_{>0}$.
- $S(x) = \#\left\{(i,j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x\right\}$ – the number of interval sums below x.
- $s_k(x) = \#\left\{(i,j) \in \mathbb{N}^2 \mid j - i = k - 1, f(i) + f(i+1) + \cdots + f(j) \leq x\right\}$ – the number of interval sums of length k below x.
- $f(x) \sim g(x), x \to a$ – meaning $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. We often omit the "$x \to a$" part, when it is clear from context.
- $f(x) \lesssim g(x)$ – meaning $\limsup \frac{f(x)}{g(x)} \leq 1$
- $f(x) \gtrsim g(x)$ – meaning $\liminf \frac{f(x)}{g(x)} \geq 1$.
- We denote sums of the form

$$\sum_{i=x}^{y} F(i). \tag{5}$$

with

$$\sum_{x \le i < y} F(i) \tag{6}$$

- $f(x) = O(g(x))$ – there exists $C > 0$ such that $|f(x)| \le Cg(x)$ for all sufficiently large x.

- $f(x) = o(g(x))$ – shorthand for

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

3 Preliminaries

Definition 1. *Function $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is **regularly varying** with index α if for all $\lambda > 0$*

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha. \tag{7}$$

Definition 2. *Function $L\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is **slowly varying** if for all $\lambda > 0$*

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1. \tag{8}$$

We assume throughout this paper that $n = n(x)$ is the unique integer satisfying

$$\sum_{i=1}^{n} f(i) \le x < \sum_{i=1}^{n+1} f(i) \tag{9}$$

where $f \in RV_\alpha^+$, that is $f(x) = x^\alpha L(x)$ (see section 1.5 in [1]), for some slowly varying function L. That is, n is the length of the largest interval.

Lemma 1. *We have*

$$x \sim \frac{n^{\alpha+1}L(n)}{\alpha+1} = \frac{nf(n)}{\alpha+1} \tag{10}$$

for $\alpha > -1$.

Proof. This is a direct consequence of Karamata's theorem (see proposition 1.5.8 in [1] or Proposition 1 in the appendix). □

Lemma 2. *Let $\varepsilon > 0$ and let $k, l > 0$ such that $k - l > \varepsilon$. If $\alpha > 1$ and*

$$\sum_{i=nl}^{nk} f(i) < x \tag{11}$$

then uniformly in k and l

$$\sum_{i=nl}^{nk} f(i) \sim \frac{1}{\alpha+1} L(n) n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}). \tag{12}$$

Proof. From the definition of n it follows that $k \leq l+1$. Since f is eventually increasing

$$\sum_{i=nl}^{nk} f(i) > n(k-l)f(nl) > n\varepsilon f(nl). \tag{13}$$

Using Potter's bound (see Theorem 1.5.6 [1] or Theorem 2 in the Appendix) and Lemma 1

$$\frac{1}{\alpha+1} n^{\alpha+1} L(n) \gtrsim n\varepsilon f(nl) \tag{14}$$

$$= n^{\alpha+1} \varepsilon l^{\alpha} L(nl) \tag{15}$$

$$\gtrsim \varepsilon n^{\alpha+1} L(n) l^{\alpha} \frac{1}{2\max(l, l^{-1})}. \tag{16}$$

Thus

$$l \lesssim \max\left(\left(\frac{2}{\varepsilon(\alpha+1)}\right)^{\frac{1}{\alpha+1}}, \left(\frac{2}{\varepsilon(\alpha+1)}\right)^{\frac{1}{\alpha-1}}\right). \tag{17}$$

Now l and k are bounded above and below, so we can use the uniform convergence theorem (see Theorem 1.2.1 in [1] or Theorem 1 in the Appendix)

$$\sum_{i=nl}^{nk} f(i) \sim \sum_{i=nl}^{nk} i^{\alpha} L(n) \tag{18}$$

$$= L(n) \sum_{i=nl}^{nk} i^{\alpha} \tag{19}$$

$$\sim L(n) \int_{nl}^{nk} t^{\alpha} dt \tag{20}$$

$$= \frac{1}{\alpha+1} L(n) n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}). \tag{21}$$

Since the uniform convergence theorem applies uniformly this convergence is also uniform. □

Lemma 3. *The function $\phi : (0, \infty) \to (0, 1)$, $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ is a strictly decreasing bijection, has an inverse and*

$$\int_0^1 \phi^{-1}(t) dt = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \tag{22}$$

for $\alpha > 1$.

Proof. Taking the derivative of ϕ

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = (1+x^{\alpha+1})^{-\frac{\alpha}{\alpha+1}} x^{\alpha} - 1 \tag{23}$$

$$= \frac{x^{\alpha}}{\left((1+x^{\alpha+1})^{\frac{1}{\alpha+1}}\right)^{\alpha}} - 1 \tag{24}$$

$$< 1 - 1 = 0 \tag{25}$$

where the last inequality follows from $(1+x^{\alpha+1})^{\frac{1}{\alpha+1}} > x$. Because the derivative of ϕ is negative everywhere, ϕ is strictly decreasing. Now ϕ is a bijection, because $\phi(0) = 1$ and $\lim_{x\to\infty} \phi(x) = 0$, hence has an inverse. Since ϕ is a strictly decreasing bijection

$$\int_0^1 \phi^{-1}(x)dx = \int_0^{\infty} \phi(x)dx. \tag{26}$$

Next we show that the integral converges. Since $x^{\frac{1}{\alpha+1}}$ is increasing with a decreasing derivative, we have $(1+y)^{\frac{1}{\alpha+1}} < y^{\frac{1}{\alpha+1}} + \frac{1}{\alpha+1} y^{-\frac{\alpha}{\alpha+1}}$. Using this we get

$$\phi(x) < x^{-\alpha}, \tag{27}$$

so the integral converges.

Doing a change of variables $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$, $x = (\frac{t}{1-t})^{\frac{1}{\alpha+1}}$, we get

$$dx = \frac{1}{\alpha+1}\left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \tag{28}$$

and

$$\phi(x) = (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \tag{29}$$

$$= \left(\frac{1}{1-t}\right)^{\frac{1}{\alpha+1}} - \left(\frac{t}{1-t}\right)^{\frac{1}{\alpha+1}} \tag{30}$$

$$= (1-t)^{-\frac{1}{\alpha+1}}(1-t^{\frac{1}{\alpha+1}}). \tag{31}$$

Putting it together

$$\int_0^{\infty} \phi(x)dx = \int_0^1 (1-t)^{-\frac{1}{\alpha+1}}(1-t^{\frac{1}{\alpha+1}})\frac{1}{\alpha+1}\left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \tag{32}$$

$$= \frac{1}{\alpha+1}\int_0^1 (1-t)^{\frac{\alpha}{\alpha+1}-\frac{1}{\alpha+1}-2}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1}-\frac{\alpha}{\alpha+1}})dt \tag{33}$$

$$= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt. \tag{34}$$

Let $g_r(t) = (1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$ and $g(t) = (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$. Since for each $r > 0$, $g_r(t) \to g(t), r \to 0$ pointwise and $|g_r(t)| < |g(t)|$, we can use the dominated convergence theorem. Thus

$$\int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \tag{35}$$

$$= \lim_{r\to 0^+} \int_0^1 (1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt. \tag{36}$$

Let B be the beta function (see NIST DLMF § 5.12.1 [2])

$$B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} \tag{37}$$

The beta function can be analytically extended to $(\mathbb{C}\setminus\mathbb{Z}, \mathbb{C}\setminus\mathbb{Z})$ using Pochhammer's integral (see NIST DLMF § 5.12.12 [2]).

$$= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \tag{38}$$

$$= \frac{1}{\alpha+1}\lim_{r\to 0^+}\int_0^1 (1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \tag{39}$$

$$= \frac{1}{\alpha+1}\lim_{r\to 0^+}\left(B\left(\frac{1}{\alpha+1}, r-\frac{2}{\alpha+1}\right) - B\left(\frac{2}{\alpha+1}, r-\frac{2}{\alpha+1}\right)\right). \tag{40}$$

Using the gamma expression for beta functions we get

$$\int_0^\infty \phi(x)dx = \frac{1}{\alpha+1}\lim_{r\to 0^+}\left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(r-\frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1}+r-\frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(r-\frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1}+r-\frac{2}{\alpha+1})}\right) \tag{41}$$

$$= \frac{1}{\alpha+1}\lim_{r\to 0^+}\left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)}\right). \tag{42}$$

Since $\Gamma(t) = \frac{\Gamma(1+t)}{t}$ we have $\Gamma(t) \sim \frac{1}{t}, t \to 0$. Thus

$$\lim_{r\to 0}\frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)} = 0 \tag{43}$$

so the expression simplifies to

$$\int_0^\infty \phi(x)dx = \frac{1}{\alpha+1}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \tag{44}$$

$$= \frac{1}{2(\alpha+1)}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \tag{45}$$

where the last equality follows from $z\Gamma(z) = \Gamma(z+1)$. □

Lemma 4. *Let $\varepsilon > 0$ and $i > \varepsilon n$, then $s_i(x)$ has the following asymptotic*

$$s_i(x) \sim n\phi^{-1}\left(\frac{i}{n}\right) \tag{46}$$

where $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$.

Proof. Let $l, k > 0$ be such that $n(k - l) = i$ and

$$\sum_{j=nl}^{nk} f(j) \leq x < \sum_{j=nl+1}^{nk+1} f(j) \tag{47}$$

Clearly it follows that l and k are unique up to producing the same range of integer indices in the sum. Using Lemma 2 and Lemma 1

$$\frac{1}{\alpha+1}L(n)n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}) \sim \sum_{j=nl}^{nk} f(j) \sim x \sim \sum_{j=1}^{n} f(j) \sim \frac{n^{\alpha+1}L(n)}{\alpha+1}. \tag{48}$$

Thus

$$\frac{n^{\alpha+1}L(n)}{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}) \sim \frac{n^{\alpha+1}L(n)}{\alpha+1} \tag{49}$$

$$k^{\alpha+1} - l^{\alpha+1} \sim 1 \tag{50}$$

$$k - l \sim (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \tag{51}$$

$$i \sim n((1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l) \tag{52}$$

$$ln \sim n\phi^{-1}\left(\frac{i}{n}\right). \tag{53}$$

Lemma 3 proves that ϕ^{-1} is well defined. Since $s_i(x)$ counts the number of intervals of length i, we get

$$s_i(x) = nl \sim n\phi^{-1}\left(\frac{i}{n}\right). \tag{54}$$

□

4 Proof of the Main Theorem

Lemma 5. *Let $\varepsilon > 0$ and $\alpha > 1$, then*

$$\sum_{i=1}^{n\varepsilon} s_i(x) = O(n^2)\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})}, \tag{55}$$

where the $O(n^2)$ term is independent of ε.

Proof. Since

$$x \geq f(i) + \cdots + f(i+k-1) \geq kf(i) \tag{56}$$

$$f^{-1}\left(\frac{x}{k}\right) \geq f(i), \tag{57}$$

it follows that $s_k(x) \leq f^{-1}\left(\frac{x}{k}\right)$. The inverse of a regularly varying function is regularly varying with index $\frac{1}{\alpha}$ (see Theorem 1.5.12 of [1] or Theorem 3 in the Appendix). Let $f^{-1}(x) = x^{\frac{1}{\alpha}}\tilde{L}(x)$. We get

$$\sum_{i=1}^{n\varepsilon} f^{-1}\left(\frac{x}{i}\right) \sim \int_1^{n\varepsilon} f^{-1}\left(\frac{x}{t}\right) dt = x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt. \tag{58}$$

By Potter's bound

$$\tilde{L}\left(\frac{x}{t}\right) \leq M\tilde{L}\left(\frac{x}{n}\right)\left(\frac{n}{t}\right)^{\delta}, \tag{59}$$

where M is the constant from Potter's bound and $\delta = \frac{1}{2}(1-\frac{1}{\alpha})$. Now we can bound the integral

$$x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt \leq x^{\frac{1}{\alpha}} M\tilde{L}\left(\frac{x}{n}\right) n^{\delta} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \tag{60}$$

$$\tag{61}$$

By definition

$$f^{-1}(f(n)) \sim n \tag{62}$$

$$n\tilde{L}(n^{\alpha}L(n))L(n)^{1/\alpha} \sim n \tag{63}$$

$$\tilde{L}(n^{\alpha}L(n))L(n)^{1/\alpha} \sim 1. \tag{64}$$

Using the relation between x and n obtained in Lemma 1

$$\tilde{L}\left(\frac{x}{n}\right) \sim \tilde{L}\left(n^{\alpha}L(n)\right). \tag{65}$$

Thus from equations 65 and 64

$$x^{\frac{1}{\alpha}} M\tilde{L}\left(\frac{x}{n}\right) n^{\delta} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \sim Mn^2 L(n)^{\frac{1}{\alpha}} \tilde{L}(n^{\alpha}L(n)) \frac{2\alpha}{\alpha-1} \varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})} \tag{66}$$

$$= O(n^2)\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})} \tag{67}$$

□

Theorem 1. *The function $S(x)$ has the asymptotic $C(\alpha)n^2$ for $\alpha > 1$, where*

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \tag{68}$$

Proof. Let $\varepsilon > 0$. From the definition of S and s_i it follows that

$$S(x) = \sum_{i=1}^{n} s_i(x). \tag{69}$$

Splitting the sum

$$S(x) = \sum_{i=1}^{n\varepsilon} s_i(x) + \sum_{i=n\varepsilon}^{n} s_i(x). \tag{70}$$

Using Lemma 4 we get

$$\sum_{i=n\varepsilon}^{n} s_i(x) \sim \sum_{i=n\varepsilon}^{n} n\phi^{-1}\left(\frac{i}{n}\right) \tag{71}$$

$$= n \sum_{i=n\varepsilon}^{n} \phi^{-1}\left(\frac{i}{n}\right) \tag{72}$$

$$\sim n \int_{n\varepsilon}^{n} \phi^{-1}\left(\frac{t}{n}\right) dt \tag{73}$$

$$= n^2 \int_{\varepsilon}^{1} \phi^{-1}(t)dt. \tag{74}$$

Since the integral converges

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{1} \phi^{-1}(t)dt = \int_{0}^{1} \phi^{-1}(t)dt \tag{75}$$

From Lemma 5 it follows that the first sum is negligible in the asymptotics, so taking $\varepsilon \to 0$ finishes the proof. □

5 Empirical Convergence of Asymptotic Formula

Figures 1 and 2 show the impact of the slowly varying part and the index on convergence, respectively. We can observe that increasing the index, slows down the convergence, and a larger growth rate in the slowly varying part corresponds to slower convergence.

The convergence is quite slow. For example $S(x)$ for $f(x) = x^3 \log(x)$ at $x = 10^{20}$ has still a relative error of $\approx 1.4\%$ compared to the asymptotic formula and for $f(x) = x^3 \log^2(x)$ the relative error is $\approx 2.8\%$.

These findings align with our analysis: the proofs rely on the uniform convergence theorem which gives slower convergence for larger growth rates and the convergence of the sum to integral is slower for larger indices.

6 Conclusion

The main theorem gives an asymptotic for the number of interval sums below x, namely $S(x) \sim C(\alpha)n^2$. It remains an open question what happens in the case $\alpha \leq 1$; our methods can't handle that case since the integral in Lemma 3 diverges. Future work could therefore investigate the case $\alpha \leq 1$, which requires a different approach as the integral for $C(\alpha)$ diverges.

References

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