Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex, twice differentiable function such that $f'(x) \geq 0$. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy*

$$f(g(x)) \sim x$$

Then

$$g(x) \sim f^{-1}(x)$$

Proof. From $f(g(x)) \sim x$ follows

$$f(g(x)) = x + o(x)$$

Taking inverse of f

$$g(x) = f^{-1}(x + o(x))$$

Since f is convex and monotonic, we get

$$f^{-1}((1+\epsilon)x) \leq (1+\epsilon)f^{-1}(x)$$
$$f^{-1}((1-\epsilon)x) \geq (1-\epsilon)f^{-1}(x)$$

Hence,

$$g(x) = f^{-1}(x + o(x)) = (1 + o(1))f^{-1}(x) \sim f^{-1}(x)$$

□

Lemma 2. *Let f be rapidly varying function. Fix $\epsilon > 0$, then for sufficiently large n*

$$\sum_{i=1}^{n} f(i) < \sum_{i=n+1}^{n(1+\epsilon)} f(i)$$

Proof. We will prove that the ratio tends to infinity. We examine the left-hand side

$$\sum_{i=1}^{n} f(i) < nf(n)$$

On the right-hand side

$$\sum_{i=n+1}^{n(1+\epsilon)} f(i) > n\frac{\epsilon}{2}f(n(1+\frac{\epsilon}{2}))$$

Inspecting the ratio

$$\frac{\sum_{i=n+1}^{n(1+\epsilon)} f(i)}{\sum_{i=1}^{n} f(i)} > \frac{n\frac{\epsilon}{2}f(n(1+\frac{\epsilon}{2}))}{nf(n)} > \frac{\epsilon}{2}\frac{f(n(1+\frac{\epsilon}{2}))}{f(n)}$$

Since f is rapidly varying function

$$\lim_{n\to\infty} \frac{f(n(1+\frac{\epsilon}{2}))}{f(n)} = \infty$$

So there exits N such that

$$\frac{f(n(1+\frac{\epsilon}{2}))}{f(n)} > \frac{2}{\epsilon}, n > N$$

$\square$

Lemma 3. *Let $f \in RV_\alpha$. The sum*

$$\sum_{i=1}^{n} f(i)$$

has asymptotic

$$\frac{1}{1+\alpha} L(n) n^{\alpha+1}$$

Proof. Using f's asymptotic

$$\sum_{i=1}^{n} f(i) \sim \sum_{i=1}^{n} i^\alpha L(i)$$

The head does not affect the asymptotic

$$\sim \sum_{i=n\epsilon}^{n} i^\alpha L(i)$$

Since L is SV, $\frac{L(\lambda x)}{L(x)} \to 1$ uniformly on compact sets. Take $\lambda \in [\epsilon, 1]$

$$\sim \sum_{i=n\epsilon}^{n} i^\alpha L(n)$$

Take the $L(n)$ out of the sum

$$\sim L(n) \sum_{i=n\epsilon}^{n} i^\alpha$$

Clearly

$$\sim L(n) \int_{n\epsilon}^{n} t^\alpha dt$$

Calculate the integral

$$= \frac{1}{1+\alpha} L(n) n^{\alpha+1} (1 - \epsilon^{1+\alpha})$$

Follows

$$\sim \frac{1}{1+\alpha} L(n) n^{\alpha+1}$$

$\square$

Lemma 4. *Given monotonic regularly varying $f \sim x^\alpha L(x), \alpha > 1$. Let n be such that*

$$\sum_{i=1}^{n} f(i) \le x < \sum_{i=1}^{n+1} f(i)$$

If $\frac{L(xL^\beta(x))}{L(x)} \sim 1$ locally uniformly $\beta \in \mathbb{R}$ and L is increasing and unbounded, then

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k})) \sim C(\alpha) n^2$$

Where $C(\alpha) = \frac{\alpha}{(\alpha-1)(1+\alpha)^{\frac{1}{\alpha}}}$

Proof. From corollary 2.3.4 in [1] it follows that

$$f^{-1}(x) \sim x^{\frac{1}{\alpha}} \tilde{L}^{-\frac{1}{\alpha}}(x^{\frac{1}{\alpha}}) \sim x^{\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})}$$

Now we inspect

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k})$$

Using f^{-1} asymptotic

$$\sim \sum_{k=1}^{n} (\frac{x}{k})^{\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}(\frac{x}{k})}$$

Take the x out of the sum

$$= x^{\frac{1}{\alpha}} \sum_{k=1}^{n} k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{k})^{\frac{1}{\alpha}})}$$

The head of sum does not affect the asymptotic

$$\sim x^{\frac{1}{\alpha}} \sum_{k=n\epsilon}^{n} k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{k})^{\frac{1}{\alpha}})}$$

Since $L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})$ is SV, $\frac{L^{\frac{1}{\alpha}}((\lambda x)^{\frac{1}{\alpha}})}{L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})} \to 1$ uniformly on compact sets. Take $\lambda \in [\epsilon, 1]$

$$\sim x^{\frac{1}{\alpha}} \sum_{k=n\epsilon}^{n} k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}$$

Take $\frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}$ out of the sum

$$\sim \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} \sum_{k=n\epsilon}^{n} k^{-\frac{1}{\alpha}}$$

Clearly

$$\sim \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} \int_{n\epsilon}^{n} t^{-\frac{1}{\alpha}} dt$$

Calculate the integral

$$= \frac{1}{1-\frac{1}{\alpha}} \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} (n^{1-\frac{1}{\alpha}} - (n\epsilon)^{1-\frac{1}{\alpha}})$$

Factor n out

$$= \frac{1}{1-\frac{1}{\alpha}} \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}} (1 - \epsilon^{1-\frac{1}{\alpha}})$$

We get

$$\sim \frac{1}{1-\frac{1}{\alpha}} \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}} \tag{1}$$

From lemma 3

$$x \sim \frac{1}{1+\alpha} L(n) n^{\alpha+1} \tag{2}$$

Using (2) to (1)

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{1}{1-\frac{1}{\alpha}} \frac{(\frac{1}{1+\alpha} L(n) n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{1}{1+\alpha} n^{\alpha} L(n))^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}}$$

Simplify

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{1}{1-\frac{1}{\alpha}} \frac{(\frac{1}{1+\alpha} L(n) n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}(\frac{1}{1+\alpha} n L(n)^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}}$$

Using $\frac{L(xL^{\beta}(x))}{L(x)} \sim 1$ it follows

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{1}{1-\frac{1}{\alpha}} \frac{(\frac{1}{1+\alpha} L(n) n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}(n)} n^{1-\frac{1}{\alpha}}$$

Simplify

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{\alpha}{(\alpha-1)(1+\alpha)^{\frac{1}{\alpha}}} n^2$$

□

Lemma 5. *Same as 4, but $\alpha = 1$.*

Proof. From proof of 4 we get

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{x}{L(x)} \int_{n\epsilon}^{n} \frac{1}{t} dt \sim \frac{x}{L(x)} \log(n)$$

Since $\frac{x}{L(x)} \gtrsim n^2$, the lemma follows. □

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.