

1 Notation

- Let RV_α^+ denote eventually monotonic regularly varying functions with index α (see [1]). We assume the domain and range are $\mathbb{R}_{>0}$
- Let $S(x) = \#\{(i, j) \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x\}$
- Let $s_k(x) = \#\{(i, j) \mid j - i = k, f(i) + f(i+1) + \cdots + f(j) \leq x\}$
- Let \sim denote asymptotic equivalence. That is,

$$f(x) \sim g(x), x \rightarrow a \iff \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1.$$

We often omit the " $x \rightarrow a$ " part, when it is clear from context.

- Denote $f(x) \lesssim g(x) \iff \limsup \frac{f(x)}{g(x)} \leq 1$ and similarly

$$f(x) \gtrsim g(x) \iff \limsup \frac{f(x)}{g(x)} \geq 1$$

2 Preliminaries

We assume throughout this section that $n = n(x)$ is the unique integer satisfying

$$\sum_{i=1}^n f(i) \leq x < \sum_{i=1}^{n+1} f(i).$$

Where $f \in RV_\alpha^+$.

Lemma 1. *We have*

$$x \sim \frac{n^{\alpha+1} L(n)}{\alpha + 1}$$

Proof. Let $\varepsilon > 0$ and split the sum

$$\sum_{i=1}^n f(i) = \sum_{i=1}^{n\varepsilon} f(i) + \sum_{i=n\varepsilon}^n f(i).$$

We will first examine the second sum.

$$\sum_{i=n\varepsilon}^n f(i) = \sum_{i=n\varepsilon}^n i^\alpha L(i)$$

Using uniform convergence theorem (see Theorem 1.2.1 in [1])

$$\begin{aligned}\sum_{i=n\varepsilon}^n i^\alpha L(i) &\sim L(n) \sum_{i=n\varepsilon}^n i^\alpha \\ &\sim L(n) \int_{n\varepsilon}^n t^\alpha dt \\ &= \frac{1}{\alpha+1} L(n) n^{\alpha+1} (1 - \varepsilon^{\alpha+1})\end{aligned}$$

Examining the first sum

$$0 < \sum_{i=1}^{n\varepsilon} f(i) < M + n\varepsilon f(n\varepsilon),$$

for some constant M . Since f is eventually increasing

$$M + n\varepsilon f(n\varepsilon) < M + n\varepsilon f(n) = M + \varepsilon n^{\alpha+1} L(n).$$

Combining these we get

$$\begin{aligned}\frac{1}{\alpha+1} n^{\alpha+1} L(n) (1 - \varepsilon^{\alpha+1}) &\lesssim \sum_{i=1}^n f(i) \lesssim M + \frac{1}{\alpha+1} n^{\alpha+1} L(n) (1 + \varepsilon - \varepsilon^{\alpha+1}) \\ \sum_{i=1}^n f(i) &\sim \frac{1}{\alpha+1} n^{\alpha+1} L(n)\end{aligned}$$

□

Lemma 2. *Let $\varepsilon > 0$ given $k, l > 0$ such that $k - l > \varepsilon$. If*

$$\sum_{i=nl}^{nk} f(i) < x$$

then

$$\sum_{i=nl}^{nk} f(i) \sim \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}).$$

Proof. From definition of n it follows $k \leq l + 1$. Since f is eventually increasing

$$\sum_{i=nl}^{nk} f(i) > n(k-l)f(nl) > n\varepsilon f(nl).$$

Using Potter's bound (see Theorem 1.5.6 [1]) and Lemma 1

$$\begin{aligned}\frac{1}{\alpha+1} n^{\alpha+1} L(n) &\gtrsim n\varepsilon f(nl) \\ &= n^{\alpha+1} \varepsilon l^\alpha L(nl) \\ &\gtrsim 2\varepsilon n^{\alpha+1} L(n) l^{\alpha+1}.\end{aligned}$$

Follows

$$l \lesssim \left(\frac{1}{2\varepsilon(\alpha+1)}\right)^{\frac{1}{\alpha+1}}.$$

Now l and k are bounded above and below, so we can use uniform convergence theorem (see Theorem 1.2.1 in [1])

$$\begin{aligned} \sum_{i=nl}^{nk} f(i) &\sim \sum_{i=nl}^{nk} (i)^\alpha L(n) \\ &= L(n) \sum_{i=nl}^{nk} i^\alpha \\ &\sim L(n) \int_{nl}^{nk} t^\alpha dt \\ &= \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}). \end{aligned}$$

□

Lemma 3. *Let $\varepsilon > 0$ and $i > \varepsilon n$, then $s_i(x)$ has the following asymptotic*

$$s_i(x) \sim n\phi^{-1}\left(\frac{i}{n}\right)$$

where $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$.

Proof. Let $l, k > 0$ be such that $n(k - l) = i$ and

$$\sum_{i=nl}^{nk} f(i) \leq x < \sum_{i=nl}^{nk+1} f(i)$$

Clearly it follows that l and k are unique. Using Lemma 2 and Lemma 1

$$\frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) \sim \sum_{i=nl}^{nk} f(i) \sim x \sim \sum_{i=1}^n f(i) \sim \frac{n^{\alpha+1} L(n)}{\alpha+1}.$$

Follows

$$\begin{aligned} \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) &\sim \frac{n^{\alpha+1} L(n)}{\alpha+1} \\ k^{\alpha+1} - l^{\alpha+1} &\sim 1 \\ k - l &\sim (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \\ i &\sim n((1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l) \\ ln &\sim n\phi^{-1}\left(\frac{i}{n}\right) \end{aligned}$$

Since $s_i(x)$ counts the number of blocks whose sum is less than x of length i , we get

$$s_i(x) \sim nl \sim n\phi^{-1}\left(\frac{i}{n}\right)$$

□

Lemma 4. *The function $\phi : (0, \infty) \rightarrow (0, 1)$, $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ is strictly decreasing bijection hence has inverse and*

$$\int_0^1 \phi^{-1}(t) dt = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}$$

Proof. Taking the derivative of ϕ

$$\begin{aligned} \frac{d\phi}{dx} &= (1 + x^{\alpha+1})^{-\frac{\alpha}{\alpha+1}} x^{\alpha} - 1 \\ &= \frac{x^{\alpha}}{((1 + x^{\alpha+1})^{\frac{1}{\alpha+1}})^{\alpha}} - 1 \\ &< 1 - 1 = 0 \end{aligned} \tag{1}$$

Where last inequality follows from $(1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} > x$. Because ϕ 's derivative is negative everywhere, ϕ is strictly decreasing. Now ϕ is bijection, because $\phi(0) = 1$ and $\lim_{x \rightarrow \infty} \phi(x) = 0$, hence has inverse.

Since ϕ is strictly decreasing bijection

$$\int_0^1 \phi^{-1}(x) dx = \int_0^{\infty} \phi(x) dx.$$

Doing change of variables $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$, $x = (\frac{t}{1-t})^{\frac{1}{\alpha+1}}$. We get

$$dx = \frac{1}{\alpha+1} \left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt$$

Changing $\phi(x)$ to t terms

$$\begin{aligned} \phi(x) &= (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \\ &= \left(\frac{1}{1-t}\right)^{\frac{1}{\alpha+1}} - \left(\frac{t}{1-t}\right)^{\frac{1}{\alpha+1}} \\ &= (1-t)^{-\frac{1}{\alpha+1}} (1-t^{\frac{1}{\alpha+1}}) \end{aligned}$$

Putting it together

$$\begin{aligned}
\int_0^\infty \phi(x)dx &= \int_0^1 (1-t)^{-\frac{1}{\alpha+1}} (1-t^{\frac{1}{\alpha+1}}) \frac{1}{\alpha+1} \left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \\
&= \frac{1}{\alpha+1} \int_0^1 (1-t)^{\frac{\alpha}{\alpha+1} - \frac{1}{\alpha+1} - 2} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1} - \frac{\alpha}{\alpha+1}}) dt \\
&= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt \\
&= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}} t^{-\frac{\alpha}{\alpha+1}} dt - \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}} t^{\frac{1-\alpha}{\alpha+1}} dt \\
&= \frac{1}{\alpha+1} \lim_{\varepsilon \rightarrow 0^+} \int_0^1 (1-t)^{\varepsilon - \frac{\alpha+3}{\alpha+1}} t^{-\frac{\alpha}{\alpha+1}} dt - \frac{1}{\alpha+1} \int_0^1 (1-t)^{\varepsilon - \frac{\alpha+3}{\alpha+1}} t^{\frac{1-\alpha}{\alpha+1}} dt \\
&= \frac{1}{\alpha+1} \lim_{\varepsilon \rightarrow 0^+} \left(B\left(\frac{1}{\alpha+1}, \varepsilon - \frac{2}{\alpha+1}\right) - B\left(\frac{2}{\alpha+1}, \varepsilon - \frac{2}{\alpha+1}\right) \right)
\end{aligned}$$

Where B is the analytically extended beta function (see NIST DLMF § 5.12 [2])

$$B(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

Using gamma expression for beta functions we get

$$\begin{aligned}
\int_0^\infty \phi(x)dx &= \frac{1}{\alpha+1} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\varepsilon - \frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1} + \varepsilon - \frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(\varepsilon - \frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1} + \varepsilon - \frac{2}{\alpha+1})} \right) \\
&= \frac{1}{\alpha+1} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\varepsilon)} \right)
\end{aligned}$$

Since $\Gamma(t) = \frac{\Gamma(1+t)}{t}$ we have $\Gamma(t) \sim \frac{1}{t}, t \rightarrow 0$. Follows

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\varepsilon)} = 0.$$

So the expression simplifies to

$$\begin{aligned}
\int_0^\infty \phi(x)dx &= \frac{1}{\alpha+1} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \\
&= \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}
\end{aligned}$$

Where last equality uses $z\Gamma(z) = \Gamma(z+1)$. □

Lemma 5. *Let $\varepsilon > 0$ then*

$$\sum_{i=1}^{n\varepsilon} s_i(x) < \dots$$

Proof. Clearly $s_i(x) \leq f^{-1}(\frac{x}{i})$. Regularly varying functions inverse is regularly varying with index $\frac{1}{\alpha}$ (see Theorem 1.5.12 in [1]). Let $f^{-1}(x) = x^{\frac{1}{\alpha}} \tilde{L}(x)$.

$$\begin{aligned} \sum_{i=1}^{n\varepsilon} f^{-1}\left(\frac{x}{i}\right) &= \sum_{i=1}^{n\varepsilon} \left(\frac{x}{i}\right)^{\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{i}\right) \\ &= x^{\frac{1}{\alpha}} \sum_{i=1}^{n\varepsilon} i^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{i}\right) \\ &\leq x^{\frac{1}{\alpha}} \sum_{j=1}^K \sum_{i=n\varepsilon^{j+1}}^{n\varepsilon^j} i^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{i}\right) \\ &\sim x^{\frac{1}{\alpha}} \sum_{j=1}^K \int_{n\varepsilon^{j+1}}^{n\varepsilon^j} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt \end{aligned}$$

Now we can use uniform convergence theorem on \tilde{L}

$$\begin{aligned} x^{\frac{1}{\alpha}} \sum_{j=1}^K \int_{n\varepsilon^{j+1}}^{n\varepsilon^j} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt &\sim x^{\frac{1}{\alpha}} \sum_{j=1}^K \int_{n\varepsilon^{j+1}}^{n\varepsilon^j} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{n\varepsilon^j}\right) dt \\ &= \frac{\alpha}{\alpha-1} x^{\frac{1}{\alpha}} \sum_{j=1}^K \tilde{L}\left(\frac{x}{n\varepsilon^j}\right) ((n\varepsilon^j)^{1-\frac{1}{\alpha}} - (n\varepsilon^{j+1})^{1-\frac{1}{\alpha}}) \\ &= \frac{\alpha}{\alpha-1} x^{\frac{1}{\alpha}} (1 - \varepsilon^{1-\frac{1}{\alpha}}) \sum_{j=1}^K \tilde{L}\left(\frac{x}{n\varepsilon^j}\right) (n\varepsilon^j)^{1-\frac{1}{\alpha}} \\ &= \frac{\alpha}{\alpha-1} x^{\frac{1}{\alpha}} n^{1-\frac{1}{\alpha}} (1 - \varepsilon^{1-\frac{1}{\alpha}}) \sum_{j=1}^K \tilde{L}\left(\frac{x}{n\varepsilon^j}\right) \varepsilon^{j(1-\frac{1}{\alpha})} \end{aligned}$$

Clearly

$$\frac{\alpha}{\alpha-1} x^{\frac{1}{\alpha}} n^{1-\frac{1}{\alpha}} (1 - \varepsilon^{1-\frac{1}{\alpha}}) = O(n^2 L(n)^{\frac{1}{\alpha}}).$$

So it is enough to show that

$$\tilde{L}\left(\frac{x}{n\varepsilon^j}\right) L(n)^{\frac{1}{\alpha}} \sim 1$$

We know that

$$\begin{aligned} f^{-1}(f(n)) &= n \\ n\tilde{L}(n^\alpha L(n)) L(n)^{1/\alpha} &\sim n \\ \tilde{L}(n^\alpha L(n)) L(n)^{1/\alpha} &\sim 1 \end{aligned}$$

□

Theorem 1. The function $S(x)$ has asymptotic $C(\alpha)n^2$, where $C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}$.

Proof. From definition of S and s_i it follows that

$$S(x) = \sum_{i=1}^n s_i(x)$$

□

References

- [1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.
- [2] *NIST Digital Library of Mathematical Functions*. <https://dlmf.nist.gov/>, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.