Appendix: Selected Results on Regular Variation

For convenience, we reproduce several standard theorems from [1] we assume through this section that f and l is measurable.

Theorem 1 (Uniform Convergence Theorem)**.** *If l is slowly varying then*

$$\frac{l(\lambda x)}{l(x)} \to 1 \qquad (x \to \infty)$$

uniformly in each compact λ-set in $(0, \infty)$.

Proof. See Theorem 1.2.1 in [1]. $\square$

Theorem 2 (Potters Theorem)**.** *(i) If l is slowly varying function then for any chosen constants $A > 1$, $\delta > 0$ there exists $X = X(A, \delta)$ such that*

$$\frac{l(y)}{l(x)} \le A \max\left\{ \left(\frac{y}{x}\right)^{\delta}, \left(\frac{x}{y}\right)^{\delta} \right\} \qquad (x \ge X, y \ge X)$$

(ii) If further, l is bounded away from 0 and ∞ in every compact subset of $[0, \infty)$, then for every $\delta > 0$ there exists $A' = A'(\delta) > 1$ such that

$$\frac{l(y)}{l(x)} \le A' \max\left\{ \left(\frac{y}{x}\right)^{\delta}, \left(\frac{x}{y}\right)^{\delta} \right\} \qquad (x \ge 0, y \ge 0)$$

(iii) If f is regularly varying of index ρ then for any chosen $A > 1$, $\delta > 0$ there exists $X = X(A, \delta)$ such that

$$\frac{f(y)}{f(x)} \le A \max\left\{ \left(\frac{y}{x}\right)^{\rho+\delta}, \left(\frac{x}{y}\right)^{\rho+\delta} \right\} \qquad (x \ge X, y \ge X)$$

Proof. See Theorem 1.5.6 in [1]. $\square$

Proposition 1 (Karamata's Theorem)**.** *If l is slowly varying, X is so large that $l(x)$ is locally bounded in $[X, \infty)$, and $\alpha > -1$, then*

$$\int_X^x t^{\alpha} l(t) dt \sim \frac{x^{\alpha+1} l(x)}{\alpha + 1} \qquad (x \to \infty)$$

Proof. See Proposition 1.5.8 in [1]. $\square$

Definition 1 (Generalized inverse)**.** *Generalized inverse of f is defined by*

$$f^{\leftarrow}(x) = \inf \left\{ x \mid f(x) = y \right\} \tag{1}$$

Theorem 3. *If $f \in RV_{\alpha}$ with $\alpha > 0$, there exits $g \in RV_{\frac{1}{\alpha}}$ with*

$$f(g(x)) \sim g(f(x)) \sim x \qquad (x \to \infty).$$

Here g (an asymptotic inverse of f) is determined uniquely to within asymptotic equivalence, and one version of g is $f^{\leftarrow}$.

Proof. See theorem 1.5.12 in [1]. $\square$

Plot of $C(\alpha)$

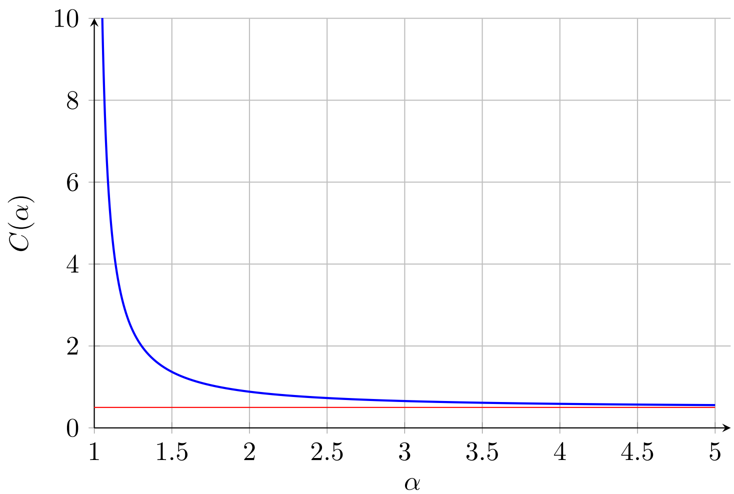

Figure 1: The blue graph shows $C(\alpha)$ and the red one shows the asymptote at $\frac{1}{2}$.

Figures on convergence

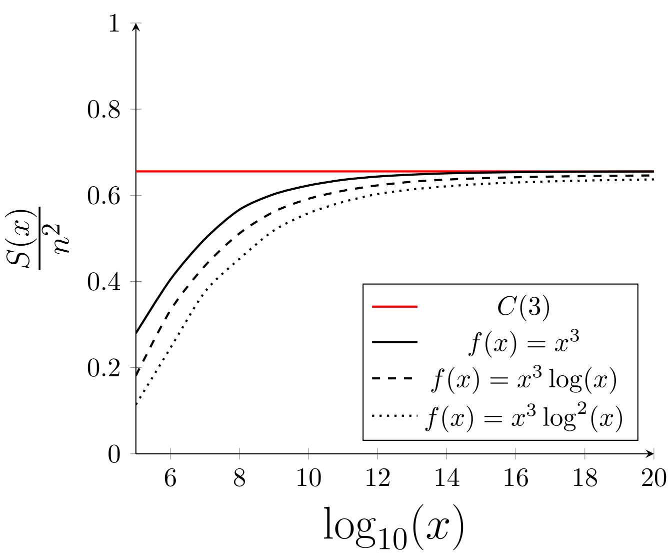

Figure 2: Example of how the slowly varying part affects convergence

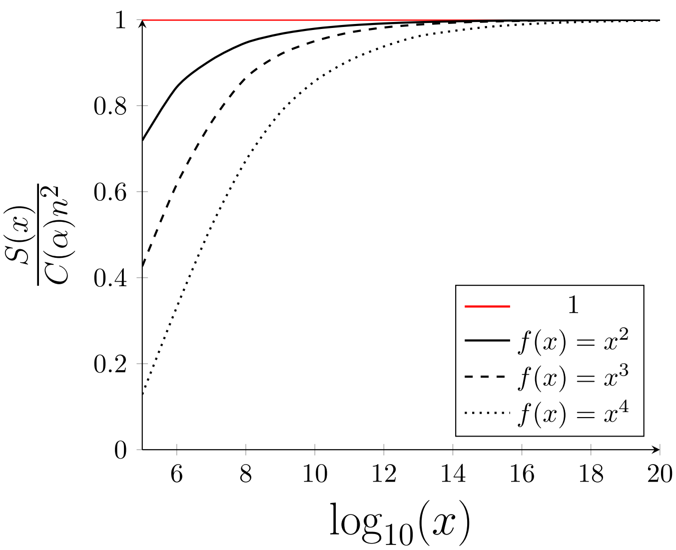

Figure 3: Example of how the index affects convergence

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.