

**lemma 1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex, twice differentiable function such that  $f'(x) \geq 0$ . Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  satisfy

$$f(g(x)) \sim x$$

Then

$$g(x) \sim f^{-1}(x)$$

*Proof.* From  $f(g(x)) \sim x$  follows

$$f(g(x)) = x + o(x)$$

Taking inverse of  $f$

$$g(x) = f^{-1}(x + o(x))$$

Since  $f$  is convex and monotonic, we get

$$f^{-1}((1 + \epsilon)x) \leq (1 + \epsilon)f^{-1}(x)$$

$$f^{-1}((1 - \epsilon)x) \geq (1 - \epsilon)f^{-1}(x)$$

Hence,

$$g(x) = f^{-1}(x + o(x)) = (1 + o(1))f^{-1}(x) \sim f^{-1}(x)$$

□

**lemma 2.** Let  $f$  be rapidly varying function. Fix  $\epsilon > 0$ , then for sufficiently large  $n$

$$\sum_{i=1}^n f(i) < \sum_{i=n+1}^{n(1+\epsilon)} f(i)$$

*Proof.* We will prove that the ratio tends to infinity. We examine the left-hand side

$$\sum_{i=1}^n f(i) < nf(n)$$

On the right-hand side

$$\sum_{i=n+1}^{n(1+\epsilon)} f(i) > n \frac{\epsilon}{2} f(n(1 + \frac{\epsilon}{2}))$$

Inspecting the ratio

$$\frac{\sum_{i=n+1}^{n(1+\epsilon)} f(i)}{\sum_{i=1}^n f(i)} > \frac{n \frac{\epsilon}{2} f(n(1 + \frac{\epsilon}{2}))}{nf(n)} > \frac{\epsilon}{2} \frac{f(n(1 + \frac{\epsilon}{2}))}{f(n)}$$

Since  $f$  is rapidly varying function

$$\lim_{n \rightarrow \infty} \frac{f(n(1 + \frac{\epsilon}{2}))}{f(n)} = \infty$$

So there exists  $N$  such that

$$\frac{f(n(1 + \frac{\epsilon}{2}))}{f(n)} > \frac{2}{\epsilon}, n > N$$

□

**lemma 3.** *Let  $f \in RV_\alpha$ . The sum*

$$\sum_{i=1}^n f(i)$$

*has asymptotic*

$$\frac{1}{1 + \alpha} L(n) n^{\alpha+1}$$

*Proof.* Using  $f$ 's asymptotic

$$\sum_{i=1}^n f(i) \sim \sum_{i=1}^n i^\alpha L(i)$$

The head does not affect the asymptotic

$$\sim \sum_{i=n\epsilon}^n i^\alpha L(i)$$

Since  $L$  is SV,  $\frac{L(\lambda x)}{L(x)} \rightarrow 1$  uniformly on compact sets. Take  $\lambda \in [\epsilon, 1]$

$$\sim \sum_{i=n\epsilon}^n i^\alpha L(n)$$

Take the  $L(n)$  out of the sum

$$\sim L(n) \sum_{i=n\epsilon}^n i^\alpha$$

Clearly

$$\sim L(n) \int_{n\epsilon}^n t^\alpha dt$$

Calculate the integral

$$= \frac{1}{1 + \alpha} L(n) n^{\alpha+1} (1 - \epsilon^{1+\alpha})$$

Follows

$$\sim \frac{1}{1 + \alpha} L(n) n^{\alpha+1}$$

□

**lemma 4.** *Given monotonic regularly varying  $f \sim x^\alpha L(x)$ ,  $\alpha > 1$ . Let  $n$  be such that*

$$\sum_{i=1}^n f(i) \leq x < \sum_{i=1}^{n+1} f(i)$$

*If  $\frac{L(xL^\beta(x))}{L(x)} \sim 1$  locally uniformly  $\beta \in \mathbb{R}$  and  $L$  is increasing and unbounded, then*

$$\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right) \sim C(\alpha)n^2$$

Where  $C(\alpha) = \frac{\alpha}{(\alpha-1)(\alpha+1)^{\frac{1}{\alpha}}}$

*Proof.* From corollary 2.3.4 in [1] it follows that

$$f^{-1}(x) \sim x^{\frac{1}{\alpha}} \tilde{L}^{-\frac{1}{\alpha}}(x^{\frac{1}{\alpha}}) \sim x^{\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})}$$

Now we inspect

$$\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right)$$

Using  $f^{-1}$  asymptotic

$$\sim \sum_{k=1}^n \left(\frac{x}{k}\right)^{\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}\left(\left(\frac{x}{k}\right)^{\frac{1}{\alpha}}\right)}$$

Take the  $x$  out of the sum

$$= x^{\frac{1}{\alpha}} \sum_{k=1}^n k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}\left(\left(\frac{x}{k}\right)^{\frac{1}{\alpha}}\right)}$$

The head of sum does not affect the asymptotic

$$\sim x^{\frac{1}{\alpha}} \sum_{k=n\epsilon}^n k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}\left(\left(\frac{x}{k}\right)^{\frac{1}{\alpha}}\right)}$$

Since  $L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})$  is SV,  $\frac{L^{\frac{1}{\alpha}}((\lambda x)^{\frac{1}{\alpha}})}{L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})} \rightarrow 1$  uniformly on compact sets. Take  $\lambda \in [\epsilon, 1]$

$$\sim x^{\frac{1}{\alpha}} \sum_{k=n\epsilon}^n k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}\left(\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}\right)}$$

Take  $\frac{1}{L^{\frac{1}{\alpha}}\left(\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}\right)}$  out of the sum

$$\sim \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}\left(\left(\frac{x}{n}\right)^{\frac{1}{\alpha}}\right)} \sum_{k=n\epsilon}^n k^{-\frac{1}{\alpha}}$$

Clearly

$$\sim \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} \int_{n\epsilon}^n t^{-\frac{1}{\alpha}} dt$$

Calculate the integral

$$= \frac{1}{1 - \frac{1}{\alpha}} \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} (n^{1-\frac{1}{\alpha}} - (n\epsilon)^{1-\frac{1}{\alpha}})$$

Factor n out

$$= \frac{1}{1 - \frac{1}{\alpha}} \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}} (1 - \epsilon^{1-\frac{1}{\alpha}})$$

We get

$$\sim \frac{1}{1 - \frac{1}{\alpha}} \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}} \quad (1)$$

From lemma 3

$$x \sim \frac{1}{1 + \alpha} L(n) n^{\alpha+1} \quad (2)$$

Using (2) to (1)

$$\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right) \sim \frac{1}{1 - \frac{1}{\alpha}} \frac{(\frac{1}{1+\alpha} L(n) n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{1}{1+\alpha} n^{\alpha} L(n))^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}}$$

Simplify

$$\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right) \sim \frac{1}{1 - \frac{1}{\alpha}} \frac{(\frac{1}{1+\alpha} L(n) n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}(\frac{1}{1+\alpha} n L(n)^{\frac{1}{\alpha}})} n^{1-\frac{1}{\alpha}}$$

Using  $\frac{L(xL^{\beta}(x))}{L(x)} \sim 1$  it follows

$$\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right) \sim \frac{1}{1 - \frac{1}{\alpha}} \frac{(\frac{1}{1+\alpha} L(n) n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}(n)} n^{1-\frac{1}{\alpha}}$$

Simplify

$$\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right) \sim \frac{\alpha}{(\alpha - 1)(\alpha + 1)^{\frac{1}{\alpha}}} n^2$$

□

**lemma 5.** *Same assumptions as 4, but  $\alpha = 1$  then*

$$n^2 = o\left(\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right)\right)$$

*Proof.* From proof of 4 we get

$$\sum_{k=1}^n f^{-1}\left(\frac{x}{k}\right) \sim \frac{x}{L(x)} \int_{n\epsilon}^n \frac{1}{t} dt \sim \frac{x}{L(x)} \log(n)$$

Using 3 we get

$$\sim n^2 \log(n)$$

The claim follows.  $\square$

**lemma 6.** *Let  $\epsilon > 0$  and  $i$  such that  $n\epsilon \leq i$ . Given  $k, l \geq \epsilon, k - l \geq \epsilon$  and eventually monotonic  $f \in RV_\alpha$*

$$\sum_{j=nl}^{nk} f(j) \sim \frac{L(n)}{\alpha + 1} n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1})$$

*Proof.* We will first prove that  $l$  is bounded above.

$$\sum_{j=ln}^{kn} f(j) > n(k - l)f(ln) \geq n\epsilon f(ln)$$

Using the sums asymptotic from 3

$$\begin{aligned} 1 &\geq \liminf \frac{n\epsilon f(ln)}{\frac{L(n)}{\alpha+1} n^{\alpha+1}} \\ 1 &\geq \liminf \frac{(\alpha+1)\epsilon L(n)l^\alpha}{L(n)} \end{aligned} \tag{3}$$

Using Potter's theorem (see 1.5.6 [1])

$$1 \geq \liminf 2(\alpha+1)\epsilon l^{\alpha+1}$$

Follows  $l \leq \left(\frac{1}{(\alpha+1)\epsilon}\right)^{\frac{1}{\alpha+1}}$ . Clearly  $k \leq l + 1$ . Now since  $l, k$  are in some compact interval independent of  $n$ , we can use uniform convergence theorem to get

$$\begin{aligned} \sum_{j=nl}^{nk} f(j) &\sim \sum_{j=nl}^{nk} L(j)j^\alpha \\ &\sim \sum_{j=nl}^{nk} L(n)j^\alpha \\ &\sim \frac{L(n)}{\alpha+1} n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) \end{aligned}$$

$\square$

**theorem 1.** Let  $f \in RV_\alpha, \alpha > 1$ , then  $S(x) \sim C(\alpha)n^2$ . Where

$$\sum_{i=1}^n f(i) \leq x < \sum_{i=1}^{n+1} f(i)$$

$$\text{and } C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}$$

*Proof.* Let  $l \geq \epsilon$ . Define  $x$  such that

$$\sum_{i=1}^n f(i) \leq x < \sum_{i=1}^{n+1} f(i)$$

Define  $k$  such that

$$\sum_{nl}^{nk} f(i) < x < \sum_{nl}^{nk+1} f(i)$$

From 3 and the fact that  $\frac{k}{l} \in [1, \frac{1}{\epsilon}]$  we get using uniform convergence of  $\frac{L(\lambda x)}{L(x)} \rightarrow 1$  in compact sets

$$\sum_{nl}^{nk} f(i) \sim \frac{L(n)}{\alpha+1} n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1})$$

On other hand we know that

$$x \sim \frac{L(n)}{1+\alpha} n^{\alpha+1}$$

Combining these

$$\frac{L(n)}{1+\alpha} n^{\alpha+1} \sim \frac{L(n)}{\alpha+1} n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1})$$

Follows

$$k^{\alpha+1} - l^{\alpha+1} = 1$$

Solving for  $k$

$$k = (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}}$$

Finally we get

$$k - l = (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \tag{4}$$

Let  $\phi(t) = (1 + t^{\alpha+1})^{\frac{1}{\alpha+1}} - t$ . Now we would like to count the  $s_i(x)$ . Calculating  $s_i(x)$  asymptotic Let  $l$  be such that  $i = n\phi(l)$ . Solving for  $nl$  we get

$$nl = n\phi^{-1}\left(\frac{i}{n}\right)$$

So

$$s_i(x) = nl = n\phi^{-1}\left(\frac{i}{n}\right)$$

We can now count  $S(x)$  asymptotic.

$$S(x) = \sum_{i=1}^n s_i(x)$$

The head doesn't affect the asymptotic

$$S(x) \sim \sum_{i=n\epsilon}^n s_i(x)$$

Using  $s_i$  formula

$$S(x) \sim \sum_{i=n\epsilon}^n n\phi^{-1}\left(\frac{i}{n}\right)$$

Since  $\phi$  is monotonic we have

$$S(x) \sim n \int_{n\epsilon}^n \phi^{-1}\left(\frac{t}{n}\right) dt$$

Doing change of variables to  $t \rightarrow un$  we get

$$S(x) \sim n^2 \int_{\epsilon}^1 \phi^{-1}(t) dt$$

Since  $\phi : (0, \infty) \rightarrow (0, 1)$  is strictly decreasing bijection

$$\int_{\epsilon}^1 \phi^{-1}(t) dt \sim \int_0^{\infty} \phi(t) dt$$

We get

$$S(x) \sim n^2 \int_0^{\infty} \phi(t) dt$$

Calculating the integral

$$\int_0^{\infty} (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x dx$$

Doing change of variables  $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$ ,  $x = \left(\frac{t}{1-t}\right)^{\frac{1}{\alpha+1}}$ . We get

$$dx = \frac{1}{\alpha+1} \left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt$$

Changing  $\phi(x)$  to  $t$  terms

$$\begin{aligned} \phi(x) &= (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \\ &= \left(\frac{1}{1-t}\right)^{\frac{1}{\alpha+1}} - \left(\frac{t}{1-t}\right)^{\frac{1}{\alpha+1}} \\ &= (1-t)^{-\frac{1}{\alpha+1}} (1 - t^{\frac{1}{\alpha+1}}) \end{aligned}$$

Putting it together

$$\begin{aligned}
\int_0^\infty \phi(t) dt &= \int_0^1 (1-t)^{-\frac{1}{\alpha+1}} (1-t^{\frac{1}{\alpha+1}}) \frac{1}{\alpha+1} \left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \\
&= \frac{1}{\alpha+1} \int_0^1 (1-t)^{\frac{\alpha}{\alpha+1} - \frac{1}{\alpha+1} - 2} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1} - \frac{\alpha}{\alpha+1}}) dt \\
&= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt \\
&= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}} t^{-\frac{\alpha}{\alpha+1}} dt - \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}} t^{\frac{1-\alpha}{\alpha+1}} dt \\
&= \frac{1}{\alpha+1} \lim_{\epsilon \rightarrow 0^+} \int_0^1 (1-t)^{\epsilon - \frac{\alpha+3}{\alpha+1}} t^{-\frac{\alpha}{\alpha+1}} dt - \frac{1}{\alpha+1} \int_0^1 (1-t)^{\epsilon - \frac{\alpha+3}{\alpha+1}} t^{\frac{1-\alpha}{\alpha+1}} dt \\
&= \frac{1}{\alpha+1} \lim_{\epsilon \rightarrow 0^+} \left( B\left(\frac{1}{\alpha+1}, \epsilon - \frac{2}{\alpha+1}\right) - B\left(\frac{2}{\alpha+1}, \epsilon - \frac{2}{\alpha+1}\right) \right)
\end{aligned}$$

Where  $B$  is the analytically extended beta function (see NIST DLMF § 5.12 [2])

$$B(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

Using gamma expression for beta functions we get

$$\begin{aligned}
\int_0^\infty \phi(t) dt &= \frac{1}{\alpha+1} \lim_{\epsilon \rightarrow 0^+} \left( \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\epsilon - \frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1} + \epsilon - \frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(\epsilon - \frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1} + \epsilon - \frac{2}{\alpha+1})} \right) \\
&= \frac{1}{\alpha+1} \lim_{\epsilon \rightarrow 0^+} \left( \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\epsilon)} \right)
\end{aligned}$$

Using  $\Gamma(t) \sim \frac{1}{t}, t \rightarrow 0$

$$\lim_{\epsilon \rightarrow 0^+} \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\epsilon)} = 0$$

So the expression simplifies to

$$\begin{aligned}
\int_0^\infty \phi(t) dt &= \frac{1}{\alpha+1} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \\
&= \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \\
&= C(\alpha)
\end{aligned}$$

Where second equality follows from  $\Gamma(z+1) = z\Gamma(z)$ .

□

## References

- [1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.
- [2] *NIST Digital Library of Mathematical Functions*. <https://dlmf.nist.gov/>, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.