Asymptotic Behavior of the Number of Interval Sums

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Abstract

In this paper we extend the results of O'Sullivan et al. on interval sums of prime powers to the broader class of regularly varying functions that are eventually monotonic and locally bounded away from 0 and ∞. We find the asymptotic formula for the number of interval sums below x. Our approach relies on approximating the number of fixed length interval sums using the function $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ that comes naturally from inverting the asymptotic relation. A negligible contribution comes from the number of interval sums of small length, allowing us to obtain the full asymptotic. This extends previous results and, in certain cases makes them more precise.

1 Introduction

Determining the asymptotic behavior of the count of interval sums less than x has been investigated for primes and prime powers [3][4][5]. However the asymptotic for more general sequences remains unknown to our knowledge. In this paper we find an asymptotic formula for a class of sequences. We study the sequences $\big(f(i)\big)_{i=1}^{\infty}$, where f is locally bounded away from 0 and ∞ and eventually monotonic with index $\alpha > 1$. This generalizes previous findings.

Our main theorem provides an asymptotic for the interval sum counting function

$$S(x) = \# \left\{ (i, j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x \right\}. \tag{1}$$

That is, we have

$$S(x) \sim n^2 C(\alpha), n = n(x) \to \infty \tag{2}$$

where $x \sim \frac{nf(n)}{\alpha+1}, x \to \infty$ and

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \tag{3}$$

for a regularly varying function f with index $\alpha > 1$ that satisfies certain assumptions.

The proof relies on classical results from the theory of regular variation, in particular the uniform convergence theorem and Potter's bound [1]. We can split the $S(x)$ into sum of s_i that count the number of intervals of length i. The main idea in the proof is that we can relate s_i to the inverse of

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \tag{4}$$

which allows us to calculate the exact coefficient in the asymptotic.

This paper is organized as follows. Section 2 is on notation. Section 3 first proves basic lemmas about asymptotics, then lemma that calculates the integral of the inverse of ϕ, and finally lemma that relates ϕ and s_i. Section 4 proves the main theorem 1 and one technical lemma bounding sum of s_i for small i's.

2 Notation

- RV_α^+ – measurable, eventually monotonic regularly varying (see definition 1 or [1]) functions locally bounded away from 0 and ∞ with index α (see [1]). We assume the domain and range are $\mathbb{R}_{>0}$.

- $S(x) = \#\left\{(i,j) \in \mathbb{N}^2 \mid i \le j, f(i) + f(i+1) + \cdots + f(j) \le x\right\}$ – the number of interval sums below x.

- $s_k(x) = \#\left\{(i,j) \in \mathbb{N}^2 \mid j - i = k - 1, f(i) + f(i+1) + \cdots + f(j) \le x\right\}$ – the number of interval sums of length k below x.

- $f(x) \sim g(x), x \to a$ – meaning $\lim_{x \to a} \frac{f(x)}{g(x)} = 1$. We often omit the "$x \to a$" part, when it is clear from context.

- $f(x) \lesssim g(x)$ – meaning $\limsup \frac{f(x)}{g(x)} \le 1$

- $f(x) \gtrsim g(x)$ – meaning $\liminf \frac{f(x)}{g(x)} \ge 1$.

- We denote sums of the form

$$\sum_{i=x}^{y} F(i). \tag{5}$$

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with

$$\sum_{x \leq i < y} F(i) \tag{6}$$

- $f(x) = O(g(x))$ – there exists $C > 0$ such that $\big|f(x)\big| \leq Cg(x)$ for all sufficiently large x.

- $f(x) = o(g(x))$ – shorthand for

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

3 Preliminaries

Definition 1. *Function $f \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is regularly varying with index α if for all $\lambda > 0$*

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha. \tag{7}$$

Definition 2. *Function $L \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is slowly varying if for all $\lambda > 0$*

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1. \tag{8}$$

We assume throughout this paper that $n = n(x)$ is the unique integer satisfying

$$\sum_{i=1}^{n} f(i) \leq x < \sum_{i=1}^{n+1} f(i) \tag{9}$$

where $f \in RV_\alpha^+$ that is $f(x) = x^\alpha L(x)$ (see section 1.5 in [1]), for some slowly varying function L.

Lemma 1. *We have*

$$x \sim \frac{n^{\alpha+1} L(n)}{\alpha + 1} = \frac{n f(n)}{\alpha + 1} \tag{10}$$

for $\alpha > -1$.

Proof. This is direct consequence of Karamata's theorem (see proposition 1.5.8 in [1] or Proposition 1 in the appendix). $\qquad\square$

Lemma 2. *Let $\varepsilon > 0$ given $k, l > 0$ such that $k - l > \varepsilon$. If $\alpha > -1$ and*

$$\sum_{i=nl}^{nk} f(i) < x \tag{11}$$

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then

$$\sum_{i=nl}^{nk} f(i) \sim \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}). \tag{12}$$

Proof. From the definition of n it follows $k \le l+1$. Since f is eventually increasing

$$\sum_{i=nl}^{nk} f(i) > n(k-l)f(nl) > n\varepsilon f(nl). \tag{13}$$

Using Potter's bound (see Theorem 1.5.6 [1] or Theorem 2 in the Appendix) and Lemma 1

$$\frac{1}{\alpha+1} n^{\alpha+1} L(n) \gtrsim n\varepsilon f(nl) \tag{14}$$

$$= n^{\alpha+1} \varepsilon l^\alpha L(nl) \tag{15}$$

$$\gtrsim \varepsilon n^{\alpha+1} L(n) l^\alpha \frac{1}{2\max(l, l^{-1})}. \tag{16}$$

Thus

$$l \lesssim \max\left(\left(\frac{2}{\varepsilon(\alpha+1)}\right)^{\frac{1}{\alpha+1}}, \left(\frac{2}{\varepsilon(\alpha+1)}\right)^{\frac{1}{\alpha-1}} \right). \tag{17}$$

Now l and k are bounded above and below, so we can use uniform convergence theorem (see Theorem 1.2.1 in [1] or Theorem 1 in the Appendix)

$$\sum_{i=nl}^{nk} f(i) \sim \sum_{i=nl}^{nk} i^\alpha L(n) \tag{18}$$

$$= L(n) \sum_{i=nl}^{nk} i^\alpha \tag{19}$$

$$\sim L(n) \int_{nl}^{nk} t^\alpha dt \tag{20}$$

$$= \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}). \tag{21}$$

$\square$

Lemma 3. *The function $\phi : (0,\infty) \to (0,1)$, $\phi(x) = (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ is strictly decreasing bijection hence has inverse and*

$$\int_0^1 \phi^{-1}(t)dt = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \tag{22}$$

for $\alpha > 1$.

Proof. Taking the derivative of ϕ

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = (1 + x^{\alpha+1})^{-\frac{\alpha}{\alpha+1}} x^\alpha - 1 \tag{23}$$

$$= \frac{x^\alpha}{\left((1 + x^{\alpha+1})^{\frac{1}{\alpha+1}}\right)^\alpha} - 1 \tag{24}$$

$$< 1 - 1 = 0 \tag{25}$$

where last inequality follows from $(1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} > x$. Because derivative of ϕ is negative everywhere, ϕ is strictly decreasing. Now ϕ is bijection, because $\phi(0) = 1$ and $\lim_{x\to\infty} \phi(x) = 0$, hence has inverse. Since ϕ is strictly decreasing bijection

$$\int_0^1 \phi^{-1}(x)dx = \int_0^\infty \phi(x)dx. \tag{26}$$

Next we show that the integral converges. Since $x^{\frac{1}{\alpha+1}}$ is increasing with decreasing derivative, we have $(1 + y)^{\frac{1}{\alpha+1}} < y^{\frac{1}{\alpha+1}} + \frac{1}{\alpha+1} y^{-\frac{\alpha}{\alpha+1}}$. Using this we get

$$\phi(x) < x^{-\alpha}, \tag{27}$$

so the integral converges.

Doing change of variables $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$, $x = \left(\frac{t}{1-t}\right)^{\frac{1}{\alpha+1}}$. We get

$$dx = \frac{1}{\alpha+1}\left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt. \tag{28}$$

Changing $\phi(x)$ to t terms

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \tag{29}$$

$$= \left(\frac{1}{1-t}\right)^{\frac{1}{\alpha+1}} - \left(\frac{t}{1-t}\right)^{\frac{1}{\alpha+1}} \tag{30}$$

$$= (1-t)^{-\frac{1}{\alpha+1}}(1 - t^{\frac{1}{\alpha+1}}). \tag{31}$$

Putting it together

$$\int_0^\infty \phi(x)dx = \int_0^1 (1-t)^{-\frac{1}{\alpha+1}}(1 - t^{\frac{1}{\alpha+1}})\frac{1}{\alpha+1}\left(\frac{t}{1-t}\right)^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt \tag{32}$$

$$= \frac{1}{\alpha+1}\int_0^1 (1-t)^{\frac{\alpha}{\alpha+1} - \frac{1}{\alpha+1} - 2}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1} - \frac{\alpha}{\alpha+1}})dt \tag{33}$$

$$= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{2}{\alpha+1} - 1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt. \tag{34}$$

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Let $g_r(t) = (1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$ and $g(t) = (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$. Since for each $r > 0$,

$g_r(t) \to g(t), r \to 0$ pointwise and $|g_r(t)| < |g(t)|$, we can use dominated convergence theorem. Thus

$$\int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \tag{35}$$

$$= \lim_{r \to 0^+} \int_0^1 (1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt. \tag{36}$$

Let B be the beta function (see NIST DLMF § 5.12.1 [2])

$$B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)} \tag{37}$$

The beta function can be analytically extended to $(\mathbb{C} \setminus \mathbb{Z}, \mathbb{C} \setminus \mathbb{Z})$ using Pochhammer's integral (see NIST

DLMF § 5.12.12 [2]).

$$= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \tag{38}$$

$$= \frac{1}{\alpha+1} \lim_{r \to 0^+} \int_0^1 (1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \tag{39}$$

$$= \frac{1}{\alpha+1} \lim_{r \to 0^+} \left(B\left(\frac{1}{\alpha+1}, r - \frac{2}{\alpha+1}\right) - B\left(\frac{2}{\alpha+1}, r - \frac{2}{\alpha+1}\right) \right). \tag{40}$$

Using gamma expression for beta functions we get

$$\int_0^\infty \phi(x)dx = \frac{1}{\alpha+1} \lim_{r \to 0^+} \left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(r - \frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1} + r - \frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(r - \frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1} + r - \frac{2}{\alpha+1})} \right) \tag{41}$$

$$= \frac{1}{\alpha+1} \lim_{r \to 0^+} \left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)} \right). \tag{42}$$

Since $\Gamma(t) = \frac{\Gamma(1+t)}{t}$ we have $\Gamma(t) \sim \frac{1}{t}, t \to 0$. Thus

$$\lim_{r \to 0} \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)} = 0 \tag{43}$$

so the expression simplifies to

$$\int_0^\infty \phi(x)dx = \frac{1}{\alpha+1} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \tag{44}$$

$$= \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \tag{45}$$

where last equality uses $z\Gamma(z) = \Gamma(z+1)$. $\qquad \square$

Lemma 4. *Let $\varepsilon > 0$ and $i > \varepsilon n$, then $s_i(x)$ has the following asymptotic*

$$s_i(x) \sim n\phi^{-1}\left(\frac{i}{n}\right) \tag{46}$$

where $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$.

Proof. Let $l, k > 0$ be such that $n(k - l) = i$ and

$$\sum_{j=nl}^{nk} f(j) \le x < \sum_{j=nl+1}^{nk+1} f(j) \tag{47}$$

Clearly it follows that l and k are unique up to producing the same range of integer indices in the sum.

Using Lemma 2 and Lemma 1

$$\frac{1}{\alpha+1}L(n)n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}) \sim \sum_{j=nl}^{nk} f(j) \sim x \sim \sum_{j=1}^{n} f(j) \sim \frac{n^{\alpha+1}L(n)}{\alpha+1}. \tag{48}$$

Thus

$$\frac{n^{\alpha+1}L(n)}{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}) \sim \frac{n^{\alpha+1}L(n)}{\alpha+1} \tag{49}$$

$$k^{\alpha+1} - l^{\alpha+1} \sim 1 \tag{50}$$

$$k - l \sim (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \tag{51}$$

$$i \sim n((1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l) \tag{52}$$

$$ln \sim n\phi^{-1}\left(\frac{i}{n}\right). \tag{53}$$

Lemma 3 proves that ϕ^{-1} is well defined. Since $s_i(x)$ counts the number of intervals whose sum is less than x of length i, we get

$$s_i(x) = nl \sim n\phi^{-1}\left(\frac{i}{n}\right). \tag{54}$$

$\square$

4 Proof of the Main Theorem

Lemma 5. *Let $\varepsilon > 0$ and $\alpha > 1$ then*

$$\sum_{i=1}^{n\varepsilon} s_i(x) = O(n^2)\varepsilon^{\frac{1}{2}(1 - \frac{1}{\alpha})}. \tag{55}$$

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Proof. Clearly $s_i(x) \leq f^{-1}\left(\frac{x}{i}\right)$. Regularly varying functions inverse is regularly varying with index $\frac{1}{\alpha}$ (see Theorem 1.5.12 of [1] or Theorem 3 in the Appendix). Let $f^{-1}(x) = x^{\frac{1}{\alpha}} \tilde{L}(x)$. We get

$$\sum_{i=1}^{n\varepsilon} f^{-1}\left(\frac{x}{i}\right) \sim \int_1^{n\varepsilon} f^{-1}\left(\frac{x}{t}\right) dt = x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt. \tag{56}$$

By Potter's bound

$$\tilde{L}\left(\frac{x}{t}\right) \leq M\tilde{L}\left(\frac{x}{n}\right)\left(\frac{n}{t}\right)^\delta, \tag{57}$$

where M is the constant from Potter's bound and $\delta = \frac{1}{2}(1 - \frac{1}{\alpha})$. Now we can bound the integral

$$x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt \leq x^{\frac{1}{\alpha}} M\tilde{L}\left(\frac{x}{n}\right) n^\delta \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \tag{58}$$

$$\tag{59}$$

By definition

$$f^{-1}(f(n)) \sim n \tag{60}$$

$$n\tilde{L}(n^\alpha L(n))L(n)^{1/\alpha} \sim n \tag{61}$$

$$\tilde{L}(n^\alpha L(n))L(n)^{1/\alpha} \sim 1. \tag{62}$$

Thus

$$x^{\frac{1}{\alpha}} M\tilde{L}\left(\frac{x}{n}\right) n^\delta \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \sim Mn^2 L(n)^{\frac{1}{\alpha}} \tilde{L}(n^\alpha L(n)) \frac{2\alpha}{\alpha-1} \varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})} \tag{63}$$

$$= O(n^2)\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})} \tag{64}$$

$\square$

Theorem 1. *The function $S(x)$ has asymptotic $C(\alpha)n^2$ for $\alpha > 1$, where*

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \tag{65}$$

Proof. Let $\varepsilon > 0$. From the definition of S and s_i it follows that

$$S(x) = \sum_{i=1}^{n} s_i(x). \tag{66}$$

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Splitting the sum

$$S(x) = \sum_{i=1}^{n\varepsilon} s_i(x) + \sum_{i=n\varepsilon}^{n} s_i(x). \tag{67}$$

Using lemma 4 we get

$$\sum_{i=n\varepsilon}^{n} s_i(x) \sim \sum_{i=n\varepsilon}^{n} n\phi^{-1}\left(\frac{i}{n}\right) \tag{68}$$

$$= n \sum_{i=n\varepsilon}^{n} \phi^{-1}\left(\frac{i}{n}\right) \tag{69}$$

$$\sim n \int_{n\varepsilon}^{n} \phi^{-1}\left(\frac{t}{n}\right) dt \tag{70}$$

$$= n^2 \int_{\varepsilon}^{1} \phi^{-1}(t) dt. \tag{71}$$

Since the integral converges

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \phi^{-1}(t) dt = \int_{0}^{1} \phi^{-1}(t) dt \tag{72}$$

From Lemma 5 it follows that the first sum is negligible in the asymptotic so taking $\varepsilon \to 0$ finishes the proof. $\square$

5 Empirical Convergence of Asymptotic Formula

Figures 1 and 2 show the impact of slowly varying part and index on convergence, respectively. We can observe that increasing the index, slows down the convergence and larger growth rate in the slowly varying part corresponds to slower convergence.

The convergence is quite slow. For example $S(x)$ for $f(x) = x^3 \log(x)$ at $x = 10^{20}$ has still relative error of $\approx 1.4\%$ compared to the asymptotic formula and for $f(x) = x^3 \log^2(x)$ the relative error is $\approx 2.8\%$.

These findings align with our analysis: the proofs rely on uniform convergence theorem which gives slower convergence for larger growth rates and the sum's asymptotic to integral has slower convergence for larger indices.

6   Conclusion

The main theorem gives asymptotic for number of interval sums below x, that is $S(x) \sim C(\alpha)n^2$. It remains open question what happens in the case $\alpha \leq 1$, our methods can't handle that case since the integral in lemma 3 diverges. Future work could refine the result by dropping assumptions or by investigating a different class of functions. The result may have applications in additive and analytic number theory.

References

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