1 Notation

- Let RV_α^+ denote eventually monotonic regularly varying functions with index α (see [1]). We assume the domain and range are $\mathbb{R}_{>0}$
- Let $S(x) = \#\{(i,j) \mid i \le j, f(i) + f(i+1) + \cdots + f(j) \le x\}$
- Let $s_k(x) = \#\{(i,j) \mid j - i = k, f(i) + f(i+1) + \cdots + f(j) \le x\}$
- Let $\sim$ denote asymptotic equivalence. That is,

$$f(x) \sim g(x), x \to a \iff \lim_{x \to a} \frac{f(x)}{g(x)} = 1.$$

  We often omit the "$x \to a$" part, when it is clear from context.

- Denote $f(x) \lesssim g(x) \iff \limsup \frac{f(x)}{g(x)} \le 1$ and similarly

$$f(x) \gtrsim g(x) \iff \limsup \frac{f(x)}{g(x)} \ge 1$$

  .

2 Preliminaries

We assume throughout this section that $n = n(x)$ is the unique integer satisfying

$$\sum_{i=1}^{n} f(i) \le x < \sum_{i=1}^{n+1} f(i).$$

Where $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$.

Lemma 1. *Given $f \in RV_\alpha^+$ we have*

$$x \sim \frac{n^{\alpha+1} L(n)}{\alpha + 1}$$

Proof. Let $\varepsilon > 0$ and split the sum

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n\varepsilon} f(i) + \sum_{i=n\varepsilon}^{n} f(i).$$

We will first examine the second sum.

$$\sum_{i=n\varepsilon}^{n} f(i) = \sum_{i=n\varepsilon}^{n} i^\alpha L(i)$$

Using uniform convergence theorem (see Theorem 1.2.1 in [1])

$$\begin{aligned}\sum_{i=n\varepsilon}^{n} i^\alpha L(i) &\sim L(n) \sum_{i=n\varepsilon}^{n} i^\alpha \\ &\sim L(n) \int_{n\varepsilon}^{n} t^\alpha dt \\ &= \frac{1}{\alpha+1} L(n) n^{\alpha+1}(1-\varepsilon^{\alpha+1})\end{aligned}$$

Examining the first sum

$$0 < \sum_{i=1}^{n\varepsilon} f(i) < M + n\varepsilon f(n\varepsilon),$$

for some constant M. Since f is eventually increasing

$$M + n\varepsilon f(n\varepsilon) < M + n\varepsilon f(n) = M + \varepsilon n^{\alpha+1} L(n).$$

Combining these we get

$$\begin{aligned}\frac{1}{\alpha+1} n^{\alpha+1} L(n)(1-\varepsilon^{\alpha+1}) \lesssim \sum_{i=1}^{n} f(i) &\lesssim M + \frac{1}{\alpha+1} n^{\alpha+1} L(n)(1+\varepsilon-\varepsilon^{\alpha+1}) \\ \sum_{i=1}^{n} f(i) &\sim \frac{1}{\alpha+1} n^{\alpha+1} L(n)\end{aligned}$$

□

Lemma 2. *Let $\varepsilon > 0$ given $k, l > 0$ such that $k - l > \varepsilon$ and $f \in RV_\alpha^+$. If*

$$\sum_{i=nl}^{nk} f(i) < x$$

then

$$\sum_{i=nl}^{nk} f(i) \sim \frac{1}{\alpha+1} L(n) n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}).$$

Proof. From definition of n it follows $k \leq l + 1$. Since f is eventually increasing

$$\sum_{i=nl}^{nk} f(i) > n(k-l) f(nl) > n\varepsilon f(nl).$$

Using Potter's bound (see Theorem 1.5.6 [1]) and Lemma 1

$$\begin{aligned}\frac{1}{\alpha+1} n^{\alpha+1} L(n) &\gtrsim n\varepsilon f(nl) \\ &= n^{\alpha+1} \varepsilon l^\alpha L(nl) \\ &\gtrsim 2\varepsilon n^{\alpha+1} L(n) l^{\alpha+1}.\end{aligned}$$

Follows

$$l \lesssim (\frac{1}{2\varepsilon(\alpha+1)})^{\frac{1}{\alpha+1}}.$$

Now l and k are bounded above and below, so we can use uniform convergence theorem (see Theorem 1.2.1 in [1])

$$\begin{aligned}
\sum_{i=nl}^{nk} f(i) &\sim \sum_{i=nl}^{nk} (i)^\alpha L(n) \\
&= L(n) \sum_{i=nl}^{nk} i^\alpha \\
&\sim L(n) \int_{nl}^{nk} t^\alpha dt \\
&= \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}).
\end{aligned}$$

□

Lemma 3. *Let $\varepsilon > 0$ and $i > \varepsilon n$, then $s_i(x)$ has the following asymptotic*

$$s_i(x) \sim n\phi^{-1}(\frac{i}{n})$$

where $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$.

Proof. Let $l, k > 0$ be such that $n(k - l) = i$ and

$$\sum_{i=nl}^{nk} f(i) \leq x < \sum_{i=nl}^{nk+1} f(i)$$

Clearly it follows that l and k are unique. Using Lemma 2 and Lemma 1

$$\frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) \sim \sum_{i=nl}^{nk} f(i) \sim x \sim \sum_{i=1}^{n} f(i) \sim \frac{n^{\alpha+1} L(n)}{\alpha+1}.$$

Follows

$$\begin{aligned}
\frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) &\sim \frac{n^{\alpha+1} L(n)}{\alpha+1} \\
k^{\alpha+1} - l^{\alpha+1} &\sim 1 \\
k - l &\sim (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \\
i &\sim n((1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l) \\
ln &\sim n\phi^{-1}(\frac{i}{n})
\end{aligned}$$

Since $s_i(x)$ counts the number of blocks whose sum is less that x of length i, we get

$$s_i(x) \sim nl \sim n\phi^{-1}(\frac{i}{n})$$

$\square$

Lemma 4. *The function* $\phi : (0,\infty) \to (0,1)$, $\phi(x) = (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ *is strictly decreasing bijection hence has inverse and*

$$\int_0^1 \phi^{-1}(t)dt = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}$$

Proof. Taking the derivative of ϕ

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}x} &= (1+x^{\alpha+1})^{-\frac{\alpha}{\alpha+1}} x^{\alpha} - 1 \\ &= \frac{x^{\alpha}}{((1+x^{\alpha+1})^{\frac{1}{\alpha+1}})^{\alpha}} - 1 \\ &< 1 - 1 = 0 \end{aligned} \tag{1}$$

Where last inequality follows from $(1+x^{\alpha+1})^{\frac{1}{\alpha+1}} > x$. Because ϕ's derivative is negative everywhere, ϕ is strictly decreasing. Now ϕ is bijection, because $\phi(0) = 1$ and $\lim_{x\to\infty} \phi(x) = 0$, hence has inverse.

Since ϕ is strictly decreasing bijection

$$\int_0^1 \phi^{-1}(x)dx = \int_0^{\infty} \phi(x)dx.$$

Doing change of variables $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$, $x = (\frac{t}{1-t})^{\frac{1}{\alpha+1}}$. We get

$$dx = \frac{1}{\alpha+1}\Big(\frac{t}{1-t}\Big)^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt$$

Changing $\phi(x)$ to t terms

$$\begin{aligned}\phi(x) &= (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \\ &= (\frac{1}{1-t})^{\frac{1}{\alpha+1}} - (\frac{t}{1-t})^{\frac{1}{\alpha+1}} \\ &= (1-t)^{-\frac{1}{\alpha+1}}(1-t^{\frac{1}{\alpha+1}})\end{aligned}$$

Putting it together

$$\begin{aligned}
\int_0^\infty \phi(x)dx &= \int_0^1 (1-t)^{-\frac{1}{\alpha+1}}(1-t^{\frac{1}{\alpha+1}})\frac{1}{\alpha+1}(\frac{t}{1-t})^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt \\
&= \frac{1}{\alpha+1}\int_0^1 (1-t)^{\frac{\alpha}{\alpha+1}-\frac{1}{\alpha+1}-2}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1}-\frac{\alpha}{\alpha+1}})dt \\
&= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \\
&= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}}t^{-\frac{\alpha}{\alpha+1}}dt - \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}}t^{\frac{1-\alpha}{\alpha+1}}dt \\
&= \frac{1}{\alpha+1}\lim_{\varepsilon\to 0^+}\int_0^1 (1-t)^{\varepsilon-\frac{\alpha+3}{\alpha+1}}t^{-\frac{\alpha}{\alpha+1}}dt - \frac{1}{\alpha+1}\int_0^1 (1-t)^{\varepsilon-\frac{\alpha+3}{\alpha+1}}t^{\frac{1-\alpha}{\alpha+1}}dt \\
&= \frac{1}{\alpha+1}\lim_{\varepsilon\to 0^+}\big(B(\frac{1}{\alpha+1},\varepsilon-\frac{2}{\alpha+1}) - B(\frac{2}{\alpha+1},\varepsilon-\frac{2}{\alpha+1})\big)
\end{aligned}$$

Where B is the analytically extended beta function(see NIST DLMF § 5.12 [2])

$$B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

Using gamma expression for beta functions we get

$$\begin{aligned}
\int_0^\infty \phi(x)dx &= \frac{1}{\alpha+1}\lim_{\varepsilon\to 0^+}(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\varepsilon-\frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1}+\varepsilon-\frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(\varepsilon-\frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1}+\varepsilon-\frac{2}{\alpha+1})}) \\
&= \frac{1}{\alpha+1}\lim_{\varepsilon\to 0^+}(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\varepsilon)})
\end{aligned}$$

Since $\Gamma(t) = \frac{\Gamma(1+t)}{t}$ we have $\Gamma(t) \sim \frac{1}{t}, t \to 0$. Follows

$$\lim_{\varepsilon\to 0^+}\frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\varepsilon)} = 0.$$

So the expression simplifies to

$$\begin{aligned}
\int_0^\infty \phi(x)dx &= \frac{1}{\alpha+1}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \\
&= \frac{1}{2(\alpha+1)}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}
\end{aligned}$$

Where last equality uses $z\Gamma(z) = \Gamma(z+1)$. □

Theorem 1. *The function $S(x)$ has asymptotic $C(\alpha)n^2$, where $C(\alpha) = \frac{1}{2(\alpha+1)}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}$.*

Proof. From definition of S and s_i it follows that

$$S(x) = \sum_{i=1}^n s_i(x)$$

□

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

[2] *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.