

Appendix: Selected Results on Regular Variation

For convenience, we reproduce several standard theorems from [1] we assume through this section that f and l is measurable.

Theorem 1 (Uniform Convergence Theorem). *If l is slowly varying then*

$$\frac{l(\lambda x)}{l(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

uniformly in each compact λ -set in $(0, \infty)$.

Theorem 2 (Potters Theorem). *(i) If l is slowly varying function then for any chosen constants $A > 1$, $\delta > 0$ there exists $X = X(A, \delta)$ such that*

$$\frac{l(y)}{l(x)} \leq A \max \left\{ \left(\frac{y}{x} \right)^\delta, \left(\frac{x}{y} \right)^\delta \right\} \quad (x \geq X, y \geq X)$$

(ii) If further, l is bounded away from 0 and ∞ in every compact subset of $[0, \infty)$, then for every $\delta > 0$ there exists $A' = A'(\delta) > 1$ such that

$$\frac{l(y)}{l(x)} \leq A' \max \left\{ \left(\frac{y}{x} \right)^\delta, \left(\frac{x}{y} \right)^\delta \right\} \quad (x \geq 0, y \geq 0)$$

(iii) If f is regularly varying of index ρ then for any chosen $A > 1$, $\delta > 0$ there exists $X = X(A, \delta)$ such that

$$\frac{f(y)}{f(x)} \leq A \max \left\{ \left(\frac{y}{x} \right)^{\rho+\delta}, \left(\frac{x}{y} \right)^{\rho+\delta} \right\} \quad (x \geq X, y \geq X)$$

Proof. See Theorem 1.5.6 in [1]. □

Proposition 1 (Karamata's Theorem (direct half)). *If l is slowly varying, X is so large that $l(x)$ is locally bounded in $[X, \infty)$, and $\alpha > -1$, then*

$$\int_X^x t^\alpha l(t) dt \sim \frac{x^{\alpha+1} l(x)}{\alpha+1} \quad (x \rightarrow \infty)$$

Proof. See Proposition 1.5.8 in [1]. □

Definition 1 (Generalized inverse). *Generalized inverse of f is defined by*

$$f^{\leftarrow}(x) = \inf \{x \mid f(x) = y\} \quad (1)$$

Theorem 3. *If $f \in RV_\alpha$ with $\alpha > 0$, there exists $g \in RV_{\frac{1}{\alpha}}$ with*

$$f(g(x)) \sim g(f(x)) \sim x \quad (x \rightarrow \infty).$$

Here g (an asymptotic inverse of f) is determined uniquely to within asymptotic equivalence, and one version of g is f^{\leftarrow} .

Proof. See theorem 1.5.12 in [1]. □

Figures on convergence

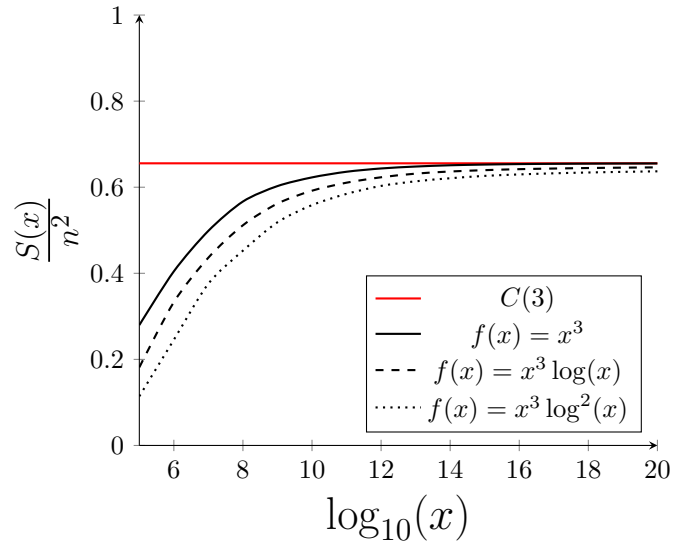


Figure 1: Example of how the slowly varying part affects convergence

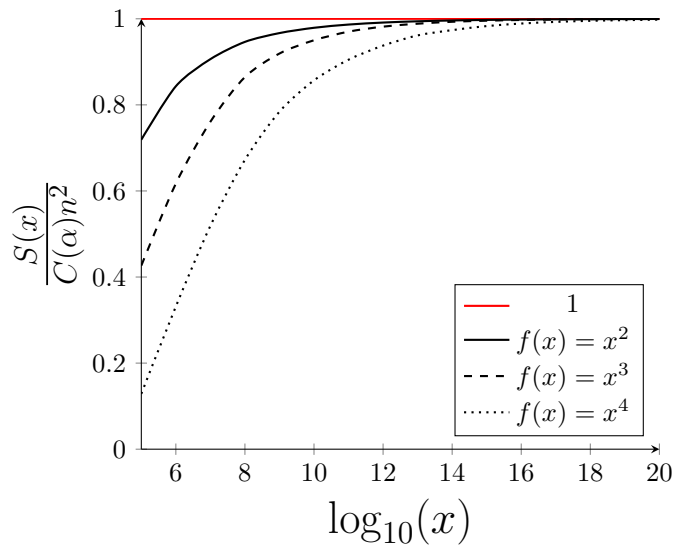


Figure 2: Example of how the index affects convergence

References

- [1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.