

Asymptotic Behavior of the Number of Interval Sums

Abstract

An asymptotic formula is found for the number of interval sums for a class of sequences $(f(i))_{i=1}^{\infty}$, where f is measurable, locally bounded away from 0 and ∞ , eventually increasing, and regularly varying with index $\alpha > 1$. This generalizes and extends results of O'Sullivan et al. on interval sums of prime powers.

The main theorem provides an asymptotic formula for the interval sum counting function

$$S_f(x) = \# \left\{ (i, j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x \right\}.$$

That is

$$S_f(x) \sim n^2 C(\alpha) \quad n = n_f(x) \rightarrow \infty$$

where $x \sim \frac{n f(n)}{\alpha+1}$, $x \rightarrow \infty$ and

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1}) \Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}.$$

The proof relies on classical results from the theory of regular variation, in particular the uniform convergence theorem and Potter's bound. The function $S_f(x)$ can be split into a sum of $s_{i,f}$ that counts the number of intervals of length i . The main idea in the proof is that $s_{i,f}$ can be related to the inverse of

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$$

which allows us to relate $S_f(x)$ asymptotically to

$$n^2 \int_0^1 \phi^{-1}(t) dt.$$

The integral can be calculated using standard substitution and using the analytically extended beta function, which completes the proof.

1 Introduction

Interval sum is the sum of consecutive terms in a sequence. Determining the asymptotic behavior of the count of interval sums less than x has been investigated for primes and prime powers [3, 4, 5]. Moser [3] determined the asymptotic behavior of interval sums of primes. Tongsomporn et al. [5] derived strict bounds for the number of interval sums of prime squares. O’Sullivan et al. [4] found asymptotic upper and lower bounds for the number of interval sums of prime powers.

To the best of our knowledge, no asymptotic formula is known for more general sequences. An asymptotic formula is found for the number of interval sums for a class of sequences $(f(i))_{i=1}^{\infty}$, where f is measurable, locally bounded away from 0 and ∞ , eventually increasing, and regularly varying with index $\alpha > 1$. This generalizes and extends O’Sullivan et al.’s work and provides exact asymptotics for the number of interval sums of prime powers.

Our main theorem provides an asymptotic for the interval sum counting function

$$S_f(x) = \# \left\{ (i, j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x \right\}. \quad (1)$$

That is

$$S_f(x) \sim n^2 C(\alpha) \quad n = n_f(x) \rightarrow \infty \quad (2)$$

where $x \sim \frac{n^2}{\alpha+1}$, $x \rightarrow \infty$ and

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1}) \Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \quad (3)$$

The proof relies on classical results from the theory of regular variation, in particular the uniform convergence theorem and Potter’s bound [1]. The function $S_f(x)$ can be split into a sum of $s_{i,f}$ that counts the number of intervals of length i . The main idea in the proof is that $s_{i,f}$ can be related to the inverse of

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \quad (4)$$

which allows us to calculate the exact coefficient in the asymptotic.

This paper is organized as follows. Section 2 defines our notation. Section 3 first proves basic lemmas about asymptotics, then a lemma that relates ϕ and $s_{i,f}$, and finally a lemma that calculates the integral of the inverse of ϕ . Section 4 proves the main theorem 1 and one technical lemma bounding the sum of $s_{i,f}$ for small i .

2 Notation

- The function $f: [0, \infty) \rightarrow [0, \infty)$ is assumed to satisfy following criterion: measurable, eventually increasing, locally bounded away from 0 and ∞ , and regularly varying with index α (see definition 1 or [1]).
- $S_f(x) = \#\{(i, j) \in \mathbb{N}^2 \mid i \leq j, f(i) + f(i+1) + \dots + f(j) \leq x\}$ – the number of interval sums below x .
- $s_{k,f}(x) = \#\{(i, j) \in \mathbb{N}^2 \mid j - i = k - 1, f(i) + f(i+1) + \dots + f(j) \leq x\}$ – the number of interval sums of length k below x .
- $f(x) \sim g(x)$, $x \rightarrow a$ – meaning $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$. The part " $x \rightarrow a$ " is often omitted when it is clear from context.
- $f(x) \lesssim g(x)$ – meaning $\limsup \frac{f(x)}{g(x)} \leq 1$
- $f(x) \gtrsim g(x)$ – meaning $\liminf \frac{f(x)}{g(x)} \geq 1$.
- sums of the form $\sum_{i=x}^y F(i)$ are denoted with $\sum_{x \leq i < y} F(i)$.
- $f(x) = O(g(x))$ – there exists $C > 0$ such that $|f(x)| \leq Cg(x)$ for all sufficiently large x .
- $f(x) = o(g(x))$ – shorthand for

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$$

- Define $n := n_f(x)$, where

$$n_f(x) = \max \left\{ n \in \mathbb{N} \mid \sum_{i=1}^n f(i) \leq x \right\} \quad (5)$$

that is, length of the largest interval.

3 Preliminaries

Definition 1. Function $f: [0, \infty) \rightarrow [0, \infty)$ is **regularly varying** with **index** α if for all $\lambda > 0$

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha. \quad (6)$$

Definition 2. Function $L: [0, \infty) \rightarrow [0, \infty)$ is **slowly varying** if for all $\lambda > 0$

$$\lim_{x \rightarrow \infty} \frac{L(\lambda x)}{L(x)} = 1. \quad (7)$$

Since f is measurable and regularly varying with index α , we have $f(x) = x^\alpha L(x)$, for slowly varying L (see Theorem 1.4.1 in [1]).

Lemma 1. We have

$$x \sim \frac{n^{\alpha+1} L(n)}{\alpha+1} = \frac{nf(n)}{\alpha+1} \quad (8)$$

for $\alpha > -1$.

Proof. This is a direct consequence of Karamata's theorem (see proposition 1.5.8 in [1] or Proposition 1 in the appendix). \square

Lemma 2. Let $\varepsilon > 0$ and let $k, l > 0$ such that $k - l > \varepsilon$. If $\alpha > 1$ and

$$\sum_{i=nl}^{nk} f(i) \sim x \quad (9)$$

then uniformly in k and l

$$\sum_{i=nl}^{nk} f(i) \sim \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}). \quad (10)$$

Proof. From the definition of n it follows that $k \leq l + 1$. Since f is eventually increasing

$$\sum_{i=nl}^{nk} f(i) > n(k-l)f(nl) > n\varepsilon f(nl). \quad (11)$$

Using Lemma 1 and hypothesis 9

$$\frac{1}{\alpha+1} n^{\alpha+1} L(n) \sim x \sim \sum_{i=nl}^{nk} f(i) \gtrsim n\varepsilon f(nl) = n^{\alpha+1} \varepsilon l^\alpha L(nl). \quad (12)$$

Using Potter's bound (i) (see Theorem 1.5.6 [1] or Theorem 2 in the Appendix)

$$n^{\alpha+1} \varepsilon l^\alpha L(nl) \gtrsim \varepsilon n^{\alpha+1} L(n) l^\alpha \frac{1}{2 \max(l, l^{-1})}. \quad (13)$$

Thus

$$\frac{1}{\alpha+1} n^{\alpha+1} L(n) \gtrsim \varepsilon n^{\alpha+1} L(n) l^\alpha \frac{1}{2 \max(l, l^{-1})} \quad (14)$$

$$l \lesssim \max \left(\left(\frac{2}{\varepsilon(\alpha+1)} \right)^{\frac{1}{\alpha+1}}, \left(\frac{2}{\varepsilon(\alpha+1)} \right)^{\frac{1}{\alpha-1}} \right). \quad (15)$$

Now l and k are bounded above and below, so we can use the uniform convergence theorem (see Theorem 1.2.1 in [1] or Theorem 1 in the Appendix)

$$\sum_{i=nl}^{nk} f(i) \sim \sum_{i=nl}^{nk} i^\alpha L(n) \quad (16)$$

$$= L(n) \sum_{i=nl}^{nk} i^\alpha \quad (17)$$

$$\sim L(n) \int_{nl}^{nk} t^\alpha dt \quad (18)$$

$$= \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}). \quad (19)$$

Since the uniform convergence theorem applies uniformly, this convergence is also uniform. \square

Lemma 3. Let $\varepsilon > 0$ and $i > \varepsilon n$, then $s_{i,f}(x)$ has the following asymptotic

$$s_{i,f}(x) \sim n\phi^{-1}\left(\frac{i}{n}\right) \quad (20)$$

where $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$.

Proof. Let $l, k > 0$ be such that $n(k - l) = i$ and

$$\sum_{j=nl}^{nk} f(j) \leq x < \sum_{j=nl+1}^{nk+1} f(j) \quad (21)$$

Clearly it follows that l and k are unique up to producing the same range of integer indices in the sum.

Using Lemma 2 and Lemma 1

$$\frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) \sim \sum_{j=nl}^{nk} f(j) \sim x \sim \sum_{j=1}^n f(j) \sim \frac{n^{\alpha+1} L(n)}{\alpha+1}. \quad (22)$$

Thus

$$\frac{n^{\alpha+1} L(n)}{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) \sim \frac{n^{\alpha+1} L(n)}{\alpha+1} \quad (23)$$

(24)

By definition we get

$$\lim_{n \rightarrow \infty} \frac{\frac{n^{\alpha+1} L(n)}{\alpha+1} (k^{\alpha+1} - l^{\alpha+1})}{\frac{n^{\alpha+1} L(n)}{\alpha+1}} = 1 \quad (25)$$

$$\lim_{n \rightarrow \infty} k^{\alpha+1} - l^{\alpha+1} = 1. \quad (26)$$

Since k and l are constants independent of n it follows

$$k^{\alpha+1} - l^{\alpha+1} = 1 \quad (27)$$

$$k - l = (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \quad (28)$$

$$i = n((1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l) \quad (29)$$

$$\ln = n\phi^{-1}\left(\frac{i}{n}\right). \quad (30)$$

Lemma 4 proves that ϕ^{-1} is well defined. Since $s_{i,f}(x)$ counts the number of intervals of length i , we get

$$s_{i,f}(x) = nl \sim n\phi^{-1}\left(\frac{i}{n}\right). \quad (31)$$

□

Lemma 4. *The function $\phi : (0, \infty) \rightarrow (0, 1)$, $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ is a strictly decreasing bijection, has an inverse, and*

$$\int_0^1 \phi^{-1}(t) dt = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \quad (32)$$

for $\alpha > 1$.

Proof. Taking the derivative of ϕ

$$\frac{d\phi}{dx} = (1 + x^{\alpha+1})^{-\frac{\alpha}{\alpha+1}} x^\alpha - 1 \quad (33)$$

$$= \frac{x^\alpha}{\left((1 + x^{\alpha+1})^{\frac{1}{\alpha+1}}\right)^\alpha} - 1 \quad (34)$$

$$< 1 - 1 = 0 \quad (35)$$

where the last inequality follows from $(1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} > x$. Because the derivative of ϕ is negative, when $x > 0$, ϕ is strictly decreasing. Now ϕ is a bijection, because $\phi(0) = 1$ and $\lim_{x \rightarrow \infty} \phi(x) = 0$, hence it has an inverse. Since ϕ is a strictly decreasing bijection

$$\int_0^1 \phi^{-1}(x) dx = \int_0^\infty \phi(x) dx. \quad (36)$$

Next we show that the integral converges. Since $x^{\frac{1}{\alpha+1}}$ is increasing with a decreasing derivative, we have

$(1 + y)^{\frac{1}{\alpha+1}} < y^{\frac{1}{\alpha+1}} + \frac{1}{\alpha+1}y^{-\frac{\alpha}{\alpha+1}}$. Thus

$$\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x < \frac{1}{1+\alpha}x^{-\alpha}, \quad (37)$$

so the integral converges.

Doing a change of variables $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$, $x = (\frac{t}{1-t})^{\frac{1}{\alpha+1}}$, we get

$$dx = \frac{1}{\alpha+1} \left(\frac{t}{1-t} \right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \quad (38)$$

and

$$\phi(x) = (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \quad (39)$$

$$= (1-t)^{-\frac{1}{\alpha+1}} (1-t^{\frac{1}{\alpha+1}}). \quad (40)$$

Putting it together

$$\int_0^\infty \phi(x) dx = \int_0^1 (1-t)^{-\frac{1}{\alpha+1}} (1-t^{\frac{1}{\alpha+1}}) \frac{1}{\alpha+1} \left(\frac{t}{1-t} \right)^{-\frac{\alpha}{\alpha+1}} (1-t)^{-2} dt \quad (41)$$

$$= \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt. \quad (42)$$

Let $g_r(t) = (1-t)^{r-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$ and $g(t) = (1-t)^{-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})$. Since for each $r > 0$,

$g_r(t) \rightarrow g(t), r \rightarrow 0$ pointwise and $|g_r(t)| < |g(t)|$, we can use the dominated convergence theorem. Thus

$$\int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt = \lim_{r \rightarrow 0^+} \int_0^1 (1-t)^{r-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt. \quad (43)$$

Let B be the beta function (see NIST DLMF § 5.12.1 [2])

$$B(z_1, z_2) = \int_0^1 t^{z_1-1} (1-t)^{z_2-1} dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}. \quad (44)$$

The beta function can be analytically extended using Pochhammer's integral (see NIST DLMF § 5.12.12 [2]).

$$\int_0^\infty \phi(x) dx = \frac{1}{\alpha+1} \int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt \quad (45)$$

$$= \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \int_0^1 (1-t)^{r-\frac{2}{\alpha+1}-1} (t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}}) dt \quad (46)$$

$$= \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \left(B\left(\frac{1}{\alpha+1}, r - \frac{2}{\alpha+1}\right) - B\left(\frac{2}{\alpha+1}, r - \frac{2}{\alpha+1}\right) \right). \quad (47)$$

Using the gamma expression for beta functions, we get

$$\int_0^\infty \phi(x) dx = \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(r - \frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1} + r - \frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(r - \frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1} + r - \frac{2}{\alpha+1})} \right) \quad (48)$$

$$= \frac{1}{\alpha+1} \lim_{r \rightarrow 0^+} \left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)} \right). \quad (49)$$

Since $\Gamma(t) = \frac{\Gamma(1+t)}{t}$, we have $\Gamma(t) \sim \frac{1}{t}$, $t \rightarrow 0$. Thus

$$\lim_{r \rightarrow 0} \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)} = 0 \quad (50)$$

so the expression simplifies to

$$\int_0^\infty \phi(x)dx = \frac{1}{\alpha+1} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \quad (51)$$

$$= \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \quad (52)$$

where the last equality follows from $z\Gamma(z) = \Gamma(z+1)$. \square

One can see the graph of $C(\alpha)$ in the appendix figure 1.

4 Proof of the Main Theorem

Lemma 5. *Let $\varepsilon > 0$ and $\alpha > 1$, then*

$$\sum_{i=1}^{n\varepsilon} s_{i,f}(x) = O(n^2)\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})}, \quad (53)$$

where the $O(n^2)$ term is independent of ε .

Proof. Since

$$x \geq f(i) + \cdots + f(i+k-1) \geq kf(i) \quad (54)$$

$$f^{-1}\left(\frac{x}{k}\right) \geq i, \quad (55)$$

it follows that $s_{k,f}(x) \leq f^{-1}\left(\frac{x}{k}\right)$. The inverse of a regularly varying function is regularly varying with index $\frac{1}{\alpha}$ (see Theorem 1.5.12 of [1] or Theorem 3 in the Appendix). Let $f^{-1}(x) = x^{\frac{1}{\alpha}}\tilde{L}(x)$. We get

$$\sum_{i=1}^{n\varepsilon} f^{-1}\left(\frac{x}{i}\right) \sim \int_1^{n\varepsilon} f^{-1}\left(\frac{x}{t}\right) dt = x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt. \quad (56)$$

By Potter's bound (ii)

$$\tilde{L}\left(\frac{x}{t}\right) \leq M\tilde{L}\left(\frac{x}{n}\right) \left(\frac{n}{t}\right)^\delta, \quad (57)$$

where M is the constant from Potter's bound and $\delta = \frac{1}{2}(1 - \frac{1}{\alpha})$. Now we can bound the integral

$$x^{\frac{1}{\alpha}} \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}} \tilde{L}\left(\frac{x}{t}\right) dt \leq x^{\frac{1}{\alpha}} M\tilde{L}\left(\frac{x}{n}\right) n^\delta \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta}. \quad (58)$$

By definition

$$f^{-1}(f(n)) \sim n \quad (59)$$

$$n\tilde{L}(n^\alpha L(n))L(n)^{1/\alpha} \sim n \quad (60)$$

$$\tilde{L}(n^\alpha L(n))L(n)^{1/\alpha} \sim 1. \quad (61)$$

Using the relation between x and n obtained in Lemma 1

$$\tilde{L}\left(\frac{x}{n}\right) \sim \tilde{L}(n^\alpha L(n)). \quad (62)$$

Thus, from equations 62 and 61

$$x^{\frac{1}{\alpha}} M \tilde{L}\left(\frac{x}{n}\right) n^\delta \int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \sim M n^2 L(n)^{\frac{1}{\alpha}} \tilde{L}(n^\alpha L(n)) \frac{2\alpha}{\alpha-1} \varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})} \quad (63)$$

$$= O(n^2) \varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})}. \quad (64)$$

□

Theorem 1. Under the assumptions and notation in 2, the function $S_f(x)$ has the asymptotic $n^2 C(\alpha)$ for $\alpha > 1$, where

$$C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1}) \Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}. \quad (65)$$

Proof. Let $\varepsilon > 0$. From the definition of S and $s_{i,f}$ it follows that

$$S_f(x) = \sum_{i=1}^n s_{i,f}(x). \quad (66)$$

Splitting the sum

$$S_f(x) = \sum_{i=1}^{n\varepsilon} s_{i,f}(x) + \sum_{i=n\varepsilon}^n s_{i,f}(x). \quad (67)$$

Using Lemma 3, we get

$$\sum_{i=n\varepsilon}^n s_{i,f}(x) \sim \sum_{i=n\varepsilon}^n n\phi^{-1}\left(\frac{i}{n}\right) \quad (68)$$

$$= n \sum_{i=n\varepsilon}^n \phi^{-1}\left(\frac{i}{n}\right) \quad (69)$$

$$\sim n \int_{n\varepsilon}^n \phi^{-1}\left(\frac{t}{n}\right) dt \quad (70)$$

$$= n^2 \int_\varepsilon^1 \phi^{-1}(t) dt. \quad (71)$$

Since the integral converges

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \phi^{-1}(t) dt = \int_0^1 \phi^{-1}(t) dt \quad (72)$$

From Lemma 5, it follows that the first sum in 67 is negligible in the asymptotic, hence taking $\varepsilon \rightarrow 0$ finishes the proof. \square

Corollary 1. *The number of interval sums for $f(x) = x^\alpha, \alpha > 1$ is asymptotic to*

$$x^{\frac{2}{1+\alpha}} (\alpha + 1)^{\frac{2}{1+\alpha}} C(\alpha) \quad (73)$$

Proof. From theorem 1 and inverting lemma 1, it follows that

$$S_f(x) \sim n^2 C(\alpha) \sim ((\alpha + 1)^{\frac{1}{1+\alpha}} x^{\frac{1}{1+\alpha}})^2 C(\alpha) \sim x^{\frac{2}{1+\alpha}} (\alpha + 1)^{\frac{2}{1+\alpha}} C(\alpha) \quad (74)$$

\square

Corollary 2. *Number of interval sums for prime powers $p^\alpha, \alpha > 1$ is asymptotic to*

$$(1 + \alpha)^2 C(\alpha) \left(\frac{x}{\log(x)} \right)^{\frac{2}{\alpha+1}} \quad (75)$$

Proof. From the prime number theorem, it follows that $f(x) \sim (x \log(x))^\alpha$. Now using lemma 1 it follows that $x \sim \frac{1}{\alpha+1} n^{\alpha+1} \log^\alpha(n)$. It is easy to see that $n = (1 + \alpha) \left(\frac{x}{\log^\alpha(x)} \right)^{\frac{1}{1+\alpha}}$ is the asymptotic inverse. By theorem 1 the corollary follows. \square

5 Convergence of the Asymptotic Formula

Figures 2 and 3 show the impact of the slowly varying part and the index on convergence, respectively. We can observe that increasing the index slows down the convergence, and a larger absolute value in growth rate in the slowly varying part seem to corresponds to slower convergence.

See the table 1 below for relative error of the asymptotic at $x = 10^{20}$ for varying functions.

$f(x)$	relative error at $x = 10^{20}$
x^2	0.037%
x^3	0.045%
x^4	0.194%
x^5	0.771%
$x^3 \log(x)$	1.433%
$x^3 \log(x)^2$	2.836%
$x^4 / \sqrt{\log(x)}$	0.215%
$x^4 / \log(x)$	0.622%

Table 1: Relative error of the asymptotic at $x = 10^{20}$ for various functions.

These findings align with our analysis: the proofs rely on the uniform convergence theorem, which gives slower convergence for larger growth rates, and the convergence of the sum to integral is slower for larger indices.

6 Conclusion

The main theorem gives an asymptotic for the number of interval sums below x , namely $S_f(x) \sim n^2 C(\alpha)$. It remains an open question what happens in the case $\alpha \leq 1$; our methods can't handle that case since the integral in Lemma 4 diverges. Future work could therefore investigate the case $\alpha \leq 1$, it seems fruitful to use methods of de Haan theory to investigate the case $\alpha = 1$.

7 Use of AI

AI models GPT-5 mini, GPT-5.1, Gemini 3 Flash and Gemini 3 Pro were used to improve clarity and grammar. GPT-5.1 was also used to come up with substitution ideas for the integral in lemma 4. All AI-generated content was carefully and independently verified by the author.

References

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