1 Introduction

2 Notation

- Let RV_α^+ denote eventually monotonic regularly varying functions bounded away from 0 and ∞ in any closed interval with index α (see [1]). We assume the domain and range are $\mathbb{R}_{>0}$
- Let $S(x) = \#\{(i,j) \mid i \leq j, f(i) + f(i+1) + \cdots + f(j) \leq x\}$
- Let $s_k(x) = \#\{(i,j) \mid j - i = k, f(i) + f(i+1) + \cdots + f(j) \leq x\}$
- Let $\sim$ denote asymptotic equivalence. That is,

$$f(x) \sim g(x), x \to a \iff \lim_{x \to a} \frac{f(x)}{g(x)} = 1.$$

  We often omit the "$x \to a$" part, when it is clear from context.
- Denote $f(x) \lesssim g(x) \iff \limsup \frac{f(x)}{g(x)} \leq 1$ and similarly

$$f(x) \gtrsim g(x) \iff \liminf \frac{f(x)}{g(x)} \geq 1$$

  .

3 Preliminaries

We assume throughout this paper that $n = n(x)$ is the unique integer satisfying

$$\sum_{i=1}^{n} f(i) \leq x < \sum_{i=1}^{n+1} f(i).$$

Where $f \in RV_\alpha^+$.

Lemma 1. *We have*

$$x \sim \frac{n^{\alpha+1} L(n)}{\alpha + 1}$$

Proof. Let $\varepsilon > 0$ and split the sum

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n\varepsilon} f(i) + \sum_{i=n\varepsilon}^{n} f(i).$$

We will first examine the second sum.

$$\sum_{i=n\varepsilon}^{n} f(i) = \sum_{i=n\varepsilon}^{n} i^\alpha L(i)$$

Using uniform convergence theorem (see Theorem 1.2.1 in [1])

$$\begin{aligned}\sum_{i=n\varepsilon}^{n} i^{\alpha}L(i) &\sim L(n)\sum_{i=n\varepsilon}^{n} i^{\alpha}\\ &\sim L(n)\int_{n\varepsilon}^{n} t^{\alpha}dt\\ &= \frac{1}{\alpha+1}L(n)n^{\alpha+1}(1-\varepsilon^{\alpha+1})\end{aligned}$$

Examining the first sum

$$0 < \sum_{i=1}^{n\varepsilon} f(i) < n\varepsilon f(n\varepsilon).$$

Since f is eventually increasing

$$n\varepsilon f(n\varepsilon) < n\varepsilon f(n) = \varepsilon n^{\alpha+1}L(n).$$

Combining these we get

$$\begin{aligned}\frac{1}{\alpha+1}n^{\alpha+1}L(n)(1-\varepsilon^{\alpha+1}) \lesssim \sum_{i=1}^{n} f(i) &\lesssim \frac{1}{\alpha+1}n^{\alpha+1}L(n)(1+\varepsilon(\alpha+1)-\varepsilon^{\alpha+1})\\ \sum_{i=1}^{n} f(i) &\sim \frac{1}{\alpha+1}n^{\alpha+1}L(n)\end{aligned}$$

□

Lemma 2. *Let $\varepsilon > 0$ given $k, l > 0$ such that $k - l > \varepsilon$. If*

$$\sum_{i=nl}^{nk} f(i) < x$$

then

$$\sum_{i=nl}^{nk} f(i) \sim \frac{1}{\alpha+1}L(n)n^{\alpha+1}(k^{\alpha+1}-l^{\alpha+1}).$$

Proof. From definition of n it follows $k \leq l+1$. Since f is eventually increasing

$$\sum_{i=nl}^{nk} f(i) > n(k-l)f(nl) > n\varepsilon f(nl).$$

Using Potter's bound (see Theorem 1.5.6 [1]) and Lemma 1

$$\begin{aligned}\frac{1}{\alpha+1}n^{\alpha+1}L(n) &\gtrsim n\varepsilon f(nl)\\ &= n^{\alpha+1}\varepsilon l^{\alpha}L(nl)\\ &\gtrsim \varepsilon n^{\alpha+1}L(n)l^{\alpha}\frac{1}{2\max(l,l^{-1})}.\end{aligned}$$

Follows

$$l \lesssim (\frac{1}{2\varepsilon(\alpha+1)})^{\frac{1}{\alpha+1}}.$$

Now l and k are bounded above and below, so we can use uniform convergence theorem (see Theorem 1.2.1 in [1])

$$\begin{aligned}
\sum_{i=nl}^{nk} f(i) &\sim \sum_{i=nl}^{nk} i^{\alpha} L(n) \\
&= L(n) \sum_{i=nl}^{nk} i^{\alpha} \\
&\sim L(n) \int_{nl}^{nk} t^{\alpha} dt \\
&= \frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}).
\end{aligned}$$

□

Lemma 3. *Let $\varepsilon > 0$ and $i > \varepsilon n$, then $s_i(x)$ has the following asymptotic*

$$s_i(x) \sim n\phi^{-1}\left(\frac{i}{n}\right)$$

where $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$.

Proof. Let $l, k > 0$ be such that $n(k-l) = i$ and

$$\sum_{j=nl}^{nk} f(j) \leq x < \sum_{j=nl}^{nk+1} f(j)$$

Clearly it follows that l and k are unique. Using Lemma 2 and Lemma 1

$$\frac{1}{\alpha+1} L(n) n^{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) \sim \sum_{j=nl}^{nk} f(j) \sim x \sim \sum_{j=1}^{n} f(j) \sim \frac{n^{\alpha+1} L(n)}{\alpha+1}.$$

Follows

$$\begin{aligned}
\frac{n^{\alpha+1} L(n)}{\alpha+1} (k^{\alpha+1} - l^{\alpha+1}) &\sim \frac{n^{\alpha+1} L(n)}{\alpha+1} \\
k^{\alpha+1} - l^{\alpha+1} &\sim 1 \\
k - l &\sim (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \\
i &\sim n((1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l) \\
ln &\sim n\phi^{-1}(\frac{i}{n})
\end{aligned}$$

Since $s_i(x)$ counts the number of blocks whose sum is less than x of length i, we get

$$s_i(x) = nl \sim n\phi^{-1}(\frac{i}{n}).$$

$\square$

4 Case $\alpha = 1$

For $\alpha = 1$ the crude bounds work. Clearly

$$\sum_{i=1}^{n} f^{-1}(\frac{x}{i}) - i < S(x) < \sum_{i=1}^{n} f^{-1}(\frac{x}{i}).$$

Doing change of variables we get

$$\begin{aligned}\sum_{i=1}^{n} f^{-1}(\frac{x}{i}) &\sim x\int_{1}^{n} t^{-1}L(\frac{x}{t})dt \\ &= x\int_{x/n}^{x} \frac{\tilde{L}(u)}{u}du \\ &> x\int_{1}^{x} \frac{\tilde{L}(u)}{u}du\end{aligned}$$

From Karamata's Theorem (direct half) (see proposition 1.5.8 [1]) it follows

$$\int_{1}^{x} \frac{\tilde{L}(u)}{u}du > M\tilde{L}(x),$$

given large enough x.

5 Case $\alpha > 1$

Lemma 4. *The function $\phi : (0, \infty) \to (0, 1)$, $\phi(x) = (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x$ is strictly decreasing bijection hence has inverse and*

$$\int_{0}^{1} \phi^{-1}(t)dt = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}$$

Proof. Taking the derivative of ϕ

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}x} &= (1 + x^{\alpha+1})^{-\frac{\alpha}{\alpha+1}} x^{\alpha} - 1 \\ &= \frac{x^{\alpha}}{((1 + x^{\alpha+1})^{\frac{1}{\alpha+1}})^{\alpha}} - 1 \\ &< 1 - 1 = 0\end{aligned}$$

Where last inequality follows from $(1+x^{\alpha+1})^{\frac{1}{\alpha+1}} > x$. Because ϕ's derivative is negative everywhere, ϕ is strictly decreasing. Now ϕ is bijection, because $\phi(0) = 1$ and $\lim_{x\to\infty} \phi(x) = 0$, hence has inverse.

Since ϕ is strictly decreasing bijection

$$\int_0^1 \phi^{-1}(x)dx = \int_0^\infty \phi(x)dx.$$

Next we show that the integral converges. Since $x^{\frac{1}{\alpha+1}}$ is increasing with decreasing derivative, we have $(1+y)^{\frac{1}{\alpha+1}} < y^{\frac{1}{\alpha+1}} + \frac{1}{\alpha+1}y^{-\frac{\alpha}{\alpha+1}}$. Using this we get

$$\phi(x) < x^{-\alpha},$$

so the integral converges.

Doing change of variables $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$, $x = (\frac{t}{1-t})^{\frac{1}{\alpha+1}}$. We get

$$dx = \frac{1}{\alpha+1}(\frac{t}{1-t})^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt.$$

Changing $\phi(x)$ to t terms

$$\begin{aligned}\phi(x) &= (1+x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \\ &= (\frac{1}{1-t})^{\frac{1}{\alpha+1}} - (\frac{t}{1-t})^{\frac{1}{\alpha+1}} \\ &= (1-t)^{-\frac{1}{\alpha+1}}(1-t^{\frac{1}{\alpha+1}}).\end{aligned}$$

Putting it together

$$\begin{aligned}\int_0^\infty \phi(x)dx &= \int_0^1 (1-t)^{-\frac{1}{\alpha+1}}(1-t^{\frac{1}{\alpha+1}})\frac{1}{\alpha+1}(\frac{t}{1-t})^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt \\ &= \frac{1}{\alpha+1}\int_0^1 (1-t)^{\frac{\alpha}{\alpha+1}-\frac{1}{\alpha+1}-2}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1}-\frac{\alpha}{\alpha+1}})dt \\ &= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt\end{aligned}$$

Since $\lim_{r\to 0}(1-t)^r = 1, t \in (0,1)$, we can take any $\varepsilon, \delta > 0$ such that $\left|(1-t)^r - 1\right| < \varepsilon, t \in [\delta, 1-\delta]$ for small enough $r > 0$. Since the integral converges

$$\begin{aligned}&\int_0^{1-\delta} (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \\ &= \int_0^1 (1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt + h(\delta)\end{aligned}$$

Where $\lim_{\delta\to 0} h(\delta) = 0$.

$$\left|\int_0^{1-\delta}(1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt-\int_0^{1}(1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt\right|$$
$$=\left|\int_0^{1-\delta}(1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt-\int_0^{1-\delta}(1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt+h(\delta)\right|$$
$$=\left|\int_0^{1-\delta}((1-t)^{r}-1)(1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt+h(\delta)\right|$$
$$<\varepsilon\int_0^{1-\delta}(1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt+\left|h(\delta)\right|$$

Letting $\varepsilon, \delta \to 0$, gives that the limit can be taken out of the integral

$$\int_0^{1}(1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt$$
$$=\lim_{r\to 0}\int_0^{1}(1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt$$

Let B be the beta function (see NIST DLMF § 5.12.1 [2])

$$B(z_1,z_2)=\int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt=\frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}$$

The beta function can be analytically extended to $\big(\mathbb{C}\setminus\mathbb{Z},\mathbb{C}\setminus\mathbb{Z}\big)$ using Pochhammer's integral (see NIST DLMF § 5.12.12).

$$=\frac{1}{\alpha+1}\int_0^{1}(1-t)^{-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt$$
$$=\frac{1}{\alpha+1}\lim_{r\to 0}\int_0^{1}(1-t)^{r-\frac{2}{\alpha+1}-1}(t^{-\frac{\alpha}{\alpha+1}}-t^{\frac{1-\alpha}{\alpha+1}})dt$$
$$=\frac{1}{\alpha+1}\lim_{r\to 0^+}\left(B(\frac{1}{\alpha+1},r-\frac{2}{\alpha+1})-B(\frac{2}{\alpha+1},r-\frac{2}{\alpha+1})\right).$$

Using gamma expression for beta functions we get

$$\int_0^{\infty}\phi(x)dx=\frac{1}{\alpha+1}\lim_{r\to 0}\left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(r-\frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1}+r-\frac{2}{\alpha+1})}-\frac{\Gamma(\frac{2}{\alpha+1})\Gamma(r-\frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1}+r-\frac{2}{\alpha+1})}\right)$$
$$=\frac{1}{\alpha+1}\lim_{r\to 0}\left(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})}-\frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)}\right).$$

Since $\Gamma(t)=\frac{\Gamma(1+t)}{t}$ we have $\Gamma(t)\sim\frac{1}{t}, t\to 0$. Follows

$$\lim_{r\to 0}\frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(r)}=0.$$

So the expression simplifies to

$$\int_0^\infty \phi(x)dx = \frac{1}{\alpha+1}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})}$$
$$= \frac{1}{2(\alpha+1)}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})},$$

where last equality uses $z\Gamma(z) = \Gamma(z+1)$. □

Lemma 5. *Let* $\varepsilon > 0$ *then*

$$\sum_{i=1}^{n\varepsilon} s_i(x) = O(n^2)\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})}.$$

Proof. Clearly $s_i(x) \le f^{-1}(\frac{x}{i})$. Regularly varying functions inverse is regularly varying with index $\frac{1}{\alpha}$ (see Theorem 1.5.12 in [1]). Let $f^{-1}(x) = x^{\frac{1}{\alpha}}\tilde{L}(x)$.

$$\sum_{i=1}^{n\varepsilon} f^{-1}(\frac{x}{i}) \sim \int_1^{n\varepsilon} f^{-1}(\frac{x}{t})dt = x^{\frac{1}{\alpha}}\int_1^{n\varepsilon} t^{-\frac{1}{\alpha}}\tilde{L}(\frac{x}{t})dt$$

By Potter's bound

$$\tilde{L}(\frac{x}{t}) \le M\tilde{L}(\frac{x}{n})(\frac{n}{t})^\delta,$$

where M is the constant from Potter's bound and $\delta = \frac{1}{2}(1-\frac{1}{\alpha})$. Now we can bound the integral

$$x^{\frac{1}{\alpha}}\int_1^{n\varepsilon} t^{-\frac{1}{\alpha}}\tilde{L}(\frac{x}{t})dt \le x^{\frac{1}{\alpha}}M\tilde{L}(\frac{x}{n})n^\delta\int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \tag{1}$$

We know that

$$f^{-1}(f(n)) = n$$
$$n\tilde{L}(n^\alpha L(n))L(n)^{1/\alpha} \sim n$$
$$\tilde{L}(n^\alpha L(n))L(n)^{1/\alpha} \sim 1$$

Follows

$$x^{\frac{1}{\alpha}}M\tilde{L}(\frac{x}{n})n^\delta\int_1^{n\varepsilon} t^{-\frac{1}{\alpha}-\delta} \sim Mn^2L(n)^{\frac{1}{\alpha}}\tilde{L}(n^\alpha L(n))\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})}$$
$$= O(n^2)\varepsilon^{\frac{1}{2}(1-\frac{1}{\alpha})}$$

□

Theorem 1. *The function* $S(x)$ *has asymptotic* $C(\alpha)n^2$*, where*

$$C(\alpha) = \frac{1}{2(\alpha+1)}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}.$$

Proof. Let $\varepsilon > 0$. From definition of S and s_i it follows that

$$S(x) = \sum_{i=1}^{n} s_i(x)$$

Splitting the sum

$$S(x) = \sum_{i=1}^{n\varepsilon} s_i(x) + \sum_{i=n\varepsilon}^{n} s_i(x)$$

Using lemma 5 we can control the first sum, so we will focus on the second sum. Using lemma 3 we get

$$\begin{aligned}
\sum_{i=n\varepsilon}^{n} s_i(x) &\sim \sum_{i=n\varepsilon}^{n} n\phi^{-1}(\frac{i}{n}) \\
&= n \sum_{i=n\varepsilon}^{n} \phi^{-1}(\frac{i}{n}) \\
&\sim n \int_{n\varepsilon}^{n} \phi^{-1}(\frac{t}{n})dt \\
&= n^2 \int_{\varepsilon}^{1} \phi^{-1}(t)dt.
\end{aligned}$$

We can bound ϕ^{-1}, using the bound derived in Lemma 4

$$\begin{aligned}
\phi(x) &< x^{-\alpha} \\
\phi(x^{-\frac{1}{\alpha}}) &< x \\
x^{-\frac{1}{\alpha}} &< \phi^{-1}(x)
\end{aligned}$$

Now we can bound

$$\begin{aligned}
\int_{0}^{\varepsilon} \phi^{-1}(t)dt &= \varepsilon\phi^{-1}(\varepsilon) + \int_{\phi^{-1}(\varepsilon)}^{\infty} \phi(t)dt \\
&< \varepsilon^{1-\frac{1}{\alpha}} + \phi^{-1}(\varepsilon)^{1-\alpha}.
\end{aligned}$$

We get

$$\begin{aligned}
n^2 \int_{\varepsilon}^{1} \phi^{-1}(t)dt &= n^2(\int_{0}^{1} \phi^{-1}(t)dt - \int_{0}^{\varepsilon} \phi^{-1}(t)dt) \\
&> n^2(C(\alpha) - \varepsilon^{1-\frac{1}{\alpha}} + \phi^{-1}(\varepsilon)^{1-\alpha})
\end{aligned}$$

and

$$n^2 \int_{\varepsilon}^{1} \phi^{-1}(t)dt < n^2 C(\alpha).$$

Combining this and bound from Lemma 5 the Theorem follows. $\square$

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

[2] *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.