lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex, twice differentiable function such that $f'(x) \geq 0$. Let $g : \mathbb{R} \to \mathbb{R}$ satisfy*

$$f(g(x)) \sim x$$

Then

$$g(x) \sim f^{-1}(x)$$

Proof. From $f(g(x)) \sim x$ follows

$$f(g(x)) = x + o(x)$$

Taking inverse of f

$$g(x) = f^{-1}(x + o(x))$$

Since f is convex and monotonic, we get

$$f^{-1}((1+\epsilon)x) \leq (1+\epsilon)f^{-1}(x)$$
$$f^{-1}((1-\epsilon)x) \geq (1-\epsilon)f^{-1}(x)$$

Hence,

$$g(x) = f^{-1}(x + o(x)) = (1 + o(1))f^{-1}(x) \sim f^{-1}(x)$$

□

lemma 2. *Let f be rapidly varying function. Fix $\epsilon > 0$, then for sufficiently large n*

$$\sum_{i=1}^{n} f(i) < \sum_{i=n+1}^{n(1+\epsilon)} f(i)$$

Proof. We will prove that the ratio tends to infinity. We examine the left-hand side

$$\sum_{i=1}^{n} f(i) < nf(n)$$

On the right-hand side

$$\sum_{i=n+1}^{n(1+\epsilon)} f(i) > n\frac{\epsilon}{2}f(n(1+\frac{\epsilon}{2}))$$

Inspecting the ratio

$$\frac{\sum_{i=n+1}^{n(1+\epsilon)} f(i)}{\sum_{i=1}^{n} f(i)} > \frac{n\frac{\epsilon}{2}f(n(1+\frac{\epsilon}{2}))}{nf(n)} > \frac{\epsilon}{2}\frac{f(n(1+\frac{\epsilon}{2}))}{f(n)}$$

Since f is rapidly varying function

$$\lim_{n\to\infty} \frac{f(n(1+\frac{\epsilon}{2}))}{f(n)} = \infty$$

So there exits N such that

$$\frac{f(n(1+\frac{\epsilon}{2}))}{f(n)} > \frac{2}{\epsilon}, n > N$$

$\square$

lemma 3. *Let $f \in RV_\alpha$. The sum*

$$\sum_{i=1}^{n} f(i)$$

has asymptotic

$$\frac{1}{1+\alpha} L(n) n^{\alpha+1}$$

Proof. Using f's asymptotic

$$\sum_{i=1}^{n} f(i) \sim \sum_{i=1}^{n} i^\alpha L(i)$$

The head does not affect the asymptotic

$$\sim \sum_{i=n\epsilon}^{n} i^\alpha L(i)$$

Since L is SV, $\frac{L(\lambda x)}{L(x)} \to 1$ uniformly on compact sets. Take $\lambda \in [\epsilon, 1]$

$$\sim \sum_{i=n\epsilon}^{n} i^\alpha L(n)$$

Take the $L(n)$ out of the sum

$$\sim L(n) \sum_{i=n\epsilon}^{n} i^\alpha$$

Clearly

$$\sim L(n) \int_{n\epsilon}^{n} t^\alpha dt$$

Calculate the integral

$$= \frac{1}{1+\alpha} L(n) n^{\alpha+1} (1 - \epsilon^{1+\alpha})$$

Follows

$$\sim \frac{1}{1+\alpha} L(n) n^{\alpha+1}$$

$\square$

lemma 4. *Given monotonic regularly varying $f \sim x^\alpha L(x), \alpha > 1$. Let n be such that*

$$\sum_{i=1}^{n} f(i) \le x < \sum_{i=1}^{n+1} f(i)$$

If $\frac{L(xL^\beta(x))}{L(x)} \sim 1$ locally uniformly $\beta \in \mathbb{R}$ and L is increasing and unbounded, then

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k})) \sim C(\alpha) n^2$$

Where $C(\alpha) = \frac{\alpha}{(\alpha-1)(\alpha+1)^{\frac{1}{\alpha}}}$

Proof. From corollary 2.3.4 in [1] it follows that

$$f^{-1}(x) \sim x^{\frac{1}{\alpha}} \tilde{L}^{-\frac{1}{\alpha}}(x^{\frac{1}{\alpha}}) \sim x^{\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})}$$

Now we inspect

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k})$$

Using f^{-1} asymptotic

$$\sim \sum_{k=1}^{n} (\frac{x}{k})^{\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}(\frac{x}{k})}$$

Take the x out of the sum

$$= x^{\frac{1}{\alpha}} \sum_{k=1}^{n} k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{k})^{\frac{1}{\alpha}})}$$

The head of sum does not affect the asymptotic

$$\sim x^{\frac{1}{\alpha}} \sum_{k=n\epsilon}^{n} k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{k})^{\frac{1}{\alpha}})}$$

Since $L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})$ is SV, $\frac{L^{\frac{1}{\alpha}}((\lambda x)^{\frac{1}{\alpha}})}{L^{\frac{1}{\alpha}}(x^{\frac{1}{\alpha}})} \to 1$ uniformly on compact sets. Take $\lambda \in [\epsilon, 1]$

$$\sim x^{\frac{1}{\alpha}} \sum_{k=n\epsilon}^{n} k^{-\frac{1}{\alpha}} \frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}$$

Take $\frac{1}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}$ out of the sum

$$\sim \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})} \sum_{k=n\epsilon}^{n} k^{-\frac{1}{\alpha}}$$

Clearly

$$\sim \frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}\int_{n\epsilon}^{n} t^{-\frac{1}{\alpha}}dt$$

Calculate the integral

$$= \frac{1}{1-\frac{1}{\alpha}}\frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}(n^{1-\frac{1}{\alpha}} - (n\epsilon)^{1-\frac{1}{\alpha}})$$

Factor n out

$$= \frac{1}{1-\frac{1}{\alpha}}\frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}n^{1-\frac{1}{\alpha}}(1 - \epsilon^{1-\frac{1}{\alpha}})$$

We get

$$\sim \frac{1}{1-\frac{1}{\alpha}}\frac{x^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{x}{n})^{\frac{1}{\alpha}})}n^{1-\frac{1}{\alpha}} \tag{1}$$

From lemma 3

$$x \sim \frac{1}{1+\alpha}L(n)n^{\alpha+1} \tag{2}$$

Using (2) to (1)

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{1}{1-\frac{1}{\alpha}}\frac{(\frac{1}{1+\alpha}L(n)n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\frac{1}{1+\alpha}n^{\alpha}L(n))^{\frac{1}{\alpha}})}n^{1-\frac{1}{\alpha}}$$

Simplify

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{1}{1-\frac{1}{\alpha}}\frac{(\frac{1}{1+\alpha}L(n)n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}(\frac{1}{1+\alpha}nL(n)^{\frac{1}{\alpha}})}n^{1-\frac{1}{\alpha}}$$

Using $\frac{L(xL^{\beta}(x))}{L(x)} \sim 1$ it follows

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{1}{1-\frac{1}{\alpha}}\frac{(\frac{1}{1+\alpha}L(n)n^{\alpha+1})^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}(n)}n^{1-\frac{1}{\alpha}}$$

Simplify

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{\alpha}{(\alpha-1)(\alpha+1)^{\frac{1}{\alpha}}}n^{2}$$

□

lemma 5. *Same assumptions as 4, but $\alpha = 1$ then*

$$n^2 = o(\sum_{k=1}^{n} f^{-1}(\frac{x}{k}))$$

Proof. From proof of 4 we get

$$\sum_{k=1}^{n} f^{-1}(\frac{x}{k}) \sim \frac{x}{L(x)} \int_{n\epsilon}^{n} \frac{1}{t} dt \sim \frac{x}{L(x)} \log(n)$$

Using 3 we get

$$\sim n^2 log(n)$$

The claim follows. ∎

lemma 6. *Let $\epsilon > 0$ and i such that $n\epsilon \leq i$. Given $k, l \geq \epsilon, k - l \geq \epsilon$ and eventually monotonic $f \in RV_\alpha$*

$$\sum_{j=nl}^{nk} f(j) \sim \frac{L(n)}{\alpha+1} n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1})$$

Proof. We will first prove that l is bounded above.

$$\sum_{j=ln}^{kn} f(j) > n(k-l)f(ln) \geq n\epsilon f(ln)$$

Using the sums asymptotic from 3

$$\begin{aligned} 1 &\geq \liminf \frac{n\epsilon f(ln)}{\frac{L(n)}{\alpha+1} n^{\alpha+1}} \\ 1 &\geq \liminf \frac{(\alpha+1)\epsilon L(nl) l^\alpha}{L(n)} \end{aligned} \tag{3}$$

Using Potter's theorem (see 1.5.6 [1])

$$1 \geq \liminf 2(\alpha+1)\epsilon l^{\alpha+1}$$

Follows $l \leq (\frac{1}{(\alpha+1)\epsilon})^{\frac{1}{\alpha+1}}$. Clearly $k \leq l+1$. Now since l, k are in some compact interval independent of n, we can use uniform convergence theorem to get

$$\begin{aligned} \sum_{j=nl}^{nk} f(j) &\sim \sum_{j=nl}^{nk} L(j) j^\alpha \\ &\sim \sum_{j=nl}^{nk} L(n) j^\alpha \\ &\sim \frac{L(n)}{\alpha+1} n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1}) \end{aligned}$$

∎

theorem 1. *Let $f \in RV_\alpha, \alpha > 1$, then $S(x) \sim C(\alpha)n^2$. Where*

$$\sum_{i=1}^{n} f(i) \leq x < \sum_{i=1}^{n+1} f(i)$$

and $C(\alpha) = \frac{1}{2(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})}$

Proof. Let $l \geq \epsilon$. Define x such that

$$\sum_{i=1}^{n} f(i) \leq x < \sum_{i=1}^{n+1} f(i)$$

Define k such that

$$\sum_{nl}^{nk} f(i) < x < \sum_{nl}^{nk+1} f(i)$$

From 3 and the fact that $\frac{k}{l} \in [1, \frac{1}{\epsilon}]$ we get using uniform convergence of $\frac{L(\lambda x)}{L(x)} \to 1$ in compact sets

$$\sum_{nl}^{nk} f(i) \sim \frac{L(n)}{\alpha+1} n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1})$$

On other hand we know that

$$x \sim \frac{L(n)}{1+\alpha} n^{\alpha+1}$$

Combining these

$$\frac{L(n)}{1+\alpha} n^{\alpha+1} \sim \frac{L(n)}{\alpha+1} n^{\alpha+1}(k^{\alpha+1} - l^{\alpha+1})$$

Follows

$$k^{\alpha+1} - l^{\alpha+1} = 1$$

Solving for k

$$k = (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}}$$

Finally we get

$$k - l = (1 + l^{\alpha+1})^{\frac{1}{\alpha+1}} - l \tag{4}$$

Let $\phi(t) = (1+t^{\alpha+1})^{\frac{1}{\alpha+1}} - t$. Now we would like to count the $s_i(x)$. Calculating $s_i(x)$ asymptotic Let l be such that $i = n\phi(l)$. Solving for nl we get

$$nl = n\phi^{-1}(\frac{i}{n})$$

So

$$s_i(x) = nl = n\phi^{-1}(\frac{i}{n})$$

We can now count $S(x)$ asymptotic.

$$S(x) = \sum_{i=1}^{n} s_i(x)$$

The head doesn't affect the asymptotic

$$S(x) \sim \sum_{i=n\epsilon}^{n} s_i(x)$$

Using s_i formula

$$S(x) \sim \sum_{i=n\epsilon}^{n} n\phi^{-1}(\frac{i}{n})$$

Since ϕ is monotonic we have

$$S(x) \sim n \int_{n\epsilon}^{n} \phi^{-1}(\frac{t}{n})dt$$

Doing change of variables to $t \to un$ we get

$$S(x) \sim n^2 \int_{\epsilon}^{1} \phi^{-1}(t)dt$$

Since $\phi : (0, \infty) \to (0, 1)$ is strictly decreasing bijection

$$\int_{\epsilon}^{1} \phi^{-1}(t)dt \sim \int_{0}^{\infty} \phi(t)dt$$

We get

$$S(x) \sim n^2 \int_{0}^{\infty} \phi(t)dt$$

Calculating the integral

$$\int_{0}^{\infty} (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - xdx$$

Doing change of variables $t = \frac{x^{\alpha+1}}{1+x^{\alpha+1}}$, $x = (\frac{t}{1-t})^{\frac{1}{\alpha+1}}$. We get

$$dx = \frac{1}{\alpha+1}(\frac{t}{1-t})^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt$$

Changing $\phi(x)$ to t terms

$$\begin{aligned}
\phi(x) &= (1 + x^{\alpha+1})^{\frac{1}{\alpha+1}} - x \\
&= (\frac{1}{1-t})^{\frac{1}{\alpha+1}} - (\frac{t}{1-t})^{\frac{1}{\alpha+1}} \\
&= (1-t)^{-\frac{1}{\alpha+1}}(1 - t^{\frac{1}{\alpha+1}})
\end{aligned}$$

Putting it together

$$
\begin{aligned}
\int_0^\infty \phi(t)dt &= \int_0^1 (1-t)^{-\frac{1}{\alpha+1}}(1-t^{\frac{1}{\alpha+1}})\frac{1}{\alpha+1}\big(\frac{t}{1-t}\big)^{-\frac{\alpha}{\alpha+1}}(1-t)^{-2}dt \\
&= \frac{1}{\alpha+1}\int_0^1 (1-t)^{\frac{\alpha}{\alpha+1}-\frac{1}{\alpha+1}-2}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1}{\alpha+1}-\frac{\alpha}{\alpha+1}})dt \\
&= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}}(t^{-\frac{\alpha}{\alpha+1}} - t^{\frac{1-\alpha}{\alpha+1}})dt \\
&= \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}}t^{-\frac{\alpha}{\alpha+1}}dt - \frac{1}{\alpha+1}\int_0^1 (1-t)^{-\frac{\alpha+3}{\alpha+1}}t^{\frac{1-\alpha}{\alpha+1}}dt \\
&= \frac{1}{\alpha+1}\lim_{\epsilon\to 0^+}\int_0^1 (1-t)^{\epsilon-\frac{\alpha+3}{\alpha+1}}t^{-\frac{\alpha}{\alpha+1}}dt - \frac{1}{\alpha+1}\int_0^1 (1-t)^{\epsilon-\frac{\alpha+3}{\alpha+1}}t^{\frac{1-\alpha}{\alpha+1}}dt \\
&= \frac{1}{\alpha+1}\lim_{\epsilon\to 0^+}\big(B(\frac{1}{\alpha+1},\epsilon-\frac{2}{\alpha+1}) - B(\frac{2}{\alpha+1},\epsilon-\frac{2}{\alpha+1})\big)
\end{aligned}
$$

Where B is the analytically extended beta function(see NIST DLMF § 5.12 [2])

$$
B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}
$$

Using gamma expression for beta functions we get

$$
\begin{aligned}
\int_0^\infty \phi(t)dt &= \frac{1}{\alpha+1}\lim_{\epsilon\to 0^+}\Big(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\epsilon-\frac{2}{\alpha+1})}{\Gamma(\frac{1}{\alpha+1}+\epsilon-\frac{2}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(\epsilon-\frac{2}{\alpha+1})}{\Gamma(\frac{2}{\alpha+1}+\epsilon-\frac{2}{\alpha+1})}\Big) \\
&= \frac{1}{\alpha+1}\lim_{\epsilon\to 0^+}\Big(\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} - \frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\epsilon)}\Big)
\end{aligned}
$$

Using $\Gamma(t) \sim \frac{1}{t}, t \to 0$

$$
\lim_{\epsilon\to 0^+}\frac{\Gamma(\frac{2}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(\epsilon)} = 0
$$

So the expression simplifies to

$$
\begin{aligned}
\int_0^\infty \phi(t)dt &= \frac{1}{\alpha+1}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(-\frac{2}{\alpha+1})}{\Gamma(-\frac{1}{\alpha+1})} \\
&= \frac{1}{2(\alpha+1)}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{\alpha-1}{\alpha+1})}{\Gamma(\frac{\alpha}{\alpha+1})} \\
&= C(\alpha)
\end{aligned}
$$

Where second equality follows from $\Gamma(z+1) = z\Gamma(z)$.

□

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

[2] *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.2.4 of 2025-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.