

## Appendix: Selected Results on Regular Variation

For convenience, we reproduce several standard theorems from [1] we assume through this section that  $f$  and  $l$  is measurable.

**Theorem 1** (Uniform Convergence Theorem). *If  $l$  is slowly varying then*

$$\frac{l(\lambda x)}{l(x)} \rightarrow 1 \quad (x \rightarrow \infty)$$

*uniformly in each compact  $\lambda$ -set in  $(0, \infty)$ .*

*Proof.* See Theorem 1.2.1 in [1]. □

**Theorem 2** (Potters Theorem). *(i) If  $l$  is slowly varying function then for any chosen constants  $A > 1$ ,  $\delta > 0$  there exists  $X = X(A, \delta)$  such that*

$$\frac{l(y)}{l(x)} \leq A \max \left\{ \left( \frac{y}{x} \right)^\delta, \left( \frac{x}{y} \right)^\delta \right\} \quad (x \geq X, y \geq X)$$

*(ii) If further,  $l$  is bounded away from 0 and  $\infty$  in every compact subset of  $[0, \infty)$ , then for every  $\delta > 0$  there exists  $A' = A'(\delta) > 1$  such that*

$$\frac{l(y)}{l(x)} \leq A' \max \left\{ \left( \frac{y}{x} \right)^\delta, \left( \frac{x}{y} \right)^\delta \right\} \quad (x \geq 0, y \geq 0)$$

*(iii) If  $f$  is regularly varying of index  $\rho$  then for any chosen  $A > 1$ ,  $\delta > 0$  there exists  $X = X(A, \delta)$  such that*

$$\frac{f(y)}{f(x)} \leq A \max \left\{ \left( \frac{y}{x} \right)^{\rho+\delta}, \left( \frac{x}{y} \right)^{\rho+\delta} \right\} \quad (x \geq X, y \geq X)$$

*Proof.* See Theorem 1.5.6 in [1]. □

**Proposition 1** (Karamata's Theorem). *If  $l$  is slowly varying,  $X$  is so large that  $l(x)$  is locally bounded in  $[X, \infty)$ , and  $\alpha > -1$ , then*

$$\int_X^x t^\alpha l(t) dt \sim \frac{x^{\alpha+1} l(x)}{\alpha+1} \quad (x \rightarrow \infty)$$

*Proof.* See Proposition 1.5.8 in [1]. □

**Definition 1** (Generalized inverse). *Generalized inverse of  $f$  is defined by*

$$f^{\leftarrow}(x) = \inf \{x \mid f(x) = y\} \quad (1)$$

**Theorem 3.** *If  $f \in RV_\alpha$  with  $\alpha > 0$ , there exists  $g \in RV_{\frac{1}{\alpha}}$  with*

$$f(g(x)) \sim g(f(x)) \sim x \quad (x \rightarrow \infty).$$

*Here  $g$  (an asymptotic inverse of  $f$ ) is determined uniquely to within asymptotic equivalence, and one version of  $g$  is  $f^{\leftarrow}$ .*

*Proof.* See theorem 1.5.12 in [1]. □

## Plot of $C(\alpha)$

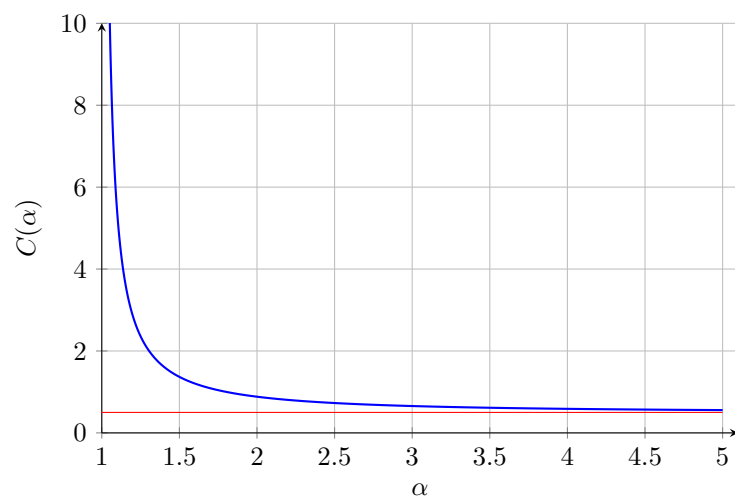


Figure 1: The blue graph shows  $C(\alpha)$  and the red one shows the asymptote at  $\frac{1}{2}$ .

## Figures on convergence

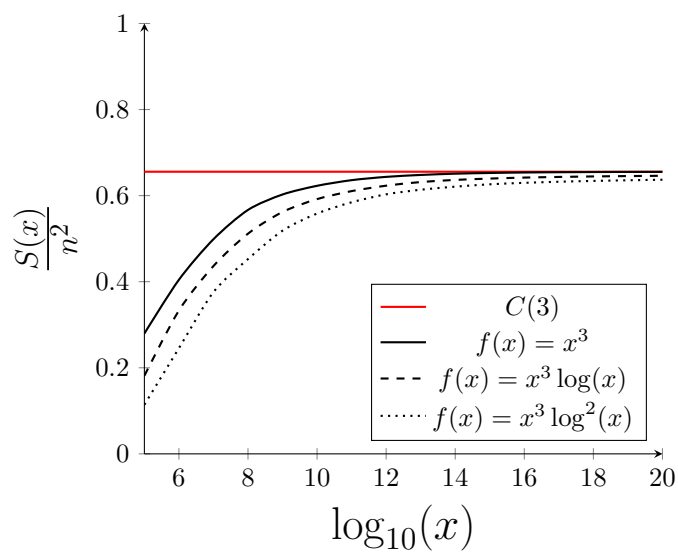


Figure 2: Example of how the slowly varying part affects convergence

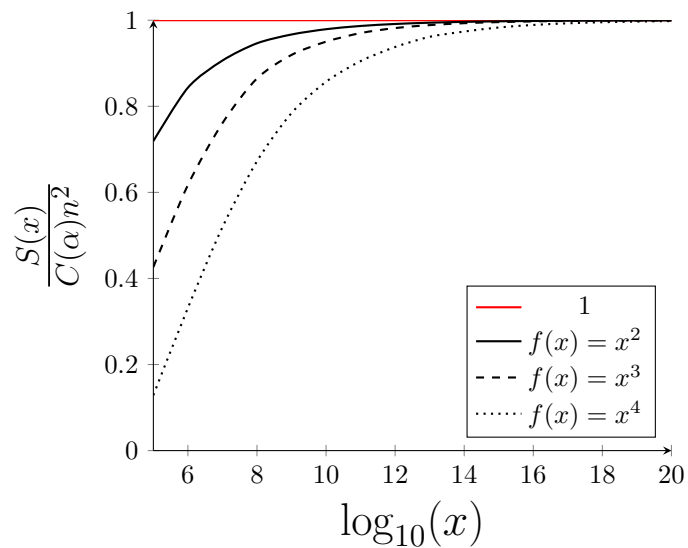


Figure 3: Example of how the index affects convergence

## References

- [1] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.