

Chapter 4 – Higher-Order Differential Equations

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Chapter 4.1 – Preliminary Theory for Linear Equations

4.1.7

$$\begin{aligned}x(t) &= x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \\x'(t) &= -x_0 \omega \sin(\omega t) + x_1 \cos(\omega t)\end{aligned}$$

$$\begin{aligned}x(0) = x_0 &\Rightarrow x_0 = x_0 \cos(0) + \frac{x_1}{\omega} \sin(0) = x_0 \cdot 1 + \frac{x_1}{\omega} \cdot 0 = x_0 \\x'(0) = x_1 &\Rightarrow x_1 = x_1 \cos(0) - x_0 \omega \sin(0) = x_1 \cdot 1 - x_0 \omega \cdot 0 = x_1\end{aligned}$$

4.1.10

$$\begin{aligned}y'' + (\tan x)y &= e^x \\ \frac{y''}{\tan x} + y &= \frac{e^x}{\tan x}\end{aligned}$$

For the equation to have a solution the coefficient of y'' needs to not be 0 on the interval

$$\text{The interval around 0 where } \frac{1}{\tan x} \neq 0 \text{ is } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

4.1.13

$$\begin{aligned}y &= c_1 e^x \cos x + c_2 e^x \sin x \\y' &= c_1 e^x \cos x - c_1 e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x\end{aligned}$$

(a)

$$\begin{aligned}y(0) = 1 &\Rightarrow c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = 1 \Rightarrow c_1 = 1 \\y'(\pi) = 0 &\Rightarrow e^\pi \cos(\pi) - e^\pi \sin \pi + c_2 e^\pi \sin(\pi) + c_2 e^\pi \cos \pi = -e^\pi - c_2 e^\pi = 0 \Rightarrow c_2 = -1\end{aligned}$$

$$y = e^x \cos x - e^x \sin x$$

(b)

$$\begin{aligned}y(0) = 1 &\Rightarrow c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = 1 \Rightarrow c_1 = 1 \\y(\pi) = -1 &\Rightarrow e^\pi \cos \pi + c_2 e^\pi \sin \pi = -e^\pi \neq -1\end{aligned}$$

No solution

(c)

$$\begin{aligned}y(0) = 1 &\Rightarrow c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = 1 \Rightarrow c_1 = 1 \\y\left(\frac{\pi}{2}\right) = 1 &\Rightarrow e^{\frac{\pi}{2}} \cos \frac{\pi}{2} + c_2 e^{\frac{\pi}{2}} \sin \frac{\pi}{2} = c_2 e^{\frac{\pi}{2}} = 1 \Rightarrow c_2 = e^{-\frac{\pi}{2}}\end{aligned}$$

$$y = e^x \cos x - e^{-\frac{\pi}{2}} e^x \sin x$$

(d)

$$\begin{aligned}y(0) = 0 &\Rightarrow c_1 e^0 \cos 0 + c_2 e^0 \sin 0 = 0 \Rightarrow c_1 = 0 \\y(\pi) = 0 &\Rightarrow +c_2 e^\pi \sin \pi = 0 \Rightarrow c_2 \in \mathbb{R}\end{aligned}$$

$$y = c_2 e^x \sin x$$

4.1.17

$$\frac{1}{5}f_1(x) - f_2(x) = 1 - \cos^2 x = \sin^2 x = f_3(x)$$

4.1.18

$$2f_3(x) - f_2(x) = 2\cos^2 x - 1 = 2\cos^2 x - \cos^2 x - \sin^2 x = \cos^2 x - \sin^2 x = \cos 2x$$

4.1.19

$$4f_1(x) - 3f_2(x) = 4x - 3(x - 1) = 4x - 3x + 3 = x + 3 = f_3(x)$$

4.1.20

Independent

4.1.21

Independent, can't create x^2

4.1.23

Both functions satisfy the differential equation.

$$W(e^{-3x}, e^{4x}) = \begin{pmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{pmatrix} = 7e^x \neq 0$$

4.1.25

Both functions satisfy the differential equation.

$$W(e^x \cos 2x, e^x \sin 2x) = \begin{pmatrix} e^x \cos 2x & e^x \sin 2x \\ e^x \cos 2x - 2e^x \sin 2x & e^x \sin 2x + 2e^x \cos 2x \end{pmatrix} \quad x = 0 \Rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 2 \neq 0$$

4.1.26

Both functions satisfy the differential equation.

$$W(e^{\frac{x}{2}}, xe^{\frac{x}{2}}) = \begin{pmatrix} e^{\frac{x}{2}} & xe^{\frac{x}{2}} \\ \frac{e^{\frac{x}{2}}}{2} & e^{\frac{x}{2}} + \frac{xe^{\frac{x}{2}}}{2} \end{pmatrix} \quad x = 0 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \neq 0$$

4.1.29

All of the functions satisfy the differential equation.

$$W(x, x^{-2}, x^{-2} \ln x) = \begin{pmatrix} x & x^{-2} & x^{-2} \ln x \\ 1 & -2x^{-3} & \frac{-2 \ln(x)+1}{x^3} \\ 0 & 6x^{-4} & -\frac{6 \ln(x)+5}{x^4} \end{pmatrix} \quad x = 1 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 6 & -5 \end{pmatrix} = 9 \neq 0$$

4.1.35

(a)

$$LHS = 12e^{2x} - 6 \cdot 6e^{2x} + 5 \cdot 3e^{2x} = e^{2x}(12 - 36 + 15) = -9e^{2x} = RHS$$

$$LHS = 2 - 6(2x + 3) + 5(x^2 + 3x) = 5x^2 + 15x - 12x + 2 - 18 = 5x^2 + 3x - 16 = RHS$$

(b)

$$y_{p_1} + y_{p_2} = 5x^2 + 3x - 16 - 3e^{2x}$$

$$-\frac{y_{p_1}}{9} - 2y_{p_2} = -2x^2 - 6x - \frac{e^{2x}}{3}$$

Chapter 4.2 – Reduction of Order

4.2.9

$$x^2 y'' + 2xy' - 6y = 0; \quad y_1 = x^4$$

$$y'' - \frac{7y'}{x} + \frac{16y}{x^2} = 0; \quad P = -\frac{7}{x}$$

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

$$y_2 = x^4 \int \frac{e^{-\int -\frac{7}{x} dx}}{x^8} dx$$

$$y_2 = x^4 \int \frac{e^{7 \int \frac{1}{x} dx}}{x^8} dx$$

$$y_2 = x^4 \int \frac{e^{7 \ln|x|}}{x^8} dx$$

$$y_2 = x^4 \int \frac{e^{7 \ln|x|}}{x^8} dx$$

$$y_2 = x^4 \int \frac{e^{\ln|x^7|}}{x^8} dx$$

$$y_2 = x^4 \int \frac{x^7}{x^8} dx$$

$$y_2 = x^4 \int \frac{1}{x} dx$$

$$y_2 = x^4 \ln|x|$$

4.2.10

$$x^2 y'' + 2xy' - 6y = 0; \quad y_1 = x^2$$

$$y'' + \frac{2y'}{x} - \frac{6y}{x^2} = 0; \quad P = \frac{2}{x}$$

$$y_2 = x^2 \int \frac{e^{-\int \frac{2}{x} dx}}{x^4} dx$$

$$y_2 = x^2 \int \frac{e^{\ln|x^{-2}|}}{x^4} dx$$

$$y_2 = x^2 \int \frac{1}{x^6} dx$$

$$y_2 = x^2 \cdot \frac{-1}{5x^5}$$

$$y_2 = -\frac{1}{5x^3}$$

4.2.19

$$\begin{aligned}
 y'' - 3y' + 2y &= 5e^{3x}; \quad y_1 = e^x \\
 y_2 &= e^x \int \frac{e^{-\int -3 \, dx}}{e^{2x}} \, dx \\
 y_2 &= e^x \int e^x \, dx \\
 y_2 &= e^{2x} \\
 y_p &= Ae^{3x} \\
 9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} &= 5e^{3x} \\
 2A &= 5 \\
 A &= \frac{5}{2} \\
 y_p &= \frac{5e^{3x}}{2}
 \end{aligned}$$

4.2.20

$$\begin{aligned}
 y'' - 4y' + 3y &= x; \quad y_1 = e^x \\
 y_2 &= e^x \int \frac{e^{-\int -4 \, dx}}{e^{2x}} \, dx \\
 y_2 &= e^{3x} \\
 y_p &= ax^2 + bx + c \\
 2a - 4(2ax + b) + 3(ax^2 + bx + c) &= x \\
 \begin{pmatrix} 2a & -4b & 3c & | & 0 \\ 8a & 3b & 0 & | & 1 \\ 3a & 0 & 0 & | & 0 \end{pmatrix} \\
 a = 0 \quad b = \frac{1}{3} \quad c = \frac{4}{9} \\
 y_p &= \frac{x}{3} + \frac{4}{9}
 \end{aligned}$$

Chapter 4.6 – Variation of Parameters

4.6.1

$$y'' + y = \sec x$$

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 + \sin^2 = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\tan x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$u'_1 = \frac{W_1}{W} = \frac{-\tan x}{1} \Rightarrow u_1 = \ln |\cos x|$$

$$u'_2 = \frac{W_2}{W} = \frac{1}{1} \Rightarrow u_2 = x$$

$$y_p = \cos x \ln |\cos x| + x \sin x$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln |\cos x| + x \sin x$$

4.6.6

$$y'' + y = \sec^2 x$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 + \sin^2 = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix} = -\sin x \sec^2 x = -\tan x \sec x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix} = \cos x \sec^2 x = \sec x$$

$$u'_1 = \frac{W_1}{W} = \frac{-\tan x \sec x}{1} = -\tan x \sec x \Rightarrow u_1 = -\sec x$$

$$u'_2 = \frac{W_2}{W} = \frac{\sec x}{1} = \sec x \Rightarrow u_2 = \ln |\sec x + \tan x|$$

$$y_p = -\sec x \cos x + \ln |\sec x + \tan x| = -1 + \ln |\sec x + \tan x|$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x - 1 + \ln |\sec x + \tan x| \sin x$$

4.6.11

$$y'' + 3y' + 2y = \frac{1}{1+e^x}$$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$W(e^x, e^{2x}) = \begin{vmatrix} e^{-x} & e^{-2x} \\ e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix} = \frac{e^{-2x}}{1+e^x}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ e^{-x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-x}}{1+e^x}$$

$$u'_1 = \frac{W_1}{W} = \frac{\frac{e^{-2x}}{1+e^x}}{e^{-3x}} = \frac{e^x}{1+e^x} \Rightarrow u_1 = \ln(1+e^x)$$

$$u'_2 = \frac{W_2}{W} = \frac{\frac{e^{-x}}{1+e^x}}{e^{-3x}} = \frac{e^{2x}}{1+e^x} \Rightarrow u_2 = \ln(1+e^x) - 1 - e^x$$

$$y_p = e^{-x} \ln(1+e^x) - e^{-x} - e^{-2x} + e^{-2x} \ln(1+e^x)$$

(Ignore e^{-x} and e^{-2x} when adding to solution because they are already present in y_c)

$$y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + e^{-x} \ln(1+e^x) + e^{-2x} \ln(1+e^x)$$

4.6.27

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{\frac{3}{2}}$$

$$y_1 = x^{-\frac{1}{2}} \cos x, \quad y_2 = x^{-\frac{1}{2}} \sin x$$

$$W(x^{-\frac{1}{2}} \cos x, x^{-\frac{1}{2}} \sin x) = \begin{vmatrix} x^{-\frac{1}{2}} \cos x & x^{-\frac{1}{2}} \sin x \\ -\frac{1}{2}x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x & -\frac{1}{2}x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \end{vmatrix} = \frac{1}{x}$$

$$W_1 = \begin{vmatrix} 0 & x^{-\frac{1}{2}} \sin x \\ x^{-\frac{1}{2}} & -\frac{1}{2}x^{-\frac{3}{2}} \sin x + x^{-\frac{1}{2}} \cos x \end{vmatrix} = -\frac{\sin x}{x}$$

$$W_2 = \begin{vmatrix} x^{-\frac{1}{2}} \cos x & 0 \\ -\frac{1}{2}x^{-\frac{3}{2}} \cos x - x^{-\frac{1}{2}} \sin x & x^{-\frac{1}{2}} \end{vmatrix} = \frac{\cos x}{x}$$

$$u'_1 = \frac{W_1}{W} = -\sin x \Rightarrow u_1 = \cos x$$

$$u'_2 = \frac{W_2}{W} = \cos x \Rightarrow u_2 = \sin x$$

$$y_p = x^{-\frac{1}{2}} \cos^2 x + x^{-\frac{1}{2}} \sin^2 x = \frac{1}{\sqrt{x}} (\cos^2 x + \sin^2 x) = \frac{1}{\sqrt{x}}$$

$$y = y_c + y_p = c_1 x^{-\frac{1}{2}} \cos x + c_2 x^{-\frac{1}{2}} \sin x + \frac{1}{\sqrt{x}}$$

4.6.28

$$x^2 y'' + xy' + y = \sec(\ln x)$$

$$y'' + \frac{y'}{x} + \frac{y}{x^2} = \frac{\sec(\ln x)}{x^2}$$

$$W(\cos(\ln x), \sin(\ln x)) = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} = \frac{\cos^2(\ln x)}{x} + \frac{\sin^2(\ln x)}{x} = \frac{1}{x}$$

$$W_1 = \begin{vmatrix} 0 & \sin(\ln x) \\ \frac{\sec(\ln x)}{x^2} & \frac{\cos(\ln x)}{x} \end{vmatrix} = -\frac{\tan(\ln x)}{x^2}$$

$$W_2 = \begin{vmatrix} \cos(\ln x) & 0 \\ -\frac{\sin(\ln x)}{x} & \frac{\sec(\ln x)}{x^2} \end{vmatrix} = \frac{1}{x^2}$$

$$u'_1 = -\frac{\tan(\ln x)}{x} \Rightarrow u_2 = \ln |\cos(\ln x)|$$

$$u'_2 = \frac{1}{x} \Rightarrow u_1 = \ln |x|$$

$$y_p = \cos(\ln x) \ln |\cos(\ln x)| + \sin(\ln x) \ln |x|$$

$$y = y_c + y_p = c_1 \cos(\ln x) + c_2 \sin(\ln x) + \cos(\ln x) \ln |\cos(\ln x)| + \sin(\ln x) \ln |x|$$