Chapter 4 – Higher-Order Differential Equations

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Contents

Chapter 4.1 – Preliminary Theory for Linear Equations

4.1.7

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega}\sin(\omega t)$$
$$x'(t) = -x_0\omega\sin(\omega t) + x_1\cos(\omega t)$$

$$x(0) = x_0 \Rightarrow x_0 = x_0\cos(0) + \frac{x_1}{\omega}\sin(0) = x_0 \cdot 1 + \frac{x_1}{\omega}\cdot 0 = x_0$$
$$x'(0) = x_1 \Rightarrow x_1 = x_1\cos(0) - x_0\omega\sin(0) = x_1 \cdot 1 - x_0\omega \cdot 0 = x_1$$

4.1.10

$$y'' + (\tan x)y = e^x$$
$$\frac{y''}{\tan x} + y = \frac{e^x}{\tan x}$$

For the equation to have a solution the coefficient of y'' needs to not be 0 on the interval

The interval around 0 where $\frac{1}{\tan x} \neq 0$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

4.1.13

$$y = c_1e^x\cos x + c_2e^x\sin x$$
$$y' = c_1e^x\cos x - c_1e^x\sin x + c_2e^x\sin x + c_2e^x\cos x$$

(a)

$$y(0) = 1 \Rightarrow c_1e^0\cos 0 + c_2e^0\sin 0 = 1 \Rightarrow c_1 = 1$$
$$y'(\pi) = 0 \Rightarrow e^\pi\cos(\pi) - e^\pi\sin\pi + c_2e^\pi\sin(\pi) + c_2e^\pi\cos\pi = -e^\pi - c_2e^\pi = 0 \Rightarrow c_2 = -1$$

$$y = e^x\cos x - e^x\sin x$$

(b)

$$y(0) = 1 \Rightarrow c_1e^0\cos 0 + c_2e^0\sin 0 = 1 \Rightarrow c_1 = 1$$
$$y(\pi) = -1 \Rightarrow e^\pi\cos\pi + c_2e^\pi\sin\pi = -e^\pi \neq -1$$

No solution

(c)

$$y(0) = 1 \Rightarrow c_1e^0\cos 0 + c_2e^0\sin 0 = 1 \Rightarrow c_1 = 1$$
$$y\left(\frac{\pi}{2}\right) = 1 \Rightarrow e^{\frac{\pi}{2}}\cos\frac{\pi}{2} + c_2e^{\frac{\pi}{2}}\sin\frac{\pi}{2} = c_2e^{\frac{\pi}{2}} = 1 \Rightarrow c_2 = e^{-\frac{\pi}{2}}$$

$$y = e^x\cos x - e^{-\frac{\pi}{2}}e^x\sin x$$

(d)

$$y(0) = 0 \Rightarrow c_1e^0\cos 0 + c_2e^0\sin 0 = 0 \Rightarrow c_1 = 0$$
$$y(\pi) = 0 \Rightarrow +c_2e^\pi\sin\pi = 0 \Rightarrow c_2 \in \mathbb{R}$$

$$y = c_2e^x\sin x$$

4.1.17

$$\frac{1}{5}f_1(x) - f_2(x) = 1 - \cos^2 x = \sin^2 x = f_3(x)$$

4.1.18

$$2f_3(x) - f_2(x) = 2\cos^2 x - 1 = 2\cos^2 x - \cos^2 x - \sin^2 x = \cos^2 x - \sin^2 x = \cos 2x$$

4.1.19

$$4f_1(x) - 3f_2(x) = 4x - 3(x-1) = 4x - 3x + 3 = x + 3 = f_3(x)$$

4.1.20

Independent

4.1.21

Independent, can't create x^2

4.1.23

Both functions satisfy the differential equation.

$$W(e^{-3x}, e^{4x}) = \begin{pmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{pmatrix} = 7e^x \neq 0$$

4.1.25

Both functions satisfy the differential equation.

$$W(e^x \cos 2x, e^x \sin 2x) = \begin{pmatrix} e^x \cos 2x & e^x \sin 2x \\ e^x \cos 2x - 2e^x \sin 2x & e^x \sin 2x + 2e^x \cos 2x \end{pmatrix} \quad x = 0 \Rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = 2 \neq 0$$

4.1.26

Both functions satisfy the differential equation.

$$W(e^{\frac{x}{2}}, xe^{\frac{x}{2}}) = \begin{pmatrix} e^{\frac{x}{2}} & xe^{\frac{x}{2}} \\ \frac{e^{\frac{x}{2}}}{2} & e^{\frac{x}{2}} + \frac{xe^{\frac{x}{2}}}{2} \end{pmatrix} \quad x = 0 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \neq 0$$

4.1.29

All of the functions satisfy the differential equation.

$$W(x, x^{-2}, x^{-2}\ln x) = \begin{pmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & \frac{-2\ln(x)+1}{x^3} \\ 0 & 6x^{-4} & -\frac{-6\ln(x)+5}{x^4} \end{pmatrix} \quad x = 1 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 6 & -5 \end{pmatrix} = 9 \neq 0$$

4.1.35

(a)

$$LHS = 12e^{2x} - 6 \cdot 6e^{2x} + 5 \cdot 3e^{2x} = e^{2x}(12 - 36 + 15) = -9e^{2x} = RHS$$

$$LHS = 2 - 6(2x + 3) + 5(x^2 + 3x) = 5x^2 + 15x - 12x + 2 - 18 = 5x^2 + 3x - 16 = RHS$$

(b)

$$y_{p_1} + y_{p_2} = 5x^2 + 3x - 16 - 3e^{2x}$$

$$-\frac{y_{p_1}}{9} - 2y_{p_2} = -2x^2 - 6x - \frac{e^{2x}}{3}$$

Chapter 4.2 – Reduction of Order

4.2.9

$$\begin{aligned}
x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^4 \\
y'' - \frac{7y'}{x} + \frac{16y}{x^2} = 0; \quad P = -\frac{7}{x}
\end{aligned}$$

$$\begin{aligned}
y_2 &= y_1(x) \int \frac{e^{-\int P(x)\,\mathrm{d}x}}{y_1^2(x)}\,\mathrm{d}x \\
y_2 &= x^4 \int \frac{e^{-\int -\frac{7}{x}\,\mathrm{d}x}}{x^8}\,\mathrm{d}x \\
y_2 &= x^4 \int \frac{e^{7\int \frac{1}{x}\,\mathrm{d}x}}{x^8}\,\mathrm{d}x \\
y_2 &= x^4 \int \frac{e^{7\ln|x|}}{x^8}\,\mathrm{d}x \\
y_2 &= x^4 \int \frac{e^{7\ln|x|}}{x^8}\,\mathrm{d}x \\
y_2 &= x^4 \int \frac{e^{\ln|x^7|}}{x^8}\,\mathrm{d}x \\
y_2 &= x^4 \int \frac{x^7}{x^8}\,\mathrm{d}x \\
y_2 &= x^4 \int \frac{1}{x}\,\mathrm{d}x \\
y_2 &= x^4 \ln|x|
\end{aligned}$$

4.2.10

$$x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$$
$$y'' + \frac{2y'}{x} - \frac{6y}{x^2} = 0; \quad P = \frac{2}{x}$$
$$y_2 = x^2 \int \frac{e^{-\int \frac{2}{x}\, \mathrm{d}x}}{x^4}\, \mathrm{d}x$$
$$y_2 = x^2 \int \frac{e^{\ln|x^{-2}|}}{x^4}\, \mathrm{d}x$$
$$y_2 = x^2 \int \frac{1}{x^6}\, \mathrm{d}x$$
$$y_2 = x^2 \cdot \frac{-1}{5x^5}$$
$$y_2 = -\frac{1}{5x^3}$$

4.2.19

$$
\begin{aligned}
y'' - 3y' + 2y &= 5e^{3x}; \quad y_1 = e^x \\
y_2 &= e^x \int \frac{e^{-\int -3 \, \mathrm{d}x}}{e^{2x}} \, \mathrm{d}x \\
y_2 &= e^x \int e^x \, \mathrm{d}x \\
y_2 &= e^{2x} \\
y_p &= Ae^{3x} \\
9Ae^{3x} - 9Ae^{3x} + 2Ae^{3x} &= 5e^{3x} \\
2A &= 5 \\
A &= \frac{5}{2} \\
y_p &= \frac{5e^{3x}}{2}
\end{aligned}
$$

4.2.20

$$
\begin{aligned}
y'' - 4y' + 3y &= x; \quad y_1 = e^x \\
y_2 &= e^x \int \frac{e^{-\int -4 \, \mathrm{d}x}}{e^{2x}} \, \mathrm{d}x \\
y_2 &= e^{3x} \\
y_p &= ax^2 + bx + c \\
2a - 4(2ax + b) + 3(ax^2 + bx + c) &= x \\
&\begin{pmatrix} 2a & -4b & 3c| & 0 \\ 8a & 3b & 0| & 1 \\ 3a & 0 & 0| & 0 \end{pmatrix} \\
a = 0 \quad b = \frac{1}{3} \quad c &= \frac{4}{9} \\
y_p &= \frac{x}{3} + \frac{4}{9}
\end{aligned}
$$

Chapter 4.6 – Variation of Parameters

4.6.1

$$
\begin{aligned}
y'' + y &= \sec x \\
m^2 + 1 = 0 &\Rightarrow m = \pm i \\
y_c &= c_1 \cos x + c_2 \sin x
\end{aligned}
$$

$$
\begin{aligned}
W(\cos x, \sin x) &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 + \sin^2 = 1 \\
W_1 &= \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\tan x \\
W_2 &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1 \\
u_1' &= \frac{W_1}{W} = \frac{-\tan x}{1} \Rightarrow u_1 = \ln|\cos x| \\
u_2' &= \frac{W_2}{W} = \frac{1}{1} \Rightarrow u_2 = x \\
y_p &= \cos x \ln|\cos x| + x \sin x
\end{aligned}
$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$$

4.6.6

$$
\begin{aligned}
y'' + y &= \sec^2 x \\
y_c &= c_1 \cos x + c_2 \sin x
\end{aligned}
$$

$$
\begin{aligned}
W(\cos x, \sin x) &= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 + \sin^2 = 1 \\
W_1 &= \begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix} = -\sin x \sec^2 x = -\tan x \sec x \\
W_2 &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix} = \cos x \sec^2 x = \sec x \\
u_1' &= \frac{W_1}{W} = \frac{-\tan x \sec x}{1} = -\tan x \sec x \Rightarrow u_1 = -\sec x \\
u_2' &= \frac{W_2}{W} = \frac{\sec x}{1} = \sec x \Rightarrow u_2 = \ln|\sec x + \tan x| \\
y_p &= -\sec x \cos x + \ln|\sec x + \tan x| = -1 + \ln|\sec x + \tan x|
\end{aligned}
$$

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x - 1 + \ln|\sec x + \tan x| \sin x$$

4.6.11

$$
\begin{aligned}
y'' + 3y' + 2y &= \frac{1}{1+e^x} \\
y_c &= c_1 e^{-x} + c_2 e^{-2x}
\end{aligned}
$$

$$
\begin{aligned}
W(e^x, e^{2x}) &= \begin{vmatrix} e^{-x} & e^{-2x} \\ e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x} \\
W_1 &= \begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix} = \frac{e^{-2x}}{1+e^x} \\
W_2 &= \begin{vmatrix} e^{-x} & 0 \\ e^{-x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-x}}{1+e^x} \\
u_1' &= \frac{W_1}{W} = \frac{\frac{e^{-2x}}{1+e^x}}{e^{-3x}} = \frac{e^x}{1+e^x} \Rightarrow u_1 = \ln(1+e^x) \\
u_2' &= \frac{W_1}{W} = \frac{\frac{e^{-x}}{1+e^x}}{e^{-3x}} = \frac{e^{2x}}{1+e^x} \Rightarrow u_2 = \ln(1+e^x) - 1 - e^x \\
y_p &= e^{-x}\ln(1+e^x) - e^{-x} - e^{-2x} + e^{-2x}\ln(1+e^x)
\end{aligned}
$$

(Ignore e^{-x} and e^{-2x} when adding to solution because they are already present in y_c)

$$
y = y_c + y_p = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln(1+e^x) + e^{-2x}\ln(1+e^x)
$$

4.6.27

$$
\begin{aligned}
x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y &= x^{\frac{3}{2}} \\
y_1 = x^{-\frac{1}{2}}\cos x, \quad y_2 &= x^{-\frac{1}{2}}\sin x \\
W(x^{-\frac{1}{2}}\cos x, x^{-\frac{1}{2}}\sin x) &= \begin{vmatrix} x^{-\frac{1}{2}}\cos x & x^{-\frac{1}{2}}\sin x \\ -\frac{1}{2}x^{-\frac{3}{2}}\cos x - x^{-\frac{1}{2}}\sin x & -\frac{1}{2}x^{-\frac{3}{2}}\sin x + x^{-\frac{1}{2}}\cos x \end{vmatrix} = \frac{1}{x} \\
W_1 &= \begin{vmatrix} 0 & x^{-\frac{1}{2}}\sin x \\ x^{-\frac{1}{2}} & -\frac{1}{2}x^{-\frac{3}{2}}\sin x + x^{-\frac{1}{2}}\cos x \end{vmatrix} = -\frac{\sin x}{x} \\
W_2 &= \begin{vmatrix} x^{-\frac{1}{2}}\cos x & 0 \\ -\frac{1}{2}x^{-\frac{3}{2}}\cos x - x^{-\frac{1}{2}}\sin x & x^{-\frac{1}{2}} \end{vmatrix} = \frac{\cos x}{x} \\
u_1' &= \frac{W_1}{W} = -\sin x \Rightarrow u_1 = \cos x \\
u_2' &= \frac{W_2}{W} = \cos x \Rightarrow u_2 = \sin x \\
y_p &= x^{-\frac{1}{2}}\cos^2 x + x^{-\frac{1}{2}}\sin^2 x = \frac{1}{\sqrt{x}}(\cos^2 x + \sin^2 x) = \frac{1}{\sqrt{x}} \\
y &= y_c + y_p = c_1 x^{-\frac{1}{2}}\cos x + c_2 x^{-\frac{1}{2}}\sin x + \frac{1}{\sqrt{x}}
\end{aligned}
$$

4.6.28

$$
\begin{aligned}
x^2y'' + xy' + y &= \sec(\ln x) \\
y'' + \frac{y'}{x} + \frac{y}{x^2} &= \frac{\sec(\ln x)}{x^2} \\
W(\cos(\ln x), \sin(\ln x)) &= \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ \frac{-\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \end{vmatrix} = \frac{\cos^2(\ln x)}{x} + \frac{\sin^2(\ln x)}{x} = \frac{1}{x} \\
W_1 &= \begin{vmatrix} 0 & \sin(\ln x) \\ \frac{\sec(\ln x)}{x^2} & \frac{\cos(\ln x)}{x} \end{vmatrix} = -\frac{\tan(\ln x)}{x^2} \\
W_2 &= \begin{vmatrix} \cos(\ln x) & 0 \\ \frac{-\sin(\ln x)}{x} & \frac{\sec(\ln x)}{x^2} \end{vmatrix} = \frac{1}{x^2} \\
u_1' &= -\frac{\tan(\ln x)}{x} \Rightarrow u_2 = \ln|\cos(\ln x)| \\
u_2' &= \frac{1}{x} \Rightarrow u_1 = \ln|x| \\
y_p &= \cos(\ln x)\ln|\cos(\ln x)| + \sin(\ln x)\ln|x|
\end{aligned}
$$

$$
y = y_c + y_p = c_1\cos(\ln x) + c_2\sin(\ln x) + \cos(\ln x)\ln|\cos(\ln x)| + \sin(\ln x)\ln|x|
$$