Introduction to Numerical Optimization – Project

Randomized Condorcet Voting System

Electing a representative from a group of candidates is the problem of aggregating the individual preferences of voters to determine a winner among the group. Depending on how the preferences of the voters are collected and aggregated, we obtain a mechanism to elect a representative with more or less good properties with respect to our perception of democracy.

The quality of an election is measured through different criteria. Notable criteria are Pareto efficiency, independence of irrelevant alternatives and stategyproofness. In short, if every voter prefers a candidate over an alternative one, then a Pareto efficient election method also prefers this candidate over the alternative one. An election is said to be independent of irrelevant alternatives if the winner of the election would remain the winner of the same election in which other candidates would have been removed. Finally, an election is stategyproof if the voters have no incentives to lie about their preferences. These last two criteria are linked to the dilemma of the useful vote that plagues our democracies. A major observation is that there is no perfect electoral system. Different theorems show the impossibility of constructing an election that would combine all the previous criteria in a democracy.

In this project, we will study elections that apply the Condorcet principle. This principle states that if a candidate is preferred to any other by a majority, then that candidate must be elected. By accepting this principle as a warrant for democracy, we can construct an election mechanism that has very good properties in terms of Pareto efficiency and independence of irrelevant alternatives. The election proceeds as follows. Voters are asked to give their preference between each pair of candidates. Then, we aggregate the voters' preferences and identify a candidate who in a duel would be elected by a majority against each of his opponents. This last candidate is the Condorcet winner. The problem is that such a winner does not always exist.

A natural extension of Condorcet elections that guarantees the existence of a winner is the randomized Condorcet voting system (RCVS). This ballot elects a probability law on the candidates rather than a single winner. A winning candidate can be drawn from this probability distribution afterwards. Let us assume a first candidate is drawn from the RCVS winning probability law. Let us furthermore assume we draw an opponent from an alternative distribution over candidates. On average over the two distributions of candidates, the first candidate is preferred over the second, and this whatever the distribution over opponents. Such a law always exists and if a Condorcet winner exists, then this law concentrates all its mass on him. Formally, let $X = \{1, 2, \dots\}$ be the finite set of candidates. Let $\pi : X \times X \to \mathbb{Z}$ be the preference function between a pair of candidates (x, x') such that:

$$\pi(x, x') = \text{number of voters preferring } x \text{ over } x' - \text{number of voters preferring } x' \text{ over } x\ . \tag{1}$$

In practice, there often is a different number of voters for each pair of candidates. A distribution $p \in \mathcal{P}(X)$ over the candidates X is a vector $(p_1, p_2, \dots)$ of probabilities such that:

$$\sum_{x \in X} p_x = 1\ . \tag{2}$$

Finally, a distribution $p \in \mathcal{P}(X)$ is a RCVS winner if it is such that:

$$\forall q \in \mathcal{P}(X) : \sum_{x \in X} \sum_{x' \in X} p_x q_{x'} \pi(x, x') \geq 0\ . \tag{3}$$

Model

In the first part of the project, the RCVS is formulated as a linear program and used for electing winners.

1. Given the definition of the RCVS, provide a linear program that computes the winning distribution law.

   Hint: let us consider the set $F_n = \{x \to \sum_i^n a_i x_i | \, x \in \mathbb{R}^n, a \in \mathbb{R}_+^n\}$ *of linear functions of real variables with non negative coefficients. Each function* $f \in F_n$ *is non negative if and only if the variable is non negative too :*

   $$\forall f \in F_n : f(x) \geq 0 \Leftrightarrow x \geq 0 \, .$$

2. Figure 1 represents the duel graph of an election. Nodes are candidates and edges represent the results of duels. The direction of an edge goes from the loser to the winner and is annotated with the result of the voters. Will there be a Condorcet winner in this election? Justify. Relying on your LP formulation, compute the RCVS winning lottery.

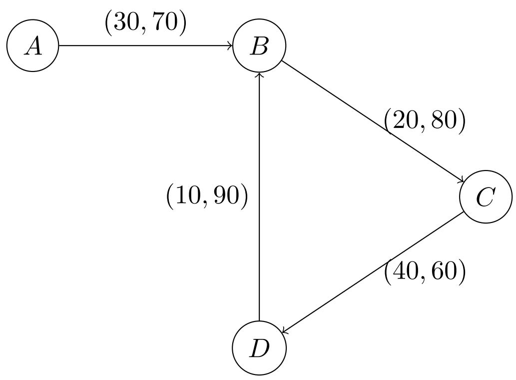

Figure 1: Duel graph between four candidates.

3. Discuss the values of the optimal dual variables and the optimal dual basis.

4. Assuming that there exists a unique RCVS winning lottery, show that the solution of the linear program can be obtained solving a linear system.

   Hint : write the equations of complementarity slackness.

5. `Bonus` : Compare the RCVS with an alternative election scheme and discuss the theoretical advantages of both methods.

Linear Robust Formulation

Let us assume that several voters change their vote. The second part of the project is dedicated to the construction of a linear optimization problem providing a robust solution to such changes.

6. Let us assume at least n votes change in the pairwise comparison of the candidates. Depending on the changes, a different election has to be conducted for electing a RCVS winning distribution law. Compute the number E of possible elections and provide a procedure for building the optimization problem corresponding to each possible election.

7. In this setting, relying on the l_0-norm, formulate an optimization problem that computes a probability law that respects the RCVS winning criterion for as many of the E elections as possible. Is this problem linear, conic, convex, or none of the previous choices?

   Reminder: The l_0 *norm of a vector is the number of non-zero entries of this vector.*

8. Provide a linear program in standard form computing an approximation of the previous optimization problem.

9. Compare the speed of the primal simplex, dual simplex, and barrier method as a function of n.

10. Provide an interpretation for the dual variables and discuss their values.

Quadratic Robust Formulation

The last part of the project consists in providing a SOCP for finding a robust solution to the RCVS.

11. Provide a SOCP approximating the solution of the previous l_0-norm robust formulation.

12. What algorithm is used for solving the optimization problem? Discuss its speed as a function of the number of changes in the votes n.

13. Provide a method for approximating the solution of the previous problem for large values of n. Compare the execution time and the solution with the exact method.

    Hint: Long summations may be approximated by Monte-Carlo.

14. `Bonus` : For the linear robust formulation and for the SOCP formulation, discuss the evolution of the winning distribution law as a function of n (for large values).

Deliverables

Students will work on this project in pairs or individually. For organizational reasons, each group is expected to send an email to Adrien Bolland (`adrien.bolland@uliege.be`) indicating the names of group members.

Each group is expected to turn in a short report describing the problem formulations used and discussing findings. The report and code shall be sent by December 8 by email. Each group will also present its methods and findings on December 13. The exact format of the presentation will be given in due course.

In case of non respect of the instructions, the group will be penalized.