Dynamics of Mechanical Systems with Mecanum Wheels

Klaus Zimmermann, Igor Zeidis, and Mohamed Abdelrahman

Abstract The kinematics and dynamics of a mechanical system with mecanum wheels is studied. A mecanum wheel is a wheel with rollers attached to its circumference. Each roller rotates about an axis that forms an angle with the plane of the disk (for the omni-wheels, the axes of the rollers lie in the plane of the wheel and in an ideal case are tangent to the outer circumference of the wheel). Such a design provides additional kinematic advantages for the mecanum wheels in comparison with the conventional wheels. Within the framework of non-holonomic mechanics, the equations of motion are derived for the case of an arbitrary angle at which the rollers are attached (usually, this angle is assumed to be equal to 45°). In robotics, a simplified approach, in which the equations of non-holonomic kinematic constraints are solved approximately by means of a pseudo-inverse matrix, is frequently applied. Such an approximate approach leads to "holonomization" of the system and allows Lagrange's equations of the second kind to be used. In the present paper, the equations of motion obtained on the basis of the principles of non-holonomic mechanics are compared with the approximate equations. It is shown that for translational motions and for the rotation of the system about its center of mass, both these approaches lead to the same result.

1 Introduction

Mobility is an important property for people. Thus, its generation, conservation, and, if necessary, recovery are an important goal and a great challenge for life scientists and engineers. Some tasks encompass fields from the design of faster and—at the same time—energy-saving transportation systems in the air to the

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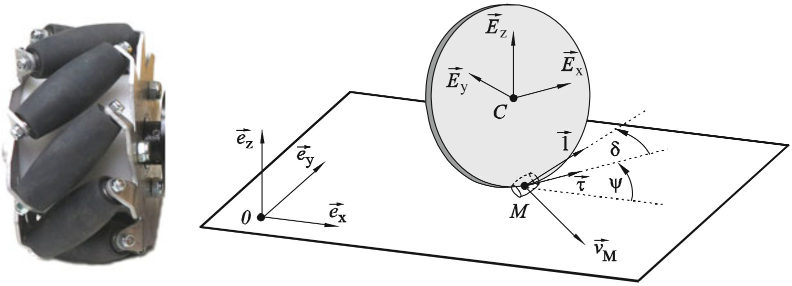

Fig. 1 A mecanum wheel (*left*) and the mechanical model (*right*)

recovery of upright walking through exoskeletons. For service robots and equipment for physically disabled persons, mobile systems with a high maneuverability play an important role. The conventional wheel moves back into the focus of interest. Seemingly, its basic mechanical function "rolling" is easy to understand and all mechanics are adequately described. But, there exist some new developments, which connect the advantages of wheels and legs in a whole system (e.g., "whegs"). Furthermore, the so-called omnidirectional wheels (e.g., "mecanum wheels"), generating constraints different from the conventional wheel, lead to investigations, based on the mechanics and control of non-holonomic systems. At the present time, vehicles with mecanum wheels (Fig. 1, left) are gaining ground for different applications. The wheels of such vehicles have rollers that are arranged along the rim at an angle to the wheel plane [5]. As a rule, this angle is equal to 45°. Such wheels have additional kinematical possibilities in comparison with conventional wheels. Due to these possibilities, a vehicle with mecanum wheels can move forward-backward and leftward-rightward and rotate in an arbitrary way. Usually a mecanum-wheeled vehicle has four wheels. By varying the rate and the direction of rotation of each wheel, one can implement, for example, a translational motion of the vehicle in any direction, as well as arbitrary turns and rotations on the spot. The kinematics of wheeled systems, including those with mecanum wheels, are reviewed in [2]. The issues of kinematics, dynamics, and control of systems with mecanum wheels in a non-holonomic treatment are considered in [6, 10, 11] for a number of particular cases. There are a great number of studies on robotics, in which the kinematics and dynamics of robots with four mecanum wheels is approximately treated in terms of holonomic mechanics (see, e.g., [3, 4, 7–9]). In these studies, pseudo-inverse matrices are used to resolve the kinematic constraint relations.

In the present study, the kinematics of a vehicle with four mecanum wheels are considered in terms of non-holonomic mechanics for an arbitrary orientation of the rollers of the mecanum wheels. The equations obtained are compared with the equations implied by an approximate holonomic treatment, and the types of motion for which both treatments coincide are found.

2 Non-holonomic Model of a Mechanical System with Mecanum Wheels

Using a non-holonomic model of a mecanum wheel, we will obtain the kinematic relations and the equation of motion for a mechanical system with such wheels.

2.1 Kinematics of a Mecanum Wheel

For a conventional wheel, the contact between the wheel and the supporting plane is characterized by the condition that the wheel is rolling without slip. This means that the velocity of the point by which the wheel contacts the plane at each current instant is equal to zero. Then the projections of the velocity of the contact point onto the direction lying in the wheel plane, as well as onto the direction perpendicular to this plane, are equal to zero. For a mecanum wheel, there is only one direction the projection onto which of the velocity of the point of contact of the wheel with the supporting plane vanishes. This direction can be arbitrary, but it is fixed relative to the wheel.

As a model of a mecanum wheel we will consider the rolling of a disk of radius R centered at the point C on a horizontal plane. The plane of the disk is always vertical. Let $\vec{l}$ denote the unit vector along the axis of attachment of the rollers, $\vec{\tau}$ the unit vector lying in the wheel plane tangent to the rim at the point of contact, and δ the angle between the plane of the wheel and the roller axis (between the vectors $\vec{\tau}$ and $\vec{l}$). The angle δ is constant. The kinematic constraint relation for a mecanum wheel implies that the vector of velocity $\vec{v}_M$ of the point M of contact of the wheel with the plane points along the line perpendicular to the axis of the roller, i.e., the projection of the velocity of the point M onto the roller axis is equal to zero (see Fig. 1, right).

The kinematic constraint relation has the form

$$\vec{v}_M \cdot \vec{l} = 0. \tag{1}$$

The velocity $\vec{v}_M$ is defined by the equation

$$\vec{v}_M = \vec{v}_C + \vec{\omega} \times \overrightarrow{CM} \tag{2}$$

where $\vec{v}_C$ is the velocity of the center C and $\vec{\omega}$ is the angular velocity of the disk.

Let $\left\{O, \vec{e}_x, \vec{e}_y, \vec{e}_z\right\}$ be a fixed reference frame (inertial system) and let C be the origin of a movable reference frame (body fixed frame) $\left\{C, \vec{E}_x, \vec{E}_y, \vec{E}_z\right\}$. The unit vectors $\vec{E}_x$ and $\vec{E}_y$ are parallel to the horizontal plane, the vector $\vec{E}_z$

lying in the disk plane and the vector $\overrightarrow{E}_y$ being perpendicular to this plane. Let φ be the angle of rotation of the disk about the axis passing through the point C perpendicular to the plane of the disk, and ψ the angle formed by the disk plane with a line parallel to the vector $\overrightarrow{E}_x$ (the angle between the vector $\overrightarrow{\tau}$ and the vector $\overrightarrow{e}_x$, see Fig. 1, right). The vectors $\vec{\omega}$ and $\overrightarrow{CM}$ are defined by

$$\overrightarrow{\omega} = \dot{\varphi}\,\overrightarrow{E}_y + \dot{\psi}\,\overrightarrow{E}_z,\ \overrightarrow{CM} = -R\,\overrightarrow{E}_z. \tag{3}$$

Let x_C, y_C, R be the coordinates of the point C in the reference frame $\left\{O, \overrightarrow{e}_x, \overrightarrow{e}_y, \overrightarrow{e}_z\right\}$. Then

$$\begin{aligned} \overrightarrow{v}_C &= (\dot{x}_C \cos\psi + \dot{y}_C \sin\psi)\,\overrightarrow{E}_x + (-\dot{x}_C \sin\psi + \dot{y}_C \cos\psi)\,\overrightarrow{E}_y, \\ \vec{\omega}\times\overrightarrow{CM} &= -R\dot{\varphi}\,\overrightarrow{E}_x. \end{aligned} \tag{4}$$

The vector $\overrightarrow{l}$ is expressed by

$$\overrightarrow{l} = \cos\delta\,\overrightarrow{E}_x + \sin\delta\,\overrightarrow{E}_y. \tag{5}$$

Finally, the kinematic relation (1) becomes

$$\dot{x}_C \cos(\psi + \delta) + \dot{y}_C \sin(\psi + \delta) = R\dot{\varphi}\cos\delta. \tag{6}$$

On the basis of the analysis of the kinematic constraints of type (6) it is shown [1] that if a mechanical system is based on n mecanum wheels in such a way that (a) $n \geq 3$; (b) not all vectors $\overrightarrow{l}_i$ are parallel to each other; and (c) the points of contact of the wheels with the plane do not lie on one line, then it is always possible to find control functions φ_i $(i = 1, \ldots, n)$ that implement any prescribed motion of the system's center of mass.

2.2 Kinematics of a Vehicle with Four Mecanum Wheels

Consider a model of a four-wheeled vehicle with mecanum wheels (Fig. 2).

Let now C be the center of mass of the system (the body together with the wheels). The coordinates of the point C in the reference frame $\left\{O, \overrightarrow{e}_x, \overrightarrow{e}_y, \overrightarrow{e}_z\right\}$ are x_C, y_C, and R; the quantities $\left|\overrightarrow{CO_1}\right| = \rho_1$ and $\left|\overrightarrow{CO_2}\right| = \rho_2$ are the distances from the center of mass to the axes of the respective wheel pairs, and $2\,l$ is the distance between the centers of the wheels of one axis.

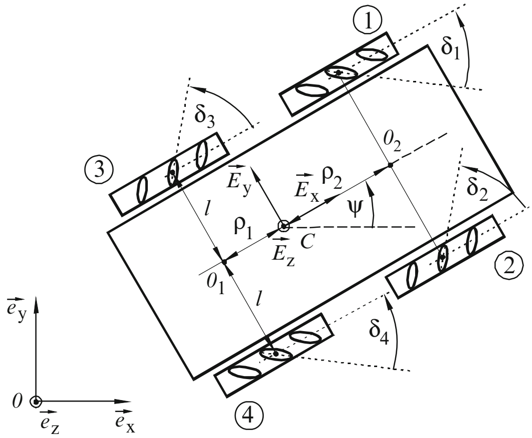

Fig. 2 A vehicle with four mecanum wheels

The corresponding kinematic relations are obtained on the basis of Eq. (6), in which δ should be replaced by $-\delta_1$ and $-\delta_4$ for wheels 1 and 4 and by δ_2 and δ_3 for wheels 2 and 3, respectively. These relations have the form

$$\dot{x}_C \cos(\psi - \delta_1) + \dot{y}_C \sin(\psi - \delta_1) - \dot{\psi}\,(l \cos\delta_1 + \rho_2 \sin\delta_1) = R\dot{\varphi}_1 \cos\delta_1, \quad (7)$$

$$\dot{x}_C \cos(\psi + \delta_2) + \dot{y}_C \sin(\psi + \delta_2) + \dot{\psi}\,(l \cos\delta_2 + \rho_2 \sin\delta_2) = R\dot{\varphi}_2 \cos\delta_2, \quad (8)$$

$$\dot{x}_C \cos(\psi + \delta_3) + \dot{y}_C \sin(\psi + \delta_3) - \dot{\psi}\,(l \cos\delta_3 + \rho_1 \sin\delta_3) = R\dot{\varphi}_3 \cos\delta_3, \quad (9)$$

$$\dot{x}_C \cos(\psi - \delta_4) + \dot{y}_C \sin(\psi - \delta_4) + \dot{\psi}\,(l \cos\delta_4 + \rho_1 \sin\delta_4) = R\dot{\varphi}_4 \cos\delta_4, \quad (10)$$

where φ_i is the angle of rotation of the disk about the rotation axis and δ_i is the angle between the wheel plane and the roller axis ($i = 1, \ldots, 4$).

Equations (7)–(10) define four non-holonomic constraints. Note that if only translational motions ($\dot{\psi} = 0$) or only rotations about the center of mass ($\dot{x}_C = \dot{y}_C = 0$) are allowed, then the constraints become holonomic.

2.3 Dynamics of a Vehicle with Mecanum Wheels

The configuration of the mechanical system under consideration is defined by seven parameters, which involve the coordinates $q^1 = x_C$, $q^2 = y_C$ of the system's center of mass C, the angle $q^3 = \psi$ between the abscissa axis of the fixed reference frame and a straight line perpendicular to the axis of the wheel pair, and the angles of rotation of the wheels $q^{i+3} = \varphi_i$ ($i = 1, \ldots, 4$).

To describe the motion of the system we will use Lagrange's equations with undetermined multipliers corresponding to the non-holonomic constraints:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_C} - \frac{\partial T}{\partial x_C} - \lambda_1 \cos(\psi - \delta_1) - \lambda_2 \cos(\psi + \delta_2) - \lambda_3 \cos(\psi + \delta_3)$$
$$- \lambda_4 \cos(\psi - \delta_4) = 0, \tag{11}$$

$$\frac{d}{d\,t}\frac{\partial T}{\partial \dot{y}_C} - \frac{\partial T}{\partial y_C} - \lambda_1 \sin(\psi - \delta_1) - \lambda_2 \sin(\psi + \delta_2) - \lambda_3 \sin(\psi + \delta_3)$$
$$- \lambda_4 \sin(\psi - \delta_4) = 0, \tag{12}$$

$$\frac{d}{d\,t}\frac{\partial T}{\partial \dot{\psi}} - \frac{\partial T}{\partial \psi} + \lambda_1 (l\cos\delta_1 + \rho_2 \sin\delta_1) - \lambda_2 (l\cos\delta_2 + \rho_2 \sin\delta_2)$$
$$+ \lambda_3 (l\cos\delta_3 + \rho_1 \sin\delta_2) - \lambda_4 (l\cos\delta_4 + \rho_1 \sin\delta_4) = 0, \tag{13}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}_i} - \frac{\partial T}{\partial \varphi_i} + \lambda_i R\cos\delta_i = M_i, i = 1, \ldots, 4. \tag{14}$$

Here, T is the kinetic energy of the system, M_i are the torques applied to the respective wheels, and λ_i are Lagrange's undetermined multipliers ($i = 1, \ldots, 4$).

The kinetic energy T is defined as the sum of the kinetic energies of the body and the wheels and is expressed as follows:

$$\begin{aligned} T = {} & \frac{1}{2}m_0\left(\dot{x}_C^2 + \dot{y}_C^2\right) + \frac{1}{2}J_0\dot{\psi}^2 + \frac{1}{2}m_1\big[4\left(\dot{x}_C^2 + \dot{y}_C^2\right) \\ & + 4\dot{x}_C\,\dot{y}_C\,(\rho_1 - \rho_2)\sin\psi - 4\dot{y}_C\,\dot{\psi}\,(\rho_1 - \rho_2)\cos\psi \\ & + 4\dot{y}_C\,\dot{\psi}(\rho_1 - \rho_2)\cos\psi + 2\dot{\psi}^{\,2}\left(\rho_1^2 + \rho_2^2 + 2l^2\right)\big] \\ & + 2J_2\dot{\psi}^{\,2} + \frac{1}{2}J_1\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + \dot{\varphi}_4^2\right). \end{aligned} \tag{15}$$

Here m_0 is the mass of the body, m_1 is the mass of each of the wheels, J_0 is the mass moment of inertia of the body about the vertical axis passing through the center of mass, J_1 is the mass moment of inertia of each wheel about its axis of rotation, and J_2 is the moment of inertia of each wheel about the vertical axis passing through the center of the wheel.

Consider a practically important case, for which $\delta_i = \delta$ ($i = 1, \ldots, 4$). Then, eliminating the multipliers λ_i in Eqs. (11)–(14), we obtain

$$\begin{aligned} & (m_0 + 4m_1)\,\ddot{x}_C + 2m_1\ddot{\psi}\,(\rho_1 - \rho_2)\sin\psi + 2m_1\dot{\psi}^2\,(\rho_1 - \rho_2)\cos\psi \\ & = -\frac{2J_1}{R^2\cos^2\delta}\big[\left(\ddot{x}_C + \dot{y}_C\dot{\psi}\right)(1 + \cos 2\psi\cos 2\delta) \\ & \quad + \left(\ddot{y}_C - \dot{x}_C\dot{\psi}\right)\sin 2\psi\cos 2\delta + \ddot{\psi}\,(\rho_1 - \rho_2)\sin\psi\sin^2\delta\big] \\ & \quad + \frac{1}{R\cos\delta}\left[(M_1 + M_4)\cos(\psi - \delta) + (M_2 + M_3)\cos(\psi + \delta)\right], \end{aligned} \tag{16}$$

$$
\begin{aligned}
&(m_0 + 4m_1)\, \ddot{y}_C - 2m_1 \ddot{\psi}\ \ (\rho_1 - \rho_2) \cos\psi\ + 2m_1 \dot{\psi}^{\,2} (\rho_1 - \rho_2) \sin\psi \\
&= -\frac{2J_1}{R^2 \cos^{2}\delta} \left[\left(\ddot{x}_C\ + \dot{y}_C \dot{\psi} \right) \sin 2\psi \cos 2\delta \right. \\
&\quad + \left. \left(\ddot{y}_C - \dot{x}_C \dot{\psi} \right) (1 - \cos 2\psi \cos 2\delta) - \ddot{\psi}\, (\rho_1 - \rho_2) \cos\psi \sin^2\delta \right] \\
&\quad + \frac{1}{R\cos\delta} \left[(M_1 + M_4) \sin(\psi - \delta) + (M_2 + M_3) \sin(\psi + \delta) \right],
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&\left(J_0 + 4J_2 + 2m_1(\rho_1^2 + \rho_2^2 + 2l^2) \right) \ddot{\psi} \\
&\quad + 2m_1 \ddot{x}_c (\rho_1 - \rho_2) \sin\psi - 2m_1 \ddot{y}_c (\rho_1 - \rho_2) \cos\ \psi \\
&\quad + \frac{2J_1}{R^2\ \cos^2\delta} \left[\ddot{\psi} (l^2 + 0.5(\rho_1^2 + \rho_2^2) + (l^2 - 0.5(\rho_1^2 + \rho_2^2)) \cos\ 2\delta \right. \\
&\qquad + l(\rho_1 + \rho_2) \sin 2\delta) \\
&\quad + \ddot{x}_c (\rho_1 - \rho_2) \sin\psi \sin^2\delta - \ddot{y}_c (\rho_1 - \rho_2) \cos\ \psi \sin^2\delta \\
&\quad \left. + \dot{x}_C \dot{\psi}\, (\rho_1 - \rho_2) \cos\psi \sin^{2}\delta + \dot{y}_C \dot{\psi}\, (\rho_1 - \rho_2) \sin\psi \sin^2\delta \right] \\
&\quad = \frac{1}{R\cos\delta} \left[l\, (M_2 - M_1 + M_4 - M_3) \cos\delta + (\rho_1\, (M_4 - M_3) \right. \\
&\qquad \left. + \rho_2\, (M_2 - M_1)) \sin\delta \right].
\end{aligned} \tag{18}
$$

As a result, for given torques M_i $(i = 1, \ldots, 4)$, we have a system of three equations for three variables, x_C, y_C, and ψ. Then the angles φ_i of rotation of the wheels can be found from Eqs. (7)–(10).

In the vehicles that are usually considered, the angle between the plane of the mecanum wheel and the roller axis is equal to 45° ($\delta = \pi/4$), and the center of mass of the system coincides with its geometric center ($\rho_1 = \rho_2 = \rho$). For this case, Eqs. (16)–(18) are simplified and become

$$
\begin{aligned}
&\left(m + \frac{4J_1}{R^2} \right) \ddot{x}_C + \frac{4J_1}{R^2} \dot{y}_C \dot{\psi} = \frac{1}{R} \left[(M_1 + M_2 + M_3 + M_4) \cos\psi \right. \\
&\quad \left. + (M_1 - M_2 - M_3 + M_4) \sin\psi \right],
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&\left(m + \frac{4J_1}{R^2} \right) \ddot{y}_C - \frac{4J_1}{R^2} \dot{x}_C \dot{\psi} = \frac{1}{R} \left[(M_1 + M_2 + M_3 + M_4) \sin\psi \right. \\
&\quad \left. - (M_1 - M_2 - M_3 + M_4) \cos\psi \right],
\end{aligned} \tag{20}
$$

$$
\left[J_C + \frac{4J_1}{R^2} (l + \rho)^2 \right] \ddot{\psi} = -\ \frac{l + \rho}{R} (M_1 - M_2 + M_3 - M_4), \tag{21}
$$

where $m = m_0 + 4\ m_1$ is the total mass of the system and $J_C = J_0 + 4J_2 + 4m_1$ $(\rho^2 + l^2)$ is the moment of inertia of the entire system relative to the center of mass.

3 Approximate Model

Let v_x and v_y be the projections of the velocity of the center of mass of the vehicle onto the axes of the coordinate system $\left\{C, \vec{E}_x, \vec{E}_y, \vec{E}_z\right\}$ that is attached to the vehicle's body and is turned by the angle ψ relative to the fixed reference frame $\left\{O, \vec{e}_x, \vec{e}_y, \vec{e}_z\right\}$. Since

$$v_x = \dot{x}_C \cos\psi + \dot{y}_C \sin\psi, v_y = -\dot{x}_C \sin\psi + \dot{y}_C \cos\psi, \tag{22}$$

the kinematic constraint equations (7)–(10) become

$$v_x \cos\delta_1 - v_y \sin\delta_1 - \dot{\psi}\,(l\cos\delta_1 + \rho_2 \sin\delta_1) = R\dot{\varphi}_1 \cos\delta_1, \tag{23}$$

$$v_x \cos\delta_2 + v_y \sin\delta_2 + \dot{\psi}\,(l\cos\delta_2 + \rho_2 \sin\delta_2) = R\dot{\varphi}_2 \cos\delta_2, \tag{24}$$

$$v_x \cos\delta_3 + v_y \sin\delta_3 - \dot{\psi}\,(l\cos\delta_3 + \rho_1 \sin\delta_3) = R\dot{\varphi}_3 \cos\delta_3, \tag{25}$$

$$v_x \cos\delta_4 - v_y \sin\delta_4 + \dot{\psi}\,(l\cos\delta_4 + \rho_1 \sin\delta_4) = R\dot{\varphi}_4 \cos\delta_4. \tag{26}$$

In the matrix form, Eqs. (23)–(26) are represented as follows:

$$J \cdot \vec{v}_C = \vec{v}_W,\ \vec{v}_C = \left(v_x, v_y, \dot{\psi}\right)^T,\ \vec{v}_W = (R\dot{\varphi}_1, R\dot{\varphi}_2, R\dot{\varphi}_3, R\dot{\varphi}_4)^T, \tag{27}$$

$$J = \begin{pmatrix} \cos\delta_1 & -\sin\delta_1 & -(l\cos\delta_1 + \rho_2\sin\delta_1) \\ \cos\delta_2 & \sin\delta_2 & l\cos\delta_2 + \rho_2\sin\delta_2 \\ \cos\delta_3 & \sin\delta_3 & -(l\cos\delta_3 + \rho_1\sin\delta_3) \\ \cos\delta_4 & -\sin\delta_4 & l\cos\delta_4 + \rho_1\sin\delta_4 \end{pmatrix}. \tag{28}$$

Following [9], we solve the matrix equation (27) for the vector $\vec{v}_C$ using the pseudo-inverse matrix J^+ to obtain

$$\vec{v}_C = J^+ \cdot \vec{v}_W, J^+ = \left(J^T \cdot J\right)^{-1} \cdot J^T. \tag{29}$$

The pseudo-inverse matrix gives an approximate solution of the matrix equation (27) that minimizes the Euclidean norm $\left\| J \cdot \vec{v}_C - \vec{v}_W \right\|$ of the residual.

In the particular case, for which all δ_i $(i = 1, \ldots, 4)$ are equal to $\pi/4$ and $\rho_1 = \rho_2 = \rho$, the solution (29) becomes

$$v_x = \frac{R}{4}\left(\dot{\varphi}_1 + \dot{\varphi}_2 + \dot{\varphi}_3 + \dot{\varphi}_4\right), \tag{30}$$

$$v_y = -\ \frac{R}{4}\left(\dot{\varphi}_1 - \dot{\varphi}_2 - \dot{\varphi}_3 + \dot{\varphi}_4\right) \tag{31}$$

$$\dot{\psi} = -\ \frac{R}{4\,(l+\rho)}\left(\dot{\varphi}_1 - \dot{\varphi}_2 + \dot{\varphi}_3 - \dot{\varphi}_4\right). \tag{32}$$

The expression (15) for the kinetic energy becomes

$$T = \frac{1}{2}m\ \left(v_x^2 + v_y^2\right) + \frac{1}{2}J_C\,\dot{\psi}^2 + \frac{1}{2}J_1\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + \dot{\varphi}_4^2\right). \tag{33}$$

Substituting relations (30)–(32) into expression (33) yields

$$\begin{aligned} T = {} & \frac{1}{2}\left[m\frac{R^2}{8} + \frac{R^2}{16(\rho+l)^2}J_C + J_1\right]\left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}_3^2 + \dot{\varphi}_4^2\right) \\ & + \left[m\frac{R^2}{8} - \frac{R^2}{16(\rho+l)^2}J_C\right]\left(\dot{\varphi}_1\,\dot{\varphi}_4 + \dot{\varphi}_2\dot{\varphi}_3\right) \\ & - \frac{R^2}{16(\rho+l)^2}J_C\left(\dot{\varphi}_1\dot{\varphi}_2 - \dot{\varphi}_1\,\dot{\varphi}_3 - \dot{\varphi}_2\dot{\varphi}_4 + \dot{\varphi}_3\,\dot{\varphi}_4\right). \end{aligned} \tag{34}$$

For this kinetic energy, write out Lagrange's equations of the second kind for 4 variables $q^i = \varphi_i$

$$\frac{d}{d\ t}\frac{\partial T}{\partial \dot{\varphi}_i} - \frac{\partial T}{\partial \varphi_i} = M_i,\ \ i = 1,\ldots,4. \tag{35}$$

Notice that this procedure is valid only for systems subject to holonomic constraints. By differentiating Eqs. (30)–(32) with respect to time and substituting into the resulting relations the expressions for $\ddot{\varphi}_i$ found from Eq. (35) we obtain the equations for the coordinates x_C and y_C of the center of mass and the angle ψ of rotation of the vehicle:

$$\begin{aligned} \left(m + \frac{4J_1}{R^2}\right)\left(\ddot{x}_C + \dot{y}_C\,\dot{\psi}\right) = {} & \frac{1}{R}\left[(M_1 + M_2 + M_3 + M_4)\cos\psi\right. \\ & \left. + \ (M_1 - M_2 - M_3 + M_4)\sin\psi\right], \end{aligned} \tag{36}$$

$$\begin{aligned} \left(m + \frac{4J_1}{R^2}\right)\left(\ddot{y}_C - \dot{x}_C\,\dot{\psi}\right) = {} & \frac{1}{R}\left[(M_1 + M_2 + M_3 + M_4)\sin\psi\right. \\ & \left. - \ (M_1 - M_2 - M_3 + M_4)\cos\psi\right], \end{aligned} \tag{37}$$

$$\left[\left(J_C + \frac{4J_1}{R^2}(l+\rho)^2\right]\ \ddot{\psi} = -\frac{l+\rho}{R}\left(M_1 - M_2 + M_3 - M_4\right). \tag{38}$$

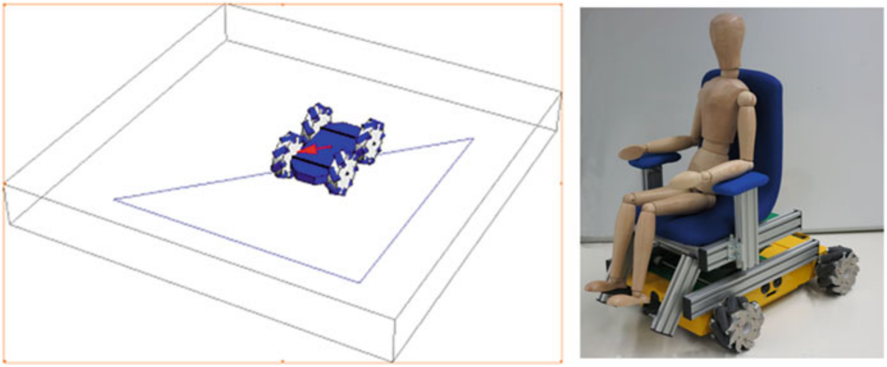

Fig. 3 Simulation of the vehicle motion using Mathematica® (realized by B. Adamov) (*left*) and the prototype of a vehicle with four mecanum wheels (*right*)

4 Conclusions

Equations of motion for a vehicle with four mecanum wheels were derived proceeding from a non-holonomic model for mecanum wheels. By comparing the non-holonomic model with an approximate model used in robotics we established that both models lead to the same result in the particular cases for which the vehicle either moves translationally or rotates about its center of mass. In both these cases, the constraints become holonomic. Based on the model considered in this study, an experimental verification of simulation results (Fig. 3 left) was realized, using a prototype of a mobile robot with four mecanum wheels (Fig. 3, right).

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