§2.2

3、解：$F'(x) = \int_0^x f(y)\,dy + 2x f(x)$

$F''(x) = f(x) + \left(2f(x) + 2x f'(x)\right) = 3f(x) + 2x f'(x)$

4、证明：$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{1}{2}\left(a\varphi'(x+at) - a\varphi'(x-at)\right) + \frac{1}{2a}\left(a\psi(x+at) + a\psi(x-at)\right)\right]$

$= \frac{a^2}{2}\left(\varphi''(x+at) + \varphi''(x-at)\right) + \frac{a}{2}\left(\psi'(x+at) - \psi'(x-at)\right)$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\varphi'(x+at) + \varphi'(x-at)\right) + \frac{1}{2a}\left(\psi(x+at) - \psi(x-at)\right)\right]$

$= \frac{1}{2}\left(\varphi''(x+at) + \varphi''(x-at)\right) + \frac{1}{2a}\left(\psi'(x+at) - \psi'(x-at)\right)$

$\Rightarrow \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

5、(1) 由 $\int_0^1 \frac{1}{1+x^2y^2}\,dy = \frac{\arctan x}{x}$ 得：

原式 $= \int_0^1\int_0^1 \frac{1}{1+x^2y^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dy\,dx = \lim_{b\to 1}\int_0^b\int_0^1 \frac{1}{1+x^2y^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dy\,dx$

(被积函数在 $[0,b]\times[0,1]$ 上连续，$\forall b<1$；$\left|\frac{1}{1+x^2y^2}\cdot\frac{1}{\sqrt{1-x^2}}\right| \le \frac{1}{\sqrt{1-x^2}}$，$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx < +\infty$)

不借助 b 也可以

$= \lim_{b\to 1}\int_0^1\int_0^b \frac{1}{1+x^2y^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\,dy$

$= \int_0^1\int_0^1 \frac{1}{1+x^2y^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx\,dy$

$\overset{x=\sin t}{=\!=\!=} \int_0^1\int_0^{\frac{\pi}{2}} \frac{1}{1+y^2\sin^2 t}\cdot\frac{1}{\cos t}\,d\sin t\,dy$

$\overset{u=\tan t}{=\!=\!=} \int_0^1\int_0^{+\infty} \frac{1+u^2}{1+u^2(1+y^2)}\cdot\frac{1}{1+u^2}\,du\,dy$

$\overset{s=\sqrt{y^2+1}\,u}{=\!=\!=} \int_0^1\int_0^{+\infty} \frac{1}{\sqrt{1+y^2}}\cdot\frac{1}{1+s^2}\,ds\,dy$

$= \int_0^1 \frac{\pi}{2}\cdot\frac{1}{\sqrt{1+y^2}}\,dy$

$\overset{y=\tan t}{=\!=\!=} \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \sec t\,dt$

$= \frac{\pi}{2}\left(\ln|\sec t + \tan t|\right)\Big|_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\ln(\sqrt{2}+1)$

(2) $\int_0^1 \underbrace{\frac{x^b - x^a}{\ln x}\sin(\ln\frac{1}{x})}_{=g(x)} dx \quad (a,b>0)$.

$\left(\lim_{x\to 0} g(x) = 0,\ \lim_{x\to 1} g(x) = 0.\right.$ 令 $g(0)=g(1)=0$ 则 $g(x)$ 在 $[0,1]$ 上连续$\left.\right)$

原式 $= \int_0^1 [\sin(\ln\frac{1}{x})\int_a^b x^y dy]\, dx$

记 $f(x,y) = \begin{cases} [\sin(\ln\frac{1}{x})]x^y, & x>0 \\ 0, & x=0 \end{cases}$ 则其在 $[0,1]\times[a,b]$ 上连续

因此原式 $= \int_a^b \int_0^1 x^y \sin(\ln\frac{1}{x})\, dx\, dy$

$\underline{\underline{x=e^{-t}}} \int_a^b \int_0^{+\infty} e^{-ty}\sin t \cdot e^{-t}\, dt\, dy$

$= \int_a^b \frac{1}{1+(y+1)^2} dy$

$= \arctan(b+1) - \arctan(a+1)$

§2.3 $\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx \quad (a,b>0)$

1. (1) $\left(\frac{\partial}{\partial y}\left(\frac{e^{-yx^2}}{x}\right) = -x e^{-yx^2}\right)$

$\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x} dx = \int_0^{+\infty}\int_a^b x e^{-yx^2} dy\, dx$

$|x e^{-yx^2}| \le x e^{-\min\{a,b\}x^2}$，由比较判别法知 $\int_0^{+\infty} x e^{-yx^2} dx$ 在 $y\in[a,b]$ 一致收敛

$\Rightarrow$ 原式 $= \int_a^b \int_0^{+\infty} x e^{-yx^2} dx\, dy = \int_a^b \frac{1}{2y} dy = \frac{1}{2}\ln\frac{b}{a}$.

(2) 由例2.3.2得：

$$\int_0^{+\infty} x e^{-ax^2} \sin yx\, dx = -I'(y) = \frac{y}{2a} I(y) = \frac{y}{4a}\sqrt{\frac{\pi}{a}}\, e^{-\frac{y^2}{4a}}, \quad a>0.$$

(3) $\int_0^{+\infty} \frac{\cos ax - \cos bx}{x^2} dx \quad (a,b>0)$

$$= \int_0^{+\infty} \int_a^b \frac{\sin yx}{x} dy\, dx$$

$\left| \int_0^A \sin yx\, dx \right| \le \frac{2}{y} \le \frac{2}{\min\{a,b\}}$，$\forall A>0$，$\forall y \in [a,b]$（或 $[b,a]$）

$\frac{1}{x}$ 关于 x 单调，$\lim_{x\to+\infty} \frac{1}{x} = 0$ 关于 $y\in[a,b]$ 一致成立

由Dirichlet判别法 $\Rightarrow \int_0^{+\infty} \frac{\sin yx}{x} dx$ 一致收敛

故原式 $= \int_a^b \int_0^{+\infty} \frac{\sin yx}{x} dx\, dy = \int_a^b \frac{\pi}{2} dy = \frac{\pi}{2}(b-a)$

2.(1) $\int_0^{+\infty} e^{-tx^2} x^{2n} dx \quad (t>0) \qquad \left(\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right)$

由上式在任意 $[a,b]$，$(a>0)$ 上一致收敛（关于 t）

$$\frac{d^n}{dt^n}\int_0^{+\infty} e^{-tx^2} dx = \int_0^{+\infty} \frac{d^n}{dt^n} e^{-tx^2} dx = (-1)^n \int_0^{+\infty} e^{-tx^2} x^{2n} dx$$

又 $\frac{d^n}{dt^n}\int_0^{+\infty} e^{-tx^2} dx = \frac{d^n}{dt^n} \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{t}} = (-1)^n \frac{\sqrt{\pi}}{2}\cdot \frac{(2n-1)!!}{2^n} t^{-n-\frac{1}{2}}$

$\Rightarrow$ 原式 $= \frac{\sqrt{\pi}}{2}\cdot\frac{(2n-1)!!}{2^n} t^{-(n+\frac{1}{2})}$

(2) $\int_0^{+\infty} \frac{dx}{(y+x^2)^{n+1}} = \frac{\pi(2n-1)!!}{2n!!} y^{-(n+\frac{1}{2})} \quad (y>0)$

$$\int_0^{+\infty} \frac{dx}{y+x^2} = \frac{\pi}{2}\cdot\frac{1}{\sqrt{y}}$$

$\int_0^{+\infty} \frac{dx}{(y+x^2)^{n+1}}$ 在任意 $[a,b]$ $(a>0)$ 上关于 y 一致收敛.

$$\frac{d^n}{dy^n}\int_0^{+\infty} \frac{dx}{x^2+y} = \int_0^{+\infty} \frac{d^n}{dy^n}\left(\frac{1}{x^2+y}\right) dx = (-1)^n n! \int_0^{+\infty} \frac{dx}{(x^2+y)^{n+1}}$$

$$\frac{d^n}{dy^n}\int_0^{+\infty} \frac{dx}{x^2+y} = \frac{d^n}{dy^n} \frac{\pi}{2}\cdot\frac{1}{\sqrt{y}} = (-1)^n \frac{\pi}{2}\cdot\frac{(2n-1)!!}{2^n} y^{-(n+\frac{1}{2})}$$

$\Rightarrow$ 结论✓.