1.6 作业解答

4. 证明：由于 $u=u(x,y,z)$ 由 $f(u^2-x^2,u^2-y^2,u^2-z^2)=0$ 确定.

∴有：$\frac{\partial f}{\partial x}=(f_1'+f_2'+f_3')u\frac{\partial u}{\partial x}-f_1'x=0$

$$\begin{cases}\frac{\partial f}{\partial y}=(f_1'+f_2'+f_3')u\frac{\partial u}{\partial y}-f_2'y=0\\ \frac{\partial f}{\partial z}=(f_1'+f_2'+f_3')u\frac{\partial u}{\partial z}-f_3'z=0\end{cases}$$

∴有：$\left(\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}\right)(f_1'+f_2'+f_3')=(f_1'+f_2'+f_3')\cdot\frac{1}{u}$

即：$\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=\frac{1}{u}$

证毕

5. 解：能确定：

$$\begin{cases}x=u+v\\ y=u-v\\ z=u^2v^2\end{cases}\Rightarrow\begin{cases}u=\frac{x+y}{2}\\ v=\frac{x-y}{2}\end{cases}\Rightarrow\begin{cases}x=u+v\\ y=u-v\\ z=\frac{(x^2-y^2)^2}{16}\end{cases}$$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\left(\frac{(x^2-y^2)^2}{16}\right)=\frac{x}{4}(x^2-y^2)$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(\frac{(x^2-y^2)^2}{16}\right)=\frac{y}{4}(y^2-x^2).$$

7. 解：将方程组化简

① 将 y 消去 构造 $f(z)=x$

② 将 x 消去 构造 $f(z)=y$

① $\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases} \Rightarrow x^2+[2-(x+z)]^2=\frac{1}{2}z^2$

∴有 $\frac{dx}{dz}$，$2x\frac{dx}{dz}+2[2-(x+z)]\cdot\left[-(1+\frac{dx}{dz})\right]=z$ ①

在点 $(1,-1,2)$ 处有 $x=1$ $z=2$

∴有 $2\frac{dx}{dz}+2(2-3)\left[-(1+\frac{dx}{dz})\right]=2$

∴有 $\frac{dx}{dz}=0$

$\frac{d^2x}{dz^2}$：对①两侧再对 z 求导.

$2\left(\frac{dx}{dz}\right)^2+2x\frac{d^2x}{dz^2}+2\left[-(1+\frac{dx}{dz})\right]^2+2[2-(x+z)]\cdot\left[-\frac{d^2x}{dz^2}\right]=1$

将 $\frac{dx}{dz}=0$，$x=1$，$z=2$ 代入有

$2\frac{d^2x}{dz^2}+2+2\frac{d^2x}{dz^2}=1$

∴有 $\frac{d^2x}{dz^2}=-\frac{1}{4}$

② $\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases} \Rightarrow [2-(y+z)]^2+y^2=\frac{1}{2}z^2$

有 $\frac{dy}{dz}$：$2y\frac{dy}{dz}+2[2-(y+z)]\left[-(1+\frac{dy}{dz})\right]=z$

在点 $(1,-1,2)$ 处有 $y=-1$ $z=2$

得 $\frac{dy}{dz}=-1$　　同理可得 $\frac{d^2y}{dz^2}=\frac{1}{4}$

9: 首先求各向量值函数的逆映射

$(u,v)=f(x,y)$ $\quad f^{-1}$ 为逆映射 $\quad J=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

(1) $|J(f)|=\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}=4(x^2+y^2)$

$\therefore$ 有 $J(f^{-1})=|J(f)|^{-1}\cdot\begin{bmatrix} \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix}=\frac{1}{2(x^2+y^2)}\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$

同理可得: 其余向量值函数的逆映射的 Jacobi 行列式和矩阵

(2): $J=\begin{bmatrix} e^{-x}\cos y & e^{-x}\sin y \\ -e^{-x}\sin y & e^{-x}\cos y \end{bmatrix}$ $\quad |J|=e^{-2x}$ (注)

(3): $J=\frac{1}{6x^3y+3y^4}\begin{bmatrix} 2xy & 3y^2 \\ -y^2 & 3x^2 \end{bmatrix}$ $\quad |J|=\frac{1}{6x^3y+3y^4}$

(4) $J=\frac{1}{\mathrm{ch}x\,\mathrm{ch}y+\mathrm{sh}x\,\mathrm{sh}y}\begin{bmatrix} \mathrm{ch}y & -\mathrm{sh}y \\ \mathrm{sh}x & \mathrm{ch}x \end{bmatrix}$ $\quad |J|=\frac{1}{\mathrm{ch}x\,\mathrm{ch}y+\mathrm{sh}x\,\mathrm{sh}y}$

(5) $J=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad |J|=\frac{1}{ad-bc}$

(6) $J=\frac{1}{9x^2y^2+1}\begin{bmatrix} 3y^2 & 1 \\ -1 & 3x^2 \end{bmatrix}$ $\quad |J|=\frac{1}{9x^2y^2+1}$

注 (2) 中题干 $v=e^y\sin y$ 改为 $v=e^x\sin y$ 方可得答案

若仍按题干计算, 则有

$J=\begin{bmatrix} \frac{1}{e^x\cos y} & \frac{\sin y}{e^y\cos y(\sin y+\cos y)} \\ 0 & \frac{1}{e^y(\sin y+\cos y)} \end{bmatrix}$ $\quad |J|=\frac{1}{e^x e^y\cos y\cdot(\sin y+\cos y)}$