



Review

- 极限与连续
- 判断函数在一点没有极限的方法
- 连续函数在有界闭集上的性质
- (n 重)极限与累次极限
- $o(\|x - x_0\|^k)$ 与 $O(\|x - x_0\|^k)$, $x \rightarrow x_0$ 时.



§ 4. 多元函数的偏导数与全微分

1. 偏导数

Def. $u = f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$ 在 $\mathbf{x}_0 = (x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(n)}) \in \mathbb{R}^n$ 的某个邻域中有定义, 若极限

$$\lim_{\Delta x_i \rightarrow 0} \frac{\Delta_{x_i} u}{\Delta x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_0^{(1)}, \dots, x_0^{(i-1)}, \mathbf{x}_0^{(i)} + \Delta x_i, x_0^{(i+1)}, \dots, x_0^{(n)}) - f(\mathbf{x}_0)}{\Delta x_i}$$

存在, 则称之为 $f(\mathbf{x})$ 在 \mathbf{x}_0 关于 x_i 的偏导数, 记作 $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$,

$$\frac{\partial u}{\partial x_i}(\mathbf{x}_0), \left. \frac{\partial f}{\partial x_i} \right|_{\mathbf{x}_0}, \left. \frac{\partial u}{\partial x_i} \right|_{\mathbf{x}_0}, u'_{x_i}(\mathbf{x}_0) \text{ 或 } f'_{x_i}(\mathbf{x}_0).$$



Remark: 二元函数 $f(x, y)$ 偏导数的几何意义.

$$f'_x(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

Remark: 1) 对某个变量求偏导数时, 视其余变量为常数, 按一元函数求导法则和公式去求.

2) 求分段函数的偏导函数时, 用定义求分界点处的偏导数, 用1) 中方法求其它点处的偏导数. 一般地, 分段函数的偏导函数仍为分段函数.

Remark: 视 x_0 为变量, 得偏导函数 $\frac{\partial f}{\partial x_i}(x), i = 1, 2, \dots, n.$



例. 设 $f(x, y) = \begin{cases} (x+y)^2 \sin \frac{1}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$, 求 $f'_x(x, y)$.

解: $f'_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x^2 \sin \frac{1}{x^2}}{x} = 0.$

$x^2 + y^2 \neq 0$ 时,

$$f'_x(x, y) = 2(x+y) \sin \frac{1}{x^2+y^2} - \frac{2x(x+y)^2}{(x^2+y^2)^2} \cos \frac{1}{x^2+y^2}. \quad \square$$



例. $f(x, y) = x^2 e^y + (x-1) \arctan \frac{y}{x}$, 求 $f'_x(1, 0)$.

解法一: $f(x, 0) = x^2$, 所以 $f'_x(1, 0) = 2$.

解法二:

$$f'_x(x, y) = 2xe^y + \arctan \frac{y}{x} + (x-1) \cdot \frac{\frac{-y}{x^2}}{1 + \left(\frac{y}{x}\right)^2}$$
$$= 2xe^y + \arctan \frac{y}{x} + \frac{y(1-x)}{x^2 + y^2}.$$

所以 $f'_x(1, 0) = 2$. \square

Remark: 求具体点处的偏导数时, 第一种方法较好.



Remark: 多元函数偏导数存在与连续性互不蕴含.

例: 设 $f(x, y) = \begin{cases} 1 & y = x^2, x > 0 \\ 0 & \text{其它情形} \end{cases}$, 则 $f(x, 0) \equiv f(0, y) \equiv 0$,

$f'_x(0, 0) = f'_y(0, 0) = 0$, 但 f 在 $(0, 0)$ 不连续.

例: $f(x, y) = \sqrt{x^2 + y^2}$ 在原点连续, 但 $f'_x(0, 0), f'_y(0, 0)$

都不存在. 事实上, $\lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{\sqrt{x^2}}{x}$ 与

$\lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{\sqrt{y^2}}{y}$ 都不存在. \square



例. $z = f(x, y), \frac{\partial z}{\partial y} = x^2 + 2y, f(x, x^2) = 1$, 求 $f(x, y)$.

解: 由 $\frac{\partial z}{\partial y} = x^2 + 2y$, 将 x 看成常数, 两边对 y 积分, 得

$$z = f(x, y) = \int (x^2 + 2y) dy = x^2 y + y^2 + g(x),$$

其中 $g(x)$ 为待定函数. 由 $f(x, x^2) = 1$, 有

$$g(x) = 1 - 2x^4,$$

$$f(x, y) = 1 + x^2 y + y^2 - 2x^4. \square$$



2. 全微分

1) 一元函数的微分

以直代曲
近似计算

$$\begin{aligned}\Delta f(x_0) &= f(x_0 + \Delta x) - f(x_0) \\ &= f'(x_0)\Delta x + \alpha,\end{aligned}$$

其中 $\lim_{\Delta x \rightarrow 0} \frac{\alpha}{\Delta x} = 0$, 即 $\alpha = o(\Delta x)$, 当 $\Delta x \rightarrow 0$ 时. 记

$$df(x_0) = f'(x_0)\Delta x = f'(x_0)dx.$$



2) 二元函数的微分

推广一元微分的概念, 形式上应该有,

$$\begin{aligned}\Delta f(x_0, y_0) &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &= (a, b) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + \alpha,\end{aligned}$$

几何直观?

其中 $\lim_{\Delta x \rightarrow 0, \Delta y \rightarrow 0} \frac{\alpha}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$, 即

$$\alpha = o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2}\right), \quad \Delta x \rightarrow 0, \Delta y \rightarrow 0 \text{ 时.}$$

这里的 a, b 应该与 f 在 (x_0, y_0) 的两个一阶偏导数有关.



3) n 元函数的微分

Def. $u = f(\mathbf{x})$ 在 $\mathbf{x}_0 \in \mathbb{R}^n$ 的某邻域中定义, 若存在常数 a_1, a_2, \dots, a_n , s.t. 当 $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n) \rightarrow 0$ 时,

$$\begin{aligned}\Delta u(\mathbf{x}_0) &= \Delta f(\mathbf{x}_0) \triangleq f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) \\ &= a_1 \Delta x_1 + a_2 \Delta x_2 + \dots + a_n \Delta x_n + o(\|\Delta \mathbf{x}\|),\end{aligned}$$

则称 $u = f(\mathbf{x})$ 在点 \mathbf{x}_0 **可微**, 称 $a_1 \Delta x_1 + a_2 \Delta x_2 + \dots + a_n \Delta x_n$ 为 f 在 \mathbf{x}_0 的**(全)微分**, 记作

$$du(\mathbf{x}_0) = df(\mathbf{x}_0) = a_1 \Delta x_1 + a_2 \Delta x_2 + \dots + a_n \Delta x_n,$$

或
$$du(\mathbf{x}_0) = df(\mathbf{x}_0) = a_1 dx_1 + a_2 dx_2 + \dots + a_n dx_n.$$



Remark: f 在 x_0 可微 $\Leftrightarrow \exists$ 常数 $a_1, a_2, \dots, a_n \in \mathbb{R}, s.t.$

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - a_1 \Delta x_1 - a_2 \Delta x_2 - \dots - a_n \Delta x_n}{\|x - x_0\|} = 0.$$

例. f 为有界函数, 即 $\exists M > 0$, 使得 $|f(x, y)| \leq M, \forall (x, y)$, 则

$g(x, y) = (x^2 + y^2)^{3/2} f(x, y)$ 在 $(0, 0)$ 可微.

Proof.
$$\frac{|g(x, y) - g(0, 0)|}{\sqrt{x^2 + y^2}} = \frac{|(x^2 + y^2)^{3/2} f(x, y)|}{\sqrt{x^2 + y^2}}$$

$$= (x^2 + y^2) |f(x, y)| \leq M(x^2 + y^2) \rightarrow 0, \text{ 当 } (x, y) \rightarrow (0, 0) \text{ 时. } \square$$



Thm. $u = f(\mathbf{x})$ 在 $\mathbf{x}_0 \in \mathbb{R}^n$ 可微, 则

- 1) f 在 \mathbf{x}_0 连续,
- 2) $\frac{\partial f}{\partial x_i}(\mathbf{x}_0)$ 存在, $i = 1, 2, \dots, n$, 且 f 在 \mathbf{x}_0 的全微分为

$$du = \frac{\partial f}{\partial x_1}(\mathbf{x}_0)dx_1 + \frac{\partial f}{\partial x_2}(\mathbf{x}_0)dx_2 + \dots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0)dx_n.$$

Proof: 记 f 在 \mathbf{x}_0 的全微分为 $du = a_1 dx_1 + a_2 dx_2 + \dots + a_n dx_n$.

$$1) f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = a_1 \Delta x_1 + a_2 \Delta x_2 + \dots + a_n \Delta x_n + o(\|\Delta \mathbf{x}\|) \\ \rightarrow 0, \text{ 当 } \Delta \mathbf{x} \rightarrow 0 \text{ 时.}$$

故 f 在 \mathbf{x}_0 连续.



2) 当 $\Delta \mathbf{x} = (\Delta x_1, 0, 0, \dots, 0)$ 时, 由可微的定义,

$$f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = a_1 \Delta x_1 + a_2 \Delta x_2 + \dots + a_n \Delta x_n + o(\|\Delta \mathbf{x}\|)$$

$$\Delta_{x_1} f(\mathbf{x}_0) = a_1 \Delta x_1 + o(|\Delta x_1|),$$

$$\text{于是, } f'_{x_1}(\mathbf{x}_0) = \lim_{\Delta x_1 \rightarrow 0} \frac{\Delta_{x_1} f(\mathbf{x}_0)}{\Delta x_1} = a_1.$$

$$\text{同理, } f'_{x_i}(\mathbf{x}_0) = a_i, i = 1, 2, \dots, n.$$

故 f 在 \mathbf{x}_0 的全微分为

$$du = \frac{\partial f}{\partial x_1}(\mathbf{x}_0) dx_1 + \frac{\partial f}{\partial x_2}(\mathbf{x}_0) dx_2 + \dots + \frac{\partial f}{\partial x_n}(\mathbf{x}_0) dx_n. \square$$



Remark: 函数的连续性与偏导数的存在性不蕴含函数的可微性.

例. 讨论 $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ 在原点的

连续性、可微性、偏导数的存在性与连续性.

解: 1) f 在 $(0, 0)$ 连续.

$$\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq |x| \rightarrow 0 = f(0, 0), \quad (x, y) \rightarrow (0, 0) \text{ 时.}$$



$$2) \quad \frac{\partial f(x, y)}{\partial x} = \begin{cases} \frac{y^3}{(x^2 + y^2)^{3/2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$$

f'_x 在 $(0, 0)$ 存在但不连续. 同理,

$$\frac{\partial f(x, y)}{\partial y} = \begin{cases} \frac{x^3}{(x^2 + y^2)^{3/2}}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0), \end{cases}$$

f'_y 在 $(0, 0)$ 存在但不连续.



3) f 在 $(0,0)$ 不可微. 若可微, 则

$$\Delta f(0,0) = \frac{\Delta x \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= 0 \cdot \Delta x + 0 \cdot \Delta y + o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2}\right), (\Delta x, \Delta y) \rightarrow (0,0) \text{ 时.}$$

$$\text{故 } \lim_{(\Delta x, \Delta y) \rightarrow (0,0)} \frac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2} = 0.$$

$$\text{这与 } \lim_{\Delta x \rightarrow 0, \Delta y = \Delta x} \frac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2} = \frac{1}{2} \text{ 矛盾. } \square$$



例. $f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$ 在原点的

连续性、可微性、偏导数的存在性与偏导函数的连续性.

解: 1) f 在 $(0, 0)$ 连续.

$$2) f(x, 0) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

$$f'_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} x \sin \frac{1}{x^2} = 0;$$



当 $(x, y) \neq (0, 0)$ 时,

$$f'_x(x, y) = 2x \sin \frac{1}{x^2 + y^2} - \frac{2x}{x^2 + y^2} \cos \frac{1}{x^2 + y^2}.$$

$\lim_{(x, y) \rightarrow (0, 0)} f'_x(x, y)$ 不存在, 故 $f'_x(x, y)$ 在 $(0, 0)$ 不连续.

同理 $f'_y(0, 0) = 0$, 但 f'_y 在 $(0, 0)$ 不连续.

3) f 在 $(0, 0)$ 可微. 事实上,

$$\begin{aligned} \Delta f(0, 0) - f'_x(0, 0)\Delta x - f'_y(0, 0)\Delta y \\ &= ((\Delta x)^2 + (\Delta y)^2) \sin \frac{1}{(\Delta x)^2 + (\Delta y)^2} \\ &= o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2}\right), (\Delta x, \Delta y) \rightarrow (0, 0) \text{ 时. } \square \end{aligned}$$



例: $f = \sqrt{\sin|xy|}$ 在原点的连续性、偏导数与可微性.

解: • f 在原点连续.

• 因 $f(x, 0) \equiv 0, f(0, y) \equiv 0$, 故 $f'_x(0, 0) = f'_y(0, 0) = 0$.

• 若 f 在原点可微, 则 $\lim_{(x,y) \rightarrow (0,0)} \frac{\sqrt{\sin|xy|}}{\sqrt{x^2 + y^2}} = 0$. 而 $k \neq 0$ 时,

$$\lim_{\substack{y=kx \\ x \rightarrow 0}} \frac{\sqrt{\sin|xy|}}{\sqrt{x^2 + y^2}} = \lim_{\substack{y=kx \\ x \rightarrow 0}} \frac{\sqrt{\sin|xy|}}{\sqrt{|xy|}} \lim_{\substack{y=kx \\ x \rightarrow 0}} \frac{\sqrt{|xy|}}{\sqrt{x^2 + y^2}} = \frac{\sqrt{|k|}}{\sqrt{1+k^2}} \neq 0,$$

矛盾, 故 f 在原点不可微. \square



3. 函数在一点可微的充要条件

Thm. n 元函数 $f(\mathbf{x})$ 在 $\mathbf{x}_0 \in \mathbb{R}$ 可微的充要条件是

$$\Delta f(\mathbf{x}_0) = f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = \sum_{i=1}^n f'_{x_i}(\mathbf{x}_0) \Delta x_i + \sum_{i=1}^n \varepsilon_i \Delta x_i,$$

其中 ε_i 为 $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$ 的 n 元函数, $i = 1, 2, \dots, n$, 且

$$\lim_{\Delta \mathbf{x} \rightarrow 0} \varepsilon_i = 0, \quad i = 1, 2, \dots, n.$$

Proof: (必要性) 若 $f(\mathbf{x})$ 在 \mathbf{x}_0 可微, 则

$$f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = \sum_{i=1}^n f'_{x_i}(\mathbf{x}_0) \Delta x_i + \alpha,$$

其中 $\alpha = o(\|\Delta \mathbf{x}\|)$, 当 $\Delta \mathbf{x} \rightarrow 0$ 时.



则

$$\alpha = \sum_{i=1}^n \frac{\alpha \cdot \operatorname{sgn}(\Delta x_i)}{|\Delta x_1| + \cdots + |\Delta x_n|} \Delta x_i = \sum_{i=1}^n \varepsilon_i \Delta x_i,$$

$$\text{其中 } \varepsilon_i = \frac{\alpha \cdot \operatorname{sgn}(\Delta x_i)}{|\Delta x_1| + \cdots + |\Delta x_n|}, i = 1, 2, \cdots, n.$$

$$\begin{aligned} |\varepsilon_i| &= \frac{|\alpha|}{|\Delta x_1| + \cdots + |\Delta x_n|} \\ &= \frac{|\alpha|}{\|\Delta \mathbf{x}\|} \cdot \frac{\|\Delta \mathbf{x}\|}{|\Delta x_1| + \cdots + |\Delta x_n|} \leq \frac{|\alpha|}{\|\Delta \mathbf{x}\|} \end{aligned}$$

$$\text{故 } \lim_{\Delta \mathbf{x} \rightarrow 0} |\varepsilon_i| = 0, \lim_{\Delta \mathbf{x} \rightarrow 0} \varepsilon_i = 0, i = 1, 2, \cdots, n.$$



(充分性) 设

$$f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = \sum_{i=1}^n f'_{x_i}(\mathbf{x}_0) \Delta x_i + \sum_{i=1}^n \varepsilon_i \Delta x_i,$$

其中 ε_i 为 $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$ 的 n 元函数, $i = 1, 2, \dots, n$, 且

$$\lim_{\Delta \mathbf{x} \rightarrow 0} \varepsilon_i = 0, \quad i = 1, 2, \dots, n.$$

则

$$\begin{aligned} \frac{\left| \sum_{i=1}^n \varepsilon_i \Delta x_i \right|}{\|\Delta \mathbf{x}\|} &\leq \frac{(|\varepsilon_1| + \dots + |\varepsilon_n|)(|\Delta x_1| + \dots + |\Delta x_n|)}{\|\Delta \mathbf{x}\|} \\ &\leq n(|\varepsilon_1| + \dots + |\varepsilon_n|), \rightarrow 0, \text{ 当 } \Delta \mathbf{x} \rightarrow 0 \text{ 时.} \end{aligned}$$

故 $f(\mathbf{x})$ 在 \mathbf{x}_0 可微. \square



Thm. f'_{x_i} 在 $\mathbf{x}_0 \in \mathbb{R}^n$ 连续, $i = 1, 2, \dots, n$, 则 f 在 \mathbf{x}_0 可微.

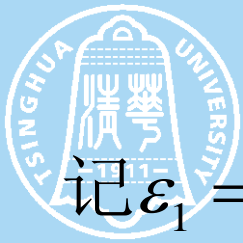
Proof: 记 $e_i, i = 1, \dots, n$, 为 \mathbb{R}^n 的自然基, $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$,

则

$$\begin{aligned} & f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) \\ &= f(\mathbf{x}_0 + \Delta x_1 e_1) - f(\mathbf{x}_0) \\ &\quad + f(\mathbf{x}_0 + \Delta x_1 e_1 + \Delta x_2 e_2) - f(\mathbf{x}_0 + \Delta x_1 e_1) \\ &\quad + \dots + f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0 + \Delta x_1 e_1 + \dots + \Delta x_{n-1} e_{n-1}) \end{aligned}$$

由一元函数的微分中值定理, $\exists \theta_i \in (0, 1), i = 1, 2, \dots, n, s.t.$

$$\begin{aligned} & f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) \\ &= f'_{x_1}(\mathbf{x}_0 + \theta_1 \Delta x_1 e_1) \Delta x_1 + f'_{x_2}(\mathbf{x}_0 + \Delta x_1 e_1 + \theta_2 \Delta x_2 e_2) \Delta x_2 \\ &\quad + \dots + f'_{x_n}(\mathbf{x}_0 + \Delta x_1 e_1 + \dots + \Delta x_{n-1} e_{n-1} + \theta_n \Delta x_n e_n) \Delta x_n \end{aligned}$$



$$\text{记 } \varepsilon_1 = f'_{x_1}(\mathbf{x}_0 + \theta_1 \Delta x_1 \mathbf{e}_1) - f'_{x_1}(\mathbf{x}_0),$$

$$\varepsilon_2 = f'_{x_2}(\mathbf{x}_0 + \Delta x_1 \mathbf{e}_1 + \theta_2 \Delta x_2 \mathbf{e}_2) - f'_{x_2}(\mathbf{x}_0),$$

$$\vdots$$

$$\varepsilon_n = f'_{x_n}(\mathbf{x}_0 + \Delta x_1 \mathbf{e}_1 + \cdots + \Delta x_{n-1} \mathbf{e}_{n-1} + \theta_n \Delta x_n \mathbf{e}_n) - f'_{x_n}(\mathbf{x}_0),$$

$$\text{则 } f(\mathbf{x}_0 + \Delta \mathbf{x}) - f(\mathbf{x}_0) = \sum_{i=1}^n f'_{x_i}(\mathbf{x}_0) \Delta x_i + \sum_{i=1}^n \varepsilon_i \Delta x_i,$$

其中 ε_i 为 $\Delta \mathbf{x} = (\Delta x_1, \Delta x_2, \cdots, \Delta x_n)$ 的 n 元函数, $i = 1, 2, \cdots, n$.

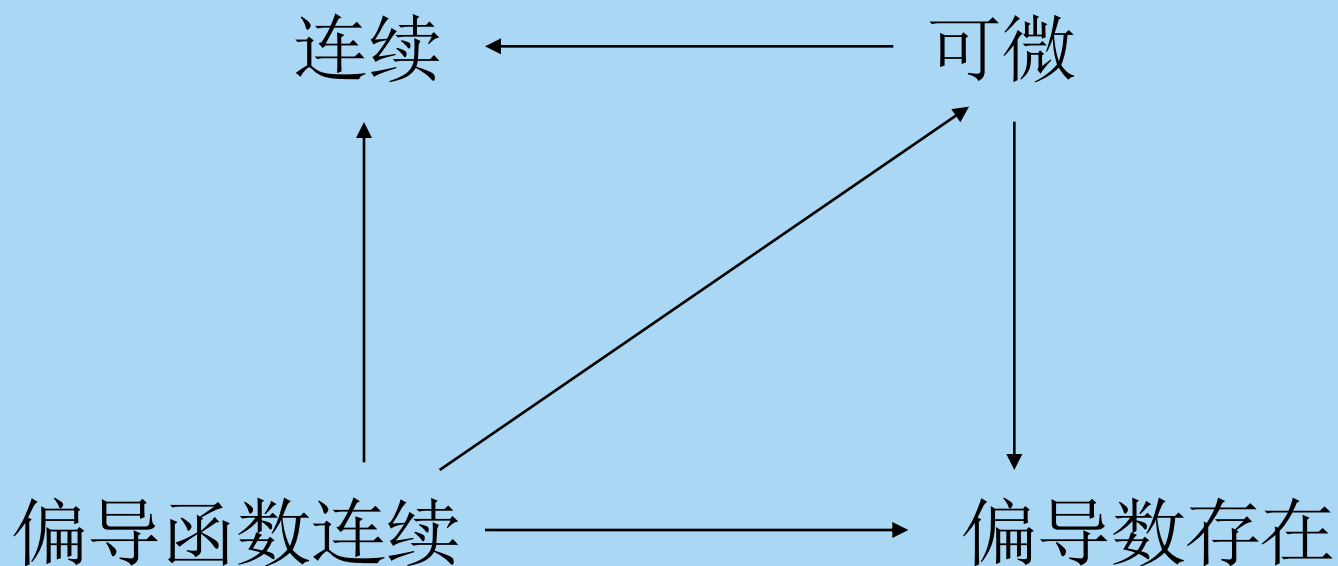
由 f'_{x_i} 在 \mathbf{x}_0 的连续性,

$$\lim_{\Delta \mathbf{x} \rightarrow 0} \varepsilon_i = 0, i = 1, 2, \cdots, n.$$

因此 f 在 \mathbf{x}_0 可微. \square



Remark: 函数的连续性、可微性、偏导数存在性与偏导数连续性之间的蕴含关系图.





4. 高阶偏导数

视 $f'_x(x, y), f'_y(x, y)$ 为 x, y 的二元函数, 有时也记为 f'_1, f'_2 , 考虑它们的偏导数, 即高阶偏导数. 例如,

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad f''_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right)$$

分别为 f 关于 x 和关于 y 的二阶偏导数, 也记为 f''_{11}, f''_{22} . 而

$$f''_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right), \quad f''_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

为 f 关于 x, y 的二阶混合偏导数, 也记为 f''_{12}, f''_{21} .



例. $z = \frac{1}{x} f(xy) + yf(x+y)$, 求 $\frac{\partial^2 z}{\partial x \partial y}$.

解: $\frac{\partial z}{\partial y} = f'(xy) + f(x+y) + yf'(x+y)$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right)$$

$$= yf''(xy) + f'(x+y) + yf''(x+y). \square$$



例. $f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases},$

求 $f''_{xy}(0, 0)$ 和 $f''_{yx}(0, 0)$.

解: $\frac{\partial f(x, y)}{\partial x} = \begin{cases} y \frac{x^4 - y^4 + 4x^2 y^2}{(x^2 + y^2)^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$

$$f''_{yx}(0, 0) = \lim_{x \rightarrow 0} \frac{f'_x(0, y) - f'_x(0, 0)}{y} = \lim_{x \rightarrow 0} \frac{-y}{y} = -1.$$



$$\frac{\partial f(x, y)}{\partial y} = \begin{cases} x \frac{x^4 - y^4 - 4x^2 y^2}{(x^2 + y^2)^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases}$$

$$f''_{xy}(0, 0) = \lim_{x \rightarrow 0} \frac{f'_y(x, 0) - f'_y(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x}{x} = 1. \square$$

Question: 要求 $f''_{xx}(0, 0)$, 是否必须计算出 $f'_x(x, y)$?

$$f''_{xx}(0, 0) = \lim_{x \rightarrow 0} \frac{f'_x(x, 0) - f'_x(0, 0)}{x}, \text{ 只需计算出 } f'_x(x, 0).$$

Remark: 混合偏导数一般情况下与求导顺序有关.



Thm. 若 f''_{xy}, f''_{yx} 都在 (x_0, y_0) 连续, 则 $f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0)$.

Proof. 令 $\Delta = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)$
 $- f(x_0, y_0 + \Delta y) + f(x_0, y_0),$

$$\varphi(t) = f(x_0 + t\Delta x, y_0 + \Delta y) - f(x_0 + t\Delta x, y_0),$$

则 $\varphi'(t) = f'_x(x_0 + t\Delta x, y_0 + \Delta y)\Delta x - f'_x(x_0 + t\Delta x, y_0)\Delta x,$

$$\Delta = \varphi(1) - \varphi(0) = \varphi'(\theta_1)$$

$$= (f'_x(x_0 + \theta_1\Delta x, y_0 + \Delta y) - f'_x(x_0 + \theta_1\Delta x, y_0))\Delta x$$

$$= f''_{yx}(x_0 + \theta_1\Delta x, y_0 + \theta_2\Delta y)\Delta x\Delta y, \quad \text{其中 } \theta_1, \theta_2 \in (0, 1).$$



同理, 令 $\psi(t) = f(x_0 + \Delta x, y_0 + t\Delta y) - f(x_0, y_0 + t\Delta y)$, 则

$$\psi'(t) = f'_y(x_0 + \Delta x, y_0 + t\Delta y)\Delta y - f'_y(x_0, y_0 + t\Delta y)\Delta y,$$

$$\Delta = \psi(1) - \psi(0) = \psi'(\theta_3)$$

$$= (f'_y(x_0 + \Delta x, y_0 + \theta_3\Delta y) - f'_y(x_0, y_0 + \theta_3\Delta y))\Delta y$$

$$= f''_{xy}(x_0 + \theta_4\Delta x, y_0 + \theta_3\Delta y)\Delta x\Delta y, \quad \theta_3, \theta_4 \in (0, 1).$$

于是 $f''_{yx}(x_0 + \theta_1\Delta x, y_0 + \theta_2\Delta y) = f''_{xy}(x_0 + \theta_4\Delta x, y_0 + \theta_3\Delta y)$,

由于 f''_{xy}, f''_{yx} 在 (x_0, y_0) 连续, 上式中令 $(\Delta x, \Delta y) \rightarrow (0, 0)$, 得

$$f''_{yx}(x_0, y_0) = f''_{xy}(x_0, y_0). \quad \square$$



5. 方向导数、梯度

Question: 用过点 $(x_0, y_0, f(x_0, y_0))$ 且平行于OZ轴的平面去截曲面 $z = f(x, y)$, 所得的交线在点 $(x_0, y_0, f(x_0, y_0))$ 处的斜率如何刻画?

点 (x_0, y_0) 及单位向量 $\boldsymbol{v} = (v_1, v_2)^T \in \mathbb{R}^2$ 确定直线

$$\ell = \{(x, y) \mid x = x_0 + v_1 t, y = y_0 + v_2 t\}.$$

其中 t 表示点 (x_0, y_0) 沿方向 \boldsymbol{v} 到点 (x, y) 的有向距离. 把直线 ℓ 类比为 x 轴, 方向 \boldsymbol{v} 类比为 x 轴正向. 则可以得到方向导数的定义.



Def. f 在 $x_0 \in \mathbb{R}^n$ 的邻域中有定义, $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ 为非零向量, l 为过 x_0 沿 \vec{v} 方向的射线, 若 t 的函数

$$g(t) = f\left(x_0 + \frac{\vec{v}}{\|\vec{v}\|} t\right) = f\left(x_0^{(1)} + \frac{v_1}{\|\vec{v}\|} t, \dots, x_0^{(n)} + \frac{v_n}{\|\vec{v}\|} t\right)$$

在 $t = 0$ 存在右导数, 即极限

$$\lim_{\substack{x \rightarrow x_0 \\ x \in l}} \frac{f(x) - f(x_0)}{\|x - x_0\|} = \lim_{t \rightarrow 0^+} \frac{g(t) - g(0)}{t}$$

存在, 则称该极限为 $f(x)$ 在 x_0 沿方向 \vec{v} 的方向导数, 记作

$$\frac{\partial f(x_0)}{\partial \vec{v}}, \left. \frac{\partial f}{\partial \vec{v}} \right|_{x_0} \text{ 或 } f'_{\vec{v}}(x_0).$$



Remark.

$\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}}$ 是函数 $f(\mathbf{x})$ 在点 \mathbf{x}_0 沿方向 \vec{v} 的变化率.

第 i 个分量

Remark. $\frac{\partial f(\mathbf{x}_0)}{\partial x_i}$ 为 f 在 \mathbf{x}_0 沿 $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ 的方向导数.

Thm. 设 f 在 $\mathbf{x}_0 \in \mathbb{R}^n$ 可微, $\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ 为非零向量, 则方向导数 $\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}}$ 存在, 且

$$\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}} = \frac{\partial f(\mathbf{x}_0)}{\partial x_1} \frac{v_1}{\|\vec{v}\|} + \frac{\partial f(\mathbf{x}_0)}{\partial x_2} \frac{v_2}{\|\vec{v}\|} + \dots + \frac{\partial f(\mathbf{x}_0)}{\partial x_n} \frac{v_n}{\|\vec{v}\|}.$$



Proof. f 在 $\mathbf{x}_0 \in \mathbb{R}^n$ 可微, 则

$$f(\mathbf{x}_0 + \frac{\vec{v}}{\|\vec{v}\|}t) - f(\mathbf{x}_0) = \sum_{i=1}^n \frac{\partial f(\mathbf{x}_0)}{\partial x_i} \frac{v_i}{\|\vec{v}\|} t + o(t).$$

$$\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}} = \lim_{t \rightarrow 0^+} \frac{f(\mathbf{x}_0 + \frac{\vec{v}}{\|\vec{v}\|}t) - f(\mathbf{x}_0)}{t}$$

$$= \sum_{i=1}^n \frac{\partial f(\mathbf{x}_0)}{\partial x_i} \frac{v_i}{\|\vec{v}\|} + \lim_{t \rightarrow 0^+} \frac{o(t)}{t}$$

$$= \sum_{i=1}^n \frac{\partial f(\mathbf{x}_0)}{\partial x_i} \frac{v_i}{\|\vec{v}\|}. \quad \square$$



例. 求 $f = x^2 + y^2$ 在 $M_0(2,1)$ 沿 $\vec{w} = (3, -4)$ 的方向导数.

解法一. $\vec{w}/\|\vec{w}\| = (3/5, -4/5)^T$,

$$g(t) = f\left(2 + \frac{3}{5}t, 1 - \frac{4}{5}t\right) = \left(2 + \frac{3}{5}t\right)^2 + \left(1 - \frac{4}{5}t\right)^2,$$

$$\frac{\partial f(2,1)}{\partial \vec{w}} = g'(0) = \frac{4}{5}.$$

解法二. $\vec{w}/\|\vec{w}\| = (3/5, -4/5)^T$, $f'_x(2,1) = 4, f'_y(2,1) = 2$.

$$\frac{\partial f(2,1)}{\partial \vec{w}} = f'_x(2,1) \cdot \frac{3}{5} + f'_y(2,1) \cdot \frac{-4}{5} = \frac{4}{5}. \square$$



Remark: 即使 $f(x, y)$ 在某点存在所有的方向导数, 也不能推断 f 在该点连续.

例. $f(x, y) = \begin{cases} 1, & y = x^2, x > 0, \\ 0, & \text{其它,} \end{cases}$ 在原点不连续, 但沿任何

非零向量 $\vec{v} \in \mathbb{R}^2$, $\frac{\partial f(0,0)}{\partial \vec{v}} = 0$.

Def. n 元函数 f 在 $\mathbf{x}_0 \in \mathbb{R}^n$ 可微, 称

$$\text{grad} f(\mathbf{x}_0) \triangleq \left(\frac{\partial f(\mathbf{x}_0)}{\partial x_1}, \frac{\partial f(\mathbf{x}_0)}{\partial x_2}, \dots, \frac{\partial f(\mathbf{x}_0)}{\partial x_n} \right)$$

为数量场 $u = f(\mathbf{x})$ 在点 \mathbf{x}_0 的梯度.



Thm. f 在 $\mathbf{x}_0 \in \mathbb{R}^n$ 可微, 记 $\vec{w} = \text{grad}f(\mathbf{x}_0)$ 则

$$\max_{\vec{v} \in \mathbb{R}^n, \vec{v} \neq 0} \frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}} = \frac{\partial f(\mathbf{x}_0)}{\partial \vec{w}} = \|\text{grad}f(\mathbf{x}_0)\|,$$

即 f 在 \mathbf{x}_0 沿梯度方向的方向导数最大, 且最大方向导数的值为梯度的模.

Proof. f 在 \mathbf{x}_0 可微, 则
$$\frac{\partial f(\mathbf{x}_0)}{\partial \vec{v}} = \text{grad}f(\mathbf{x}_0) \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \|\text{grad}f(\mathbf{x}_0)\| \cdot \cos \langle \text{grad}f(\mathbf{x}_0), \vec{v} \rangle \leq \|\text{grad}f(\mathbf{x}_0)\|,$$

当且仅当 $\vec{v} = k \cdot \text{grad}f(\mathbf{x}_0), k > 0$ 时 " $=$ " 成立. \square



作业： 习题1.4
No. 1(单),2(单),
8,11(单) ,12 (单),15 (单)