习题 3.4 作业参考解答

《高等微积分教程（下）》

12. 计算下列三重积分的值.

(1) $\iiint\limits_{\Omega} xy^2z^3dxdydz$, Ω 是由马鞍面 $z=xy$ 与平面 $y=x, x=1, z=0$ 所围成的空间区域.

解：

$$\begin{aligned}\iiint\limits_{\Omega} xy^2z^3dxdydz &= \int_0^1 dx\int_0^x dy\int_0^{xy} xy^2z^3dz\\ &= \int_0^1 dx\int_0^x xy^2\cdot\frac{1}{4}(xy)^4dy\\ &= \int_0^1 \frac{1}{28}x^{12}dx = \frac{1}{364}.\end{aligned}$$

(3) $\iiint\limits_{\Omega} x\cos(y+z)dxdydz$, Ω 是由曲面 $x=\sqrt{y}$ 与平面 $x=0, z=0, y+z=\frac{\pi}{2}$ 围成的区域.

解：

$$\begin{aligned}\iiint\limits_{\Omega} x\cos(y+z)dxdydz &= \int_0^{\frac{\pi}{2}} dz \int_0^{\frac{\pi}{2}-z} dy \int_0^{\sqrt{y}} x\cos(y+z)dx \\ &= \int_0^{\frac{\pi}{2}} dz \int_0^{\frac{\pi}{2}-z} \frac{1}{2}y\cos(y+z)dy \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\pi}{2}-z-\cos z dz \\ &= \frac{\pi^2}{16}-\frac{1}{2}.\end{aligned}$$

(5) $\iiint\limits_{\Omega} \dfrac{\sin z}{z}dxdydz$, $\Omega=\{(x,y,z)|\sqrt{x^2+y^2}\le z\le 4\}$.

解：令 $x=r\cos\theta, y=r\sin\theta$，则

$$\begin{aligned}\iiint\limits_{\Omega} \frac{\sin z}{z}dxdydz &= \int_0^4 dz \int_0^{2\pi} d\theta \int_0^z \frac{\sin z}{z}rdr \\ &= 2\pi\int_0^4 \frac{\sin z}{z}\cdot\frac{1}{2}z^2dz \\ &= (-4\cos 4+\sin 4)\pi.\end{aligned}$$

6. 计算累次积分 $I=\int_0^1 dx\int_0^x dy\int_0^y \dfrac{\cos z}{(1-z)^2}dz$ 的值.

解：积分的区域为 $(x,y,z): 0\le z\le y\le x\le 1$，可以另一种次序计算累次积分.

$$\begin{aligned}I &= \int_0^1 dz\int_z^1 dy\int_y^1 \frac{\cos z}{(1-z)^2}dx \\ &= \int_0^1 dz\int_z^1 \frac{\cos z}{(1-z)^2}(1-y)dy \\ &= \int_0^1 \frac{1}{2}\cos z dz = \frac{1}{2}\sin 1.\end{aligned}$$

7. 计算下列三重积分的值.

(1) $\iiint\limits_{\Omega} \sqrt{x^2+y^2}dxdydz$, $\Omega=\{(x,y,z)|\sqrt{x^2+y^2}\leq z\leq 1\}$.

解： 令 $x=r\cos\theta, y=r\sin\theta$，则

$$\begin{aligned}\iiint\limits_{\Omega} \sqrt{x^2+y^2}dxdydz &= \int_0^1 dz\int_0^{2\pi} d\theta\int_0^z r\cdot rdr\\ &=2\pi\int_0^1 \frac{1}{3}z^3dz\\ &=\frac{\pi}{6}.\end{aligned}$$

(3) $\iiint\limits_{\Omega} \frac{z}{x^2+y^2}dxdydz$, $\Omega=\{(x,y,z)|0\leq z\leq x^2+y^2, x+y\leq 1, x,y\geq 0\}$.

解：

$$\begin{aligned}\iiint\limits_{\Omega} \frac{z}{x^2+y^2}dxdydz &= \int_0^1 dx\int_0^{1-x} dy\int_0^{x^2+y^2} \frac{z}{x^2+y^2}dz\\ &=\int_0^1 dx\int_0^{1-x} \frac{1}{2}(x^2+y^2)dy\\ &=\int_0^1 \frac{1}{6}-\frac{1}{2}x+x^2-\frac{2}{3}x^3dx\\ &=\frac{1}{12}.\end{aligned}$$

(5) $\iiint\limits_{\Omega} xyzdxdydz$, $\Omega=\{(x,y,z)|x^2+y^2+z^2\leq 4, x^2+y^2+(z-2)^2\leq 4, x\geq 0, y\geq 0\}$.

解： 令 $x=r\cos\theta, y=r\sin\theta$.

则当 $z\in[0,1]$ 时，$r\in[0,\sqrt{4z-z^2}]$，当 $z\in(1,2]$ 时，$r\in[0,\sqrt{4-z^2}]$.

从而

$$
\begin{aligned}
&\iiint\limits_{\Omega} xyzdxdydz \\
&=\int_0^1 dz\int_0^{\frac{\pi}{2}} d\theta\int_0^{\sqrt{4z-z^2}} r^2\cos\theta\sin\theta\cdot z\cdot rdr+\int_1^2 dz\int_0^{\frac{\pi}{2}} d\theta\int_0^{\sqrt{4-z^2}} r^2\cos\theta\sin\theta\cdot z\cdot rdr \\
&=\frac{1}{2}(\int_0^1\frac{1}{4}(4z-z^2)^2zdz+\int_1^2\frac{1}{4}(4-z^2)^2zdz) \\
&=\frac{53}{60}.
\end{aligned}
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8. 作适当的变量代换，计算下列三重积分.

(1) $\iiint\limits_{\Omega}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dxdydz$, $\Omega=\{(x,y,z)|\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\}$.

解: 令 $x=ar\sin\phi\cos\theta, y=br\sin\phi\sin\theta, z=cr\cos\phi$,则 $\left|\det\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}\right|=abcr^2\sin\phi$，且积分区域为

$$E=\{(r,\phi,\theta):r\in[0,1],\phi\in[0,\pi],\theta\in[0,2\pi)\}$$

从而

$$
\begin{aligned}
&\iiint\limits_{\Omega}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dxdydz \\
&=\int_0^{2\pi}d\theta\int_0^{\pi}d\phi\int_0^1\sqrt{1-r^2}abcr^2\sin\phi dr \\
&=4\pi abc\int_0^1\sqrt{1-r^2}r^2dr \\
&=\frac{\pi^2}{4}abc.
\end{aligned}
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