微积分 A2 · 第八周作业 · 乙班

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习题 3.4

9. 求下列空间曲面包围几何体的体积

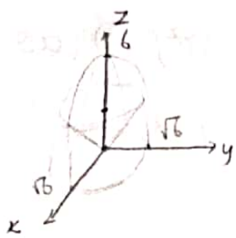
(1) $z=6-x^2-y^2$，$z=\sqrt{x^2+y^2}$

(3) $x^2+y^2=a^2$，$x^2+z^2=a^2$，$x,y,z\geqslant 0\ (a>0)$

(5) $(x^2+y^2+z^2)^2=a^3 z\ (a>0)$

(7) $(a_{11}x+a_{12}y+a_{13}z)^2+(a_{21}x+a_{22}y+a_{23}z)^2+(a_{31}x+a_{32}y+a_{33}z)^2=r^2$，其中 $A=(a_{ij})_{3\times3}$ 可逆

解：(1)




如图：当 $6-x^2-y^2=\sqrt{x^2+y^2}$ 且 $z>0$，$z=2$.

$\therefore\ 2\leqslant z\leqslant 6$ 与 $0\leqslant z\leqslant 2$ 两段

$\therefore\ V=V_1+V_2$.

$V_1=\int_0^2 dz\iint_{D_1}\sqrt{x^2+y^2}\,dx\,dy$

$D_1$：$0\leqslant\rho\leqslant 2$，$0\leqslant\theta\leqslant 2\pi$.

$V_1=\pi\int_0^2 z^2\,dz$

$=\frac{8}{3}\pi$

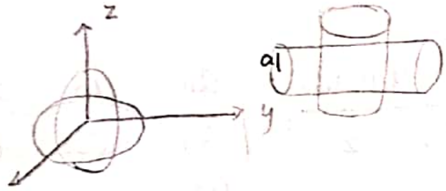
$V_2=\pi\int_2^6 dz \int(6-z)dz$

$=\pi\left(6z-\frac{1}{2}z^2\right)\Big|_2^6=8\pi$

$V=V_1+V_2=\frac{32\pi}{3}$

$\therefore\ V=\frac{32}{3}\pi$

(3)




$\because x,y,z\geqslant 0 \Rightarrow$ 第一卦限

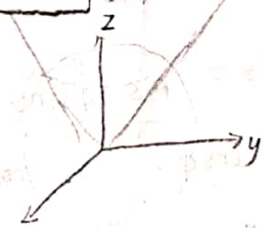
$\therefore\ D=\{(x,y)\mid 0\leqslant y\leqslant\sqrt{a^2-x^2},\ 0\leqslant x\leqslant R\}$

$z=\sqrt{R^2-x^2}$

$\therefore\ V=\iint_D\sqrt{R^2-x^2}\,d\sigma$

$=\int_0^R(R^2-x^2)\,dx=\frac{2}{3}R^3$

(5) 作球坐标变换：




$$\begin{cases}x=\rho\sin\phi\cos\theta\\ y=\rho\sin\phi\sin\theta\\ z=\rho\cos\phi>0\end{cases}$$

$\rho^4=a^3\rho\cos\phi$

$\therefore\ \rho^3=a^3\cos\phi$

$0\leqslant\theta\leqslant 2\pi$，$0\leqslant\phi\leqslant\frac{\pi}{2}$

$V=\int_0^{\frac{\pi}{2}}d\phi\int_0^{2\pi}d\theta\int_0^{a\sqrt[3]{\cos\phi}}a^3\rho\cos\phi\cdot\rho^2\sin\phi\,d\rho$

$=\int_0^{a\sqrt[3]{\cos\phi}}d\rho\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}a^3\rho\cos\phi\,\rho^2\sin\phi\,d\phi$

$=\frac{a^3}{2}\int_0^{a\sqrt[3]{\cos\phi}}d\rho\int_0^{2\pi}\rho^3\sin 2\phi$

$=\frac{\pi}{3}a^3$

Pa

(7) 令 $a_{11}x + a_{12}y + a_{13}z = \rho\sin\phi\cos\theta = u$

$a_{21}x + a_{22}y + a_{23}z = \rho\sin\phi\sin\theta = v$

$a_{31}x + a_{32}y + a_{33}z = \rho\cos\phi = w$

$0 \le \rho \le r$ , $0 \le \theta \le 2\pi$ , $0 \le \phi \le \pi$.

又∵ $dx\,dy\,dz = \left|\frac{D(x,y,z)}{D(\rho,\phi,\theta)}\right| d\rho\, d\phi\, d\theta$

又∵ $\frac{D(u,v,w)}{D(x,y,z)} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

又∵ $\frac{D(u,v,w)}{D(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta, & \rho\cos\theta\cos\phi, & \rho\sin\phi \overset{-\sin\theta}{} \\ \sin\phi\sin\theta, & \rho\sin\theta\cos\phi, & \rho\sin\phi\cos\theta \\ \cos\phi, & \rho, & 0 \end{vmatrix}$

∴ $\left|\frac{D(x,y,z)}{D(\rho,\phi,\theta)}\right| = \frac{分子}{分母}$

分子 $= \underbrace{\rho\sin\phi\sin\theta\cdot(-\rho\sin\theta\sin\phi)}_{-\rho^2\sin^2\phi\sin^2\theta} + \underbrace{\cos\phi\cdot\rho\cos\theta\cos\phi\cdot\rho\sin\phi\cos\theta}_{\rho^2\cos^2\theta\cos^2\phi\sin\phi}$

$- [\underbrace{-\rho^2\sin\theta\cos^2\phi\cdot\sin\theta\sin\phi}_{-\rho^2\sin^2\theta\cos^2\phi\sin\phi} + \rho^2\sin^2\phi\cos^2\theta]$

$= -\rho^2\sin^2\phi + \rho^2 \cancel{\rho^2} \cos^2\phi\sin\phi$

分母 $= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$

$- (a_{31}a_{22}a_{13} + a_{21}a_{12}a_{33} + a_{11}a_{32}a_{23})$

∴原式 $= \int_0^{2\pi} d\theta \int_0^{\pi} \int_0^{r} \checkmark\, d\rho$

$= \frac{1}{(a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}) - (a_{31}a_{22}a_{13} + a_{21}a_{12}a_{33} + a_{11}a_{32}a_{23})} \times \frac{4\pi r^3}{3}$

$\| \quad |\det A|$

P3

10. 设 $f(t)$ 在 $(-\infty,+\infty)$ 连续. $f(t) = 3\iiint\limits_{x^2+y^2+z^2\le t^2} f(\sqrt{x^2+y^2+z^2})\,dx\,dy\,dz + |t^3|$

求 $f(t)$

解: $f(t) = 3\iiint\limits_{x^2+y^2+z^2\le t^2} f(\sqrt{x^2+y^2+z^2})\,dx\,dy\,dz + |t^3|$

令 $x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

① 当 $t > 0$ 时.

令 $3\iiint\limits_{\Omega} f(\sqrt{x^2+y^2+z^2})\,dv = A$

∴ $A = 12\pi\int_0^t f(r)r^2\,dr$

$f(t) = 12\pi\int_0^t f(r)r^2\,dr + t^3$

∴ $f'(t) = 12\pi t^2 f(t) + 3t^2$

∴ $f(t) = -\frac{1}{4\pi} + Ce^{4\pi t^3}$

又∵ $t=0$, $f(t)=0 \Rightarrow C = \frac{1}{4\pi}$

∴ $f(t) = \frac{1}{4\pi}(e^{4\pi t^3} - 1)$

② 当 $t < 0$ 时. 偶函数

$f(t) = 12\pi\int_0^{-t} f(r)r^2\,dr - t^3$.

∴ $f'(t) = 12\pi(-1)t^2 f(-t) - 3t^2$

$= -12\pi t^2 f(-t) - 3t^2$

.....

$f(t) = f(-t) = \frac{1}{4\pi}(e^{-4\pi t^3} - 1)$

P4

数 $f(x,y,z)$ 连续，计算极限：

$$\lim_{r\to 0^+} \frac{1}{r^3} \iiint_{x^2+y^2+z^2\le r^2} f(x,y,z)\,dxdydz$$

解： 球坐标变换：

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$\therefore\quad 0\le \rho\le r,$

原式 $= \frac{1}{r^3} \int_0^r d\rho \int_0^{2\pi} d\phi \int_0^{2\pi} f(\cdot\cdot,\cdot\cdot,\cdot\cdot)\rho^2\sin\phi\, d\phi$

$= \frac{1}{r^3} \int_0^r d\rho \int_0^{2\pi} 2\rho^2 f(\cdot\cdot,\cdot\cdot,\cdot\cdot)\, d\theta$

$= \frac{4\pi}{r^3} \int_0^r f(\cdot\cdot,\cdot\cdot,\cdot\cdot)\cdot\rho^2 d\rho$

$= \frac{4\pi}{3} f(\cdot\cdot,\cdot\cdot,\cdot\cdot)$

$\therefore$ 当 $r\to 0^+$ 时.

$\lim = \frac{4\pi}{3} f(0,0,0)$

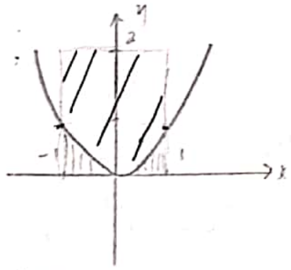
## 章总复习题

5. 计算二重积分

(1) $\iint_D |y-x^2|\,dx\,dy$, $D=\{(x,y)\mid -1\le x\le 1,\ 0\le y\le 2\}$

(2) $\iint_D \left|\frac{x+y}{\sqrt{2}}-x^2-y^2\right| dx\,dy$, $D=\{(x,y)\mid x^2+y^2\le 1\}$

解：(1)



① 当 $y\geq x^2$ 时

$-1\le x\le 1,\ x^2\le y\le 2$.

$\therefore V_1=\int_{-1}^{1}dx\int_{x^2}^{2}(y-x^2)\,dy=\frac{43}{15}$

② 当 $y\le x^2$ 时

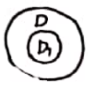
$-1\le x\le 1,\ 0\le y\le x^2$

$\therefore V_2=\int_{-1}^{1}dx\int_0^{x^2}(x^2-y)\,dy=\frac{1}{5}$

$\therefore V=V_1+V_2=\frac{46}{15}$

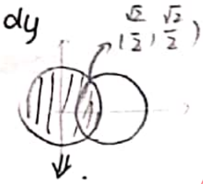
(2) $\because \frac{x+y}{\sqrt{2}}-x^2-y^2=-\left(x-\frac{1}{2\sqrt{2}}\right)^2-\left(y-\frac{1}{2\sqrt{2}}\right)^2+\frac{1}{4}$

$\therefore D_1: \left(x-\frac{1}{2\sqrt{2}}\right)^2+\left(y-\frac{1}{2\sqrt{2}}\right)^2=\frac{1}{4}$



$\therefore$ 在 $D_1$ 内 $\le 0$；在 $D_1$ 外 $\geq 0$

$\therefore$ 原式 $=\iint_{D_1}\left(x^2+y^2-\frac{x+y}{\sqrt{2}}\right)dx\,dy+\iint_{D_2}\left(\frac{x+y}{\sqrt{2}}-x^2-y^2\right)dx\,dy$



$=\iint_{D_1}f(x,y)\,dx\,dy-\iint_{D\setminus D_1}f(x,y)\,dx\,dy$

$=2\iint_{D_1}f(x,y)\,dx\,dy-\iint_{D}f(x,y)\,dx\,dy$

$=2I_1+I_2$

而 $I_1=\iint_{D_1}(\cdots)\,dx\,dy$

$\therefore$ 原式 $=\frac{9}{16}\pi$



7. 计算广义二重积分

(2) $\iint_D e^{-x^2-y^2}\sin(x^2+y^2)\,dx\,dy$, $D=R^2$

解：$\rho^2=x^2+y^2\in(0,+\infty)$.

$\therefore \iint_D e^{-x^2-y^2}\sin(x^2+y^2)\,dx\,dy$

$=\int_0^{2\pi}d\theta\int_0^{A}e^{-\rho^2}\sin\rho^2\cdot\rho\,d\rho$

$=2\pi\int_0^{A}\rho e^{-\rho^2}\sin\rho^2\,d\rho$, $A\to+\infty$

y

又 $\because (\cos\rho^2)'=-2\rho\sin\rho^2$

$\therefore$ 原式 $=-\pi\int_0^{A}e^{-\rho^2}(\cos\rho^2)'\,d\rho$

$=-\pi\left[e^{-\rho^2}\cdot\cos\rho^2\Big|_0^{A}-\int_0^{A}\cos\rho^2\cdot e^{-\rho^2}(-2\rho)\right]$

$=-\pi\left[(0-1)+2\int_0^{A}\rho\cos\rho^2 e^{-\rho^2}\right]$

$[\sin\rho^2]'=2\rho\cos\rho^2$

$=\pi-\pi\left[2\int_0^{A}\rho\cos\rho^2 e^{-\rho^2}\right]$

$=\pi-\pi\left[\sin\rho^2\cdot e^{-\rho^2}\Big|_0^{A}+2\int_0^{A}\rho\sin\rho^2\cdot e^{-\rho^2}d\rho\right]$

$\therefore 2\pi y=\pi-\pi[2\cdot y]$

$\therefore y=\frac{1}{4}$

$\therefore$ 原式 $=\frac{\pi}{2}$

8. 设常数 a.b不全为 0. 证明:

$$\iint\limits_{x^2+y^2\le 1} f(ax+by+c)\,dx\,dy = 2\int_{-1}^{1}\sqrt{1-t^2}\, f(t\sqrt{a^2+b^2}+c)\,dt$$

证明: 作正交变换

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore I = \iint\limits_{u^2+v^2\le 1} f(u\sqrt{a^2+b^2}+c)\,du\,dv$$

$$= \int_{-1}^{1} du \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f(u\sqrt{a^2+b^2}+c)\,dv$$

$$= 2\int_{-1}^{1} f(u\sqrt{a^2+b^2}+c)\sqrt{1-u^2}\,du$$

$$= 2\int_{-1}^{1} f(t\sqrt{a^2+b^2}+c)\sqrt{1-t^2}\,dt$$



12. 计算累次积分

$$I = \int_0^1 dx \int_0^{1-x} dz \int_0^{1-z-x} (1-y)\, e^{-(1-y-z)^2}\, dy.$$

解:
$$\begin{cases} 0 \le x \le 1 \\ 0 \le z \le 1-x \\ 0 \le y \le 1-z-x \end{cases}$$

$$\therefore \begin{array}{l} 0 \le y \le 1 \\ 0 \le z \le 1-y \\ 0 \le x \le 1-z-y \end{array}$$

$$原式 = \int_0^1 dy \int_0^{1-y} dz \int_0^{1-y-z} (1-y)\, e^{-(1-y-z)^2}\, dx$$

$$= \int_0^1 dy \int_0^{1-y} (1-y)(1-y-z)\, e^{-(1-y-z)^2}\, dz$$

$$= \int_0^1 dy\,(1-y) \int_0^{1-y} \frac{[e^{-(1-y-z)^2}]'}{+2}\, dz$$

$$= \int_0^1 +\frac{(1-y)}{2}\, dy \cdot \left\{ e^{-(1-y-z)^2} \Big|_{z=0}^{z=1-y} \right\}$$

$$= \int_0^1 + \frac{(1-y)}{2} \cdot [1 - e^{-(1-y)^2}]\, dy$$

$$= +\frac{1}{2}\int_0^1 (1-y)\,dy - \frac{1}{2}\int_0^1 (1-y)\, e^{-(1-y)^2}\, dy$$

$$= +\frac{1}{2}\left(y - \frac{1}{2}y^2\right)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{[e^{-(1-y)^2}]'}{+2}\, dy$$

$$= +\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot(1-e^{-1})$$

$$= \frac{1}{4e}$$

$f(t)$ 在 $(-\infty, +\infty)$ 上连续，证明：

$$\int_0^1 dx \int_x^1 f(x)f(y)\,dy = \frac{1}{2}\left(\int_0^1 f(x)\,dx\right)^2$$

(2) $$\int_0^a dx \int_0^x dy \int_0^y f(x)f(y)f(z)\,dz = \frac{1}{6}\left(\int_0^a f(x)\,dx\right)^3$$

证明：（轮换对称性）

(1) $\because$ $0 \le x \le 1$

$x \le y \le 1$

$\Rightarrow$ $0 \le x, y \le 1$

$\therefore$ $\iint_D f(x)\,dxdy = \iint_D f(y)\,dx\,dy$

$\therefore$ $\int_0^1 dx \int_x^1 f(x)f(y)\,dy$

$= \int_0^1 dy \int_0^y f(x)f(y)\,dx$

$= \frac{1}{2}\left[\int_0^1 dx \int_0^x f(x)f(y)\,dy + \int_0^1 dy \int_0^y f(x)f(y)\,dx\right]$

$= \frac{1}{2}\int_0^1\int_0^1 f(x)f(y)\,dx\,dy$

$= \frac{1}{2}\left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 f(y)\,dy\right)$

$= \frac{1}{2}\left(\int_0^1 f(x)\,dx\right)^2$

(2) $0 \le x \le a$

$0 \le y \le x$ $\Rightarrow$ $0 \le x, y, z \le a$

$0 \le z \le y$

$\therefore$ $\int_0^a dx \int_0^x dy \int_0^y f(x)f(y)f(z)\,dz$

$\therefore$ $\iiint_D f(x)\,dxdydz = \iiint_D f(y)\,dxdydz$

$= \iiint f(z)\,dxdydz$

P13

$\therefore$ $\int_0^a dx \int_0^x dy \int_0^y f(x)f(y)f(z)\,dz$

$= \frac{1}{6}\Big(\int_0^a dx \int_0^x dy \int_0^y f(x)f(y)f(z)\,dz$

$+ \int_0^a dy \int_0^y dz \int_z^a f(x)f(y)f(z)\,dx$

$+ \int_0^a dz \int_y^a dx \int_z^x f(x)f(y)f(z)\,dy$

$+ \cdots + \cdots + \cdots\Big)$

$= \frac{1}{6}\int_0^a\int_0^a\int_0^a f(x)f(y)f(z)\,dv$

$= \frac{1}{6}\left(\int_0^a f(x)\,dx\right)^3$

Excellent!

P14