2020.6.8 的微积分A2 考试.

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1. 解: $\because$ $y=y(x)$, $z=z(x)$

又$\because$ $\begin{cases} x^3+y^3-z^3=10 & \cdots ① \\ x+y+z=0 & \cdots ② \end{cases}$

① 两边对 x 求导: $3x^2+3y^2\cdot y'(x)-3z^2\cdot z'(x)=0 \cdots ③$

② 两边对 x 求导: $1+y'(x)+z'(x)=0 \cdots ④$

又$\because$ $(1,1,-2) \Rightarrow x=1, y=1, z=-2$

$\therefore$ $\begin{cases} 3+3y'(x)-12z'(x)=0 \\ 1+y'(x)+z'(x)=0 \end{cases}$

得: $\begin{cases} y'(1)=-1 \\ z'(1)=0 \end{cases}$

2. 解：$z = f(x^2+xy+y^2)$ 在 $(1,1)$ 处，设 $u = x^2+xy+y^2$

$= 1+1+1 = 3$

$$\frac{\partial z}{\partial y} = f'(u) \times \frac{\partial u}{\partial y}$$

$$= f'(u) \times (x+2y)$$

$$= f'(3) \times 3 = 3f'(3)$$

又∵ $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[f'(u)(x+2y)]$$

$$= \frac{\partial f'(u)}{\partial x}(x+2y) + f'(u) \times 1$$

$$= f''(u) \times (2x+y)(x+2y) + f'(u)$$

$$= f''(3) \times 3 \times 3 + f'(3)$$

$$= f'(3) + 9f''(3)$$

3. 解:

$$u=(\sin x)(\sin y)(\sin z)$$

$$st:\ x+y+z=\frac{\pi}{2}$$

$$x>0,\ y>0,\ z>0.$$

设 $L=(\sin x)(\sin y)(\sin z)-\lambda(x+y+z-\frac{\pi}{2})$

$$\therefore \begin{cases} L_x=(\cos x)(\sin y)(\sin z)-\lambda=0 \\ L_y=(\cos y)(\sin x)(\sin z)-\lambda=0 \\ L_z=(\cos z)(\sin x)(\sin y)-\lambda=0 \\ x+y+z=\frac{\pi}{2} \quad 且\ x,y,z>0. \end{cases}$$

$$\therefore \begin{cases} \cos x\sin y=\cos y\sin x \Rightarrow \sin(x-y)=0 \\ \cos y\sin z=\cos z\sin y \Rightarrow \sin(y-z)=0 \\ \cos x\sin z=\cos z\sin x \Rightarrow \sin(x-z)=0. \end{cases}$$

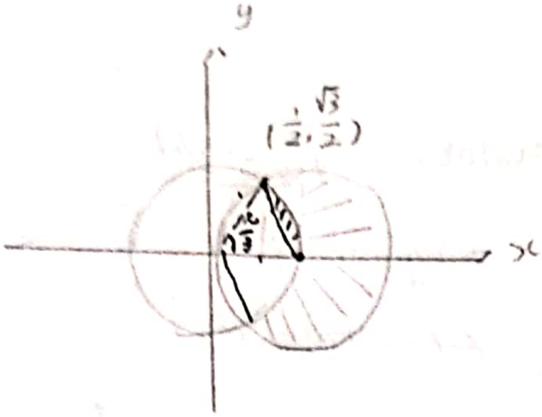
$$\therefore\ x=y=z=\frac{1}{6}\pi.$$

$$u=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

$$又\because\ H=\begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix}=\begin{vmatrix} -\frac{1}{8}, & \frac{3}{8}, & \frac{3}{8} \\ \frac{3}{8} & ,-\frac{1}{8}, & \frac{3}{8} \\ \frac{3}{8} & ,\frac{3}{8} & ,-\frac{1}{8} \end{vmatrix}\ 负定.$$

$$\therefore\ 极大值为\ \frac{1}{8}$$

④ 解：



如右图：

交点 $(\frac{1}{2},\frac{\sqrt{3}}{2})$

$(\frac{1}{2},-\frac{\sqrt{3}}{2})$

设 S_1： $x^2+y^2\leq 2x$ 即： $\rho\leq 2\cos\theta$ ，$\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

S_2： $\rho=1$，$-\frac{\pi}{3}\leq\theta\leq\frac{\pi}{3}$

$$\iint_{S_1}\left|\frac{y}{x}\right|dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}|\tan\theta|\,\rho\,d\rho$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{4\cos^2\theta}{2}|\tan\theta|\,d\theta$$

$$=2\int_{-\frac{\pi}{2}}^{0}(-\cos^2\theta\tan\theta)d\theta+2\int_0^{\frac{\pi}{2}}\cos^2\theta\tan\theta\,d\theta$$

$$=-2\int_{-\frac{\pi}{2}}^{0}\cos\theta\sin\theta\,d\theta+2\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,d\theta$$

$$=2$$

$$S_3:\ \iint_{S_3}\left|\frac{y}{x}\right|dxdy=\iint_{S_3}\left|\frac{\sin\theta}{1+\cos\theta}\right|dxdy=\iint=\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}d\theta\int_0^1|\ \ |\rho\,d\rho$$

$$=\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left|\frac{\sin\theta}{1+\cos\theta}\right|\cancel{dxdy}\,d\theta=\ln\frac{4}{3}$$

$$\therefore\ \iint_D\left|\frac{y}{x}\right|dxdy=\ln\frac{4}{3}-\Big[\underbrace{\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}d\theta\int_0^1|\tan\theta|\rho\,d\rho}_{}^{\ln 2}-2\underset{\triangle}{\iint}\Big]$$

$$=\frac{7}{4}\ln 2-\ln 3$$

5. 解：

(1) $\because$ $X = \dfrac{-x}{x^2+2y^2}$，$Y = \dfrac{-y}{x^2+2y^2}$.

$$\therefore X_x = \frac{-1\times(x^2+2y^2)-(-x)\times 2x}{(x^2+2y^2)^2} = \frac{x^2-2y^2}{(x^2+2y^2)^2}$$

$$Y_y = \frac{-1\times(x^2+2y^2)-(-y)\times 4y}{(x^2+2y^2)^2} = \frac{-x^2+2y^2}{(x^2+2y^2)}$$

$\therefore$ $X_x + Y_y = 0$.

故：与路径无关.

设 $x^2+2y^2 = \varepsilon^2(\varepsilon>0)$，则 $\int_{(A)}^{(B)} = -2\pi$.

(2) 存在 $Z(x,y) = \dfrac{\sqrt{2}}{2}\arctan\dfrac{x}{\sqrt{2}y} + C$（C为常数）

设 $Z = Z(x,y)$，$dz = \dfrac{y\,dx}{x^2+2y^2} - \dfrac{x\,dy}{x^2+2y^2}$

$\therefore$ $Z_x = \dfrac{y}{x^2+2y^2}$，$Z_y = \dfrac{-x}{x^2+2y^2}$

$$\therefore Z(x,y) = \int \frac{y}{x^2+2y^2}\,dx$$

$$= \int \frac{1}{2y\left(\frac{x^2}{2y^2}+1\right)}\,dx$$

$$= \frac{\sqrt{2}y}{2y}\cdot\int \frac{1}{\left(\frac{x}{\sqrt{2}y}\right)^2+1}\,d\frac{x}{\sqrt{2}y}$$

$$= \frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}y} + C$$

又$\because$ $Z_y = \dfrac{\sqrt{2}}{2}\times\dfrac{1\times\frac{x}{\sqrt{2}}\times\left(\frac{1}{-y^2}\right)}{1+\frac{x^2}{2y^2}} + C_y$

$$= \frac{1}{2}\,\frac{-x}{y^2+\frac{x^2}{2}} + C_y$$

$$= \frac{-x}{x^2+2y^2} + C_y \Rightarrow C_y = 0$$

故：$Z(x,y) = \dfrac{\sqrt{2}}{2}\arctan\dfrac{x}{\sqrt{2}y} + C$

（C为常数）

6. 解: 作球坐标变换:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$r\in[0,1]$, $\theta\in[0,\pi]$, $\varphi\in[0,2\pi]$

$$\therefore \text{原式} = \int_0^1 d\theta\int_0^{2\pi} d\varphi\int_0^1 \sqrt{1-r^3}\cdot r^2\sin\theta\, dr$$

$$= \int_0^1 d\theta\int_0^{2\pi} d\varphi\int_0^1 \frac{\sqrt{1-r^3}}{3}\sin\theta\, dr^3$$

$$= 2\pi\times(-\cos\theta)\Big|_0^1 \times\int_0^1 \frac{\sqrt{1-a}}{3}\, da$$

$$= \frac{2\pi}{3}\times(1-\cos1)\times\frac{(1-a)^{\frac{3}{2}}}{-\frac{3}{2}}\Big|_0^1$$

$$= \frac{2\pi}{3}\times(1-\cos1)\times\frac{2}{3} = \frac{4\pi}{9}(1-\cos1)$$

7. 解:

$$L^{+}: \begin{cases} x^2+y^2+z^2=2ax \\ x^2+y^2=2x \end{cases} \quad (z \geq 0).$$

由 stokes 公式:

$$X=y^2+z^2, \quad Y=z^2+x^2, \quad Z=x^2+y^2.$$

$$\therefore \quad Z_y-Y_z=2y-2z$$

$$X_z-Z_x=2z-2x$$

$$Y_x-X_y=2x-2y$$

$$\text{又}\because \quad \vec{n}=\frac{(x-a,\ y,\ z)}{\sqrt{2ax}}$$

$$\therefore \int_{L^+}=\iint_S \frac{1}{\sqrt{2ax}}\Big[\underbrace{(2y-2z)(x-a)+(2z-2x)y+(2x-2y)z}_{2a(z-y)}\Big]ds$$

$$\text{又}\because \quad ds=\sqrt{1+z_x^2+z_y^2}\,dx\,dy \qquad = \iint_{D_{xy}} \frac{1}{\sqrt{2ax}}\times 2az\times\frac{a}{z}\,dxdy$$

S在xy平面: $(x-1)^2+y^2=1 \Rightarrow \rho=2\cos\theta$

$$\text{故: 原式}=\sqrt{2}a^{\frac{3}{2}}\iint_{D_{xy}}\frac{1}{\sqrt{x}}dxdy=\sqrt{2}a^{\frac{3}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}\frac{1}{\sqrt{\rho\cos\theta}}\rho\,d\rho$$

$$=\frac{16}{3}a^{\frac{3}{2}}$$

8. 解：

(1) $T=2\pi$, $f(x)=\begin{cases} 0, & -\pi<x\le 0 \\ x, & 0<x\le\pi \end{cases}$

$\therefore a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)=\frac{1}{\pi}\times\int_0^{\pi} x\,dx=\frac{1}{2}\pi$

$a_n=\frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$

$=\frac{1}{\pi}\times\int_0^{\pi} x\cdot[\frac{\sin nx}{n}]'dx$

$=\frac{1}{\pi}\times\frac{1}{n}\cdot[x\sin nx\Big|_0^{\pi}-\int_0^{\pi}\sin nx\,dx]$

$=\frac{1}{n\pi}\quad[0+\frac{\cos nx}{n}\Big|_0^{\pi}]=\frac{1}{n^2\pi}[(-1)^n-1]$

$b_n=\frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx=\frac{1}{\pi}\times\int_0^{\pi} x\cdot[-\frac{\cos nx}{n}]'dx$

$=-\frac{1}{n\pi}\quad(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx\,dx]$

$=\frac{1}{-n\pi}\times\pi\times(-1)^n=\frac{(-1)^{n+1}}{n}$

得: $f(x)\sim\frac{1}{4}\pi+\sum_{n=1}^{+\infty}\{\frac{1}{\pi n^2}[(-1)^n-1]\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\}$

(2) 由(1)可知: $f(x)\sim\frac{1}{4}\pi+\sum_{n=1}^{+\infty}\{\frac{1}{\pi n^2}[(-1)^n-1]\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\}$

设 $x=0$, $n'=2n-1$

$\therefore f(0)=\frac{1}{4}\pi+\sum_{n=1}^{+\infty}\frac{1}{\pi(2n-1)^2}(-2)$

$=\frac{1}{4}\pi-\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}\Rightarrow\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

$\therefore\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}=\frac{1}{8}\pi^2$

9. 证明： $\cos\langle\vec{r},\vec{n}\rangle = \dfrac{\vec{r}\cdot\vec{n}}{r}$

由 Gauss 公式：

$$\text{左边} = \frac{1}{2}\iint_{\partial\Omega} \cos\langle\vec{r},\vec{n}\rangle \, ds$$

$$= \frac{1}{2}\iint_{\partial\Omega^+} \frac{1}{r} \quad (x,y,z)\cdot\vec{n} \, ds$$

又∵ $r = (x^2+y^2+z^2)^{\frac{1}{2}}$

且 $\vec{F} = \frac{1}{r}(x,y,z)$, $\quad , \operatorname{div}\vec{F} = 0$.

$$\therefore \quad \text{左边} = \frac{1}{2}\iint_{\partial\Omega^+} \frac{1}{r}(x,y,z)\,\vec{n}\,ds$$

$$= \lim_{\varepsilon\to 0^+} \iiint_{\Omega} \frac{1}{r}\,dxdydz$$

则原命题成立

10. 证明:

(1) $\because \sum_{n=0}^{\infty} a_n n!$ 收敛

且一般项为 $u_n = a_n n!$

$\therefore \lim_{n \to +\infty} a_n n! = 0$

$\therefore \lim_{n \to +\infty} a_n = 0$

$\therefore \lim_{n \to +\infty} \sqrt[n]{a_n} = 0$

$\therefore R = \frac{1}{\sqrt[n]{a_n}} = +\infty$ 成立.

(2) 判断 $\sum_{n=0}^{+\infty} e^{x} f(x)$ 的敛散性

由(1) $\because R = +\infty$.

$\therefore f(x)$ 在R上收敛

又$\because$ e^{-x} 对任意固定 x 都单调 $\downarrow \to 0$

由 Abel 判别法: $\sum_{n=0}^{+\infty} e^{-x} f(x)$ 收敛

$\Rightarrow \int_0^{+\infty} e^{-x} f(x) dx$ 收敛.

由 $f(x) = \sum_{n=0}^{+\infty} a_n x^n$

$$\int_0^{+\infty} e^{-x} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n \int_0^{+\infty} e^{-x} x^n dx$$

$$= \sum_{n=0}^{+\infty} a_n n! \text{ 成立}$$