第十次作业参考解答

《高等微积分教程（上）》

习题 5.7

§5.7

2.(5) 求三叶线 $\rho=a\sin 3\theta\ (a>0)$ 所围成图形的面积.

$x=\rho\cos\theta=a\sin 3\theta\cos\theta,\ y=\rho\sin\theta=a\sin 3\theta\sin\theta$

$S=3\int_{0}^{\frac{\pi}{3}}\frac{1}{2}\rho^2(\theta)\,d\theta=3\int_{0}^{\frac{\pi}{3}}\frac{1}{2}a^2\sin^2 3\theta\,d\theta=\frac{3}{2}\int_0^{\frac{\pi}{3}}a^2\sin^2 3\theta\,d\theta=\frac{3}{2}\int_0^{\frac{\pi}{3}}a^2\left(\frac{1-\cos 6\theta}{2}\right)d\theta$

$=\frac{3a^2}{4}\int_0^{\frac{\pi}{3}}(1-\cos 6\theta)\,d\theta=\frac{3a^2}{4}\left(\theta-\frac{1}{6}\sin 6\theta\right)\Big|_0^{\frac{\pi}{3}}=\frac{a^2}{4}\pi.$

3.(1) 求曲线 $y=\int_{-\frac{\pi}{2}}^{x}\sqrt{\cos t}\,dt\ (x\in[-\frac{\pi}{2},\frac{\pi}{2}])$ 的弧长.

$y'=\sqrt{\cos x}\Rightarrow l=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1+\cos x}\,dx=2\int_0^{\frac{\pi}{2}}\sqrt{2\cos^2\frac{x}{2}}\,dx=2\sqrt{2}\int_0^{\frac{\pi}{2}}\cos\frac{x}{2}\,dx=4\sqrt{2}\int_0^{\frac{\pi}{2}}\cos\frac{x}{2}\,d\frac{x}{2}$

$=4\sqrt{2}\sin\frac{x}{2}\Big|_0^{\frac{\pi}{2}}=4\sqrt{2}\left(\frac{1}{\sqrt{2}}-0\right)=4.$

7.(4) 求圆 $x^2+(y-b)^2=a^2\ (b>a>0)$ 绕x轴~~旋转~~生成的旋转体的体积.

$y_1=b+\sqrt{a^2-x^2},\ y_2=b-\sqrt{a^2-x^2}.$

$V=\pi\int_{-a}^{a}y_1^2\,dx-\pi\int_{-a}^{a}y_2^2\,dx=4b\pi\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=4b\pi\times\frac{\pi}{2}a^2=2\pi^2a^2b$

☆8.(2) 求旋转轮线 $x=a(t-\sin t),\ y=a(1-\cos t)$ 在 $0\le t\le 2\pi$ 的部分,绕直线 $x=a\pi$ 旋转生成的旋转面的表面积.

$\Rightarrow$ $x=a(t-\sin t-\pi),\ y=a(1-\cos t)$ 绕x轴旋转一周的表面积.

$\Rightarrow S=2\pi\int_0^{2\pi}a\,\overset{|\pi-t+\sin t|}{\cancel{(1-\cos t)}}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt=2\pi a^2\int_0^{2\pi}\overset{|\pi-t+\sin t|}{\cancel{(1-\cos t)}}\sqrt{2(1-\cos t)}\,dt$

$\cancel{=2\sqrt{2}\pi a^2\int_0^{2\pi}(\sqrt{1-\cos t})^3\,dt=2\sqrt{2}\pi a^2\int_0^{2\pi}2\sqrt{2}\left|\sin^3\frac{t}{2}\right|dt=8\pi a^2\int_0^{2\pi}\left|\sin^3\frac{t}{2}\right|dt}$

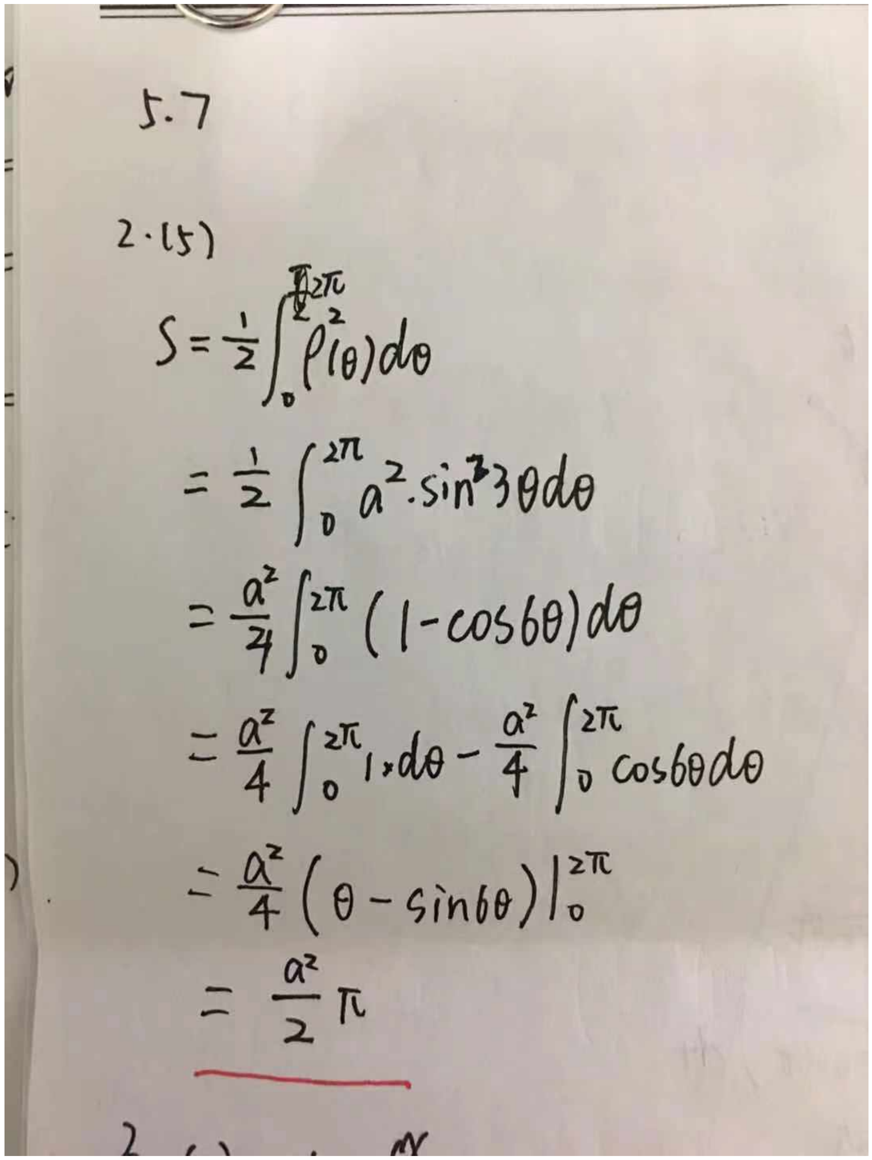
$\cancel{=8\pi}=4\pi a^2\int_0^{\pi}(\pi-t+\sin t)\sin\frac{t}{2}\,dt\overset{\varphi=\pi-t}{=\!=}4\pi a^2\int_0^{\pi}(\varphi+\sin\varphi)\cos\frac{\varphi}{2}\,d\varphi$

$\int_0^{\pi}\varphi\cos\frac{\varphi}{2}\,d\varphi+\int_0^{\pi}\sin\varphi\cos\frac{\varphi}{2}\,d\varphi=2\varphi\sin\frac{\varphi}{2}\Big|_0^{\pi}-\int_0^{\pi}2\sin\frac{\varphi}{2}\,d\varphi+2\int_0^{\pi}\sin\frac{\varphi}{2}\cos^2\frac{\varphi}{2}\,d\varphi$

$=2\pi+4\cos\frac{\varphi}{2}\Big|_0^{\pi}-4\int_0^{\pi}\cos^2\frac{\varphi}{2}\,d\cos\frac{\varphi}{2}=2\pi-4-\frac{4}{3}\cos^3\frac{\varphi}{2}\Big|_0^{\pi}=2\pi-\frac{8}{3}=2\left(\pi-\frac{4}{3}\right)$

$\Rightarrow S=4\pi a^2\cdot 2\left(\pi-\frac{4}{3}\right)=8\pi a^2\left(\pi-\frac{4}{3}\right)$

错误：



5.7

2.(5)

$$S=\frac{1}{2}\int_0^{\cancel{\frac{\pi}{2}}2\pi}\rho^2(\theta)d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}a^2\cdot\sin^2 3\theta d\theta$$

$$=\frac{a^2}{4}\int_0^{2\pi}(1-\cos 6\theta)d\theta$$

$$=\frac{a^2}{4}\int_0^{2\pi}1\cdot d\theta-\frac{a^2}{4}\int_0^{2\pi}\cos 6\theta d\theta$$

$$=\frac{a^2}{4}(\theta-\sin 6\theta)\Big|_0^{2\pi}$$

$$=\frac{a^2}{2}\pi$$

错误原因：注意此曲线当 $\rho \geq 0$ 时才有意义，由此可得出对 θ 应在何范围内进行积分。

☆9.12) 求星形线 $x=a\cos^3 t$, $y=a\sin^3 t$ 第一象限的部分(线密度 ρ 为常数)的质心

$$m=\rho S=\rho\int_0^{\frac{\pi}{2}}\sqrt{x'(t)^2+y'(t)^2}\,dt=\rho\sqrt{(-3a\sin t\cos^2 t)^2+(3a\cos t\sin^2 t)^2}\,dt$$

$$=3a\rho\int_0^{\frac{\pi}{2}}\sin t\cos t\,dt=\frac{3}{2}\int_0^{\frac{\pi}{2}}d\sin^2 t=\frac{3}{2}a\rho$$

$$\bar{x}=\frac{1}{m}\int_0^{\frac{\pi}{2}}a\cos^3 t\cdot\rho\sin t\cos t\,dt=-2a\int_0^{\frac{\pi}{2}}\cos^4 t\,d\cos t=2a\rho\cdot\frac{1}{5}\cos^5 t\Big|_0^{\frac{\pi}{2}}=\frac{2}{5}a$$

$$\bar{y}=\frac{1}{m}\cdot 3a\int_0^{\frac{\pi}{2}}a\sin^3 t\cdot\rho\sin t\cos t\,dt=2a\int_0^{\frac{\pi}{2}}\sin^4 t\,d\sin t$$

$$=2a\cdot\frac{1}{5}\sin^5 t\Big|_0^{\frac{\pi}{2}}=\frac{2}{5}a$$

$\Rightarrow$ 质心为 $(\frac{2}{5}a,\frac{2}{5}a)$

习题 6.1

§6.2

2、(1) $\int_1^{+\infty}\frac{1}{x(x+1)}dx=\lim\limits_{A\to+\infty}\int_1^A\left(\frac{1}{x}-\frac{1}{x+1}\right)dx=\lim\limits_{A\to+\infty}\left(\ln|x|\Big|_1^A-\ln|x+1|\Big|_1^A\right)=$

$=\cancel{\lim\limits_{A\to+\infty}\ln\left(1-\frac{1}{A+1}\right)=0.}\ \lim\limits_{A\to+\infty}\ln 2\left(1-\frac{1}{A+1}\right)=\ln 2$

(3) $\int_1^{+\infty}\frac{1}{x(1+x^2)}dx=\lim\limits_{A\to+\infty}\int_1^A\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$

$\int_1^A\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx=\ln|x|\Big|_1^A-\cancel{x\arctan x\Big|_1^A}+\frac{1}{2}\int_1^A\frac{dx^2}{x^2+1}$

$=\ln|x|\Big|_1^A-\frac{1}{2}\ln(x^2+1)\Big|_1^A=\ln A-\frac{1}{2}\ln(A^2+1)+\frac{1}{2}\ln 2.$

$=\frac{1}{2}\ln\frac{2A^2}{A^2+1}=\frac{1}{2}\ln\left(2-\frac{2}{A^2+1}\right)$

$\lim\limits_{A\to+\infty}\int_1^A\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx=\lim\limits_{A\to+\infty}\frac{1}{2}\ln\left(2-\frac{2}{A^2+1}\right)=\frac{1}{2}\ln 2$

(8) $\int_1^e\frac{dx}{x\sqrt{1-\ln^2x}}=\int_1^e\frac{d\ln x}{\sqrt{1-\ln^2x}}=\int_0^1\frac{dt}{\sqrt{1-t^2}}=\arcsin t\Big|_0^1=\frac{\pi}{2}.$

3、(2) $\int_0^{+\infty}\frac{\arctan x}{(1+x^2)^{\frac{3}{2}}}dx\ \overset{\tan y=x}{=\!=\!=}\ \int_0^{\frac{\pi}{2}}y|\cos y|dy=y\sin y\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin y\,dy$

$=\frac{\pi}{2}+\cos y\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}-1.$

☆(5) $\int_{-1}^1\sqrt{\frac{1-x}{1+x}}dx\ \overset{t=\sqrt{\frac{1-x}{1+x}}}{=\!=\!=}\ \int_0^{+\infty}\frac{4t^2}{(t^2+1)^2}dt=-\frac{2t}{t^2+1}\Big|_0^{+\infty}+2\int_0^{+\infty}\frac{1}{t^2+1}dt=\frac{-2t}{t^2+1}\Big|_0^{+\infty}+2\arctan t\Big|_0^{+\infty}$

$=(0-0)+\left(2\times\frac{\pi}{2}-0\right)=\pi$

(6) $\int_0^1\frac{dx}{(2+x)\sqrt{1-x}}\ \overset{t=\sqrt{1-x}}{=\!=\!=}\ \int_1^0\frac{-2t\,dt}{(3-t^2)t}=\int_1^0\frac{2}{t^2-3}dt=\frac{1}{\sqrt3}\int_1^0\left(\frac{1}{t-\sqrt3}-\frac{1}{t+\sqrt3}\right)dt$

$=\frac{1}{\sqrt3}\left[\ln|t-\sqrt3|-\ln|t+\sqrt3|\right]\Big|_1^0=\frac{1}{\sqrt3}\ln\frac{\sqrt3+1}{\sqrt3-1}$