清华大学2021春季学期

电路原理C

第16讲

谐振

内容

1 谐振

2 谐振电路的品质因数

本讲重难点

- 求谐振频率
- 求谐振入端电阻
- 定性画LC一端口频率特性

1、谐振 (resonance)

resonance

The increase in amplitude of oscillation of an electric or mechanical system exposed to a periodic force whose frequency is equal or very close to the natural undamped frequency of the system.

19世纪的
垮桥悲剧
法、德、俄

Tacoma大桥垮塌事件

Washington, USA 1980 米长

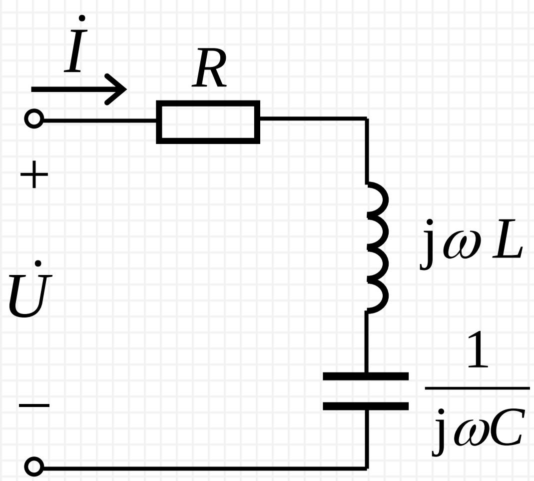
July 1, 1940 ～ November 7, 1940

虎门大桥1997年6月9日建成通车，全长15.76千米，主桥全长4.6千米，桥面为双向六车道高速公路，设计速度120千米/小时。2020年5月5日发生竖向弯曲振动，5月15日恢复通车

(1) 电路中谐振的定义

当 ω，L，C 满足一定条件，恰好使一端口网络的端口电压、电流出现同相位。一端口网络的这种状态称为**谐振**。

*RLC*串联



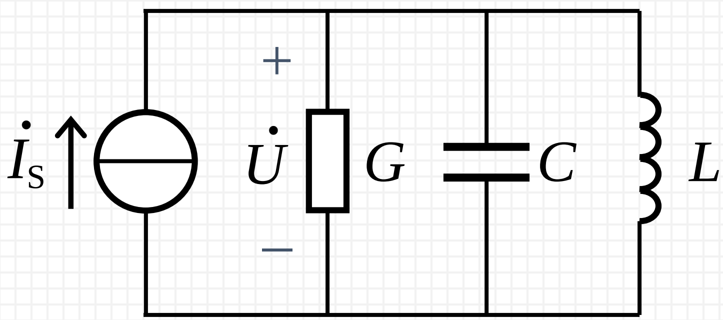
$$Z = R + \mathrm{j}(\omega L - \frac{1}{\omega C})$$

$\omega L > \dfrac{1}{\omega C}$ 感性

$\omega L < \dfrac{1}{\omega C}$ 容性

串联谐振 $\omega L = \dfrac{1}{\omega C}$ **阻性**

RLC 并联



$$Y = G + \mathrm{j}(\omega C - \frac{1}{\omega L})$$

$\omega C > \frac{1}{\omega L}$ 容性

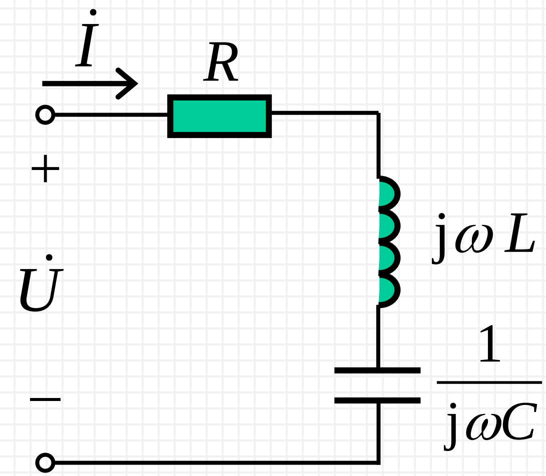
$\omega C < \frac{1}{\omega L}$ 感性

并联谐振 $\omega C = \frac{1}{\omega L}$ **阻性**

(1) RLC串联谐振

(a) 串联谐振的谐振条件和谐振时端口入端电阻

① L、C 不变，改变 ω，使 $X_L=|X_C|$



谐振时 $$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

谐振角频率 (*resonant angular frequency*)

$$\boxed{Z_0 = R}$$

谐振时端口入端阻抗(入端电阻)

② 电源频率不变，改变 L 或 C (常改变C)，使 $X_L=|X_C|$ 。

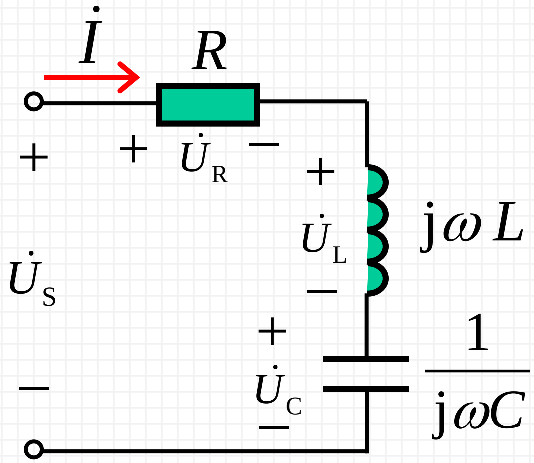
(b) 串联谐振时的电压和电流

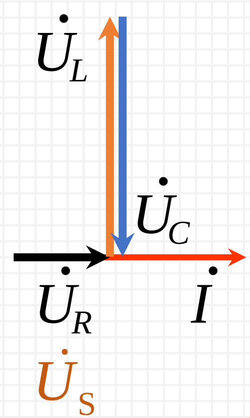
$$\dot{U}_R = R\dot{I} = \dot{U}_S \qquad \dot{I} = \frac{\dot{U}_S}{R}$$

$$\dot{U}_L = \mathrm{j}\omega_0 L\dot{I} = \mathrm{j}\frac{\omega_0 L}{R}\dot{U}_S$$

$$\dot{U}_C = \frac{\dot{I}}{\mathrm{j}\omega_0 C} = -\mathrm{j}\frac{1}{\omega_0 CR}\dot{U}_S$$

$$\omega_0 L = \frac{1}{\sqrt{LC}}L = \sqrt{\frac{L}{C}} = \frac{1}{\omega_0 C}$$

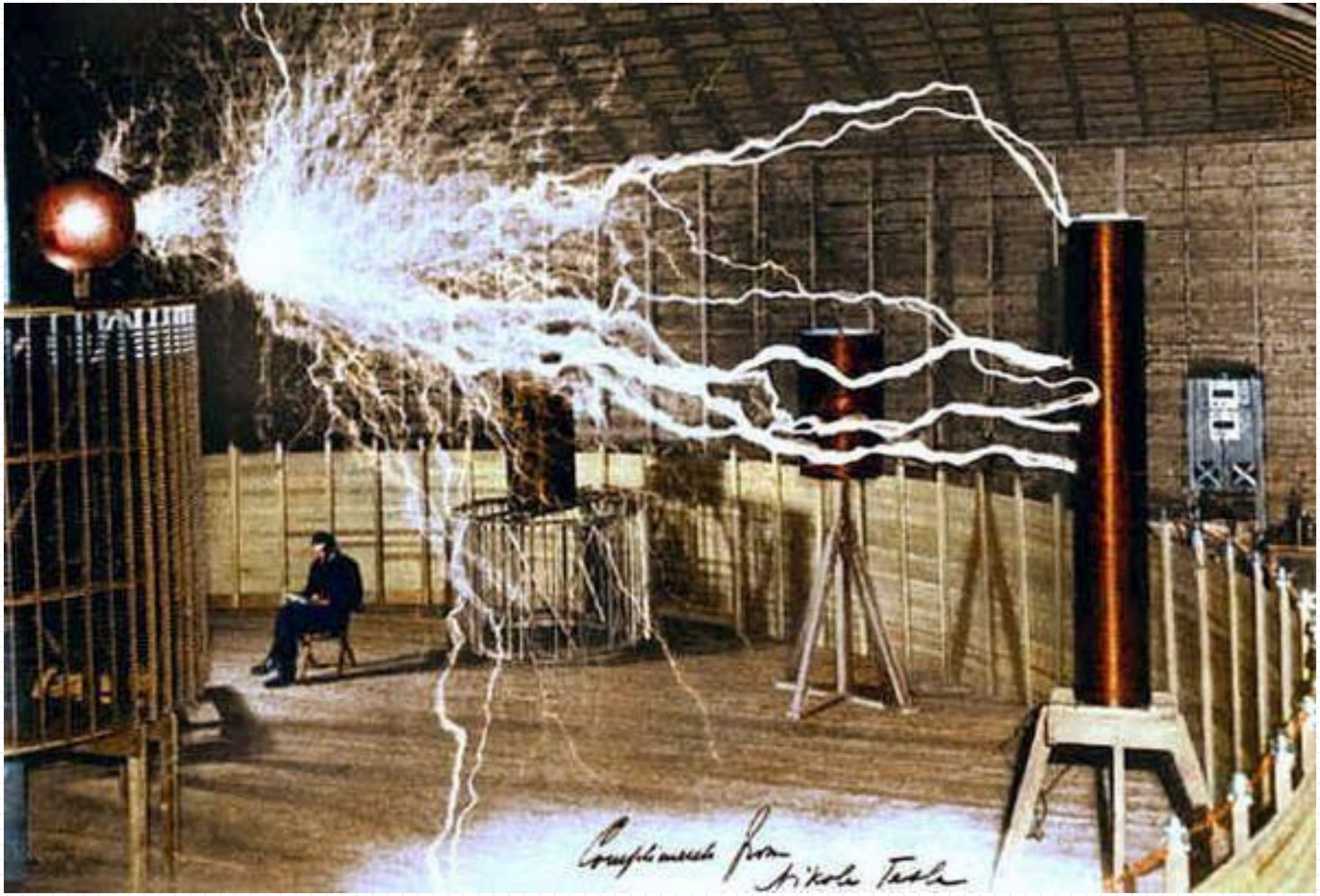




L 和 C 上可能出现比端口电压更高的电压

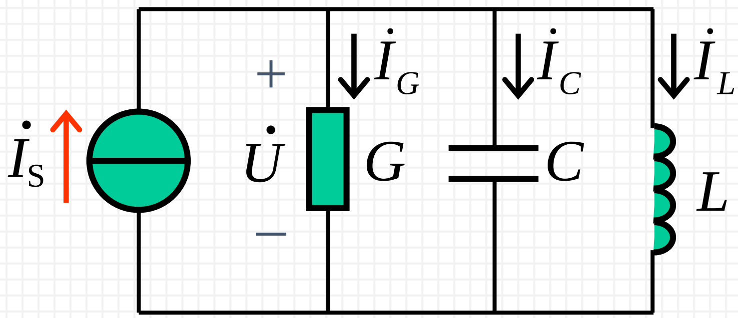
谐振时的相量图

串联谐振又称**电压谐振**



尼古拉.特斯拉在他简陋的人工闪电实验室闪电弧光下阅读

(2) GCL并联谐振



$$Y = G + \mathrm{j}(\omega C - \frac{1}{\omega L})$$

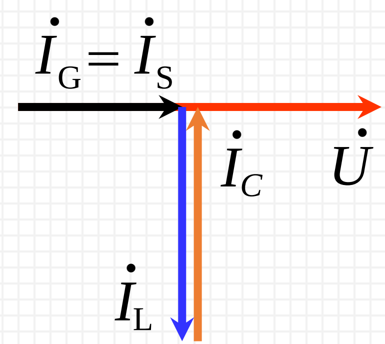
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Z_0 = 1/G$$

$$\dot{I}_G = G\dot{U} = \dot{I}_S \qquad \dot{U} = \frac{\dot{I}_S}{G}$$

$$\dot{I}_L = \frac{\dot{U}}{\mathrm{j}\omega_0 L} = -\mathrm{j}\frac{1}{\omega_0 LG}\dot{I}_S$$

$$\dot{I}_C = \mathrm{j}\omega_0 C\dot{U} = \mathrm{j}\frac{\omega_0 C}{G}\dot{I}_S$$



L 和 C 上可能出现比端口电流更大的电流

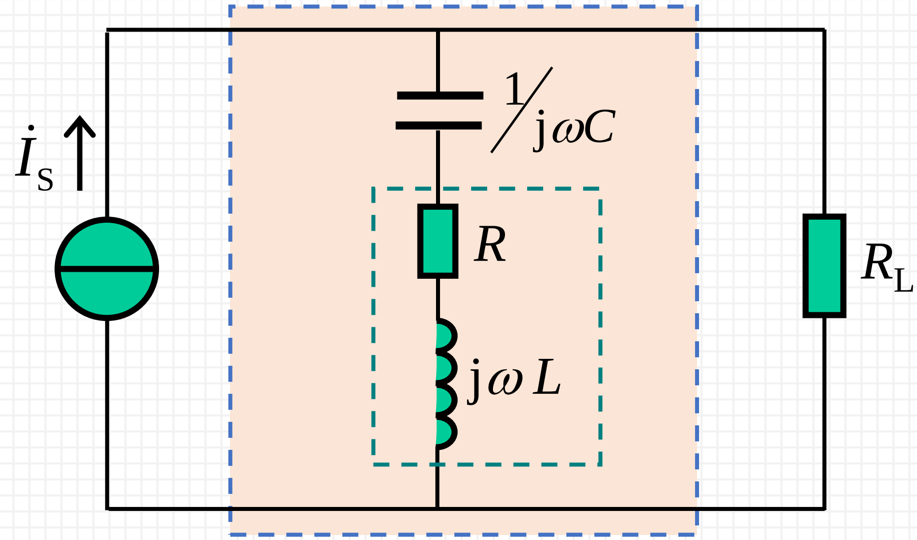
谐振时的相量图

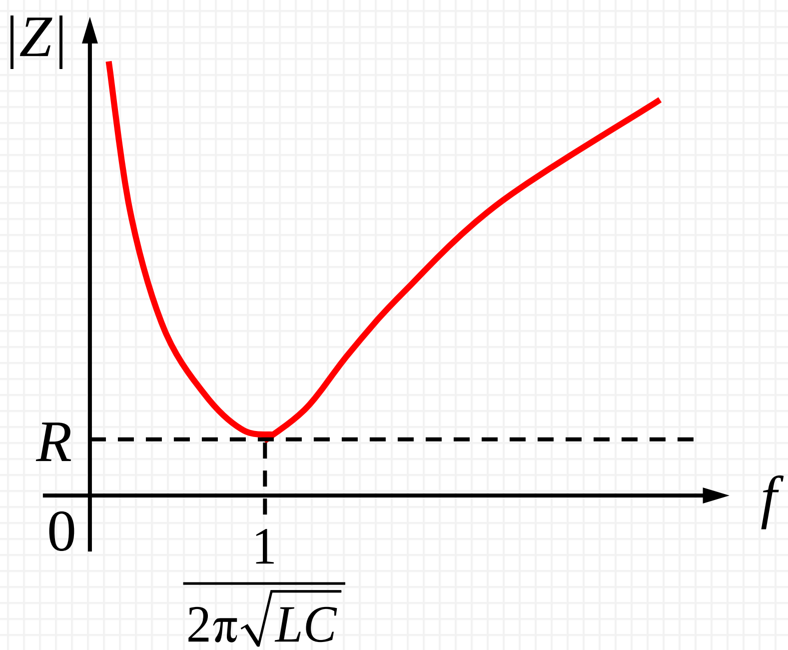
并联谐振又称**电流谐振**

(4) 谐振可视为某种滤波器

电力谐振滤波器

$$Z = R + \mathrm{j}(\omega L - \frac{1}{\omega C})$$

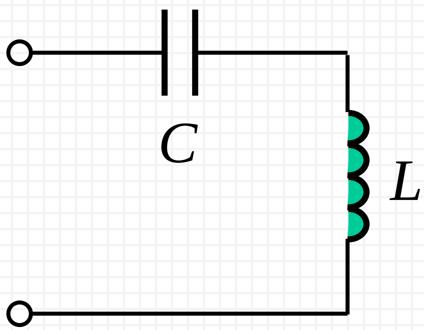




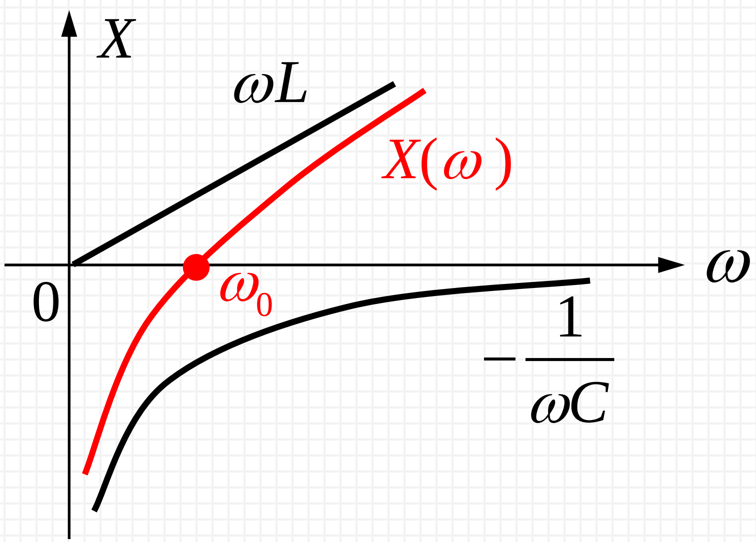
带阻滤波器

(5) LC谐振电路

(a) 串联谐振

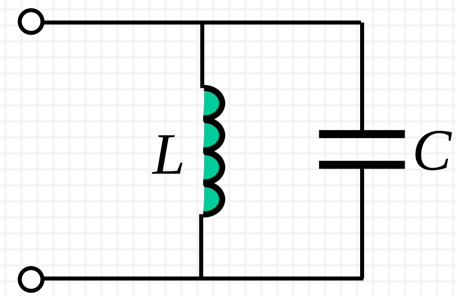


$$\mathrm{j}X = \mathrm{j}(\omega L - \frac{1}{\omega C})$$



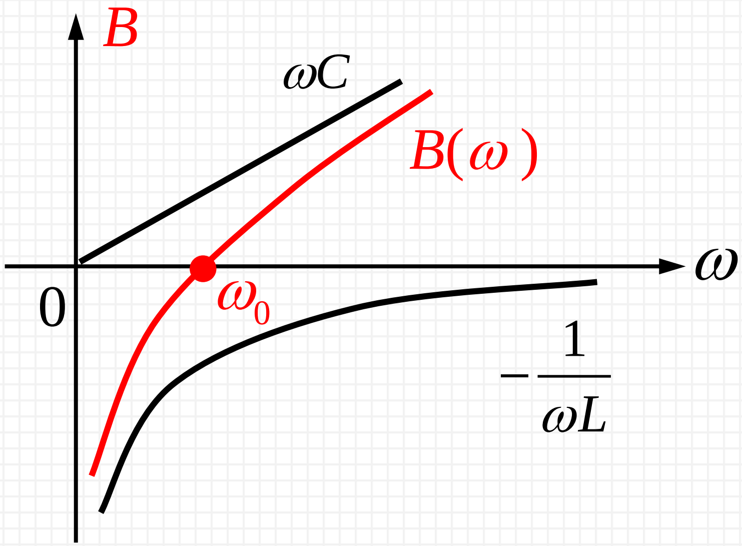
$\omega = \omega_0$ 时，端口相当于短路

(b) 并联谐振

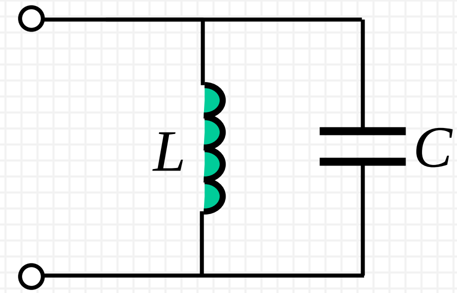


$$jB = \frac{1}{j\omega L} + j\omega C = j(\omega C - \frac{1}{\omega L})$$

$$jX = \frac{1}{jB} = j(-\frac{1}{B})$$

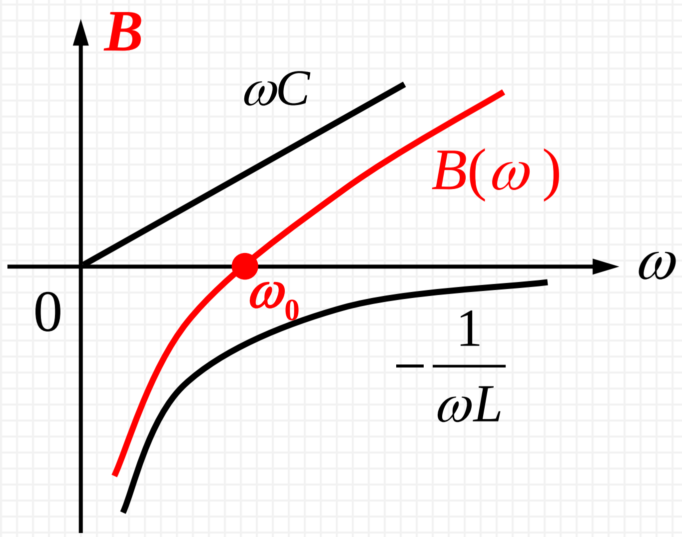


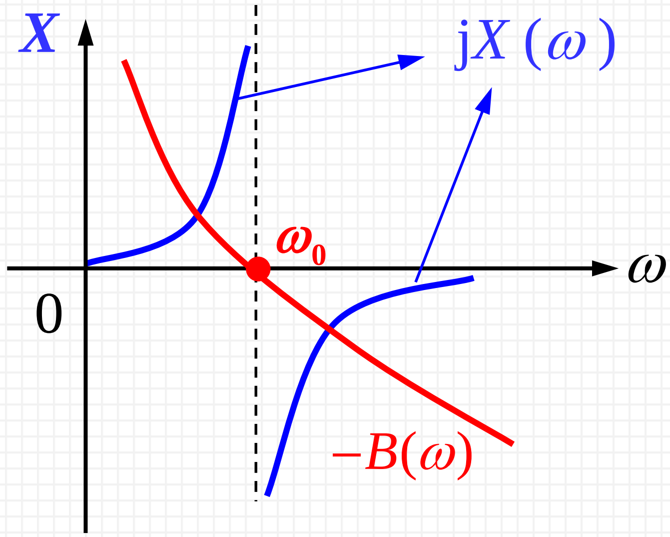
(b) 并联谐振



$$\mathrm{j}B = \frac{1}{\mathrm{j}\omega L} + \mathrm{j}\omega C = \mathrm{j}(\omega C - \frac{1}{\omega L})$$

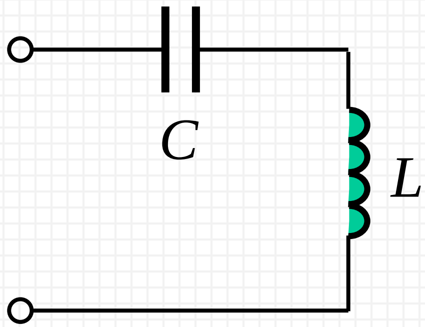
$$\mathrm{j}X = \frac{1}{\mathrm{j}B} = \mathrm{j}(-\frac{1}{B})$$

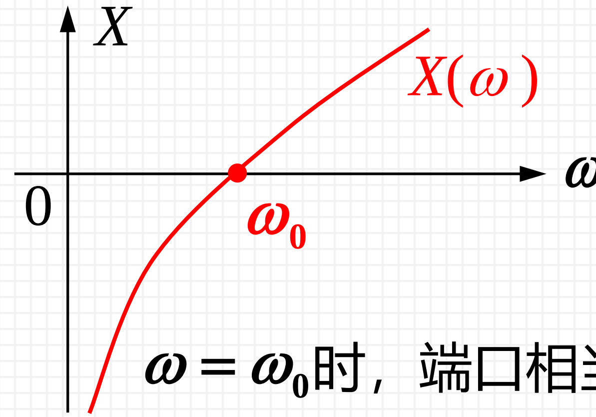




$\omega = \omega_0$ 时，端口相当于开路

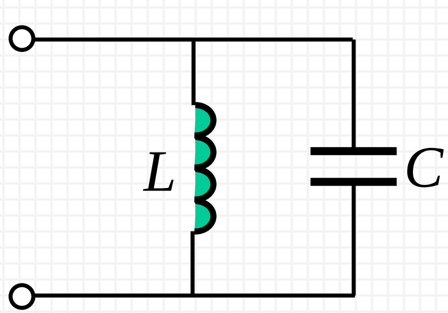
LC串联谐振

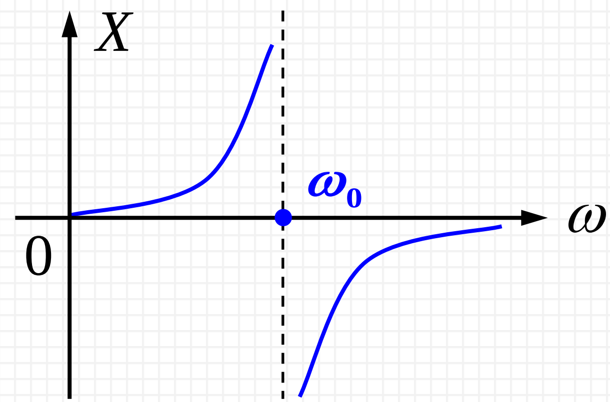




$\omega = \omega_0$时，端口相当于短路

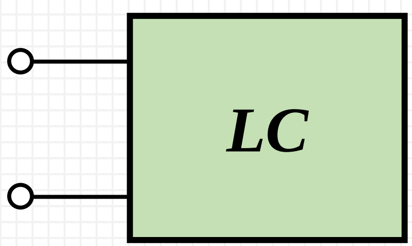
LC并联谐振





$$Z(\omega) = \mathrm{j}\frac{f_1(\omega)}{f_2(\omega)}$$

$\omega = \omega_0$时，端口相当于开路

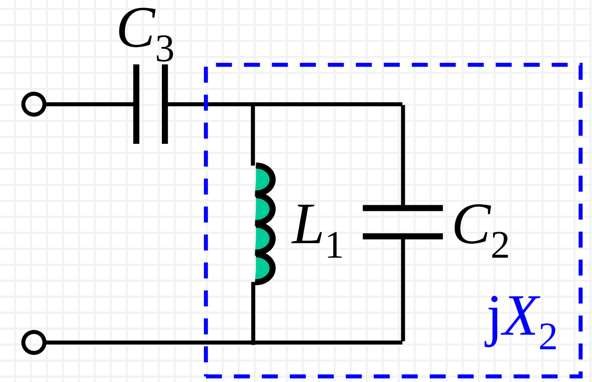


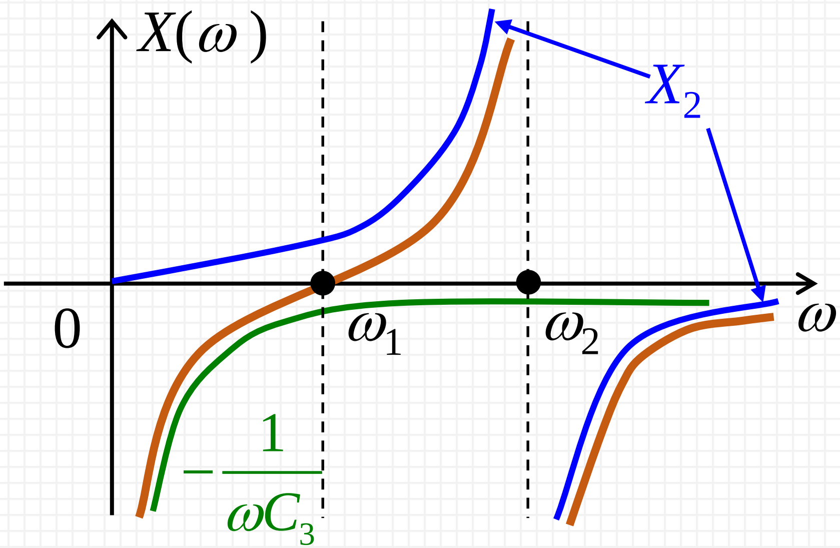
$f_1(\omega_0) = 0$ 时，电路发生串联谐振

$f_2(\omega_0) = 0$ 时，电路发生并联谐振

(c) 混联谐振

$$\mathrm{j}X=\frac{1}{\mathrm{j}\omega C_3}+\mathrm{j}X_2=\mathrm{j}(-\frac{1}{\omega C_3}+X_2)$$



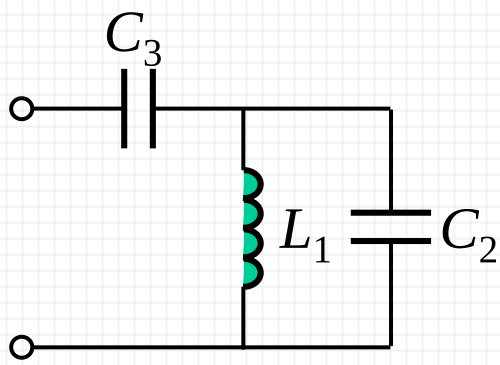


L_1、C_2并联，在某一角频率 ω_2 下发生**并联**谐振。

将虚线端口视为一个元件X_2，它和C_3**串联**的电抗频率特性是怎样的?

$\omega>\omega_2$时，并联部分呈容性，$\omega<\omega_2$时，并联部分呈感性，在某一角频率 ω_1下可与 C_3发生串联谐振。

定量分析



$$Z(\omega)=\frac{1}{\mathrm{j}\omega C_3}+\frac{\mathrm{j}\omega L_1\dfrac{1}{\mathrm{j}\omega C_2}}{\mathrm{j}\omega L_1+\dfrac{1}{\mathrm{j}\omega C_2}}$$

$$=\frac{1}{\mathrm{j}\omega C_3}+\frac{\mathrm{j}\omega L_1}{1-\omega^2 L_1 C_2}$$

分别令分子、分母为零，可得:

$$=-\mathrm{j}\frac{1-\omega^2 L_1(C_2+C_3)}{\omega C_3(1-\omega^2 L_1 C_2)}$$

$$\boxed{\omega_1=\frac{1}{\sqrt{L_1(C_2+C_3)}}}$$

发生串联谐振 $\boxed{Z_0=0}$

$$\boxed{\omega_2=\frac{1}{\sqrt{L_1 C_2}}}$$

发生并联谐振 $\boxed{Z_0=\infty}$

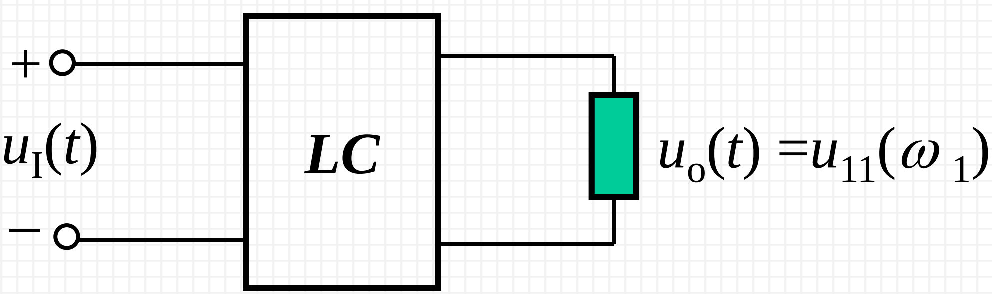
思考

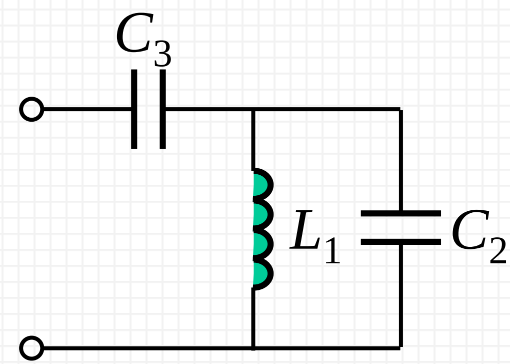
激励 $u_I(t)$，包含两个频率ω_1、 ω_2分量 ($\omega_1 < \omega_2$):

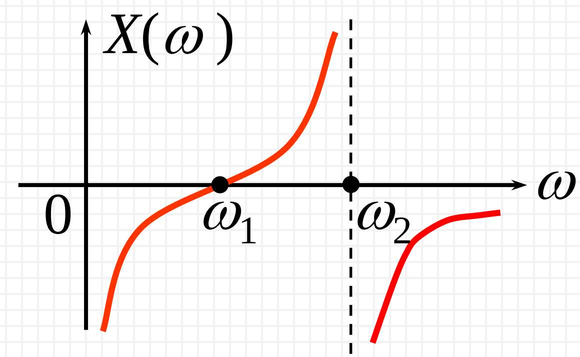
$$u_I(t) = u_{11}(\omega_1) + u_{12}(\omega_2)$$

要求负载电压 $u_o(t)$ **只有$u_{11}(\omega_1)$** 频率电压，(**无ω_2频率电压**) 。

如何实现?







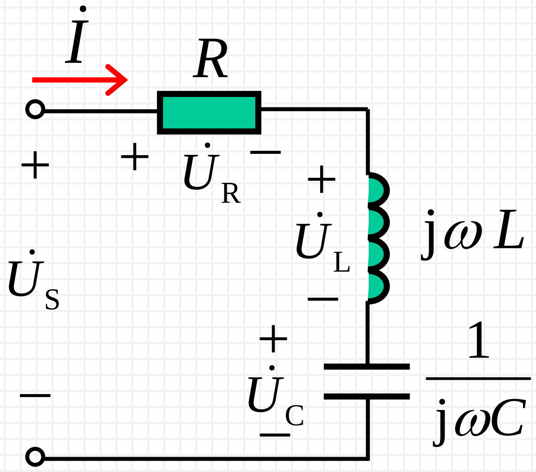
若$\omega_1 > \omega_2$， 仍要只得到ω_1频率电压，

如何设计电路?

2、谐振电路的品质因数（Quality Factor）

(1) 从支路量幅值角度考虑

以串联谐振为例



$$Z = R + j(\omega L - \frac{1}{\omega C})$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$U_{L0} = \frac{\omega_0 L}{R} U_S$$

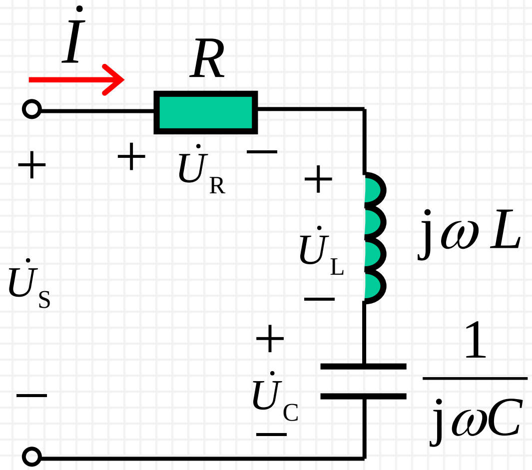
$$U_{C0} = \frac{1}{\omega_0 CR} U_S$$

二者相等

$$Q \overset{\text{def}_1}{=} \frac{U_{L0}}{U_S} = \frac{U_{C0}}{U_S} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Q 大 → 谐振时储能元件上的电压(电流）大

无量纲



$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

品质因数 Q

与 ω 无关

特性阻抗　　单位：Ω

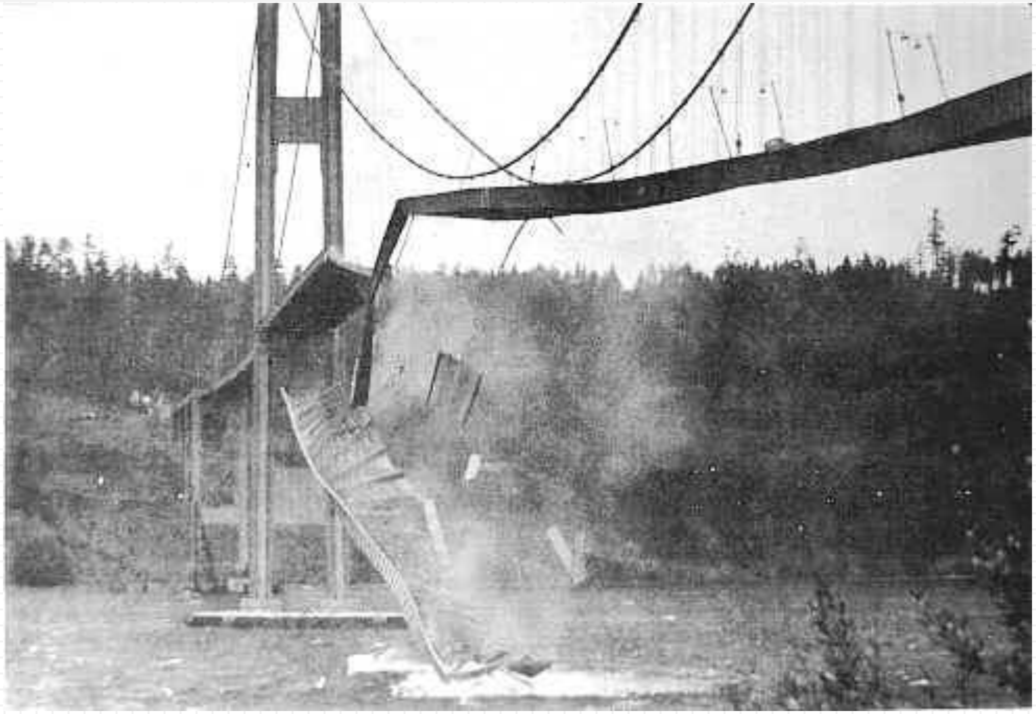
(*characteristic impedance*)

$$\boxed{\rho \overset{\text{def}}{=} \sqrt{\frac{L}{C}}}$$

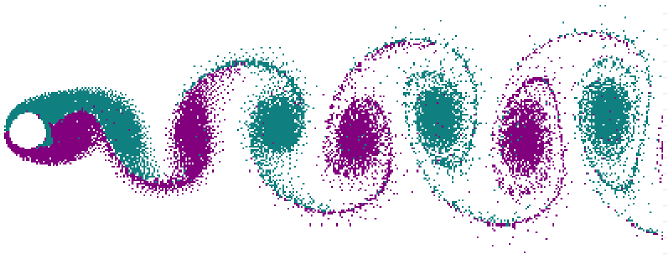
$$\dot{U}_R = \dot{U}_S \qquad \dot{U}_L = \mathrm{j}Q\dot{U}_S \qquad \dot{U}_C = -\mathrm{j}Q\dot{U}_S$$

L 和 C 上可能出现高电压 → **利用** / **避免**

Tacoma大桥为什么会垮掉?



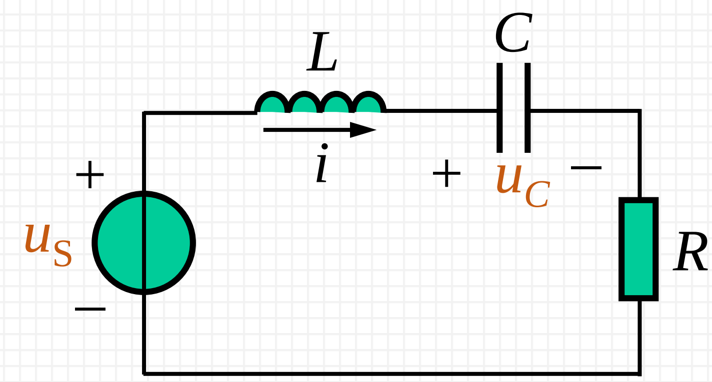
卡门涡街



原因：风的频率 $\approx$ 桥的自振频率

桥自振的 Q 大

(2) 从能量角度考虑



设 $u_S = U_m \sin\omega_0 t$

则 $i = \dfrac{U_m}{R}\sin\omega_0 t = I_m \sin\omega_0 t$

电感存储的磁场能量 $w_L = \dfrac{1}{2}Li^2 = \dfrac{1}{2}LI_m^2\sin^2\omega_0 t$

$$u_C = U_{Cm}\sin(\omega_0 t - 90^\circ) = \frac{1}{\omega_0 C} I_m \sin(\omega_0 t - 90^\circ) = -\sqrt{\frac{L}{C}} I_m \cos\omega_0 t$$

U_{Cm}

$\sqrt{\dfrac{L}{C}}$

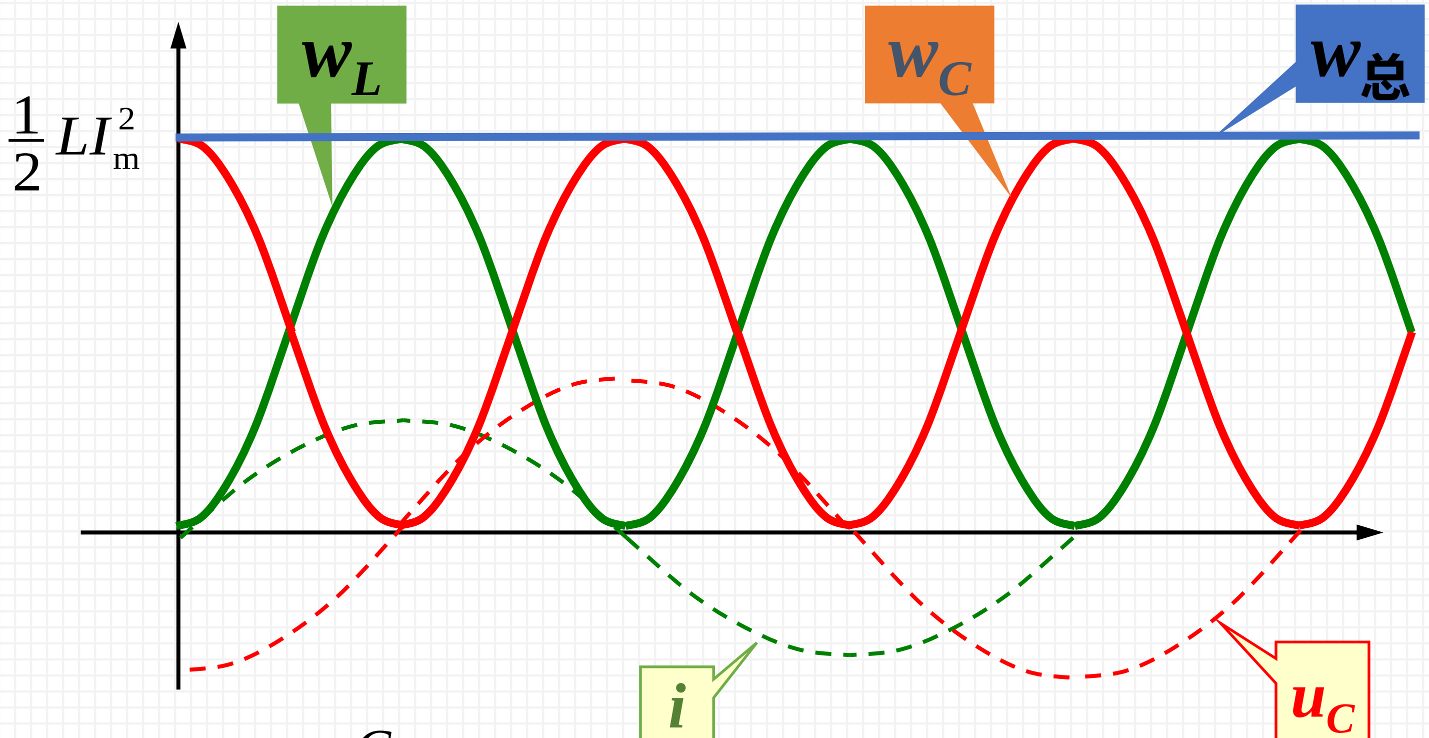
电容存储的电场能量

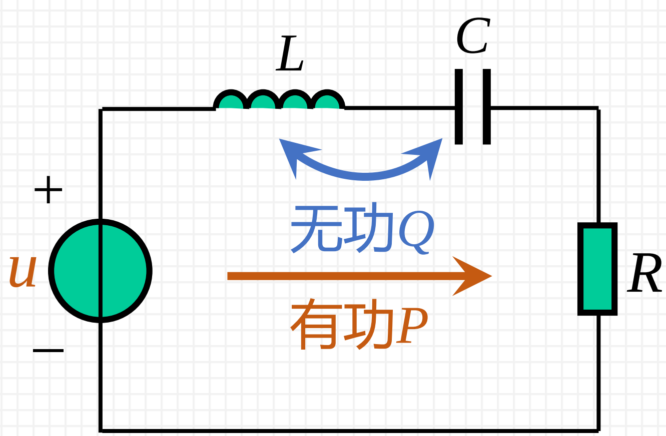
$$w_C = \frac{1}{2}Cu_C^2 = \frac{1}{2}LI_m^2\cos^2\omega_0 t$$

电感和电容能量按2倍频正弦规律变化，最大值相等 $w_{Lm}=w_{Cm}$。

$$\boxed{w_{Total} = w_L + w_C = \frac{1}{2}LI_m^2 = \frac{1}{2}CU_{Cm}^2}$$

磁场能量 $w_L=\frac{1}{2}LI_m^2\sin^2\omega_0 t$ **电场能量** $w_C=\frac{1}{2}LI_{Lm}^2\cos^2\omega_0 t=\frac{1}{2}CU_{Cm}^2\cos^2\omega_0 t$





电磁场总能量

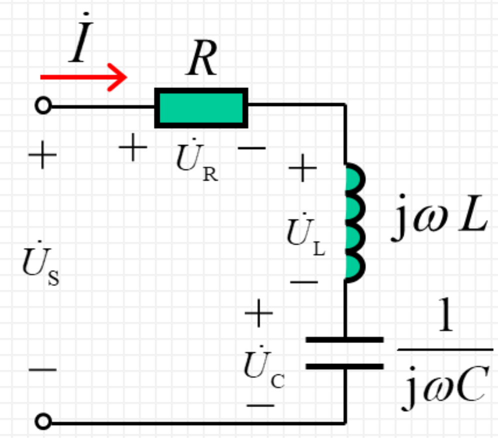
$$w_{\text{Total}}=w_C+w_L=\frac{1}{2}LI_m^2=\frac{1}{2}CU_{Cm}^2$$

谐振时：有功 ***P*** 和无功 ***Q*** 解耦

$$Q \overset{\text{def}_2}{=} 2\pi \frac{\text{电路中储存的电磁场总能量}}{\text{谐振时一个周期内电路消耗的能量}}$$

Q 大 ⟶ 谐振时储能大，消耗能量少。

Q 是反映谐振回路中电磁振荡程度的量



$$Q \overset{\text{def}_2}{=} 2\pi \frac{\text{电路中储存的电磁场总能量}}{\text{谐振时一个周期内电路消耗的能量}}$$

$$= 2\pi \frac{LI^2}{RI^2 T_0} = \frac{\omega_0 L}{R}$$

$$\begin{aligned} w_{\text{Total}} &= w_C + w_L \\ &= \frac{1}{2} C U_{\text{Cm}}^2 \\ &= \frac{1}{2} L I_{\text{m}}^2 \\ &= LI^2 \end{aligned}$$

Q 的定义 1 和定义 2 吻合

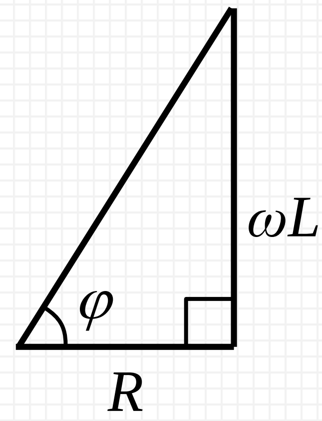
谐振电路的品质因数

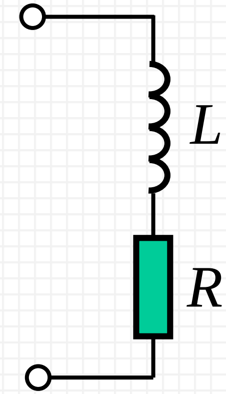
$$Q \overset{\text{def}_2}{=} 2\pi \frac{\text{电路中储存的电磁场总能量}}{\text{谐振时一个周期内电路消耗的能量}}$$

电感线圈的品质因数 Q_L (某个工作频率下)

$$Q_L \overset{\text{def}}{=} 2\pi \frac{\text{线圈中储存的最大磁场能量}}{\text{一个周期内线圈电阻消耗的能量}}$$

$$= 2\pi \frac{\frac{1}{2}L(\sqrt{2}I)^2}{I^2RT} = \frac{\omega L}{R}$$



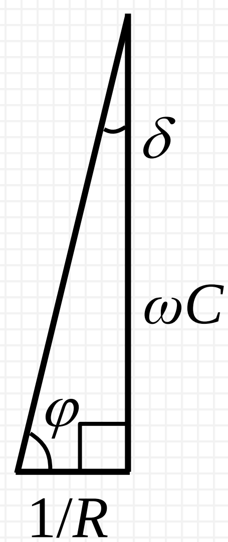


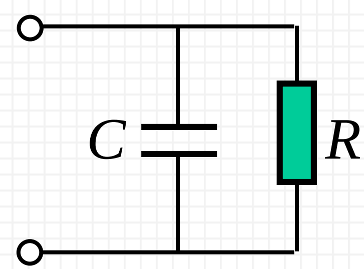
电容器的介质损耗角正切(某个工作频率下)

注意:
这不是谐振！！

$$\tan\delta = \frac{1}{Q_C} \overset{\text{def}}{=} \frac{1}{2\pi} \frac{\text{一个周期内电容消耗的能量}}{\text{电容中储存的最大电场能量}}$$

$$= \frac{(U^2/R)T}{2\pi \frac{1}{2}C(\sqrt{2}U)^2} = \frac{1}{\omega CR}$$

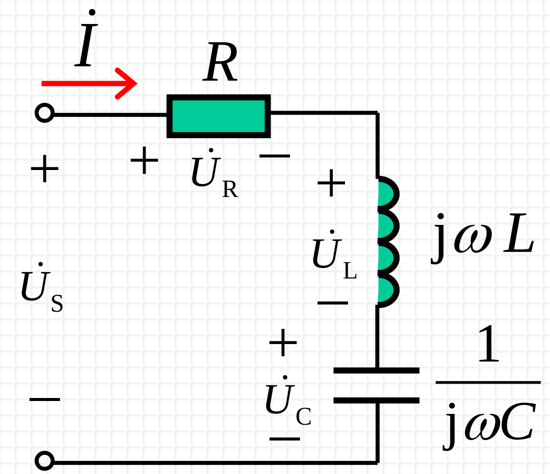


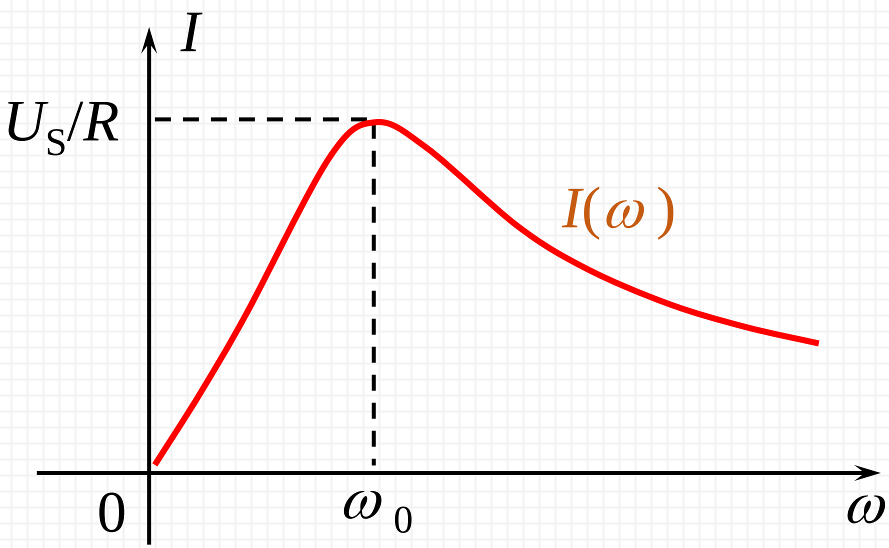


(3) 从频率特性角度考虑

电流频率特性 $\dot{I}=\dfrac{\dot{U}_S}{R+\mathrm{j}(\omega L-\dfrac{1}{\omega C})}$

幅值关系 $I(\omega)=\dfrac{U_S}{\sqrt{R^2+(\omega L-\dfrac{1}{\omega C})^2}}\le\dfrac{U_S}{R}$

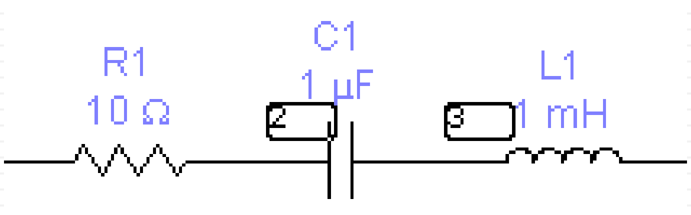




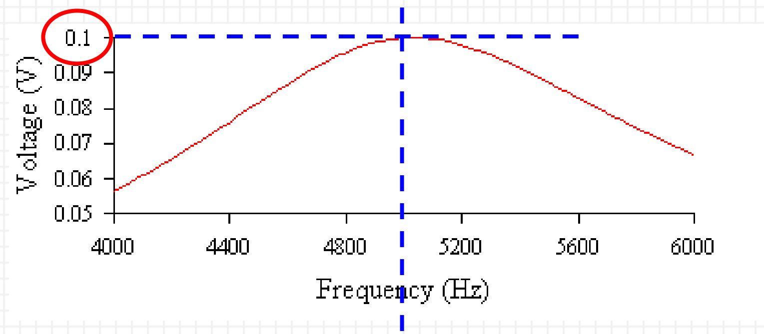
电流谐振曲线

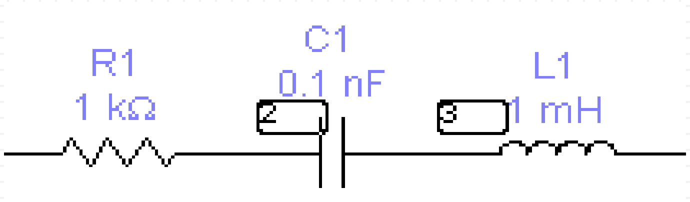
如何从电流谐振曲线看出Q来?

$$I(\omega)=\frac{U_S}{\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}}$$

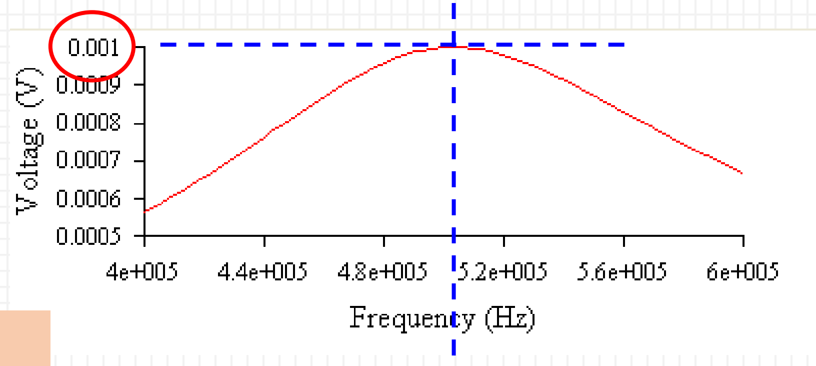


$Q=3.16$ $\omega_0=3.16\times10^4$ rad/s





$Q=3.16$ $\omega_0=3.16\times10^6$ rad/s



如何比较谐振频率不同、幅频特性最大幅值不同的两个谐振电路的Q?

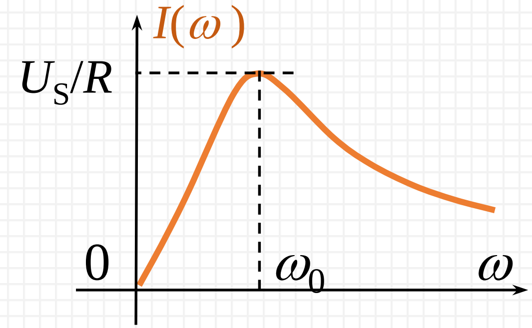
希望：谐振点处幅频特性的**幅值**都为**1**。在同一点**发生谐振**。

进行**归一化**处理!

纵轴变量的归一化

$$\frac{I(\omega)}{I(\omega_0)}=\frac{U_S/|Z|}{U_S/R}$$

$$I(\omega)=\frac{U_S}{\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}}$$



横轴变量的归一化

$$=\frac{R}{\sqrt{R^2+(\omega L-\frac{1}{\omega C})^2}}=\frac{1}{\sqrt{1+(\frac{\omega L}{R}-\frac{1}{\omega RC})^2}}$$

$$=\frac{1}{\sqrt{1+(\frac{\omega_0 L}{R}\cdot\frac{\omega}{\omega_0}-\frac{1}{\omega_0 RC}\cdot\frac{\omega_0}{\omega})^2}}=\frac{1}{\sqrt{1+(Q\frac{\omega}{\omega_0}-Q\frac{\omega_0}{\omega})^2}}$$

令 $\boxed{\eta=\frac{\omega}{\omega_0}}$

Q的定义1

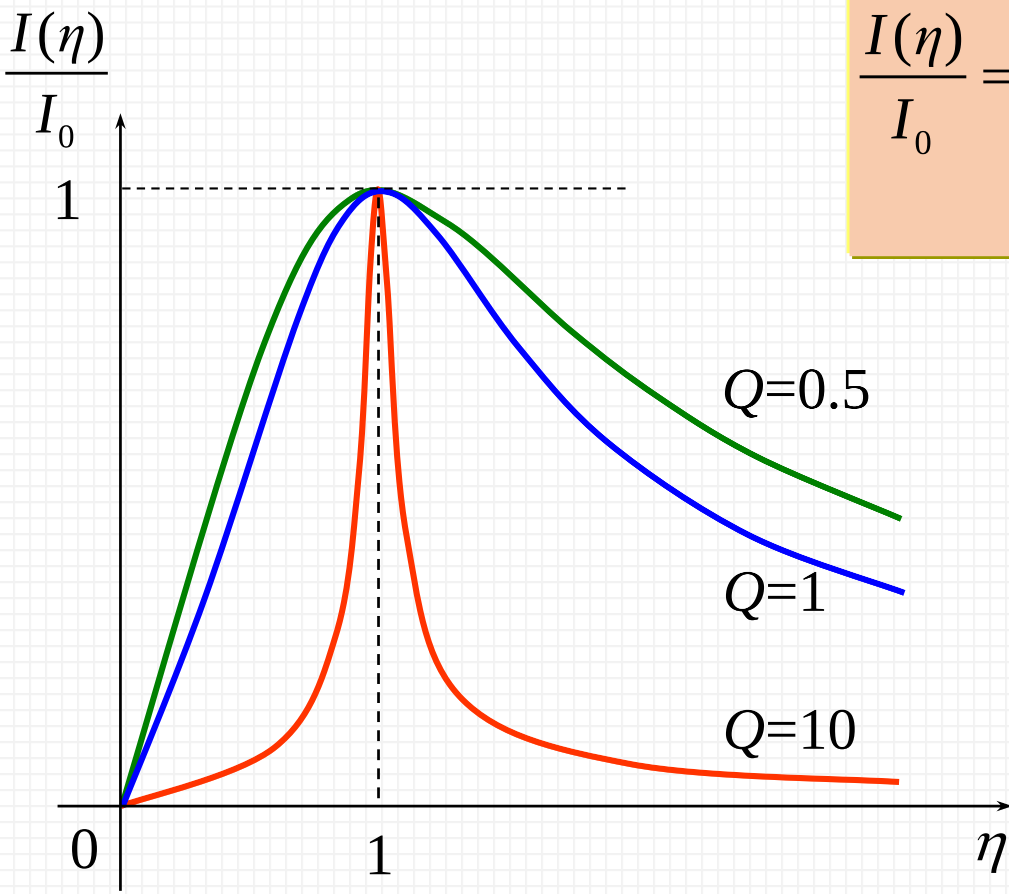
$$\boxed{\frac{I(\eta)}{I_0}=\frac{1}{\sqrt{1+Q^2(\eta-\frac{1}{\eta})^2}}}$$

$$\omega=\omega_0 \longrightarrow \eta=1 \longrightarrow \frac{I(\eta)}{I_0}=1$$

任何谐振，都在 $\eta=1$ 处发生，
谐振点处幅频特性的幅值都为1。

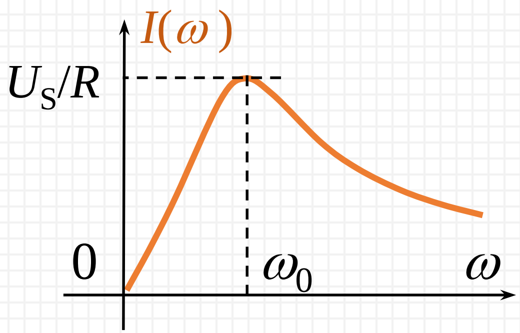
归一化完成！

通用谐振频率特性



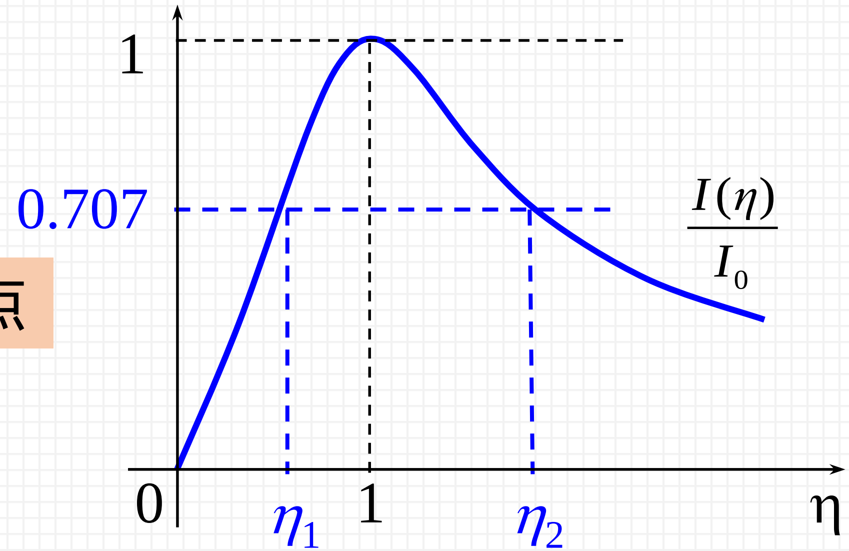
$$\frac{I(\eta)}{I_0}=\frac{1}{\sqrt{1+Q^2(\eta-\frac{1}{\eta})^2}}$$

Q 大 → 归一化后电流谐振曲线尖；谐振电路的选择性好



–3dB

半功率点



$$\frac{I(\eta)}{I_0}=\frac{1}{\sqrt{1+Q^2(\eta-\frac{1}{\eta})^2}}=\frac{1}{\sqrt{2}}$$

$$\eta_1=-\frac{1}{2Q}+\sqrt{\frac{1}{4Q^2}+1}$$

$$\eta_2=\frac{1}{2Q}+\sqrt{\frac{1}{4Q^2}+1}$$

$$\eta_2-\eta_1=\frac{1}{Q}$$

$$Q\overset{\text{def}_3}{=}\frac{1}{\eta_2-\eta_1}=\frac{\omega_0}{\omega_2-\omega_1}=\frac{f_0}{f_2-f_1}$$

$$\eta=\frac{\omega}{\omega_0}$$

可利用频率特性求Q

带宽 Band Width (**BW**)

品质因数 Q 定义的归纳

- 从信号幅值的变化来衡量

$$Q \overset{\text{def}_1}{=} \frac{U_{L0}}{U_S} = \frac{U_{C0}}{U_S}$$

Q 大 ⟶ 谐振时电容电压和电感电压大。

- 从电磁能量的转换来衡量

$$Q \overset{\text{def}_2}{=} 2\pi \frac{\text{电路中储存的电磁场总能量}}{\text{谐振时一个周期内电路消耗的能量}}$$

Q 大 ⟶ 谐振时储能大，消耗能量少。

- 从频率特性的形状来衡量

$$Q \overset{\text{def}_3}{=} \frac{\omega_0}{\omega_2 - \omega_1}$$

Q 大 ⟶ 谐振电路的选择性好