1. $\|\boldsymbol{x}\|_2^{1.5}$是否是光滑的凸函数？判断并证明。其中$\boldsymbol{x}$为 n 维实数向量，$\|\cdot\|_2^{1.5}$为 2 范数的 1.5 次方。

$$f(\lambda x_1+(1-\lambda)x_2)=\|\lambda x_1+(1-\lambda)x_2\|_2^{3/2}$$
$$\le \lambda^{3/2}\|x_1\|_2^{3/2}+(1-\lambda)^{3/2}\|x_2\|_2^{\frac{3}{2}}$$
$$\le \lambda\|x_1\|_2^{3/2}+(1-\lambda)\|x_2\|_2^{\frac{3}{2}}$$

故$f(x)$为凸函数.

显然$\|x\|_2^{1.5}$处处可导.

$$\nabla f(x)=\nabla(x^Tx)^{3/4}=\frac{3}{4}(x^Tx)^{-\frac{1}{4}}\cdot 2x=\frac{3}{2}(x^Tx)^{-1/4}x.$$

$$\nabla^2 f(x)=\frac{3}{2}(x^Tx)^{-1/4}-\frac{3}{4}(x^Tx)^{-\frac{5}{4}}xx^T$$

可发现二阶导不连续，故不光滑.

综上，$f(x)$为不光滑的凸函数.

2. 判别下列函数哪些是凸函数，哪些是凹函数，哪些是非凸非凹函数，并简述理由。

a) 函数$f(x_1,x_2)=x_1x_2+x_1$，定义域为$R_{++}^2=\{(x_1,x_2)\in R^2|x_1>0,x_2>0\}$；

b) 函数$f(x_1,x_2,x_3)=x_1^2+3x_2^2+12x_3^2-2x_1x_2+6x_2x_3+2x_1x_3$，定义域为$R^3$；

c) 函数$f(x_1,x_2)=-x_1^2-2x_2^2+2x_1x_2+10x_1-10x_2$，定义域为$R^2$；

a)
$$\nabla^2 f(x_1,x_2)=\begin{pmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2}\\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$\lambda_1=1\quad \lambda_2=-1$ 是不定矩阵，故f是非凸非凹函数.

b)
$$\nabla^2 f(x_1,x_2,x_3)=\begin{pmatrix}2 & -2 & 2\\ -2 & 6 & 6\\ 2 & 6 & 24\end{pmatrix}$$

顺序主子式：$|2|>0\quad \begin{vmatrix}2 & -2\\ -2 & 6\end{vmatrix}>0\quad \begin{vmatrix}2 & -2 & 2\\ -2 & 6 & 6\\ 2 & 6 & 24\end{vmatrix}>0$

故f为凸函数.

c)
$$\nabla^2 f(x_1 x_2 x_3)=\begin{pmatrix}-2 & 2\\ 2 & -4\end{pmatrix}$$

$|-2|<0\ ,\ \begin{vmatrix}-2 & 2\\ 2 & -4\end{vmatrix}>0$

故f为凹函数.

3. 求函数

$$f(x) = -3x^2 + 21.6x + 1$$

在区间[0,25]上的极大点和极大值，要求缩短后区间长度不大于原区间长度的 8%，用斐波那契法、黄金分割法（0.618 法）、折半搜索法、牛顿法分别进行求解。

Fibonacci :

$\delta = 25 \times 8\% = 2$

取 $\min F_n \geq \frac{b-a}{\delta} = 12.5$

$\because F_5 = 8 \quad F_6 = 13$

$\therefore n = 6$

$a_0 = 0 \qquad b_0 = 25$

$k=1$: $t_1 = a_0 + \frac{F_5}{F_6}(b_0 - a_0) = \frac{200}{13}$

$t_1' = b_0 - \frac{F_5}{F_6}(b_0 - a_0) = \frac{125}{13}$

$\because f(t_1') > f(t_1) \quad \therefore a_1 = a_0 = 0 \ , \ b_1 = t_1 = \frac{200}{13}$

$k=2$: $t_2 = t_1' = \frac{125}{13} \qquad t_2' = b_1 - \frac{F_4}{F_5}(b_1 - a_1) = \frac{75}{13}$

$f(t_2') > f(t_2) \quad a_2 = a_1 = 0 \quad b_2 = t_2 = \frac{125}{13}$

$k=3$: $t_3 = t_2' = \frac{75}{13}$, $t_3' = b_2 - \frac{F_3}{F_4}(b_2 - a_2) = \frac{50}{13}$

$f(t_3') > f(t_3) \quad a_3 = a_2 = 0 \quad b_3 = t_3 = \frac{75}{13}$

$k=4$: $t_4 = t_3' = \frac{50}{13}$, $t_4' = b_3 - \frac{F_2}{F_3}(b_3 - a_3) = \frac{25}{13}$

$f(t_4') < f(t_4) \quad a_4 = t_4' = \frac{25}{13}$, $b_4 = b_3 = \frac{75}{13}$

$k=5$: $t_5' = t_4 = \frac{50}{13}$, $t_5 = a_4 + \frac{F_1}{F_2}(b_4 - a_4) = \frac{50}{13}$

$f(t_5') > f(t_5) \quad a_5 = a_4 = \frac{25}{13} \quad b_5 = t_5 = \frac{30}{13}$

故局部极大值点, $x^* = \frac{a_5 + b_5}{2} = \frac{75}{26} \qquad f(x^*) \approx 38.3447$.

0.618 法

$\delta = 25 \times 8\% = 2$

$\because 0.618^6 (b-a) < \delta < 0.618^5 (b-a)$

$\therefore n-1 = 6$, $n = 7$

$a_0 = 0 \quad b_0 = 25 \quad 0.618 = w$

$k=1 \quad t_1 = a_0 + w(b_0 - a_0) = 15.45 \qquad t_1' = b_0 - w(b_0 - a_0) = 9.55$

$f(t_1') > f(t_1) \quad a_1 = a_0 = 0 \quad b_1 = t_1 = 15.45$

$k=2$： $t_2 = t_1'$ $t_2' = b_1 - w(b_1 - a_1) = 5.902$.

$f(t_2') > f(t_2)$ $a_2 = a_1 = 0$ $b_2 = t_2 = 9.550$

$k=3$： $t_3 = t_2'$ $t_3' = b_2 - w(b_2 - a_2) = 3.648$.

$f(t_3') > f(t_3)$ $a_3 = a_2 = 0$ $b_3 = t_3 = 5.902$

$k=4$： $t_4 = t_3'$ $t_4' = b_3 - w(b_3 - a_3) = 2.255$

$f(t_4') < f(t_4)$ $a_4 = t_4' = 2.255$ $b_4 = b_3 = 5.902$

$k=5$： $t_5' = t_4 = 3.648$, $t_5 = a_4 + w(b_4 - a_4) = 4.509$.

$f(t_5') > f(t_5)$ $a_5 = a_4 = 2.255$ $b_5 = t_5 = 4.509$.

$k=6$： $t_6 = t_5'$, $t_6' = b_5 - w(b_5 - a_5) = 3.116$

$f(t_6') < f(t_6)$ $a_6 = t_6' = 3.116$ $b_6 = b_5 = 4.509$

故局部极大 $x^* = \dfrac{a_5 + b_6}{2} = 3.813$. $f(x^*) = 39.744$.

附加题：比较 0.618 法和斐波那契法的运算速度，简述理由。

比较上两方法：0.618法比Fibonacci多一次迭代，是以后者分数数列替代每个分数值的方法，不用引入ε来分析.

折半法： $\delta = 2$. $\delta \geq \dfrac{b-a}{2^n} \Rightarrow n = 4$

$k=1$： $f(a_0) > f\left(\dfrac{a_0 + b_0}{2}\right) > f(b_0)$ $a_1 = 0$ $b_1 = 12.5$

$k=2$： $f(a_1) > f\left(\dfrac{a_1 + b_1}{2}\right) > f(b_1)$ $a_2 = 0$ $b_2 = 6.25$

$k=3$： $f\left(\dfrac{a_2 + b_2}{2}\right) > f(b_2) > f(a_2)$ $a_3 = 3.125$ $b_3 = 6.25$

$k=4$： $f(a_3) > f\left(\dfrac{a_3 + b_3}{2}\right) > f(b_3)$ $a_4 = 3.125$ $b_4 = 4.6875$.

故 $x^* = \dfrac{a_4 + b_4}{2} = 3.90625$ $f(x^*) = 39.6$

牛顿法： $f'(x) = -6x + 21.6$ $f''(x) = -6$

取 $x_0 = 0$ $x_1 = x_0 - \dfrac{f'(0)}{f''(0)} = 3.6$ $f(x_1) = 39.88$.