1. 给定线性规划问题：

$$\begin{aligned} \min\ & 5x_1 + 21x_3 \\ \text{s.t.}\ & x_1 - x_2 + 6x_3 \geq b_1 \\ & x_1 + x_2 + 2x_3 \geq 1 \\ & x_j \geq 0,\ j = 1,2,3 \end{aligned}$$

其中b_1是某一个正数，已知这个问题的一个最优解为$(0.5,\ 0,\ 0.25)^{\mathrm{T}}$。

(1) 写出对偶问题；

(2) 求对偶问题的最优解.

(1) 对偶问题如下：

$$\begin{aligned} \max\ & b_1 y_1 + y_2 \\ \text{s.t.}\ & y_1 + y_2 \leq 5 \\ & -y_1 + y_2 \leq 0 \\ & 6y_1 + 2y_2 \leq 21 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

(2) 由互补松弛性条件

$$\hat{x}^{\mathrm{T}}(A^{\mathrm{T}}\hat{y} - c) = 0$$

即 $\begin{cases} \frac{1}{2}(y_1 + y_2 - 5) = 0 \\ \frac{1}{4}(6y_1 + 2y_2 - 21) = 0 \end{cases} \Rightarrow \begin{cases} y_1 = \frac{11}{4} \\ y_2 = \frac{9}{4} \end{cases}$

$$\hat{y}^{\mathrm{T}}(b - A\hat{x}) = 0$$

即 $\begin{cases} \frac{1}{2} + 6 \times \frac{1}{4} - b_1 = 0 \\ \frac{1}{2} + 2 \times \frac{1}{4} - 1 = 0 \end{cases} \Rightarrow b_1 = 2$

∴最优解 $(y_1, y_2) = (\frac{11}{4}, \frac{9}{4})$ 最优函数值为$\frac{31}{4}$.

2. 求解以下带参数的线性规划问题，并给出$z(\lambda)$与λ的变化关系：

(1) $$\min\ z=(6-\lambda)x_1+(5-\lambda)x_2+(-3+\lambda)x_3+(-4+\lambda)x_4$$
$$\begin{aligned}\text{s.t. }\ & x_1-x_2-x_3\le 1\\ & -x_1+x_2-x_4\le 1\\ & -x_2+x_3\le 1\\ & x_j\ge 0,\ j=1,2,3,4\end{aligned}$$

(2) $$\min\ z=2x_1+6x_2+15x_3$$
$$\begin{aligned}\text{s.t. }\ & -2x_1-3x_2-5x_3\le 6-\lambda\\ & x_1+x_2+x_3\le -2+\lambda\\ & x_2+2x_3\le -3+2\lambda\\ & x_j\ge 0,\ j=1,2,3\end{aligned}$$

(1)

引入松弛变量：

$$\min\ (6-\lambda,\ 5-\lambda,\ \lambda-3,\ \lambda-4)^T\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$

$$\begin{aligned}\text{s.t. }\ & x_1-x_2-x_3+x_5=1\\ & -x_1+x_2-x_4+x_6=1\\ & -x_2+x_3+x_7=1\\ & x_j\ge 0,\quad j=1,2,\cdots,6.\end{aligned}$$

可得单纯型表如下：

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	1	-1	-1	0	1	0	0	1
x_6	-1	1	0	-1	0	1	0	1
x_7	0	-1	1	0	0	0	1	1
	$6-\lambda$	$5-\lambda$	$\lambda-3$	$\lambda-4$	0	0	0	z

$\lambda<4$ 时，x_4 不能进基，此时原问题无界即 $z\to-\infty$

$4\le\lambda<5$ 时，已满足最优，$z=0$

$5\le\lambda<6$ 时，x_2 进基，x_6 出基.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	0	-1	-1	1	1	0	2
x_2	-1	1	0	-1	0	1	0	1
x_7	-1	0	1	-1	0	1	1	2
	$6-\lambda$	$5-\lambda$	$\lambda-3$	$\lambda-4$	0	0	0	$z+\lambda-5$

$5\le\lambda\le5.5$，已达最优，$z=5-\lambda$

$5.5<\lambda$，x_1 不能进基，此时原问题 $z\to-\infty$

综上所述：$$z=\begin{cases}-\infty & \lambda<4\ \text{or}\ \lambda>5.5\\ 0 & 4\le\lambda<5\\ 5-\lambda & 5\le\lambda\le5.5.\end{cases}$$

(2).

化为对偶问题：

$$\max\ z=(\lambda-6)y_1+(2-\lambda)y_2+(3-2\lambda)y_3$$

$$\begin{pmatrix}2&-1&0\\3&-1&-1\\5&-1&-2\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\le\begin{pmatrix}2\\6\\15\end{pmatrix}$$

引入松弛变量列单纯型表如下：

	y_1	y_2	y_3	y_4	y_5	y_6	
y_4	2	-1	0	1	0	0	2
y_5	3	-1	-1	0	1	0	6
y_6	5	-1	-2	0	0	1	15
	$\lambda-6$	$2-\lambda$	$3-2\lambda$	0	0	0	$z-(\lambda-6)$

$\lambda<2$　y_2不能进基，原问题无界.

$2\le\lambda\le6$　最优，$z=0$

$\lambda>6$　y_1进，y_4出，单纯型表如下：

	y_1	y_2	y_3	y_4	y_5	y_6	
y_4	2	-1	0	1	0	0	2
y_5	3	-1	-1	0	1	0	6
y_6	5	-1	-2	0	0	1	15
	$\lambda-6$	$2-\lambda$	$3-2\lambda$	0	0	0	$z-(\lambda-6)$

最优解为：$\lambda-6$.

综上：$$z=\begin{cases}\lambda-6 & ,\ \lambda>6.\\ 0 & ,\ 2\le\lambda\le6\\ \infty & ,\ \lambda<2\end{cases}$$

3. 已知线性规划问题 A 和 B 如下

问题 A

$$\begin{array}{lll}\max & \sum_{j=1}^{n} c_j x_j & \text{影子价格} \\ \text{s.t.} & \sum_{j=1}^{n} a_{1j} x_j = b_1 & y_1 \\ & \sum_{j=1}^{n} a_{2j} x_j = b_2 & y_2 \\ & \sum_{j=1}^{n} a_{3j} x_j = b_3 & y_3 \\ & x_j \geq 0, j = 1, 2, \cdots, n & \end{array}$$

问题 B

$$\begin{array}{lll}\max & \sum_{j=1}^{n} c_j x_j & \text{影子价格} \\ \text{s.t.} & \sum_{j=1}^{n} k_1 a_{1j} x_j = k_1 b_1 & \hat{y}_1 \\ & \sum_{j=1}^{n} k_2 a_{2j} x_j = k_2 b_2 & \hat{y}_2 \\ & \sum_{j=1}^{n} (a_{3j} + k_3 a_{1j}) x_j = b_3 + k_3 b_1 & \hat{y}_3 \\ & x_j \geq 0, j = 1, 2, \cdots, n & \end{array}$$

求 y_i 与 $\hat{y}_i (i = 1, 2, 3)$ 的关系。

A: $\min \ b_1 y_1 + b_2 y_2 + b_3 y_3$

s.t. $a_{1j} y_1 + a_{2j} y_2 + a_{3j} y_3 \geq c_j \quad j = 1, 2, \cdots, n$

$y_i \geq 0 \quad i = 1, 2, 3$

B: $\min \ k_1 b_1 \hat{y}_1 + k_2 b_2 \hat{y}_2 + (k_3 b_1 + b_3) \hat{y}_3$

s.t. $k_1 a_{1j} \hat{y}_1 + k_2 a_{2j} \hat{y}_2 + (a_{3j} + k_3 a_{1j}) \hat{y}_3 \geq c_j \quad j = 1, 2, \cdots, n$

$\hat{y}_i \geq 0 \quad i = 1, 2, 3$

即 $(k_1 \hat{y}_1 + k_3 \hat{y}_3) a_{1j} + k_2 \hat{y}_2 a_{2j} + \hat{y}_3 a_{3j} \geq c_j \quad j = 1, 2, \cdots, n$

故 y_i 与 $\hat{y}_i$ 关系为：

$$\begin{cases} y_1 = k_1 \hat{y}_1 + k_3 \hat{y}_3 \\ y_2 = k_2 \hat{y}_2 \\ y_3 = \hat{y}_3 \end{cases}$$

4. 用对偶单纯形算法求解以下线性规划问题

$$
\begin{aligned}
\min\ z &= 6x_1 + 4x_2 + 8x_3 \\
\text{s.t.}\quad & 3x_1 + 2x_2 + x_3 \geq 2 \\
& 4x_1 + x_2 + 3x_3 \geq 4 \\
& 2x_1 + 2x_2 + 2x_3 \geq 3 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

将其转化为标准型：

$$
\begin{aligned}
-\max\ w &= -z = -6x_1 - 4x_2 - 8x_3 \\
\text{s.t.}\quad & 3x_1 + 2x_2 + x_3 - x_4 = 2 \\
& 4x_1 + x_2 + 3x_3 - x_5 = 4 \\
& 2x_1 + 2x_2 + 2x_3 - x_6 = 3 \\
& x_j \geq 0,\ j = 1, 2, \cdots, 6
\end{aligned}
$$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	-3	-2	-1	1	0	0	-2
x_5	-4	-1	-3	0	1	0	-4
x_6	-2	-2	-2	0	0	1	-3
	-6	-4	-8	0	0	0	z

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	-5/4	5/4	1	-3/4	0	1
x_1	1	1/4	3/4	0	-1/4	0	1
x_6	0	-3/2	-1/2	0	-1/2	1	-1
	0	-5/2	-7/2	0	-3/2	0	$z+6$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	5/3	1	-1/3	-5/6	11/6
x_1	1	0	2/3	0	-1/3	1/6	5/6
x_2	0	1	1/3	0	1/3	-2/3	2/3
	0	0	-8/3	0	-2/3	-5/3	$z+\frac{23}{3}$

$$x^* = \left(\frac{5}{6}, \frac{2}{3}, 0, \frac{11}{6}, 0, 0\right)^T$$

$$z^* = -w^* = \frac{23}{3}$$

5. 用单纯形法求解以下线性规划问题

$$\min\ 24y_1 + 6y_2 + 5y_3 + 3y_4 + 3y_5 + 2y_6 + 6y_7 + y_8 + y_9$$

$$\begin{aligned} \text{s.t.}\quad & 7y_1 + 6y_2 + 2.5y_3 + y_4 + 0.75y_5 + 0.4y_6 + y_7 + y_8 = 6 \\ & 7y_1 + y_2 + y_3 + 0.75y_4 + y_5 + y_6 + 6y_7 + y_9 = 7 \\ & y_i \geq 0, i = 1,2,\dots,9 \end{aligned}$$

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
y_8	7	6	2.5	1	0.75	0.4	1	1	0	6
y_9	7	1	1	0.75	1	1	6	0	1	7
	10	-1	1.5	1.25	1.25	0.6	-1	0	0	z-13

y_2 进基，y_8 出基.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
y_2	7/6	1	5/12	1/6	1/8	1/15	1/6	1/6	0	1
y_9	35/6	0	7/12	7/12	7/8	14/15	35/6	-1/6	1	6
	67/6	0	23/12	17/12	11/8	2/3	-5/6	1/6	0	z-12

y_7 进基，y_9 出基.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
y_2	7/6	1	2/5	3/20	1/10	1/23	0	6/35	-1/35	29/35
y_7	35/6	0	1/10	1/10	3/20	4/25	1	-1/35	6/35	36/35
	67/6	0	2	3/2	3/2	4/5	0	1/7	1/7	$z-\frac{78}{7}$

故最优：$y = (0, \frac{29}{35}, 0, 0, 0, 0, \frac{36}{35}, 0, 0)$

$\min z = \frac{78}{7}$.