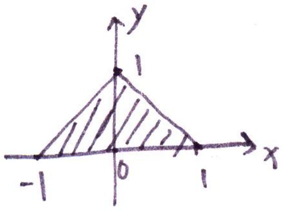
习题3.3.

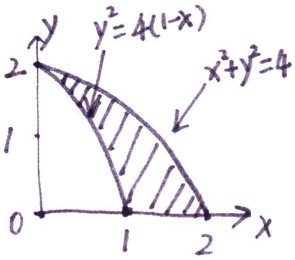
5. (1)



交换积分次序: $\int_{-1}^{0}dx\int_{0}^{1+x}f(x,y)dy+\int_{0}^{1}dx\int_{0}^{1-x}f(x,y)dy$

$$=\int_{0}^{1}dy\int_{y-1}^{1-y}f(x,y)dx.$$

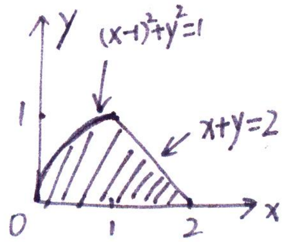
(2)



交换积分次序: $\int_{0}^{1}dx\int_{2\sqrt{1-x}}^{\sqrt{4-x^2}}f(x,y)dy+\int_{1}^{2}dx\int_{0}^{\sqrt{4-x^2}}f(x,y)dy$

$$=\int_{0}^{2}dy\int_{1-\frac{y^2}{4}}^{\sqrt{4-y^2}}f(x,y)dx.$$

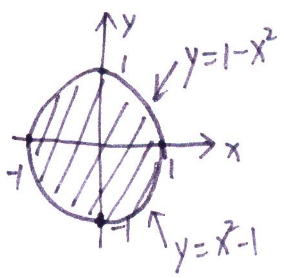
(3)



交换积分次序: $\int_{0}^{1}dx\int_{0}^{\sqrt{2x-x^2}}f(x,y)dy+\int_{1}^{2}dx\int_{0}^{2-x}f(x,y)dy$

$$=\int_{0}^{1}dy\int_{1-\sqrt{1-y^2}}^{2-y}f(x,y)dx.$$

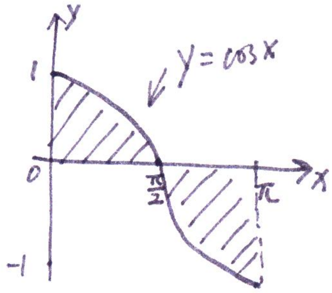
(4)



交换积分次序: $\int_{-1}^{1}dx\int_{x^2-1}^{1-x^2}f(x,y)dy$

$$=\int_{-1}^{0}dy\int_{-\sqrt{1+y}}^{\sqrt{1+y}}f(x,y)dx+\int_{0}^{1}dy\int_{-\sqrt{1-y}}^{\sqrt{1-y}}f(x,y)dx.$$

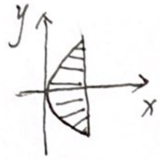
(5)



交换积分次序: $\int_{0}^{\pi}dx\int_{0}^{\cos x}f(x,y)dy$

$$=\int_{-1}^{0}dy\int_{\arccos y}^{\pi}f(x,y)dx+\int_{0}^{1}dy\int_{0}^{\arccos y}f(x,y)dx.$$

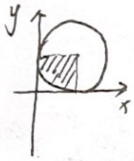
6. (1). $\iint\limits_D xy^2\,dxdy \qquad D=\{(x,y)\mid 4x\geq y^2,\ x\leq 1\};$



$$\iint\limits_D xy^2\,dxdy=\int_{-2}^{2}dy\int_{\frac{y^2}{4}}^{1}xy^2\,dx=\int_{-2}^{2}\frac{y^2}{2}-\frac{y^6}{32}\,dy$$

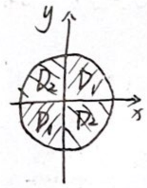
$$=2\int_0^2\frac{y^2}{2}-\frac{y^6}{32}\,dy=\frac{32}{21}$$

(2) $\iint\limits_D \frac{1}{\sqrt{2a-x}}\,dxdy=\int_0^a dx\int_{a-\sqrt{a^2-(x-a)^2}}^{a}\frac{1}{\sqrt{2a-x}}\,dy=\int_0^a\frac{\sqrt{x(2a-x)}}{\sqrt{2a-x}}\,dx$



$$=\int_0^a\sqrt{x}\,dx=\frac{2}{3}a^{\frac{3}{2}}$$

(3).

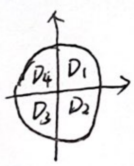


$\cancel{\iint\limits_D\frac{1}{\sqrt{2a-x}}dxdy}=\iint\limits_D|xy|\,dxdy=\iint\limits_{D_1}xy\,dxdy-\iint\limits_{D_2}xy\,dxdy$

$$=\int_0^R dx\int_0^{\sqrt{R^2-x^2}}xy\,dy+\int_{-R}^{0}dx\int_{-\sqrt{R^2-x^2}}^{0}xy\,dy-\int_{-R}^{0}dx\int_0^{\sqrt{R^2-x^2}}xy\,dy$$

$$-\int_0^R dx\int_{-\sqrt{R^2-x^2}}^{0}xy\,dy=4\int_0^R dx\int_0^{\sqrt{R^2-x^2}}xy\,dy=\frac{R^4}{2}$$

(4).



$$\iint\limits_D x\cos(xy)\,dxdy=\iint\limits_{D_1}x\cos(xy)\,dxdy+\iint\limits_{D_2}x\cos(xy)\,dxdy$$

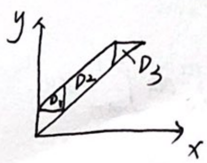
$$+\iint\limits_{D_3}x\cos(xy)\,dxdy+\iint\limits_{D_4}x\cos(xy)\,dxdy$$

$$\iint\limits_{D_1}x\cos(xy)\,dxdy=\int_0^R dx\int_0^{\sqrt{R^2-x^2}}x\cos(xy)\,dy=\int_0^R dx\left(\sin xy\Big|_{y=0}^{y=\sqrt{R^2-x^2}}\right)$$

$=\int_0^R\sin x\sqrt{R^2-x^2}\,dx$ 注意到，$\sin x\sqrt{R^2-x^2}$ 为奇函数.

同理，$\iint\limits_D x\cos(xy)\,dxdy=2\int_0^R\sin x\sqrt{R^2-x^2}\,dx+2\int_{-R}^{0}\sin x\sqrt{R^2-x^2}\,dx=0$

(5).



$$\iint\limits_D(x^2+y^2)\,dxdy=\iint\limits_{D_1}(x^2+y^2)\,dxdy+\iint\limits_{D_2}(x^2+y^2)\,dxdy+\iint\limits_{D_3}(x^2+y^2)\,dxdy$$

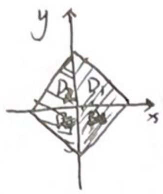
$$=\int_0^1dx\int_1^{x+1}(x^2+y^2)\,dy+\int_1^3dx\int_x^{x+1}(x^2+y^2)\,dy+\int_3^4dx\int_x^4(x^2+y^2)\,dy$$

$$=\int_0^1x^3+\frac{(x+1)^3}{3}-\frac{1}{3}\,dx+\int_1^3x^2+\frac{(x+1)^3}{3}-\frac{x^3}{3}\,dx+\int_3^4 4x^2-\frac{4}{3}x^3+\frac{4^3}{3}\,dx$$

$$=x^4+\frac{(x+1)^4}{12}-\frac{x}{3}\Big|_0^1+\frac{x^3}{3}+\frac{(x+1)^4}{12}-\frac{x^4}{12}\Big|_1^3+\frac{4}{3}x^3-\frac{x^4}{3}+\frac{4^3}{3}x\Big|_3^4$$

$$=\frac{71}{2}$$

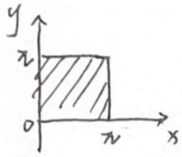
(6).



$$\iint_D e^{x+y}dxdy = \int_0^1 dx\int_{x-1}^{1-x} e^{x+y}dy + \int_{-1}^0 dx\int_{-x-1}^{x+1} e^{x+y}\cdot dy$$

$$= e+e^{-1}$$

(7).

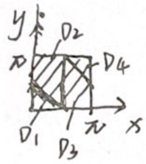


$$\iint_D \cos(x+y)\,dy\,dx = \int_0^\pi dx\int_0^\pi \cos(x+y)\cdot dy$$

$$= \int_0^\pi[\sin(x+\pi)-\sin x]\,dx = \int_0^\pi -2\sin x\,dx$$

$$= 2\cos x\Big|_0^\pi = -4.$$

(8).



$$\iint_D |\cos(x+y)|\,dxdy = \int_0^{\frac{\pi}{2}}dx\int_0^{-x+\frac{\pi}{2}}\cos(x+y)\,dy - \int_0^{\frac{\pi}{2}}dx\int_{-x+\frac{\pi}{2}}^{\pi}\cos(x+y)\,dy$$

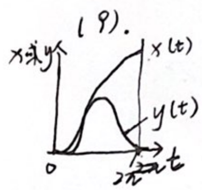
$$- \int_{\frac{\pi}{2}}^{\pi}dx\int_0^{-x+\frac{3}{2}\pi}\cos(x+y)\,dxdy + \int_{\frac{\pi}{2}}^{\pi}dx\int_{-x+\frac{3}{2}\pi}^{\pi}\cos(x+y)\,dy$$

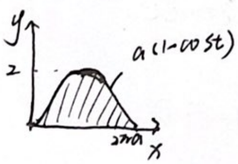
$$= \int_0^{\frac{\pi}{2}} 1-\sin x\,dx - \int_0^{\frac{\pi}{2}} -1+\sin(x+\pi)\,dx - \int_{\frac{\pi}{2}}^{\pi} -1-\sin x\,dx$$

$$+\int_{\frac{\pi}{2}}^{\pi} 1+\sin(x+\pi)\,dx$$

$$= \int_0^\pi +2\,dx = \cancel{2\pi}\ 2\pi$$

(9).





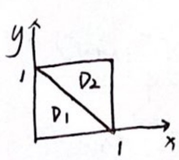
$$\iint_D y^2dxdy = \int_0^{2\pi a}dx\int_0^{a(1-\cos t)} y^2dy = \frac{1}{3}\int_0^{2\pi a} a^3(1-\cos t)^3dx$$

$$=\frac{1}{3}\int_0^{2\pi a} a^3(1-\cos t)^3\, d[a(t-\sin t)] = \frac{a^4}{3}\int_0^{2\pi}(1-\cos t)^4dt.$$

$$=\frac{2^4a^4}{3}\int_0^{2\pi}\sin^8\frac{t}{2}\,dt = \frac{2^5a^4}{3}\int_0^\pi \sin^8 u\,du = \frac{2^6a^4}{3}\int_0^{\frac{\pi}{2}}\sin^8u\,du$$

$$=\frac{2^6a^4}{3}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{35\pi a^4}{12}$$

(10)

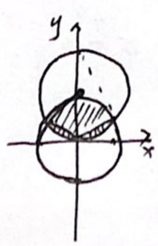


$$\iint_D [x+y]\,dxdy = \iint_{D_1}0\,dxdy + \iint_{D_2}1\,dxdy = \int_0^1dx\int_{-x+1}^1dy$$

$$=\frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

相交时: $2\sin\theta=1$ $\theta=\frac{\pi}{6}$

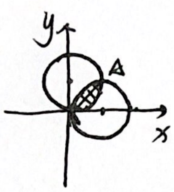
11. (1).



故 $\iint f(x,y)\,dxdy = \int_0^{\frac{\pi}{6}}d\theta\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\,r\,dr$

$$+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}d\theta\int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr + \int_{\frac{5}{6}\pi}^{\pi}d\theta\int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\,r\,dr$$

(2)



易知，相交点 A 时，$\theta = \frac{\pi}{4}$

$$\iint_D f(x,y)\,dxdy = \int_0^{\frac{\pi}{4}} d\theta \int_0^{a\sin\theta} f(r\cos\theta, r\sin\theta)\,r\,dr + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{a\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr$$