习题 1.8

1.(1). $z = \cos(x^2+y^2) = f(x,y)$.

解: $f'_x = \cancel{2x\cos}\ -2x\sin(x^2+y^2)$.

$f'_y = -2y\sin(x^2+y^2)$.

$f''_{xx} = -2\sin(x^2+y^2) - 4x^2\cos(x^2+y^2)$.

$f''_{yy} = -2\sin(x^2+y^2) - 4y^2\cos(x^2+y^2)$

$f''_{xy} = -4xy\cos(x^2+y^2)$.

$f'''_{xxx} = -4x\cos(x^2+y^2) - 8x\cos(x^2+y^2) + 8x^3 \cancel{\cos x}\ \sin(x^2+y^2)$.

$f'''_{yyy} = -12y\cos(x^2+y^2) + 8y^3\cos(x^2+y^2)$.

$f'''_{xxy} = -4y\cos(x^2+y^2) + 8x^2y\sin(x^2+y^2)$.

$f'''_{xyy} = -4x\cancel{\cos}(x^2+y^2) + 8xy^2\sin(x^2+y^2)$.

① 带 Peano 余项的 2 阶 Taylor 公式:

$$f(x,y) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + o(h^2+k^2)$$

$$= 1 + (hf'_x + kf'_y)\big|_{(x_0,y_0)} + \frac{1}{2!}(h^2f''_{xx} +$$

$$k^2f''_{yy} + 2hkf''_{xy})\big|_{(x_0,y_0)} + o(h^2+k^2).$$

$$= 1 + o(x^2+y^2)$$

② 带有 Lagrange 余项的2阶 Taylor公式：

$$f(x,y)=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta h,\ y_0+\theta k)\qquad 0<\theta<1$$

$$=1+0+0+\frac{1}{3!}\left(h^3 f'''_{xxx}+k^3 f'''_{yyy}+3h^2k f'''_{x^2y}\right.$$

$$\left.+3hk^2 f'''_{xy^2}\right)\Big|_{(x_0+\theta h,\ y_0+\theta k)}$$

$\because h=x-x_0=x\quad k=y-y_0=y$

$$\therefore f(x,y)=1+\frac{1}{3!}\left[h^3(-12x\cos(x^2+y^2)+8x^3\sin(x^2+y^2))\right.$$

$$+k^3(-12y\cos(x^2+y^2)+8y^3\sin(x^2+y^2))+3h^2k(-4y\cos(x^2+y^2)$$

$$\left.+8x^2y\sin(x^2+y^2))+3hk^2(-4x\cos(x^2+y^2)+8xy^2\sin(x^2+y^2))\right]$$

中括号内 $=(-12\theta x^4-12\theta y^4-12\theta x^2y^2-12\theta x^2y^2)\cos(\theta^2x^2+\theta^2y^2)$

$$+(8\theta^3x^6+8\theta^3y^6+24\theta^3x^4y^2+24\theta^3x^2y^4)\sin(\theta^2x^2+\theta^2y^2)$$

$$t_\theta=-12\theta(x^2+y^2)^2\cos(\theta^2x^2+\theta^2y^2)$$

$$+8\theta^3(x^2+y^2)^3\sin(\theta^2x^2+\theta^2y^2)$$

$$\therefore f(x,y)=1+\frac{1}{3!}(t_\theta)\qquad 0<\theta<1$$

(2). $z=e^{x^2-y^2}$

解: $J(z_0)=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)\Big|_{(0,0)}=(2xe^{x^2-y^2},-2ye^{x^2-y^2})$

$=(0,0)$

$\frac{\partial^2 z}{\partial x\partial y}=-4xy\,e^{x^2-y^2}$ $\qquad$ $\frac{\partial^2 z}{\partial x^2}=2e^{x^2-y^2}+4x^2e^{x^2-y^2}$

$\frac{\partial^2 z}{\partial y^2}=-2e^{x^2-y^2}+4y^2e^{x^2-y^2}$

$$H(X_0)=\begin{bmatrix}2&0\\0&-2\end{bmatrix}$$

∴带 Peano 余项的二阶 Taylor 公式:

$$z=f(X_0)+J(f(X_0))\Delta X+\frac{1}{2!}(\Delta X)^T H(X_0)\Delta X+o(\Delta X)$$

$$=1+0+\frac{1}{2!}(x,y)\begin{bmatrix}2&0\\0&-2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}+o(x^2+y^2)$$

$$=1+x^2-y^2+o(x^2+y^2)$$

~~$H(X_0+\theta\Delta X)=H$~~

$$X_0+\theta\Delta X=\begin{pmatrix}0\\0\end{pmatrix}+\theta\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\theta x\\\theta y\end{pmatrix}$$

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$$\frac{\partial^3 z}{\partial x^3}=4xe^{x^2-y^2}+8xe^{x^2-y^2}+8x^3e^{x^2-y^2}$$

$$=(12x+8x^3)e^{x^2-y^2}$$

$$\frac{\partial^3 z}{\partial y^3}=(4y+8y-8y^3)e^{x^2-y^2}=(12y-8y^3)e^{x^2-y^2}$$

$$\frac{\partial^3 z}{\partial x^2\partial y}=(-4y-8x^2y)e^{x^2-y^2}$$

$$\frac{\partial^3 z}{\partial x\partial y^2}=(-4x+8xy^2)e^{x^2-y^2}$$

$h=x-x_0=x \quad k=y-y_0=y$

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3=\sum_{i=0}^{3}C_3^i h^i k^{3-i}\frac{\partial^3}{\partial x^i\partial y^{3-i}}$$

$$=y^3\frac{\partial^3 z}{\partial y^3}+3xy^2\frac{\partial^3 z}{\partial x\partial y^2}+3x^2y\frac{\partial^3 z}{\partial x^2\partial y}$$

$$+x^3\frac{\partial^3 z}{\partial x^3}=(12x^4+8x^6-24x^2y^2+24x^2y^4-24x^4y^2$$
$$-8y^6+12y^4)e^{x^2-y^2}.$$

则在 $x_0+\theta\Delta x$ 处，上式为：

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta\Delta x)=(12\theta^4x^4+8\theta^6x^6-24\theta^4x^2y^2$$

$$+24\theta^6 x^2 y^4 - 24\theta^6 x^4 y^2 - 8\theta^6 y^6 + 12\theta^4 y^4) e^{\theta^2 (x^2 - y^2)} = t_\theta$$

∴ 带 Lagrange 余项的 n 阶 Taylor 公式为：

$$z = 1 + x^2 - y^2 + \cancel{\frac{1}{3!}(\ldots} \ t\ ) \quad (0<\theta<1)$$

$$+\frac{1}{3!}(t_\theta)$$

13). $u=\ln(1+x+y+z)$

解: 设 $t=x+y+z$, $u=f(t)=\ln(1+t)$

① 带 Peano 余项的 2 阶 Taylor 公式:

$$f(t)=f(t_0)+f'(t_0)(t-t_0)+\frac{1}{2!}f''(t_0)(t-t_0)^2+o((t-t_0)^2)$$

$$=0+t+\frac{1}{2!}(-1)t^2+o(t^2)$$

$$f(x,y,z)=x+y+z-\frac{1}{2}(x+y+z)^2+o((x+y+z)^2)$$

② 带 Lagrange 余项的 2 阶 Taylor 公式:

$$f(t)=f(t_0)+f'(t_0)(t-t_0)+\frac{1}{2!}f''(t_0)(t-t_0)^2+\frac{1}{3!}f'''(t_0+\theta(t-t_0))(t-t_0)^3$$

$$=0+t+\frac{1}{2!}(-1)t^2+\frac{1}{3!}\frac{2}{(1+\theta t)^3}t^3$$

$$f(x,y,z)=x+y+z-\frac{(x+y+z)^2}{2}+\frac{(x+y+z)^3}{3(1+\theta x+\theta y+\theta z)^3}$$

$(0<\theta<1)$

2, 12). $z = \frac{\cos x}{\cos y}$

解: $f'_x = -\frac{\sin x}{\cos y}$ $\quad f'_y = \frac{\cos x \sin y}{\cos^2 y}$

$f''_{xx} = -\frac{\cos x}{\cos y}$ $\quad f''_{yy} = \frac{\cos x(\sin^2 y + 1)}{\cos^3 y}$ $\quad f''_{xy} = -\frac{\sin x \sin y}{\cos^2 y}$

$f'''_{xxx} = \frac{\sin x}{\cos y}$ $\quad f'''_{yyy} = \frac{\cos x\,(5\sin y + \sin^3 y)}{\cos^4 y}$

$f'''_{xxy} = \frac{-\cos x \sin y}{\cos^2 y}$ $\quad f'''_{xyy} = \frac{-\sin x(\sin^2 y + 1)}{\cos^3 y}$

① 带 Peano 余项的2阶 Taylor 公式:

$$z = f(x,y) = f(x_0, y_0) + (h f'_{x_0} + k f'_{y_0}) + \frac{1}{2!}(h^2 f''_{x_0x_0} + k^2 f''_{y_0y_0} + 2hk f''_{x_0y_0}) + o(h^2+k^2).$$

$$= 1 + 0 + \frac{1}{2!}(-h^2 + k^2) + o(h^2+k^2).$$

$$= 1 + \frac{1}{2}(-x^2 + y^2) + o(x^2+y^2).$$

② 带 Lagrange 余项的2阶 Taylor 公式:

$=$

$$z=f(x,y)=1+\frac{1}{2}(-x^2+y^2)+\frac{1}{3!}(h^3f'''_{xxx}+k^3f'''_{yyy}$$

$$+3h^2kf'''_{x^2y}+3hk^2f'''_{xy^2})\Big|_{(x_0+\theta h,\ y_0+\theta k)} \qquad 0<\theta<1.$$

$$=1+\frac{1}{2}(-x^2+y^2)+\frac{1}{6}\Big[x^3\frac{\sin\theta x}{\cos\theta y}+y^3\frac{\cos\theta x(5\sin\theta y+\sin^3\theta y)}{\cos^4\theta y}$$

$$-3x^2y\frac{\cos\theta x\sin\theta y}{\cos^2\theta y}-3xy^2\frac{\sin\theta x(\sin^2\theta y+1)}{\cos^3\theta y}\Big]$$