习题4.2

3. (1): $x=3t$, $y=3t^2$, $z=2t^3$ 从 $(0.0.0)$ 到 $A(3,3,2)$

$$L_{弧长}=\int_L dl=\int_0^1\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

$$=\int_0^1\sqrt{3^2+(6t)^2+(6t^2)^2}\,dt=\int_0^1 3(2t^2+1)\,dt$$

$$=5$$

(2): $x=e^{-t}\cos t$, $y=e^{-t}\sin t$, $z=e^{-t}$ $(0\leq t\leq+\infty)$

$$L_{弧长}=\int_L dL=\int_0^{+\infty}\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

$$=\int_0^{+\infty}\sqrt{[-e^{-t}\cos t+(-e^{-t}\sin t)]^2+[-e^{-t}\sin t+e^{-t}\cos t]^2+[-e^{-t}]^2}\,dt$$

$$=\int_0^{+\infty}\sqrt{3e^{-2t}}\,dt=\sqrt{3}$$

4. $y=\ln x$ 线密度为 $\rho(x,y)=x^2$ 在 $x=\sqrt{3}$ 到 $x=\sqrt{15}$ 之间质量

解 $$M=\int_L \rho(x,y)\,dl$$

$$=\int_{\sqrt{3}}^{\sqrt{15}} x^2\cdot\sqrt{1+[(\ln x)']^2}\,dx$$

$$=\int_{\sqrt{3}}^{\sqrt{15}} x\sqrt{x^2+1}\,dx \overset{令x^2=t}{=\!=\!=} \frac{1}{3}(t+1)^{\frac{3}{2}}\Big|_3^{15}$$

$$=\frac{56}{3}$$

5、求圆柱面 $x^2+y^2=a^2$ 介于曲面 $z=a+\frac{x^2}{a}$ 与 $z=0$ 之间面积、

解：$\int_L z\,dl = \int_0^{2\pi} z\,a\,d\theta$ 在极坐标下

$\begin{cases} x=r\cdot\cos\theta \\ y=r\cdot\sin\theta \end{cases}$ $r=a$

$= \int_0^{2\pi}(a+\frac{x^2}{a})\cdot a\,d\theta$

$= a^2\int_0^{2\pi}[\cos^2\theta+1]\,d\theta$

$= 3\pi a^2$

6. 摆线：$\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$ $0\le t\le\pi$ 的质心

解：$M=\int_L \rho\,dl = \int_0^{\pi}\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$

$= a\int_0^{\pi}\sqrt{(1-\cos t)^2+\sin^2 t}\,dt = a\int_0^{\pi}\sqrt{2-2\cos t}\,dt$

$= \cancel{8a}\ 4a$

静力矩：$M_x=\int_L y\rho\,dl = \int_0^{\pi}a^2(1-\cos t)\sqrt{2-2\cos t}\,dt$

$= \frac{16}{3}a^2$

$M_y=\int_L x\rho\,dl = \int_0^{\pi}a^2(t-\sin t)\sqrt{2-2\cos t}\,dt$

$= \frac{16}{3}a^2$

$\begin{cases} \bar{x} = \frac{M_x}{M} = \frac{4}{3}a \\ \bar{y} = \frac{M_y}{M} = \frac{4}{3}a \end{cases}$

7、螺线 $x=a\cos t$, $y=a\sin t$ $z=\frac{b}{2\pi}t$ $(0\le t\le 2\pi)$ 绕 x 轴旋转的转动惯量（线密度为1）

解：$J_x=\int_L (y^2+z^2)\,dl$

$=\int_0^{2\pi}\left[(a\sin t)^2+\left(\frac{b}{2\pi}t\right)^2\right]\cdot\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$

$=\int_0^{2\pi}\left(a^2\sin^2 t+\frac{b^2}{4\pi^2}t^2\right)\cdot\sqrt{a\sin^2 t+a\cos^2 t+\left(\frac{b}{2\pi}\right)^2}\,dt$

$=\left(\frac{a^2}{2}+\frac{b^2}{3}\right)\sqrt{4\pi^2a^2+b^2}$

4.3、

2、计算圆柱面 $x^2+y^2=ax$ 被球面 $x^2+y^2+z^2=a^2$ 所截部分的面积

解：圆柱面：$x^2+y^2=ax \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$ $|z|=\sqrt{a^2-x^2-y^2}$

$ds=\sqrt{[x'(t)]^2+[y'(t)]^2}=\frac{a}{2}dt$

~~$S=\iint$~~

圆柱面在 x, y 上投影为

$L\begin{cases} x=\frac{a}{2}(1+\cos t) \\ y=\frac{a}{2}\sin t \end{cases}$ $t\in[0,\pi]$

$A=4\int_L |z|\,ds=4\int_L\sqrt{a^2-x^2-y^2}\,ds$

$=4a^2$

3、抛物面 $2z=x^2+y^2$ 在 $z\in[0,1]$ 部分的质量，$\sigma=z$

$M=\iint_S z\,ds=\iint_{D_{xy}}\frac{1}{2}(x^2+y^2)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$

当 $z=1$ 时有 $2=x^2+y^2$. $\frac{\partial z}{\partial x}=x$; $\frac{\partial z}{\partial y}=y$

∴有：$M=\int_0^{2\pi}d\varphi\int_0^{\sqrt{2}}\frac{1}{2}r^2\sqrt{1+r^2}\cdot r\,dr=\frac{2}{15}\pi(1+6\sqrt{3})$

§4.3

6、解：设质心坐标为 $(\bar{x},\bar{y},\bar{z})$

① 第一象限：

由对称性 $\Rightarrow \bar{x}=\bar{y}=\bar{z}$

记 $S_1=\{(x,y,z)\mid x^2+y^2+z^2=a^2,\ x,y,z\geq 0\}$

则 $\bar{z}=\dfrac{\iint_{S_1} z\,dS}{\iint_{S_1} dS}=\dfrac{\iint_{S_1} z\,dS}{S_1}=\dfrac{\iint_{D_1} z\cdot\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy}{\frac{1}{2}\pi a^2}$

$=\dfrac{\iint_{D_1}\sqrt{a^2-x^2-y^2}\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy}{\frac{1}{2}\pi a^2}=\dfrac{a\cdot\frac{1}{4}\pi a^2}{\frac{1}{2}\pi a^2}=\frac{1}{2}a$

其中 D_1 为 S_1 在 xOy 平面投影.

$\Rightarrow$ 质心 $(\frac{a}{2},\frac{a}{2},\frac{a}{2})$

② 上半球面：

由对称性 $\Rightarrow \bar{x}=\bar{y}=0$

记 $S_2=\{(x,y,z)\mid x^2+y^2+z^2=a^2,\ z\geq 0\}$，$D_2$ 为 S_2 在 xOy 平面上投影

$\bar{z}=\dfrac{\iint_{S_2} z\,dS}{\iint_{S_2} dS}=\dfrac{\iint_{D_2} a\,dx\,dy}{S_2}=\dfrac{a\cdot\pi a^2}{2\pi a^2}=\dfrac{a}{2}$

$\Rightarrow$ 质心 $(0,0,\frac{a}{2})$

10、解：(x_0,y_0,z_0) 处切平面方程：$\dfrac{2x_0}{a^2}(x-x_0)+\dfrac{2y_0}{b^2}(y-y_0)+\dfrac{2z_0}{c^2}(z-z_0)=0$

原点到此平面距离：$\dfrac{\left|\frac{x_0^2}{a^2}+\frac{1}{b^2}y_0^2+\frac{1}{c^2}z_0^2\right|}{\sqrt{\frac{x_0^2}{a^4}+\frac{y_0^2}{b^4}+\frac{z_0^2}{c^4}}}=\dfrac{1}{\sqrt{\frac{1}{a^4}x_0^2+\frac{1}{b^4}y_0^2+\frac{1}{c^4}z_0^2}}$

记 S 的上半平面部分为 S_1

则 $\iint_{S_1} L(x,y,z)\,dS=\iint_{D_{xy}}\dfrac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\cdot\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$

$=\iint_{D_{xy}}|c|\cdot\dfrac{1}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\,dx\,dy \xlongequal[y=|b|r\sin\theta]{x=|a|r\cos\theta}\int_0^1\int_0^{2\pi}|c|\cdot\dfrac{1}{\sqrt{1-r^2}}\cdot|ab|r\,d\theta\,dr$

$=2\pi|abc|$

$\Rightarrow \iint_S L(x,y,z)\,dS=4\pi|abc|$

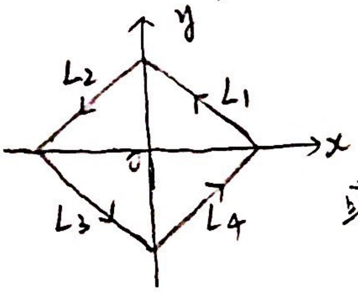
§4.4

2、(1) 原式 $= \int_0^2 \left[x^2 - (x^2)^2\right] dx = \left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = -\frac{56}{15}$

(2) 原式 $\overset{x=a\cos\theta}{\underset{y=a\sin\theta}{=\!=}} \int_0^{2\pi} \frac{(a\cos\theta + a\sin\theta)\cdot(-a\sin\theta) + (a\sin\theta - a\cos\theta)\cdot(a\cos\theta)}{a^2} d\theta$

$= \int_0^{2\pi} -1 \, d\theta = -2\pi$

(3) 原式 $= \oint_{L^+} \frac{1 + \frac{dy}{dx}}{|x|+|y|} dx = \int_{L_2^+} \frac{2}{|x|+|y|} dx + \int_{L_4^+} \frac{2}{|x|+|y|} dx$

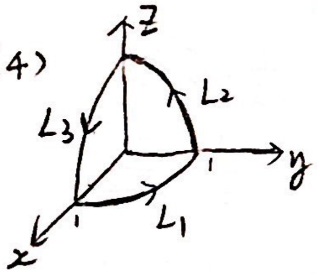


$= \int_0^{-1} \frac{2}{-x+(x+1)} dx + \int_0^1 \frac{2}{x-(x-1)} dx = -2+2=0$

或：由对称性，原式 $= \left[\left(\int_{L_1^+} + \int_{L_3^+}\right) + \left(\int_{L_2^+} + \int_{L_4^+}\right)\right] \frac{dx+dy}{|x|+|y|}$

$= 0 + 0 = 0$

(4)



原式 $= 3\int_{L_1^+} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz$ (对称性)

$\overset{x=\cos\theta,\ y=\sin\theta,\ z=0}{=\!=} 3\int_0^{\frac{\pi}{2}} \left[\sin^2\theta(-\sin\theta) + (-\cos^2\theta)(\cos\theta)\right] d\theta$

$= 3\int_0^{\frac{\pi}{2}} (1-\cos^2\theta) d\cos\theta + 3\int_0^{\frac{\pi}{2}} (\sin^2\theta - 1) d\sin\theta$

$= 3 \times \left(-\frac{4}{3}\right) = -4$

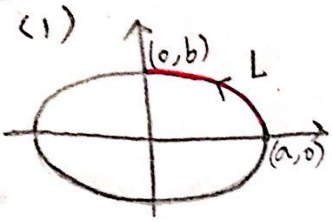
(5) 曲线参数方程为 $x = \cos\theta$，$y = \frac{\sqrt{2}}{2}\sin\theta$，$z = \frac{\sqrt{2}}{2}\sin\theta$，$\theta \in [0, 2\pi]$

$\int_{L^+} xyz \, dz = \int_0^{2\pi} \frac{1}{2}\cos\theta\sin^2\theta \cdot \frac{\sqrt{2}}{2}\cos\theta \, d\theta$

$= \frac{\sqrt{2}}{4}\int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta = \frac{\sqrt{2}}{16}\pi$

$F(x,y) = \sqrt{x^2+y^2}\left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}\right) = (-x,-y)$

4、(1)



$$W_1 = \int_{L^+} -x\,dx - y\,dy$$

$$\xlongequal[0\le\theta\le\frac{\pi}{2}]{\begin{cases}x=a\cos\theta\\ y=b\sin\theta\end{cases}} \int_0^{\frac{\pi}{2}} (-a\cos\theta)(-a\sin\theta) - (b\sin\theta)(b\cos\theta)\,d\theta$$

$$= (a^2-b^2)\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\,d\theta$$

$$= \frac{a^2-b^2}{2}$$

(2) $W_2 = 0$ (对称性)