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微积分A(1) [redacted] [redacted] 3-1. [redacted]

1.5.1 $\exists \varepsilon>0, \forall N\in N_+, \exists n,m>N$，使 $|a_n-a_m|\geq\varepsilon$.

1.5.2(3) $\forall n,p\in N_+$, $|a_{n+p}-a_n|=\left|\sum\limits_{k=n+1}^{n+p}c_kq^k\right|\leq\sum\limits_{k=n+1}^{n+p}|c_k||q|^k$.

$\because \{c_k\}$有界 $\therefore \forall k\in N_+\ |c_k|\leq|M|$ $\therefore \sum\limits_{k=n+1}^{n+p}|c_k||q|^k\leq|M|\sum\limits_{k=n+1}^{n+p}|q|^k=\frac{1-|q|^p}{1-|q|}\cdot|M|\cdot|q|^{n+1}\leq\frac{|M||q|^{n+1}}{1-|q|}$

$\therefore \exists\ N=\left|\log_{|q|}\left(\frac{1-|q|}{\cancel{1-|q|^p}}\cdot\frac{\varepsilon}{|M|}\right)\right|+1$，对 $\forall n,p\in N_+, n>N$，有：

$\frac{1-\cancel{|q|^p}}{1-|q|}|M||q|^{n+1}<\varepsilon \Rightarrow |a_{n+p}-a_n|<\varepsilon$ $\therefore \{a_n\}$为柯西列，故 $\lim\limits_{n\to\infty}a_n$ 存在.

1.5.2(6) $\forall n,p\in N_+$ $|a_{n+p}-a_n|=\left|\sum\limits_{k=n+1}^{n+p}\frac{(-1)^{k-1}}{k^\alpha}\right|$

当p为偶数. $\left|\sum\limits_{k=n+1}^{n+p}\frac{(-1)^{k-1}}{k^\alpha}\right|=\left|\frac{1}{(n+1)^\alpha}-\frac{1}{(n+2)^\alpha}+\cdots+\frac{1}{(n+p-1)^\alpha}-\frac{1}{(n+p)^\alpha}\right|$

$\leq\left|\frac{1}{n^\alpha}-\frac{1}{(n+2)^\alpha}+\frac{1}{(n+3)^\alpha}-\frac{1}{(n+4)^\alpha}+\cdots+\frac{1}{(n+p-1)^\alpha}-\frac{1}{(n+p)^\alpha}\right|<\frac{1}{n^\alpha}$

当p为奇数 $\left|\sum\limits_{k=n+1}^{n+p}\frac{(-1)^{k-1}}{k^\alpha}\right|<\left|\frac{1}{n^\alpha}-\frac{1}{(n+2)^\alpha}+\cdots+\frac{1}{(n+p-3)^\alpha}-\frac{1}{(n+p-1)^\alpha}\right|+\frac{1}{(n+p)^\alpha}<\frac{2}{n^\alpha}$

$\therefore \forall\varepsilon>0$ $\exists N=\left[\left(\frac{2}{\varepsilon}\right)^{\alpha}\right]+1$ $\forall n,p\in N_+, n>N$，有 $|a_{n+p}-a_n|<\frac{2}{n^\alpha}<\varepsilon$

$\therefore \{a_n\}$为柯西列. $\lim\limits_{n\to\infty}a_n$ 存在.

1.5.3(1) a_n 可记为 $a_n=\begin{cases}-\frac{1}{2}, & n\neq 3k, k\in Z\\ 1, & n=3k, k\in Z.\end{cases}$

对 $\varepsilon=1$，$\forall N\in N_+$，总有 $n>N$ 且 $n=3k$ 则 $|a_n-a_{n+1}|=\frac{3}{2}>1$

$\therefore \{a_n\}$ 不是柯西列，则 $\lim\limits_{n\to\infty}a_n$ 不存在.

6. 由Bolzano定理，$\{a_n\}$ 必有一个子列 $\{a_n^{(1)}\}$ 收敛，记收敛于A

$\because \{a_n\}$不收敛 $\therefore \exists\varepsilon>0$. $N_1=1$，$\exists n_1>N_1$，使 $|a_{n_1}-A|\geq\varepsilon$

$N_2=n_1+1$，$\exists n_2>N_2$，使 $|a_{n_2}-A|\geq\varepsilon\cdots$

以此类推，得到数列 $\{a_{n_k}\}$ 不收敛于A. 其子列 $\{a_n^{(2)}\}$ 中任一元素须有 $|a_n^{(2)}-A|\geq\varepsilon$

故也不收敛于A. 但 $\{a_{n_k}\}$ 有界，故必有子列 $\{a_n^{(3)}\}$ 收敛到不同于A的B.

8. 由题: $|a_{n+1}-a_n| \le q^{n-1}|a_2-a_1|$ $\forall p\in N_+$, $n\in N_+$ 且 $n\ge 2$

$|a_{n+p}-a_{n+p-1}+a_{n+p-1}-\cdots+a_{n+1}-a_n| \le \sum_{k=n+1}^{n+p}|a_k-a_{k-1}| \le |a_2-a_1|\sum_{k=n+1}^{n+p} q^{k-2} = |a_2-a_1|\frac{q^{n-1}(q^p-1)}{q-1} \le |a_2-a_1|\frac{q^{n-1}}{1-q}$

∴ $\forall \varepsilon>0\ \exists N=\left[\log_q\left(\frac{\varepsilon(1-q)}{q\cancel{(1-q^p)}|a_2-a_1|}\right)\right]+2$, $\forall n,p\in N_+$, $n>N$

由 $|a_{n+p}-a_n|\le|a_2-a_1|\frac{q^{n-1}(q^p-1)}{q-1}<\varepsilon$. ∴$\{a_n\}$为柯西列，故$\{a_n\}$收敛.

总复习1.12.

(1) $|x_{n+p}-x_n| = |x_{n+p}-x_{n+p-1}+x_{n+p-1}-x_{n+p-2}+\cdots+x_{n+1}-x_n| \le \sum_{k=n}^{n+p-1}|x_{k+1}-x_k| \le \sum_{k=n}^{n+p-1}\frac{1}{k} \le \frac{p}{n}$

~~先证~~ $\cancel{\sum_{k=1}^{n}\frac{1}{k}<\ln(n+1)}$ ~~即证~~ $\cancel{\frac{1}{n}<\ln(\frac{n+1}{n})}$ 当设 $x_n=\sum_{k=1}^{n}\frac{1}{k}$.

证 $x_n>\ln(n+1)$ 即证 $\frac{1}{n}>\ln(1+\frac{1}{n})$ 易证,略

$\forall M>0$ $\exists N=[e^M]+1$ $n>N$时 $x_n>\ln(n+1)>M$ ∴$\{x_n\}$为无穷大数列

∴$\{x_n\}$此时不收敛，$\{x_n\}$不是柯西列.

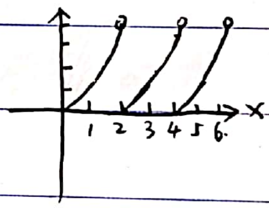
(2) 类似的, $|x_{n+p}-x_n| \le \sum_{k=n}^{n+p-1}\frac{1}{k^2}$. $n\ge2$时. $|x_{n+p}-x_n|\le\sum_{k=n}^{n+p-1}\frac{1}{k(k-1)} = \frac{1}{n-1}-\frac{1}{n+p-1}\le\frac{1}{n-1}$

$\forall\varepsilon>0$, $\exists N=[\frac{1}{\varepsilon}]+2$, 对~~n≥N~~ $n,p\in N_+$, $n>N$时 $|x_{n+p}-x_n|\le\frac{1}{n-1}<\varepsilon$.

∴$\{x_n\}$为柯西列.

2.1.14 $x\in[4,6)$时, $f(x)=f(x-4)=(x-4)^2$.

$x\in[2,4)$时, $f(x)=f(x-2)=(x-2)^2$.



18 (1). $y-2=\ln(x-1)$ $x=e^{y-2}+1$ ∴$f^{-1}(x)=e^{x-2}+1$. 定义域R

(3) ∵$x\in[0,\pi]$ ∴$\cos x\in[-1,1]$ $\cos^3x\in[-1,1]$ ∴$y\in[0,2]$ 定义域$[0,2]$.

$\cos^3x=y-1$ ⇒ $\cos x=\sqrt[3]{y-1}$ $x=\arccos\sqrt[3]{y-1}$ ∴$f^{-1}(x)=\arccos\sqrt[3]{y-1}$.

(5) $x<-1$时. $2x^2=1-y$ ⇒ $x=-\sqrt{\frac{1-y}{2}}$ ∵$x<-1$ ∴$y<-1$.

$-1\le x\le 2$时. $x=\sqrt[3]{y}$ $-1\le y\le 8$.

$x>2$时. $x=\frac{y+16}{12}$ $y>8$ 综上 $f^{-1}(x)=\begin{cases}-\sqrt{\frac{1-x}{2}}, & x<-1\\ \sqrt[3]{x} & -1\le x\le 8\\ \frac{x+16}{12} & x>8\end{cases}$

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总复习 2.1

(1) $f: (0,1) \to R$ $\quad f(x) = \cancel{\arctan}\ \tan(\pi x - \frac{\pi}{2})$

(2) $f: N \to Z$ $\quad f(x) = \begin{cases} \frac{x}{2}, & x\text{是偶数}. \\ -\frac{x+1}{2}, & x\text{是奇数}. \end{cases}$

[redacted]

1. 设 $x_0 > 0$，证 $\lim\limits_{x \to x_0} \ln x = \ln x_0$.

pf: $\forall \varepsilon > 0$，~~∃ δ…~~ ∃ δ满足 $0 < \delta < \min\{x_0(e^{\varepsilon}-1), x_0(1-e^{-\varepsilon})\}$

当 $0<|x-x_0|<\delta$ 时，有 $x_0(e^{-\varepsilon}-1) < x - x_0 < x_0(e^{\varepsilon}-1)$

$\Rightarrow x_0 e^{-\varepsilon} < x < x_0 e^{\varepsilon} \Rightarrow e^{-\varepsilon} < \frac{x}{x_0} < e^{\varepsilon} \Rightarrow -\varepsilon < \ln x - \ln x_0 < \varepsilon \Rightarrow |\ln x - \ln x_0| < \varepsilon$

$\therefore \lim\limits_{x \to x_0} \ln x = \ln x_0$. #

2.2.2.

(3) 等价. 先证 $\lim\limits_{x \to x_0} f(x) = A$ 对 (3) 充分:

因 $\forall \varepsilon > 0$，$\exists \delta_\varepsilon > 0$，只要 ~~在~~ $0<|x-x_0|<\delta_\varepsilon$，就有 $|f(x)-A|<\varepsilon$

对任意 k，取 $\varepsilon < \frac{1}{2^k}$，就有 $|f(x)-A| < \varepsilon < \frac{1}{2^k}$，不难由定义知成立.

再证必要: ~~因~~ 对 $\forall \varepsilon > 0$，$\exists k$ 满足 $k > \log_2 \frac{1}{\varepsilon}$

由(3)，$\exists \delta_k > 0$ 只要 ~~在~~ $0<|x-x_0|<\delta_k$，就有 $|f(x)-A| < \frac{1}{2^k} < \varepsilon$ 故成立.

综上，二者等价. #

(4). 不等价. 先证 $\lim\limits_{x \to x_0} f(x) = A$ 对 (4) ~~充分~~ 不充分 举反例说明 如 $f(x)=2x$

因 $\forall \varepsilon > 0$，$\exists \delta_\varepsilon > 0$，只要 $0<|x-x_0|<\delta_\varepsilon$，就有 $|f(x)-A| < \varepsilon$

取 $\varepsilon = \frac{1}{n}$ 若 $0<\delta_\varepsilon<\frac{1}{n}$，则 (4) 不成立.

2.2.4.

pf：已知 ~~$\forall \lim_{x\to x_0} f(x)=A, \exists \varepsilon$~~ $\forall \varepsilon>0$，$\exists \delta_\varepsilon>0$ st. 只要 $0<|x-x_0|<\delta_\varepsilon$，就有 $|f(x)-A|<\varepsilon$.

由绝对值不等式：$\big||f(x)|-|A|\big|<|f(x)-A|<\varepsilon$ 由定义得 $\lim_{x\to x_0}|f(x)|=|A|$. #

2.2.6(1)

记 f 在 (a,b) 的上确界为 A.

$\forall \varepsilon>0$ $\exists x$，st. $A-\varepsilon<f(x)<A$ $\because f(x)$ 递增

$\therefore \exists \delta>0$，满足 $\delta<b-a$ 且 $A-\varepsilon<f(b-\delta)<A$ 此时 $|f(x)-A|<\varepsilon$

$\therefore$ 由定义，$\lim_{x\to b^-} f(x)$ 存在

2.3.6

(4) $\lim_{x\to 1^-}\frac{\sqrt{(x-1)^2}}{x-1}=\lim_{x\to 1^-}\frac{|x-1|}{x-1}$ $\because x<1$ $\therefore \lim_{x\to 1^-}\frac{\sqrt{(x-1)^2}}{x-1}=\lim_{x\to 1^-}\frac{-(x-1)}{(x-1)}=-1$.

(11) $m=1$ 时 $\lim_{x\to 1}\frac{x^m-1}{x-1}=1$. $m>1$ 时 $\lim_{x\to 1}\frac{x^m-1}{x-1}=$ ~~$\frac{(x-1)\sum x}{x-1}$~~ $\lim_{x\to 1}\frac{(x-1)\sum_{k=1}^{m}x^{k-1}}{x-1}$

$=\lim_{x\to 1}\sum_{k=1}^{m}x^{k-1}=m$

综上，$\lim_{x\to 1}\frac{x^m-1}{x-1}=m$.

(17) 易知：$x(\frac{1}{x}-1)\le x[\frac{1}{x}]\le x\frac{1}{x}=1$

$\because \lim_{x\to 0}(\frac{1}{x}-1)x=\lim_{x\to 0}(1-x)=1$ 由夹逼定理，$\lim_{x\to 0}x[\frac{1}{x}]=1$