

3.3

(2)

$$\begin{aligned}
 & 3.3 \\
 & \text{ii) } y' = \frac{1}{2} x^{-\frac{1}{2}} \quad y'' = \frac{1}{2} \times (-\frac{1}{2}) x^{-\frac{3}{2}} \quad y''' = \frac{1}{2} \times (-\frac{1}{2}) \times (-\frac{3}{2}) \cdot x^{-\frac{5}{2}} \quad \dots \\
 & y^{(n)} = \frac{-(2n-3)!!}{2^{10}} x^{-\frac{19}{2}} \quad (n=10) = \frac{-17!!}{2^{10}} x^{-\frac{19}{2}}
 \end{aligned}$$

$$13) \text{ 由 } y = \frac{1}{x} \cdot \ln x$$

$$\left(\frac{1}{x}\right)^{(n)} = (-1)^n n! \frac{1}{x^{n+1}}$$

$$(\ln x)^{(n)} = \left(\frac{1}{x}\right)^{(n-1)}$$

$$\begin{aligned} \therefore y^{(5)} &= \sum_{k=0}^5 C_5^k \left(\frac{1}{x}\right)^{(k)} (\ln x)^{(5-k)} \\ &= \frac{274 - 120 \ln x}{x^6} \end{aligned}$$

$$15) y^{(100)} = \sum_{k=0}^{100} C_{100}^k x^{(k)} (\sinh x)^{(100-k)}$$

$$= x \sinh x + 100 \cosh x$$

$$17) \text{ 由 } (e^{ax})^n = a^n e^{ax}$$

$$(\sin bx)^n = b^n \sin\left(bx + \frac{n\pi}{2}\right)$$

$$\begin{aligned} \therefore y^{(n)} &= \sum_{k=0}^n C_n^k (e^{ax})^k (\sin bx)^{n-k} \\ &= \sum_{k=0}^n C_n^k a^k b^{n-k} e^{ax} \sin\left[bx + \frac{(n-k)\pi}{2}\right] \end{aligned}$$

$$19) \text{ 取 } f(x) = x^3, g(x) = e^x$$

$$y^{(n)} = C_n^0 x^3 e^x + C_n^1 \cdot 3x^2 e^x + C_n^2 \cdot 6x e^x + C_n^3 e^x$$

$$y^{(n)} = x^3 e^x + 3n x^2 e^x + 3n(n-1) x e^x + n(n-1)(n-2) e^x$$

(4)

$$\begin{aligned}
 4.13) \quad y''_{xx} &= y'_{xt} = \frac{y'}{x'} = \frac{f'(t) + t f''(t) - f'(t)}{f''(t)} = t \\
 y''_{xx} &= \frac{d^2 y}{dx^2} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{f''(t)} \\
 y'''_{xx} &= \frac{\frac{d}{dt} \left(\frac{1}{f''(t)} \right)}{\frac{dx}{dt}} = \frac{-[f''(t)]^2 \cdot f'''(t)}{f''(t)} = -\frac{f'''(t)}{[f''(t)]^3}
 \end{aligned}$$

(5)

$$\begin{aligned}
 5.13) \quad y'-2 &= (1-y')/\ln(x-y) + \frac{1-y'}{x-y} (x-y) \\
 y' &= \frac{\ln(x-y)+3}{2+\ln(x-y)} = 1 + \frac{1}{2+\ln(x-y)} \\
 y'' &= \frac{-\frac{1}{x-y} (1-y')}{[2+\ln(x-y)]^2} = \frac{1}{[2+\ln(x-y)]^3 (x-y)}
 \end{aligned}$$

(6)

$$\begin{aligned}
 6. \quad f'(x) &= \frac{1}{1+x^2}, \quad (1+x^2)f'(x) = 1 \\
 \text{求 } (n+2) \text{ 阶导} &= \sum_{k=0}^{n+1} C_{n+1}^k (1+x^2)^{(k)} f^{(n+1-k)}(x) = 0 \\
 \text{即 } (1+x^2)f^{(n+2)}(x) &+ 2(n+1)x f^{(n+1)}(x) + n(n+1)f^{(n)}(x) = 0 \\
 \text{令 } x=0 \text{ 则 } f^{(n+2)}(0) &= -n(n+1)f^{(n)}(0) \\
 f^{(0)}(0) &= 1, \quad f^{(2)}(0) = 0 \quad \text{即 } f^{(n)}(0) = \begin{cases} 0, & n=2k \\ (-1)^k (-1)^k, & n=2k+1 \end{cases} \quad k \in \mathbb{N} \\
 \text{利用归纳法 } y_{(n)}^{(1)} &= \frac{2 \arcsin x}{\sqrt{1-x^2}}
 \end{aligned}$$

$f^{(n)}(0) = \begin{cases} 0, & n \text{ 为偶数} \\ (-1)^{\frac{n-1}{2}} (n-1)!, & n \text{ 为奇数} \end{cases}$

$\text{法二: } y' = \frac{2 \arcsin x}{\sqrt{1-x^2}}$

(7)

$$7. f(x) = 2 \arcsin x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} f'(x) = 2 \arcsin x$$

$$\text{对 } x \text{ 求导: } \sqrt{1-x^2} f''(x) - \frac{2x}{\sqrt{1-x^2}} \cdot \frac{1}{2} f'(x) = \frac{2}{\sqrt{1-x^2}}$$

$$\text{即 } (1-x^2) f''(x) - x f'(x) = 2$$

对 x 求 n 阶导.

$$\sum_{k=0}^n C_n^k (1-x^2)^{(k)} [f''(x)]^{(n-k)} - \sum_{k=0}^n C_n^k x^{(k)} [f'(x)]^{(n-k)} = 0$$

分别保留 3 项, 2 项得

$$(1-x^2) f^{(n+2)}(x) - (2n+1)x f^{(n+1)}(x) - n^2 f^{(n)}(x) = 0$$

得证.

$$\text{代入 } x=0, \quad f^{(n+2)}(0) = n^2 f^{(n)}(0)$$

$$\text{由 } f'(0)=0 \quad f''(0)=2$$

$$\text{得 } f^{(n)}(0) = \begin{cases} 0, & n \text{ 为奇数} \\ 2^{n-1} \left[\left(\frac{n}{2} - 1 \right)! \right]^2, & n \text{ 为偶数} \end{cases}$$

4.1

(2)

习题4.1.

2. 设常数 a_0, a_1, \dots, a_n 满足 $\frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \dots + \frac{a_1}{2} + a_0 = 0$. 证明: 多项式

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ 在 $(0, 1)$ 内有一零点.

解: 令 $f(x) = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \dots + \frac{a_1}{2} x^2 + a_0 x$.

$f(0) = 0$, $f(1) = \frac{a_n}{n+1} + \dots + a_0 = 0 = f(0)$.

由罗尔中值定理, $\exists \xi \in (0, 1)$ 使 $f'(\xi) = 0$.

而 $f'(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$. \therefore 多项式 $a_n x^n + \dots + a_0$ 在 $(0, 1)$ 上有零点 ξ .

(5)

5. $\forall x_0, x_0 + \Delta x \in I$, s.t.

$$0 \leq \left| \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right| \leq M |\Delta x|$$

$$\text{当 } \Delta x \rightarrow 0, \lim_{\Delta x \rightarrow 0} \left| \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right| = f'(x_0)$$

$$\lim_{\Delta x \rightarrow 0} M |\Delta x| = 0$$

由夹逼原理, $f'(x_0) = 0$.

$$\text{则 } f'(x) \equiv 0$$

由 Lagrange 中值定理, 对 $\forall a, b \in I$ 且 $a < b$,

$$\exists \xi \in (a, b) \text{ s.t. } f'(\xi) = \frac{f(b) - f(a)}{b - a} = 0$$

$$\therefore f(b) = f(a)$$

$\therefore f(x)$ 恒为常数

年 月 日 星期
 设 $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow b^-} f(x) = A$
 11. 1) 补充定义 $f(a) = f(b) = A$

则 $f(x)$ 在 a 处右连续, b 处左连续

$\therefore f \in C[a, b]$, f 在 (a, b) 可导,

由罗尔定理, $\exists \xi \in (a, b)$ s.t.

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} = 0$$

(2) 设 $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow +\infty} f(x) = A$

则 $f \in C[a, +\infty)$ f 在 $(a, +\infty)$ 可导.

对 $\forall \varepsilon > 0, \exists M > 0$ s.t. $\forall x > M$, $0 < x - a < 3A$

有 $|f(x) - A| < \varepsilon$

$$\therefore \left| \frac{f(x) - f(a)}{x - a} \right| < \frac{\varepsilon}{x - a}$$

由夹挤原理 $0 \leq \lim_{x \rightarrow +\infty} \left| \frac{f(x) - f(a)}{x - a} \right| \leq \lim_{x \rightarrow +\infty} \frac{\varepsilon}{x - a} = 0$

$$\therefore \lim_{x \rightarrow +\infty} \frac{f(x) - f(a)}{x - a} = 0.$$

故 $\exists \xi \in (a, +\infty)$ s.t. $f'(\xi) = \lim_{x \rightarrow +\infty} \frac{f(x) - f(a)}{x - a} = 0$

有同样的结论成立

(12)

12.

$$\therefore \lim_{x \rightarrow \infty} \frac{f(x)}{|x|} = +\infty \quad \forall a \in \mathbb{R}$$
$$\text{令 } g(x) = f(x) - ax$$
$$\therefore \lim_{x \rightarrow \infty} \frac{g(x)}{|x|} = \frac{f(x)}{|x|} - a = +\infty$$
$$\therefore \lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow +\infty} g(x) = +\infty$$
$$\exists m > 0. \text{ s.t. } |x| > m \text{ 时, } g(x) > m$$
$$\therefore g(x), x \in C[-m, m].$$

存在最大值和最小值 A, B.

$$\exists \xi \in [-m, m]. \text{ s.t. } g(\xi) = \min_{x \in [-m, m]} g(x)$$

$g(x)$ 在 \mathbb{R} 上可导,

$$\text{有 } g'(\xi) = 0 \Rightarrow f'(\xi) = a.$$

(13)

$$13. \text{ 令 } H(x) = [f(a)g(b) - f(b)g(a)][h(x) - h(a)] + [g(a)h(b) - g(b)h(a)][f(x) - f(a)] + (h(a)f(b) - h(b)f(a))(g(x) - g(a))$$

$$H(a) = H(b) = 0 \quad \therefore \exists \xi \in (a, b) \text{ s.t. } H'(\xi) = 0$$

$$\text{即 } \begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(\xi) & g'(\xi) & h'(\xi) \end{vmatrix} = 0$$

(14)

$$14. \text{ (1) 设 } g(x) = f(x)e^x, \text{ 则 } g(a) = g(b) = 0.$$

$$g'(x) = [f'(x) + f(x)]e^x$$

$$g''(x) = [f''(x) + 2f'(x) + f(x)]e^x$$

$$\text{有 } g'_+(a) = f'_+(a)e^a$$

$$g'_-(b) = f'_-(b)e^b \quad \text{且 } f_+(a)f'_-(b) > 0$$

$$\therefore g'_+(a)g'_-(b) > 0$$

$$\text{设 } g'_+(a) > 0, g'_-(b) > 0.$$

$$\text{则 } \exists \delta > 0, \text{ s.t.}$$

$$g(x) > g(a) = 0, \forall x \in (a, a + \delta);$$

$$g(x) < g(b) = 0, \forall x \in (b - \delta, b)$$

$$\therefore \exists x_0 \in (a, b) \text{ s.t. } g(x_0) = 0$$

由 Lagrange 中值定理.

$$\exists x_1 \in (a, x_0) \text{ s.t. } g'(x_1) = \frac{g(x_0) - g(a)}{x_0 - a} = 0$$

$$\exists x_2 \in (x_0, b) \text{ s.t. } g'(x_2) = \frac{g(b) - g(x_0)}{b - x_0} = 0$$

由 $g'(x)$ 可导.

$$\text{知 } \exists \xi \in (x_1, x_2), \text{ s.t. } g''(\xi) = \frac{g'(x_2) - g'(x_1)}{x_2 - x_1} = 0$$

$$\text{即 } f''(\xi) + 2f'(\xi) + f(\xi) = 0.$$

(2) 设 $g(x) = f(x)e^{-x}$

$$\text{则 } g_+'(a)g_-'(b) > 0 \quad g''(x) = [f''(x) - 2f'(x) + f(x)]e^{-x}$$

由 (1), $\exists x_0 \in (a, b)$ s.t. $g(x_0) = 0$.

由 Lagrange 中值定理:

$$\exists x_1 \in (a, x_0) \text{ s.t. } g'(x_1) = \frac{g(x_0) - g(a)}{x_0 - a} = 0$$

$$\exists x_2 \in (x_0, b) \text{ s.t. } g'(x_2) = \frac{g(b) - g(x_0)}{b - x_0} = 0$$

$$\exists \theta \in (x_1, x_2) \text{ s.t. } g''(\theta) = \frac{g'(x_2) - g'(x_1)}{x_2 - x_1} = 0$$

$$\text{即 } f''(\theta) - 2f'(\theta) + f(\theta) = 0.$$

(3) 令 $g(x) = e^{-x} f(x)$, 则 $g'(x) = e^{-x} (f'(x) - f(x))$

$g(a) = g(b) = 0$ 则 $\exists x_0 \in (a, b)$

s.t. $g'(x_0) = 0$ (罗尔定理)

即 $f'(x_0) - f(x_0) = 0$

令 $G(x) = \cancel{f(x_0) - f(x)}_2 f'(x) - f(x)$

~~$G(a) = G(b)$~~ 则 $G'(x) = f''(x) - f'(x)$

且 $\lim_{x \rightarrow a} G(x) = \lim_{x \rightarrow b} G(x) = 0$

若 $\exists x_1 \in (a, b), x_1 \neq x_0$

s.t. $G(x_1) = 0$

则由罗尔定理,

$\exists \eta \in (a, b)$ s.t. $G'(\eta) = 0$

若 $G(x)$ 在 (a, b) 仅一零点 x_0 , 则

x_0 为 $G(x)$ 极值点. 由费马定理:

$$G'(x_0) = 0$$

综上, $\exists \eta \in (a, b)$, s.t. $G'(\eta) = 0$ 即

$$f'(\eta) = f(\eta)$$

(4) 由 (3) 知:

$$\exists x_0 \in (a, b), \text{ s.t. } f'(x_0) - f(x_0) = 0$$

$$\text{令 } M(x) = e^x (f'(x) - f(x))$$

$$\text{则 } M'(x) = e^x (f''(x) - f(x))$$

$$\underline{\text{且}} \lim_{x \rightarrow a} M(x) = \lim_{x \rightarrow b} M(x) = 0, \quad M(x_0) = 0$$

$$\text{若 } \exists x_2 \in (a, b), \text{ ~~且~~ } x_2 \neq x_0,$$

$$\text{s.t. } M(x_2) = 0$$

则由罗尔定理:

$$\exists \xi \in (a, b), \text{ s.t. } M'(\xi) = 0$$

若 (a, b) 上 $M(x)$ 仅 x_0 一个零点, 则其为极值点,

$$\text{由费马定理: } M'(x_0) = 0$$

$$\text{综上, } \exists \xi \in (a, b), \text{ s.t. } M'(\xi) = 0 \text{ 即}$$

$$f''(\xi) = f(\xi)$$