微积分A　第三周　　　　新雅93 章越闻 2019013397

3.2

习题1.5

4. 已知函数 $z=u\ln(u-v)$，其中 $u=e^{-x}$，$v=\ln x$，求 $\frac{dz}{dx}$.

解：$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}=\left[\frac{u}{u-v}+\ln(u-v)\right]\cdot(-e^{-x})-\frac{u}{u-v}\cdot\frac{1}{x}$

$=-u\left[\frac{u}{u-v}+\ln(u-v)\right]+\frac{u}{e^{v}(v-u)}$.

将 $u=e^{-x}$，$v=\ln x$ 代入得：$\frac{dz}{dx}=-e^{-x}\left[\frac{1}{1-e^{x}\ln x}+\ln(e^{-x}-\ln x)\right]+\frac{1}{x(e^{x}\ln x-1)}$

5. 已知函数 $u=f(x,y)$，其中 $x=r\cos\theta$，$y=r\sin\theta$，$f$ 可微，证明：

$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}\right)^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2.$$

证明：记 $(x,y)=g(r,\theta)=(r\cos\theta,r\sin\theta)$，$h(r,\theta)=f(g(r,\theta))=f(r\cos\theta,r\sin\theta)$，则

$Dh(r,\theta)=Df(x,y)\,Dg(r,\theta)=\left[\frac{\partial u}{\partial x}\ \ \frac{\partial u}{\partial y}\right]\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}=\left[\frac{\partial u}{\partial r}\ \ \frac{\partial u}{\partial\theta}\right]$

于是 $\left[\frac{\partial u}{\partial r}\ \ \frac{\partial u}{\partial\theta}\right]=\left[\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\quad -\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta\right]$

故 $\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}\right)^2=\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)^2+\left(-\frac{\partial u}{\partial x}\sin\theta+\frac{\partial u}{\partial y}\cos\theta\right)^2$

$=\left(\frac{\partial u}{\partial x}\right)^2\cos^2\theta+\left(\frac{\partial u}{\partial y}\right)^2\sin^2\theta+2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\sin\theta\cos\theta$

$+\left(\frac{\partial u}{\partial x}\right)^2\sin^2\theta+\left(\frac{\partial u}{\partial y}\right)^2\cos^2\theta-2\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\sin\theta\cos\theta$

$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$.　得证.

6. 设 $f$ 可微，$u=xy+xf\left(\frac{y}{x}\right)$，证明：$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u+xy$

证明：$\frac{\partial u}{\partial x}=y+f\left(\frac{y}{x}\right)+x\left[f\left(\frac{y}{x}\right)\right]_x=y+f\left(\frac{y}{x}\right)+xf'\left(\frac{y}{x}\right)\left[\frac{y}{x}\right]_x=y+f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)$ ……①

$\frac{\partial u}{\partial y}=x+x\left[f\left(\frac{y}{x}\right)\right]_y=x+xf'\left(\frac{y}{x}\right)\left[\frac{y}{x}\right]_y=x+f'\left(\frac{y}{x}\right)$ ……②

由①②式得：$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\left[y+f\left(\frac{y}{x}\right)-\frac{y}{x}f'\left(\frac{y}{x}\right)\right]+y\left[x+f'\left(\frac{y}{x}\right)\right]=2xy+xf\left(\frac{y}{x}\right)=u+xy$.

证毕.

7. 设 $f \in C^2(\mathbb{R}^2)$ 满足 Laplace 方程 $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$，证明：$u(x,y)=f\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)$ 也满足 Laplace 方程.

证明：记 $a=\frac{x}{x^2+y^2}$，$b=\frac{y}{x^2+y^2}$，由于 $f$ 满足 Laplace 方程，故 $f_{aa}+f_{bb}=0$ ⋯⋯ ①

$$Du(x,y)=[f_a\ \ f_b]\begin{bmatrix} a_x & a_y \\ b_x & b_y \end{bmatrix}=[f_a\ \ f_b]\begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2} \\ -\frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{bmatrix} \cdots ②$$

于是 $u_x=a_xf_a+b_xf_b$，$u_y=a_yf_a+b_yf_b$

故 $u_{xx}=[a_x]_xf_a+a_x[f_a]_x+[b_x]_xf_b+b_x[f_b]_x=a_{xx}f_a+a_x(f_{aa}a_x+f_{ab}b_x)+b_{xx}f_b+b_x(f_{ba}a_x+f_{bb}b_x)$

同理 $u_{yy}=a_{yy}f_a+a_y(f_{aa}a_y+f_{ab}b_y)+b_{yy}f_b+b_y(f_{ba}a_y+f_{bb}b_y)$

故 $u_{xx}+u_{yy}=(a_{xx}+a_{yy})f_a+(b_{xx}+b_{yy})f_b+(a_x^2+a_y^2)f_{aa}+(b_x^2+b_y^2)f_{bb}+2f_{ab}(a_xb_x+a_yb_y)$ ⋯ ③

由②可知：$a_x=-b_y$，$a_y=b_x$，于是 $a_x^2+a_y^2=b_x^2+b_y^2$，$a_xb_x+a_yb_y=0$. 再结合①式得：

$u_{xx}+u_{yy}=(a_{xx}+a_{yy})f_a+(b_{xx}+b_{yy})f_b$ ⋯ ③

由②知：$a_{xx}=\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]_x=\frac{2x^3-6xy^2}{(x^2+y^2)^3}$，由对称性：$b_{yy}=\frac{2y^3-6x^2y}{(x^2+y^2)^3}$

$a_{yy}=\left[-\frac{2xy}{(x^2+y^2)^2}\right]_y=\frac{6xy^2-2x^3}{(x^2+y^2)^3}$，由对称性 $b_{xx}=\frac{6x^2y-2y^3}{(x^2+y^2)^3}$

故 $a_{xx}+a_{yy}=b_{xx}+b_{yy}=0$，代入③中得：$u_{xx}+u_{yy}=0$.

即 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$，故 $u(x,y)$ 满足 Laplace 方程. 证毕.

8. 已知变换 $\begin{cases} w=x+y+z \\ u=x \\ v=x+y \end{cases}$，化简方程 $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}=0$，以 $w$ 为因变量，$u,v$ 为自变量.

解：原方程 $\Leftrightarrow [z_x-z_y]_x-[z_x-z_y]_y+(z_x-z_y)=0$ ⋯ ①

而 $z_x=[w(u,v)-v]_x=w_u\cdot u_x+w_v\cdot v_x-v_x=w_u+w_v-1$

$z_y=[w(u,v)-v]_y=w_u\cdot u_y+w_v\cdot v_y-v_y=w_v-1$. 故 $z_x-z_y=w_u$

代入①式得：$[w_u]_x-[w_u]_y+w_u=0$ ⋯ ②

而 $[w_u]_x=w_{uu}\cdot u_x+w_{uv}\cdot v_x=w_{uu}+w_{uv}$

$[w_u]_y=w_{uu}\cdot u_y+w_{uv}\cdot v_y=w_{uv}$

代入②式得：$w_{uu}+w_u=0$，即 $\frac{\partial^2 w}{\partial u^2}+\frac{\partial w}{\partial u}=0$.

习题 1.6

2. (1) $(x^2+y^2)^2 = a^2(y^2-x^2)$.

解：记 $F(x,y) = (x^2+y^2)^2 - a^2(y^2-x^2)$，则 $F_y(x,y) = 4y(x^2+y^2) - 2a^2y = 4y(x^2+y^2-\frac{a^2}{2})$

故当 $F_y(x_0,y_0) \neq 0$，即 $y_0 \neq 0$ 且 $x_0^2+y_0^2 \neq \frac{a^2}{2}$ 时，点 $(x_0,y_0)$ 附近存在唯一确定的隐函数 $y=y(x)$.

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{4x(x^2+y^2)+2a^2x}{4y(x^2+y^2)-2a^2y} = -\frac{x(x^2+y^2+\frac{a^2}{2})}{y(x^2+y^2-\frac{a^2}{2})}$$

(2) $e^{-(x+y+z)} = x+y+z$

解：记 $F(x,y,z) = e^{-(x+y+z)} - (x+y+z)$，则 $F_z(x,y,z) = -e^{-(x+y+z)} - 1 < 0$

故 $\forall (x_0,y_0,z_0) \in \mathbb{R}^3$，其附近均存在 $z = z(x,y)$，且

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{-e^{-(x+y+z)}-1}{-e^{-(x+y+z)}-1} = -1，\quad 同理 \frac{\partial z}{\partial y} = -1.$$

(3) $\sin xy + \sin yz + \sin zx = 0$

解：记 $F(x,y,z) = \sin xy + \sin yz + \sin zx$，则 $F_z(x,y,z) = y\cos yz + x\cos xz$.

故当 $x_0\cos x_0z_0 + y_0\cos y_0z_0 \neq 0$ 时，可确定 $z = z(x,y)$，且

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{y\cos xy + z\cos zx}{y\cos yz + x\cos zx}，\quad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = -\frac{x\cos xy + z\cos yz}{y\cos yz + x\cos zx}.$$

3. (1) $f(ax-cz, ay-bz) = 0$，$f$ 可微，计算：$c\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y}$.

解：记 $u = ax-cz$，$v = ay-bz$，则

$$\frac{\partial z}{\partial x} = -\frac{f_u\cdot u_x + f_v\cdot v_x}{f_u\cdot u_z + f_v\cdot v_z} = \frac{af_u}{cf_u+bf_v}，\quad \frac{\partial z}{\partial y} = -\frac{f_u\cdot u_y + f_v\cdot v_y}{f_u\cdot u_z + f_v\cdot v_z} = \frac{af_v}{cf_u+bf_v}$$

故 $c\cdot\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = \frac{acf_u + abf_v}{cf_u+bf_v} = a$.

(3) $f(x, x+y, x+y+z) = 0$，$f$ 二阶可微，计算：$\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$，$\frac{\partial^2 z}{\partial x^2}$.

解：记 $a = x$，$b = x+y$，$c = x+y+z$，则

$$\frac{\partial z}{\partial x} = -\frac{f_a\cdot a_x + f_b\cdot b_x + f_c\cdot c_x}{f_a\cdot a_z + f_b\cdot b_z + f_c\cdot c_z} = -\frac{f_a+f_b+f_c}{f_c}$$

$$\frac{\partial z}{\partial y} = -\frac{f_a\cdot a_y + f_b\cdot b_y + f_c\cdot c_y}{f_a\cdot a_z + f_b\cdot b_z + f_c\cdot c_z} = -\frac{f_b+f_c}{f_c}$$

$$\frac{\partial^2 z}{\partial x^2} = \left[-\frac{f_a+f_b+f_c}{f_c}\right]_x = -\left[\frac{f_a+f_b}{f_c}\right]_x = -\frac{1}{f_c^2}\left[(f_a+f_b)_x f_c - (f_a+f_b)(f_c)_x\right]$$

$$= -\frac{1}{f_c^2}\left[(f_{aa}+f_{ab}+f_{ac}+f_{ba}+f_{bb}+f_{bc})f_c - (f_a+f_b)(f_{ca}+f_{cb}+f_{cc})\right]$$

$$= -\frac{1}{f_c^2}\left(f_{aa}f_c + 2f_{ab}f_c - 2f_{ac}f_a - 2f_{ac}f_b + f_{bb}f_c - 2f_{bc}f_a - 2f_{bc}f_b + f_{cc}(f_a+f_b)^2/f_c\right)$$

4. 设方程 $f(u^2-x^2, u^2-y^2, u^2-z^2)=0$ 确定了函数 $u=u(x,y,z)$，其中 $f$ 可微，证明：

$$\frac{1}{x}\cdot\frac{\partial u}{\partial x}+\frac{1}{y}\cdot\frac{\partial u}{\partial y}+\frac{1}{z}\cdot\frac{\partial u}{\partial z}=\frac{1}{u}.$$

证明：记 $a=u^2-x^2$，$b=u^2-y^2$，$c=u^2-z^2$，则

$$\frac{\partial u}{\partial x}=-\frac{f_a\cdot a_x+f_b\cdot b_x+f_c\cdot c_x}{f_a\cdot a_u+f_b\cdot b_u+f_c\cdot c_u}=\frac{2xf_a}{2uf_a+2uf_b+2uf_c}=\frac{x}{u}\cdot\frac{f_a}{f_a+f_b+f_c}$$

同理可得：$\dfrac{\partial u}{\partial y}=\dfrac{y}{u}\cdot\dfrac{f_b}{f_a+f_b+f_c}$，$\dfrac{\partial u}{\partial z}=\dfrac{z}{u}\cdot\dfrac{f_c}{f_a+f_b+f_c}$

故 $\dfrac{1}{x}\cdot\dfrac{\partial u}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial u}{\partial y}+\dfrac{1}{z}\cdot\dfrac{\partial u}{\partial z}=\dfrac{1}{u}\left(\dfrac{f_a}{f_a+f_b+f_c}+\dfrac{f_b}{f_a+f_b+f_c}+\dfrac{f_c}{f_a+f_b+f_c}\right)=\dfrac{1}{u}.$

证毕.

5. 方程组 $\begin{cases}x=u+v\\y=u-v\\z=u^2v^2\end{cases}$ 能否确定 $z$ 是 $x,y$ 的函数？如果能，求 $\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$；如果不能，说明理由.

解：由于 $\begin{cases}u+v=x\\u-v=y\end{cases}$，故 $\begin{cases}u=\dfrac{x+y}{2}\\v=\dfrac{x-y}{2}\end{cases}$，代入得：$z=u^2v^2=\left(\dfrac{x+y}{2}\right)^2\cdot\left(\dfrac{x-y}{2}\right)^2=\dfrac{(x^2-y^2)^2}{16}$

故 $z$ 能确定为 $x,y$ 的函数，$z(x,y)=\dfrac{(x^2-y^2)^2}{16}$.

$\dfrac{\partial z}{\partial x}=\dfrac{x^3-y^2x}{4}$，$\dfrac{\partial z}{\partial y}=\dfrac{y^3-x^2y}{4}$.

6. 方程组 $\begin{cases}x+y+z+z^2=0\\x+y^2+z+z^3=0\end{cases}$ 在点 $P(-1,1,0)$ 附近能否确定向量值函数 $\begin{pmatrix}y\\z\end{pmatrix}=f(x)$，如果能确定，求出 $y'(-1)$，$z'(-1)$.

解：记方程组为 $\begin{cases}h(x,y,z)=x+y+z+z^2\\g(x,y,z)=x+y^2+z+z^3\end{cases}$，则 $(-1,1,0)$ 为方程组的一个解.

考虑映射 $F=(h,g):R^3\to R^2$ 在 $P(-1,1,0)$ 处的 Jacobian 矩阵

$$\begin{bmatrix}1&1&1+2z\\1&2y&1+3z^2\end{bmatrix}=\begin{bmatrix}1&1&1\\1&2&1\end{bmatrix}$$

因矩阵 $D_{(y,z)}F\big|_P=\begin{bmatrix}1&1\\2&1\end{bmatrix}$ 非奇，故由 IFT 知存在映射 $f(x)=\begin{pmatrix}y\\z\end{pmatrix}:B_\delta\subset R\to R^2$，使得

$$\begin{cases}y(x_0)=y_0\\z(x_0)=z_0\end{cases}，\quad 其中 B_\delta=\{x\mid x_0-\delta<x<x_0+\delta\}$$

即上述向量值函数 $\begin{pmatrix}y\\z\end{pmatrix}=f(x)$ 存在，且 $\begin{bmatrix}y_x\\z_x\end{bmatrix}_{x_0}=-\begin{bmatrix}h_y&h_z\\g_y&g_z\end{bmatrix}_P^{-1}\begin{bmatrix}h_x\\g_x\end{bmatrix}_P$

$$=-\begin{bmatrix}1&1\\2&1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$$

故 $y'(-1)=0$，$z'(-1)=-1$

补充习题

设三元函数 $F(x,y,z)$ 在开集 $\Omega\subset\mathbb{R}^3$ 是 $C^1$ 的，设点 $P_0(x_0,y_0,z_0)\in\Omega$，使得 $F(x_0,y_0,z_0)=0$ 且 $F_y(x_0,y_0,z_0)\neq 0$. 于是由IFT可知由方程 $F(x,y,z)=0$ 在点 $P_0$ 附近解出唯一的隐函数 $z=z(x,y)$，$(x,y)\in B_\delta$，这里 $B_\delta$ 代表以点 $(x_0,y_0)$ 为心，以 $\delta>0$ 为半径的开球域. 求 $z_{xy}$，$z_{yy}$.

解：$(z_x,z_y)=-\dfrac{(F_x,F_y)}{F_z}\Big|_{(x,y,z(x,y))}$，$(x,y)\in B_\delta$.

$$
\begin{aligned}
\text{于是 } z_{xy} &= -\left(\frac{F_x}{F_z}\right)_y = -\frac{1}{F_z^2}\left[(F_x)_y F_z - F_x (F_z)_y\right]\\
&= -\frac{1}{F_z^2}\left[F_z\left(F_{xy}+F_{xz}\cdot z_y\right) - F_x\left(F_{zy}+F_{zz}\cdot z_y\right)\right]\\
&= -\frac{1}{F_z^2}\left[F_z\left(F_{xy}+F_{xz}\cdot\left(-\frac{F_y}{F_z}\right)\right) - F_x\left(F_{zy}+F_{zz}\cdot\left(-\frac{F_y}{F_z}\right)\right)\right]\\
&= \frac{1}{F_z^3}\left(F_yF_zF_{xz}+F_xF_zF_{zy}-F_z^2F_{xy}-F_xF_yF_{zz}\right)
\end{aligned}
$$

$$
\begin{aligned}
z_{yy} &= -\left(\frac{F_y}{F_z}\right)_y = -\frac{1}{F_z^2}\left[(F_y)_y F_z - F_y (F_z)_y\right]\\
&= -\frac{1}{F_z^2}\left[F_z\left(F_{yy}+F_{yz}\cdot\left(-\frac{F_y}{F_z}\right)\right) - F_y\left(F_{zy}+F_{zz}\cdot\left(-\frac{F_y}{F_z}\right)\right)\right]\\
&= \frac{1}{F_z^3}\left(2F_yF_zF_{yz}-F_z^2F_{yy}-F_y^2F_{zz}\right).
\end{aligned}
$$

3.4

习题1.6

7. 已知 $\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases}$ 在 $(1,-1,2)$ 附近确定了向量值函数 $\begin{pmatrix} x \\ y \end{pmatrix}=f(z)$，在 $(1,-1,2)$ 处求 $\frac{dx}{dz}$, $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}$, $\frac{d^2y}{dz^2}$.

解：设 $F=(h,g): R^3 \to R^2$，$\begin{cases} h(x,y,z) = x^2+y^2-\frac{1}{2}z^2 \\ g(x,y,z) = x+y+z-2 \end{cases}$，故 $F(1,-1,z)=0$

考虑F在该点处的Jacobian矩阵 $\begin{bmatrix} 2x & 2y & -z \\ 1 & 1 & 1 \end{bmatrix}_{(1,-1,2)} = \begin{bmatrix} 2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

因 $D_{(x,y)}F = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ 非奇，由IFT知存在 $C^1$ 映射 $f(z)=\begin{pmatrix} x \\ y \end{pmatrix}: B_\delta \subset R \to R^2$，且

$\begin{bmatrix} x'(z) \\ y'(z) \end{bmatrix} = -\begin{bmatrix} h_x & h_y \\ g_x & g_y \end{bmatrix}^{-1} \begin{bmatrix} h_z \\ g_z \end{bmatrix} = -\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$，故 $\frac{dx}{dz}=0$，$\frac{dy}{dz}=-1$

对方程组 $\begin{cases} x^2+y^2-\frac{1}{2}z^2=0 \\ x+y+z-2=0 \end{cases}$ 对z求导：$\begin{cases} 2x\cdot\frac{dx}{dz}+2y\cdot\frac{dy}{dz}=z \\ \frac{dx}{dz}+\frac{dx}{dy}+1=0 \end{cases}$

再对z求导得：$\begin{cases} 2\left(\frac{dx}{dz}\right)^2+2x\cdot\frac{d^2x}{dz^2}+2\left(\frac{dy}{dz}\right)^2+2y\cdot\frac{d^2y}{dz^2}=1 \\ \frac{d^2x}{dz^2}+\frac{d^2y}{dz^2}=0 \end{cases}$ 将 $\frac{dx}{dz}=0$，$\frac{dy}{dz}=-1$ 代入：

$\begin{cases} \frac{d^2x}{dz^2}-\frac{d^2y}{dz^2}=-\frac{1}{2} \\ \frac{d^2x}{dz^2}+\frac{d^2y}{dz^2}=0 \end{cases}$ 故 $\frac{d^2x}{dz^2}=-\frac{1}{4}$，$\frac{d^2y}{dz^2}=\frac{1}{4}$

9. (1) $\begin{cases} u=x^2-y^2 \\ v=2xy \end{cases}$

解：$DF(x,y)=\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}=\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$，设逆映射为 $\begin{pmatrix} x \\ y \end{pmatrix}=g(u,v)$，

则 $Dg(u,v)=[DF(x,y)]^{-1}=\frac{1}{2(x^2+y^2)}\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$，故 $|Dg(u,v)|=\frac{1}{4(x^2+y^2)}$

(3) $\begin{cases} u=x^3-y^3 \\ v=xy^2 \end{cases}$

解：$DF(x,y)=\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}=\begin{bmatrix} 3x^2 & -3y^2 \\ y^2 & 2xy \end{bmatrix}$，设逆映射为 $\begin{pmatrix} x \\ y \end{pmatrix}=g(u,v)$，

则 $Dg(u,v)=[DF(x,y)]^{-1}=\frac{1}{6x^3y+3y^4}\begin{bmatrix} 2xy & 3y^2 \\ -y^2 & 3x^2 \end{bmatrix}$，故 $|Dg(u,v)|=\frac{1}{6x^3y+3y^4}$

10. (1) $\begin{cases} u=\xi^2-\eta^2 \\ v=2\xi\eta \end{cases}$，$\begin{cases} \xi=e^x\cos y \\ \eta=e^x\sin y \end{cases}$，$(x_0,y_0)=(1,0)$.

解：$g(\xi,\eta)=\begin{pmatrix} u \\ v \end{pmatrix}$，$f(x,y)=\begin{pmatrix} \xi \\ \eta \end{pmatrix}$，而 $Dg(\xi,\eta)=\begin{bmatrix} 2\xi & -2\eta \\ 2\eta & 2\xi \end{bmatrix}$，$Df(x,y)=\begin{bmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{bmatrix}$

于是 $|Dg|=4(\xi^2+\eta^2)\geq 0$，$|Df|=e^{2x}>0$. $(x_0,y_0)=(1,0)$ 时 $(\xi_0,\eta_0)=(e,0)$

故 $|Dg|\neq 0$ 且 $|Df|\neq 0$，从而g存在逆映射 $g^{-1}:(u,v)\mapsto(\xi,\eta)$，f存在逆映射 $f^{-1}:(\xi,\eta)\mapsto(x,y)$

于是在 $(x_0,y_0)$ 邻域内能确定 $g\circ f$ 的逆映射 $(g\circ f)^{-1}=(f^{-1})\circ(g^{-1}):(u,v)\mapsto(x,y)$.

且由于f, g为 $C^1$ 映射，故 $f^{-1}$, $g^{-1}$ 为 $C^1$ 映射，从而 $(g\circ f)^{-1}$ 为 $C^1$ 映射，即 $(g\circ f)^{-1}$ 可微.

习题1-7

1.(1) $z=x^2+y^2$，点 $P(1,2,5)$

解：$\frac{\partial z}{\partial x}=2x=2$，$\frac{\partial z}{\partial y}=2y=4$. 故切平面：$z=2(x-1)+4(y-2)+5$，即 $2x+4y-z=5$

法线：$\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-5}{-1}$

(3) $(2a^2-z^2)x^2=a^2y^2$，点 $P(a,a,a)$.

解：$F(x,y,z)\triangleq(2a^2-z^2)x^2-a^2y^2$，$F_x(x,y,z)=2(2a^2-z^2)x$，$F_y(x,y,z)=-2a^2y$，$F_z(x,y,z)=-2x^2z$

故切平面：$2a^3(x-a)+(-2a^3)(y-a)+(-2a^3)(z-a)=0$，即 $x-y-z+a=0$

法线：$\frac{x-a}{1}=\frac{y-a}{-1}=\frac{z-a}{-1}$

(5) $\begin{cases}x=u\cos v\\ y=u\sin v\\ z=av\end{cases}$，点 $(u,v)=(u_0,v_0)$

解：记以上方程组为 $F:(u,v)\mapsto(x,y,z)$，则 $F$ 的 Jacobian 矩阵 $DF(u,v)=\begin{bmatrix}x_u & x_v\\ y_u & y_v\\ z_u & z_v\end{bmatrix}=\begin{bmatrix}\cos v & -u\sin v\\ \sin v & u\cos v\\ 0 & a\end{bmatrix}$

故所求切平面向量形式为：$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}u_0\cos v_0\\ u_0\sin v_0\\ av_0\end{bmatrix}+(u-u_0)\begin{bmatrix}\cos v_0\\ \sin v_0\\ 0\end{bmatrix}+(v-v_0)\begin{bmatrix}-u_0\sin v_0\\ u_0\cos v_0\\ a\end{bmatrix}$

而 $(\cos v_0,\sin v_0,0)\times(-u_0\sin v_0,u_0\cos v_0,a)=(a\sin v_0,-a\cos v_0,u_0)$

故法线：$\frac{x-u_0\cos v_0}{a\sin v_0}=\frac{y-u_0\sin v_0}{-a\cos v_0}=\frac{z-av_0}{u_0}$

2. 在椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 上求一点 $P$，使得过 $P$ 点的法线与坐标轴正方向成等角.

解：$F(x,y,z)\triangleq\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$，$F_x(x,y,z)=\frac{2x}{a^2}$，$F_y(x,y,z)=\frac{2y}{b^2}$，$F_z(x,y,z)=\frac{2z}{c^2}$.

故法向量（梯度）$\nabla F^0=\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right)$，故 $\nabla F^0 \parallel (1,1,1)$

故 $x:y:z=a^2:b^2:c^2$，结合 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 知：

$P\left(\frac{a^2}{\sqrt{a^2+b^2+c^2}},\frac{b^2}{\sqrt{a^2+b^2+c^2}},\frac{c^2}{\sqrt{a^2+b^2+c^2}}\right)$ 或 $P\left(-\frac{a^2}{\sqrt{a^2+b^2+c^2}},-\frac{b^2}{\sqrt{a^2+b^2+c^2}},-\frac{c^2}{\sqrt{a^2+b^2+c^2}}\right)$

3. 求曲面 $x^2+2y^2+3z^2=21$ 上平行于 $x+4y+6z=0$ 的切平面.

解：$F(x,y,z)\triangleq x^2+2y^2+3z^2-21$，$F_x(x,y,z)=2x$，$F_y(x,y,z)=4y$，$F_z(x,y,z)=6z$.

切平面 $\left[(x_0,y_0,z_0)\text{处}\right]$：$2x_0(x-x_0)+4y_0(y-y_0)+6z_0(z-z_0)=0$.

故 $(2x_0):(4y_0):(6z_0)=1:4:6$，结合 $x^2+2y^2+3z^2=21$，得 $(x_0,y_0,z_0)=(1,2,2)$ 或 $(-1,-2,-2)$

故切平面：$x+4y+6z=\pm 21$

4.(1) 曲面$xyz=a^3$上任一点的切平面与坐标平面围成的四面体的体积为定值.

证明: $F(x,y,z)\triangleq xyz-a^3$, 则$F_x=yz$, $F_y=xz$, $F_z=xy$. 取曲面上任一点$(x_0,y_0,z_0)$则,

切平面: $y_0z_0(x-x_0)+x_0z_0(y-y_0)+x_0y_0(z-z_0)=0$, 即$\frac{x}{x_0}+\frac{y}{y_0}+\frac{z}{z_0}=3$.

该平面分别与$x,y,z$轴交于$(3x_0,0,0)$, $(0,3y_0,0)$, $(0,0,3z_0)$.

故四面体体积 $V=\frac{1}{6}\cdot(3x_0)\cdot(3y_0)\cdot(3z_0)=\frac{9}{2}x_0y_0z_0=\frac{9}{2}a^3$ 为定值. 证毕.

(3) 曲面$z=x^2+y^2$与直线$l$: $\begin{cases}x+2z=1\\y+2z=2\end{cases}$ 垂直的切平面.

解: 曲面上该点设为$P_0(x_0,y_0,z_0)$, 其中$z_0=x_0^2+y_0^2$. $\frac{\partial z}{\partial x}\Big|_{P_0}=2x\Big|_{P_0}=2x_0$, $\frac{\partial z}{\partial y}\Big|_{P_0}=2y\Big|_{P_0}=2y_0$.

切平面: $z=z_0+2x_0(x-x_0)+2y_0(y-y_0)$ ⋯ ①

而法线$l'$: $\frac{x-x_0}{-2x_0}=\frac{y-y_0}{-2y_0}=\frac{z-z_0}{1}$, 即$\begin{cases}x+2x_0z=2x_0z_0+x_0\\y+2y_0z=2y_0z_0+y_0\end{cases}$ ⋯ ②

由题设知法线$l'\parallel l$, 故$x_0=1$, $y_0=1$, 从而$z_0=2$.

代入①中得切平面: $z=2+2(x-1)+2(y-1)$, 即$2x+2y-z-2=0$

(5) 设$f$可微, 曲面$z=yf(\frac{x}{y})$的所有~~#~~切平面相交于一个定点.

解: 任取曲面上一点$P_0(x_0,y_0,z_0)$, 其中$z_0=y_0f(\frac{x_0}{y_0})$.

$\frac{\partial z}{\partial x}\Big|_{P_0}=yf'(\frac{x}{y})\cdot\frac{1}{y}\Big|_{P_0}=f'(\frac{x_0}{y_0})$, $\frac{\partial z}{\partial y}\Big|_{P_0}=\left[f(\frac{x}{y})+y\cdot f'(\frac{x}{y})\cdot(-\frac{x}{y^2})\right]\Big|_{P_0}=f(\frac{x_0}{y_0})-\frac{x_0}{y_0}f'(\frac{x_0}{y_0})$.

切平面: $z=y_0f(\frac{x_0}{y_0})+f'(\frac{x_0}{y_0})(x-x_0)+\left[f(\frac{x_0}{y_0})-\frac{x_0}{y_0}f'(\frac{x_0}{y_0})\right](y-y_0)$

即 $z=yf(\frac{x_0}{y_0})+(x-\frac{x_0}{y_0}y)f'(\frac{x_0}{y_0})$, 该平面恒过点$(0,0,0)$. (与$x_0,y_0,z_0$无关)

因此, 该曲面的所有切平面交于原点. 证毕.

5. 求曲线$l$: $\begin{cases}x^2+y^2+z^2=6\\x+y+z=0\end{cases}$ 在点$P(1,-2,1)$处的切线方程与法平面方程.

解: 记$F(x,y,z)=x^2+y^2+z^2-6$, $G(x,y,z)=x+y+z$, 考虑在$P$点$(1,-2,1)$处梯度

则$\begin{cases}\nabla F=(2x,2y,2z)=(2,-4,2)\\\nabla G=(1,1,1)\end{cases}$ 故切线: $\begin{cases}2(x-1)-4(y+2)+2(z-1)=0\\x-1+y+2+z-1=0\end{cases}$

化简得切线: $\begin{cases}x-2y+z=6\\x+y+z=0\end{cases}$ $\Leftrightarrow$ $\frac{x-1}{-1}=\frac{y+2}{0}=\frac{z-1}{1}$

故切平面: $-(x-1)+(z-1)=0$, 即$x-z=0$.

6. 证明：螺旋线 $\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$ 的切线与 z 轴形成定角.

证明：$\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right) = (-a\sin t, a\cos t, b)$，故 $\forall t = t_0$，曲线在 $(x(t_0), y(t_0), z(t_0))$ 处

切线为 $l$: $\dfrac{x - a\cos t_0}{-a\sin t_0} = \dfrac{y - a\sin t_0}{a\cos t_0} = \dfrac{z - bt_0}{b}$，其方向向量 $\vec{l} = (-a\sin t_0, a\cos t_0, b)$.

考虑 z 轴正方向的单位向量 $\vec{n} = (0,0,1)$ 与 $\vec{l}$ 的夹角：

$$\cos\theta = \frac{\vec{n}\cdot\vec{l}}{\|\vec{n}\|\cdot\|\vec{l}\|} = \frac{b}{\sqrt{a^2\sin^2 t_0 + a^2\cos^2 t_0 + b^2}} = \frac{b}{\sqrt{a^2+b^2}}$$，故 $\theta$ 为定值.

因此，该曲线的任意切线与 z 轴成定角.

7. 已知函数 $f$ 可微. 若 $T$ 为曲面 $S$: $f(x,y,z) = 0$ 在点 $P(x_0, y_0, z_0)$ 处的切平面，$l$ 为 $T$ 上任意一条过 $P$ 的直线，求证：在 $S$ 上存在一条曲线，该曲线在 $P$ 处的切线恰好为 $l$.

证明：可知平面 $T$: $f_x^0(x-x_0) + f_y^0(y-y_0) + f_z^0(z-z_0) = 0$.

设 $l$ 为平面 $T$ 与任意过 $P$ 的平面 $V$ 的交线，即 $l = T \cap V$.

其中 $V$: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$，$a, b, c \in \mathbb{R}$

故 $l$: $\begin{cases} f_x^0(x-x_0) + f_y^0(y-y_0) + f_z^0(z-z_0) = 0 \\ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \end{cases}$ ---- $(*)$

考虑 $V$ 与 $S$ 的交线 $k$: $\begin{cases} f(x,y,z) = 0 \\ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \end{cases}$. 考虑在 $P(x_0, y_0, z_0)$ 处的梯度:

(记 $g(x,y,z) = a(x-x_0) + b(y-y_0) + c(z-z_0)$)

$\begin{cases} \nabla f = (f_x^0, f_y^0, f_z^0) \\ \nabla g = (a, b, c) \end{cases}$，故曲线 $k$ 在 $P$ 处的切线为:

$l'$: $\begin{cases} f_x^0(x-x_0) + f_y^0(y-y_0) + f_z^0(z-z_0) = 0 \\ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \end{cases}$ $\cdots$ $(**)$

对比 $(*)$ 式及 $(**)$ 式可知：$l = l'$.

因此，$\forall l \in T$，$P \in l$，存在曲线 $k \in S$ 使得 $k$ 在 $P$ 处的切线恰为 $l$. 证毕.