练习样题

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1. 解:

(1) ∵ 当 $(x,y)\neq(0,0)$ 时

$$f(x,y) = \frac{x^3}{x^2+y^2} + \frac{y^3}{x^2+y^2}$$

$$= \frac{x^2}{x^2+y^2}\cdot x + \frac{y^2}{x^2+y^2}\cdot y$$

∴ $|f(x,y)| = \left|\frac{x^2}{x^2+y^2}\cdot x + \frac{y^2}{x^2+y^2}\cdot y\right|$

$$\leq |x| + |y|$$

∴ 当 $(x,y)\to 0$ 时

$f(x,y) = 0$

故: 在 $(0,0)$ 处连续

(2) ∵ $f_x = \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$ $(\Delta x\to 0)$

$= \frac{f(\Delta x,0)}{\Delta x}$ $(\Delta x\to 0)$

$= 1$

同理: $f_y = 1$

(3) 不可微:

假设可微, 则

$$\frac{x^3+y^3}{x^2+y^2} = x+y+o(\rho).$$

∴ $o(\rho) = \frac{-xy(x+y)}{x^2+y^2}$

即 $\lim\limits_{(x,y)\to(0,0)} \frac{-xy(x+y)}{(x^2+y^2)^{\frac{3}{2}}} = 0$

又∵ 该极限 若y沿 $y=kx$ 近 $(0,0)$

则 $\lim = \frac{-x\cdot kx\,(1+k)x}{(k^2+1)^{\frac{3}{2}}\cdot x^3}$

与k关

⇒ 故矛盾!!

2. 解: ∵ $z=z(x,y)$ 由

$x^3+y^3+z^3 = x+y+z$ 确定.

令 $F(x,y,z) = x^3+y^3+z^3-x-y-z = 0$.

∴ $\frac{\partial F}{\partial z} = 3z^2-1$

$\frac{\partial F}{\partial x} = 3x^2-1$

$\frac{\partial F}{\partial y} = 3y^2-1$

∴ $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$

$= -\frac{3x^2-1}{3z^2-1}$ 且 $z = z(x,y)$

∴ $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}$

$= -\frac{0-(3x^2-1)\times 6z\times z_y}{(3z^2-1)^2}$

又∵ $z_y = \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{3y^2-1}{3z^2-1}$

故: $\frac{\partial^2 z}{\partial x\partial y} = \frac{-6z(3x^2-1)(3y^2-1)}{(3z^2-1)^3}$

3. 解：设该点为 (x_0, y_0, z_0)

设 $F = x^2+y^2+\frac{z^2}{4}-1=0$

$\frac{\partial F}{\partial x} = 2x_0$

$\frac{\partial F}{\partial y} = 2y_0$

$\frac{\partial F}{\partial z} = \frac{z_0}{2}$

∴ 切面：$2x_0(x-x_0)+2y_0(y-y_0)+\frac{z_0}{2}(z-z_0)=0$

∴ $2x_0x+2y_0y+\frac{z_0}{2}z = 2x_0^2+2y_0^2+\frac{1}{2}z_0^2$

$= 2.$

又∵ 与坐标轴的交点：

$A_1(\frac{1}{x_0}, 0, 0)$ $A_2(0, \frac{1}{y_0}, 0)$

$A_3(0, 0, \frac{4}{z_0})$

转化为条件极值：

$$\begin{cases} \min & \frac{1}{x_0^2}+\frac{1}{y_0^2}+\frac{16}{z_0^2} \\ \text{st.} & x_0^2+y_0^2+\frac{z_0^2}{4}=1 \end{cases}$$

令 $L = \frac{1}{x_0^2}+\frac{1}{y_0^2}+\frac{16}{z_0^2} - \lambda(x_0^2+y_0^2+\frac{z_0^2}{4}-1)$

$L_x = \frac{-2x_0}{x_0^4} - \lambda\cdot 2x_0 = 0$

$L_y = \frac{-2y_0}{y_0^4} - \lambda\cdot 2y_0 = 0$

$L_z = \frac{-16\times 2z_0}{z_0^4} - \lambda\cdot\frac{2z_0}{4} = 0$

且 $x_0^2+y_0^2+\frac{z_0^2}{4}=1$

∴ 在 $\left((-\frac{1}{\lambda})^{\frac{1}{4}}, (-\frac{1}{\lambda})^{\frac{1}{4}}, (-\frac{64}{\lambda})^{\frac{1}{4}}\right)$ 处

∴ ~~$(\frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{10}\times 2\sqrt{2})$ 处~~

$(\frac{1}{2}, \frac{1}{2}, \sqrt{2})$ 处

经检验

H矩阵为正定

故：

最小值 $= \frac{1}{(\frac{1}{2})^2}+\frac{1}{(\frac{1}{2})^2}+\frac{16}{(\sqrt{2})^2}$

$= 8+4+4$

$= 16$

该点为 $(\frac{1}{2}, \frac{1}{2}, \sqrt{2})$

4. 证明：

∵ 对 $\forall (x,y)\in R^2$

f 都连续可微

令 $g(t) = f(x-t, y+t)$

则 $g(t)$ 也连续可微

又∵ $g'(t) = f_x(x-t, x+t)\cdot(-1)$

$+ f_y(x-t, x+t)\cdot(1)$

且 $f'_x(x,y) = f'_y(x,y)$

∴ $g'(t)=0$

∴ $g(t)=C$.

∴ $g(0) = f(x,y) = g(-y)$

$= f(x+y, 0) > 0$

则 对于 $\forall (x,y)\in R^2$，$f(x,y) > 0$

5. 解:

$$\int_0^{+\infty} \frac{\arctan bx - \arctan ax}{x} dx, \quad b>a>0.$$

$\because$ $$\frac{\arctan bx - \arctan ax}{x} = \int_a^b \frac{1}{1+(xy)^2} dy$$

验证合理性证明积分

$\int_0^{+\infty} \frac{dx}{1+(xy)^2}$ 关于 $y \in [a,b]$ 一致收敛

$\because$ $0 < \frac{1}{1+(xy)^2} \leq \frac{1}{1+(ax)^2}$, $\forall x \geq 0$.

$\therefore$ 成立.

故原式 $= \int_a^b dy \int_0^{+\infty} \frac{1}{1+(xy)^2} dx$

$= \frac{\pi}{2}(\ln b - \ln a)$

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