1. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

解：令 $x=\frac{1-t}{1+t}$，$dx=\frac{-2}{(1+t)^2}dt$ 　原式 $=I$

$I=\int_1^0 \frac{\ln(1+\frac{1-t}{1+t})}{1+(\frac{1-t}{1+t})^2}\cdot\frac{-2}{(1+t)^2}dt=\int_0^1\frac{2}{(1-t)^2+(1+t)^2}\cdot\ln\frac{2}{1+t}dt$

$=\int_0^1\frac{1}{1+t^2}[\ln2-\ln(1+t)]dt=\int_0^1\frac{\ln2}{1+t^2}dt-I$

于是 $2I=\int_0^1\frac{\ln2}{1+t^2}dt=\ln2\cdot\arctan t|_0^1=\frac{\pi}{4}\ln2$ 　$\therefore I=\frac{\pi}{8}\ln2$

2. $\int\frac{dx}{x^4\sqrt{x^2+1}}$

此题要注意，可用 $x=\tan t$ 代换，但是十分复杂．分母次数高于分子，倒代换

设 $x=\frac{1}{t}$，$dx=-\frac{1}{t^2}dt$ 　原式 $=\int\frac{t^4}{\sqrt{1+\frac{1}{t^2}}}(-\frac{1}{t^2})dt=\int\frac{t^3}{\sqrt{1+t^2}}dt$

$=\frac{1}{2}[\int\sqrt{1+t^2}d(t^2+1)-\int\frac{1}{\sqrt{1+t^2}}d(t^2+1)]$

$=-\frac{1}{2}\cdot\frac{2}{3}(1+t^2)^{\frac{3}{2}}+\sqrt{1+t^2}+C$ 　$=-\frac{1}{3}(1+\frac{1}{x^2})^{\frac{3}{2}}+\sqrt{1+\frac{1}{x^2}}+C$

3. 设 $\sqrt[6]{x}=t$，则 $x=t^6$，$dx=6t^5dt$

原式 $=\int\frac{6t^8}{t^3-t^2}dt=6\int(t^5+t^4+t^3+t^2+t+\frac{1}{t-1})dt$

$=6(\frac{t^6}{6}+\frac{t^5}{5}+\frac{t^4}{4}+\frac{t^3}{3}+\frac{t^2}{2}+t+\ln|t-1|)+C=x+\frac{6}{5}x^{\frac{5}{6}}+\frac{3}{2}x^{\frac{2}{3}}+\frac{2}{\sqrt{x}}+3x^{\frac{1}{3}}$
$+6x^{\frac{1}{6}}+6\ln|x^{\frac{1}{6}}-1|+C$

4. $\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$

解：设 $\sqrt[3]{\frac{x-1}{x+1}}=t$ 　则 $x=\frac{1+t^3}{1-t^3}$ 　$x+1=\frac{2}{1-t^3}$，$x-1=\frac{2t^3}{1-t^3}$，$dx=\frac{6t^2}{(1-t^3)^2}dt$

原式 $=\int(x+1)(x-1)\sqrt[3]{\frac{x-1}{x+1}}dx=\int\left(\frac{2}{1-t^3}\cdot\frac{2t^3}{1-t^3}\cdot t\right)^{-1}\frac{6t^2}{(1-t^3)^2}dt=\frac{3}{2}\int\frac{1}{t^2}dt$

$=-\frac{3}{2t}+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$

5. 设 $x=\frac{1}{t}$，则 $dx=-\frac{1}{t^2}dt$ $\quad I=\int_0^{+\infty}\frac{dx}{(1+x^2)(1+x^\alpha)}=\int_{+\infty}^{0}\frac{(-\frac{1}{t^2})dt}{(1+\frac{1}{t^2})(1+\frac{1}{t^\alpha})}$

$=\int_0^{+\infty}\frac{t^\alpha}{(1+t^2)(1+t^\alpha)}dt$

于是 $I=\frac{1}{2}\left[\int_0^{+\infty}\frac{dx}{(1+x^2)(1+x^\alpha)}+\int_0^{+\infty}\frac{x^\alpha}{(1+x^2)(1+x^\alpha)}dx\right]=\frac{1}{2}\int_0^{+\infty}\frac{dx}{1+x^2}=\frac{1}{2}\arctan x\Big|_0^{+\infty}=\frac{\pi}{4}$

6. 令 $x=\sin t$，$I=\int_0^{\frac{\pi}{2}}\frac{\cos t}{2\sin t-\cos t}dt=\frac{1}{5}\left[-\int_0^{\frac{\pi}{2}}dt+2\int_0^{\frac{\pi}{2}}\frac{d(2\sin t-\cos t)}{2\sin t-\cos t}\right]$

$=\frac{1}{5}(2\ln 2-\frac{\pi}{2})$

错误：将1，0代入发现分母不为0，则非广义 实际上令分母=0，得 $x=\frac{1}{\sqrt5}$

$I=\lim\limits_{\varepsilon\to0^+}\int_0^{\frac{1}{\sqrt5}-\varepsilon}\frac{dx}{2x-\sqrt{1-x^2}}+\lim\limits_{\varepsilon\to0^+}\int_{\frac{1}{\sqrt5}+\varepsilon}^{1}\frac{dx}{2x-\sqrt{1-x^2}}$

其中 $\lim\limits_{\varepsilon\to0^+}\int_0^{\frac{1}{\sqrt5}-\varepsilon}\frac{dx}{2x-\sqrt{1-x^2}}=\lim\limits_{\varepsilon\to0^+}\frac{1}{5}\left[2\ln\left|2(\frac{1}{\sqrt5}-\varepsilon)-\sqrt{1-(\frac{1}{\sqrt5}-\varepsilon)^2}\right|-\arcsin(\frac{1}{\sqrt5}-\varepsilon)\right]$

$\to\infty$

故原广义积分发散

7. 1. $I=-\frac{1}{2}\int_0^2\frac{d(1-x^2)}{1-x^2}=-\frac{1}{2}\ln|1-x^2|\Big|_0^2=-\frac{\ln3}{2}$

2. $I=\int_0^1\frac{x}{1-x^2}dx+\int_1^2\frac{x}{1-x^2}dx=-\frac{1}{2}\lim\limits_{\varepsilon\to0^+}\int_0^{1-\varepsilon}\frac{d(1-x^2)}{1-x^2}-\frac{1}{2}\lim\limits_{\varepsilon\to0^+}\int_{1+\varepsilon}^{2}\frac{d(1-x^2)}{1-x^2}$

$=-\frac{1}{2}\lim\limits_{\varepsilon\to0^+}\left[\ln|1-x^2|\Big|_0^{1-\varepsilon}+\ln|1-x^2|\Big|_{1+\varepsilon}^{2}\right]=-\frac{\ln3}{2}+\frac{1}{2}\lim\limits_{\varepsilon\to0^+}\ln\left|\frac{2+\varepsilon}{2-\varepsilon}\right|=-\frac{\ln3}{2}$

正确解法：$I=\int_0^1\frac{x}{1-x^2}dx+\int_1^2\frac{x}{1-x^2}dx$

其中 $\int_0^1\frac{x}{1-x^2}dx=\lim\limits_{\varepsilon\to0^+}-\frac{1}{2}\int_0^{1-\varepsilon}\frac{d(1-x^2)}{1-x^2}=-\frac{1}{2}\lim\limits_{\varepsilon\to0^+}\ln|1-x^2|\Big|_0^{1-\varepsilon}\to+\infty$

故原广义积分发散

8. $\int\frac{\arcsin e^x}{e^x}dx=-\int\arcsin e^x\,d(e^{-x})=-e^{-x}\arcsin e^x+\int\frac{dx}{\sqrt{1-e^{2x}}}$

令 $e^x=\sin t$，$\int\frac{dx}{\sqrt{1-e^{2x}}}=\int\csc t\,dt=\ln|\csc t-\cot t|+C$

原式 $=-e^{-x}\arcsin e^x-x+\ln(1-\sqrt{1-e^{2x}}]+C$

9. $\int\frac{2x}{(x+1)(x^2+1)^2}dx$　　令 $\frac{2x}{(x+1)(x^2+1)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$

由待定系数法，$A=C=-\frac{1}{2}$，$B=\frac{1}{2}$，$D=E=1$

于是 $\frac{2x}{(x+1)(x^2+1)^2}=-\frac{1}{2}\frac{1}{x+1}+\frac{1}{2}\frac{x-1}{x^2+1}+\frac{x+1}{(x^2+1)^2}$

$\int\frac{2x}{(x+1)(x^2+1)^2}dx=\int\left(-\frac{1}{2}\frac{1}{x+1}+\frac{1}{2}\frac{x-1}{x^2+1}+\frac{x+1}{(x^2+1)^2}\right)dx=+\frac{1}{4}\ln\frac{1+x^2}{(1+x)^2}+\frac{x-1}{2(x^2+1)}+C$

10. 当 $x\geq1$ 时，$\int|x-1|dx=\int(x-1)dx=\frac{x^2}{2}-x+C_1$

当 $x<1$ 时，$\int|x-1|dx=-\int(x-1)dx=-\frac{x^2}{2}+x+C_2$

而 $\int|x-1|dx$ 在 $x=1$ 处连续，$C_1=1+C_2$　故 $\int|x-1|dx=\begin{cases}\frac{x^2}{2}-x+C+1, & x\geq1\\ -\frac{x^2}{2}+x+C, & x<1\end{cases}$

11. (凑微分)　原式 $=\frac{1}{2}\int\ln\frac{1+x}{1-x}\,d\left(\ln\frac{1+x}{1-x}\right)=\frac{1}{4}\left(\ln\frac{1+x}{1-x}\right)^2+C$

12. 原式 $=\frac{1}{2}\int\frac{d(\sin^2 x)}{\sqrt{(a^2-b^2)\sin^2 x+b^2}}=\begin{cases}\frac{\sqrt{(a^2-b^2)\sin^2 x+b^2}}{a^2-b^2}+C, & a^2\neq b^2\\ \frac{1}{2|b|}\sin^2 x+C, & a^2=b^2\end{cases}$

13. $\int\frac{x^4}{(x+1)^{100}}dx\overset{x+1=t}{=\!=}\int\frac{(t-1)^4}{t^{100}}dt=\int\left(\frac{1}{t^{96}}-\frac{4}{t^{97}}+\frac{6}{t^{98}}-\frac{4}{t^{99}}+\frac{1}{t^{100}}\right)dt$

$=-\frac{1}{95(x+1)^{95}}+\frac{1}{24(x+1)^{96}}-\frac{6}{97(x+1)^{97}}+\frac{2}{49(x+1)^{98}}-\frac{1}{99(x+1)^{99}}+C$

14. 原式 $=\int\frac{f(x)}{f'(x)}\frac{[f'(x)]^2-f(x)f''(x)}{[f'(x)]^2}dx=\int\frac{f(x)}{f'(x)}d\left[\frac{f(x)}{f'(x)}\right]=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C$

15. $\int\frac{x}{\sqrt{(1+x^2)^3}}e^{-\frac{1}{\sqrt{1+x^2}}}dx=e^{-\frac{1}{\sqrt{1+x^2}}}+C$

16. $\int\frac{dx}{(x-a)\sqrt{(x-a)(x-b)}}=\int\frac{dx}{(x-a)\sqrt{(x-a)(x-a+a-b)}}=-\int\frac{d\left(\frac{1}{x-a}\right)}{\sqrt{1+\frac{a-b}{x-a}}}=\begin{cases}\frac{2}{b-a}\sqrt{1+\frac{a-b}{x-a}}+C, & a\neq b\\ \frac{1}{a-x}+C, & a=b\end{cases}$

17. $\int\frac{1+\sin x}{1+\cos x}dx=\int\frac{1}{2\cos^2\frac{x}{2}}dx+\int\frac{\sin x}{1+\cos x}dx=\tan\frac{x}{2}-\ln(1+\cos x)+C$

或者：$\int\frac{1+\sin x}{1+\cos x}dx=\int\frac{(1+\sin x)(1-\cos x)}{1-\cos^2 x}dx=\int(\csc^2 x+\csc x-\csc x\cot x-\cot x)dx$

18. $\int\frac{7\cos x-3\sin x}{5\cos x+2\sin x}dx=\int\frac{5\cos x+2\sin x+(-5\sin x+2\cos x)}{5\cos x+2\sin x}dx=x+\ln|5\cos x+2\sin x|+C$

19. $\int\frac{\sqrt{x(1+x)}}{\sqrt{x}+\sqrt{1+x}}dx=\int(\sqrt{1+x}-\sqrt{x})\sqrt{x(1+x)}\,dx=\int[(1+x)\sqrt{x}-x\sqrt{1+x}]dx$

$=\frac{2}{3}x\sqrt{x}+\frac{2}{5}x^2\sqrt{x}+\frac{2}{3}(1+x)^{\frac{3}{2}}-\frac{2}{5}(1+x)^{\frac{5}{2}}+C$

20. $\int \frac{\sqrt{2x^2+3}}{x}dx \xlongequal{x=\sqrt{\frac{3}{2}}\tan t} \int \sqrt{2}\frac{\sqrt{\frac{3}{2}}\sec t}{\sqrt{\frac{3}{2}}\tan t}\sqrt{\frac{3}{2}}\sec^2 t\,dt = \sqrt{3}\int \frac{dt}{\sin t\cos^2 t}$

$= \sqrt{3}\int [\frac{\sin t}{\cos^2 t} + \frac{1}{\sin t}]dt = \sqrt{3}[\sec t - \ln|\csc t + \cot t|] + C$

$= \sqrt{2x^2+3} - \sqrt{3}\ln\frac{\sqrt{2x^2+3}+\sqrt{3}}{\sqrt{2}x} + C$

21. $\int \frac{dx}{\sqrt{(x-2)(x-3)}} = \int \frac{dx}{\sqrt{(x-\frac{5}{2})^2 - \frac{1}{4}}} = \ln(x - \frac{5}{2} + \sqrt{x^2-5x+6}) + C$

22. $\int \frac{e^{\arctan x}}{(1+x^2)\sqrt{1+x^2}}dx \xlongequal{\arctan x = t} \int e^t \cos t\,dt = \frac{e^t}{2}(\cos t + \sin t) + C = \frac{(1+x)e^{\arctan x}}{\sqrt{1+x^2}} + C$

23. $\int \frac{\cos x - \sin x}{1+\sin x\cos x}dx = \int \frac{\cos x - \sin x}{\frac{1}{2}+\frac{1}{2}+\sin x\cos x}dx = \int \frac{2(\cos x - \sin x)}{1+(\cos x+\sin x)^2}dx$

$= 2\int \frac{d(\cos x+\sin x)}{1+(\cos x+\sin x)^2} = 2\arctan(\cos x+\sin x) + C$

24. $\int \frac{\sin x+\cos x}{1+\sin x\cos x}dx = \int \frac{\cos x+\sin x}{\frac{3}{2}-\frac{1}{2}+\sin x\cos x}dx = \int \frac{2(\cos x+\sin x)}{3-(\sin x-\cos x)^2}dx$

$= 2\int \frac{d(\sin x - \cos x)}{3-(\sin x-\cos x)^2} = \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x} + C$

25. 由 $\int \frac{\cos x-\sin x}{\sqrt{2+\sin 2x}}dx = \int \frac{d(\sin x+\cos x)}{\sqrt{1+(\sin x+\cos x)^2}} = \ln(\sin x+\cos x+\sqrt{2+\sin 2x}) + C$

$\int \frac{\cos x+\sin x}{\sqrt{2+\sin 2x}}dx = \int \frac{d(\sin x-\cos x)}{\sqrt{3-(\sin x-\cos x)^2}} = \arcsin(\frac{\sin x-\cos x}{\sqrt{3}}) + C$

故原式 $= \frac{1}{2}\left[\arcsin\left(\frac{\sin x-\cos x}{\sqrt{3}}\right)-\ln\left(\sin x+\cos x+\sqrt{2+\sin 2x}\right)\right]+C$

26. 当 $a\neq 0$ 时, $\int \frac{\tan x}{a^2\sin^2 x+b^2\cos^2 x}dx=\int\frac{\tan x}{b^2+a^2\tan^2 x}d(\tan x)=\frac{1}{2a^2}\ln(b^2+a^2\tan^2 x)+C$

当 $a=0$ 时, $\int\frac{\tan x}{a^2\sin^2 x+b^2\cos^2 x}dx=\frac{1}{b^2}\int\frac{\sin x}{\cos^3 x}dx=\frac{1}{2b^2}\sec^2 x+C$

27. $\int\frac{1+x}{x(1+xe^x)}dx=\int\frac{(1+x)e^x}{xe^x(1+xe^x)}dx=\int\frac{d(xe^x)}{xe^x(1+xe^x)}$ 令 $xe^x=t$

$=\int\frac{dt}{t(1+t)}=\int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt=\ln\left|\frac{t}{1+t}\right|+C=\ln\left|\frac{xe^x}{1+xe^x}\right|+C$

28. $\int\sqrt{\frac{e^x-1}{e^x+1}}dx=\int\frac{e^x-1}{\sqrt{e^{2x}-1}}dx=\int\frac{e^x}{\sqrt{e^{2x}-1}}dx-\int\frac{1}{\sqrt{e^{2x}-1}}dx$

$=\int\frac{de^x}{\sqrt{e^{2x}-1}}+\int\frac{de^{-x}}{\sqrt{1-e^{-2x}}}=\ln\left(e^x+\sqrt{e^{2x}-1}\right)+\arcsin e^{-x}+C$

29. $\int\frac{\ln\tan x}{\sin 2x}dx=\int\frac{\ln\tan x}{2\frac{\sin x}{\cos x}\cos^2 x}dx=\frac{1}{2}\int\frac{\ln\tan x}{\tan x}d(\tan x)$

$=\frac{1}{2}\int\ln\tan x\,d(\ln\tan x)=\frac{1}{4}(\ln\tan x)^2+C$

30. $\int\frac{dx}{x^4\sqrt{1+x^2}}\xlongequal{x=\tan t}\int\frac{\cos^3 t}{\sin^4 t}dt=\int\frac{1-\sin^2 t}{\sin^4 t}d(\sin t)$

$=-\frac{1}{3}\frac{1}{\sin^3 t}+\frac{1}{\sin t}+C=-\frac{1}{3}\frac{\sqrt{(1+x^2)^3}}{x^3}+\frac{\sqrt{1+x^2}}{x}+C$

31. $\int\frac{\sqrt{a^2-x^2}}{x^4}dx\xlongequal{x=a\sin t}\frac{1}{a^2}\int(\csc^4 t-\csc^2 t)dt=\frac{\cot t}{a^2}+\frac{1}{a^2}\int\csc^4 t\,dt$

$$\int \csc^4 t\,dt = -\cot t\cdot\csc^2 t - 2\int \csc^2 t\cdot\cot^2 t\,dt = -\cot t\cdot\csc^2 t - 2\int\csc^4 t\,dt + 2\int\csc^2 t\,dt$$

$$= -(2+\csc^2 t)\cot t - 2\int \csc^4 t\,dt$$

得 $\int \csc^4 t\,dt = -\frac{1}{3}(2+\csc^2 t)\cot t + C$

故 $\int \frac{\sqrt{a^2-x^2}}{x^4}dx = \frac{\cot t}{a^2} - \frac{\cot t}{3a^2}(2+\csc^2 t) + C = \frac{\sqrt{a^2-x^2}}{3a^2 x}\left(1-\frac{a^2}{x^2}\right) + C$

另解 $\int \frac{\sqrt{a^2-x^2}}{x^4}dx \overset{x=\frac{1}{t}}{=} -\int t\sqrt{a^2t^2-1}\,dt = -\frac{1}{3a^2}(a^2t^2-1)\sqrt{a^2t^2-1} + C = \frac{\sqrt{a^2-x^2}}{3a^2x}\left(1-\frac{a^2}{x^2}\right) + C$

32. $\int x e^x \sin x\,dx = x\frac{e^x(\sin x - \cos x)}{2} - \frac{1}{2}\int e^x(\sin x - \cos x)dx = x\frac{e^x(\sin x-\cos x)}{2} + \frac{e^x}{2}\cos x + C$

33. 设 $\frac{x^3}{(x+1)^2(x^2+x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+x+1}$

则 $A(x+1)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x+1)^2 = x^3$

令 $x=-1$，得 $B=-1$，令 $x=0$，得 $A+D=1$

又 x^3 的系数得 $A+C=1$，比较 x 的系数且 $B=-1$，得 $2A+C+2D=1$

解得 $A=2$，$C=D=-1$

$\therefore \int \frac{x^3}{(x+1)^2(x^2+x+1)}dx = 2\ln|x+1| + \frac{1}{x+1} - \int\frac{x+1}{x^2+x+1}dx$

而 $\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int\frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int\frac{dx}{x^2+x+1}$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\int\frac{d\frac{2x+1}{\sqrt{3}}}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + C$$

故得 $\int \frac{x^3}{(x+1)^2(x^2+x+1)}dx = 2\ln|x+1| + \frac{1}{x+1} - \frac{1}{2}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\arctan\frac{2x+1}{\sqrt{3}} + C$

34. $\int_0^{\pi}\sqrt{\sin x - \sin^3 x}\,dx = \int_0^{\frac{\pi}{2}}|\cos x|\sqrt{\sin x}\,dx = \int_0^{\frac{\pi}{2}}\cos x\sqrt{\sin x}\,dx - \int_{\frac{\pi}{2}}^{\pi}\cos\sqrt{\sin x}\,dx$

$= \int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,d(\sin x) - \int_{\frac{\pi}{2}}^{\pi}\sqrt{\sin x}\,d(\sin x) = \frac{2}{3}(\sin x)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}} - \frac{2}{3}(\sin x)^{\frac{3}{2}}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{4}{3}$

35. $\int_0^{k\pi}\sqrt{1-\sin^2 x}\,dx = \int_0^{k\pi}\sqrt{1-\sin^2 x}\,dx = k\int_0^{\pi}\sqrt{1-\sin^2 x}\,dx = k\pi\int_0^{\pi}|\cos x|dx = 2k$

36. $\int_0^{\pi}\frac{\sin nx}{\sin x}dx = I_n = \int_0^{\pi}\frac{\sin[(n-1)+1]}{\sin x}dx = \int_0^{\pi}\frac{\sin(n-1)x\cos x}{\sin x}dx + \int_0^{\pi}\frac{\cos(n-1)x\sin x}{\sin x}dx$

$= \int_0^{\pi}\frac{\sin(n-1)x\cos x}{\sin x}dx = \int_0^{\pi}\frac{\sin(n-2)x\cos^2 x}{\sin x}dx + \int_0^{\pi}\frac{\cos(n-2)x\sin x\cos x}{\sin x}dx$

$= \int_0^{\pi}\frac{\sin(n-2)x}{\sin x} + \int_0^{\pi}[\cos(n-2)x\cos x - \sin(n-2)x\sin x]dx$

$= I_{n-2} + \int_0^{\pi}\cos(n-1)x\,dx = I_{n-2}$

$\therefore I_{2n} = I_2 = 0,\quad I_{2n+1} = I_1 = \pi$

37. 原式 $= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x\sec x\,d\left(\frac{-1}{x\sin x + \cos x}\right) = \frac{-x\sec x}{x\sin x + \cos x}\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{(x\sin x + \cos x)\sec^2 x}{x\sin x + \cos x}dx$

$= \frac{4\pi}{\sqrt{3}\pi + 18} - \frac{4\pi}{6\sqrt{3}\pi + 3} + \frac{2\sqrt{3}}{3}$

38. $\int_0^{\frac{\pi}{4}}\ln(1+\tan x)dx = \int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos x + \sin x}{\cos x}\right)dx = \int_0^{\frac{\pi}{4}}\ln(\cos x + \sin x)dx - \int_0^{\frac{\pi}{4}}\ln\cos x\,dx$

$=\int_0^{\frac{\pi}{4}}\ln\sqrt{2}\,dx+\int_0^{\frac{\pi}{4}}\ln\cos(x-\frac{\pi}{4})dx-\int_0^{\frac{\pi}{4}}\ln\cos x\,dx=\frac{\pi}{8}\ln 2$

39. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^4 x}{1+e^{-x}}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{(1+e^x-1)\sin^4 x}{1+e^x}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4 x\,dx-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^4 x}{1+e^x}dx$

令 $x=-t$，则 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^4 x}{1+e^x}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^4 t}{1+e^{-t}}dt=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^4 x}{1+e^{-x}}dx=I$

$\therefore I=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^4 x\,dx=\int_0^{\frac{\pi}{2}}\sin^4 x\,dx=\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{16}\pi$

40. $\int_0^3\arcsin\sqrt{\frac{x}{x+1}}dx=x\arcsin\sqrt{\frac{x}{x+1}}\Big|_0^3-\int_0^3 x[\arcsin\sqrt{\frac{x}{x+1}}]'dx$

$=x\arcsin\sqrt{\frac{x}{x+1}}\Big|_0^3-\frac{1}{2}\int_0^3\frac{\sqrt{x}}{x+1}dx=3\arcsin\frac{\sqrt{3}}{2}-\frac{1}{2}\int_0^3\frac{\sqrt{x}}{1+x}dx$

令 $\sqrt{x}=t$，$\int_0^3\frac{\sqrt{x}}{x+1}dx=\int_0^{\sqrt{3}}\frac{2t^2}{1+t^2}dt=2\int_0^{\sqrt{3}}\frac{t^2+1-1}{t^2+1}dt$

$=2(t-\arctan t)\Big|_0^{\sqrt{3}}=2(\sqrt{3}-\arctan\sqrt{3})=2\sqrt{3}-\frac{2}{3}\pi$

$\therefore\int_0^3\arcsin\sqrt{\frac{x}{x+1}}dx=3\cdot\frac{\pi}{3}-\sqrt{3}+\frac{\pi}{3}=\frac{4}{3}\pi-\sqrt{3}$