习题5.5

1. (2) $\int \frac{1}{x(1+x^2)}dx$

$= \int \frac{1}{x}dx + \int \frac{-x}{1+x^2}dx$

$= \ln|x| - \frac{1}{2}\int \frac{dx^2}{1+x^2}$

$= \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

(6) $\int \frac{1}{1+x^3}dx$

$= \int \frac{dx}{(1+x)(x^2-x+1)}$

$= \frac{1}{3}\int \frac{dx}{1+x} - \frac{1}{3}\int \frac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}dx$

$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\int \frac{d(x-\frac{1}{2})^2}{(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{\sqrt{3}}\int \frac{d\frac{2x-1}{\sqrt{3}}}{x^2-x+1}$

$= \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$

2. (4) $\int \frac{1+\tan x}{\sin 2x}dx$

$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

设$t=\tan x$. $x=\arctan t$, $dx=\frac{1}{1+t^2}dt$

$\int \frac{(1+t)(1+t^2)}{2t}\cdot\frac{1}{1+t^2}dt = \int (\frac{1}{2t}+\frac{1}{2})dt$

$= \frac{1}{2}\ln|t| + \frac{1}{2}t = \frac{1}{2}\ln|\tan x| + \frac{1}{2}\tan x + C$

(7) $\int \frac{\sin x}{\sin x+\cos x}dx$

$= \int \frac{\tan x}{\tan x+1}dx$ 设$t=\tan x$. $x=\arctan t$ $dx=\frac{1}{1+t^2}dt$.

$= \int \frac{t}{(1+t)(1+t^2)}dt$

$= -\frac{1}{2}\int \frac{1}{1+t}dt + \frac{1}{2}\int \frac{t+1}{1+t^2}dt$

$= -\frac{1}{2}\ln|1+t| + \frac{1}{4}\ln(1+t^2) + \frac{1}{2}\arctan t$

$= -\frac{1}{2}\ln|\overset{1+\tan x}{\cancel{\arctan x}}| + \frac{1}{4}\ln(1+\cancel{\text{arc}}\tan^2 x) + \frac{1}{2}x + C$

3. (3) $\int x\sqrt{x+2}\,dx$.

设$t=\sqrt{x+2}$, 则 $x=t^2-2$. $dx=2t\,dt$

$\int x\sqrt{x+2}\,dx = \int 2t^2(t^2-2)dt = \int 2t^4 dt - \int 4t^2 dt$

$= \frac{2}{5}t^5 - \frac{4}{3}t^3 + C$

(6) $\int x\sqrt{\frac{1+x}{1-x}}\,dx$.

设$t=\sqrt{\frac{1+x}{1-x}}$. $t^2=\frac{1+x}{1-x}$ 则 $x=\frac{t^2-1}{t^2+1}$, $dx=\frac{4t}{(t^2+1)^2}dt$

$\int x\sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{(t^2-1)\cdot 4t^2}{(t^2+1)^3}dt$

$= 4\int \frac{dt}{t^2+1} - 12\int \frac{dt}{(t^2+1)^2} + 8\int \frac{dt}{(t^2+1)^3}$

$\int \frac{dt}{(t^2+1)^2} = \frac{t}{(t^2+1)^2} - \int t\cdot\frac{-4t}{(t^2+1)^3}dt$

$= \frac{t}{(t^2+1)^2} + 4\int \frac{t^2+1-1}{(t^2+1)^3}dt$

$= \frac{t}{(t^2+1)^2} + 4\int \frac{1}{(t^2+1)^2}dt - 4\int \frac{1}{(t^2+1)^3}dt$

$4\int \frac{1}{(t^2+1)^3}dt = 3\int \frac{1}{(t^2+1)^2}dt + \frac{t}{(t^2+1)^2}$

$\int \frac{dt}{(t^2+1)} = \frac{t}{t^2+1} - \int t\cdot\frac{-2t}{(t^2+1)^2}dt = \frac{t}{t^2+1} + 2\int \frac{t^2dt}{(t^2+1)^2}$

$= \frac{t}{t^2+1} + 2\int \frac{1}{t^2+1}dt - 2\int \frac{dt}{(t^2+1)^2}$

$2\int \frac{1}{(t^2+1)^2}dt = \int \frac{1}{t^2+1}dt + \frac{t}{t^2+1}$

原式 $= 4\int \frac{dt}{t^2+1} - 12\int \frac{dt}{(t^2+1)^2} + 6\int \frac{dt}{(t^2+1)^2} + \frac{2t}{(t^2+1)^2}$

$= 4\int \frac{dt}{(t^2+1)} - 3\int \frac{1}{t^2+1}dt - \frac{3t}{t^2+1} + \frac{2t}{(t^2+1)^2}$

$= \arctan t - \frac{3t}{t^2+1} + \frac{2t}{(t^2+1)^2}$

$= \arctan\sqrt{\frac{1+x}{1-x}} - \frac{3\sqrt{\frac{1+x}{1-x}}}{\frac{2}{1-x}} + \frac{2\sqrt{\frac{1+x}{1-x}}}{(\frac{2}{1-x})^2}$

$= \arctan\sqrt{\frac{1+x}{1-x}} - \frac{3}{2}\sqrt{\frac{1+x}{1-x}}(1-x) + \frac{1}{2}\sqrt{\frac{1+x}{1-x}}(1-x)^2$.

4. (6) $\int \frac{1+\sin x}{1+\cos x} e^x dx$

$= \int \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} e^x dx$

$= \int \frac{1}{2\cos^2\frac{x}{2}} e^x dx + \int \tan\frac{x}{2} e^x dx$

$= \tan\frac{x}{2}\cdot e^x - \int \tan\frac{x}{2} e^x dx + \int \tan\frac{x}{2} e^x dx$

$= \tan\frac{x}{2} e^x + C.$

$\frac{1}{\cos^2 x} \to (\tan x)'$ ☆.

(8) $\int \frac{x\ln x}{(x^2+1)^2} dx$

$\int \frac{x}{(x^2+1)^2} dx = \frac{1}{2}\int \frac{dx^2}{(x^2+1)^2} = -\frac{1}{2(x^2+1)}$

$\int \frac{x\ln x}{(x^2+1)^2} dx = -\frac{1}{2(x^2+1)}\ln x + \int \frac{1}{2(x^2+1)} d\ln x$

$= -\frac{1}{2(x^2+1)}\ln x + \frac{1}{2}\int \frac{1}{x(x^2+1)} dx$

$\int \frac{1}{x(x^2+1)} dx = \int \frac{1}{x} dx - \int \frac{x}{x^2+1} dx$

$= \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

$\int \frac{x\ln x}{(x^2+1)^2} dx = -\frac{\ln x}{2(x^2+1)} + \frac{1}{2}\ln x - \frac{1}{4}\ln(1+x^2) + C$

(9) $\int \frac{\arctan x}{x^2(1+x^2)} dx$

$\int \frac{\arctan x}{x^2(1+x^2)} dx = \int \frac{\arctan x}{x^2} dx - \int \frac{\arctan x}{1+x^2} dx$

$= -\int \arctan x \, d\frac{1}{x} - \int \arctan x \, d\arctan x$

$= -\arctan x \cdot \frac{1}{x} + \int \frac{1}{x}\cdot\frac{1}{x^2+1} dx - \frac{\arctan^2 x}{2}$

$= -\frac{\arctan x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) - \frac{\arctan^2 x}{2} + C$

习题5.6

1. (12) $\int_0^{\pi} \sqrt{1-\sin 2x}\, dx$

$= \int_0^{\pi} \sqrt{\sin^2 x - 2\sin x\cos x + \cos^2 x}\, dx$

$= \int_0^{\pi} \sqrt{(\sin x - \cos x)^2}\, dx$

$= \int_0^{\pi} |\sin x - \cos x|\, dx$

$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x) dx$

$= \sin x\Big|_0^{\frac{\pi}{4}} + \cos x\Big|_0^{\frac{\pi}{4}} - \cos x\Big|_{\frac{\pi}{4}}^{\pi} - \sin x\Big|_{\frac{\pi}{4}}^{\pi}$

$= \frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{2} - 1) - (-1 - \frac{\sqrt{2}}{2}) - (-\frac{\sqrt{2}}{2})$

$= 2\sqrt{2}$

(11) $\int_0^1 \frac{1}{(x+1)\sqrt{1+x^2}} dx$

令 $x=\tan t$ $\quad t=\arctan x$

$= \int_0^{\frac{\pi}{4}} \frac{1}{(\tan t+1)\cdot\frac{1}{\cos t}}\cdot\frac{1}{\cos^2 t} dt$

$= \int_0^{\frac{\pi}{4}} \frac{1}{\sin t+\cos t} dt = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{1}{\sin(\frac{\pi}{4}+x)} dx$ (令 $t=x$)

$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{1}{\cos x} dx = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos^2 x} dx$

$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{d\sin x}{1-\sin^2 x} = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{4}} (\frac{1}{1-\sin x} + \frac{1}{1+\sin x}) d\sin x$

$= \frac{1}{2\sqrt{2}}(-\ln(1-\sin x))\Big|_0^{\frac{\pi}{4}} + \frac{1}{2\sqrt{2}}\ln(1+\sin x)\Big|_0^{\frac{\pi}{4}}$

$= -\frac{1}{2\sqrt{2}}\ln(1-\frac{\sqrt{2}}{2}) + \frac{1}{2\sqrt{2}}\ln(1+\frac{\sqrt{2}}{2})$

$= \frac{1}{2\sqrt{2}}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}$

(14) $\int_0^{\ln 2}\sqrt{1+e^x}\,dx$

令 $t=\sqrt{1+e^x}$. $x=\ln(t^2-1)$

$\int_{\sqrt2}^{\sqrt3} t\,d\ln(t^2-1)$

$=\int_{\sqrt2}^{\sqrt3}\frac{2t^2}{t^2-1}dt$

$=\int_{\sqrt2}^{\sqrt3}(2+\frac{1}{t-1}-\frac{1}{t+1})dt$

$=2t\Big|_{\sqrt2}^{\sqrt3}+\ln|t-1|\Big|_{\sqrt2}^{\sqrt3}-\ln|t+1|\Big|_{\sqrt2}^{\sqrt3}$

$=2(\sqrt3-\sqrt2)+\ln(\sqrt3-1)-\ln(\sqrt2-1)-\ln(\sqrt3+1)+\ln(\sqrt2+1)$

$=2(\sqrt3-\sqrt2)+\ln(2-\sqrt3)+\ln(3+2\sqrt2)$

2. (12) $\int_0^1 x^2e^{-2x}dx$

$=\int_0^1 x^2\cdot(-\frac12)de^{-2x}$

$=-\frac{x^2}{2}e^{-2x}\Big|_0^1+\int_0^1 xe^{-2x}dx$

$=-\frac{x^2}{2}e^{-2x}\Big|_0^1+\int_0^1 x\cdot(-\frac12)de^{-2x}$

$=-\frac{x^2}{2}e^{-2x}\Big|_0^1-\frac{x}{2}e^{-2x}\Big|_0^1+\frac12\int_0^1 e^{-2x}dx$

$=-\frac{1}{2e^2}-\frac{1}{2e^2}-\frac14e^{-2x}\Big|_0^1$

$=-\frac{1}{e^2}-\frac{1}{4e^2}+\frac14$

$=\frac14-\frac{5}{4e^2}$

(4) $\int_0^{\sqrt3}x\arctan x\,dx$

$=\int_0^{\sqrt3}\frac12\arctan x\,dx^2$

$=\frac12x^2\arctan x\Big|_0^{\sqrt3}-\frac12\int_0^{\sqrt3}x^2\cdot\frac{1}{1+x^2}dx.$

$=\frac12x^2\arctan x\Big|_0^{\sqrt3}-\frac12\int_0^{\sqrt3}(1-\frac{1}{x^2+1})dx$

$=\frac12x^2\arctan x\Big|_0^{\sqrt3}-\frac12x\Big|_0^{\sqrt3}+\frac12\arctan x\Big|_0^{\sqrt3}$

$=\frac32\cdot\frac\pi3-\frac{\sqrt3}{2}+\frac12\cdot\frac\pi3$

$=\frac\pi2+\frac\pi6-\frac{\sqrt3}2=\frac23\pi-\frac{\sqrt3}2$

(18) $\int_0^{\frac\pi2}e^{2x}\sin^2x\,dx$

$=\frac12\int_0^{\frac\pi2}\sin^2x\,de^{2x}$

$=\frac12\sin^2x\,e^{2x}\Big|_0^{\frac\pi2}-\frac12\int_0^{\frac\pi2}e^{2x}\cdot2\sin x\cos x\,dx$

$=\frac12\sin^2x\,e^{2x}\Big|_0^{\frac\pi2}-\frac14\int_0^{\pi}e^{2x}\sin2x\,d2x$

令换 $t=2x$ $I=\int_0^\pi e^t\sin t\,dt$

$I=e^t\sin t\Big|_0^\pi-\int_0^\pi e^t\cos t\,dt$

$=e^t\sin t\Big|_0^\pi-e^t\cos t\Big|_0^\pi-\int_0^\pi e^t\sin t\,dt$

$I=\frac12e^t\sin t\Big|_0^\pi-\frac12e^t\cos t\Big|_0^\pi$

$=\frac12e^\pi+\frac12$

原式$=\frac12e^\pi-\frac18e^\pi-\frac18=\frac38e^\pi-\frac18$

3. (1) $\int_0^{\frac\pi2}\sin^4x\,dx=I$

$=\int_0^{\frac\pi2}\sin^4(\frac\pi2-x)dx=\int_0^{\frac\pi2}\cos^4x\,dx$

$2I=\int_0^{\frac\pi2}[(\sin^2x+\cos^2x)^2-2\sin^2x\cos^2x]dx$

$=\int_0^{\frac\pi2}(1-\frac12\sin^22x)dx$

$=x\Big|_0^{\frac\pi2}-\frac14\int_0^{\frac\pi2}\sin^22x\,d2x$

$\int_0^{\frac\pi2}\sin^22x\,d2x$ 令 $t=2x$. $=\int_0^\pi\sin^2t\,dt=I_2$

$I_2=\int_0^\pi\sin x\,d(-\cos x)=-\sin x\cos x\Big|_0^\pi-\int_0^\pi(-\cos^2x)dx$

$=-\sin x\cos x\Big|_0^\pi+\int_0^\pi dx-\int_0^\pi\sin^2x\,dx$

$I_2=-\frac12\sin x\cos x\Big|_0^\pi+\frac12\int_0^\pi dx=\frac\pi2$

$I=\frac\pi4-\frac18\cdot\frac\pi2=\frac{3}{16}\pi$

(4) $\int_0^{\pi}\cos^7 x\,dx$

$=\int_0^{\pi}\cos^7(\pi-x)\,dx$

$=-\int_0^{\pi}\cos^7 x\,dx$

$\int_0^{\pi}\cos^7 x\,dx=0$

(8) $\int_{-a}^{a}(1-x)\sqrt{a^2-x^2}\,dx$

$=\int_{-a}^{a}\sqrt{a^2-x^2}\,dx-\int_{-a}^{a}x\sqrt{a^2-x^2}\,dx$

$=\int_{-a}^{a}\sqrt{a^2-x^2}\,dx+\int_{a}^{a}x\sqrt{a^2-x^2}\,dx$

$=2\int_0^{a}\sqrt{a^2-x^2}\,dx$

令 $x=a\sin t$ $\quad t=\arcsin\frac{x}{a}$

$=2\int_0^{\frac{\pi}{2}}a^2\cos^2 t\,dt=2a^2\int_0^{\frac{\pi}{2}}\cos^2 t\,dt=I$

$I=2a^2\int_0^{\frac{\pi}{2}}\sin^2 t\,dt$

$2I=2a^2\int_0^{\frac{\pi}{2}}dt \quad I=a^2t\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}a^2$

则 $\int_{-a}^{a}(1-x)\sqrt{a^2-x^2}\,dx=\frac{\pi}{2}a^2$

5. $\int_0^{\pi}(\pi-x)f[\sin(\pi-x)]\,dx=\int_0^{\pi}xf(\sin x)\,dx$

$=\int_0^{\pi}\pi f(\sin x)\,dx-\int_0^{\pi}xf(\sin x)\,dx$

$\int_0^{\pi}xf(\sin x)\,dx=\frac{1}{2}\int_0^{\pi}\pi f(\sin x)\,dx.$

$=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx.$

8. $\int_0^1 xf(x)\,dx=\frac{1}{2}\int_0^1 f(x)\,dx^2$

$=\frac{1}{2}f(x)x^2\Big|_0^1-\frac{1}{2}\int_0^1 x^2\,df(x)$

$df(x)=2x\cdot e^{-x^4}dx$

$\int_0^1 x^2\,df(x)=\int_0^1 2x^3e^{-x^4}dx$

$=-\frac{1}{2}\int_0^1(-4x^3)e^{-x^4}dx$

$=-\frac{1}{2}e^{-x^4}\Big|_0^1$

$=-\frac{1}{2e}+\frac{1}{2}$

$\frac{1}{2}f(x)x^2\Big|_0^1=\frac{1}{2}\int_1^1 e^{-t^2}dt\cdot 1-\frac{1}{2}\int_1^0 e^{-t^2}dt\cdot 0=0$

$\int_0^1 xf(x)\,dx=\frac{1}{4e}-\frac{1}{4}$

11. $\int_{-a}^{a}f(x)\,dx$ 令 $t=-x$

$\int_{a}^{-a}f(-t)\,d(-t)=-\int_{a}^{a}f(-t)\,dt=\int_{-a}^{a}f(-t)\,dt$

令 $t=x$. $\int_{-a}^{a}f(-t)\,dt=\int_{-a}^{a}f(-x)\,dx.$

则 $\int_{-a}^{a}f(x)\,dx=\int_{-a}^{a}f(-x)\,dx$

$\int_{-a}^{a}f(x)\,dx=\frac{1}{2}\int_{-a}^{a}[f(x)+f(-x)]\,dx$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2x}{1+e^{-x}}dx=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\sin^2x}{1+e^{x}}+\frac{e^x\sin^2x}{1+e^x}\right)dx$

$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2x\,dx$

$=\int_0^{\frac{\pi}{2}}\sin^2x\,dx.=\int_0^{\frac{\pi}{2}}\cos^2x\,dx.$

$\int_0^{\frac{\pi}{2}}\sin^2x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin^2x+\cos^2x)\,dx$

$=\frac{1}{2}x\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{4}$

则 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2x}{1+e^{-x}}dx=\frac{\pi}{4}$

13. (1) 设 $F(x)=\int_0^x f(t)\,dt$, $F'(x)=f(x)$.

$F(0)=0$. $F(1)=0$

$F(x)+xF'(x)=[xF(x)]'$

设 $g(x)=x\cdot F(x)$ $\quad g(0)=0$. $g(1)=0$ (背面)

$\exists\,\xi\in(0,1)$, $g(\xi)=0$ 即 $F(\xi)+\xi F'(\xi)=0$

即 $\int_0^{\xi} f(x)dx = -\xi f(\xi)$

(2) $F(x)=\int_0^x f(t)dt$. $F'(x)=f(x)$

$F(0)=0$. $F(1)=0$ 设 $g(x)=xF(x)$

$g'(x)=[xF(x)]'=F(x)+xF'(x)$

$g'(0)=F(0)+0=0$、由1)知 $\exists\,\xi\in(0,1)$. $g'(\xi)=0$

由Rolle定理，$\exists\,\eta\in(0,\xi)$. $g''(\eta)=0$

$$\begin{aligned} g''(x) &= F'(x)+F'(x)+xF''(x) \\ &= 2F'(x)+xF''(x) \\ &= 2f(x)+xf'(x) \end{aligned}$$

则 $g''(\eta)=2f(\eta)+\eta f'(\eta)=0$