Review

- Lagrange乘子法

$$\max(\min)\ \ f(\mathrm{x}) = f(x_1,\cdots,x_n)$$

$$s.t.\quad g_i(\mathrm{x}) = g_i(x_1,\cdots,x_n) = 0,\quad i=1,\cdots,m.$$

其中rank $\dfrac{\partial(g_1,\cdots,g_m)}{\partial(x_1,\cdots,x_n)} = m$ (正则性条件).

结论: x_0是条件极值问题的最大(小)值点, 则$\exists\lambda_0$,s.t. (x_0,λ_0)是

$$L(\mathrm{x},\lambda) = L(x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_m)$$

$$= f(x_1,\cdots,x_n) + \sum_{i=1}^{m}\lambda_i g_i(x_1,\cdots,x_n)$$

的驻点.

第二章 含参积分与广义含参积分

$$I(t)=\int_a^b f(t,x)dx$$

$$I(t)=\int_a^{+\infty} f(t,x)dx$$

$$I(t)=\int_{-\infty}^b f(t,x)dx$$

$x:$积分变量

$t:$参变量

Question: $I(t)$的连续性、可微性、可积性？

Question: 研究含参积分的意义？

§ 1. 含参(定)积分的性质

1. 多元函数的一致连续性

Def. 设$f:\Omega\subset\mathbb{R}^n\to\mathbb{R}$,若$\forall\varepsilon>0,\exists\delta>0,s.t.$

$$\left|f(\mathrm{x})-f(\mathrm{x}')\right|<\varepsilon,\quad \forall\mathrm{x},\mathrm{x}'\in\Omega,\left\|\mathrm{x}-\mathrm{x}'\right\|<\delta,$$

则称f在Ω上一致连续.

Thm. f在$\Omega\subset\mathbb{R}^n$上一致连续的充要条件是:

对Ω中任意两个点列$\{\mathrm{x}_k\},\{\mathrm{y}_k\}$,当$\lim\limits_{n\to\infty}\left\|\mathrm{x}_k-\mathrm{y}_k\right\|=0$时,有

$$\lim_{k\to\infty}(f(\mathrm{x}_k)-f(\mathrm{y}_k))=0.$$

Thm. $\Omega\subset\mathbb{R}^n$为有界闭集,$f\in C(\Omega)$,则$f$在$\Omega$上一致连续.

2. 含参积分的连续性

Thm1. 设二元函数$g(t,x)$在$D=[a,b]\times[\alpha,\beta]$上连续, 则

$$f(t)=\int_\alpha^\beta g(t,x)dx \text{在}[a,b]\text{上一致连续.}$$

Proof: $g(t,x)$在有界闭区域D上连续, 从而一致连续. $\forall\varepsilon>0$, $\exists\delta>0, \forall(t,x),(t_0,x_0)\in D$, 只要$\sqrt{(t-t_0)^2+(x-x_0)^2}\le\delta$, 就有

$$|g(t,x)-g(t_0,x_0)|<\varepsilon.$$

特别地, $\forall t,t_0\in[a,b], \forall x\in[\alpha,\beta]$, 只要$|t-t_0|<\delta$, 就有

$$|g(t,x)-g(t_0,x)|<\varepsilon,$$

于是 $|f(t)-f(t_0)|=\left|\int_\alpha^\beta g(t,x)dx-\int_\alpha^\beta g(t_0,x)dx\right|$

$$\le\int_\alpha^\beta|g(t,x)-g(t_0,x)|dx\le\varepsilon(\beta-\alpha).\square$$

Remark:定理中$f(t)=\int_{\alpha}^{\beta} g(t,x)dx$在$[a,b]$上连续. 即

$$\lim_{t\to t_0} f(t)=f(t_0),\ \forall t_0\in[a,b].$$

而
$$\lim_{t\to t_0} f(t)=\lim_{t\to t_0}\int_{\alpha}^{\beta} g(t,x)dx,$$

$$f(t_0)=\int_{\alpha}^{\beta} g(t_0,x)dx=\int_{\alpha}^{\beta}\lim_{t\to t_0} g(t,x)dx,$$

于是
$$\lim_{t\to t_0}\int_{\alpha}^{\beta} g(t,x)dx=\int_{\alpha}^{\beta}\lim_{t\to t_0} g(t,x)dx.$$

对这一等式的解释是: $g(t,x)$在$[a,b]\times[\alpha,\beta]$上连续时,对参变量$t$的极限运算$\lim\limits_{t\to t_0} f(t)$与对积分变量$x$的积分运算$\int_{\alpha}^{\beta} g(t,x)dx$可以交换顺序.□

例. 计算 $\lim\limits_{y\to 0+}\int_0^1 \dfrac{1}{1+(1+xy)^{1/y}}dx.$

解: $\lim\limits_{y\to 0+}(1+xy)^{1/y} = e^x,\quad \forall x\in[0,1].$ 令

$$f(x,y)=\begin{cases}\dfrac{1}{1+(1+xy)^{1/y}}, & 0\le x\le 1, 0<y\le 1,\\ \dfrac{1}{1+e^x}, & 0\le x\le 1, y=0.\end{cases}$$

$f(x,y)$在$(x,y)\in[0,1]\times[0,1]$上连续,因此

$$\lim_{y\to 0+}\int_0^1 \frac{1}{1+(1+xy)^{1/y}}dx=\lim_{y\to 0+}\int_0^1 f(x,y)dx$$

$$=\int_0^1 \lim_{y\to 0+} f(x,y)dx=\int_0^1\frac{1}{1+e^x}dx=\ln\frac{2e}{1+e}.\square$$

例. $\lim\limits_{y\to 0}\int_0^1 \frac{x}{y^2}e^{\frac{-x^2}{y^2}}dx \overset{?}{=} \int_0^1 \lim\limits_{y\to 0}\frac{x}{y^2}e^{\frac{-x^2}{y^2}}dx$, 为什么？

解: $\lim\limits_{y\to 0}\int_0^1 \frac{x}{y^2}e^{-\frac{x^2}{y^2}}dx = \lim\limits_{y\to 0}\frac{1}{2}\int_0^1 e^{\frac{-x^2}{y^2}}d\frac{x^2}{y^2} = \lim\limits_{y\to 0}\frac{1}{2}(1-e^{\frac{-1}{y^2}}) = \frac{1}{2}.$

$\int_0^1 \lim\limits_{y\to 0}\frac{x}{y^2}e^{\frac{-x^2}{y^2}}dx = 0 \neq \lim\limits_{y\to 0}\int_0^1 \frac{x}{y^2}e^{\frac{-x^2}{y^2}}dx.$ 这是因为

$$f(x,y)=\begin{cases}\frac{x}{y^2}e^{\frac{-x^2}{y^2}}, & x\in[0,1], y\neq 0\\ 0, & x\in[0,1], y=0\end{cases}$$ 在(0,0)不连续. 事实上,

$$\lim_{x=y^2, y\to 0} f(x,y) = \lim_{x=y^2, y\to 0}\frac{x}{y^2}e^{\frac{-x^2}{y^2}} = \lim_{y\to 0} e^{-y^2} = 1 \neq f(0,0) = 0.\square$$

交换极限需谨慎！

例. $f\in C[0,1],\ f(x)>0,\ \alpha>0.$研究$g(y)=\int_0^1\frac{y^\alpha f(x)}{x^2+y^2}\mathrm{d}x\ (y\geq 0)$的连续性,并证明你的结论.

结论: $0<\alpha\leq 1$时, $f\in C(0,+\infty)$, 在$y=0$处不连续;

$\alpha>1$时, $f\in C[0,+\infty)$.

证明: 对$(0,+\infty)$的任意闭子区间$[\delta,N]$, $\frac{y^\alpha f(x)}{x^2+y^2}\ (\alpha>0)$在$[0,1]\times[\delta,N]$上连续, 因此$g(y)$在$(0,+\infty)$上连续.

以下讨论$g(y)$在$y_0=0$处的连续性. $g(0)=0.$

$f\in C[0,1],\ f(x)>0,$ 因此 f在$[0,1]$上有最大值M和最小值m, 且$0<m\leq f(x)\leq M,\ \forall x\in[0,1].$

$$\frac{my^{\alpha}}{x^2+y^2} \le \frac{y^{\alpha} f(x)}{x^2+y^2} \le \frac{My^{\alpha}}{x^2+y^2}, \forall y>0, 0\le x\le 1.$$

$$\int_0^1 \frac{y^{\alpha}}{x^2+y^2}\mathrm{d}x = y^{\alpha-1}\int_0^1 \frac{y}{x^2+y^2}\mathrm{d}x = y^{\alpha-1}\arctan\frac{1}{y}, \forall y>0.$$

当$0<\alpha\le 1$时,

$$g(y)=\int_0^1 \frac{y^{\alpha} f(x)}{x^2+y^2}\mathrm{d}x \ge my^{\alpha-1}\arctan\frac{1}{y} \to +\infty, y\to 0^+\text{时}.$$

$g(y)$在$y_0=0$处不连续.

当$\alpha>1$时, $0\le g(y)\le My^{\alpha-1}\arctan\dfrac{1}{y}$, $\lim\limits_{y\to 0^+} g(y)=0=g(0)$,

$g(y)$在$y_0=0$处连续.□

3. 含参积分的可微性

Thm2. 设$D=[a,b]\times[\alpha,\beta]$, 且$g(t,x), g_t'(t,x)\in C(D)$, 则

$f(t)=\int_\alpha^\beta g(t,x)dx$在$[a,b]$上连续可导, 且

$$f'(t)=\frac{d}{dt}\int_\alpha^\beta g(t,x)dx=\int_\alpha^\beta g_t'(t,x)dx.$$

Proof: $\forall t\in[a,b]$, 设$t+\Delta t\in[a,b]$($f(t+\Delta t)$有意义), 则

$$\frac{f(t+\Delta t)-f(t)}{\Delta t}=\int_\alpha^\beta \frac{1}{\Delta t}\left[g(t+\Delta t,x)-g(t,x)\right]dx$$

$$=\int_\alpha^\beta g_t'(t+\theta\Delta t,x)dx,\qquad \theta(t,\Delta t,x)\in(0,1).$$

$g_t'(t,x)\in C(D)$, 从而一致连续, $\forall\varepsilon>0, \exists\delta>0, s.t.$

$$\left|g_t'(t_1,x)-g_t'(t_2,x)\right|<\varepsilon,\quad \forall(t_i,x)\in D, i=1,2, \left|t_1-t_2\right|<\delta.$$

于是,当$|\Delta t|<\delta$时,

$$\left|g_t'(t+\theta\Delta t,x)-g_t'(t,x)\right|<\varepsilon,$$

从而有

$$\left|\frac{f(t+\Delta t)-f(t)}{\Delta t}-\int_\alpha^\beta g_t'(t,x)dx\right|$$

$$=\left|\int_\alpha^\beta g_t'(t+\theta\Delta t,x)dx-\int_\alpha^\beta g_t'(t,x)dx\right|$$

$$\le\int_\alpha^\beta\left|g_t'(t+\theta\Delta t,x)-g_t'(t,x)\right|dx\ \le(\beta-\alpha)\varepsilon.$$

故 $f'(t)=\int_\alpha^\beta g_t'(t,x)dx.$ □

Remark:当二元函数$g(t,x), g_t'(t,x)$在$[a,b]\times[\alpha,\beta]$上连续时,含参变量$t$的积分$\int_\alpha^\beta g(t,x)dt$,对于参变量$t$求导的运算与对于积分变量$x$的积分运算可以交换次序:

$$\frac{d}{dt}\int_\alpha^\beta g(t,x)dx = \int_\alpha^\beta g_t'(t,x)dx.$$

于是,当积分$f(t)=\int_\alpha^\beta g(t,x)dx$难以计算,而$f(t_0)$容易计算时, 可尝试先求出$f'(t)=\int_\alpha^\beta g_t'(t,x)dx$,再对$t$积分:

$$f(t)=f(t_0)+\int_{t_0}^t f'(t)dt.$$

例. $F(x)=\int_0^{2\pi}e^{x\cos\theta}\cos(x\sin\theta)d\theta$, 证明$F(x)\equiv 2\pi$.

Proof. 令$f(x,\theta)=e^{x\cos\theta}\cos(x\sin\theta)$, 则$\forall r>0$, $f(x,\theta)$, $f_x'(x,\theta)$在$[-r,r]\times[0,2\pi]$上连续. 因此

$$F'(x)=\int_0^{2\pi}f_x'(x,\theta)d\theta=\int_0^{2\pi}e^{x\cos\theta}\cos\theta\cos(x\sin\theta)d\theta$$
$$-\int_0^{2\pi}e^{x\cos\theta}\sin(x\sin\theta)\sin\theta d\theta\triangleq I-J.$$

$$I=\int_0^{2\pi}\frac{1}{x}e^{x\cos\theta}d\sin(x\sin\theta)$$
$$=\frac{1}{x}e^{x\cos\theta}\sin(x\sin\theta)\Big|_{\theta=0}^{2\pi}-\frac{1}{x}\int_0^{2\pi}\sin(x\sin\theta)de^{x\cos\theta}=J,\forall x\neq 0.$$

于是, $F'(x)\equiv 0,\forall|x|\le r.$由$F(0)=2\pi$及$r$的任意性, $F(x)\equiv 2\pi$.□

Thm3.设$g(t,x), g_t'(t,x)\in C([a,b]\times[c,d]), \alpha(t), \beta(t)$在$[a,b]$上可导,且

$$c\le\alpha(t),\beta(t)\le d,\quad \forall t\in[a,b],$$

则

$$f(t)=\int_{\alpha(t)}^{\beta(t)} g(t,x)dx$$

在区间$[a,b]$上可导,且

$$f'(t)=\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} g(t,x)dx$$

$$=\int_{\alpha(t)}^{\beta(t)} g_t'(t,x)dx+g(t,\beta(t))\beta'(t)-g(t,\alpha(t))\alpha'(t).$$

Proof. 令$J(t,\alpha,\beta)=\int_{\alpha}^{\beta} g(t,x)dx$,由$g(t,x)$, $g_t'(t,x)$的连续性,

$$J_t'=\int_{\alpha}^{\beta} g_t'(t,x)dx,\ J_{\alpha}'=-g(t,\alpha),\ J_{\beta}'=g(t,\beta)$$

均在$(t,\alpha,\beta)\in D=[a,b]\times[c,d]\times[c,d]$上连续.因此$J(t,\alpha,\beta)$在$D$上可微,复合函数

$$f(t)=\int_{\alpha(t)}^{\beta(t)} g(t,x)dx=J(t,\alpha(t),\beta(t))$$

在$t\in[a,b]$上可微,且

$$\begin{aligned} f'(t)&=J_t'+J_{\alpha}'\cdot\alpha'(t)+J_{\beta}'\cdot\beta'(t)\\ &=\int_{\alpha(t)}^{\beta(t)} g_t'(t,x)dx+g(t,\beta(t))\beta'(t)-g(t,\alpha(t))\alpha'(t).\square \end{aligned}$$

例. $z(x,y)=\int_0^1 f(t)\,|xy-t|\,dt, 0\le x,y\le 1, f\in C([0,1]).$

求证$z''_{xx}=2y^2 f(xy).$

证明: 首先要去绝对值.

$$z=\int_0^{xy} f(t)(xy-t)dt+\int_{xy}^1 f(t)(t-xy)dt$$

$$z'_x=\int_0^{xy} f(t)y\,dt+\int_{xy}^1 f(t)(-y)dt$$

$$z''_{xx}=2y^2 f(xy).\square$$

例. $f(x)=\int_0^x\left[\int_t^x e^{-s^2}ds\right]dt$. 求$f'(x)$.

解: 令$g(x,t)=\int_t^x e^{-s^2}ds$, 则

$$g'_x(x,t)=e^{-x^2}, \quad f(x)=\int_0^x g(x,t)dt.$$

$g(x,t), g'_x(x,t)$均为$\mathbb{R}^2$上连续函数, $\alpha(x)=0, \beta(x)=x$均为可导函数.于是

$$f'(x)=\int_0^x g'_x(x,t)dt+g(x,\beta(x))\beta'(x)-g(x,\alpha(x))\alpha'(x)$$
$$=\int_0^x e^{-x^2}dt+0-0=xe^{-x^2}.$$

注意到$f(0)=0$,有

$$f(x)=\int_0^x f'(x)dx=\int_0^x xe^{-x^2}dx=\frac{1}{2}(1-e^{-x^2}).\square$$

引入参变量y

例. 计算$\int_0^{\pi}\ln(1+\frac{1}{2}\cos x)dx.$

解: 令$I(y)=\int_0^{\pi}\ln(1+y\cos x)dx,\ y\in[0,3/4]$, 则$I(0)=0$,

$$f(x,y)=\ln(1+y\cos x),\ f_y'(x,y)=\frac{\cos x}{1+y\cos x}$$

均在$(x,y)\in[0,\pi]\times[0,3/4]$上连续. 因此

$$I'(y)=\int_0^{\pi}\frac{\cos x}{1+y\cos x}dx=\frac{1}{y}\int_0^{\pi}(1-\frac{1}{1+y\cos x})dx$$

$$=\frac{\pi}{y}-\frac{1}{y}\int_0^{\pi}\frac{dx}{1+y\cos x}=\frac{\pi}{y}-\frac{1}{y}\int_0^{+\infty}\frac{2dt}{1+y+(1-y)t^2}$$

$t=\tan(x/2)$

$$=\frac{\pi}{y}-\frac{2}{y\sqrt{1-y^2}}\arctan\left(\sqrt{\frac{1-y}{1+y}}t\right)\Bigg|_{t=0}^{+\infty}$$

$$=\frac{\pi}{y}\left(1-\frac{1}{\sqrt{1-y^2}}\right),\ \ (y>0).$$

积分得 $I(y)=\pi\ln(1+\sqrt{1-y^2})-\pi\ln 2.$

故 $\int_0^{\pi}\ln(1+\frac{1}{2}\cos x)dx=I(\frac{1}{2})=\pi\ln\frac{2+\sqrt{3}}{4}.$ □

4. 含参积分的可积性

Thm4. (累次积分交换次序的充分条件)

设$g(t,x)$在$(t,x)\in D=[a,b]\times[\alpha,\beta]$上连续, 则$\int_\alpha^\beta g(t,x)dx$

在$t\in[a,b]$上可积, $\int_a^b g(t,x)dt$在$x\in[\alpha,\beta]$上可积, 且

$$\int_a^b\left(\int_\alpha^\beta g(t,x)dx\right)dt=\int_\alpha^\beta\left(\int_a^b g(t,x)dt\right)dx,$$

简记为 $$\int_a^b dt\int_\alpha^\beta g(t,x)dx=\int_\alpha^\beta dx\int_a^b g(t,x)dt.$$

Proof. 由$g(t,x)$的连续性及Thm1, $\int_\alpha^\beta g(t,x)dx$在$t\in[a,b]$上

连续, 从而可积. 同理, $\int_a^b g(t,x)dt$在$x\in[\alpha,\beta]$上可积.

关于累次积分交换次序, 我们可以证明更一般的结论:

$$\int_a^z \left(\int_\alpha^\beta g(t,x)dx\right)dt = \int_\alpha^\beta \left(\int_a^z g(t,x)dt\right)dx, \quad \forall z \in [a,b].$$

事实上, $z = a$时, 上式左右两边相等. 下面只要证两边对z的导函数存在且相等.

先看右边. $\forall z, z_0 \in [a,b], x, x_0 \in [\alpha, \beta]$, 有

$$\left|\int_a^z g(t,x)dt - \int_a^{z_0} g(t,x_0)dt\right|$$

$$\leq \int_a^{z_0} \left|g(t,x) - g(t,x_0)\right| dt + \left|\int_{z_0}^z \left|g(t,x)\right| dt\right|,$$

由$g(t,x)$的连续性, $\int_a^z g(t,x)dt$在$(z,x) \in D$上连续.

而 $\frac{\partial}{\partial z}\int_a^z g(t,x)dt = g(z,x)$ 也在 D 上连续, 于是

$$\frac{d}{dz}\int_\alpha^\beta \left(\int_a^z g(t,x)dt\right)dx = \int_\alpha^\beta \left(\frac{\partial}{\partial z}\int_a^z g(t,x)dt\right)dx$$

$$= \int_\alpha^\beta g(z,x)dx.$$

再看左边, 有

$$\frac{d}{dz}\int_a^z \left(\int_\alpha^\beta g(t,x)dx\right)dt = \int_\alpha^\beta g(z,x)dx,$$

故左右两边的导函数也相等, 命题得证. □

例. $I=\int_0^1 \frac{x^2-x}{\ln x}dx.$

化定积分为重积分

解: x^t在$(t,x)\in[1,2]\times[0,1]$上连续, 且

$$\int_1^2 x^t dt = \left.\frac{x^t}{\ln x}\right|_{t=1}^{2} = \frac{x^2-x}{\ln x}.$$

$$I=\int_0^1 \frac{x^2-x}{\ln x}dx = \int_0^1\left(\int_1^2 x^t dt\right)dx = \int_1^2\left(\int_0^1 x^t dx\right)dt$$

$$= \int_1^2 \left.\left(\frac{x^{t+1}}{t+1}\right)\right|_{x=0}^{1} dt = \int_1^2 \frac{1}{t+1}dt = \ln\frac{3}{2}. \square$$

Question. $\frac{x^2-x}{\ln x}$在$x=0,1$的连续性?

作业：习题2.2 No.1-5