

Review

- 二重积分的几何与物理意义
- 二重积分的定义

$$\iint_{[a,b]\times[c,d]} f(x,y)dx dy = \lim_{\lambda(T) \rightarrow 0} \sum_{i=1}^n \sum_{j=1}^k f(\xi_{ij}, \eta_{ij}) \Delta x_i \Delta y_j.$$

$$\iint_D f(x,y)dx dy = \iint_{I=[a,b]\times[c,d](\supset D)} f_I(x,y)dx dy.$$

- 二重积分的性质

• 可积条件

Thm. $D = [a, b] \times [c, d]$, 则

(1) $f \in R(D) \Rightarrow f$ 在 D 上有界;

(2) $f \in C(D) \Rightarrow f \in R(D)$;

(3) f 在 D 上的间断点集为零面积集 $\Rightarrow f \in R(D)$.

Thm. $D \subset \mathbb{R}^2$ 为有界闭集, f 为 D 上有界函数. 若 f 在 D 上的间断点集为零面积集, ∂D 为零面积集, 则 $f \in R(D)$.

§ 2. 二重积分的计算

- 直角坐标下二重积分的计算及例题
- 极坐标下二重积分的计算及例题
- 补充例题

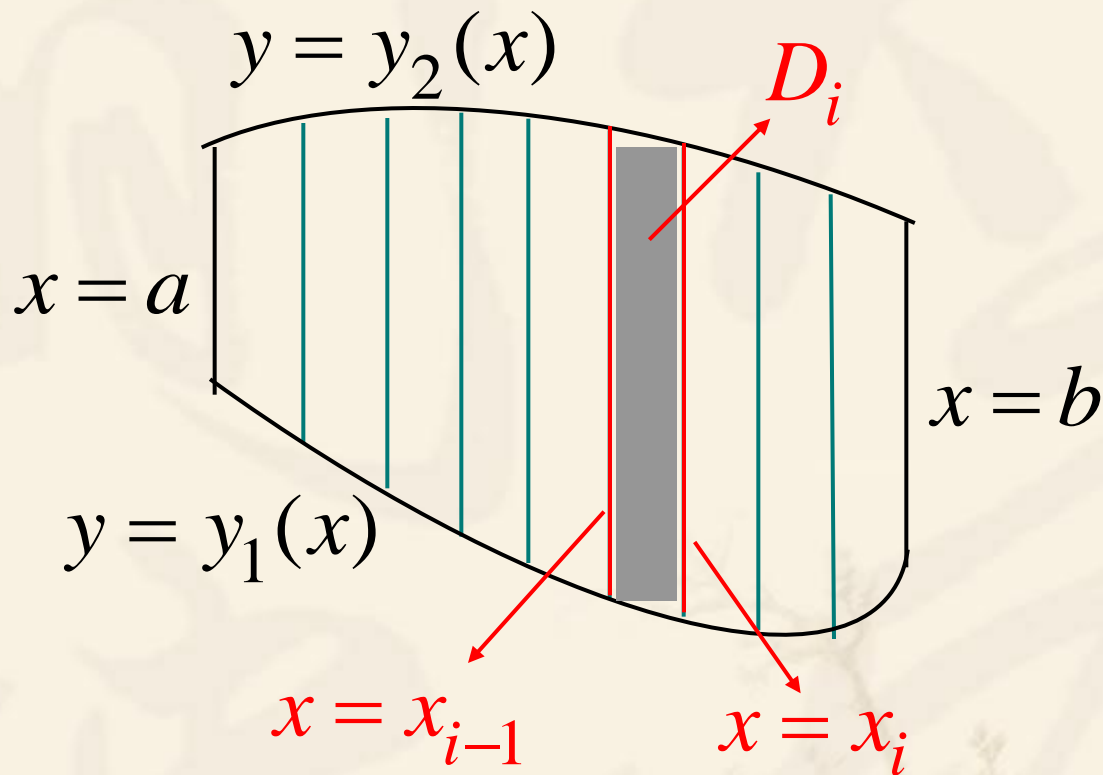
1. 用直角坐标系计算二重积分

$$S : z = f(x, y), (x, y) \in D.$$

换一个思路来计算以 D 为下底,以 S 为顶的曲顶柱体 Ω 的体积

$$V(\Omega) = \iint_D f(x, y) dx dy.$$

设 $D = \{(x, y) \mid a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\}$.



- Step 1. 对 D 进行分划: $a = x_0 < x_1 < \cdots < x_n = b$, 将 D 分成平行于 y 轴的细条 D_1, D_2, \cdots, D_n .

相应地, Ω 被平行于 OYZ 平面的平面 $x = x_i$ 切成薄片 $\Omega_1, \Omega_2, \dots, \Omega_n$.

• Step 2. 求近似和

曲顶柱体 Ω 中截面 $x = x$ 的面积为

$$A(x) = \int_{y_1(x)}^{y_2(x)} f(x, y) dy.$$

于是薄片 Ω_i 的体积近似为

$$V(\Omega_i) \approx A(x_i)(x_{i+1} - x_i) = A(x_i)\Delta x_i.$$

曲顶柱体的体积近似为 $V(\Omega) \approx \sum_{i=1}^n A(x_i)\Delta x_i.$

•Step3.取极限 当分划越来越细时,

$$\sum_{i=1}^n A(x_i) \Delta x_i \rightarrow V(\Omega).$$

综上,

$$V(\Omega) = \int_a^b A(x) dx = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right) dx,$$

即

$$\begin{aligned} \iint_D f(x, y) dx dy &= \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right) dx \\ &\triangleq \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy. \quad (*) \end{aligned}$$

Remark:等式后两项的意义是, 先固定 x (视 x 为常数), 对变量 y 求定积分

$$A(x) = \int_{y_1(x)}^{y_2(x)} f(x, y) dy,$$

再让 x 变起来, 对变量 x 求定积分

$$\int_a^b A(x) dx.$$

正因为如此, (*)式右端的积分也称为先 y 后 x 的
累次积分.

Remark: 对称地, 若区域 D 具有如下形式:

$$D = \{(x, y) \mid c \leq y \leq d, x_1(y) \leq x \leq x_2(y)\}.$$

则

$$\begin{aligned} \iint_D f(x, y) dx dy &= \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x, y) dx \right) dy \\ &\triangleq \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x, y) dx. \end{aligned}$$

Remark: 对于一般的区域 D , 可以分成若干个具有以上两种形式的区域, 并将二重积分利用区域可加性化为累次积分来计算.

Thm. 设 $f(x, y)$ 在有界闭区域 D 上连续, 若

$$D = \{(x, y) \mid a \leq x \leq b, y_1(x) \leq y \leq y_2(x)\},$$

其中 $y_1(x), y_2(x) \in C([a, b])$. 则

$$\iint_D f(x, y) dx dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy.$$

若 $D = \{(x, y) \mid c \leq y \leq d, x_1(y) \leq x \leq x_2(y)\},$

其中 $x_1(y), x_2(y) \in C([c, d])$. 则

$$\iint_D f(x, y) dx dy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x, y) dx. \quad \square$$

Remark:将二重积分化为累次积分计算时,选择不同的积分次序,难易程度可能相差很大.一般应根据被积函数和积分区域选择合适的累次积分次序.

例: 求 $I = \iint_{x^2+y^2 \leq a^2} y^2 \sqrt{a^2 - x^2} \, dx dy$.

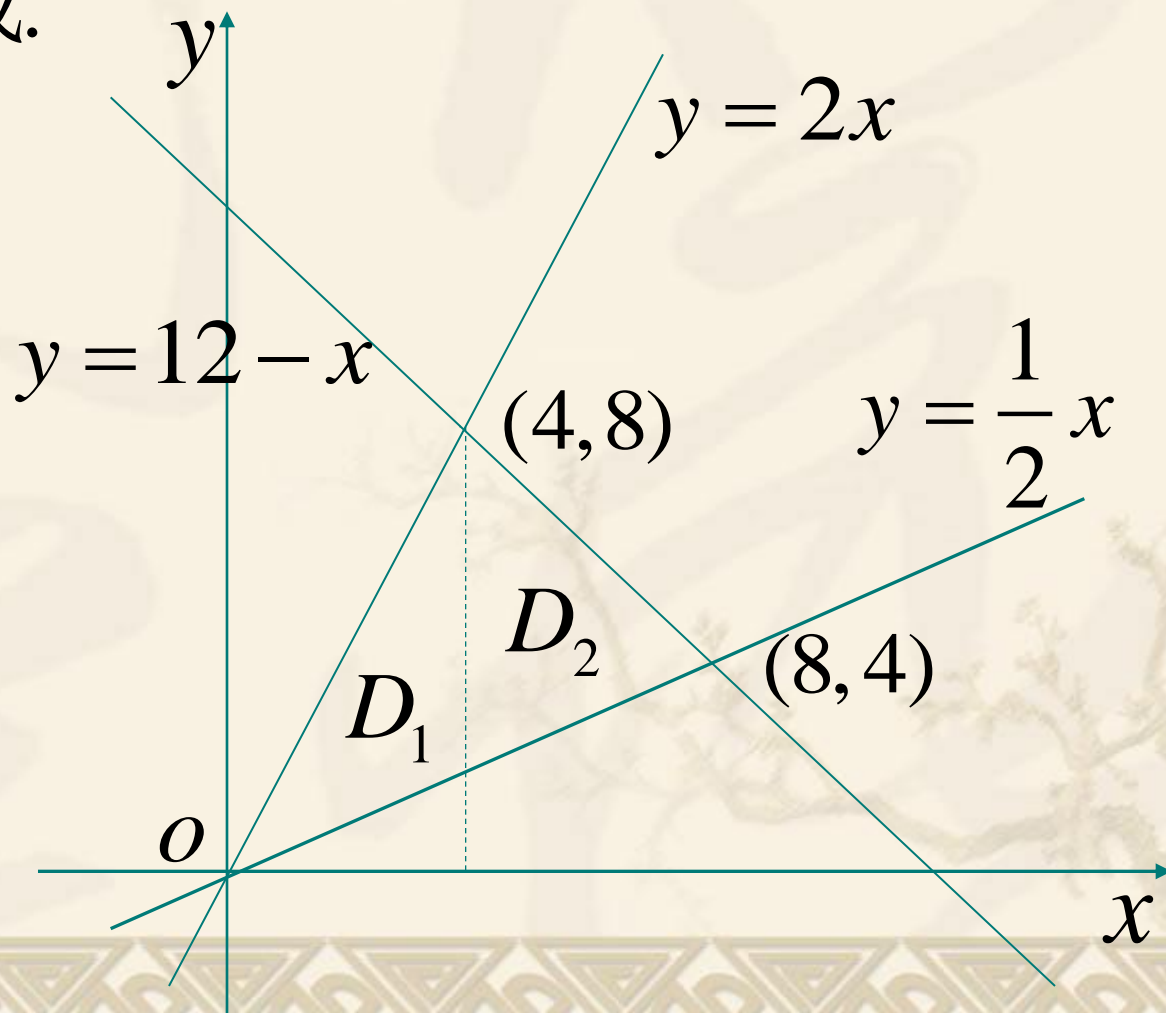
解: 积分区域为 $x \in [-a, a], y \in [-\sqrt{a^2 - x^2}, \sqrt{a^2 - x^2}]$.

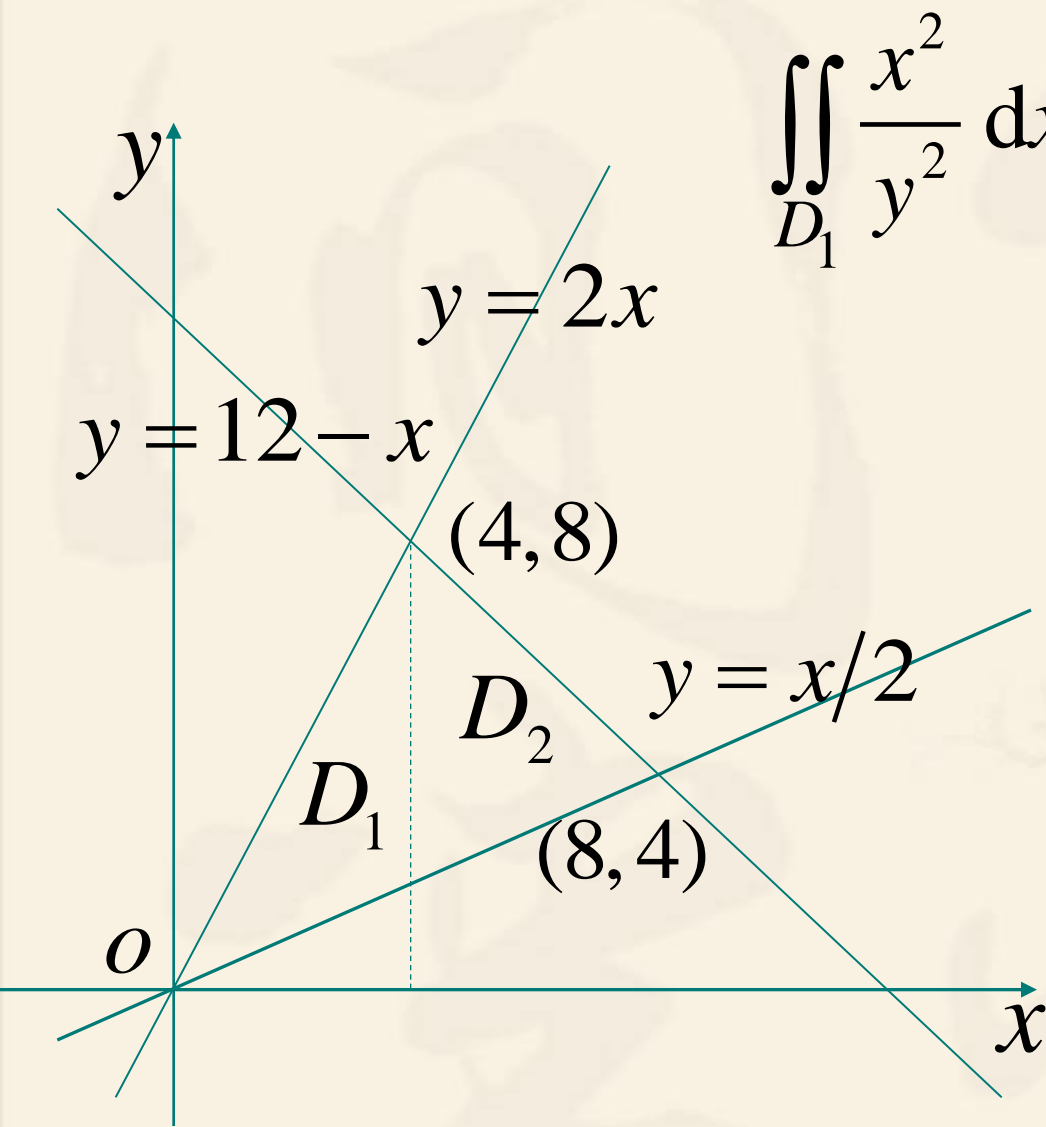
$$\begin{aligned} I &= \int_{-a}^a dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 \sqrt{a^2-x^2} dy \\ &= \int_{-a}^a \sqrt{a^2-x^2} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 dy \\ &= \int_{-a}^a \sqrt{a^2-x^2} \left(\frac{1}{3} y^3 \Big|_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \right) dx \\ &= \frac{2}{3} \int_{-a}^a (a^2 - x^2)^2 dx = \frac{32}{45} a^5. \quad \square \end{aligned}$$

例: 求 $I = \iint_D \frac{x^2}{y^2} dx dy$, 其中 D 由直线 $y = 2x$, $y = \frac{1}{2}x$

及 $y = 12 - x$ 围成.

解: 如图,
区域 D 可
以分成 D_1 ,
 D_2 两部分.





$$\iint_{D_1} \frac{x^2}{y^2} dx dy = \int_0^4 dx \int_{\frac{1}{2}x}^{2x} \frac{x^2}{y^2} dy$$

$$= \int_0^4 \left(-\frac{x^2}{y} \Big|_{y=\frac{1}{2}x}^{y=2x} \right) dx$$

$$= \int_0^4 x^2 \left(\frac{2}{x} - \frac{1}{2x} \right) dx$$

$$= 12,$$

$$\iint_{D_2} \frac{x^2}{y^2} dx dy = \int_4^8 dx \int_{\frac{1}{2}x}^{12-x} \frac{x^2}{y^2} dy$$

$$= \int_0^4 x^2 \left(\frac{2}{x} - \frac{1}{12-x} \right) dx = 120 - 144 \ln 2.$$

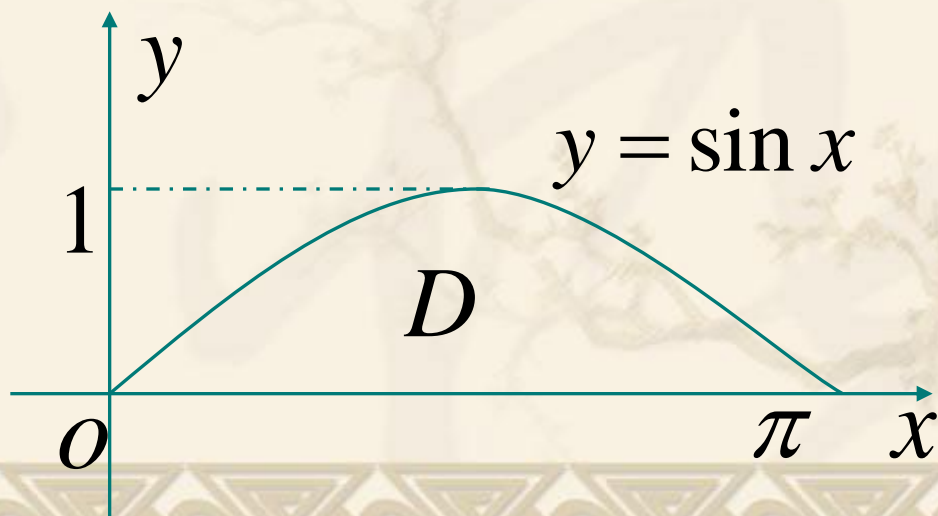
于是 $\iint_D \frac{x^2}{y^2} dx dy = \iint_{D_1} \frac{x^2}{y^2} dx dy + \iint_{D_2} \frac{x^2}{y^2} dx dy$

$$= 132 - 144 \ln 2. \quad \square$$

例: 求 $I = \int_0^1 dy \int_{\arcsin y}^{\pi - \arcsin y} x dx$.

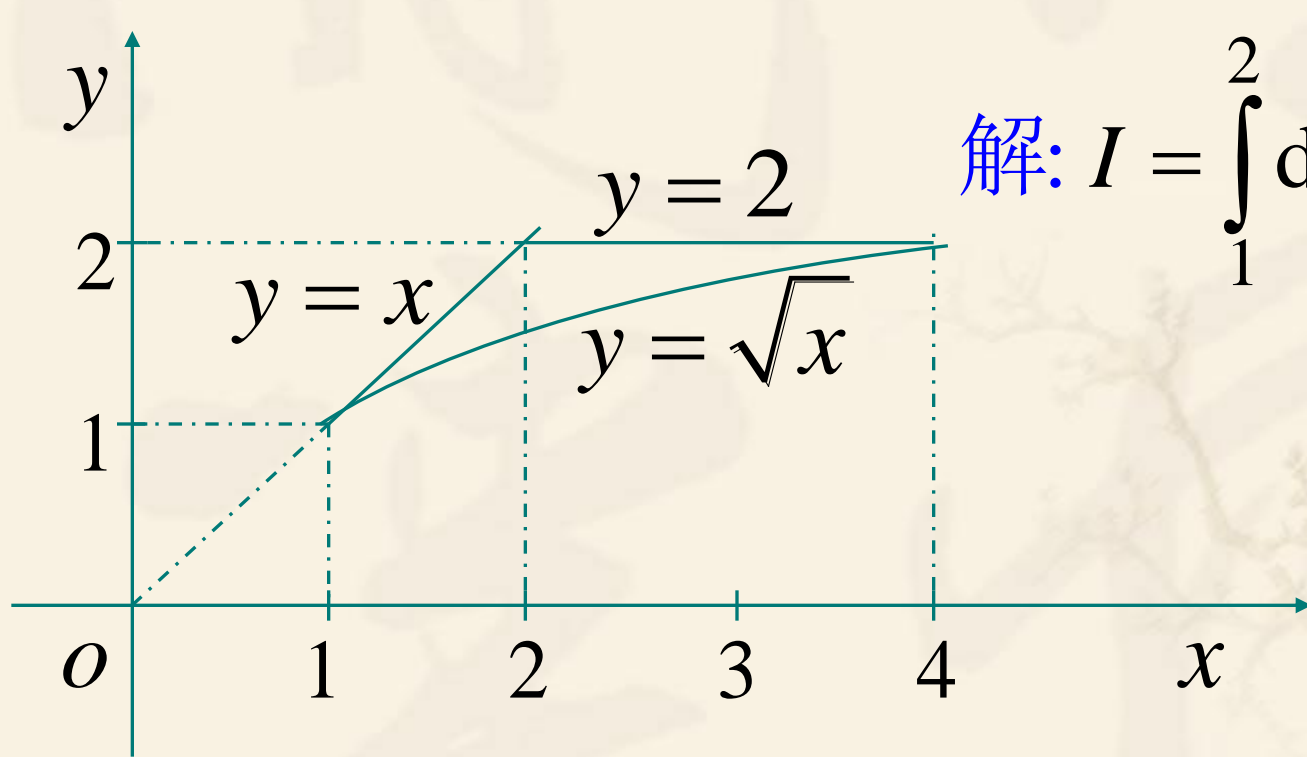
分析: 按所给积分次序, 内层积分容易求出, 但再积分就困难了. 所以尝试交换积分次序.

$$\begin{aligned} \text{解: } I &= \int_0^{\pi} x dx \int_0^{\sin x} dy \\ &= \int_0^{\pi} x \sin x dx = - \int_0^{\pi} x d \cos x \\ &= -x \cos x \Big|_{x=0}^{\pi} + \int_0^{\pi} \cos x dx = \pi. \square \end{aligned}$$



例: $I = \int_1^2 dx \int_{\sqrt{x}}^x \sin \frac{\pi x}{2y} dy + \int_2^4 dx \int_{\sqrt{x}}^2 \sin \frac{\pi x}{2y} dy.$

分析: 里层积分困难, 考虑交换积分次序.

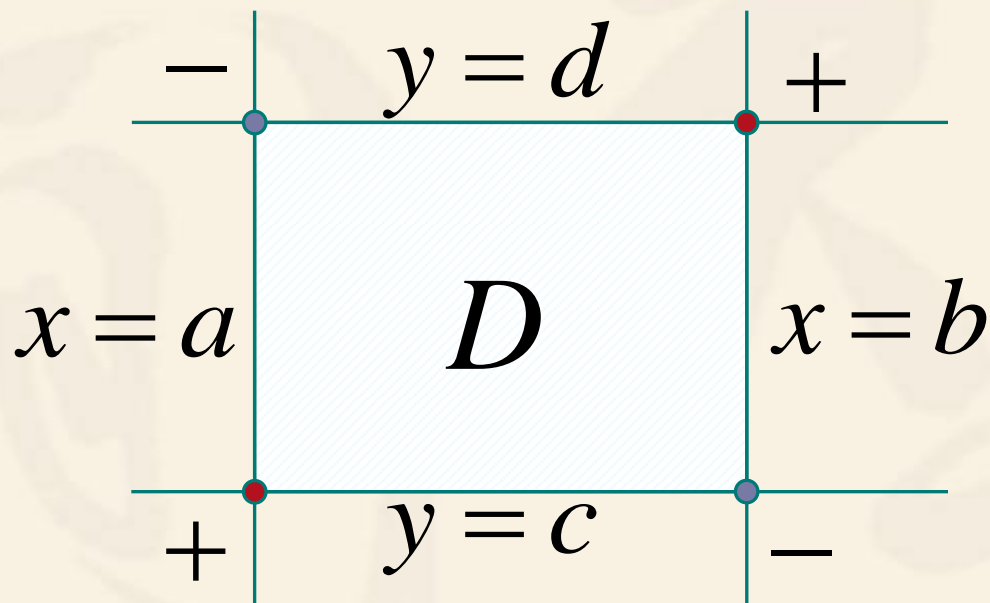


解: $I = \int_1^2 dy \int_y^{y^2} \sin \frac{\pi x}{2y} dx$

$$I = \int_1^2 dy \int_y^{y^2} \sin \frac{\pi x}{2y} dx$$

$$= \frac{2}{\pi} \int_1^2 y \left(\cos \frac{\pi}{2} - \cos \frac{\pi y}{2} \right) dy$$

$$= -\frac{2}{\pi} \int_1^2 y \cos \frac{\pi y}{2} dy = 4(2 + \pi) / \pi^3 . \square$$



例: 设 $\frac{\partial^2 f}{\partial x \partial y}$ 在 $D = [a, b] \times [c, d]$ 上可积, 则

$$\iint_D \frac{\partial^2 f}{\partial x \partial y} dx dy = f(b, d) - f(b, c) - f(a, d) + f(a, c).$$

证明: $\iint_D \frac{\partial^2 f}{\partial x \partial y} dx dy = \int_c^d dy \int_a^b \frac{\partial^2 f}{\partial x \partial y} dx$

$$= \int_c^d \left[\frac{\partial f(x, y)}{\partial y} \Big|_{x=a}^b \right] dy$$

$$= \int_c^d \frac{\partial f(b, y)}{\partial y} dy - \int_c^d \frac{\partial f(a, y)}{\partial y} dy$$

$$= f(b, y) \Big|_{y=c}^d - f(a, y) \Big|_{y=c}^d$$

$$= f(b, d) - f(b, c) - f(a, d) + f(a, c). \quad \square$$

2. 用极坐标系计算二重积分

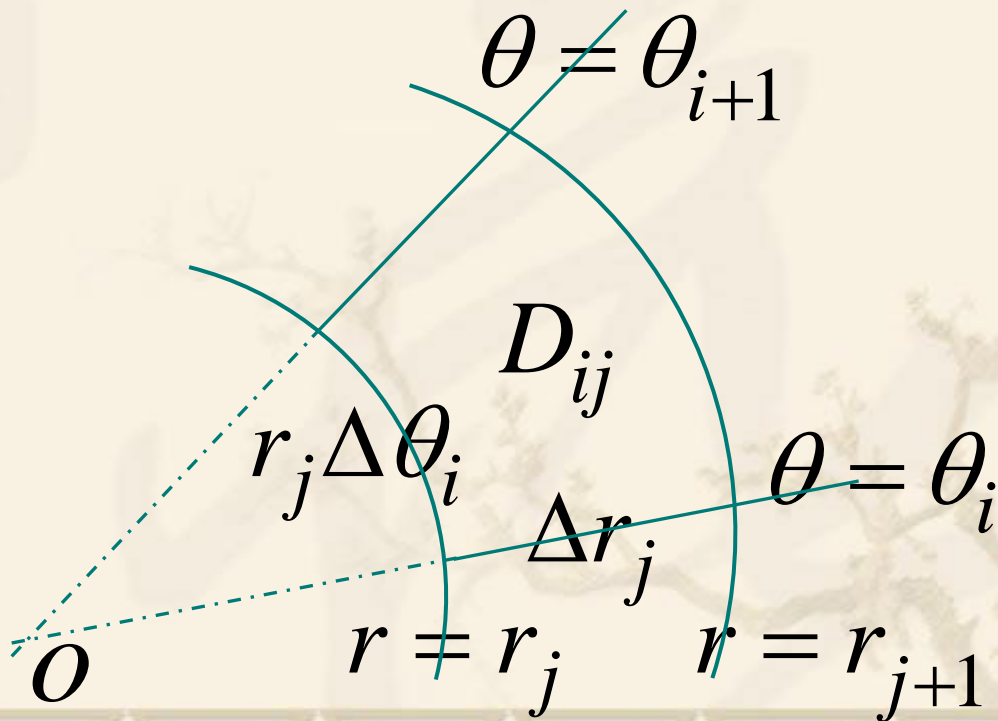
在直角坐标系下将二重积分化为累次积分来计算,如果被积区域 D 的形状不好,或者被积函数的表达式比较复杂,那么累次积分的计算将很复杂,甚至可能计算不出结果来.

再换一个思路来计算以 D 为底,以曲面 $S: z = f(x, y), (x, y) \in D$ 为顶的曲顶柱体的 Ω 体积 $V(\Omega) = \iint_D f(x, y) dx dy$.

用过原点的射线 $\theta = \theta_i (i = 1, 2, \dots, n)$ 和以原点为圆心的同心圆 $r = r_j (j = 1, 2, \dots, m)$ 对区域 D 作分划. 忽略位于区域 D 边界的那些不规则的小区域, 考虑由 $\theta = \theta_i, \theta = \theta_{i+1}, r = r_j$ 和 $r = r_{j+1}$ 围成的曲边四边形

D_{ij} . 当 $\Delta r_j = r_{j+1} - r_j$, $\Delta \theta_i = \theta_{i+1} - \theta_i$ 很小时, D_{ij} 近似为矩形, 边长分别为 Δr_j 和 $r_j \Delta \theta_i$.

$$\sigma(D_{ij}) \approx r_j \Delta \theta_i \Delta r_j$$



$$\begin{aligned}\text{于是 } V(\Omega) &\approx \sum_{1 \leq i \leq n, 1 \leq j \leq m} \sigma(D_{ij}) f(r_j \cos \theta_i, r_j \sin \theta_i) \\ &\approx \sum_{1 \leq i \leq n, 1 \leq j \leq m} r_j \Delta \theta_i \Delta r_j f(r_j \cos \theta_i, r_j \sin \theta_i).\end{aligned}$$

当分划越来越细时,有.

$$\sum_{i,j} r_j \Delta \theta_i \Delta r_j f(r_j \cos \theta_i, r_j \sin \theta_i) \rightarrow V(\Omega).$$

设 E 是原积分区域 D 在极坐标下的表示, 即

$$E = \{(r, \theta) \mid (r \cos \theta, r \sin \theta) \in D\}.$$

$$\text{则 } V(\Omega) = \iint_E f(r \cos \theta, r \sin \theta) r dr d\theta.$$

$$\text{即 } \iint_D f(x, y) dx dy = \iint_E f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Remark: 于是在极坐标系下面积微元为 $d\sigma = r dr d\theta$.

若 $E = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)\}$, 则

$$\begin{aligned} \iint_E f(r \cos \theta, r \sin \theta) r dr d\theta \\ = \int_{\alpha}^{\beta} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr. \end{aligned}$$

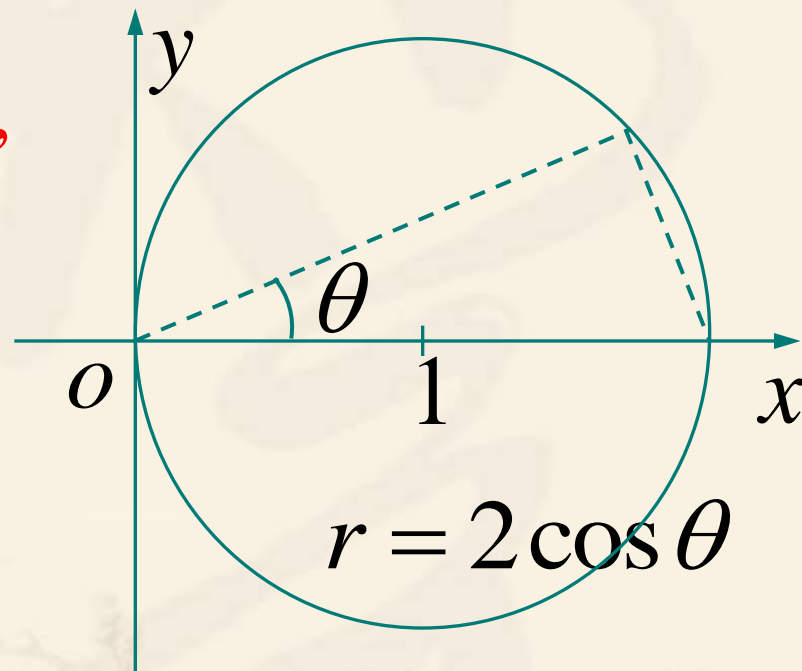
于是, 我们可以将二重积分化为极坐标下的累次积分来计算.

例: 求 $I = \iint_{x^2+y^2 \leq 2x} (y + \sqrt{x^2 + y^2}) dx dy$.

解: 积分区域关于 OX 轴对称,

故
$$\iint_{x^2+y^2 \leq 2x} y dx dy = 0,$$

$$I = \iint_{x^2+y^2 \leq 2x} \sqrt{x^2 + y^2} dx dy.$$



极坐标下, 积分区域为 $\{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, r^2 \leq 2r \cos \theta\}$.

故
$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2 \cos \theta} r^2 dr.$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} r^2 dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \left(\frac{1}{3} r^3 \right) \Big|_{r=0}^{2\cos\theta}$$

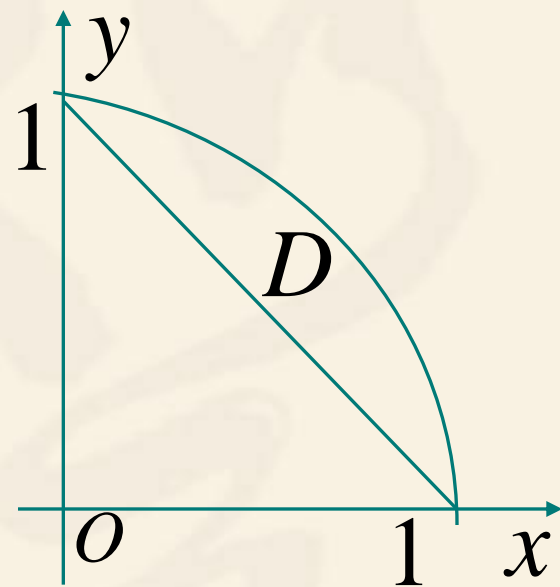
$$= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 \theta d\theta = \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 \theta) d\sin \theta$$

$$= \frac{8}{3} \left(\sin \theta - \frac{1}{3} \sin^3 \theta \right) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{32}{9}. \quad \square$$

例: 求 $I = \iint_{x^2+y^2 \leq 1, x+y > 1} \frac{x+y}{x^2+y^2} dx dy.$

解: 极坐标下积分区域为

$$0 \leq \theta \leq \frac{\pi}{2}, \frac{1}{\sin \theta + \cos \theta} \leq r \leq 1.$$



$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\sin \theta + \cos \theta}}^1 \frac{r \sin \theta + r \cos \theta}{r^2} \cdot r dr \\ &= \int_0^{\frac{\pi}{2}} (\sin \theta + \cos \theta - 1) d\theta = 2 - \frac{\pi}{2}. \quad \square \end{aligned}$$

例. 求 $I = \iint_{x^2+y^2 \leq 1} (x^2 + xy + 2y^2) dx dy$.

解: $\iint_{x^2+y^2 \leq 1} xy dx dy = 0$

$$\iint_{x^2+y^2 \leq 1} x^2 dx dy = \iint_{x^2+y^2 \leq 1} y^2 dx dy \quad (\text{轮换不变性})$$

$$I = \iint_{x^2+y^2 \leq 1} (x^2 + 2y^2) dx dy = \frac{3}{2} \iint_{x^2+y^2 \leq 1} (x^2 + y^2) dx dy$$

$$= \frac{3}{2} \int_0^{2\pi} d\theta \int_0^1 r^3 dr = \frac{3\pi}{4} \cdot \square$$

例:求Poisson积分 $I = \int_{-\infty}^{+\infty} e^{-x^2} dx$.

解:令 $I(R) = \int_{-R}^{+R} e^{-x^2} dx$, 则 $I(R) > 0$.

$$\begin{aligned} I^2(R) &= \int_{-R}^{+R} e^{-x^2} dx \int_{-R}^{+R} e^{-y^2} dy \\ &= \iint_{-R \leq x, y \leq R} e^{-(x^2+y^2)} dx dy \end{aligned}$$

$$\begin{aligned} \text{于是, } \iint_{x^2+y^2 \leq R^2} e^{-(x^2+y^2)} dx dy &\leq I^2(R) \\ &\leq \iint_{x^2+y^2 \leq 2R^2} e^{-(x^2+y^2)} dx dy \end{aligned}$$

$$\begin{aligned}\text{而} \iint_{x^2+y^2 \leq R^2} e^{-(x^2+y^2)} dx dy &= \int_0^{2\pi} d\theta \int_0^R r e^{-r^2} dr \\ &= 2\pi \cdot \left(-\frac{1}{2} e^{-r^2} \right) \Big|_{r=0}^R = \pi(1 - e^{-R^2}).\end{aligned}$$

$$\text{同理, } \iint_{x^2+y^2 \leq 2R^2} e^{-(x^2+y^2)} dx dy = \pi(1 - e^{-2R^2}).$$

$$\text{所以 } \pi(1 - e^{-R^2}) \leq I^2(R) \leq \pi(1 - e^{-2R^2}).$$

$$\text{由夹挤原理, } \lim_{R \rightarrow +\infty} I^2(R) = \pi.$$

$$\text{故 } I = \lim_{R \rightarrow \infty} I(R) = \sqrt{\pi}. \quad \square$$

3. 补充例题

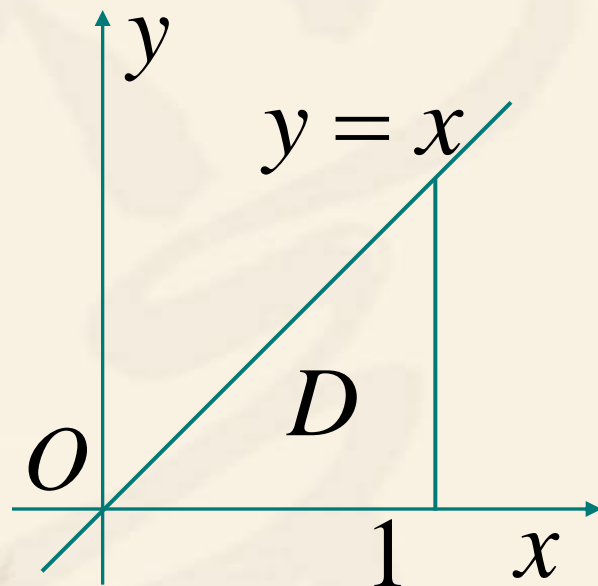
*例: 求 $I = \int_0^1 \frac{\ln(1+x)}{(2-x)^2} dx$.

解:

$$I = \int_0^1 \frac{1}{(2-x)^2} \left(\int_0^x \frac{1}{1+y} dy \right) dx$$

$$= \int_0^1 \frac{1}{(2-x)^2} dx \int_0^x \frac{1}{1+y} dy$$

$$= \int_0^1 \frac{1}{1+y} dy \int_y^1 \frac{1}{(2-x)^2} dx \quad (\text{交换积分次序})$$



$$\begin{aligned} &= \int_0^1 \frac{(1-y)dy}{(1+y)(2-y)} \\ &= \frac{2}{3} \int_0^1 \frac{dy}{1+y} + \frac{1}{3} \int_0^1 \frac{dy}{2-y} = \frac{1}{3} \ln 2. \square \end{aligned}$$

Remark: 将一元函数的定积分化成二重积分计算, 有时候可能会更简单.

*例: $\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx.$

证明: 记 $D = [a, b] \times [a, b]$.

$$\begin{aligned} 0 &\leq \iint_D [f(x)g(y) - f(y)g(x)]^2 dx dy \\ &= \iint_D f^2(x)g^2(y)dx dy + \iint_D f^2(y)g^2(x)dx dy \\ &\quad - 2 \iint_D f(x)f(y)g(x)g(y)dx dy \\ &= 2 \int_a^b f^2(x)dx \int_a^b g^2(y)dy \\ &\quad - 2 \int_a^b f(x)g(x)dx \int_a^b f(y)g(y)dy \\ &= 2 \int_a^b f^2(x)dx \int_a^b g^2(x)dx - 2 \left(\int_a^b f(x)g(x)dx \right)^2. \quad \square \end{aligned}$$

*例: $f(x) \in C[0,1], f > 0, f \downarrow$. 求证

$$\frac{\int_0^1 x f^2(x) dx}{\int_0^1 x f(x) dx} \leq \frac{\int_0^1 f^2(x) dx}{\int_0^1 f(x) dx}.$$

证明: 只要证 $I = \int_0^1 x f^2(x) dx \int_0^1 f(x) dx$
 $-\int_0^1 x f(x) dx \int_0^1 f^2(x) dx \leq 0.$

定积分与积分变量所用字母无关, 故

$$I = \int_0^1 x f^2(x) dx \int_0^1 f(\textcolor{red}{y}) d\textcolor{red}{y} - \int_0^1 x f(x) dx \int_0^1 f^2(\textcolor{red}{y}) d\textcolor{red}{y}$$

即
$$I = \iint_{0 \leq x, y \leq 1} x f^2(x) f(y) dx dy - \iint_{0 \leq x, y \leq 1} x f(x) f^2(y) dx dy$$
$$= \iint_{0 \leq x, y \leq 1} x f(x) f(y) [f(x) - f(y)] dx dy$$

由于积分区域关于直线 $y = x$ 对称,

$$I = \iint_{0 \leq x, y \leq 1} y f(x) f(y) [f(y) - f(x)] dx dy$$

两式相加, 由 $f > 0, f \downarrow$, 得

$$2I = \iint_{0 \leq x, y \leq 1} (x - y) f(x) f(y) [f(x) - f(y)] dx dy \leq 0. \square$$

*例: 设 $D = \{(x, y) \mid 0 \leq x, y \leq 1\}$, $z = f(x, y) \in C^2(D)$. 若

$$\left| \frac{\partial^2 f(x, y)}{\partial x \partial y} \right| \leq 4, \quad \forall (x, y) \in D$$

$$f(x, y) \equiv f'_x(x, y) \equiv 0, \quad \forall (x, y) \in \partial D,$$

$$\text{则} \left| \iint_D f(x, y) dx dy \right| \leq 1.$$

证明: $\iint_D f(x, y) dx dy = \int_0^1 dy \int_0^1 f(x, y) dx$

(分部积分) $= \int_0^1 \left[x f(x, y) \Big|_{x=0}^1 - \int_0^1 x \frac{\partial f(x, y)}{\partial x} dx \right] dy$

0 $= - \int_0^1 dy \int_0^1 x \frac{\partial f}{\partial x} dx = - \int_0^1 x dx \int_0^1 \frac{\partial f}{\partial x} dy$

(分部积分) $= - \int_0^1 x \left[y \frac{\partial f}{\partial x} \Big|_{y=0}^1 - \int_0^1 y \frac{\partial^2 f}{\partial x \partial y} dy \right] dx$

$$= \int_0^1 x dx \int_0^1 y \frac{\partial^2 f}{\partial x \partial y} dy = \iint_D xy \frac{\partial^2 f}{\partial x \partial y} dx dy$$

于是

$$\left| \iint_D f(x, y) dx dy \right| = \left| \iint_D xy \frac{\partial^2 f}{\partial x \partial y} dx dy \right|$$

$$\leq \iint_D \left| xy \frac{\partial^2 f}{\partial x \partial y} \right| dx dy \leq 4 \iint_D xy dx dy$$

$$= 4 \int_0^1 x dx \int_0^1 y dy = 1. \square$$

作业：习题3.3 No. 5, 6, 11

$$\text{No.6(2)} \ D = \left\{ (x, y) \left| \begin{array}{l} (x-a)^2 + (y-a)^2 \leq a^2, \\ 0 \leq x, y \leq a \end{array} \right. \right\}$$

$$\text{No.6(7)} \ D = \{ (x, y) \mid 0 \leq x, y \leq \pi \}$$