2020.6.8 的微积分A2考试.

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1. 解: $\because$ $y=y(x), z=z(x)$

又$\because$ $\begin{cases} x^3+y^3-z^3=10 & \cdots ① \\ x+y+z=0 & \cdots ② \end{cases}$

① 两边对x求导: $3x^2+3y^2\cdot y'(x)-3z^2\cdot z'(x)=0 \cdots ③$

② 两边对x求导: $1+y'(x)+z'(x)=0 \cdots ④$

又$\because$ $(1,1,-2) \Rightarrow x=1, y=1, z=-2$

$\therefore$ $\begin{cases} 3+3y'(x)-12z'(x)=0 \\ 1+y'(x)+z'(x)=0 \end{cases}$

得: $\begin{cases} y'(1)=-1 \\ z'(1)=0 \end{cases}$

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2. 解: $z = f(x^2+xy+y^2)$ 在 $(1,1)$ 处, 设 $u = x^2+xy+y^2$

$= 1+1+1=3$

$$\frac{\partial z}{\partial y} = f'(u) \times \frac{\partial u}{\partial y}$$

$$= f'(u) \times (x+2y)$$

$$= f'(3) \times 3 = 3f'(3)$$

又∵ $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[f'(u)(x+2y)]$$

$$= \frac{\partial f'(u)}{\partial x}(x+2y) + f'(u) \times 1$$

$$= f''(u) \times (2x+y)(x+2y) + f'(u)$$

$$= f''(3) \times 3 \times 3 + f'(3)$$

$$= f'(3) + 9f''(3)$$

3. 解：

$$u=(\sin x)(\sin y)(\sin z)$$

$$\text{st}: \quad x+y+z=\frac{\pi}{2}$$

$$x>0,\ y>0,\ z>0.$$

设 $L=(\sin x)(\sin y)(\sin z)-\lambda\left(x+y+z-\frac{\pi}{2}\right)$

$$\therefore \begin{cases} L_x=(\cos x)(\sin y)(\sin z)-\lambda=0 \\ L_y=(\cos y)(\sin x)(\sin z)-\lambda=0 \\ L_z=(\cos z)(\sin x)(\sin y)-\lambda=0 \\ x+y+z=\frac{\pi}{2} \quad \text{且 } x, y, z>0. \end{cases}$$

$$\therefore \begin{cases} \cos x\sin y=\cos y\sin x \Rightarrow \sin(x-y)=0 \\ \cos y\sin z=\cos z\sin y \Rightarrow \sin(y-z)=0 \\ \cos x\sin z=\cos z\sin x \Rightarrow \sin(x-z)=0. \end{cases}$$

$$\therefore\ x=y=z=\frac{1}{6}\pi.$$

$$u=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

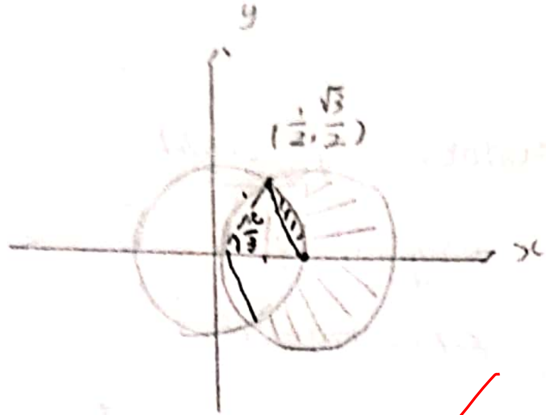
$$\text{又}\because\ H=\begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix}=\begin{vmatrix} -\frac{1}{8}, & \frac{3}{8}, & \frac{3}{8} \\ \frac{3}{8} & ,-\frac{1}{8}, & \frac{3}{8} \\ \frac{3}{8} & ,\frac{3}{8} & ,-\frac{1}{8} \end{vmatrix} \text{ 负定.}$$

-2

$\therefore$ 极大值为 $\frac{1}{8}$

④ 解:

如右图:



交点 $(\frac{1}{2}, \frac{\sqrt{3}}{2})$

$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

设 S_1 : $x^2+y^2 \le 2x$ 即: $\rho \le 2\cos\theta$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

S_2 : $\rho = 1$, $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$

:
$$\iint_{S_1} \left|\frac{y}{x}\right| dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} |\tan\theta| \rho \, d\rho$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos^2\theta}{2} |\tan\theta| \, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{0} (-\cos^2\theta \tan\theta) d\theta + 2\int_0^{\frac{\pi}{2}} \cos^2\theta \tan\theta \, d\theta$$

$$= -2\int_{-\frac{\pi}{2}}^{0} \cos\theta \sin\theta \, d\theta + 2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \, d\theta$$

$$= 2$$

S_3 : $$\iint_{S_3} \left|\frac{y}{x}\right| dxdy = \iint_{S_3} \left|\frac{\sin\theta}{1+\cos\theta}\right| dxdy = \iint = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \int_0^1 | \quad | \rho \, d\rho$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left|\frac{\sin\theta}{1+\cos\theta}\right| \cancel{dxdy}\, d\theta = \ln\frac{4}{3}$$

$$\therefore \iint_D \left|\frac{y}{x}\right| dxdy = \ln\frac{4}{3} - \left[\overset{\ln 2}{\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \int_0^1 |\tan\theta| \rho \, d\rho} - 2\iint_{\triangle}\right]$$

$$= \frac{7}{4}\ln 2 - \ln 3$$

−3

5. 解：

(1) $\because\quad X=\frac{-x}{x^2+2y^2},\ Y=\frac{-y}{x^2+2y^2}.$

$\therefore\quad X_x=\frac{-1\times(x^2+2y^2)-(-x)\times 2x}{(x^2+2y^2)^2}=\frac{x^2-2y^2}{(x^2+2y^2)^2}$

$Y_y=\frac{-1\times(x^2+2y^2)-(-y)\times 4y}{(x^2+2y^2)^2}=\frac{-x^2+2y^2}{(x^2+2y^2)}$

$\therefore\quad X_x+Y_y=0.$

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故：与路径无关.

设 $x^2+2y^2=\varepsilon^2(\varepsilon>0)$，则 $\int_{(A)}^{(B)}=-2\pi.$

(2) 存在 $Z(x,y)=\frac{\sqrt2}{2}\arctan\frac{x}{\sqrt2 y}+C$（C为常数）

设 $Z=z(x,y)$，$dz=\frac{y\,dx}{x^2+2y^2}-\frac{x\,dy}{x^2+2y^2}$

5

$\therefore\quad Z_x=\frac{y}{x^2+2y^2},\quad Z_y=\frac{-x}{x^2+2y^2}$

$\therefore\quad Z(x,y)=\int\frac{y}{x^2+2y^2}dx$

$=\int\frac{1}{2y\left(\frac{x^2}{2y^2}+1\right)}dx$

$=\frac{\sqrt2 y}{2y}\cdot\int\frac{1}{\left(\frac{x}{\sqrt2 y}\right)^2+1}\,d\frac{x}{\sqrt2 y}$

$=\frac{\sqrt2}{2}\arctan\frac{x}{\sqrt2 y}+C$

又 $\because\ Z_y=\frac{\sqrt2}{2}\times\frac{1\times\frac{x}{\sqrt2}\times\left(\frac{1}{-y^2}\right)}{1+\frac{x^2}{2y^2}}+C_y$

$=\frac12\,\frac{-x}{y^2+\frac{x^2}{2}}+C_y$

$=\frac{-x}{x^2+2y^2}+C_y\Rightarrow C_y=0$

故：$Z(x,y)=\frac{\sqrt2}{2}\arctan\frac{x}{\sqrt2 y}+C$

（C为常数）

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6. 解: 作球坐标变换:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$r\in[0,1]$, $\theta\in[0,\pi]$, $\varphi\in[0,2\pi]$

6

$\therefore$ 原式 $= \int_0^1 d\theta \int_0^{2\pi} d\varphi \int_0^1 \sqrt{1-r^3}\cdot r^2\sin\theta\, dr$

$= \int_0^{1} d\theta \int_0^{2\pi} d\varphi \int_0^1 \frac{\sqrt{1-r^3}}{3}\sin\theta\, dr^3$ $\to \pi$

$= 2\pi \times (-\cos\theta)\Big|_0^1 \cdot \int_0^1 \frac{\sqrt{1-a}}{3}\, da$

$= \frac{2\pi}{3} \times (1-\cos 1) \times \frac{(1-a)^{\frac{3}{2}}}{-\frac{3}{2}}\Big|_0^1$

$= \frac{2\pi}{3} \times (1-\cos 1) \times \frac{2}{3} = \frac{4\pi}{9}(1-\cos 1)$

7. 解:

$$L^+: \begin{cases} x^2+y^2+z^2=2ax \\ x^2+y^2=2x \end{cases} \quad (z\geq 0).$$

由 stokes 公式:

$X = y^2+z^2$, $Y = z^2+x^2$, $Z = x^2+y^2$.

$\therefore$ $Z_y - Y_z = 2y-2z$

$X_z - Z_x = 2z-2x$

$Y_x - X_y = 2x-2y$

又 $\because$ $\vec{n} = \dfrac{(x-a, y, z)}{\sqrt{2ax}}$

$\therefore$ $\int_{L^+} = \iint_S \dfrac{1}{\sqrt{2ax}}\left[\underbrace{(2y-2z)(x-a)+(2z-2x)y+(2x-2y)z}_{2a(z-y)}\right]ds$

又 $\because$ $ds = \sqrt{1+z_x^2+z_y^2}\,dx\,dy$ $\quad = \iint_{D_{xy}} \dfrac{1}{\sqrt{2ax}} \times 2az \times \dfrac{a}{z}\,dxdy$

S 在 xy 平面: $(x-1)^2+y^2=1 \Rightarrow \rho = 2\cos\theta$

故: 原式 $= \sqrt{2}a^{\frac{3}{2}} \iint_{D_{xy}} \dfrac{1}{\sqrt{x}}\,dxdy = \sqrt{2}a^{\frac{3}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \dfrac{1}{\sqrt{\rho\cos\theta}}\,\rho\,d\rho$

$= \dfrac{16}{3}a^{\frac{3}{2}}$

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8. 解：

(1) $T=2\pi$, $f(x)=\begin{cases}0, & -\pi<x\le 0\\ x, & 0<x\le\pi\end{cases}$

$\therefore a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)=\frac{1}{\pi}\times\int_0^{\pi}x\,dx=\frac{1}{2}\pi$

$a_n=\frac{1}{\pi}\int_0^{\pi}x\cos nx\,dx$

$=\frac{1}{\pi}\times\int_0^{\pi}x\cdot\left[\frac{\sin nx}{n}\right]'dx$

$=\frac{1}{\pi}\times\frac{1}{n}\cdot\left[x\sin nx\Big|_0^{\pi}-\int_0^{\pi}\sin nx\,dx\right]$

$=\frac{1}{n\pi}\left[0+\frac{\cos nx}{n}\Big|_0^{\pi}\right]=\frac{1}{n^2\pi}\left[(-1)^n-1\right]$

$b_n=\frac{1}{\pi}\int_0^{\pi}x\sin nx\,dx=\frac{1}{\pi}\times\int_0^{\pi}x\cdot\left[-\frac{\cos nx}{n}\right]'dx$

$=\frac{1}{-n\pi}\left(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx\,dx\right]$

$=\frac{1}{-n\pi}\times\pi\times(-1)^n=\frac{(-1)^{n+1}}{n}$

得：$f(x)\sim\frac{1}{4}\pi+\sum_{n=1}^{+\infty}\left\{\frac{1}{\pi n^2}\left[(-1)^n-1\right]\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\right\}$

(2) 由(1)可知：$f(x)\sim\frac{1}{4}\pi+\sum_{n=1}^{+\infty}\left\{\frac{1}{\pi n^2}\left[(-1)^n-1\right]\cos nx+\frac{(-1)^{n+1}}{n}\sin nx\right\}$

设 $x=0$，$n'=2n-1$

$\therefore f(0)=\frac{1}{4}\pi+\sum_{n=1}^{+\infty}\frac{1}{\pi(2n-1)^2}(-2)$

$=\frac{1}{4}\pi-\frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}\Rightarrow\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

$\therefore\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}=\frac{1}{8}\pi^2$

9. 证明：$\cos\langle\vec{r},\vec{n}\rangle = \dfrac{\vec{r}\cdot\vec{n}}{r}$

由 Gauss 公式：

$$\text{左边} = \frac{1}{2}\iint_{\partial\Omega} \cos\langle\vec{r},\vec{n}\rangle\, dS$$

$$= \frac{1}{2}\iint_{\partial\Omega^+} \frac{1}{r}\,(x,y,z)\cdot\vec{n}\, dS$$

5

又∵ $r = (x^2+y^2+z^2)^{\frac{1}{2}}$

且 $\vec{F} = \frac{1}{r}(x,y,z)$， $\operatorname{div}\vec{F}=0$

?

$$\therefore\ \text{左边} = \frac{1}{2}\iint_{\partial\Omega^+} \frac{1}{r}(x,y,z)\,\vec{n}\, dS$$

$$= \lim_{\varepsilon\to0^+}\iiint_{\Omega} \frac{1}{r}\,dxdydz$$

则原命题成立

10. 证明:

(1) $\because \sum_{n=0}^{\infty} a_n n!$ 收敛

且一般项为 $u_n = a_n n!$

$\therefore \lim_{n \to +\infty} a_n n! = 0$ ✓ 2

$\therefore \lim_{n \to +\infty} a_n = 0$

$\therefore \lim_{n \to +\infty} \sqrt[n]{a_n} = 0$?

$\therefore R = \frac{1}{\sqrt[n]{a_n}} = +\infty$ 成立.

(2) 判断 $\sum_{n=0}^{+\infty} e^{x} f(x)$ 的敛散性

由 (1) $\because R = +\infty$.

$\therefore f(x)$ 在R上收敛

又$\because$ e^{-x} 对任意固定 x 都单调$\downarrow \to 0$

由 Abel 判别法: $\sum_{n=0}^{+\infty} e^{-x} f(x)$ 收敛

$\Rightarrow \int_0^{+\infty} e^{-x} f(x) dx$ 收敛.

由 $f(x) = \sum_{n=0}^{+\infty} a_n x^n$

$$\int_0^{+\infty} e^{-x} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n \int_0^{+\infty} e^{-x} x^n dx$$ 2

$$= \sum_{n=0}^{+\infty} a_n n! \text{ 成立}$$