习题 1.4 作业参考解答

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1. 求下列函数的偏导数.

(1) $z = ax^2y + bxy^2$.

解：对多元函数的某个变量求偏导数时，可视其余变量为常数，对一元函数求导。

$\dfrac{\partial z}{\partial x} = 2axy + by^2$, $\dfrac{\partial z}{\partial y} = ax^2 + 2bxy$.

(3) $z = \dfrac{x}{y} + \dfrac{y}{x}$.

解：$\dfrac{\partial z}{\partial x} = \dfrac{1}{y} - \dfrac{y}{x^2}$, $\dfrac{\partial z}{\partial y} = -\dfrac{x}{y^2} + \dfrac{1}{x}$.

(5) $z = \ln(x + \sqrt{x^2 - y^2})$.

解：

$$\frac{\partial z}{\partial x} = \frac{1 + \dfrac{2x}{2\sqrt{x^2 - y^2}}}{x + \sqrt{x^2 - y^2}} = \frac{1}{\sqrt{x^2 - y^2}},$$

$$\frac{\partial z}{\partial y} = \frac{\dfrac{-2y}{2\sqrt{x^2 - y^2}}}{x + \sqrt{x^2 - y^2}} = -\frac{y}{\sqrt{x^2 - y^2}(x + \sqrt{x^2 - y^2})}.$$

(7) $z = \cos(1 + 2^{xy})$.

解：$\dfrac{\partial z}{\partial x} = -\sin(1 + 2^{xy})2^{xy}(\ln 2)y$, $\dfrac{\partial z}{\partial y} = -\sin(1 + 2^{xy})2^{xy}(\ln 2)x$.

(9) $z=\sqrt{|xy|}$.

解：当 $y=0$ 时，$z=0$，故由定义知 $\dfrac{\partial z}{\partial x}=0$.

当 $y\neq 0$ 时，$z=|y|^{\frac{1}{2}}\sqrt{|x|}$，对关于 x 的一元函数 $\sqrt{|x|}$，有

$$\sqrt{|x|}'=\begin{cases}\dfrac{1}{2}x^{-\frac{1}{2}}, & x>0,\\ \text{不存在}, & x=0,\\ -\dfrac{1}{2}(-x)^{-\frac{1}{2}}, & x<0.\end{cases}$$

从而

$$\frac{\partial z}{\partial x}=\begin{cases}\dfrac{1}{2}x^{-\frac{1}{2}}|y|^{\frac{1}{2}}, & x>0,\\ \text{不存在}, & x=0,\\ -\dfrac{1}{2}(-x)^{-\frac{1}{2}}|y|^{\frac{1}{2}}, & x<0.\end{cases}$$

从而可以得到 $\dfrac{\partial z}{\partial x}$. 对 $\dfrac{\partial z}{\partial y}$ 也类似地讨论.

综上所述有

$$\frac{\partial z}{\partial x}=\begin{cases}0, & y=0,\\ \dfrac{1}{2}x^{-\frac{1}{2}}|y|^{\frac{1}{2}}, & y\neq 0, x>0,\\ \text{不存在}, & y\neq 0, x=0,\\ -\dfrac{1}{2}(-x)^{-\frac{1}{2}}|y|^{\frac{1}{2}}, & y\neq 0, x<0.\end{cases}\quad \frac{\partial z}{\partial y}=\begin{cases}0, & x=0,\\ \dfrac{1}{2}y^{-\frac{1}{2}}|x|^{\frac{1}{2}}, & x\neq 0, y>0,\\ \text{不存在}, & x\neq 0, y=0,\\ -\dfrac{1}{2}(-y)^{-\frac{1}{2}}|x|^{\frac{1}{2}}, & x\neq 0, y<0.\end{cases}$$

2. 考查下列函数在坐标原点的可微性.

(1) $f(x,y)=\sqrt{x}\cos y$.

解：若可微，则偏导数存在。考查 $f(x,y)$ 在 $(0,0)$ 处对 x 的偏导数.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)}=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{\sqrt{x}}{x}\quad \text{不存在}.$$

所以 $f(x,y)$ 在坐标原点不可微.

(3) $f(x,y)=\begin{cases}\dfrac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$

解：由 $f(x,0)=f(0,y)=0$ 知 $\dfrac{\partial f}{\partial x}\Big|_{(0,0)}=\dfrac{\partial f}{\partial y}\Big|_{(0,0)}=0.$

若可微，则 $\lim\limits_{(x,y)\to(0,0)}\dfrac{f(x,y)-f(0,0)-0}{\sqrt{x^2+y^2}}=0,$

即 $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y^2}{(x^2+y^2)^2}=0.$

令 $y=x$，有 $\dfrac{x^2y^2}{(x^2+y^2)^2}=\dfrac{1}{4}.$

所以 $f(x,y)$ 在坐标原点不可微.

8. 设函数 $f(x,y)=\sqrt[3]{xy}$，证明：函数 f 在原点处连续、偏导数存在，但沿方向 $\mathbf{l}=(a,b)(ab\neq 0)$ 的方向导数不存在.

证明. 由

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}\sqrt[3]{xy}=0=f(0,0)$$

知 f 在原点连续.

由 $f(x,0)=f(0,y)=0$ 知 $\dfrac{\partial f}{\partial x}\Big|_{(0,0)}=\dfrac{\partial f}{\partial y}\Big|_{(0,0)}=0$，即 f 在原点处偏导数存在.

沿 $\mathbf{l}$ 方向的单位向量为 $(\dfrac{a}{\sqrt{a^2+b^2}},\dfrac{b}{\sqrt{a^2+b^2}})$.

从而

$$\frac{\partial f}{\partial \mathbf{l}}\Big|_{(0,0)}=\lim_{t\to0^+}\frac{f(t\frac{a}{\sqrt{a^2+b^2}},t\frac{b}{\sqrt{a^2+b^2}})-f(0,0)}{t}=\frac{\sqrt[3]{ab}}{\sqrt[3]{a^2+b^2}}\lim_{t\to0^+}t^{-\frac{1}{3}}$$

故 f 沿方向 $\mathbf{l}=(a,b)$ 的方向导数不存在. $\square$

11. 求下列函数在点 P_0 处沿方向 $\mathbf{l}$ 的方向导数.

(1) $z=\cos(x+y)$, $P_0=(0,\dfrac{\pi}{2})$, $\mathbf{l}=(3,-4)$.

解：沿 $\mathbf{l}$ 方向的单位向量为 $(\dfrac{3}{5},-\dfrac{4}{5})$.

$$\begin{aligned}\frac{\partial z}{\partial \mathbf{l}}\Big|_{P_0} &= \lim_{t\to 0^+}\frac{z(0+\frac{3}{5}t,\frac{\pi}{2}-\frac{4}{5}t)-z(0,\frac{\pi}{2})}{t}\\ &= \lim_{t\to 0^+}\frac{\cos(\frac{\pi}{2}-\frac{1}{5}t)}{t}=\lim_{t\to 0^+}\frac{\sin\frac{1}{5}t}{t}=\frac{1}{5}.\end{aligned}$$

(3) $z=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}x_ix_j$, $P_0=(1,1,\ldots,1)$, $\mathbf{l}=(-1,-1,\ldots,-1)$.

解：沿 $\mathbf{l}$ 方向的单位向量为 $(-\dfrac{1}{\sqrt{n}},-\dfrac{1}{\sqrt{n}},\ldots,-\dfrac{1}{\sqrt{n}})$

$$\begin{aligned}\frac{\partial z}{\partial \mathbf{l}}\Big|_{P_0} &= \lim_{t\to 0^+}\frac{z(1-\frac{1}{\sqrt{n}}t,1-\frac{1}{\sqrt{n}}t,\ldots,1-\frac{1}{\sqrt{n}}t)-z(1,1,\ldots,1)}{t}\\ &= \lim_{t\to 0^+}\frac{n^2(1-\frac{t}{\sqrt{n}})^2-n^21^2}{t}\\ &= \lim_{t\to 0^+}\frac{-2n^{\frac{3}{2}}t+nt^2}{t}=-2n^{\frac{3}{2}}.\end{aligned}$$

注：若函数在一点可微 (此时函数在这一点沿任意方向的方向导数都存在)，方向导数可由梯度与单位方向向量的内积得出.

12. 求下列数量场的梯度.

(1) $u(x,y)=\sqrt{x^2+y^2}$.

解：由

$$\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2}},\ \frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}.$$

知

$$\mathrm{grad}u(x,y)=\left(\frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}}\right).$$

(3) $u(x_1,x_2,\dots,x_n)=\sum\limits_{i=1}^{n}x_i$.

解:由

$$\frac{\partial u}{\partial x_i}=1,\quad i=1,2,\dots,n.$$

知

$$\mathrm{grad}u(x_1,x_2,\dots,x_n)=(1,1,\dots,1).$$

15. 证明下列函数满足相应的等式.

(1) $u=2\cos^2(x-\frac{y}{2})$ 满足 $2\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}=0$.

证明.

$$u=2\cos^2(x-\frac{y}{2})=\cos(2x-y)+1,$$

$$\frac{\partial u}{\partial y}=\sin(2x-y)\Rightarrow\begin{cases}\dfrac{\partial^2 u}{\partial y^2}=-\cos(2x-y)\\ \dfrac{\partial^2 u}{\partial x\partial y}=2\cos(2x-y)\end{cases}.$$

从而有 $2\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}=0$. □

(3) $\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$ 满足 Cauchy-Riemann 条件 $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$ ，且分别满足 Laplace 方程 $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

证明.

$$\frac{\partial u}{\partial x} = e^x \cos y,\ \frac{\partial u}{\partial y} = -e^x \sin y,\ \frac{\partial v}{\partial x} = e^x \sin y,\ \frac{\partial v}{\partial y} = e^x \cos y.$$

故有

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}.$$

满足 Cauchy-Riemann 条件.

又

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = e^x \cos y \\ \dfrac{\partial^2 u}{\partial y^2} = -e^x \cos y \end{cases} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = e^x \sin y \\ \dfrac{\partial^2 v}{\partial y^2} = -e^x \sin y \end{cases} \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

故 u, v 均满足 Laplace 方程. □