Review

- 偏导数的定义与几何意义
- 方向导数的定义与几何意义

$$g(t)=f\left(x_0+\frac{v}{\|\vec{v}\|}t\right)=f\left(x_0^{(1)}+\frac{v_1}{\|\vec{v}\|}t,\cdots,x_0^{(n)}+\frac{v_n}{\|\vec{v}\|}t\right)$$

$$\frac{\partial f(x_0)}{\partial\vec{v}}=\lim_{t\to0^+}\frac{g(t)-g(0)}{t}=g'_+(0)$$

$$\frac{\partial f(x_0)}{\partial\vec{v}}\overset{f\text{在}(x_0,y_0)\text{可微时}}{=\!=\!=\!=\!=}\mathrm{grad}f(x_0,y_0)\cdot\frac{\vec{v}}{\|\vec{v}\|}$$

- 梯度grad $f(x_0)$的定义与意义

•n元函数可微的定义与判别

f则x_0可微 $\Leftrightarrow$ ∃常数$a_1, a_2, \cdots, a_n \in \mathbb{R}$, s.t.

$$\lim_{\mathrm{x}\to\mathrm{x}_0} \frac{f(\mathrm{x}) - f(\mathrm{x}_0) - a_1\Delta x_1 - a_2\Delta x_2 - \cdots - a_n\Delta x_n}{\|\mathrm{x}-\mathrm{x}_0\|} = 0.$$

$$\Leftrightarrow f(\mathrm{x}_0+\Delta\mathrm{x}) - f(\mathrm{x}_0)$$

$$= \frac{\partial f}{\partial x_1}(\mathrm{x}_0)\Delta x_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathrm{x}_0)\Delta x_n + o(\|\Delta\mathrm{x}\|), \Delta\mathrm{x} \to 0 \text{时}.$$

$$\Leftrightarrow f(\mathrm{x}_0+\Delta\mathrm{x}) - f(\mathrm{x}_0) = \sum_{i=1}^{n} f'_{x_i}(\mathrm{x}_0)\Delta x_i + \sum_{i=1}^{n} \varepsilon_i \Delta x_i,$$

$$\lim_{\Delta\mathrm{x}\to 0} \varepsilon_i(\Delta\mathrm{x}) = 0, \quad i = 1, 2, \cdots, n.$$

- 多元函数连续、可微、偏导数存在、偏导函数连续之间的关系

- f''_{xy}, f''_{yx} 都在(x_0, y_0)连续

$$\Rightarrow f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0)$$

- f''_{xy}, f''_{yx} 也记为$f''_{12} = f''_{21}$

§5. 向量值函数的微分

1. 线性映射

Def. 称向量值函数$f:\mathbb{R}^n \to \mathbb{R}^m$为线性的, 若对任意$\alpha,\beta \in \mathbb{R}, x, y \in \mathbb{R}^n$, 都有$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$.

Thm. $\mathbb{R}^n \to \mathbb{R}^m$的线性映射与$\mathrm{M}_{mn}$中矩阵一一对应.

Proof. 记$\mathbb{R}^n \to \mathbb{R}^m$的线性映射构成的集合为$\mathrm{Y}$, 定义

$$\varphi:\mathrm{M}_{mn} \to \mathrm{Y}$$
$$\mathrm{A} \mapsto \varphi(\mathrm{A})$$

使得$\varphi(\mathrm{A})x=\mathrm{A}x, \forall x \in \mathbb{R}^n$. 下证$\varphi$为一一映射.

任给 $f \in \mathrm{Y}$, 记$e_1, e_2, \cdots, e_n$为$\mathbb{R}^n$中自然基, 则

$$\mathrm{A} \triangleq (f(e_1), f(e_2), \cdots, f(e_n)) \in M_{mn}.$$

对任意$x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, 有

$$f(x) = f(x_1e_1 + x_2e_2 + \cdots + x_ne_n)$$

$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n) = \mathrm{A}x.$$

因此φ为满射.

任给 $f \in \mathrm{Y}$, 若$\exists \mathrm{A}, \mathrm{B} \in \mathrm{M}_{m,n}$, $s.t. \varphi(\mathrm{A}) = \varphi(\mathrm{B}) = f$, 则 $f(x) = \mathrm{A}x = \mathrm{B}x, \forall x \in \mathbb{R}^n$, 从而$\mathrm{A} = \mathrm{B}$. 因此$\varphi$为单射. □

Thm. $\mathrm{A}\in M_{mn}$,则存在$\mathrm{C}\geq 0, s.t.$

$$\|\mathrm{A}x\|_m \leq \mathrm{C}\|x\|_n, \forall x\in\mathbb{R}^n.$$

使得此不等式成立的最小的C记为$\|\mathrm{A}\|$.

Proof. $y=\|\mathrm{A}x\|_m$ 连续, $\mathrm{C}=\max\{\|\mathrm{A}x\|_m : x\in\mathbb{R}^n, \|x\|_n=1\}$存在,

$$\|\mathrm{A}x\|_m=\left\|\mathrm{A}\left(\|x\|_n\frac{x}{\|x\|_n}\right)\right\|_m=\left\|\mathrm{A}\frac{x}{\|x\|_n}\right\|_m\|x\|_n\leq \mathrm{C}\|x\|_n, \forall x\in\mathbb{R}^n. \square$$

2.向量值函数的微分

Def. 设 $f:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^m$, $x_0\in\Omega$. 若存在 $m\times n$ 矩阵A, $s.t.$

$$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)=\mathrm{A}\Delta x+\alpha,$$

其中, $\alpha=\alpha(\Delta x)=\left(\alpha_1(\Delta x),\cdots,\alpha_m(\Delta x)\right)^{\mathrm{T}}$, 且当 $\Delta x\to 0$ 时,

$\alpha=o(\Delta x)$, 即

$$\lim_{\Delta x\to 0}\frac{\|\alpha\|_m}{\|\Delta x\|_n}=0,$$

则称 f 在点 x_0 可微, 称 $\mathrm{A}\Delta x$ 为 f 在点 x_0 的微分, 记为

$$\mathrm{d}f(x_0)=\mathrm{A}\Delta x=\mathrm{A}dx,$$

称A为 f 在 x_0 的Jacobi矩阵, 记作 $\mathrm{A}=J(f)\big|_{x_0}=J_f(x_0)$.

清華大學

Thm. $f=\left(f_1,f_2,\cdots,f_m\right)^{\mathrm{T}}:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}^m$在$x_0$可微

$\Leftrightarrow n$元函数$f_i:\Omega\left(\subset\mathbb{R}^n\right)\to\mathbb{R}$在$x_0$可微,$i=1,2,\cdots,m.$

Proof.(必要性) 设f在点x_0可微, 则

$$\Delta f(x_0)=f(x_0+\Delta x)-f(x_0)=\mathrm{A}\Delta x+\alpha,\ \lim_{\Delta x\to 0}\frac{\|\alpha\|_m}{\|\Delta x\|_n}=0,$$

比较第i个分量 $(1\le \mathrm{i}\le m)$,得

$$f_i(x_0+\Delta x)-f_i(x_0)=\sum_{j=1}^{n}a_{ij}\Delta x_j+\alpha_i(\Delta x),\ \lim_{\Delta x\to 0}\frac{\alpha_i(\Delta x)}{\|\Delta x\|_n}=0,$$

即f_i在x_0可微,$i=1,2,\cdots,m.$

(充分性) 设f_i在x_0可微, $i=1,2,\cdots,m$, 则

$$\begin{pmatrix}\Delta f_1(x_0)\\ \Delta f_2(x_0)\\ \vdots\\ \Delta f_m(x_0)\end{pmatrix}=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}\Bigg|_{x_0}\begin{pmatrix}\Delta x_1\\ \Delta x_2\\ \vdots\\ \Delta x_n\end{pmatrix}+\begin{pmatrix}\alpha_1(\Delta x)\\ \alpha_2(\Delta x)\\ \vdots\\ \alpha_n(\Delta x)\end{pmatrix},$$

且$\alpha_i(\Delta x)=o(\|\Delta x\|_n)$, $\Delta x\to 0$时, $1\le i\le m$. 故f在x_0可微.□

Remark: $f=(f_1,f_2,\cdots,f_m)^{\mathrm{T}}:\Omega(\subset\mathbb{R}^n)\to\mathbb{R}^m$在$x_0$可微,

则f在x_0的*Jacobi*矩阵为

$$\mathrm{J}f(x_0)=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n}\\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & \vdots & \vdots\\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n}\end{pmatrix}\Bigg|_{x_0}\triangleq\left.\frac{\partial(f_1,f_2,\cdots,f_m)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0},$$

也记为$\dfrac{\partial f}{\partial x}(x_0)$.

3.复合映射的微分

Thm. $u=g(x):\Omega\subset\mathbb{R}^n\to\mathbb{R}^m,\ y=f(u):g(\Omega)\subset\mathbb{R}^m\to\mathbb{R}^k$, $x_0\in\Omega, u_0=g(x_0)$, 若 $g(x)$ 在 x_0 可微, $f(u)$ 在 u_0 可微, 则复合映射 $y=(f\circ g)(x)=f\big(g(x)\big)$ 在 x_0 可微, 且

$$J(f\circ g)\big|_{x_0}=J(f)\big|_{u_0}\cdot J(g)\big|_{x_0},$$

即

$$\left.\frac{\partial(y_1,y_2,\cdots,y_k)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0}=\left.\frac{\partial(y_1,y_2,\cdots,y_k)}{\partial(u_1,u_2,\cdots,u_m)}\right|_{u_0}\cdot\left.\frac{\partial(u_1,u_2,\cdots,u_m)}{\partial(x_1,x_2,\cdots,x_n)}\right|_{x_0},$$

简记为

$$\left.\frac{\partial y}{\partial x}\right|_{x_0}=\left.\frac{\partial y}{\partial u}\right|_{u_0}\cdot\left.\frac{\partial u}{\partial x}\right|_{x_0}$$

Question. 可微能否替换成偏导存在? 否!

Proof. 记$A = \left.\dfrac{\partial(y_1, y_2, \cdots, y_k)}{\partial(u_1, u_2, \cdots, u_m)}\right|_{u_0}$, $B = \left.\dfrac{\partial(u_1, u_2, \cdots, u_m)}{\partial(x_1, x_2, \cdots, x_n)}\right|_{x_0}$.

$u = g(x)$在x_0可微, $y = f(u)$在u_0可微, 则

$$\Delta u = g(x_0 + \Delta x) - g(x_0) = B\Delta x + o(\Delta x),$$

$$\Delta y = f(u_0 + \Delta u) - f(u_0) = A\Delta u + o(\Delta u),$$

于是

$$\Delta(f \circ g)(x_0) = f(g(x_0 + \Delta x)) - f(g(x_0))$$

$$= f(u_0 + \Delta u) - f(u_0) = A\Delta u + o(\Delta u)$$

$$= A(B\Delta x + o(\Delta x)) + o(\Delta u) = AB\Delta x + Ao(\Delta x) + o(\Delta u),$$

往证: $Ao(\Delta x) + o(\Delta u) = o(\Delta x)$.

$$\frac{\|\mathrm{A}o(\Delta x)\|}{\|\Delta x\|} \le \frac{\|\mathrm{A}\|\|o(\Delta x)\|}{\|\Delta x\|} \to 0, \text{当}\Delta x \to 0\text{时}.$$

$$\frac{\|\Delta u\|}{\|\Delta x\|} = \frac{\|\mathrm{B}\Delta x + o(\Delta x)\|}{\|\Delta x\|} \le \frac{\|\mathrm{B}\Delta x\|}{\|\Delta x\|} + \frac{\|o(\Delta x)\|}{\|\Delta x\|}$$

$$\le \|\mathrm{B}\| + \frac{\|o(\Delta x)\|}{\|\Delta x\|} \to \|\mathrm{B}\|, \text{当}\Delta x \to 0\text{时}.$$

$$\frac{\|o(\Delta u)\|}{\|\Delta x\|} = \frac{\|o(\Delta u)\|}{\|\Delta u\|} \cdot \frac{\|\Delta u\|}{\|\Delta x\|} \to 0, \text{当}\Delta x \to 0\text{时}.$$

$$\frac{\|\mathrm{A}o(\Delta x) + o(\Delta u)\|}{\|\Delta x\|} \le \frac{\|\mathrm{A}o(\Delta x)\|}{\|\Delta x\|} + \frac{\|o(\Delta u)\|}{\|\Delta x\|} \to 0, \text{当}\Delta x \to 0\text{时}.\square$$

Remark. $k=1$时, $y=f(u_1,u_2,\cdots,u_m), u_i=g_i(x_1,x_2,\cdots,x_n)$, $1\le i\le m$, 则$y=y(x_1,x_2,\cdots,x_n)$的偏导数满足链式法则

$$\frac{\partial y}{\partial x_i}=\frac{\partial y}{\partial u_1}\frac{\partial u_1}{\partial x_i}+\frac{\partial y}{\partial u_2}\frac{\partial u_2}{\partial x_i}+\cdots+\frac{\partial y}{\partial u_m}\frac{\partial u_m}{\partial x_i}, i=1,2,\cdots,n.$$

例. 球坐标变换 $\begin{cases} x = r\sin\varphi\cos\theta, \\ y = r\sin\varphi\sin\theta, \\ z = r\cos\varphi, \end{cases}$

是$\Omega = \{(r,\varphi,\theta) \mid 0 \le r < \infty, 0 \le \varphi \le \pi, 0 \le \theta < 2\pi\}$到$\mathbb{R}^3$的连续可微映射.

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = \begin{pmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{pmatrix}$$

$$\frac{D(x,y,z)}{D(r,\varphi,\theta)} \triangleq \det\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = r^2\sin\varphi. \square$$

例: $\begin{cases} y_1 = u_1 u_2 - u_1 u_3, \\ y_2 = u_1 u_3 - u_2^2, \end{cases}$ $\begin{cases} u_1 = x_1 \cos x_2 + \left(x_1 + x_2\right)^2 \\ u_2 = x_1 \sin x_2 + x_1 x_2 \\ u_3 = x_1^2 - x_1 x_2 + x_2^2 \end{cases}$

求 $\left.\dfrac{\partial\left(y_1, y_2\right)}{\partial\left(x_1, x_2\right)}\right|_{(1,0)}$.

解: $(x_1, x_2) = (1, 0)$时, $(u_1, u_2, u_3) = (2, 0, 1)$,

$$\left.\frac{\partial\left(y_1, y_2\right)}{\partial(x_1, x_2)}\right|_{(1,0)} = \left.\frac{\partial\left(y_1, y_2\right)}{\partial\left(u_1, u_2, u_3\right)}\right|_{(2,0,1)} \cdot \left.\frac{\partial\left(u_1, u_2, u_3\right)}{\partial(x_1, x_2)}\right|_{(1,0)}$$

$$=\begin{pmatrix} u_2-u_3 & u_1 & -u_1 \\ u_3 & -2u_2 & u_1 \end{pmatrix}\Bigg|_{(2,0,1)}$$

$$\cdot\begin{pmatrix} \cos x_2+2(x_1+x_2) & -x_1\sin x_2+2(x_1+x_2) \\ \sin x_2+x_2 & x_1\cos x_2+x_1 \\ 2x_1-x_2 & -x_1+2x_2 \end{pmatrix}\Bigg|_{(1,0)}$$

$$=\begin{pmatrix} -1 & 2 & -2 \\ 1 & 0 & 2 \end{pmatrix}\times\begin{pmatrix} 3 & 2 \\ 0 & 2 \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} -7 & 4 \\ 7 & 0 \end{pmatrix}.\square$$

例. $z=2yf(\frac{x^2}{y},3y)$, f 二阶连续可微. 求 z''_{xy}.

解: f 二阶连续可微, 则 $z''_{xy}=z''_{yx}$. 后者求解相对容易.

设 $u=x^2/y, v=3y, z=2yf(u,v)$. 则

$$z'_x=2y\left[f'_u\cdot(2x/y)+f'_v\cdot 0\right]=4xf'_u$$

$$z''_{xy}=z''_{yx}=\frac{\partial}{\partial y}(4xf'_u)=4x\left[f''_{uu}\cdot\frac{-x^2}{y^2}+f''_{vu}\cdot 3\right]$$

$$=-\frac{4x^3}{y^2}f''_{uu}+12xf''_{vu}.\square$$

例. $u(x,y)$二阶连续可微, 且$u''_{xx}-u''_{yy}=0$.令$\xi=x-y$,

$\eta=x+y$.试证:$u''_{\xi\eta}=0$.

解: $x=(\xi+\eta)/2, y=(\eta-\xi)/2$,

$$u'_{\eta}=u'_x x'_{\eta}+u'_y y'_{\eta}=(u'_x+u'_y)/2$$

$$u''_{\xi\eta}=\frac{1}{2}\frac{\partial}{\partial\xi}\left(u'_x+u'_y\right)$$

$$=\left(u''_{xx}x'_{\xi}+u''_{yx}y'_{\xi}+u''_{xy}x'_{\xi}+u''_{yy}y'_{\xi}\right)\Big/2$$

$$=\left(u''_{xx}-u''_{yx}+u''_{xy}-u''_{yy}\right)\Big/4=\left(u''_{xx}-u''_{yy}\right)\Big/4=0.\square$$

例. $f(x,y)$可微,$f(x,x^2)=1$, $f'_x(x,x^2)=x$, 求$f'_y(x,x^2)$, $(x\neq 0)$.

分析: $f'_x(x,x^2)=\left.\dfrac{\partial f}{\partial x}\right|_{(x,y)=(x,x^2)}$.

解: 将$f(x,x^2)=1$两边同时对x求导, 有

$$f'_x(x,x^2)\cdot 1+f'_y(x,x^2)\cdot 2x=0,$$

$$x+2xf'_y(x,x^2)=0,$$

故$f'_y(x,x^2)=-\dfrac{1}{2}$, $\forall x\neq 0$. □

例. $y=(1/x)^{-(1/x)}$，求$y'(x)$.

解：令$u=1/x, v=-1/x, y=u(x)^{v(x)}$.

$$\frac{dy}{dx}=\frac{\partial y}{\partial u}\cdot\frac{du}{dx}+\frac{\partial y}{\partial v}\cdot\frac{dv}{dx}$$

$$=vu^{v-1}(-1/x^2)+u^v(\ln u)(1/x^2)$$

$$=u^{v+2}(1+\ln u)$$

$$=(1/x)^{2-\frac{1}{x}}(1-\ln x).\square$$

作业：习题1.5 No. 4,5,7,9.