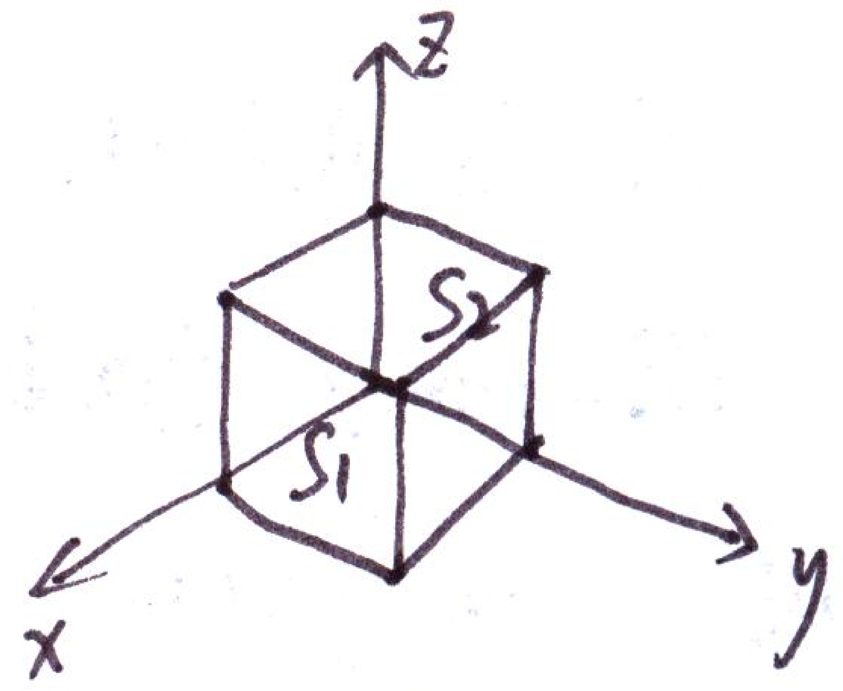
习题4.5.



3.解：(1) 由对称性（S^+关于x.y.z轴对称）

$$\iint_{S^+} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$$

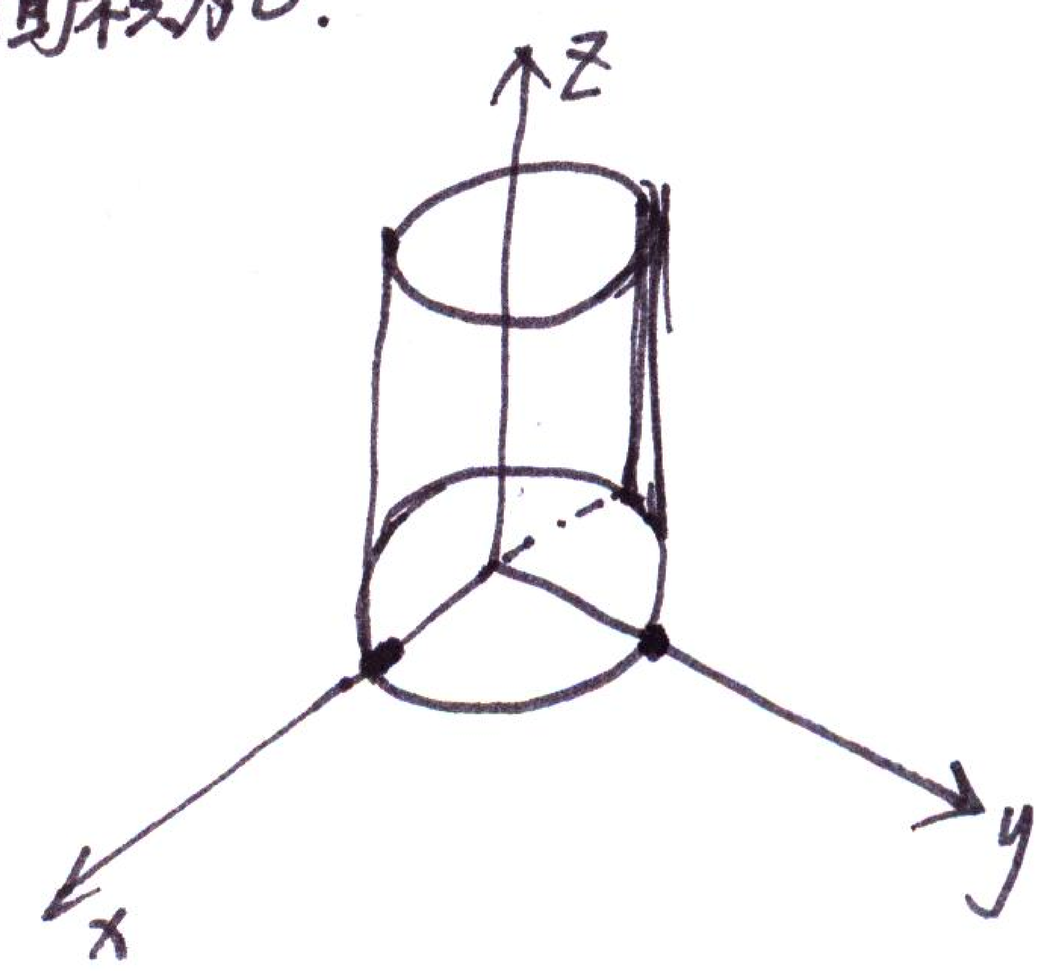
$$=3\iint_{S^+} x\,dy\wedge dz = 3\left(\iint_{S_1} x\,dydz - \iint_{S_2} x\,dydz\right)$$ （设与x轴垂直的两个面为S_1, S_2）

$$=3\iint_{S_1} dydz$$ （由于在S_2上$x=0$，在S_1上$x=1$）

$$=3\sigma(S_1)$$

$$=3$$

(2) 由于S^+在xy平面上的投影为圆周$x^2+y^2=1$，面积为0.



故 $\iint_{S^+} z^2\,dx\wedge dy = 0$.

设S^+在yz平面上的投影为D_{xy}，

S^+被yz平面割成S_1与S_2两部分.

则 $\iint_{S^+} x^2\,dy\wedge dz = \iint_{S^+} \sqrt{1-y^2}\,dy\wedge dz$

$$=\iint_{S_1} \sqrt{1-y^2}\,dy\wedge dz + \iint_{S_2} \sqrt{1-y^2}\,dy\wedge dz$$

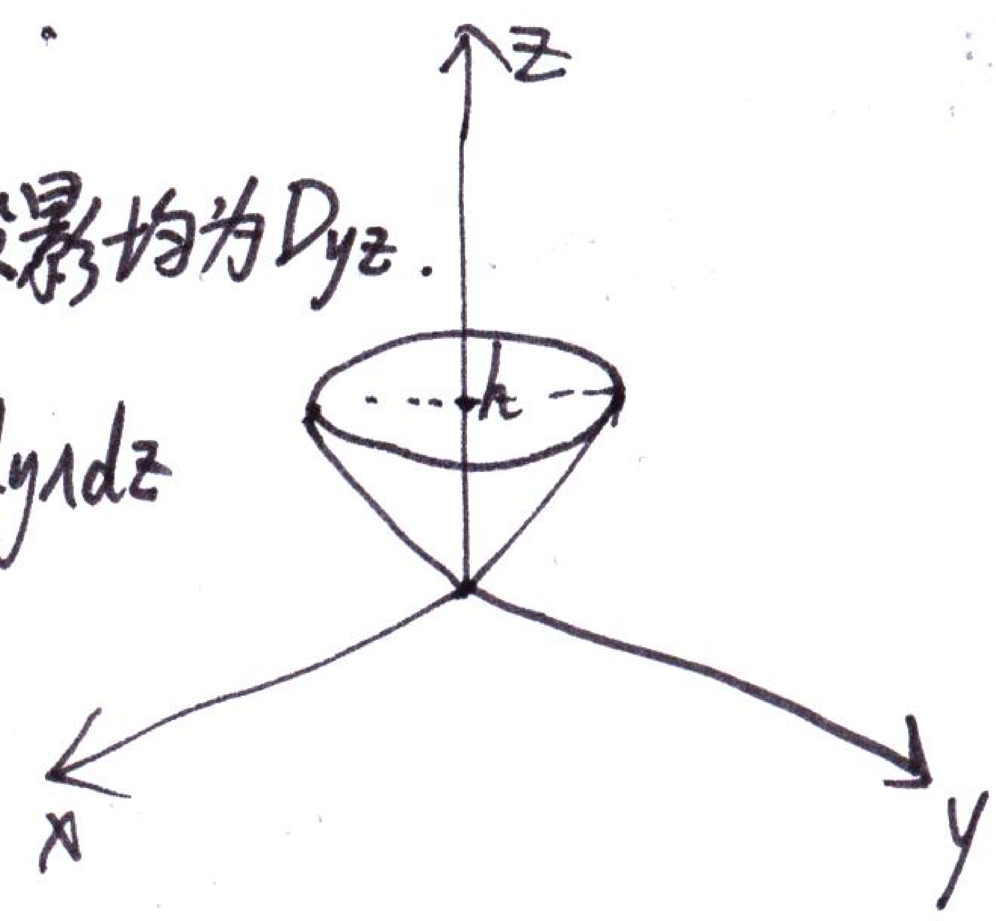
$$=\iint_{D_{xy}} \sqrt{1-y^2}\,dydz - \iint_{D_{xy}} \sqrt{1-y^2}\,dydz$$

$$=0$$

同理可证 $\iint_{S^+} y^2\,dz\wedge dx = 0$.

因此 $\iint_{S^+} x^2\,dy\wedge dz + y^2\,dz\wedge dx + z^2\,dx\wedge dy = 0$.

(3) 设 S^+ 的圆侧面为 S_1，底面为 S_2.



设 S_1 被 yz 平面割为 S_{11} 与 S_{12}，它们在 yz 平面的投影均为 D_{yz}.

则 $$\iint_{S^+}(y-z)\,dy\wedge dz=\iint_{S_{11}}(y-z)\,dy\wedge dz+\iint_{S_{12}}(y-z)\,dy\wedge dz+\iint_{S_2}(y-z)\,dy\wedge dz$$

$$=\iint_{D_{yz}}(y-z)\,dydz-\iint_{D_{yz}}(y-z)\,dydz+0\quad(S_2\perp O_{yz})$$

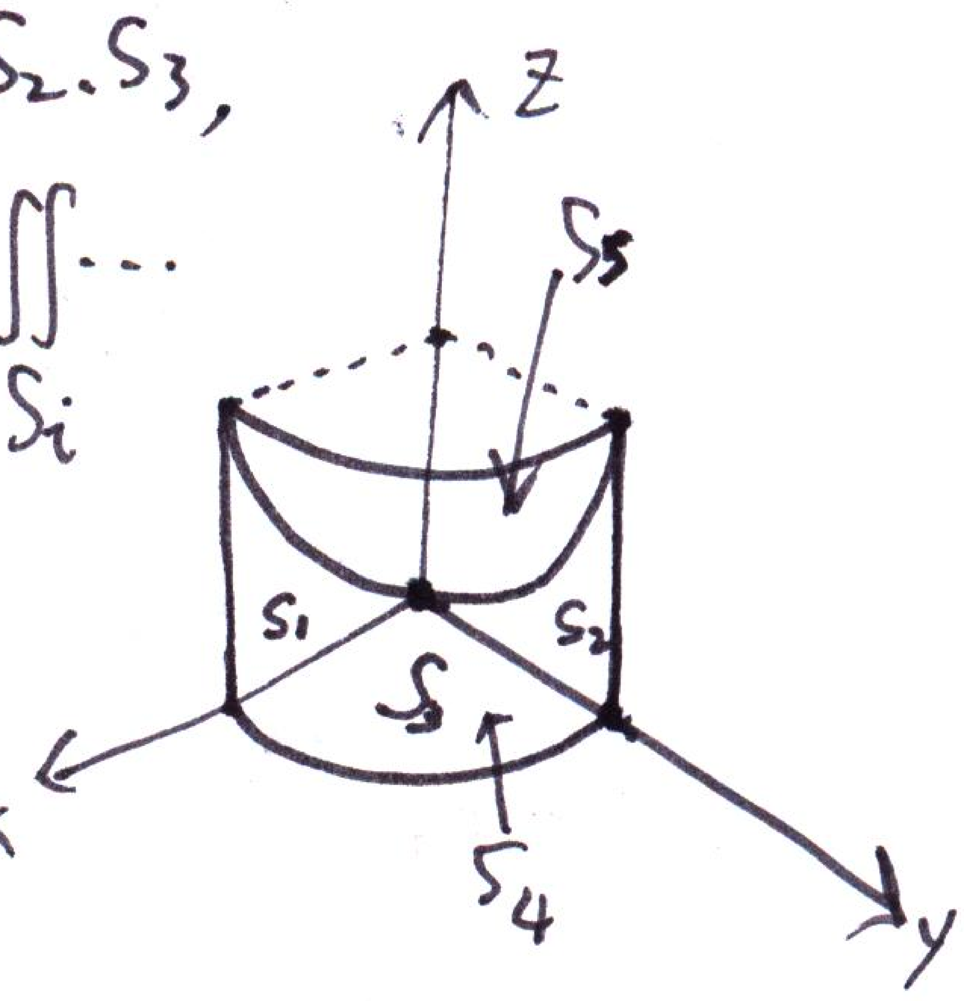
$=0$. 同理可证 $\iint_{S^+}(z-x)\,dz\wedge dx=0$.

设 S_1 与 S_2 在 xy 平面的投影均为 D_{xy}.

则 $$\iint_{S^+}(x-y)\,dx\wedge dy=\iint_{S_1}(x-y)\,dx\wedge dy+\iint_{S_2}(x-y)\,dx\wedge dy=\iint_{D_{xy}}(x-y)\,dxdy+\iint_{D_{xy}}(x-y)\,dxdy=0.$$

故 $$\iint_{S^+}(y-z)\,dy\wedge dz+(z-x)\,dz\wedge dx+(x-y)\,dx\wedge dy=0.$$

(4) 如图所示，设该立体在三个坐标平面上的部分为 S_1、S_2、S_3，



该立体柱面为 S_4，旋转抛物面为 S_5. 则 $\iint_{S^+}\cdots=\sum_{i=1}^{5}\iint_{S_i}\cdots$

由于 S_1 与 yz 平面、xy 平面均垂直.

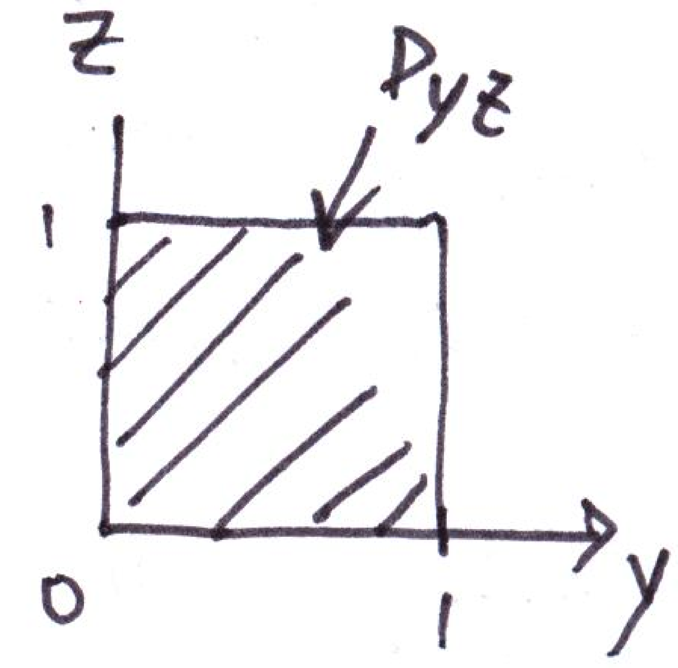
故 $$\iint_{S_1}y^2z\,dx\wedge dy+z^2x\,dy\wedge dz+x^2y\,dz\wedge dx$$

$$=\iint_{S_1}x^2y\,dz\wedge dx=0\quad(\because y=0)$$

同理可证 $\iint_{S_i}y^2z\,dx\wedge dy+z^2x\,dy\wedge dz+x^2y\,dz\cancel{dx}\wedge dx=0$，$i=2,3$.

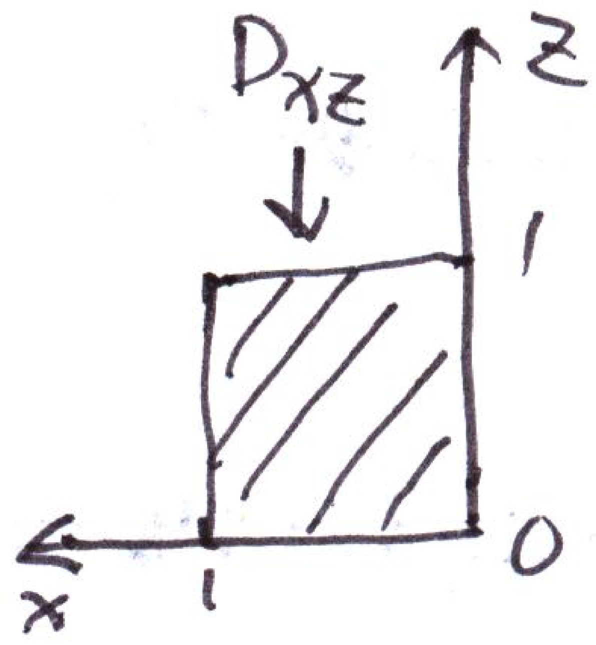
以下计算 S_4、S_5 的曲面积分.

① 因S_4与xy平面垂直，故 $\iint_{S_4} y^2 z\,dx\wedge dy = 0$.



② $\iint_{S_4} z^2 x\,dy\wedge dz = \iint_{D_{yz}} z^2\sqrt{1-y^2}\,dydz = \int_0^1 z^2 dz\int_0^1\sqrt{1-y^2}\,dy$

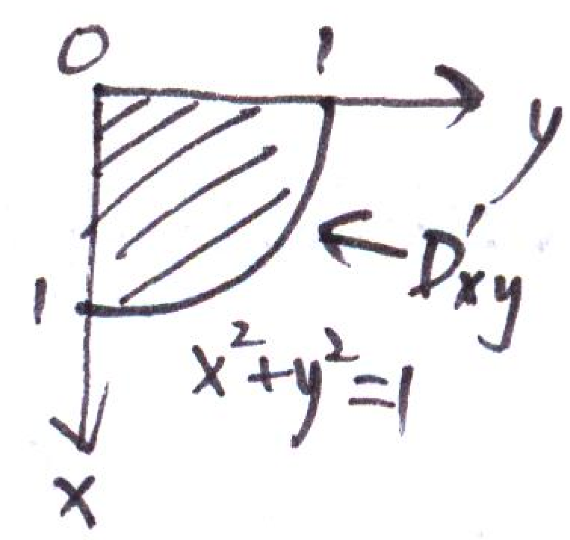
$\xlongequal[\theta\in[0,\frac{\pi}{2}]]{y=\sin\theta} \frac{1}{3}\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{1+\cos2\theta}{2}d\theta = \frac{\pi}{12}$.



③ $\iint_{S_4} x^2 y\,dz\wedge dx = \iint_{D_{xz}} x^2\sqrt{1-x^2}\,dzdx = \int_0^1 x^2\sqrt{1-x^2}\,dx$

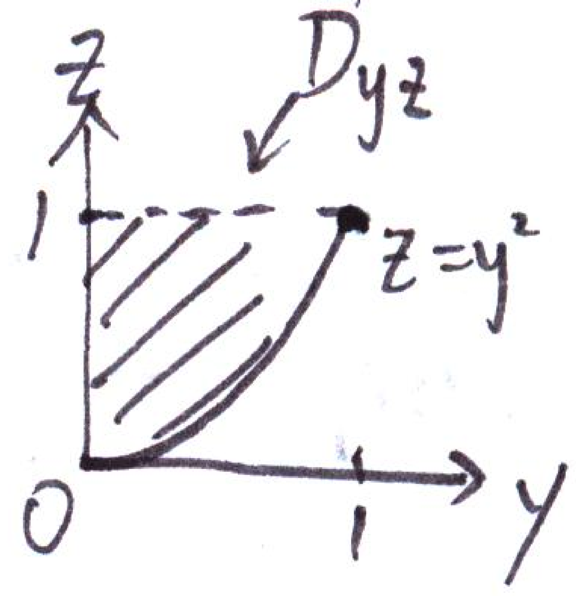
$\xlongequal[\theta\in[0,\frac{\pi}{2}]]{x=\sin\theta} \int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}}\frac{1}{4}\sin^2 2\theta\, d\theta$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1-\cos4\theta}{2}d\theta = \frac{\pi}{16}$.



④ $\iint_{S_5} y^2 z\,dx\wedge dy = \iint_{D'_{xy}} y^2(x^2+y^2)\,dxdy \xlongequal[y=r\sin\theta]{x=r\cos\theta} \iint_{D'_{r\theta}} r^5\sin^2\theta\, drd\theta$

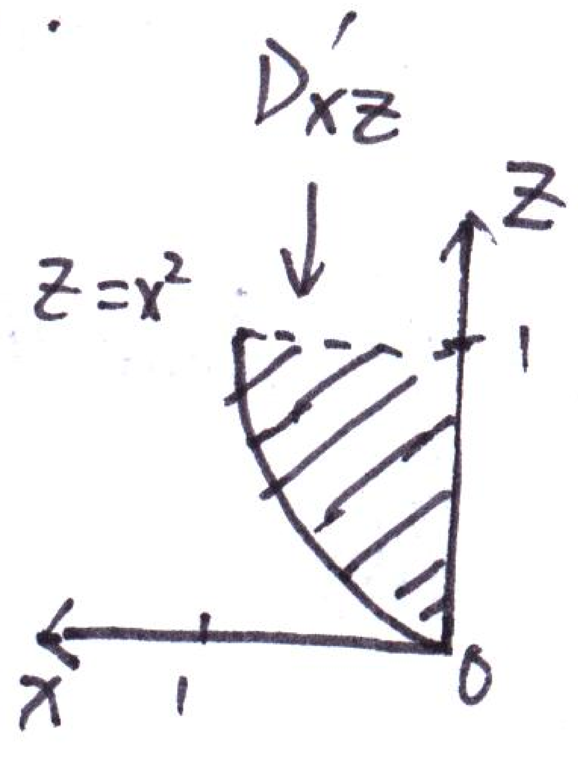
$= \int_0^1 r^5 dr\int_0^{\frac{\pi}{2}}\sin^2\theta\, d\theta = \frac{1}{6}\int_0^{\frac{\pi}{2}}\frac{1-\cos2\theta}{2}d\theta = \frac{1}{6}\cdot\frac{\pi}{4} = \frac{\pi}{24}$.



⑤ $\iint_{S_5} z^2 x\,dy\wedge dz = -\iint_{D'_{yz}} z^2\sqrt{z-y^2}\,dydz = -\int_0^1 dz\int_0^{\sqrt{z}} z^2\sqrt{z-y^2}\,dy$

令 $y=\sqrt{z}\sin\theta,\ \theta\in[0,\frac{\pi}{2}]$ $dy=\sqrt{z}\cos\theta\, d\theta$.

上式 $= -\int_0^1 dz\int_0^{\frac{\pi}{2}} z^3\cos^2\theta\, d\theta = \cancel{-\frac{1}{4}\cdot\frac{\pi}{4}} = -\int_0^1 z^3 dz\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta = -\frac{\pi}{16}$.

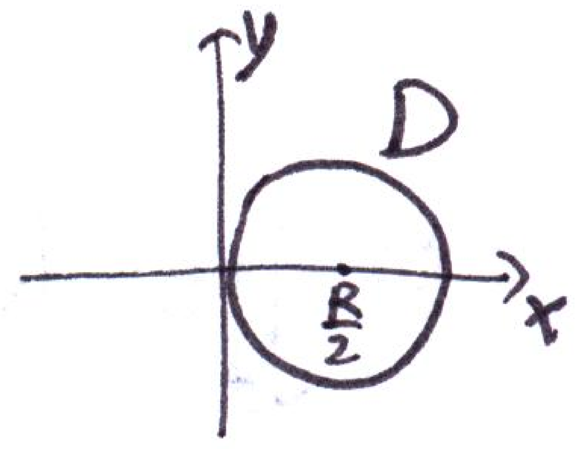


⑥ $\iint_{S_5} x^2 y\,dz\wedge dx = -\iint_{D'_{xz}} x^2\sqrt{z-x^2}\,dxdz \xlongequal{x=\sqrt{z}\sin\theta} -\int_0^1 dz\int_0^{\frac{\pi}{2}} z^2\cos^2\theta\sin^2\theta\, d\theta$

$= -\int_0^1 z^2 dz\int_0^{\frac{\pi}{2}}\frac{1}{4}\sin^2 2\theta\, d\theta = -\frac{1}{3}\cdot\frac{1}{4}\cdot\int_0^{\frac{\pi}{2}}\frac{1-\cos4\theta}{2}d\theta = -\frac{1}{48}\pi$.

综上所述，$\iint_{S^+} y^2 z\,dx\wedge dy + z^2 x\,dy\wedge dz + x^2 y\,dz\wedge dx = ①+②+③+④+⑤+⑥ = \frac{5}{48}\pi$.

(5) S^+在xy平面上的投影即为 $x^2+y^2=Rx$ 在xy平面上所围成的圆，设为D.



则 $\iint\limits_{S^+} z^2\,dx\wedge dy = \iint\limits_{S^+}(R^2-x^2-y^2)\,dx\wedge dy = \iint\limits_{D}(R^2-x^2-y^2)\,dxdy$.

$= \iint\limits_{D} R^2\,dxdy - \iint\limits_{D}(x^2+y^2)\,dxdy = R^2\cdot\sigma(D) - \iint\limits_{D}(x^2+y^2)\,dxdy$.

换元 $x=\frac{R}{2}+r\cos\theta$, $y=r\sin\theta$, $r\in[0,\frac{R}{2}]$, $\theta\in[0,2\pi]$.

则 $\left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}\right| = r$, $D=\{(x,y)\mid x^2+y^2\le Rx\}=\left\{(r,\theta)\,\middle|\,\begin{matrix}0\le r\le\frac{R}{2}\\ 0\le\theta\le 2\pi\end{matrix}\right.$

$\iint\limits_{D}(x^2+y^2)\,dxdy = \iint\limits_{D} r\left(\frac{R^2}{4}+Rr\cos\theta+r^2\right)drd\theta = \int_0^{\frac{R}{2}}dr\int_0^{2\pi}\left(\frac{R^2}{4}r+Rr^2\cos\theta+r^3\right)d\theta$

$=\int_0^{\frac{R}{2}}2\pi\left(\frac{R^2}{4}r+r^3\right)dr = 2\pi\cdot\left(\frac{1}{4}r^4+\frac{1}{8}R^2r^2\right)\Big|_0^{r=\frac{R}{2}} = \frac{3}{32}R^4\pi$.

故 $\iint\limits_{S^+} z^2\,dx\wedge dy = \frac{1}{4}\pi R^4 - \frac{3}{32}\pi R^4 = \frac{5}{32}\pi R^4$.

5. 解：$\vec{v}=(xy,yz,zx)$ 半径为1的球面上任一点 (x,y,z) 外法向量 $\vec{n}=(x,y,z)$

流量 $Q = \iint\limits_{S_1^+}\vec{v}\cdot\vec{n}\,dS = \iint\limits_{S_1^+}(x^2y+y^2z+z^2x)\,dS$，$S_1^+$表示该球面在第一象限的部分.

因 $\cos\alpha=x$, $\cos\beta=y$, $\cos\gamma=z$,

$dy\wedge dz=\cos\alpha\,dS = x\,dS$, $dz\wedge dx=\cos\beta\,dS = y\,dS$, $dx\wedge dy=\cos\gamma\,dS = z\,dS$

有 $Q=\iint\limits_{S_1^+} x^2\,dz\wedge dx + y^2\,dx\wedge dy + z^2\,dy\wedge dz = 3\iint\limits_{S_1^+} x^2\,dz\wedge dx$ (因S_1^+关于x.y.z轴对称

$= 3\iint\limits_{D_{xz}} x^2\,dxdz \xlongequal[z=r\sin\theta]{x=r\cos\theta} 3\iint\limits_{D_{xz}} r^3\cos^2\theta\,drd\theta = 3\int_0^{\frac{\pi}{2}}d\theta\int_0^1 r^3\cos^2\theta\,dr$

$= \frac{3}{4}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta = \frac{3}{4}\int_0^{\frac{\pi}{2}}\frac{1+\cos2\theta}{2}\,d\theta = \frac{3}{16}\pi$.

7. 解：$x=u\cos v,\ y=u\sin v,\ z=av$.

$$\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)=\begin{vmatrix}\cos v & -u\sin v\\ \sin v & u\cos v\end{vmatrix}=u,\quad \det\left(\frac{\partial(y,z)}{\partial(u,v)}\right)=\begin{vmatrix}\sin v & u\cos v\\ 0 & a\end{vmatrix}=a\sin v,$$

$$\det\left(\frac{\partial(z,x)}{\partial(u,v)}\right)=\begin{vmatrix}0 & a\\ \cos v & -u\sin v\end{vmatrix}=-a\cos v.$$

则 $$\iint_{S^+}(x^2+y^2)\,dx\wedge dy+y^2\,dy\wedge dz+z^2\,dz\wedge dx$$

$$=\iint_{D_{uv}}(u^3+u^2\sin^3 v\cdot a-a^3v^2\cos v)\,du\,dv.$$

$$\iint_{D_{uv}}u^3\,du\,dv=\int_0^1du\int_0^{2\pi}u^3\,dv=2\pi\int_0^1u^3\,du=\frac{\pi}{2},\quad \iint_{D_{uv}}u^2\sin^3v\cdot a\,du\,dv=a\int_0^1u^2\,du\int_0^{2\pi}\sin^3v\,dv$$

(因 $\sin 3\alpha=3\sin\alpha-4\sin^3\alpha$，故 $\sin^3\alpha=\frac{3\sin\alpha-\sin 3\alpha}{4}$) $=a\cdot\frac{1}{3}\cdot\int_0^{2\pi}\frac{3\sin v-\sin 3v}{4}\,dv=0.$

~~或者~~

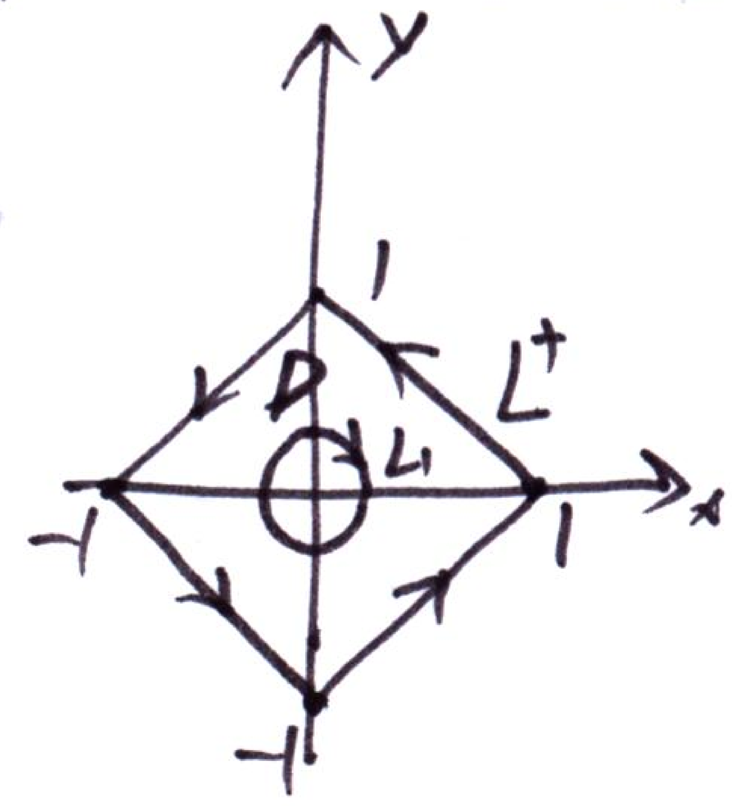
$$\iint_{D_{uv}}a^3v^2\cos v\,du\,dv=a^3\int_0^{2\pi}v^2\cos v\,dv=a^3\left[v^2\sin v\Big|_0^{v=2\pi}-\int_0^{2\pi}2v\sin v\,dv\right]=-2a^3\int_0^{2\pi}v\sin v\,dv$$

$$=-2a^3\left[-v\cos v\Big|_0^{v=2\pi}-\int_0^{2\pi}(-\cos v)\,dv\right]=-2a^3\cdot\left(-2\pi+\int_0^{2\pi}\cos v\,dv\right)=+4\pi a^3.$$

故 $$\iint_{S^+}(x^2+y^2)\,dx\wedge dy+y^2\,dy\wedge dz+z^2\,dz\wedge dx$$

$$=\iint_{D_{uv}}u^3\,du\,dv+\iint_{D_{uv}}au^2\sin^3v\,du\,dv-\iint_{D_{uv}}a^3v^2\cos v\,du\,dv=\frac{\pi}{2}+0-4\pi a^3=\left(\frac{1}{2}-4a^3\right)\pi.$$

习题4.6 2.(3) 解：设L_1为顺时针方向圆周L_1：$x^2+y^2=r^2$，r充分小，使L_1在L^+内部.



$$P=\frac{x+y}{x^2+y^2},\quad Q=\frac{y-x}{x^2+y^2}.$$

在L^+与L_1围成的区域D内$P, Q\in C^1$，且

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}.$$

由Green公式，

$$\oint_{L^+}\frac{(x+y)dx+(y-x)dy}{x^2+y^2}-\oint_{L_1}\frac{(x+y)dx+(y-x)dy}{x^2+y^2}=\iint_D 0\,dxdy=0.$$

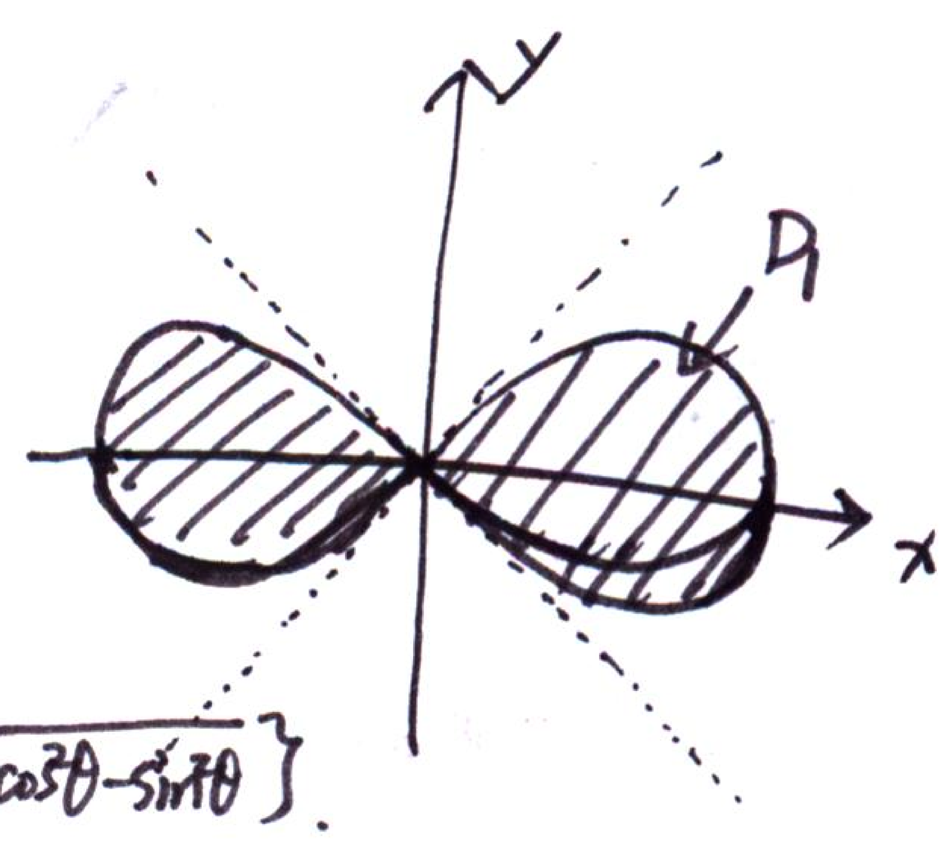
故 $$\oint_{L^+}\frac{(x+y)dx+(y-x)dy}{x^2+y^2}=\oint_{L_1}\frac{(x+y)dx+(y-x)dy}{x^2+y^2}\ \cancel{\overset{x=r\cos\theta}{\underset{y=r\sin\theta}{=\!=}}\int_0^{2\pi}\frac{-r(\cos\theta+\sin\theta)\sin\theta}{r^2}}$$

(极坐标换元) $$=\int_0^{2\pi}\frac{-r^2(\cos\theta+\sin\theta)\sin\theta+r^2(\sin\theta-\cos\theta)\cdot\cos\theta}{r^2}d\theta=\int_0^{2\pi}-1\,d\theta=-2\pi.$$

(4) 做法与(3)完全相同，答案为-2π.

4.(2) 解：设双纽线所围区域为D，在第一象限部分为D_1.

则由对称性，$\Delta D=\iint\limits_D 1\,dxdy=4\iint\limits_{D_1}dxdy$.



设$x=r\cos\theta$，$y=r\sin\theta$ 代入双纽线方程有

$$r^4=a^2r^2(\cos^2\theta-\sin^2\theta)\Rightarrow r=a\sqrt{\cos^2\theta-\sin^2\theta}.$$

因D_1有渐近线$y=x$. 故$D_1=\{(r,\theta)\mid 0\le\theta<\frac{\pi}{4},\ 0\le r\le a\sqrt{\cos^2\theta-\sin^2\theta}\}$.

$$\Delta D=4\iint\limits_{D_1}dxdy=4\iint\limits_{D_1}r\,drd\theta=4\int_0^{\frac{\pi}{4}}d\theta\int_0^{a\sqrt{\cos^2\theta-\sin^2\theta}}r\,dr$$

$$=4\int_0^{\frac{\pi}{4}}\frac{1}{2}a^2(\cos^2\theta-\sin^2\theta)d\theta=2a^2\int_0^{\frac{\pi}{4}}\cos2\theta\,d\theta=a^2\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta$$

$$=a^2$$