习题 3.3 作业参考解答

《高等微积分教程（下）》

12. 计算下列二重积分.

(1) $\iint\limits_D (x^2+y^2)dxdy$, $D=\{(x,y)|2x\le x^2+y^2\le 4x\}$.

解： 令 $x=\rho\cos\theta, y=\rho\sin\theta$，则积分区域为

$$E=\{(\rho,\theta)|\theta\in[-\frac{\pi}{2},\frac{\pi}{2}],\rho\in[2\cos\theta,4\cos\theta]\}.$$

从而

$$\begin{aligned}\iint\limits_D (x^2+y^2)dxdy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{2\cos\theta}^{4\cos\theta} \rho^2\cdot\rho d\rho \\ &= 60\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta \\ &= \frac{45}{2}\pi.\end{aligned}$$

(2) $\iint\limits_D \sqrt{(x^2+y^2)^3}dxdy$, $D=\{(x,y)|x^2+y^2\le\min\{1,2x\}\}$.

解： 令 $x=\rho\cos\theta, y=\rho\sin\theta$.

则当 $\theta\in[-\frac{\pi}{2},-\frac{\pi}{3}]\cup[\frac{\pi}{3},\frac{\pi}{2}]$ 时，$\rho\le 2\cos\theta$；当 $\theta\in[-\frac{\pi}{3},\frac{\pi}{3}]$ 时，$\rho\le 1$.

从而

$$\iint\limits_D \sqrt{(x^2+y^2)^3}dxdy = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} d\theta \int_0^{2\cos\theta} \rho^3 \cdot \rho d\rho + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho^3 \cdot \rho d\rho + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta \int_0^1 \rho^3 \cdot \rho d\rho$$
$$= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} \frac{32}{5}\cos^5\theta d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{32}{5}\cos^5\theta d\theta + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{5} d\theta$$
$$= \frac{512}{75} - \frac{98}{25}\sqrt{3} + \frac{2}{15}\pi.$$

(3) $\iint\limits_D (x+y)dxdy$，D 是由 $x^2+y^2=x+y$ 围成的平面区域.

解： 令 $x=\frac{1}{2}+r\cos\theta, y=\frac{1}{2}+r\sin\theta$，则

$$\iint\limits_D (x+y)dxdy = \int_0^{\frac{\sqrt{2}}{2}} dr \int_0^{2\pi} (1+r\cos\theta+r\sin\theta)\cdot r d\theta$$
$$= \int_0^{\frac{\sqrt{2}}{2}} 2\pi r dr$$
$$= \frac{1}{2}\pi.$$

(4) $\iint\limits_D (y-x)^2dxdy$,$D=\{(x,y)|0\le y\le x+a, x^2+y^2\le a^2\}$,$a>0$.

解：

$$\iint\limits_D (y-x)^2dxdy = \int_{-a}^0 dx \int_0^{x+a} (y-x)^2 dy + \int_0^a dx \int_0^{\sqrt{a^2-x^2}} (y-x)^2 dy$$
$$= \int_{-a}^0 \frac{1}{3}x^3 + \frac{1}{3}a^3 dx + \int_0^a \frac{1}{3}\sqrt{a^2-x^2}(a^2-x^2) - (a^2-x^2)x + x^2\sqrt{a^2-x^2}dx$$
$$= \frac{1}{4}a^4 + (\frac{\pi}{8} - \frac{1}{4})a^4$$
$$= \frac{\pi}{8}a^4.$$

(5) $\iint\limits_D \arctan\frac{y}{x}dxdy$,$D=\{(x,y)|x^2+y^2\le 1, x\le 0, y\le 0\}$.

解：令 $x=\rho\cos\theta, y=\rho\sin\theta$，则积分区域为

$$E=\{(\rho,\theta)|\theta\in[\pi,\frac{3}{2}\pi],\rho\in[0,1]\}.$$

从而

$$\begin{aligned}\iint\limits_{D}\arctan\frac{y}{x}dxdy&=\int_{\pi}^{\frac{3}{2}\pi}d\theta\int_0^1(\theta-\pi)\cdot\rho d\rho\\&=\frac{1}{16}\pi^2.\end{aligned}$$

(6) $\int_0^{\frac{1}{\sqrt{2}}}dy\int_0^y e^{-x^2-y^2}dx+\int_{\frac{1}{\sqrt{2}}}^1dy\int_0^{\sqrt{1-y^2}}e^{-x^2-y^2}dx.$

解：令 $x=\rho\cos\theta, y=\rho\sin\theta$，则积分区域为

$$E=\{(\rho,\theta)|\theta\in[\frac{\pi}{4},\frac{\pi}{2}],\rho\in[0,1]\}.$$

从而

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{2}}}dy\int_0^y e^{-x^2-y^2}dx+\int_{\frac{1}{\sqrt{2}}}^1dy\int_0^{\sqrt{1-y^2}}e^{-x^2-y^2}dx\\=&\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^1 e^{-\rho^2}\cdot\rho d\rho\\=&\frac{\pi}{8}(1-e^{-1}).\end{aligned}$$

(7) $\iint\limits_{D}f(x,y)dxdy$,$f(x,y)=\begin{cases}1, & |x|+|y|\le 1,\\ 2, & 1<|x|+|y|\le 2,\end{cases}$
$D=\{(x,y)||x|+|y|\le 2\}.$

解：令 $D_1=\{(x,y)||x|+|y|\le 1\}$，$D_2=\{(x,y)|1<|x|+|y|\le 2\}.$

则

$$\iint\limits_{D}f(x,y)dxdy=\iint\limits_{D_1}1dxdy+\iint\limits_{D_2}2dxdy=1\cdot 2+2\cdot 6=14.$$

13. 求下列曲线所围图形的面积.

(1) 双纽线 $(x^2+y^2)^2=2a^2(x^2-y^2)$ 与圆 $x^2+y^2=a^2$ 所围图形 (圆外部分) 的面积.

解： 令 $x=\rho\cos\theta, y=\rho\sin\theta$，则要求面积的区域为

$$E=\{(\rho,\theta)|\theta\in[-\frac{\pi}{6},\frac{\pi}{6}]\cup[\frac{5}{6}\pi,\frac{7}{6}\pi],\rho\in[a,a\sqrt{2\cos2\theta}]\}.$$

从而所求面积为

$$\begin{aligned}&\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta\int_a^{a\sqrt{2\cos2\theta}}\rho d\rho+\int_{\frac{5}{6}\pi}^{\frac{7}{6}\pi}d\theta\int_a^{a\sqrt{2\cos2\theta}}\rho d\rho\\=&2a^2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}(2\cos2\theta-1)d\theta\\=&(\sqrt{3}-\frac{\pi}{3})a^2.\end{aligned}$$

(3) 心脏线 $r=a(1+\cos\theta)$ 与圆 $x^2+y^2=\sqrt{3}ay$ 所围图形 (心脏线内部) 的面积 (a>0).

解： 令 $x=r\cos\theta, y=r\sin\theta$，考查两条曲线的交点.
当 $\theta\in[0,\frac{\pi}{3}]$ 时，所围图形取 $r\in[0,\sqrt{3}a\sin\theta]$；
当 $\theta\in[\frac{\pi}{3},\pi]$ 时，所围图形取 $r\in[0,a(1+\cos\theta)]$.
从而所求面积为

$$\begin{aligned}&\frac{1}{2}\int_0^{\frac{\pi}{3}}3a^2\sin^2\theta d\theta+\frac{1}{2}\int_{\frac{\pi}{3}}^{\pi}a^2(1+\cos\theta)^2d\theta\\=&\frac{3}{2}a^2(\frac{\pi}{6}-\frac{\sqrt{3}}{8})+\frac{a^2}{2}(\frac{2}{3}\pi-\sqrt{3}+\frac{\pi}{3}-\frac{\sqrt{3}}{8})\\=&(\frac{3}{4}\pi-\frac{3}{4}\sqrt{3})a^2.\end{aligned}$$

14. 通过适当的变量代换，计算下列二重积分.

(1) $\iint\limits_D x^2y^2dxdy$，$D$ 是由 $xy=2, xy=4, y=x, y=3x$ 在第一象限所围成的平面区域.

解： 令 $u=xy, v=\dfrac{y}{x}$，则 $\det\dfrac{\partial(u,v)}{\partial(x,y)}=\dfrac{2y}{x}=2v$，且积分区域为

$$E=\{(u,v):u\in[2,4],v\in[1,3]\}$$

从而

$$\begin{aligned}\iint\limits_D x^2y^2dxdy &= \int_2^4 du\int_1^3\frac{u^2}{2v}dv\\ &=\int_2^4 u^2\cdot\frac{1}{2}ln3du\\ &=\frac{28}{3}ln3.\end{aligned}$$

(2) $\iint\limits_D (x^2+y^2)dxdy$，$D$ 是由 $xy=1, xy=2, x^2-y^2=1, x^2-y^2=2$ 所围成的平面区域.

解： 令 $u=xy, v=x^2-y^2$，则 $\det\dfrac{\partial(u,v)}{\partial(x,y)}=-2(x^2+y^2)$，且积分区域为

$$E=\{(u,v):u\in[1,2],v\in[1,2]\}$$

从而

$$\iint\limits_D (x^2+y^2)dxdy=\int_1^2 du\int_1^2\frac{1}{2}dv=\frac{1}{2}.$$

(3) $\iint\limits_D (x^2+y^2)dxdy$，$D=\{(x,y)||x|+|y|\le 1\}$.

解： 令 $u=x-y, v=x+y$，则 $\det\dfrac{\partial(u,v)}{\partial(x,y)}=2$，且积分区域为

$$E=\{(u,v):u\in[-1,1],v\in[-1,1]\}$$

从而

$$\iint\limits_D (x^2+y^2)dxdy=\int_{-1}^{1}du\int_{-1}^{1}\frac{u^2+v^2}{4}dv=\frac{2}{3}.$$

(4) $\iint\limits_D (x-y^2)dxdy$，$D$ 是由 $y=2, y^2-y-x=1, y^2+2y-x=2$ 所围成的平面区域.

解： 令 $u=y^2-x$，则 $\det\dfrac{\partial(u,y)}{\partial(x,y)}=-1$，且积分区域为

$$E=\{(u,y):y\in[\frac{1}{3},2],u\in[2-2y,y+1]\}$$

从而

$$\begin{aligned}\iint\limits_D (x-y^2)dxdy&=\int_{\frac{1}{3}}^{2}dy\int_{2-2y}^{y+1}(-u)du\\&=\int_{\frac{1}{3}}^{2}\frac{3}{2}y^2-5y+\frac{3}{2}dy\\&=-\frac{175}{54}.\end{aligned}$$

17. 设函数 $f(t)$ 连续，$f(t)>0$. 求积分 $\iint\limits_{x^2+y^2\le R^2}\dfrac{af(x)+bf(y)}{f(x)+f(y)}dxdy$.

解：由于积分区域关于 x, y 对称，故

$$\begin{aligned}\iint\limits_{x^2+y^2\leq R^2} \frac{af(x)+bf(y)}{f(x)+f(y)}dxdy &= \iint\limits_{x^2+y^2\leq R^2} \frac{af(y)+bf(x)}{f(x)+f(y)}dxdy \\ &= \frac{1}{2}(\iint\limits_{x^2+y^2\leq R^2} \frac{af(x)+bf(y)}{f(x)+f(y)}dxdy + \iint\limits_{x^2+y^2\leq R^2} \frac{af(y)+bf(x)}{f(x)+f(y)}dxdy) \\ &= \frac{1}{2}\iint\limits_{x^2+y^2\leq R^2} (a+b)dxdy \\ &= \frac{1}{2}\pi R^2(a+b).\end{aligned}$$

18. 设函数 $f(t,s)$ 连续，求 $F(x) = \int_0^x \int_{t^2}^{x^2} f(t,s)dsdt$ 的导函数.

解：$\int_{t^2}^{x^2} f(t,s)ds$ 是一个关于 t, s 的函数，设为 $g(t,x)$，则

$$\begin{aligned}F'(x) &= (\int_0^x g(t,x)dt)' \\ &= g(x,x) + \int_0^x \frac{\partial g(t,x)}{\partial x}dt \\ &= \int_0^x 2x \cdot f(t,x^2)dt \\ &= 2x\int_0^x f(t,x^2)dt.\end{aligned}$$