1. (5) 解: $r_u' = (\cos v, \sin v, 0)$, $r_v' = (-u\sin v, u\cos v, a)$.

当 $(u,v)=(u_0,v_0)$ 时, $r(u_0,v_0) = (u_0\cos v_0, u_0\sin v_0, av_0)$,

$r_u'(u_0,v_0) = (\cos v_0, \sin v_0, 0)$, $r_v'(u_0,v_0) = (-u_0\sin v_0, u_0\cos v_0, a)$.

法向量 $\vec{n} \parallel (r_u' \times r_v') = \det\begin{pmatrix} i & j & k \\ \cos v_0 & \sin v_0 & 0 \\ -u_0\sin v_0 & u_0\cos v_0 & a \end{pmatrix} = (a\sin v_0, -a\cos v_0, u_0)$.

因此切平面方程为 $(x-u_0\cos v_0, y-u_0\sin v_0, z-av_0)\cdot\vec{n}=0$.

整理后得 $a\sin v_0\, x - a\cos v_0\, y + u_0 z - au_0v_0 = 0$.

法线方程为 $\begin{cases} x = u_0\cos v_0 + a\sin v_0\, t, \\ y = u_0\sin v_0 - a\cos v_0\, t, \\ z = av_0 + u_0 t. \end{cases}$

(6) 解: $r_u' = (1, 2u, 3u^2)$, $r_v' = (1, 2v, 3v^2)$.

当 $(u,v)=(1,2)$ 时, $r(1,2)=(3,5,9)$, $r_u'=(1,2,3)$, $r_v'=(1,4,12)$.

法向量 $\vec{n} \parallel (r_u' \times r_v') = \det\begin{pmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 4 & 12 \end{pmatrix} = (12,-9,2)$.

因此切平面方程为 $(x-3, y-5, z-9)\cdot\vec{n}=0$.

整理后得 $12x-9y+2z-9=0$.

~~法向量~~ 法线方程为 $\begin{cases} x = 3+12t, \\ y = 5-9t, \\ z = 9+2t. \end{cases}$

2. 解: 设 $P=(x_0,y_0,z_0)$, $F(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$.

则过 P 点的法向量为 $\mathrm{grad}F(x_0,y_0,z_0) = \left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2}\right)$.

由题知该法向量与坐标轴正方向成等角.

故 $\frac{2x_0}{a^2} = \frac{2y_0}{b^2} = \frac{2z_0}{c^2}$, 结合 $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}-1=0$.

解得 ~~$P=(\frac{\sqrt{3}}{3}a, \frac{\sqrt{3}}{3}b, \frac{\sqrt{3}}{3}c)$ 或 $(-\frac{\sqrt{3}}{3}a, -\frac{\sqrt{3}}{3}b, -\frac{\sqrt{3}}{3}c)$~~.

$P=\left(\frac{a^2}{\sqrt{a^2+b^2+c^2}}, \frac{b^2}{\sqrt{a^2+b^2+c^2}}, \frac{c^2}{\sqrt{a^2+b^2+c^2}}\right)$ 或 $\left(\frac{-a^2}{\sqrt{a^2+b^2+c^2}}, \frac{-b^2}{\sqrt{a^2+b^2+c^2}}, \frac{-c^2}{\sqrt{a^2+b^2+c^2}}\right)$.

3. 解：设$F(x,y,z)=x^2+2y^2+3z^2-21$.

设该曲面上过点$P=(x_0,y_0,z_0)$的切平面S_1平行于$S_2: x+4y+6z=0$.

则过点P的法向量为$\vec{n}=\mathrm{grad}F(x_0,y_0,z_0)=(2x_0,4y_0,6z_0)$.

切平面S_1的方程为 $(x-x_0,y-y_0,z-z_0)\cdot\vec{n}=0$.

整理后得 $x_0x+2y_0y+3z_0z-x_0^2-2y_0^2-3z_0^2=0$.

由$S_1 // S_2$知 $\begin{cases} x_0y_0z_0\neq 0, \\ x_0:2y_0:3z_0=1:4:6, \\ x_0^2+2y_0^2+3z_0^2=21. \end{cases}$

解得 $\begin{cases} x_0=1, \\ y_0=2, \\ z_0=2. \end{cases}$ 或 $\begin{cases} x_0=-1, \\ y_0=-2, \\ z_0=-2. \end{cases}$ 故所求切平面方程为：$x+4y+6z-21=0$

或 $x+4y+6z+21=0$.

5. 解：令$F(x,y,z)=x^2+y^2+z^2-6$, $G(x,y,z)=x+y+z$.

则$\mathrm{grad}F(1,-2,1)=(2,-4,2)$，$\mathrm{grad}G(1,-2,1)=(1,1,1)$.

曲线l在点$P(1,-2,1)$的切向量为

$$\vec{v}=\mathrm{grad}F(P)\times\mathrm{grad}G(P)=\begin{pmatrix} i & j & k \\ 2 & -4 & 2 \\ 1 & 1 & 1 \end{pmatrix}=(-6,0,6).$$

故曲线l在点P的切线方程为 $\begin{cases} x=1-6t, \\ y=-2, \\ z=1+6t. \end{cases}$

法平面方程为 $(x-1,y+2,z-1)\cdot\vec{v}=0$

整理得 $x-z=0$.

6. 证明：螺旋线上任一点$P=(a\cos t_0, a\sin t_0, bt_0)$处

的切向量设为$\vec{n_1}=r'(t_0)=(-a\sin t_0, a\cos t_0, b)$，$\vec{n_2}$为z轴正方向的单位向量.

则$\vec{n_1}$与$\vec{n_2}$的夹角$\theta=\arccos\left(\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}\right)=\arccos\frac{b}{\sqrt{a^2+b^2}}$.

由P的任意性知螺旋线的切线与z轴形成定角.