第5章 Riemann积分

学习材料（11）

1 Riemann积分概念及Riemann积分存在条件

2 Riemann积分的性质

3 变上限积分与原函数的存在性

4 不定积分

4.1 不定积分概念

4.2 换元积分法

4.3 分部积分法

4.4 分式函数的积分

4.5 三角函数有理式的积分

4.6 简单无理式的积分

记$R(u,v)$表示由u,v和常数进行有限次的四则运算后所得的表达式。

$$\int R\left(x,\sqrt{1-x^2}\right)dx \xlongequal[\substack{\text{令}x=\sin t,\\ t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}]{} \int \cos t\cdot R(\sin t,\cos t)dt$$

$$\int R\left(x,\sqrt{x^2+1}\right)dx \xlongequal[\substack{\text{令}x=\tan t,\\ t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}]{} \int \frac{1}{\cos^2 t}\cdot R\left(\frac{\sin x}{\cos x},\frac{1}{\cos t}\right)dt$$

$$\int R\left(x,\sqrt{x^2-1}\right)dx \xlongequal[\substack{\text{令}x=\sec t,\\ t\in\left(0,\frac{\pi}{2}\right)}]{} \int \frac{\sin t}{\cos^2 t}\cdot R\left(\frac{1}{\cos x},\frac{\sin t}{\cos t}\right)dt.$$

$$\int R(x,\sqrt[n]{ax+b})dx \xlongequal[\substack{\text{令}\sqrt[n]{ax+b}=t,\\ \text{即}x=\frac{t^n-b}{a}}]{} \int \frac{n}{a}t^{n-1}\cdot R\left(\frac{t^n-b}{a},t\right)dt$$

$$\int R\left(x,\sqrt[n]{\frac{ax+b}{cx+d}}\right)dx \xlongequal[\substack{\text{令}\sqrt[n]{\frac{ax+b}{cx+d}}=t,\\ \text{即}x=\frac{dt^n-b}{a-ct^n}}]{} \int \frac{n(ad-bc)t^{n-1}}{(a-ct^n)^2}\cdot R\left(\frac{dt^n-b}{a-ct^n},t\right)dt$$

5 定积分的计算

定理1 设$f \in C[a,b]$，$\varphi : [\alpha,\beta] \to [a,b]$可导，且$\varphi' \in R[\alpha,\beta]$，则

$$\int_\alpha^\beta f(\varphi(x))\varphi'(x)dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du.$$

注1 若$\varphi(\alpha) > \varphi(\beta)$，规定

$$\int_{(\alpha)}^{(\beta)} f(u)du = -\int_{\varphi(\beta)}^{\varphi(\alpha)} f(u)du.$$

证：由$f \in C[a,b]$知f在区间$[a,b]$上有原函数F，于是当$\varphi(\alpha) < \varphi(\beta)$时，

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du &= F(u)|_{\varphi(\alpha)}^{\varphi(\beta)} \quad \text{（Newton-Leibniz公式）} \\ &= F(\varphi(\beta)) - F(\varphi(\alpha));\end{aligned}$$

当$\varphi(\alpha) = \varphi(\beta)$时，

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du &= 0 \\ &= F(\varphi(\beta)) - F(\varphi(\alpha));\end{aligned}$$

而当$\varphi(\alpha) > \varphi(\beta)$时，

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du &= -\int_{\varphi(\beta)}^{\varphi(\alpha)} f(u)du \\ &= -F(u)|_{\varphi(\beta)}^{\varphi(\alpha)} \quad \text{（Newton-Leibniz公式）} \\ &= F(\varphi(\beta)) - F(\varphi(\alpha)).\end{aligned}$$

综上，

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du = F(\varphi(\beta)) - F(\varphi(\alpha)).$$

另一方面，由$f \in C[a,b]$，$\varphi : [\alpha,\beta] \to [a,b]$可导知$f \circ \varphi \in C[\alpha,\beta]$；易知函数

$$F \circ \varphi : [\alpha,\beta] \to R$$

是函数$(f \circ \varphi) \cdot \varphi' : [\alpha,\beta] \to R$的一个原函数，故由Newton-Leibniz公式，有

$$\begin{aligned}\int_\alpha^\beta f(\varphi(x))\varphi'(x)dx &= F(\varphi(x))|_\alpha^\beta \\ &= F(\varphi(\beta)) - F(\varphi(\alpha)),\end{aligned}$$

所以

$$\int_\alpha^\beta f(\varphi(x))\varphi'(x)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du.$$

注2 称如下计算定积分$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx$ 的过程为<u>第一换元法</u>：

$$\begin{aligned}\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx &=\!=\!=\!=\!= \int_{\alpha}^{\beta} f(\varphi(x))d\varphi(x)\\ &\underset{\text{令 } \varphi(x)=u}{=\!=\!=\!=\!=} \int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du.\end{aligned}$$

称如下计算定积分$\int_{g(\alpha)}^{g(\beta)} f(x)dx$ 的过程为<u>第二换元法</u>：

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx &\underset{\text{令 } x=\varphi(t)}{=\!=\!=\!=\!=} \int_{\alpha}^{\beta} f(\varphi(t))d\varphi(t)\\ &=\!=\!=\!=\!= \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.\end{aligned}$$

定理2 设$f,g:[a,b]\to R$可导，且$f',g'\in R[a,b]$，则

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx.$$

证：由$f',g'\in R[a,b]$知，$f\cdot g'$，$f'\cdot g\in R[a,b]$，从而$[f\cdot g]'\in R[a,b]$，于是

$$\begin{aligned}\int_a^b f(x)g'(x)dx+\int_a^b f'(x)g(x)dx &=\!=\!=\!=\!= \int_a^b [f(x)g'(x)+f'(x)g(x)]dx \quad \text{（积分的线性性质）}\\ &=\!=\!=\!=\!= \int_a^b [f(x)g(x)]'dx\\ &=\!=\!=\!=\!= f(x)g(x)|_a^b \quad \text{（}[f\cdot g]'\in R[a,b]\text{，}f\cdot g\text{ 是}(f\cdot g)'\text{的一个原函数）},\end{aligned}$$

所以

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx.$$

注3 称如下计算定积分$\int_a^b f(x)g'(x)dx$的过程为<u>分部积分法</u>：

$$\begin{aligned}\int_a^b f(x)g'(x)dx &=\!=\!=\!=\!= \int_a^b f(x)dg(x)\\ &=\!=\!=\!=\!= f(x)g(x)|_a^b - \int_a^b g(x)df(x)\\ &=\!=\!=\!=\!= f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx.\end{aligned}$$

例1 求定积分$\int_0^{\frac{\pi}{4}}\frac{1}{\cos x}dx$，$\int_0^{\frac{1}{2}}\sqrt{1-x^2}dx$

解：

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\frac{1}{\cos x}dx === \int_0^{\frac{\pi}{4}}\frac{\cos x}{\cos^2 x}dx &=====\quad \int_0^{\frac{\pi}{4}}\frac{1}{1-\sin^2 x}d\sin x\\ &\underset{令\sin x=u}{=====}\quad \int_0^{\frac{\sqrt{2}}{2}}\frac{1}{1-u^2}du\\ &=====\quad \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)\bigg|_0^{\frac{\sqrt{2}}{2}}\\ &====\quad \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) == \ln\left(\sqrt{2}+1\right).\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{1}{2}}\sqrt{1-x^2}dx \quad &\underset{\substack{令x=\sin t,\\ t\in\left[0,\arcsin\frac{1}{2}\right]}}{=====}\quad \int_0^{\arcsin\frac{1}{2}}\cos t d\sin t\\ &=====\quad \int_0^{\arcsin\frac{1}{2}}\cos t\cos t dt\\ &=====\quad \int_0^{\arcsin\frac{1}{2}}\frac{1+\cos 2t}{2}dt\\ &=====\quad \left[\frac{t}{2}+\frac{\sin 2t}{4}\right]\bigg|_0^{\arcsin\frac{1}{2}} = \frac{\arcsin\frac{1}{2}}{2}+\frac{\sqrt{3}}{8}.\end{aligned}$$

例2 求定积分 1. $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$; 2. $\int_0^{\frac{\pi}{2}}\frac{\sin^p x}{\sin^p x+\cos^p x}dx$; 3. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin^2 x}{1+e^{-x}}dx$.

解：1.

$$\begin{aligned}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \quad &\underset{令x=\pi-t}{=====}\quad \int_{\pi}^{0}\frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}d(\pi-t)\\ &=====\quad \int_0^{\pi}\frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}dt\\ &=====\quad \int_0^{\pi}\frac{(\pi-t)\sin t}{1+\cos^2 t}dt,\end{aligned}$$

故

$$\begin{aligned}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx \quad &=====\quad \frac{1}{2}\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx\\ &=====\quad -\frac{\pi}{2}\arctan(\cos x)\Big|_0^{\pi} == \frac{\pi^2}{4}.\end{aligned}$$

2.

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin^p x}{\sin^p x+\cos^p x} dx &\xlongequal[\text{令} x=\frac{\pi}{2}-t]{=====} \int_{\frac{\pi}{2}}^{0} \frac{\sin^p\left(\frac{\pi}{2}-t\right)}{\sin^p\left(\frac{\pi}{2}-t\right)+\cos^p\left(\frac{\pi}{2}-t\right)} d\left(\frac{\pi}{2}-t\right) \\
&===== \int_0^{\frac{\pi}{2}} \frac{\sin^p\left(\frac{\pi}{2}-t\right)}{\sin^p\left(\frac{\pi}{2}-t\right)+\cos^p\left(\frac{\pi}{2}-t\right)} dt \\
&===== \int_0^{\frac{\pi}{2}} \frac{\cos^p t}{\cos^p t+\sin^p t} dt,
\end{aligned}$$

故

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin^p x}{\sin^p x+\cos^p x} dx &===== \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin^p x}{\sin^p x+\cos^p x} dx+\int_0^{\frac{\pi}{2}} \frac{\cos^p t}{\cos^p t+\sin^p t} dt\right] \\
&===== \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} dx\right] == \frac{\pi}{4}.
\end{aligned}$$

3.

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1+e^{-x}} dx &\xlongequal[\text{令} x=-t]{=====} \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\sin^2 t}{1+e^{t}} d(-t) \\
&===== \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 t}{1+e^{t}} dt,
\end{aligned}$$

故

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1+e^{-x}} dx &===== \frac{1}{2}\left[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1+e^{-x}} dx+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 t}{1+e^{t}} dt\right] \\
&===== \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\left[\frac{1}{1+e^{-x}}+\frac{1}{1+e^{x}}\right] dx \\
&===== \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx \\
&===== \underline{\int_0^{\frac{\pi}{4}} \sin^2 x dx} \\
&===== \left.\left(\frac{x}{2}-\frac{\sin 2x}{4}\right)\right|_0^{\frac{\pi}{4}} = \frac{\pi}{8}-\frac{1}{4}.
\end{aligned}$$

例3 求定积分$\int_0^{\frac{\pi}{2}} x\cos x dx$，$\int_0^{\frac{\pi}{2}} \cos^n x dx$，$\int_0^{\frac{\pi}{2}} \sin^n x dx \quad (n=2,3,\cdots)$.

解：

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x\cos x dx &===== \int_0^{\frac{\pi}{2}} x d\sin x \\
&===== x\sin x|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}} \sin x dx = \frac{\pi}{2}+\cos x|_0^{\frac{\pi}{2}} == \frac{\pi}{2}-1.
\end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx \xlongequal[\text{令 } x=\frac{\pi}{2}-t]{=====} \int_{\frac{\pi}{2}}^0 \sin^n\left(\frac{\pi}{2}-t\right) d\left(\frac{\pi}{2}-t\right)$$

$$\xlongequal{=====} \int_0^{\frac{\pi}{2}} \sin^n\left(\frac{\pi}{2}-t\right) dt \xlongequal{===} \int_0^{\frac{\pi}{2}} \cos^n t dt.$$

令$I_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n x dx \quad (n=1,2,\cdots)$，则当$n \geq 2$ 时，

$$\begin{aligned}
I_n &\xlongequal{=====} \int_0^{\frac{\pi}{2}} \cos^{n-1} x d\sin x \\
&\xlongequal{=====} \cos^{n-1} x \sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x d\cos^{n-1} x \\
&\xlongequal{=====} (n-1)\int_0^{\frac{\pi}{2}} \sin x\left[\cos^{n-2} x \sin x\right] dx \\
&\xlongequal{=====} (n-1)\int_0^{\frac{\pi}{2}} \left[\cos^{n-2} x - \cos^n x\right] dx \\
&\xlongequal{=====} (n-1)\left[I_{n-2} - I_n\right],
\end{aligned}$$

故

$$I_n = \frac{n-1}{n} I_{n-2},$$

所以

$$\begin{aligned}
I_{2k} &= \frac{2k-1}{2k} I_{2(k-1)} & I_{2k+1} &= \frac{2k}{2k+1} I_{2k-1} \\
&= \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2} I_{2(k-2)} & &= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1} I_{2k-3} \\
&= \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdots\cdot\frac{1}{2} I_0 & &= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\cdot\frac{2}{3} I_1 \\
&= \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdots\cdot\frac{1}{2}\cdot\frac{\pi}{2} & &= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\cdot\frac{2}{3} \\
&=: \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}, & &=: \frac{(2k)!!}{(2k+1)!!}.
\end{aligned}$$

例4 Taylor公式的积分余项形式：设f在$[a,b]$上有$n+1$阶连续导数，$x_0 \in [a,b]$。则$\forall x \in [a,b]$，有

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{1}{n!}\int_{x_0}^x (x-u)^n f^{(n+1)}(u) du.$$

证：当$n=1$时，根据Newton-Leibnitz公式，

$$f(x) = f(x_0) + \int_{x_0}^x f'(u) du.$$

应用分部积分法，

$$\int_{x_0}^{x} f'(u)du == \int_{x_0}^{x} f'(u)d(u-x) \quad ===== \quad (u-x)f'(u)|_{x_0}^{x} - \int_{x_0}^{x}(u-x)f''(u)du$$

$$===== \quad f'(x_0)(x-x_0) + \int_{x_0}^{x}(x-u)f''(u)du,$$

结论成立。假定当$n=m$时，结论成立，即

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + \frac{1}{m!}\int_{x_0}^{x}(x-u)^m f^{(m+1)}(u)du.$$

应用分部积分法，

$$\begin{aligned}
\frac{1}{m!}\int_{x_0}^{x}(x-u)^m f^{(m+1)}(u)du \quad &=== \quad -\frac{1}{(m+1)!}\int_{x_0}^{x} f^{(m+1)}(u)d(x-u)^{m+1} \\
&=== \quad -\frac{1}{(m+1)!}(x-u)^{m+1}f^{(m+1)}(u)\Big|_{x_0}^{x} + \frac{1}{(m+1)!}\int_{x_0}^{x}(x-u)^{m+1}f^{(m+2)}(u)du \\
&=== \quad \frac{1}{(m+1)!}f^{(m+1)}(x_0)(x-x_0)^{m+1} + \frac{1}{(m+1)!}\int_{x_0}^{x}(x-u)^{m+1}f^{(m+2)}(u)du.
\end{aligned}$$

故

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{1}{(m+1)!}f^{(m+1)}(x_0)(x-x_0)^{m+1} + \frac{1}{(m+1)!}\int_{x_0}^{x}(x-u)^{m+1}f^{(m+2)}(u)du.$$

所以当$n=m+1$时，结论成立。根据数学归纳法，结论对任何自然数n成立。

6 定积分的应用

6.1 平面图形的面积

例1 求由摆线$\begin{cases} x = a(t-\sin t), \\ y = a(1-\cos t) \end{cases}$ $t \in [0, 2\pi]$与x轴所围图形D的面积A，其中a为正常数。画图。

解：

$$\begin{aligned}
A \quad &===== \quad \int_0^{2\pi a} y(x)dx \\
&\underset{\text{令 } x=a(t-\sin t)}{=====} \quad \int_0^{2\pi} a(1-\cos t)da(t-\sin t) \\
&===== \quad \int_0^{2\pi} a^2(1-\cos t)^2 dt == \int_0^{2\pi} a^2\left[1-2\cos t+\cos^2 t\right]dt \\
&===== \quad a^2\left[2\pi+\pi\right] == 3\pi a^2.
\end{aligned}$$

注1 求由极坐标曲线方程$r=\varphi(\theta)$ $(\theta\in[\alpha,\beta])$与射线$\theta=\alpha$及射线$\theta=\beta$所围图形$D$的面积$A$的公式。

画图。

1. 分割区间$[\alpha,\beta]$. 在区间$[\alpha,\beta]$中以任意方式插入一组点

$$\alpha=\theta_0<\theta_1<\cdots<\theta_{n-1}<\theta_n=\beta,$$

将区间$[\alpha,\beta]$分割为若干个子区间$[\theta_{i-1},\theta_i]$ $(i=1,2,\cdots,n)$. 此时射线$\theta=\theta_i$ $(i=0,1,2,\cdots,n)$将图形D分成n个小块D_i $(i=1,2,\cdots,n)$.

2. 局部近似. 在子区间$[\theta_{i-1},\theta_i]$上任取一点ξ_i，则小块D_i的面积A_i近似等于以$r(\xi_i)$为半径、以$\theta_i-\theta_{i-1}$为角度的扇形的面积，即

$$A_i\approx\frac{1}{2}r^2(\xi_i)(\theta_i-\theta_{i-1}),$$

从而图形D的面积A近似地表示为

$$A\approx\sum_{i=1}^{n}\frac{1}{2}r^2(\xi_i)(\theta_i-\theta_{i-1}).$$

3. 取极限. 当$\max\limits_{1\le i\le n}\{\theta_i-\theta_{i-1}\}$越来越小时，$\sum\limits_{i=1}^{n}\frac{1}{2}r^2(\xi_i)(\theta_i-\theta_{i-1})$越来接近于$A$，即

$$A=\int_\alpha^\beta\frac{1}{2}r^2(\theta)d\theta.$$

例2 求由极坐标曲线方程$r=1+\cos\theta$ $(\theta\in[0,2\pi])$所围图形D的面积A.

解：画图，称极坐标曲线方程$r=1+\cos\theta$ $(\theta\in[0,2\pi])$为心脏线。

$$\begin{aligned}A &===== \int_0^{2\pi}\frac{1}{2}\varphi^2(\theta)d\theta\\ &===== \int_0^{2\pi}\frac{1}{2}(1+\cos\theta)^2d\theta\\ &===== \frac{1}{2}\int_0^{2\pi}\left[1+2\cos t+\cos^2\theta\right]d\theta\\ &===== \frac{1}{2}\left[2\pi+\pi\right]==\frac{3\pi}{2}.\end{aligned}$$

6.2 旋转体的体积

设D是由曲线$y=f(x)$ $(x\in[a,b])$和直线$y=0$，$x=a$，$x=b$所围的图形，其中$f\ge0$。画图。
求将D绕x轴旋转所得图形D_x体积V_x的方法。画图。

1. 分割区间$[a,b]$. 在区间$[a,b]$中以任意方式插入一组点

$$a=x_0<x_1<\cdots<x_{n-1}<x_n=b,$$

将区间$[a,b]$分割为若干个子区间$[x_{i-1},x_i]$ $(i=1,2,\cdots,n)$. 此时平面$x=x_i$ $(i=0,1,2,\cdots,n)$将图形D_x分成n个小块$D_{x,i}$ $(i=1,2,\cdots,n)$.

2. 局部近似. 在子区间$[x_{i-1},x_i]$上任取一点ξ_i，则小块$D_{x,i}$的体积$V_{x,i}$近似等于以$f(\xi_i)$为半径、以x_i-x_{i-1}为长的圆柱体的体积，即

$$V_{x,i}\approx \pi f^2(\xi_i)(x_i-x_{i-1}),$$

从而图形D_x的体积V_x近似地表示为

$$V_x\approx \sum_{i=1}^{n}\pi f^2(\xi_i)(x_i-x_{i-1}).$$

3. 取极限. 当$\max\limits_{1\le i\le n}\{x_i-x_{i-1}\}$越来越小时，$\sum\limits_{i=1}^{n}\pi f^2(\xi_i)(\theta_i-\theta_{i-1})$越来接近于$V_x$，即

$$V_x=\int_a^b \pi f^2(x)dx.$$

例1 设D是由曲线$y=\sqrt{x}$ $(x\in[0,3])$和直线$y=0$，$x=3$所围的图形，求将D绕x轴旋转所得图形D_x体积V_x.

解：

$$\begin{aligned} V_x &===== \int_0^3 \pi y^2(x)dx \\ &===== \int_0^3 \pi[\sqrt{x}]^2dx \\ &===== \left.\frac{\pi x^2}{2}\right|_0^3 == \frac{9\pi}{2}. \end{aligned}$$

6.3 曲线的弧长

设曲线L有参数表示

$$L:\begin{cases} x=f(t), \\ y=g(t) \end{cases} \quad t\in[\alpha,\beta].$$

现定义曲线L的弧长。对区间$[\alpha,\beta]$进行分割

$$T:\ \alpha=t_0<t_1<\cdots<t_{n-1}<t_n=\beta,$$

记$M_i=\begin{pmatrix} f(t_i) \\ g(t_i) \end{pmatrix}$ $(i=0,1,2,\cdots,n)$，从而得到折线$\overline{M_0M_1\cdots M_n}$. 记$|\overline{M_0M_1\cdots M_n}|$为折线$\overline{M_0M_1\cdots M_n}$的长度，则

$$|\overline{M_0M_1\cdots M_n}|=\sum_{i=1}^{n}\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}.$$

定义1 若极限

$$\lim_{\lambda(T)\to 0}\sum_{i=1}^{n}\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}$$

存在，即如果存在实数l，使得$\forall\varepsilon>0$，$\exists\delta>0$，当$[\alpha,\beta]$的分割T

$$\pi:\alpha=t_0<t_1<\cdots<t_n=\beta$$

满足$\lambda(T)<\delta$时，就有

$$\left|\sum_{i=1}^{n}\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}-l\right|<\varepsilon,$$

则称曲线L是<u>可求长的</u>，其<u>弧长</u>为l.

定理1 设曲线L的参数方程

$$L:\begin{cases} x=f(t), \\ y=g(t), \end{cases}\quad t\in[\alpha,\beta]$$

满足$f,g:[\alpha,\beta]\to\mathcal{R}$可导，且$f',g'\in R[\alpha,\beta]$，则$L$是可求长曲线，且$L$的弧长为

$$\int_{\alpha}^{\beta}\sqrt{[f'(t)]^2+[g'(t)]^2}dt.$$

证：由$f',g'\in R[\alpha,\beta]$知，$\sqrt{[f']^2+[g']^2}\in R[\alpha,\beta]$. 下用定义证明$L$ 的弧长为$\int_{\alpha}^{\beta}\sqrt{[f'(t)]^2+[g'(t)]^2}dt$. $\forall\varepsilon>0$，$\exists\delta>0$，当区间$[\alpha,\beta]$的分割

$$T:\ \alpha=t_0<t_1<\cdots<t_{n-1}<t_n=\beta$$

满足$\lambda(T)<\delta$时，

$$\left|\sum_{i=1}^{n}\sqrt{[f'(\xi_i)]^2+[g'(\xi_i)]^2}(t_i-t_{i-1})-\int_{\alpha}^{\beta}\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|<\frac{\varepsilon}{2}\quad\forall\xi_i\in[t_{i-1},t_i]\ (i=1,2,\cdots,n),$$

$$\sum_{i=1}^{n}\sup_{\xi,\eta\in[t_{i-1},t_i]}|g'(\xi)-g'(\eta)|\,(t_i-t_{i-1})<\frac{\varepsilon}{2},$$

于是

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}-\int_{\alpha}^{\beta}\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|\\
=&\left|\sum_{i=1}^{n}\sqrt{[f'(\xi_i^*)(t_i-t_{i-1})]^2+[g'(\eta_i^*)(t_i-t_{i-1})]^2}-\int_{\alpha}^{\beta}\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|\\
&\qquad(\text{Lagrange微分中值定理，}\exists\xi_i^*,\eta_i^*\in(t_{i-1},t_i))\\
\le&\sum_{i=1}^{n}\underline{\left|\sqrt{[f'(\xi_i^*)]^2+[g'(\eta_i^*)]^2}-\sqrt{[f'(\xi_i^*)]^2+[g'(\xi_i^*)]^2}\right|}(t_i-t_{i-1})\\
&+\left|\sum_{i=1}^{n}\sqrt{[f'(\xi_i^*)]^2+[g'(\xi_i^*)]^2}(t_i-t_{i-1})-\int_{\alpha}^{\beta}\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|\\
\le&\sum_{i=1}^{n}\underline{|g'(\eta_i^*)-g'(\xi_i^*)|}(t_i-t_{i-1})+\frac{\varepsilon}{2}\qquad(\,|\,|\vec{a}|-|\vec{b}|\,|\le|\vec{a}-\vec{b}|\,)\\
<&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}$$

故L是可求长曲线，且L的弧长为

$$\int_\alpha^\beta \sqrt{[f'(t)]^2+[g'(t)]^2}dt.$$

注1 若曲线L的方程为

$$L: y=g(x), \quad x\in[a,b],$$

其中$g:[a,b]\to\mathcal{R}$可导，且$g'\in R[a,b]$，则L的弧长为

$$\int_a^b \sqrt{1+[g'(x)]^2}dx.$$

注2 若曲线L的极坐标下的方程为$r=\varphi(\theta)\ (\theta\in[\alpha,\beta])$，即$L$的参数方程为

$$L:\begin{cases} x=\varphi(\theta)\cos\theta, \\ y=\varphi(\theta)\sin\theta, \end{cases} \quad \theta\in[\alpha,\beta]$$

其中$\varphi:[\alpha,\beta]\to\mathcal{R}_+$可导，且$\varphi'\in R[\alpha,\beta]$，则$L$的弧长为

$$\int_\alpha^\beta \sqrt{[\varphi'(\theta)\cos\theta-\varphi(\theta)\sin\theta]^2+[\varphi'(\theta)\sin\theta+\varphi(\theta)\cos\theta]^2}d\theta,$$

即

$$\int_\alpha^\beta \sqrt{[\varphi'(\theta)]^2+[\varphi(\theta)]^2}d\theta.$$

例1 求极坐标下曲线$r=1+\cos\theta\ (\theta\in[0,2\pi])$的弧长$l$.

解：

$$\begin{aligned} l &= 2\int_0^\pi \sqrt{[\sin\theta]^2+[1+\cos\theta]^2}d\theta. \\ &= 2\int_0^\pi \sqrt{2+2\cos\theta}d\theta \\ &= 2\int_0^\pi 2\cos\frac{\theta}{2}d\theta == 8\sin\frac{\theta}{2}\Big|_0^\pi = 8. \end{aligned}$$

6.4 旋转面的面积

例1 求圆台的侧面积。如图，令线段绕x轴旋转，求该旋转面的面积A.

解：由几何关系知

$$x\theta=2\pi r,\ (x+l)\theta=2\pi R,$$

故

$$
\begin{aligned}
A &= \frac{1}{2}(x+l)^2\theta - \frac{1}{2}x^2\theta \\
&= \frac{1}{2}\,l(2x+l)\theta \\
&= \pi\,(r+R)\,l.
\end{aligned}
$$

现定义简单曲线L绕x轴旋转所得旋转面的面积。设简单曲线L有参数表示

$$
L:\begin{cases} x=f(t), \\ y=g(t) \end{cases} \quad t\in[\alpha,\beta],
$$

其中$g(t)\geq 0,\ \forall t\in[\alpha,\beta]$. 对区间$[\alpha,\beta]$进行分割

$$
T:\ \alpha=t_0<t_1<\cdots<t_{n-1}<t_n=\beta,
$$

记$M_i=\begin{pmatrix} f(t_i) \\ g(t_i) \end{pmatrix}$ $(i=0,1,2,\cdots,n)$，从而得到折线$\overline{M_0M_1\cdots M_n}$. 则此折线绕$x$轴旋转所得旋转面的面积为$n$个圆台的侧面积之和，即

$$
\sum_{i=1}^{n}\pi[g(t_{i-1})+g(t_i)]\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}.
$$

定义1 若极限

$$
\lim_{\lambda(T)\to 0}\sum_{i=1}^{n}\pi[g(t_{i-1})+g(t_i)]\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}
$$

存在，即如果存在实数A，使得$\forall\varepsilon>0$，$\exists\delta>0$，当$[\alpha,\beta]$的分割T

$$
T:\alpha=t_0<t_1<\cdots<t_n=\beta
$$

满足$\lambda(T)<\delta$时，就有

$$
\left|\sum_{i=1}^{n}\pi[g(t_{i-1})+g(t_i)]\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}-A\right|<\varepsilon,
$$

则称曲线L绕x轴旋转所得旋转面的面积为A.

定理1 设简单曲线L的参数方程

$$
L:\begin{cases} x=f(t), \\ y=g(t), \end{cases} \quad t\in[\alpha,\beta]
$$

满足$f,g:[\alpha,\beta]\to\mathcal{R}$可导，$f',g'\in R[\alpha,\beta]$，且$g(t)\geq 0,\ \forall t\in[\alpha,\beta]$，则曲线$L$绕$x$轴旋转所得旋转面的面积为

$$
\int_{\alpha}^{\beta}2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}dt.
$$

证：由$f',g'\in R[\alpha,\beta]$和第二节命题3知$\sqrt{[f']^2+[g']^2}\in R[\alpha,\beta]$，从而$g\sqrt{[f']^2+[g']^2}\in R[\alpha,\beta]$. 下用定义证明曲线$L$绕$x$轴旋转所得旋转面的面积为

$$\int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}dt.$$

$\forall\varepsilon>0$，$\exists\delta>0$，当区间$[\alpha,\beta]$的分割T

$$T:\ \alpha=t_0<t_1<\cdots<t_{n-1}<t_n=\beta$$

满足$\lambda(T)<\delta$时，

$$\left|\sum_{i=1}^n 2\pi g(\xi_i)\sqrt{[f'(\xi_i)]^2+[g'(\xi_i)]^2}(t_i-t_{i-1})-\int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|<\frac{\varepsilon}{?2}\quad \forall\xi_i\in[t_{i-1},t_i]\ (i=1,2,\cdots,n),$$

$$\sum_{i=1}^n \sup_{\xi,\eta\in[t_{i-1},t_i]}|g'(\xi)-g'(\eta)|(t_i-t_{i-1})<\frac{\varepsilon}{?2M+1},$$

其中$M=:2\pi\left[\sup_{\zeta\in[\alpha,\beta]}|g(\zeta)|+\sup_{\xi\in[\alpha,\beta]}\sqrt{[f'(\xi)]^2+[g'(\xi)]^2}\right]$, 于是

$$\begin{aligned}
&\left|\sum_{i=1}^n \pi[g(t_{i-1})+g(t_i)]\sqrt{[f(t_i)-f(t_{i-1})]^2+[g(t_i)-g(t_{i-1})]^2}-\int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|\\
=&\left|\sum_{i=1}^n 2\pi g(\zeta_i^*)\sqrt{[f'(\xi_i^*)(t_i-t_{i-1})]^2+[g'(\eta_i^*)(t_i-t_{i-1})]^2}-\int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|\\
&\qquad(\text{连续函数的介值定理，Lagrange微分中值定理，}\exists\zeta_i^*\in[t_{i-1},t_i],\ \xi_i^*,\eta_i^*\in(t_{i-1},t_i))\\
\le&\sum_{i=1}^n 2\pi|g(\zeta_i^*)|\underline{\left|\sqrt{[f'(\xi_i^*)]^2+[g'(\eta_i^*)]^2}-\sqrt{[f'(\xi_i^*)]^2+[g'(\xi_i^*)]^2}\right|}(t_i-t_{i-1})\\
&+\sum_{i=1}^n 2\pi|g(\zeta_i^*)-g(\xi_i^*)|\sqrt{[f'(\xi_i^*)]^2+[g'(\xi_i^*)]^2}(t_i-t_{i-1})\\
&+\left|\sum_{i=1}^n 2\pi g(\xi_i^*)\sqrt{[f'(\xi_i^*)]^2+[g'(\xi_i^*)]^2}(t_i-t_{i-1})-\int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}dt\right|\\
\le&\sum_{i=1}^n 2\pi|g(\zeta_i^*)|\ \underline{|g'(\eta_i^*)-g'(\xi_i^*)|}(t_i-t_{i-1})\qquad\qquad(|\,|\vec{a}|-|\vec{b}|\,|\le|\vec{a}-\vec{b}|)\\
&+\sum_{i=1}^n 2\pi|g(\zeta_i^*)-g(\xi_i^*)|\sqrt{[f'(\xi_i^*)]^2+[g'(\xi_i^*)]^2}(t_i-t_{i-1})+\frac{\varepsilon}{2}\\
\le& M\sum_{i=1}^n \sup_{\xi,\eta\in[t_{i-1},t_i]}|g'(\xi)-g'(\eta)|(t_i-t_{i-1})+\frac{\varepsilon}{2}\\
<&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}$$

故曲线L绕x轴旋转所得旋转面的面积为

$$\int_\alpha^\beta 2\pi g(t)\sqrt{[f'(t)]^2+[g'(t)]^2}dt.$$

注1 若曲线L的方程为
$$L: y=g(x),\quad x\in[a,b],$$
其中$g:[a,b]\to\mathcal{R}_+$可导，且$g'\in R[a,b]$，则曲线L绕x轴旋转所得旋转面的面积为
$$\int_a^b 2\pi g(x)\sqrt{1+[g'(x)]^2}dx.$$

注2 若曲线L的极坐标下的方程为$r=\varphi(\theta)$ $(\theta\in[\alpha,\beta])$，即$L$的参数方程为
$$L:\begin{cases} x=\varphi(\theta)\cos\theta, \\ y=\varphi(\theta)\sin\theta, \end{cases}\quad \theta\in[\alpha,\beta]$$
其中$\varphi:[\alpha,\beta]\to\mathcal{R}_+$可导，$\varphi'\in R[\alpha,\beta]$，且$\varphi(\theta)\sin\theta\ge 0,\ \forall\theta\in[\alpha,\beta]$，则曲线$L$绕极轴旋转所得旋转面的面积为
$$\int_\alpha^\beta 2\pi\varphi(\theta)\sin\theta\sqrt{[\varphi'(\theta)\cos\theta-\varphi(\theta)\sin\theta]^2+[\varphi'(\theta)\sin\theta+\varphi(\theta)\cos\theta]^2}d\theta,$$
即
$$\int_\alpha^\beta 2\pi\varphi(\theta)\sin\theta\sqrt{[\varphi'(\theta)]^2+[\varphi(\theta)]^2}d\theta.$$

例1 求极坐标下曲线$L: r=1+\cos\theta$ $(\theta\in[0,\pi])$绕极轴旋转所得旋转面的面积A.

解：
$$\begin{aligned} A &= \int_0^\pi 2\pi(1+\cos\theta)\sin\theta\sqrt{[\sin\theta]^2+[1+\cos\theta]^2}d\theta \\ &= \int_0^\pi 2\pi(1+\cos\theta)\sin\theta\sqrt{2(1+\cos\theta)}d\theta \\ &= -\int_0^\pi 2\pi(1+\cos\theta)\sqrt{2(1+\cos\theta)}d(1+\cos\theta) \\ &= -2\sqrt{2}\pi\cdot\frac{2}{5}(1+\cos\theta)^{\frac{5}{2}}\Big|_0^\pi == \frac{32\pi}{5}. \end{aligned}$$

6.5 曲率半径

为了刻画曲线的弯曲程度，我们引入曲率的概念。设曲线L有参数表示
$$L:\begin{cases} x=f(t), \\ y=g(t) \end{cases}\quad t\in[\alpha,\beta],$$
其中$f,g:[\alpha,\beta]\to\mathcal{R}$可导，且$f',g'\in R[\alpha,\beta]$. 则曲线在参数$t_0$处切线与在参数$t$处切线间的夹角为
$$\left|\arctan\left(\frac{g'(t)}{f'(t)}\right)-\arctan\left(\frac{g'(t_0)}{f'(t_0)}\right)\right|,$$

曲线从参数t_0到参数t段曲线的弧长为

$$\left|\int_{t_0}^{t}\sqrt{[f'(u)]^2+[g'(u)]^2}du\right|.$$

定义1 若极限

$$\lim_{t\to t_0}\frac{\left|\arctan\left(\frac{g'(t)}{f'(t)}\right)-\arctan\left(\frac{g'(t_0)}{f'(t_0)}\right)\right|}{\left|\int_{t_0}^{t}\sqrt{[f'(u)]^2+[g'(u)]^2}du\right|}$$

存在，则称上述极限值为曲线L在<u>参数t_0处的曲率</u>，称上述极限值的倒数为曲线L 在<u>参数t_0处的曲率半径</u>。

注1 若$f,g\in C^2[\alpha,\beta]$，$[f'(t_0)]^2+[g'(t_0)]^2\neq 0$，则曲率

$$\begin{aligned}
k &=\!=\!=\!=\!= \lim_{t\to t_0}\frac{\left|\arctan\left(\frac{g'(t)}{f'(t)}\right)-\arctan\left(\frac{g'(t_0)}{f'(t_0)}\right)\right|}{\left|\int_{t_0}^{t}\sqrt{[f'(u)]^2+[g'(u)]^2}du\right|}.\\
&\Leftarrow\!=\!=\!=\!= \left|\lim_{t\to t_0}\frac{\arctan\left(\frac{g'(t)}{f'(t)}\right)-\arctan\left(\frac{g'(t_0)}{f'(t_0)}\right)}{\int_{t_0}^{t}\sqrt{[f'(u)]^2+[g'(u)]^2}du}\right|.\\
&\overset{\Leftarrow\!=\!=\!=\!=}{\text{L'Hospital法则}} \left|\lim_{t\to t_0}\frac{\dfrac{\dfrac{g''(t)f'(t)-g'(t)f''(t)}{[f'(t)]^2}}{1+\left[\frac{g'(t)}{f'(t)}\right]^2}}{\sqrt{[f'(t)]^2+[g'(t)]^2}}\right|.\\
&=\!=\!=\!=\!= \frac{|g''(t_0)f'(t_0)-g'(t_0)f''(t_0)|}{\left[[f'(t_0)]^2+[g'(t_0)]^2\right]^{\frac{3}{2}}}.
\end{aligned}$$

特别，若曲线L的方程为

$$L: y=g(x),\quad x\in[a,b],$$

其中$g\in C^2[a,b]$，则曲线L在x_0处的曲率

$$k =\!=\!=\!=\!= \frac{|g''(x_0)|}{\left[1+[g'(x_0)]^2\right]^{\frac{3}{2}}};$$

例1 L的参数方程为

$$L:\begin{cases} x=R\cos\theta,\\ y=R\sin\theta,\end{cases}\quad \theta\in[0,2\pi]$$

求曲线L的曲率与曲率半径。

解：曲率为

$$k \quad ===== \quad \frac{|y''(\theta_0)x'(\theta_0)-y'(\theta_0)x''(\theta_0)|}{\left[[x'(\theta_0)]^2+[y'(t_0)]^2\right]^{\frac{3}{2}}}$$

$$===== \frac{1}{R} \quad .$$

曲率半径为R.

注2

若光滑曲线L在点M处的曲率半径为R，过点M作L的法线l，并在l上L凹的一侧取点O使得$|OM|=R$.以点O为圆心、以R为半径的圆称为曲线L在点M处的曲率圆，点O称为曲率中心。

例2

求抛物线$y=x^2$上任一点处的曲率、曲率半径与曲率中心。

解：抛物线上任一点$M(x,x^2)$处的曲率为

$$k \quad ===== \quad \frac{|y''(x)|}{\left[[1+[y'(x)]^2\right]^{\frac{3}{2}}} ==== \frac{2}{[1+4x^2]^{\frac{3}{2}}} \quad .$$

曲率半径为

$$R=\frac{1}{k}=\frac{1}{2}\left[1+4x^2\right]^{\frac{3}{2}}.$$

抛物线在点$M(x,x^2)$处的法线方程为

$$Y-x^2=-\frac{1}{2x}(X-x).$$

曲率中心$O(X,Y)$应满足

$$(X-x)^2+(Y-x^2)^2=R^2.$$

于是

$$(X-x)^2+\frac{1}{4x^2}(X-x)^2=\frac{1}{4}\left[1+4x^2\right]^3.$$

当$x>0$时$X<x$；当$x<0$时，$X>x$，所以

$$X=-4x^3.$$

从而$Y=\frac{1}{2}+3x^2$，故抛物线在点$M(x,x^2)$处的曲率中心为$O=\left(-4x^3,\frac{1}{2}+3x^2\right)$.