4.2

2 (1) $\lim\limits_{x\to 0^+}\frac{\ln(1-\cos x)}{\ln x}=\lim\limits_{x\to 0^+}\frac{\frac{\sin x}{1-\cos x}}{\frac{1}{x}}=\lim\limits_{x\to 0^+}\frac{x\sin x}{1-\cos x}=\lim\limits_{x\to 0^+}\frac{\sin x+x\cos x}{\sin x}=\lim\limits_{x\to 0^+}\frac{2\cos x-x\sin x}{\cos x}=2$

(2) $\lim\limits_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}=\lim\limits_{x\to 0}\frac{(e^x-1)-(e^{\sin x}-1)}{x-\sin x}=\lim\limits_{x\to 0}\frac{x-\sin x+o(x)-o(\sin x)}{x-\sin x}$

$=\lim\limits_{x\to 0}\left(1+\frac{o(x)-o(\sin x)}{x-\sin x}\right)=\lim\limits_{x\to 0}\left(1+\frac{o(x)-o(x)}{x-\sin x}\right)=1$

(3) $\lim\limits_{x\to\frac{\pi}{2}}\frac{\sqrt{1+2\cos x}-1}{x-\frac{\pi}{2}}=\lim\limits_{x\to\frac{\pi}{2}}\frac{-\sin x}{\sqrt{1+2\cos x}}=-1$

(4) $\lim\limits_{x\to+\infty}\frac{\ln(e^x+1)}{x^2}=\lim\limits_{x\to+\infty}\frac{e^x}{2x(e^x+1)}=\lim\limits_{x\to+\infty}\frac{1}{2x(1+e^{-x})}=0$

(5) $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\frac{(\arcsin x)^2-\frac{\pi^2}{16}}{2x^2-1}=\lim\limits_{x\to\frac{1}{\sqrt{2}}}\frac{\arcsin x}{2x\sqrt{1-x^2}}=\frac{\pi}{4}$

(6) $\lim\limits_{x\to 0}\frac{x-\sin x}{x^3}=\lim\limits_{x\to 0}\frac{1-\cos x}{3x^2}=\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$

(7) $\lim\limits_{x\to 0}\frac{\cos\alpha x-\cos\beta x}{\ln(x^2+1)}=\lim\limits_{x\to 0}\frac{(\beta\sin\beta x-\alpha\sin\alpha x)(x^2+1)}{2x}=\lim\limits_{x\to 0}\left(\frac{\beta^2\sin\beta x}{2\beta x}-\frac{\alpha^2\sin\alpha x}{2\alpha x}\right)(x^2+1)$

$=\frac{\beta^2-\alpha^2}{2}$

(8) $\lim\limits_{x\to 0}\frac{e^x-x-1}{x(e^x-1)}=\lim\limits_{x\to 0}\frac{e^x-1}{xe^x+e^x-1}=\lim\limits_{x\to 0}\frac{1}{\frac{e^x}{\frac{e^x-1}{x}}+1}=\frac{1}{1+1}=\frac{1}{2}$

(9) $\lim\limits_{x\to+\infty}(x-\sqrt{x^2+x})=\lim\limits_{x\to+\infty}\frac{-x}{x+\sqrt{x^2+x}}=\lim\limits_{x\to+\infty}\frac{-1}{1+\frac{2x+1}{2\sqrt{x^2+x}}}=\lim\limits_{x\to+\infty}\frac{-1}{1+\frac{2+\frac{1}{x}}{2\sqrt{1+\frac{1}{x}}}}=-\frac{1}{2}$

(10) $\lim\limits_{x\to+\infty}\frac{\ln(x+1)}{x^2}=\lim\limits_{x\to+\infty}\frac{\ln(x+1)}{x}\cdot\frac{1}{x}=0$

(11) $\lim\limits_{x\to 0}\left(\cot x-\frac{1}{x}\right)=\lim\limits_{x\to 0}\frac{x\cos x-\sin x}{x\sin x}=\lim\limits_{x\to 0}\frac{-x\sin x}{\sin x+x\cos x}=\lim\limits_{x\to 0}\frac{-x}{1+\frac{x}{\sin x}\cdot\cos x}$

$=0$

(12) $\lim\limits_{x\to\frac{\pi}{2}}(\sec x-\tan x)=\lim\limits_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}=\lim\limits_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}=0$

(13) $\lim\limits_{x\to1}(x-1)\tan\frac{\pi}{2}x=\lim\limits_{x\to1}\frac{(x-1)\sin\frac{\pi}{2}x}{\cos\frac{\pi}{2}x}=\lim\limits_{x\to1}\frac{\sin\frac{\pi}{2}x-\frac{\pi}{2}(x-1)\cos\frac{\pi}{2}x}{-\frac{\pi}{2}\sin\frac{\pi}{2}x}=-\frac{2}{\pi}$

(14) $\lim\limits_{x\to0^+}x\ln x=\lim\limits_{x\to0^+}\frac{\ln x}{\frac{1}{x}}=\lim\limits_{x\to0^+}-x=0$

(15) $\lim\limits_{x\to+\infty}(\pi-2\arctan x)\ln x=\lim\limits_{x\to+\infty}2\operatorname{arccot}x\ln x=\lim\limits_{x\to+\infty}\frac{2\ln x}{\frac{1}{\operatorname{arccot}x}}$

$=\lim\limits_{x\to+\infty}\frac{\frac{2}{x}}{\frac{-1}{x^2+1}\cdot\frac{-1}{(\operatorname{arccot}x)^2}}=\lim\limits_{x\to+\infty}\frac{2(x^2+1)(\operatorname{arccot}x)^2}{x}=\lim\limits_{x\to+\infty}\frac{2(\operatorname{arccot}x)^2}{\frac{x}{x^2+1}}$

$=\lim\limits_{x\to+\infty}\frac{-4\operatorname{arccot}x\cdot(1+x^2)}{1-x^2}=\lim\limits_{x\to+\infty}\frac{-4\operatorname{arccot}x}{1-\frac{2}{x^2+1}}=0$

(16) $\lim\limits_{x\to\infty}x\ln(1+\frac{1}{x})=\lim\limits_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}=1$

(17) $\lim\limits_{x\to\frac{\pi}{2}}(\frac{\pi}{x}-1)^{\tan x}=\lim\limits_{x\to\frac{\pi}{2}}\exp(\tan x\ln(\frac{\pi}{x}-1))=\lim\limits_{x\to\frac{\pi}{2}}\exp(\frac{\ln(\frac{\pi}{x}-1)\cdot\sin x}{\cos x})$

$=\lim\limits_{x\to\frac{\pi}{2}}\exp(\frac{\frac{\pi}{x(x-\pi)}\sin x+\cos x\cdot\ln(\frac{\pi}{x}-1)}{-\sin x})=e^{\frac{4}{\pi}}$

(18) $\lim\limits_{x\to1^-}(1-x)^{\ln x}=\lim\limits_{x\to1^-}\exp(\ln x(\ln(1-x)))=\lim\limits_{x\to1^-}\exp\left(\frac{\ln(1-x)}{\frac{1}{\ln x}}\right)=\lim\limits_{x\to1^-}\exp(\frac{-x(\ln x)^2}{1-x})$

$=\lim\limits_{x\to1^-}\exp((\ln x)^2+2\ln x)=e^0=1$

(19) $\lim\limits_{x\to\infty}(\cos\frac{1}{x})^{x^2}=\lim\limits_{x\to\infty}\exp(x^2\ln\cos\frac{1}{x})=\lim\limits_{x\to\infty}\exp(\frac{-x\sin\frac{1}{x}}{2\cos\frac{1}{x}})=e^{-\frac{1}{2}}$

(20) $\lim\limits_{n\to\infty}n[(1+\frac{1}{n})^n-e]=\lim\limits_{n\to\infty}\frac{(1+\frac{1}{n})^n-e}{\frac{1}{n}}=\lim\limits_{n\to\infty}\frac{(1+\frac{1}{n})^n\cdot[\ln(1+\frac{1}{n})-\frac{1}{n+1}]}{-\frac{1}{n^2}}$

$=\lim\limits_{n\to\infty}(1+\frac{1}{n})^n\cdot\lim\limits_{n\to\infty}\frac{n^2}{-2(n+1)^2}=-\frac{1}{2}e$

故 $\lim\limits_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)^n - e\right] = -\frac{e}{2}$

3. $\lim\limits_{h\to 0} \frac{(f(a+h)-f(a)) - (f(a)-f(a-h))}{h^2}$

$\lim\limits_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2} = \lim\limits_{h\to 0} \frac{f'(a+h)-f'(a-h)}{2h}.$

$= \lim\limits_{h\to 0} \frac{f''(a+h)+f''(a-h)}{2} =$

$= \lim\limits_{h\to 0} \left[\frac{f'(a+h)-f'(a)}{h} + \frac{f'(a)-f'(a-h)}{h}\right] \times \frac{1}{2}$

$= f''(a).$

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4. $\lim\limits_{x\to0}\frac{f(\sin x)-1}{\ln f(x)} = \lim\limits_{x\to0}\frac{f'(\sin x)\cos x\cdot f(x)}{f'(x)} = 1$

4.3

5. $\ln\lim\limits_{x\to\infty}(x-x^2\ln(1+\frac{1}{x})) = \lim\limits_{x\to\infty}(x-x^2[\frac{1}{x}-\frac{1}{2x^2}+o(\frac{1}{x^2})])$

$=\lim\limits_{x\to\infty}(x-x+\frac{1}{2}-o(1))=\frac{1}{2}$

(2) $\lim\limits_{x\to0}\frac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{(\cos x-e^{x^2})\sin x^2} = \lim\limits_{x\to0}\frac{\frac{x^2}{2}+1-(1+\frac{1}{2}x^2-\frac{1}{8}x^4+o(x^4))}{[(1-\frac{1}{2}x^2+o(x^2))-(1+x^2+o(x^2))][x^2+o(x^2)]}$

$=\lim\limits_{x\to0}\frac{\frac{1}{6}x^4+o(x^4)}{-\frac{3}{2}x^4+o(x^4)} = -\frac{1}{12}$

(3) $\lim\limits_{x\to+\infty}x^{\frac{3}{2}}(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}) = \lim\limits_{x\to+\infty}x^2(\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}-2)$

$=\lim\limits_{x\to+\infty}x^2(1+\frac{1}{2x}-\frac{1}{8}\frac{1}{x^2}+o(\frac{1}{x^2})+1-\frac{1}{2x}-\frac{1}{8}\frac{1}{x^2}+o(x^2)-2)$

$=\lim\limits_{x\to+\infty}x^2(-\frac{1}{4}\frac{1}{x^2}+o(\frac{1}{x^2}))=-\frac{1}{4}$

(4) $\lim\limits_{x\to0}\frac{\sin(\sin x)-\tan(\tan x)}{\sin x-\tan x} = \lim\limits_{x\to0}\frac{\sin x-\frac{1}{6}\sin^3x+o(x^3)-\tan x-\frac{1}{3}\tan^3x+o(x^3)}{\sin x-\tan x}$

$=\lim\limits_{x\to0}(1-\frac{\frac{1}{6}\sin^3x+\frac{1}{3}\tan^3x+o(x^3)}{\sin x-\tan x}) = \lim\limits_{x\to0}(1-\frac{\frac{1}{6}\sin^3x+\frac{1}{3}\tan^3x+o(x^3)}{1-\frac{1}{6}x^3-1-\frac{1}{3}x^3+o(x^3)})$

$=2$

No.6.

解: $x\to 0$:

$$\ln(1+\sin x^2)=\sin x^2-\frac{(\sin x^2)^2}{2}+o\left((\sin x^2)^2\right)$$

$$=\left(x^2+o(x^4)\right)-\frac{\left(x^2+o(x^4)\right)^2}{2}+o(x^4)$$

$$=x^2-\frac{1}{2}x^4+o(x^4)$$

$$\sqrt[3]{2-\cos x}=\left(1+2\sin^2\frac{x}{2}\right)^{\frac{1}{3}}=1+\frac{2}{3}\sin^2\frac{x}{2}+\frac{\frac{1}{3}\times\left(-\frac{2}{3}\right)}{2!}\left(2\sin^2\frac{x}{2}\right)^2+o\left(4\sin^4\frac{x}{2}\right)$$

$$\cancel{=1+\frac{2}{3}\left(\frac{x}{2}+o(x^3)\right)^2-\frac{4}{9}\left(\frac{x}{2}+o(x)\right)^4+o(x^4)}$$

$$\cancel{=1+\frac{2}{3}\left(\frac{x^2}{4}\right.}$$

$$=1+\frac{2}{3}\left(\frac{x}{2}-\frac{x^3}{48}+o(x^3)\right)^2-\frac{4}{9}\left(\frac{x}{2}+o(x)\right)^4+o(x^4)$$

$$=1+\frac{2}{3}\left(\frac{x^2}{4}-\frac{x^4}{48}+o(x^4)\right)-\frac{4}{9}\left(\frac{x^4}{16}+o(x^4)\right)+o(x^4)$$

$$=1+\frac{1}{6}x^2-\frac{1}{24}x^4+o(x^4)$$

则 $\ln(1+\sin x^2)+\alpha\left(\sqrt[3]{2-\cos x}-1\right)$

$$=x^2-\frac{1}{2}x^4+o(x^4)+\frac{\alpha}{6}x^2-\frac{\alpha}{24}x^4+o(x^4)$$

$$=\frac{6+\alpha}{6}x^2-\frac{12+\alpha}{24}x^4+o(x^4)$$

当 $6+\alpha=0$ 即 $\alpha=-6$:

$$\lim_{x\to 0}\frac{\ln(1+x^2)+\alpha\left(\sqrt[3]{2-\cos x}-1\right)}{x^4}=\lim_{x\to 0}\frac{-\frac{12+\alpha}{24}x^4+o(x^4)}{x^4}=-\frac{12+\alpha}{24}=-\frac{1}{4}\neq 0.$$

$\Rightarrow$ $x\to 0$: $\ln(1+\sin x^2)+\alpha\left(\sqrt[3]{2-\cos x}-1\right)$ 阶为4.

当 $6+\alpha\neq 0$, 即 $\alpha\neq -6$:

$$\lim_{x\to 0}\frac{\ln(1+x^2)+\alpha\left(\sqrt[3]{2-\cos x}-1\right)}{x^2}=\lim_{x\to 0}\frac{\frac{6+\alpha}{6}x^2-\frac{12+\alpha}{24}x^4+o(x^4)}{x^2}=\frac{6+\alpha}{6}\neq 0$$

$\Rightarrow$ $\ln(1+\sin x^2)+\alpha\left(\sqrt[3]{2-\cos x}-1\right)$ 为2阶.

$= \frac{1}{4}x^4 + o(x^4)$ ∴为4阶

8. $\ln x = \ln y + \frac{1}{y}\cdot(x-y) + \frac{f''(\xi)}{2}\cdot(x-y)^2 \quad \xi \in (x,y)$

$\therefore \left|\frac{\ln x - \ln y}{x-y} - \frac{1}{y}\right| = \left|\frac{x-y}{2}\right| |f''(\xi)|$

$\because |f''(\xi)| = \left|\frac{1}{\xi^2}\right| < 1$

$\therefore \left|\frac{\ln\frac{x}{y}}{x-y} - \frac{1}{y}\right| = \left|\frac{x-y}{2}\right| \cdot |f''(\xi)| \leqslant \left|\frac{x-y}{2}\right| \quad \square$

10. $f(x)=f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)+\frac{f''(\xi)}{2}\left(x-\frac{a+b}{2}\right)^2$

$\therefore f(a)=f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\cdot\frac{a-b}{2}+\frac{f''(\xi_1)}{2}\cdot\left(\frac{a-b}{2}\right)^2 \quad \xi_1\in\left(a,\frac{a+b}{2}\right)$

$f(b)=f\left(\frac{a+b}{2}\right)+f'\left(\frac{a+b}{2}\right)\cdot\frac{b-a}{2}+\frac{f''(\xi_2)}{2}\cdot\left(\frac{a-b}{2}\right)^2 \quad \xi_2\in\left(\frac{a+b}{2},b\right)$

$\therefore f(a)+f(b)=2f\left(\frac{a+b}{2}\right)+\frac{(a-b)^2}{4}\cdot\frac{f''(\xi_1)+f''(\xi_2)}{2}$

即 $f(a)+f(b)-2f\left(\frac{a+b}{2}\right)=\frac{(b-a)^2}{4}\cdot\frac{f''(\xi_1)+f''(\xi_2)}{2}$

由达布定理可知：$\exists\,\xi\in(\xi_1,\xi_2)$. s.t. $f''(\xi)=\frac{f''(\xi_1)+f''(\xi_2)}{2}$ □

12. $f(x)=f(a)+\frac{f''(\xi_1)}{2}(x-a)^2 \quad \therefore f\left(\frac{a+b}{2}\right)=f(a)+\frac{f''(\xi_1)}{2}\cdot\frac{(b-a)^2}{4}$

$f(x)=f(b)+\frac{f''(\xi_2)}{2}(x-b)^2 \quad f\left(\frac{a+b}{2}\right)=f(b)+\frac{f''(\xi_2)}{2}\cdot\frac{(b-a)^2}{4}$

$\therefore f(b)-f(a)=\frac{(b-a)^2}{4}\cdot\frac{f''(\xi_1)-f''(\xi_2)}{2}$

$\therefore \frac{4}{(b-a)^2}|f(b)-f(a)|=\left|\frac{f''(\xi_1)-f''(\xi_2)}{2}\right|\leqslant\frac{|f''(\xi_1)|+|f''(\xi_2)|}{2}$

$=\max\{|f''(\xi_1)|,|f''(\xi_2)|\}$

① 当 $f''(\xi_1)f''(\xi_2)>0$ 时

$$\frac{\qquad}{2} \le \frac{\qquad}{2}$$

不妨设 $|f''(\xi_1)| > |f''(\xi_2)|$

则 $\exists\, \xi_1 \in (a,b)$. s.t. $\frac{4}{(b-a)^2} \le |f''(\xi_1)|$ □