## 习题5.2

8. 证明: $\min\{f(x), g(x)\} = \dfrac{f(x)+g(x)-|f(x)-g(x)|}{2}$

$\max\{f(x), g(x)\} = \dfrac{f(x)+g(x)+|f(x)-g(x)|}{2}$

而 $f(x)$、$g(x)$、$|f(x)-g(x)| \in R[a,b]$

由积分的线性性质:

$\min\{f(x), g(x)\}$、$\max\{f(x), g(x)\} \in R[a,b]$

9. 证明: 由柯西不等式:

$(\int_a^b f(x)\,dx)(\int_a^b \frac{1}{f(x)}\,dx) \geqslant (\int_a^b \sqrt{f(x)}\cdot\frac{1}{\sqrt{f(x)}}\,dx)^2 = (b-a)^2$

得证.

## 习题5.3

4. 解: 对 $\int_0^{\sqrt{x}} f(t)\,dt = x+\sin x$ 两边同时求导得:

$\frac{1}{2\sqrt{x}} f(\sqrt{x}) = 1+\cos x$ 即 $f(\sqrt{x}) = 2\sqrt{x}(1+\cos x)$

用x代替$\sqrt{x}$得 $f(x) = 2x(1+\cos x^2)$

故 $f(x) = 2x(1+\cos x^2)$

5. 解: $F'(x) = \dfrac{x}{e^{x^2}}$ 令 $F'(x)=0$ 得 $x=0$. $x\in(-\infty,0)$ 时 $F'(x)<0$; $x\in(0,+\infty)$ 时 $F'(x)>0$

$F''(x) = \dfrac{1-2x^2}{e^{x^2}}$ 令 $F''(x)=0$ 得 $x=\pm\frac{\sqrt{2}}{2}$. $x\in(-\infty,-\frac{\sqrt{2}}{2})\cup(\frac{\sqrt{2}}{2},+\infty)$ 时 $F''(x)<0$; $x\in(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$ 时 $F''(x)>0$

故 $F(x)$ 极小值点为 $x=0$. 拐点为 $x=\pm\frac{\sqrt{2}}{2}$

7. 解: 显然 $f(x) \in R[-1,1]$. $f(x)\in C[-1,0)$. $f(x)\in C(0,1]$

由微积分基本定理知: $F(x)\in C[-1,1]$ $F(x)$在 $[-1,0)$、$(0,1]$ 上可导

而 $F'_-(0)=1$. $F'_+(0)=0$

故 $F(x)$在 $x=0$ 处不可导

12. (6). $\frac{1}{2}\ln^2 x$ 是 $\frac{\ln x}{x}$ 的一个原函数

则 $\int_1^8 \frac{\ln x}{x}\,dx = \frac{1}{2}\ln^2 x\Big|_1^8 = \frac{1}{2}\ln^2 8 = \frac{9}{2}\ln^2 2$

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13. (1). $\lim_{n\to\infty}\frac{1}{n}(\sin\frac{1}{n}\pi+\sin\frac{2}{n}\pi+\cdots+\sin\frac{n}{n}\pi) = \frac{1}{\pi}\lim_{n\to\infty}\frac{\pi}{n}(\sin\frac{\pi}{n}+\sin\frac{2}{n}\pi+\cdots+\sin\frac{n}{n}\pi)$

$= \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^{\pi} = \frac{2}{\pi}$

14. 解. $f'(x)=\ln(1+x)$

$\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)(n+2)\cdots(n+n)}}{n} = \exp\left\{\lim_{n\to\infty}\frac{1}{n}[\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\cdots+\ln(1+\frac{n}{n})]\right\}$

$= \exp\left\{\int_0^1\ln(1+x)\,dx\right\} = \exp\left\{(1+x)\ln(1+x)-x\Big|_0^1\right\} = e^{2\ln2-1} = \frac{4}{e}$

15. 证明: 由 $\sin x>\frac{2}{\pi}x$, $\forall x\in(0,\frac{\pi}{2})$ 知:

$R>0$ 时 $e^{-R\sin x}<e^{-\frac{2}{\pi}Rx}$ 则有: $\int_0^{\frac{\pi}{2}}e^{-R\sin x}dx<\int_0^{\frac{\pi}{2}}e^{-\frac{2}{\pi}Rx} = -\frac{\pi}{2R}e^{-\frac{2R}{\pi}x}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2R}(1-e^{-R})$

$R<0$ 时 $e^{-R\sin x}>e^{-\frac{2}{\pi}Rx}$ 则有: $\int_0^{\frac{\pi}{2}}e^{-R\sin x}dx>\int_0^{\frac{\pi}{2}}e^{-\frac{2}{\pi}Rx} = -\frac{\pi}{2R}e^{-\frac{2R}{\pi}x}\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2R}(1-e^{-R})$

原式得证.

习题 5.4

3. (3) 当 $a\neq\frac{1}{e}$ 时:

$\int a^xe^xdx=\int(ae)^xdx=\frac{a^xe^x}{\ln a+1}+C$

当 $a=\frac{1}{e}$ 时:

$\int a^xe^xdx = x+C$

(7). $\int(\frac{4}{\sqrt{1-x^2}}+\sin x)\,dx = 4\arcsin x-\cos x+C$

4. (1). $\int\frac{2x+1}{x^2+x+1}dx = \ln(x^2+x+1)+C$

(11). $\int\frac{1}{e^x+e^{-x}}dx=\int\frac{e^x}{e^{2x}+1}dx = \arctan e^x+C$

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5. (5). $\int \frac{2x+1}{\sqrt{4x-x^2}}\,dx = -2\int \frac{2-x}{\sqrt{4x-x^2}}\,dx + 5\int \frac{\frac{1}{2}}{\sqrt{1-(\frac{x}{2}-1)^2}}\,dx$

$= -2\sqrt{4x-x^2} + 5\arcsin(\frac{x}{2}-1) + C$

(12). $\int \frac{\sin 2x}{1+\sin^4 x}\,dx = \int \frac{d(\sin^2 x)}{1+(\sin^2 x)^2} = \arctan(\sin^2 x) + C$

6. (1). $\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx$

$= x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx - \int \frac{a^2}{\sqrt{a^2+x^2}}\,dx$

即 $2\int \frac{x^2}{\sqrt{a^2+x^2}} = x\sqrt{a^2+x^2} - a^2\ln|x+\sqrt{x^2+a^2}|$

故 $\int \frac{x^2}{\sqrt{a^2+x^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + C$

7. (3). $\int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x - \int (-x)\cos 2x\,dx$

$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$

$= -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

$= (\frac{1}{4}-\frac{1}{2}x^2)\cos 2x + \frac{1}{2}x\sin 2x + C$

(11). $\int \frac{\arcsin e^x}{e^x}\,dx = -\frac{\arcsin e^x}{e^x} - \int (-\frac{1}{e^x}\cdot\frac{e^x}{\sqrt{1-e^{2x}}})\,dx$

$= -\frac{\arcsin e^x}{e^x} + \int \frac{dx}{\sqrt{1-e^{2x}}}$

令 $t = \frac{1}{e^x}$ 则 $x = -\ln t$

$\int \frac{dx}{\sqrt{1-e^{2x}}} = \int \frac{-\frac{1}{t}\,dt}{\sqrt{1-\frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{t^2-1}} = -\ln(t+\sqrt{t^2-1})+C = -\ln(\frac{1}{e^x}+\sqrt{\frac{1}{e^{2x}}-1})+C = x-\ln(1+\sqrt{1-e^{2x}})+C$

故 $\int \frac{\arcsin e^x}{e^x}\,dx = -\frac{\arcsin e^x}{e^x} + x - \ln(1+\sqrt{1-e^{2x}}) + C$

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