§1.5 作业解答

4、解：
$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}$$
$$=\left(\ln(u-v)+\frac{u}{u-v}\right)\cdot(-e^{-x})+\left(-\frac{u}{u-v}\right)\cdot\left(\frac{1}{x}\right)$$
$$=-e^{-x}\ln(e^{-x}-\ln x)-\frac{e^{-2x}+\frac{1}{x}e^{-x}}{e^{-x}-\ln x}$$

5、
$$\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2=\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}\right)^2$$
$$=\left(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta\right)^2+\frac{1}{r^2}\left(-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}\right)^2$$
$$=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$$

7、记 $s=\frac{x}{x^2+y^2}$，$t=\frac{y}{x^2+y^2}$，则 $u=f(s,t)$

$$\frac{\partial s}{\partial x}=\frac{y^2-x^2}{(x^2+y^2)^2}=-\frac{\partial t}{\partial y},\quad \frac{\partial t}{\partial x}=\frac{-2xy}{(x^2+y^2)^2}=\frac{\partial s}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}\right)$$
$$=\frac{\partial^2 f}{\partial s^2}\cdot\left(\frac{\partial s}{\partial x}\right)^2+\frac{\partial^2 f}{\partial t\partial s}\cdot\frac{\partial s}{\partial x}\cdot\frac{\partial t}{\partial x}+\frac{\partial f}{\partial s}\cdot\frac{\partial^2 s}{\partial x^2}$$
$$+\frac{\partial^2 f}{\partial s\partial t}\cdot\frac{\partial s}{\partial x}\cdot\frac{\partial t}{\partial x}+\frac{\partial^2 f}{\partial t^2}\cdot\left(\frac{\partial t}{\partial x}\right)^2+\frac{\partial f}{\partial t}\cdot\frac{\partial^2 t}{\partial x^2}$$

$$(\text{由 } f\in C^2(\mathbb{R}^2))=\frac{\partial^2 f}{\partial s^2}\cdot\left(\frac{\partial s}{\partial x}\right)^2+2\frac{\partial^2 f}{\partial s\partial t}\cdot\frac{\partial s}{\partial x}\cdot\frac{\partial t}{\partial x}+\frac{\partial^2 f}{\partial t^2}\cdot\left(\frac{\partial t}{\partial x}\right)^2$$
$$+\frac{\partial f}{\partial s}\cdot\frac{\partial^2 s}{\partial x^2}+\frac{\partial f}{\partial t}\cdot\frac{\partial^2 t}{\partial x^2}\qquad ①$$

同理：
$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 f}{\partial s^2}\cdot\left(\frac{\partial s}{\partial y}\right)^2+2\frac{\partial^2 f}{\partial s\partial t}\cdot\frac{\partial s}{\partial y}\cdot\frac{\partial t}{\partial y}+\frac{\partial^2 f}{\partial t^2}\cdot\left(\frac{\partial t}{\partial y}\right)^2$$
$$+\frac{\partial f}{\partial s}\cdot\frac{\partial^2 s}{\partial y^2}+\frac{\partial f}{\partial t}\cdot\frac{\partial^2 t}{\partial y^2}\qquad ②$$

因 $\left(\frac{\partial s}{\partial x}\right)^2=\left(\frac{\partial t}{\partial y}\right)^2$，$\left(\frac{\partial t}{\partial x}\right)^2=\left(\frac{\partial s}{\partial y}\right)^2$，$\frac{\partial^2 f}{\partial s^2}+\frac{\partial^2 f}{\partial t^2}=0$，

$$\frac{\partial s}{\partial x}\cdot\frac{\partial t}{\partial x}=-\frac{\partial s}{\partial y}\cdot\frac{\partial t}{\partial y},\quad \frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}=-\frac{\partial^2 t}{\partial x\partial y}+\frac{\partial^2 t}{\partial x\partial y}=0$$

同理 $\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}=0$

故 ①+② 得：$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

9、解：(1) $J(Y) = J(f)\cdot J(g)$

$$= \begin{pmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \\ \frac{\partial y_3}{\partial u_1} & \frac{\partial y_3}{\partial u_2} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ u_2 & u_1 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{pmatrix} \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-x^2-2xy+y^2}{(x^2+y^2)^2} & \frac{x^2-2xy-y^2}{(x^2+y^2)^2} \\ \frac{-3x^2y+y^3}{(x^2+y^2)^3} & \frac{x^3-3xy^2}{(x^2+y^2)^3} \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}$$

全微分 $dY = J(Y)\,dX$

(2) $J(Y) = J(f)\cdot J(g)$

$$= \begin{pmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} 2u_1 & 2u_2 \\ 2u_1 & -2u_2 \end{pmatrix} \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

$$= \frac{2}{x^2+y^2} \begin{pmatrix} x\ln\sqrt{x^2+y^2} - y\arctan\frac{y}{x} & y\ln\sqrt{x^2+y^2} + x\arctan\frac{y}{x} \\ x\ln\sqrt{x^2+y^2} + y\arctan\frac{y}{x} & y\ln\sqrt{x^2+y^2} - x\arctan\frac{y}{x} \end{pmatrix}$$

全微分 $dY = J(Y)\,dX$

(3) $y_1 = \ln\sqrt{u_1^2+u_2^2} = \ln\sqrt{(e^x\cos y)^2 + (e^x\sin y)^2} = x$

$y_2 = \arctan(\cot y)$

$$J(Y) = \begin{pmatrix} \frac{\partial y_1}{\partial x} & \frac{\partial y_1}{\partial y} \\ \frac{\partial y_2}{\partial x} & \frac{\partial y_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

全微分：$dY = J(Y)\,dX$