

Review

含参广义积分的性质

$$I(y) = \int_a^{+\infty} f(x, y) dx, \quad D = [a, +\infty) \times [\alpha, \beta].$$

$$\bullet \begin{cases} f(x, y), f'_y(x, y) \in C(D); \\ \forall y \in [\alpha, \beta], I(y) = \int_a^{+\infty} f(x, y) dx \text{ 收敛}; \\ \int_a^{+\infty} f'_y(x, y) dx \text{ 关于 } y \in [\alpha, \beta] \text{ 一致收敛}; \end{cases}$$

$$\Rightarrow \in C^1[\alpha, \beta], \text{ 且 } I'(y) = \frac{d}{dy} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} f'_y(x, y) dx.$$

$$\bullet \begin{cases} f(x, y) \in C(D); \\ \int_a^{+\infty} f(x, y) dx \text{ 关于 } y \in [\alpha, \beta] \text{ 一致连续}; \end{cases}$$

$$\Rightarrow \begin{cases} I(y) \in C[\alpha, \beta], \text{ 即} \\ \lim_{y \rightarrow y_0} \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \lim_{y \rightarrow y_0} f(x, y) dx. \\ I(y) \in R[\alpha, \beta], \\ \text{且 } \int_{\alpha}^{\beta} dy \int_a^{+\infty} f(x, y) dx = \int_a^{+\infty} \int_{\alpha}^{\beta} f(x, y) dy. \end{cases}$$



§ 4. Γ 函数与B函数

Γ 函数与B函数是物理和工程技术中常见的两个函数.

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \quad (x > 0)$$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad (x > 0, y > 0)$$

1. Γ 函数 (Gamma函数)

Lemma1. Γ 函数在 $(0, +\infty)$ 上有定义、连续且有任何阶可微.

Lemma2. 设 $f, g \in C[a, b]$, $f(t) \geq 0, g(t) \geq 0, \forall t \in [a, b]$.

若 $\alpha > 0, \beta > 0, \alpha + \beta = 1$, 则

$$\int_a^b (f(t))^\alpha (g(t))^\beta dt \leq \left(\int_a^b f(t) dt \right)^\alpha \left(\int_a^b g(t) dt \right)^\beta.$$

Proof. 若 $u \geq 0, v \geq 0, \alpha > 0, \beta > 0, \alpha + \beta = 1$, 则

$$u^\alpha v^\beta \leq \alpha u + \beta v. \quad (\text{几何平均} \leq \text{代数平均})$$

记 $F = \int_a^b f(t) dt, G = \int_a^b g(t) dt, \tilde{f}(t) = f(t)/F, \tilde{g}(t) = g(t)/G$.

往证 $\int_a^b (\tilde{f}(t))^\alpha (\tilde{g}(t))^\beta dt \leq 1$. 事实上,

$$\begin{aligned} \text{左边} &\leq \int_a^b [\alpha \tilde{f}(t) + \beta \tilde{g}(t)] dt \\ &= \alpha \int_a^b \tilde{f}(t) dt + \beta \int_a^b \tilde{g}(t) dt = \alpha + \beta = 1. \square \end{aligned}$$

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \quad (x > 0)$$

Theorem 1. Γ 函数有以下基本性质

(1) $\Gamma(x) \geq 0, \forall x \in (0, +\infty)$, 且 $\Gamma(1) = 1$;

(2) $\Gamma(x+1) = x\Gamma(x), \forall x \in (0, +\infty)$;

(3) $\ln \Gamma(x)$ 在 $(0, +\infty)$ 是凸函数.

Proof. (1) $\Gamma(1) = \int_0^{+\infty} e^{-t} dt = -e^{-t} \Big|_{t=0}^{+\infty} = 1.$

$$\begin{aligned} (2) \Gamma(x+1) &= \int_0^{+\infty} t^x e^{-t} dt = - \int_0^{+\infty} t^x de^{-t} \\ &= -t^x e^{-t} \Big|_{t=0}^{+\infty} + \int_0^{+\infty} x e^{-t} t^{x-1} dt = x\Gamma(x). \end{aligned}$$

(3)欲证 $\ln \Gamma(x)$ 在 $(0, +\infty)$ 是凸函数, 只要证 $\forall \alpha > 0, \beta > 0, \alpha + \beta = 1$, 及 $x > 0, y > 0$, 都有

$$\ln \Gamma(\alpha x + \beta y) \leq \alpha \ln \Gamma(x) + \beta \ln \Gamma(y)$$

也即 $\Gamma(\alpha x + \beta y) \leq (\Gamma(x))^\alpha (\Gamma(y))^\beta.$

由Lemma2,

$$\begin{aligned} \Gamma(\alpha x + \beta y) &= \int_0^{+\infty} t^{\alpha x + \beta y - 1} e^{-t} dt \\ &= \int_0^{+\infty} (t^{x-1} e^{-t})^\alpha (t^{y-1} e^{-t})^\beta dt \\ &\leq \left(\int_0^{+\infty} t^{x-1} e^{-t} dt \right)^\alpha \left(\int_0^{+\infty} t^{y-1} e^{-t} dt \right)^\beta \\ &= (\Gamma(x))^\alpha (\Gamma(y))^\beta. \quad \square \end{aligned}$$

Corollary 1. Γ 函数可以看成阶乘函数的推广.

$$\Gamma(n+1) = n!, \quad n = 1, 2, 3, \dots. \square$$

Theorem 1 中的三条性质完全确定了 Γ 函数.

Theorem 2. (Bohr-Mollerup) 若定义于 $(0, +\infty)$ 上的函数 $f(x)$ 满足以下条件:

(1) $f(x) \geq 0, \forall x \in (0, +\infty)$, 且 $f(1) = 1$,

(2) $f(x+1) = xf(x), \forall x \in (0, +\infty)$,

(3) $\ln f(x)$ 在 $(0, +\infty)$ 是凸函数,

则 $f(x) = \Gamma(x), \forall x \in (0, +\infty)$.



Proof: 只要证明这三个条件确定的函数是唯一的.
为此, 以下证明过程分两步:

Step1. $\forall n \in N, f(n)$ 唯一确定.

Step2. $\forall x \in (0, 1), f(x)$ 唯一确定.

这是因为由性质 (2), $\forall x \in (0, 1)$ 及 $\forall n \in N$,

$$f(x+n+1) = (x+n)(x+n-1) \cdots x f(x).$$

也被唯一确定了.

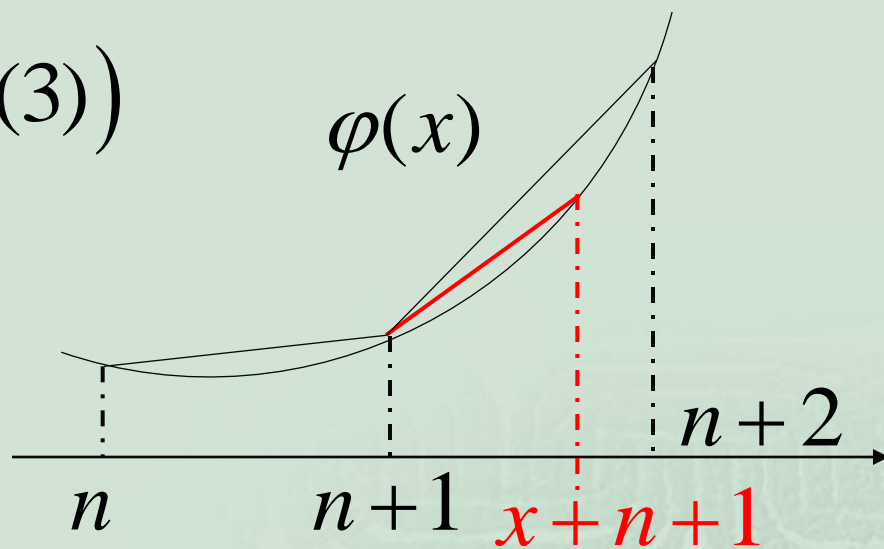
Step1. 由性质(1), $\forall n \in N, f(n+1) = n!$ 唯一确定.

Step2. $\forall x \in (0, 1)$, 令 $\varphi(x) = \ln f(x)$. 则

$$\varphi(x+n+1) = \varphi(x) + \ln[x(x+1)\cdots(x+n)].$$

由于 $\varphi(x)$ 是凸函数(条件(3))

$$\begin{aligned} & \frac{\varphi(n+1) - \varphi(n)}{(n+1) - n} \\ & \leq \frac{\varphi(x+n+1) - \varphi(n+1)}{(x+n+1) - (n+1)} \\ & \leq \frac{\varphi(n+2) - \varphi(n+1)}{(n+2) - (n+1)} \end{aligned}$$



即 $\ln n \leq \frac{\varphi(x+n+1) - \ln(n!)}{x} \leq \ln(n+1)$

$$x \ln n + \ln(n!) \leq \varphi(x+n+1) \leq x \ln(n+1) + \ln(n!)$$

至此, 我们得到了关于 $\varphi(x)$ 的两个条件

$$\varphi(x+n+1) = \varphi(x) + \ln[x(x+1)\cdots(x+n)].$$

$$x \ln n + \ln(n!) \leq \varphi(x+n+1) \leq x \ln(n+1) + \ln(n!)$$

由此得

$$\ln \frac{n^x \cdot n!}{x(x+1)\cdots(x+n)} \leq \varphi(x) \leq \ln \frac{(n+1)^x \cdot n!}{x(x+1)\cdots(x+n)}$$

于是

$$0 \leq \varphi(x) - \ln \frac{n^x \cdot n!}{x(x+1)\cdots(x+n)} \leq x \ln \left(1 + \frac{1}{n} \right)$$

由夹挤原理得

$$\varphi(x) = \lim_{n \rightarrow +\infty} \ln \frac{n^x \cdot n!}{x(x+1)\cdots(x+n)}. \quad \square$$



Corollary 2. $\Gamma(x) = \lim_{n \rightarrow +\infty} \frac{n^x \cdot n!}{x(x+1)(x+2) \cdots (x+n)}.$ \square

Theorem 3. (Γ 函数的余元公式)

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \forall x \in (0, 1).$$

Proof. $\Gamma(x) = \lim_{n \rightarrow +\infty} \frac{n^x}{x(1+x)(1+\frac{x}{2}) \cdots (1+\frac{x}{n})},$

$$\Gamma(1-x) = \lim_{n \rightarrow +\infty} \frac{n^{1-x}}{(1-x)(1-\frac{x}{2}) \cdots (1-\frac{x}{n})(n+1-x)}.$$

$$\Gamma(x)\Gamma(1-x)$$

$$= \lim_{n \rightarrow +\infty} \left[\frac{1}{x(1-x^2)(1-\frac{x^2}{2^2}) \cdots (1-\frac{x^2}{n^2})} \cdot \frac{n}{n+1-x} \right]$$

$$= \frac{1}{\lim_{n \rightarrow +\infty} x \prod_{n=1}^{+\infty} (1-\frac{x^2}{n^2})} = \frac{\pi}{\sin \pi x}.$$

最后一个等式的证明见下面的引理. \square



Lemma3. $\sin \pi x = \lim_{n \rightarrow +\infty} \pi x \prod_{n=1}^{+\infty} (1 - \frac{x^2}{n^2}), \forall x \in (0, 1).$

Proof.

$$\text{令 } \psi(x) = \ln \left[\pi \prod_{n=1}^{+\infty} (1 - \frac{x^2}{n^2}) \right] = \ln \pi + \sum_{n=1}^{+\infty} \ln(1 - \frac{x^2}{n^2}).$$

$$\text{则 } \psi(0) = \ln \pi, \psi'(x) = \sum_{n=1}^{+\infty} \frac{2x}{x^2 - n^2}.$$

给定 $x \in (0, 1)$, 函数 $\cos xt, t \in [-\pi, \pi]$ 的 Fourier 展式为

$$\cos xt = \frac{\sin x\pi}{\pi} \left(\frac{1}{x} + \sum_{n=1}^{+\infty} (-1)^n \frac{2x \cos nt}{x^2 - n^2} \right).$$

$$\text{令 } t = \pi \text{ 得, } \pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{+\infty} \frac{2x}{x^2 - n^2}.$$

因此,

$$\psi'(x) = \pi \cot \pi x - \frac{1}{x}, \quad \psi(x) = \ln \frac{\sin \pi x}{x} + C.$$

令 $x \rightarrow 0^+$ 得, $C = 0$, 即

$$\ln \left[\pi \prod_{n=1}^{+\infty} \left(1 - \frac{x^2}{n^2} \right) \right] = \ln \frac{\sin \pi x}{x},$$

$$\sin \pi x = \lim_{n \rightarrow +\infty} \pi x \prod_{n=1}^{+\infty} \left(1 - \frac{x^2}{n^2} \right), \quad \forall x \in (0, 1). \square$$

Corollary 3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \square$



2. B函数(Beta函数)

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad (x > 0, y > 0)$$

Lemma4. $B(x, y)$ 对任意 $x > 0, y > 0$ 有定义, 且满足以下性质:

(1) $B(x, y) > 0, \forall x, y > 0$, 且 $B(1, y) = 1/y$,

(2) $B(x+1, y) = \frac{x}{x+y} B(x, y), \forall x, y > 0$,

(3) 给定 $y > 0$, $\ln B(x, y)$ 关于 x 在 $(0, +\infty)$ 是凸函数. \square

Lemma5. $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \forall x, y > 0.$

Proof: 任意给定 $y > 0$, 令 $f(x) = \frac{B(x, y)\Gamma(x+y)}{\Gamma(y)},$

往证 $f(x) = \Gamma(x)$. 只要证 f 满足 Γ 函数的三条性质.
由 Lemma3 得

$$(1) f(1) = \frac{B(1, y)\Gamma(1+y)}{\Gamma(y)} = \frac{\Gamma(1+y)}{y\Gamma(y)} = 1.$$

$$\begin{aligned} (2) f(x+1) &= \frac{B(x+1, y)\Gamma(x+y+1)}{\Gamma(y)} \\ &= \frac{x}{(x+y)} B(x, y) \cdot \frac{(x+y)\Gamma(x+y)}{\Gamma(y)} = xf(x). \end{aligned}$$

(3)对于取定的 $y > 0$,由于 $\ln B(x, y)$ 和 $\ln \Gamma(x + y)$ 都是 x 的凸函数,所以

$\ln f(x) = \ln B(x, y) + \ln \Gamma(x + y) - \ln \Gamma(y)$
也是 x 的凸函数.□

Corollary 4. $B(x, y) = B(y, x), \forall x, y > 0$.□

3. 例题 例1. $I = \int_0^1 \sqrt{x - x^2} dx.$

$$\begin{aligned} \text{解: } I &= \int_0^1 x^{1/2} (1-x)^{1/2} dx = B(3/2, 3/2) \\ &= \frac{\Gamma(3/2)^2}{\Gamma(3)} = \frac{[1/2 \Gamma(1/2)]^2}{2!} = \frac{\pi}{8}. \square \end{aligned}$$



例2. $I = \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}.$

解: 令 $t = x^{1/4}$, 则 $x = t^4$, $dx = 4t^3 dt$,

$$I = 4 \int_0^1 t^3 (1-t)^{-1/2} dt = 4B(4, 1/2) = \frac{4\Gamma(4)\Gamma(1/2)}{\Gamma(4+1/2)}$$

$$= 4 \cdot \frac{3!\Gamma(1/2)}{(3+1/2)(2+1/2)(1+1/2)1/2\Gamma(1/2)} = \frac{128}{35} \square$$



例3. $I = \int_0^{\pi/2} \sqrt{\tan x} dx.$

解: $I = \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x dx.$ 令 $t = \sin^2 x$, 则

$$dt = 2 \sin x \cos x dx, \quad dx = \frac{dt}{2 \sin x \cos x}.$$

$$I = \frac{1}{2} \int_0^1 t^{-1/4} (1-t)^{-3/4} dt = \frac{1}{2} B(3/4, 1/4)$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)} = \frac{1}{2} \frac{\pi}{\sin \pi/4} = \frac{\pi}{\sqrt{2}}. \quad \square$$



例4. $I = \int_0^{+\infty} x^{2n} e^{-x^2} dx.$

解: 令 $t = x^2$, 则 $dt = 2x dx$, $dx = \frac{dt}{2\sqrt{t}}.$

$$\begin{aligned} I &= \int_0^{+\infty} t^n e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2} \int_0^{+\infty} t^{n-1/2} e^{-t} dt \\ &= \frac{1}{2} \Gamma(n+1/2) = \frac{1}{2} (n-1/2)(n-3/2)\cdots 1/2 \Gamma(1/2) \\ &= \frac{(2n-1)!!}{2^{n+1}} \sqrt{\pi}. \square \end{aligned}$$



作业:利用B函数和 Γ 函数计算积分

$$(1) \int_0^{+\infty} \frac{dx}{1+x^4}$$

$$(2) \int_0^{\pi/2} \sin^6 x \cos^4 x dx$$

$$(3) \int_0^a x^2 \sqrt{a^2 - x^2} dx.$$

答案:(1) $\frac{\pi}{2\sqrt{2}}$, (2) $\frac{3\pi}{516}$, (3) $\frac{\pi}{16} a^4$.

