1.5.2 (1) $\forall n,p\in N_+$

$$|a_{n+p}-a_n|=\sum_{k=n+1}^{n+p}\frac{\sin k}{2^k}\le\sum_{k=n+1}^{n+p}\frac{1}{2^k}<\frac{1}{2^n}$$

$\forall\varepsilon>0$ $|\frac{1}{2^n}|<\varepsilon$ 只需 $n\ge\log_2\frac{1}{\varepsilon}$ 即可

取 $N=\max\{1,[\log_2\frac{1}{\varepsilon}]\}$ 则 $|a_{n+p}-a_n|<\varepsilon$ 成立

∴得证

(2) $\forall n,p\in N_+$

$$|a_{n+p}-a_n|=\sum_{k=n+1}^{n+p}\frac{\cos k!}{k(k+1)}\le\sum_{k=n+1}^{n+p}\frac{1}{k(k+1)}=\frac{1}{n+1}-\frac{1}{n+p+1}<\frac{1}{n+1}$$

$\forall\varepsilon>0$ $|\frac{1}{n+1}|<\varepsilon$ 只需 $n>\frac{1}{\varepsilon}-1$ 即可

取 $N=\max\{1,[\frac{1}{\varepsilon}-1]\}$ 则 $|a_{n+p}-a_n|<\varepsilon$ 成立

∴得证

(3) $\forall n,p\in N_+$

$$|a_{n+p}-a_n|=|\sum_{k=n+1}^{n+p}c_kq^k|<\sum_{k=n+1}^{n+p}|c_k||q^k|$$ 令 $|c_k|$ 界为 c

$$\therefore |a_{n+p}-a_n|\le c\cdot\sum_{k=n+1}^{n+p}|q^k|=c\cdot|q|^{n+1}\cdot\frac{1-|q|^p}{1-|q|}<\frac{c}{1-|q|}\cdot|q|^{n+1}$$

$\forall\varepsilon>0$ 只需 $n\ge\log_{|q|}\left(\frac{\varepsilon(1-|q|)}{c}\right)$，即可

取 $N=\max\{1,[\log_{|q|}\frac{\varepsilon(1-|q|)}{c}]\}$ 则 $|a_{n+p}-a_n|<\varepsilon$ 成立

∴得证

(4) $\ln a_n=\sum_{k=1}^{n}\ln(1+\frac{1}{k^2})$ $|\ln a_{n+p}-\ln a_n|=\sum_{k=n+1}^{n+p}\ln(1+\frac{1}{k^2})<\sum_{k=n+1}^{n+p}\ln\frac{k^2}{(k-1)(k+1)}$

$=\ln\frac{(n+1)(n+p)}{n(n+p+1)}<\ln(1+\frac{1}{n})$

$\forall\varepsilon>0$ $|\ln(1+\frac{1}{n})|<\varepsilon$ 只需 $n>\frac{1}{1-e^{\varepsilon}}$

∴取 $N=\max\{1,[\frac{1}{1-e^{\varepsilon}}]\}$ 则 $|\ln a_{n+p}-\ln a_n|<\varepsilon$ 成立

∴ $\ln a_n$ 收敛 故 a_n 收敛

(5) $\forall n,p\in N^+$

$|a_{n+p}-a_n|=\sum_{k=n+1}^{n+p}(-1)^{k+1}\frac{1}{k}\le\frac{1}{n+1}+\frac{1}{n+p}<\frac{2}{n}$

n为奇 $\forall\varepsilon>0$ $|\frac{2}{n}|<\varepsilon$ 即 $n>\frac{2}{\varepsilon}$

取 $N=[\frac{2}{\varepsilon}]$ 则 $|a_{n+p}-a_n|<\varepsilon$ 成立

∴得证

(6) $\forall n,p\in N^+$

$|a_{n+p}-a_n|=\sum_{k=n+1}^{n+p}\frac{(-1)^{k-1}}{k^\alpha}<\frac{1}{(n+1)^\alpha}+\frac{1}{(n+p)^\alpha}<\frac{2}{(n+1)^\alpha}$

$\forall\varepsilon>0$ $|\frac{2}{(n+1)^\alpha}|<\varepsilon$ 即 $n>\sqrt[\alpha]{\frac{2}{\varepsilon}}-1$

取 $N=\max\{1,[\sqrt[\alpha]{\frac{2}{\varepsilon}}-1]\}$ 则 $|a_{n+p}-a_n|<\varepsilon$ 成立

∴得证

1.5.3 (1) $|a_{n+1}-a_n|=|\cos\frac{2n+2}{3}\pi-\cos\frac{2n}{3}\pi|$

令 $n=3k$ $k\in N^+$

∴ $|a_{n+1}-a_n|=\frac{3}{2}$

令 $\varepsilon=1$ 则 $\forall N$ $n>N$ 时总有 $|a_{n+1}-a_n|>\varepsilon$ 3n 成立

∴ a_n不收敛

(2) $|a_{n+1}-a_n|=|\sqrt[n+1]{1+3^{(-1)^{n+1}}(n+1)}-\sqrt[n]{1+3^{(-1)^n}\cdot n}|$

$2|n$ 时 $a_n>3$ $a_{n+1}<2$

∴ $|a_{n+1}-a_n|>1$

∴ 令 $\varepsilon=1$ 则 $\forall N$ $n>N$ 时 总有 $|a_{n+1}-a_n|>\varepsilon$ 3n 成立

∴ a_n不收敛

(3) $|a_{n+p}-a_n|=\sum_{k=n+1}^{n+p}\frac{1}{\sqrt{k}}>\sum_{k=n+1}^{n+p}\frac{1}{k}$

当 $p\to+\infty$ 时 $\sum_{k=n+1}^{n+p}\frac{1}{k}\to+\infty$

∴ $\forall\varepsilon, N$ 总 $\exists p$ 使 $|a_{n+p}-a_n|>\varepsilon$ $n>N$

∴ a_n不收敛

1.5.8

$$|a_{n+p}-a_n| \le |a_{n+p}-a_{n+p-1}| + \cdots + |a_{n+1}-a_n|$$

$$\le |a_{n+1}-a_n|(1+q+\cdots+q^{p-1})$$

$$< |a_{n+1}-a_n| \cdot \frac{1}{1-q}$$

$$\le \frac{a_2-a_1}{1-q} \cdot q^{n-1}$$

$\forall \varepsilon>0$ $\left|\frac{a_2-a_1}{1-q} \cdot q^{n-1}\right|<\varepsilon$ 只需 $n > \log_q \frac{1-q}{a_2-a_1} \cdot \varepsilon + 1$

$\therefore$ 取 $N=\max\left\{1, \left[\log_q \frac{1-q}{a_2-a_1} \cdot \varepsilon + 1\right]\right\}$ $|a_{n+p}-a_n|<\varepsilon$ 成立

$\therefore$ a_n 收敛

扫描全能王 创建

2.2

3. (1) 对 $\forall \varepsilon > 0$

$\left|\sqrt{5+x^2}-3\right| < \varepsilon \Leftrightarrow 3-\varepsilon < \sqrt{5+x^2} < 3+\varepsilon$

$\Leftrightarrow 4+\varepsilon^2-6\varepsilon < x^2 < \varepsilon^2+6\varepsilon+4$

而 $\varepsilon^2-6\varepsilon+4 < (2-\varepsilon)^2 < (\varepsilon+2)^2 < \varepsilon^2+6\varepsilon+4$

取 $|x-2| < \frac{\varepsilon}{2}$ 则使上式成立.

故 $\lim\limits_{x\to 2}\sqrt{x^2+5} = 3$

(3) 对 $\varepsilon > 0$ 不妨设 $|x-2| < \frac{1}{2}$

由 $\left|\frac{x^2-3}{x^2-4x+3}-(-1)\right| = \left|\frac{2x^2-4x}{x^2-4x+3}\right| = \left|\frac{2x(x-2)}{(x-1)(x-3)}\right|$

$< \frac{20}{3}|x-2|$ 在 $|x-2| < \frac{3\varepsilon}{20}$ 时 $< \varepsilon$

故令 $\delta = \min\left(\frac{3\varepsilon}{20}, \frac{1}{2}\right)$

对 $\forall \varepsilon > 0$ 由 $\forall x \in U(2,\delta)$ 有 $\left|\frac{x^2-3}{x^2-4x+3}\right| < \varepsilon$

(5) 对 $\varepsilon > 0$

由 $\left|x+\sqrt{x^2-a}-0\right| = \left|\frac{a}{\sqrt{x^2-a}-x}\right| < \left|\frac{a}{x}\right|$

故取 $M = \frac{a}{\varepsilon}$, $a < M$ 时

$\left|x+\sqrt{x^2-a}\right| < \varepsilon$

即 $\lim\limits_{x\to\infty}(x+\sqrt{x^2-a}) = 0$

(7) 对 $\forall \varepsilon > 0$

由 $\left|\sin\sqrt{x+1}-\sin\sqrt{x}\right| < \left|\sqrt{x+1}-\sqrt{x}\right| = \left|\frac{1}{\sqrt{x+1}+\sqrt{x}}\right| < \left|\frac{1}{\sqrt{x}}\right|$

当 $x > M = \frac{1}{\varepsilon^2}$ 时上式 $< \varepsilon$

故 $\lim\limits_{x\to+\infty}(\sin\sqrt{x+1}-\sin\sqrt{x}) = 0$

(9) 对 $\forall \varepsilon > 0$ 令 $\delta = \varepsilon$

$\left|\arctan x - \arctan x_0\right| = \left|\arctan\frac{x-x_0}{1+x\cdot x_0}\right| < \left|\arctan(x-x_0)\right|$

$< |x-x_0| < \varepsilon$

$\Rightarrow \lim\limits_{x\to x_0}\arctan x = \arctan x_0$

7. 取 $x_i=[a]+1, [a]+2\cdots$ 则 $x_n\to\infty$

从而 $\lim\limits_{n\to\infty} f(x_n)$ 由单调收敛原理存在. 记为 M

对 $\forall\varepsilon$, 设 $\exists n_0$ s.t. 对 $\forall n>n_0$, $|f(x_n)-M|<\varepsilon$

对应 x_{n_0}

从而对 $\forall\varepsilon$, $\exists n_1=x_{n_0}$ s.t. $\forall n>n_1$, $|f(n)-M|<\varepsilon$

又由于 $f(x)$ 单调有界 $f(x_n)<f(n)<M$. 即证.

8. 若 $f(x)\neq c$, $c\in R$

设 $f(x_1)=a$, $f(x_2)=b$

则取 $x_1, x_1+T\cdots$ 与 $y_1, y_1+T\cdots$ $(T>0)$ 则 x_1+nT 与 y_1+nT

$\lim\limits_{n\to\infty} x_1+nT=a$, $\lim\limits_{n\to\infty} y_1+nT=b$ $a\neq b$. 发散至无穷

$\Rightarrow \lim\limits_{n\to\infty} f(n)$ 不存在. 矛盾!

故 $f(x)\equiv C$

若 $c\neq 0$. 则 $\exists N$ s.t. $\forall n>N$. $|f(n)-c|>\frac{c}{2}$

$\Rightarrow f(n)\in(\frac{c}{2}, \frac{3}{2}c)$ 与 $\lim\limits_{n\to\infty} f(n)=0$ 矛盾!

故 $c=0$

从而 $f(x)\equiv 0$

习题 2.3

0

6. (1) 由函数连续性知 $\lim\limits_{x\to 2}(5-3x)(3x-1)=(5-3\times2)(3\times2-1)=-5$

(3) $\lim\limits_{x\to 2}\frac{x(x-2)}{(x-2)(x-1)}=\lim\limits_{x\to 2}\frac{x}{x-1}=2$

(5) $\lim\limits_{x\to -\infty}\frac{1-x-4x^3}{1+x^2+2x^3}=\frac{\lim\limits_{x\to-\infty}(-4-\frac{1}{x^2}+\frac{1}{x^3})}{\lim\limits_{x\to-\infty}(2+\frac{1}{x}+\frac{1}{x^3})}=-2$

(7) $\lim\limits_{h\to 0}\frac{(x+h)^3-x^3}{h}=3x^2$

(9) $\lim\limits_{x\to 0}\frac{\sqrt{x^2+p^2}-p}{\sqrt{x^2+q^2}-q}=\lim\limits_{x\to 0}\frac{x^2}{\sqrt{x^2+p^2}+p}\cdot\frac{\sqrt{x^2+q^2}+q}{x^2}=\frac{q}{p}$

(11) $\lim\limits_{x\to 1}\frac{x^m-1}{x-1}=\lim\limits_{x\to 1}\frac{(x-1)(x^{m-1}+x^{m-2}+\cdots+1)}{x-1}=m$

(13) $\lim\limits_{x\to 1}\frac{x+x^2+\cdots+x^n-n}{x-1}=\lim\limits_{x\to 1}\frac{(x-1)+(x^2-1)+\cdots+(x^n-1)}{x-1}=\lim\limits_{x\to 1}\frac{(x-1)[1+(1+x)+\cdots+(1+x+\cdots+x^{n-1})]}{x-1}$

$=1+2+\cdots+n=\frac{n(n+1)}{2}$

(15) $\lim\limits_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2}=\lim\limits_{x\to 0}\frac{(1+mx)^n-1-[(1+nx)^m-1]}{x^2}=\lim\limits_{x\to 0}\frac{\cancel{nmx-nmx}}{x^2}=0$

$=\lim\limits_{x\to 0}\frac{C_n^2m^2x^2+nmx+1-(C_m^2n^2x^2+mnx+1)}{x^2}+o(x)$

$=\frac{n(n-1)}{2}m^2-\frac{m(m-1)}{2}\cdot n^2=\frac{mn(n-m)}{2}$

(17) $\cancel{\lim\limits_{x\to 0}x[\frac{1}{x}]}$ $x\to 0^+$, $1=x\cdot\frac{1}{x}<x[\frac{1}{x}]<x(\frac{1}{x}+1)$ $\lim\limits_{x\to 0^+}x(\frac{1}{x}+1)=1$ $\therefore\lim\limits_{x\to 0^+}x[\frac{1}{x}]=1$

由 $x\to 0^-$ $x(\frac{1}{x}-1)<x[\frac{1}{x}]<x\cdot(\frac{1}{x})=1$ $\lim\limits_{x\to 0^-}x(\frac{1}{x}-1)=1$ $\therefore\lim\limits_{x\to 0^-}x[\frac{1}{x}]=1$

$\therefore\lim\limits_{x\to 0}x[\frac{1}{x}]=1$

7. (1) $\lim\limits_{x\to 0}\frac{\sin x^3}{\sin^3 6x}=\lim\limits_{x\to 0}\frac{\sin x^3}{x^3}\cdot\frac{(6x)^3}{\sin^3 6x}\cdot\frac{1}{6^3}=\frac{1}{216}$

(3) $\lim\limits_{x\to 0}\frac{\tan 3x}{x}=3\lim\limits_{x\to 0}\frac{\tan 3x}{3x}=3$

(5) 令 $u=\frac{\pi}{2^n}$ $\lim\limits_{n\to\infty}2^n\sin\frac{\pi}{2^n}=\lim\limits_{u\to 0}\pi\cdot\frac{\sin u}{u}=\pi$. 且 $n\neq n_0$ 时, $u\neq u_0$

(7) 令 $u=\frac{\pi}{2}-2x$ $\lim\limits_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\cos 2x}=\lim\limits_{u\to 0}\frac{-\sqrt{2}\sin\frac{u}{2}}{\sin u}=\lim\limits_{u\to 0}\frac{-\sqrt{2}\sin\frac{u}{2}}{2\cos\frac{u}{2}\cdot\sin\frac{u}{2}}=-\frac{\sqrt{2}}{2}$

(9) $\lim\limits_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\lim\limits_{x\to 1}(1-x)\cdot\frac{\sin\frac{\pi x}{2}}{\sin[\frac{\pi}{2}(1-x)]}=\frac{2}{\pi}$

(11) $\lim\limits_{x\to 0}\frac{\sin^2 ax-\sin^2 bx}{x\sin x}=\lim\limits_{x\to 0}\frac{\frac{1-\cos 2ax}{2}-\frac{1-\cos 2bx}{2}}{x\sin x}=\lim\limits_{x\to 0}\frac{\sin[(a+b)x]\cdot\sin[(a-b)x]}{(a+b)x(a-b)x}\cdot\frac{(a+b)(a-b)x}{\sin x}$

$=a^2-b^2$

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8.

(1) $\lim\limits_{x\to0}\left[(1+kx)^{\frac{1}{kx}}\right]^k=\left[\lim\limits_{x\to0}(1+kx)^{\frac{1}{kx}}\right]^k=e^k$

(2) $\lim\limits_{x\to\infty}\left(\frac{x+n}{x-n}\right)^x=\lim\limits_{x\to\infty}\left[\left(1+\frac{2n}{x-n}\right)^{\frac{x-n}{2n}}\right]^{2n}\left(1+\frac{2n}{x-n}\right)^n=e^{2n}$

(3) $\lim\limits_{x\to0}\left[(1+3\tan x)^{\frac{1}{3}\cot x}\right]^3=e^3$

(4) 令 $\mu=1-\tan x$ $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}=\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\frac{2\tan x}{1-\tan^2 x}}=\lim\limits_{\mu\to0}(1-\mu)^{-\frac{1}{\mu}\left[-\frac{2(1-\mu)}{2-\mu}\right]}$

$=\lim\limits_{\mu\to0}\left[\lim\limits_{\mu\to0}(1-\mu)^{-\frac{1}{\mu}}\right]^{-\frac{2(1-\mu)}{2-\mu}}=e^{-1}$

(5) 令 $\mu=\frac{1}{x-1}$ $\lim\limits_{x\to1}(2x-1)^{\frac{1}{x-1}}=\lim\limits_{\mu\to\infty}\left[\left(1+\frac{2}{\mu}\right)^{\frac{\mu}{2}}\right]^2=\left[\lim\limits_{\mu\to0}\left(1+\frac{2}{\mu}\right)^{\frac{\mu}{2}}\right]^2=e^2$

(6) $\lim\limits_{x\to0}(2\sin x+\cos x)^{\frac{1}{x}}=\lim\limits_{x\to0}(1+2\tan x)^{\frac{1}{2\tan x}\cdot\frac{2\tan x}{x}}\cdot(\cos x)^{\frac{1}{x^2}\cdot x}=\left[\lim\limits_{x\to0}(1+2\tan x)^{\frac{1}{2\tan x}}\right]^{\lim\limits_{x\to0}\frac{2\tan x}{x}}\cdot\left[\lim\limits_{x\to0}(\cos x)^{\frac{1}{x^2}}\right]^{\lim\limits_{x\to0}x}$

$=e^2\cdot\left(e^{-\frac{1}{2}}\right)^0=e^2$

9.

(1) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-x+1}-ax-b\right)=\lim\limits_{x\to-\infty}\frac{(1-a^2)x^2-(1+2ab)x+1-b^2}{\sqrt{x^2-x+1}+ax+b}=\lim\limits_{x\to-\infty}\frac{(1-a^2)x-(1+2ab)+\frac{1-b^2}{x}}{\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+a+\frac{b}{x}}$

易得 $a=-1$ 或 1（舍），原式 $=\lim\limits_{x\to-\infty}\frac{-(1+2ab)}{a}=1-2b=0$ $\therefore b=\frac{1}{2}$

$\therefore a=-1,\ b=\frac{1}{2}$

(2) $\lim\limits_{x\to+\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=\lim\limits_{x\to+\infty}\frac{(1-a)x^2-(a+b)x+1-b}{x+1}=\lim\limits_{x\to+\infty}\frac{(1-a)x-(a+b)+\frac{1-b}{x}}{1+\frac{1}{x}}$

易得 $a=1$ $=\frac{\lim\limits_{x\to+\infty}\left(-1-b+\frac{1-b}{x}\right)}{\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)}=0$ $\therefore b=-1$

$\therefore a=1,\ b=-1$