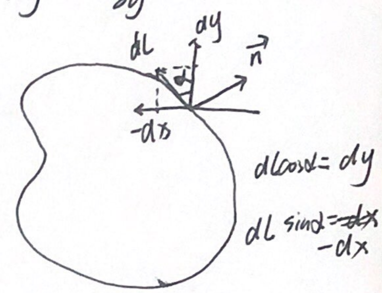
8. (1). $\displaystyle\oint_{\partial D} \frac{\partial u}{\partial \vec{n}} \cdot dL = \oint_{\partial D} \frac{\partial u}{\partial x} \cdot \cos\alpha \, dL + \frac{\partial u}{\partial y} \sin\alpha \, dL = \oint_{\partial D} \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$



$$\overset{\text{Green公式}}{=\!=} \iint_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx\,dy = \iint_D \Delta u \, dx\,dy$$

(2). $\displaystyle\oint_{\partial D} v \frac{\partial u}{\partial \vec{n}} \cdot dL = \oint_{\partial D} v \frac{\partial u}{\partial x} dy - v \frac{\partial u}{\partial y} dx$

$$\overset{\text{Green公式}}{=\!=} \iint_D \left(v \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx\,dy$$

$$= \iint_D v \Delta u \, dx\,dy + \iint_D \nabla u \cdot \nabla v \, dx\,dy$$ （题目符号有误！）

(3). $\displaystyle\oint_{\partial D} \begin{vmatrix} \frac{\partial u}{\partial \vec{n}} & \frac{\partial v}{\partial \vec{n}} \\ u & v \end{vmatrix} dL = \oint_{\partial D} v \frac{\partial u}{\partial \vec{n}} dL - \oint_{\partial D} u \frac{\partial v}{\partial \vec{n}} \cdot dL$

$$\overset{\text{(2)题结论}}{=\!=} \iint_D v \Delta u \, dx\,dy + \iint_D \nabla u \cdot \nabla v \, dx\,dy - \iint_D u \Delta v \, dx\,dy - \iint_D \nabla v \nabla u \, dx\,dy$$

$$= \iint_D v \Delta u \, dx\,dy - \iint_D u \Delta v \, dx\,dy = \iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dx\,dy$$

9. $\vec{n}=\frac{dy}{dL}\vec{i}-\frac{dx}{dL}\vec{j}$

$\cos\langle\vec{n},\vec{i}\rangle=\frac{\vec{n}\cdot\vec{i}}{|\vec{n}||\vec{i}|}=\frac{dy}{dL}$, $\cos\langle\vec{n},\vec{j}\rangle=-\frac{dx}{dL}$

$\oint_L (x\cos\langle\vec{n},\vec{i}\rangle+y\cos\langle\vec{n},\vec{j}\rangle)\,dL=\oint_L x\,dy-y\,dx=\iint_D\left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}\right)dx\,dy$

$=\iint_D 2\,dx\,dy=2S$ ，S为L所围平面区域D的面积.

10. (1). ∵ $\frac{\partial(x^2-y)}{\partial y}=-1=\frac{\partial(-x-\sin^2 y)}{\partial x}$ ∴ 存在 $u(x,y)$，使 $du=(x^2-y)dx+(-x-\sin^2 y)dy$

$\frac{\partial u}{\partial x}=x^2-y$ ， $u=\frac{x^3}{3}-xy+C(y)$，$C(y)$为只与y有关的一元函数

$\frac{\partial u}{\partial y}=-x+C'(y)=-x-\sin^2 y$ ， $C'(y)=-\sin^2 y$， $C(y)=-\frac{y}{2}+\frac{\sin 2y}{4}+C$

∴ $u(x,y)=\frac{x^3}{3}-xy-\frac{y}{2}+\frac{\sin 2y}{4}+C$ 　故解为 $\frac{x^3}{3}-xy-\frac{y}{2}+\frac{\sin 2y}{4}+C=0$.

(2). $\frac{\partial(e^y)}{\partial y}=e^y=\frac{\partial(xe^y-2y)}{\partial x}$ ∴ $du=(e^y)dx+(xe^y-2y)dy$

$\frac{\partial u}{\partial x}=e^y$ ， $u=xe^y+C(y)$， $\frac{\partial u}{\partial y}=xe^y+C'(y)=xe^y-2y$ ，

$C'(y)=-2y$ 　 $C(y)=-y^2+C$ 　∴ $u(x,y)=xe^y-y^2+C=0$，解为 $xe^y-y^2+C=0$

(3). $\frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}+\frac{x\,dy-y\,dx}{x^2}=0 \iff d(\sqrt{x^2+y^2})+d\left(\frac{y}{x}\right)=0$

解为 $\sqrt{x^2+y^2}+\frac{y}{x}+C=0$

(4). ∵ $\frac{\partial(\cos x+\frac{1}{y})}{\partial y}=-\frac{1}{y^2}=\frac{\partial(\frac{1}{y}-\frac{x}{y^2})}{\partial x}$ ∴ $du=(\cos x+\frac{1}{y})dx+(\frac{1}{y}-\frac{x}{y^2})dy$

$\frac{\partial u}{\partial x}=\cos x+\frac{1}{y}$ ， $u=\sin x+\frac{x}{y}+C(y)$， $\frac{\partial u}{\partial y}=-\frac{x}{y^2}+C'(y)=\frac{1}{y}-\frac{x}{y^2}$

$C'(y)=\frac{1}{y}$ 　 $C(y)=\ln y+C$ 　∴ $u(x,y)=\sin x+\frac{x}{y}+\ln y+C$

解为： $\sin x+\frac{x}{y}+\ln y+C=0$

11. (1). $\frac{\partial(y\cos x-x\sin x)}{\partial y}\neq\frac{\partial(y\sin x+x\cos x)}{\partial x}$，方程两边同乘 e^y 得：

$e^y(y\cos x-x\sin x)dx+e^y(y\sin x+x\cos x)dy=0$ 　此时 $\frac{\partial[e^y(y\cos x-x\sin x)]}{\partial y}$

$=\frac{\partial[e^y(y\sin x+x\cos x)]}{\partial x}$

故存在 ~~du~~ $u(x,y)$，$\frac{\partial u}{\partial x} = e^y(y\cos x - x\sin x)$，$u = e^y(y\sin x + x\cos x - \sin x) + C(y)$

$\frac{\partial u}{\partial y} = e^y(y\sin x + x\cos x - \sin x + \sin x) + C'(y) = e^y(y\sin x + x\cos x)$，$\therefore C'(y) = 0 \quad C(y) = C$

$\therefore u = e^y(y\sin x + x\cos x - \sin x) + C$，解为 $e^y(y\sin x + x\cos x - \sin x) + C = 0$

12). $x\,dx + y\,dy + y\,dx - x\,dy = 0$，$\frac{x\,dx + y\,dy}{x^2+y^2} + \frac{y\,dx - x\,dy}{x^2+y^2} = 0$

$d[\frac{1}{2}\ln(x^2+y^2)] + d[\arctan\frac{x}{y}] = 0$，解为 $\frac{1}{2}\ln(x^2+y^2) + \arctan\frac{x}{y} = C$

13). $(3x^3+y)\,dx + (2x^2y - x)\,dy = 0$ 两边同乘 $\frac{1}{x^2}$ 得

$3x\,dx + 2y\,dy + \frac{y\,dx - x\,dy}{x^2} = 0$ $\quad d(\frac{3}{2}x^2 + y^2) - d(\frac{y}{x}) = 0$

$\frac{3}{2}x^2 + y^2 - \frac{y}{x} = C$

14). $dx - dy = \frac{dx + dy}{x+y}$ $\quad d(x-y) = \frac{d(x+y)}{x+y} = d(\ln|x+y|)$

$\therefore x - y - \ln|x+y| = C$

15). $\frac{x^2 - \sin^2 y}{x^2}dx + \frac{x\sin 2y\,dy}{x^2} = (1 - \frac{\sin^2 y}{x^2})dx + \frac{\sin 2y}{x}dy = 0$

$\frac{\partial(1 - \frac{\sin^2 y}{x^2})}{\partial y} = -\frac{2\sin y\cos y}{x^2} = \frac{\partial(\frac{\sin 2y}{x})}{\partial x}$

$\therefore$ ~~du~~ $\frac{\partial u}{\partial x} = 1 - \frac{\sin^2 y}{x^2}$，$u = x + \frac{\sin^2 y}{x} + C(y)$，$\frac{\partial u}{\partial y} = \frac{2\sin x\sin y}{x} + C'(y)$

$C'(y) = 0$，$\therefore u = x + \frac{\sin^2 y}{x} + C$

$\therefore$ 解为 $x + \frac{\sin^2 y}{x} + C = 0$