第5章 Riemann积分

学习材料（10）

1 Riemann积分概念及Riemann积分存在条件

2 Riemann积分的性质

3 变上限积分与原函数的存在性

4 不定积分

4.1 不定积分概念

定义1 设I是个区间，f是定义在I上的函数. 若有函数$F: I \to R$使得

$$F'(x) = f(x),\ \forall x \in I,$$

则称在区间I上F是f的一个原函数。

注1 f有原函数，但f未必可积；f可积，但f未必有原函数。

例1 设$f(x) = [x]$，问f有原函数吗？

解：f没有原函数。反证法，若F是f的一个原函数，则

$$F'(x) = f(x) = [x],$$

因此F'有第一类间断点，但这与“导函数既无可去间断点，也无第一类间断点”的结论矛盾。因此f没有原函数。

定理1 设I是个区间，若$f \in C(I)$，则函数f在区间I上有原函数。

定义2 设I是个区间，f是定义在I上的函数，并假设f有原函数，称函数族

$$\int f(x)dx =: \{H | H\text{是}f\text{在区间}I\text{上的原函数}\}$$

为f在区间I上的不定积分。

设F是f在区间I上的一个原函数，则对任意的常数C，$F(x)+C$也是f在区间I上的原函数。另一方面，如果$G(x)$也是f在区间I上的任一个原函数，则

$$[G(x)-F(x)]'=G'(x)-F'(x)\equiv 0,$$

从而存在常数C，使得$G(x)\equiv F(x)+C.$

定理2 设I是个区间，f是定义在I上的函数。若函数F是f在区间I上的一个原函数，则

$$\int f(x)dx=F(x)+C,$$

其中C表示任意常数。

注1 不定积分的几何意义？

常用不定积分公式：

$$\begin{aligned}
&\int 0dx=C; &&\int \cos x dx=\sin x+C;\\
&\int 1dx=x+C; &&\int \sin x dx=-\cos x+C;\\
&\int x^p dx=\frac{x^{p+1}}{p+1}+C,\ p\neq -1; &&\int \frac{1}{1+x^2}dx=\arctan x+C;\\
&\int \frac{1}{x}dx=\ln|x|+C,\ x\neq 0; &&\int \frac{1}{\sqrt{1-x^2}}dx=\arcsin x+C;\\
&\int e^x dx=e^x+C; &&\int \frac{1}{\cos^2 x}dx=\tan x+C;\\
&\int a^x dx=\frac{a^x}{\ln a}+C,\ a>0,\ a\neq 1; &&\int \frac{1}{\sin^2 x}dx=-\cot x+C;
\end{aligned}$$

注2 初等函数并不一定有初等原函数，例如$\frac{\sin x}{x}$，e^{x^2}没有初等原函数，通常称这些函数积不出来。

注3 设函数F,G分别是函数f,g在区间I上的原函数，α，β是常数。若α，β不全为零，则

$$\begin{aligned}
\int [\alpha f(x)+\beta g(x)]\,dx &= \alpha\int f(x)dx+\beta\int g(x)dx\\
&= \alpha F(x)+\beta G(x)+C.
\end{aligned}$$

这种方法称为线性法。

例3 求$\int \frac{1}{1-x^2}dx$, $\int \frac{1}{\cos^2 x \sin^2 x}dx$.

解：

$$\begin{aligned}\int \frac{1}{1-x^2}dx = \int \frac{1}{2}\left[\frac{1}{1+x}+\frac{1}{1-x}\right]dx &= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + C \\ &= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C.\end{aligned}$$

$$\int \frac{1}{\cos^2 x \sin^2 x}dx = \int \left[\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}\right]dx = \tan x - \cot x + C.$$

4.2 换元积分法

若

$$\int f(u)du = F(u) + C,$$

则$[F(\varphi(x))]' = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x)$, 于是

$$\int f(\varphi(x))\varphi'(x)dx = F(\varphi(x)) + C.$$

故欲求$\int f(\varphi(x))\varphi'(x)dx$，可令$\varphi(x) = u$，将$\int f(\varphi(x))\varphi'(x)dx$变换为$\int f(u)du$，求出结果$F(u)+C$后再把$u = \varphi(x)$代入：

$$\begin{aligned}\int f(\varphi(x))\varphi'(x)dx &=\!=\!=\!=\!= \int f(\varphi(x))d\varphi(x) \\ &\underset{\text{令 } \varphi(x)=u}{=\!=\!=\!=\!=} \int f(u)du \\ &=\!=\!=\!=\!= F(u) + C \\ &\underset{\text{令 } u=\varphi(x)}{=\!=\!=\!=\!=} F(\varphi(x)) + C.\end{aligned}$$

这种方法称为第一换元法，或凑微分法。

若

$$\int f(\varphi(t))\varphi'(t)dt = H(t) + C,$$

则$[H(\varphi^{-1}(x))]' = H'(\varphi^{-1}(x))\frac{1}{\varphi'(\varphi^{-1}(x))} = f(x)$，于是

$$\int f(x)dx = H(\varphi^{-1}(x)) + C.$$

故欲求$\int f(x)dx$，可令$x=\varphi(t)$，将$\int f(x)dx$变换为$\int f(\varphi(t))\varphi'(t)dt$，求出结果$H(t)+C$后再把$t=\varphi^{-1}(x)$代入：

$$\begin{aligned}\int f(x)dx &\xlongequal[\text{令 } x=\varphi(t)]{} \int f(\varphi(t))d\varphi(t)\\ &\xlongequal{} \int f(\varphi(t))\varphi'(t)dt\\ &\xlongequal{} H(t)+C\\ &\xlongequal[\text{令 } t=\varphi^{-1}(x)]{} H(\varphi^{-1}(x))+C.\end{aligned}$$

这种方法称为<u>第二换元法</u>。

例1 求不定积分$\int \tan x dx, \int \frac{1}{\cos x}dx$

解：

$$\begin{aligned}\int \tan x dx \xlongequal{} \int \frac{\sin x}{\cos x}dx &\xlongequal{} \int \frac{-1}{\cos x}d\cos x\\ &\xlongequal[\text{令 } \cos x=u]{} \int \frac{-1}{u}du\\ &\xlongequal{} -\ln|u|+C\\ &\xlongequal[\text{令 } u=\cos x]{} -\ln|\cos x|+C.\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cos x}dx \xlongequal{} \int \frac{\cos x}{\cos^2 x}dx &\xlongequal{} \int \frac{1}{1-\sin^2 x}d\sin x\\ &\xlongequal[\text{令 } \sin x=u]{} \int \frac{1}{1-u^2}du\\ &\xlongequal{} \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+C\\ &\xlongequal[\text{令 } u=\sin x]{} \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C=\ln\left|\frac{1+\sin x}{\cos x}\right|+C.\end{aligned}$$

例2 求不定积分$\int \frac{1}{\sqrt{x^2+1}}dx, \int \frac{1}{\sqrt{x^2-1}}dx$

解：

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+1}}dx &\xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{\text{令}x=\tan t,} \int \cos t d\tan t \\ &\xlongequal{\quad} \int \cos t\cdot\frac{1}{\cos^2 t}dt = \int \frac{1}{\cos t}dt \\ &\xlongequal{\quad} \ln\left[\sec t+\tan t\right]+C \\ &\xlongequal[\text{将}t\text{用}x\text{表示}]{} \ln\left[\sqrt{x^2+1}+x\right]+C.\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-1}}dx &\xlongequal[t\in\left(0,\frac{\pi}{2}\right)]{\text{令}x=\sec t,} \int \frac{1}{\tan t}d\sec t \\ &\xlongequal{\quad} \int \frac{1}{\tan t}\cdot\sec t\cdot\tan t dt = \int \sec t dt == \int \frac{1}{\cos t}dt \\ &\xlongequal{\quad} \ln\left|\sec t+\tan t\right|+C \\ &\xlongequal[\text{将}t\text{用}x\text{表示}]{} \ln\left|x+\sqrt{x^2-1}\right|+C.\end{aligned}$$

4.3 分部积分法

若

$$\int v(x)u'(x)dx = F(x)+C,$$

则$[u(x)v(x)-F(x)]' = u(x)v'(x)+v(x)u'(x)-F'(x) = u(x)v'(x)$, 于是

$$\int u(x)v'(x)dx = u(x)v(x)-F(x)+C = u(x)v(x)-\int v(x)u'(x)dx.$$

由此得到分部积分法：

$$\begin{aligned}\int u(x)v'(x)dx &\xlongequal{\quad} \int u(x)dv(x) \\ &\xlongequal{\quad} u(x)v(x)-\int v(x)du(x) \\ &\xlongequal{\quad} u(x)v(x)-\int v(x)u'(x)dx \\ &\xlongequal{\quad} u(x)v(x)-F(x)+C.\end{aligned}$$

例1 求不定积分$\int x\cos x dx$, $\int xe^x dx$, $\int x\ln x dx$, $\int x\arcsin x dx$.

解：

$$\begin{aligned}\int x\cos x dx &===== \int x d\sin x\\ &===== x\sin x - \int \sin x dx\\ &===== x\sin x + \cos x + C.\end{aligned}$$

$$\begin{aligned}\int x e^x dx &===== \int x d e^x\\ &===== x e^x - \int e^x dx\\ &===== x e^x - e^x + C.\end{aligned}$$

$$\begin{aligned}\int x\ln x dx &===== \frac{1}{2}\int \ln x d x^2\\ &===== \frac{x^2}{2}\ln x - \frac{1}{2}\int x^2 d\ln x\\ &===== \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx\\ &===== \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.\end{aligned}$$

$$\begin{aligned}\int x\arcsin x dx &===== \frac{1}{2}\int \arcsin x d x^2\\ &===== \frac{x^2}{2}\arcsin x - \frac{1}{2}\int x^2 d\arcsin x\\ &===== \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx\\ &===== \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \left[\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right] dx\\ &===== \frac{x^2}{2}\arcsin x - \frac{1}{2}\left[\arcsin x - \frac{1}{2}\left[\arcsin x + x\sqrt{1-x^2}\right]\right] + C\\ &===== \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\arcsin x - x\sqrt{1-x^2}\right] + C.\end{aligned}$$

例2 求不定积分$\int \sqrt{1-x^2}dx$, $\int \sqrt{x^2+1}dx$, $\int \sqrt{x^2-1}dx$.

解、

$$\begin{aligned}\int\sqrt{1-x^2}dx &=\!=\!=\!=\!= x\sqrt{1-x^2}-\int xd\sqrt{1-x^2}\\ &=\!=\!=\!=\!= x\sqrt{1-x^2}-\int x\frac{-x}{\sqrt{1-x^2}}dx\\ &=\!=\!=\!=\!= x\sqrt{1-x^2}-\int\left[\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right]dx\\ &=\!=\!=\!=\!= x\sqrt{1-x^2}-\int\sqrt{1-x^2}dx+\int\frac{1}{\sqrt{1-x^2}}dx\\ &=\!=\!=\!=\!= x\sqrt{1-x^2}-\int\sqrt{1-x^2}dx+\arcsin x,\end{aligned}$$

故

$$\int\sqrt{1-x^2}dx=\frac{1}{2}\left\{x\sqrt{1-x^2}+\arcsin x\right\}+C.$$

$$\begin{aligned}\int\sqrt{x^2+1}dx &=\!=\!=\!=\!= x\sqrt{x^2+1}-\int xd\sqrt{x^2+1}\\ &=\!=\!=\!=\!= x\sqrt{x^2+1}-\int x\frac{x}{\sqrt{x^2+1}}dx\\ &=\!=\!=\!=\!= x\sqrt{x^2+1}-\int\left[\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right]dx\\ &=\!=\!=\!=\!= x\sqrt{x^2+1}-\int\sqrt{x^2+1}dx+\int\frac{1}{\sqrt{x^2+1}}dx\\ &=\!=\!=\!=\!= x\sqrt{x^2+1}-\int\sqrt{x^2+1}dx+\ln\left[\sqrt{x^2+1}+x\right],\end{aligned}$$

故

$$\int\sqrt{x^2+1}dx=\frac{1}{2}\left\{x\sqrt{x^2+1}+\ln\left[\sqrt{x^2+1}+x\right]\right\}+C.$$

$$\begin{aligned}\int\sqrt{x^2-1}dx &=\!=\!=\!=\!= x\sqrt{x^2-1}-\int xd\sqrt{x^2-1}\\ &=\!=\!=\!=\!= x\sqrt{x^2-1}-\int x\frac{x}{\sqrt{x^2-1}}dx\\ &=\!=\!=\!=\!= x\sqrt{x^2-1}-\int\left[\sqrt{x^2-1}+\frac{1}{\sqrt{x^2-1}}\right]dx\\ &=\!=\!=\!=\!= x\sqrt{x^2-1}-\int\sqrt{x^2-1}dx-\int\frac{1}{\sqrt{x^2-1}}dx\\ &=\!=\!=\!=\!= x\sqrt{x^2-1}-\int\sqrt{x^2-1}dx-\ln\left|x+\sqrt{x^2-1}\right|,\end{aligned}$$

故

$$\int\sqrt{x^2-1}dx=\frac{1}{2}\left\{x\sqrt{x^2-1}-\ln\left|x+\sqrt{x^2-1}\right|\right\}+C.$$

例3 求不定积分$\int e^x\cos xdx,\ \int e^x\sin xdx$

解：

$$\begin{aligned}\int e^x\cos xdx \quad &=\!=\!=\!=\!= \quad \int e^x d\sin x\\ &=\!=\!=\!=\!= \quad e^x\sin x-\int\sin x de^x\\ &=\!=\!=\!=\!= \quad e^x\sin x-\int e^x\sin xdx\\ &=\!=\!=\!=\!= \quad e^x\sin x+\int e^x d\cos x\\ &=\!=\!=\!=\!= \quad e^x\sin x+e^x\cos x-\int\cos x de^x\\ &=\!=\!=\!=\!= \quad e^x\sin x+e^x\cos x-\int e^x\cos xdx,\end{aligned}$$

故

$$\int e^x\cos xdx=\frac{1}{2}e^x\left\{\sin x+\cos x\right\}+C.$$
$$\int e^x\sin xdx=\frac{1}{2}e^x\left\{\sin x-\cos x\right\}+C.$$

例4 求不定积分$\int\cos^n xdx\quad(n=2,3,\cdots),\ \int\frac{1}{(x^2+1)^n}dx\quad(n=1,2,\cdots).$

解：令$I_n=\int\cos^n xdx\quad(n=1,2,\cdots)$，则当$n\geq 2$时，

$$\begin{aligned}I_n=\int\cos^n xdx==\int\cos^{n-1}xd\sin x \quad &=\!=\!=\!=\!= \quad \cos^{n-1}x\sin x-\int\sin xd\cos^{n-1}x\\ &=\!=\!=\!=\!= \quad \cos^{n-1}x\sin x+(n-1)\int\sin x\left[\cos^{n-2}x\sin x\right]dx\\ &=\!=\!=\!=\!= \quad \cos^{n-1}x\sin x+(n-1)\int\left[\cos^{n-2}x-\cos^n x\right]dx\\ &=\!=\!=\!=\!= \quad \cos^{n-1}x\sin x+(n-1)\left[I_{n-2}-I_n\right],\end{aligned}$$

故

$$nI_n=\cos^{n-1}x\sin x+(n-1)I_{n-2},$$

即

$$I_n=\frac{1}{n}\cos^{n-1}x\sin x+\frac{n-1}{n}I_{n-2},$$

也即

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx,$$

特别

$$\int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C.$$

再令$I_n = \displaystyle\int \frac{1}{(x^2+1)^n}dx \quad (n = 1, 2, \cdots)$，则

$$I_1 = \int \frac{1}{x^2+1}dx == \arctan x + C,$$

当$n \geq 1$时，

$$\begin{aligned}
I_n = \int \frac{1}{(x^2+1)^n}dx \quad &===== \quad \frac{x}{(x^2+1)^n} - \int x\left[(-n)\frac{2x}{(x^2+1)^{n+1}}\right]dx \\
&===== \quad \frac{x}{(x^2+1)^n} + 2n\int\left[\frac{1}{(x^2+1)^n} - \frac{1}{(x^2+1)^{n+1}}\right]dx \\
&===== \quad \frac{x}{(x^2+1)^n} + 2n\left[I_n - I_{n+1}\right],
\end{aligned}$$

故

$$2nI_{n+1} = \frac{x}{(x^2+1)^n} + (2n-1)I_n,$$

即

$$I_{n+1} = \frac{x}{2n(x^2+1)^n} + \frac{2n-1}{2n}I_n,$$

也即特别

$$\int \frac{1}{(x^2+1)^2}dx = \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x + C.$$

4.4 分式函数的积分

设

$$f(x) = \frac{P(x)}{Q(x)},$$

其中P, Q为x的多项式。求不定积分$\displaystyle\int f(x)dx$的步骤如下：

1. 将f表示成一个整式与一个真分式之和:

$$f(x) = S(x) + \frac{R(x)}{Q(x)},$$

其中S, R, Q仍为x的多项式，且R的方次小于Q的方次。

2. 将Q分解成如下一次因子和二次不可约因子的乘积

$$x + b,\ x^2 + px + q,$$

其中$p^2-4q<0$.

3. 将$\dfrac{R}{Q}$写成如下部分分式的和

$$\frac{A}{x+b},\quad \frac{A}{(x+b)^k},\quad \frac{Bx+D}{x^2+px+q},\quad \frac{Bx+D}{(x^2+px+q)^k}.$$

4.

$$\int\frac{1}{x+b}dx==\ln|x+b|+C,\quad \int\frac{1}{(x+b)^k}dx==\frac{1}{(1-k)\,(x+b)^{k-1}}+C\ (k>1)$$

5.

$$\begin{aligned}\int\frac{Bx+D}{(x^2+px+q)^k}dx \quad \underset{令x=t-\frac{p}{2}}{=====} \quad & \int\frac{Bt+D-\frac{Bp}{2}}{(t^2+q-\frac{p^2}{4})^k}dt\\ ===== \quad & B\int\frac{t}{(t^2+q-\frac{p^2}{4})^k}dt+\left(D-\frac{Bp}{2}\right)\int\frac{1}{(t^2+q-\frac{p^2}{4})^k}dt,\end{aligned}$$

其中$\displaystyle\int\frac{t}{\left(t^2+q-\frac{p^2}{4}\right)^k}dt===\begin{cases}\frac{1}{2}\ln\left(t^2+q-\frac{p^2}{4}\right), & k=1,\\ \frac{1}{2(1-k)\left(t^2+q-\frac{p^2}{4}\right)^{k-1}}, & k\neq 1.\end{cases}$ 而$\displaystyle\int\frac{1}{(t^2+q-\frac{p^2}{4})^k}dt$ 满足3节的递推公式。

注1 分式函数的原函数是初等函数。

例1 求$\displaystyle\int\frac{x^2+2x+2}{(x^2+1)^2}dx$.

解:

$$\frac{x^2+2x+2}{(x^2+1)^2}=\frac{1}{x^2+1}+\frac{2x+1}{(x^2+1)^2},$$

故

$$\begin{aligned}\int\frac{x^2+2x+2}{(x^2+1)^2}dx \quad ===== \quad & \int\frac{1}{x^2+1}dx+\int\frac{2x+1}{(x^2+1)^2}dx\\ ===== \quad & \arctan x+\int\frac{2x}{(x^2+1)^2}dx+\int\frac{1}{(x^2+1)^2}dx\\ ===== \quad & \arctan x-\frac{1}{x^2+1}+\frac{x}{2(x^2+1)}+\frac{1}{2}\arctan x+C\\ ===== \quad & \frac{3}{2}\arctan x-\frac{1}{x^2+1}+\frac{x}{2(x^2+1)}+C.\end{aligned}$$

4.5 三角函数有理式的积分

记$R(u,v)$表示由u,v和常数进行有限次的四则运算后所得的表达式，称$R(\sin x,\cos x)$为三角有理式。三角有理式的积分总可以化为分式函数的积分，过程如下：

$$\int R(\sin x,\cos x)dx \quad ==== \quad \int R\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}},\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\right)dx$$

$$==== \quad \int R\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)dx$$

$$\underset{\substack{\text{令}\tan\frac{x}{2}=t,\\ \text{即}x=2\arctan t}}{=====} \quad \int R\left(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt.$$

注1 三角有理式函数的原函数是初等函数。

例1 求$\int\frac{dx}{4+4\sin x+\cos x}$.

解：

$$\int\frac{dx}{4+4\sin x+\cos x} \quad \underset{\substack{\text{令}\tan\frac{x}{2}=t,\\ \text{即}x=2\arctan t}}{=====} \quad \int\frac{1}{4+4\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$===== \quad 2\int\frac{1}{(t+1)(3t+5)}dt$$

$$===== \quad \int\left(\frac{1}{t+1}-\frac{3}{3t+5}\right)dt$$

$$===== \quad \ln\left|\frac{t+1}{3t+5}\right|+C$$

$$\underset{\text{令}t=\tan\frac{x}{2}}{=====} \quad \ln\left|\frac{\tan\frac{x}{2}+1}{3\tan\frac{x}{2}+5}\right|+C.$$

特殊形式可用特殊变换

a. 若$R(-u,\ v)=-R(u,\ v)$，则有$R(u,v)=u\ \widetilde{R}(u^2,\ v)$，故

$$\int R(\sin x,\ \cos x)dx \quad ===== \quad \int\sin x\ \widetilde{R}(\sin^2 x,\ \cos x)dx=\int\sin x\ \widetilde{R}(1-\cos^2 x,\ \cos x)dx$$

$$\underset{\text{令}\cos x=t}{=====} \quad -\int\widetilde{R}(1-t^2,\ t)dt$$

b. 若$R(u,\ -v)=-R(u,\ v)$，则有$R(u,v)=v\ \widetilde{R}(u,\ v^2)$，故

$$\int R(\sin x,\ \cos x)dx \quad ===== \quad \int \cos x\ \widetilde{R}(\sin x,\ \cos^2 x)dx = \int \cos x\ \widetilde{R}(\sin x,\ 1-\sin^2 x)dx$$

$$\underset{\text{令}\sin x=t}{=====} \quad \int \widetilde{R}(t,\ 1-t^2)dt$$

c. 若$R(-u,\ -v)=R(u,\ v)$，则有$R(u,v)=\widetilde{R}\left(\frac{u}{v},\ v^2\right)$或者$R(u,v)=\widehat{R}\left(u^2,\ \frac{u}{v}\right)$，故

$$\int R(\sin x,\ \cos x)dx \quad ===== \quad \int \widetilde{R}\left(\tan x,\ \cos^2 x\right)dx = \int \widetilde{R}\left(\tan x,\ \frac{1}{1+\tan^2 x}\right)dx$$

$$\underset{\text{令}\tan x=t}{=====} \quad \int \widetilde{R}\left(t,\ \frac{1}{1+t^2}\right)\frac{1}{1+t^2}dt$$

或者

$$\int R(\sin x,\ \cos x)dx \quad ===== \quad \int \hat{R}\left(\sin^2 x,\ \tan x\right)dx = \int \hat{R}\left(\frac{\tan^2 x}{1+\tan^2 x},\ \tan x\right)dx$$

$$\underset{\text{令}\tan x=t}{=====} \quad \int \hat{R}\left(\frac{t^2}{1+t^2},\ t\right)\frac{1}{1+t^2}dt$$

例2 求$\int \frac{dx}{a\cos x+b\sin x}$.

解：

$$\int \frac{dx}{a\cos x+b\sin x} \quad \underset{\substack{\text{令}\cos\varphi=\frac{a}{\sqrt{a^2+b^2}},\\ \sin\varphi=\frac{b}{\sqrt{a^2+b^2}}}}{=====} \quad \int \frac{dx}{\sqrt{a^2+b^2}\left[\cos\varphi\cos x+\sin\varphi\sin x\right]}$$

$$===== \quad \int \frac{dx}{\sqrt{a^2+b^2}\cos(\varphi-x)}$$

$$===== \quad \int \frac{dx}{\sqrt{a^2+b^2}\cos(x-\varphi)}$$

$$===== \quad \frac{1}{2\sqrt{a^2+b^2}}\ln\left|\frac{1+\sin(x-\varphi)}{1-\sin(x-\varphi)}\right|+C$$

$$===== \quad \frac{1}{2\sqrt{a^2+b^2}}\ln\left|\frac{1+\sin x\cos\varphi-\cos x\sin\varphi}{1-\sin x\cos\varphi+\cos x\sin x}\right|+C$$

$$===== \quad \frac{1}{2\sqrt{a^2+b^2}}\ln\left|\frac{\sqrt{a^2+b^2}+a\sin x-b\cos x}{\sqrt{a^2+b^2}-a\sin x+b\cos x}\right|+C.$$

4.6 简单无理式的积分

记$R(u,v)$表示由u,v和常数进行有限次的四则运算后所得的表达式。

1. $$\int R(x, \sqrt[n]{ax+b})dx \xlongequal[\text{即}x=\frac{t^n-b}{a}]{\text{令}\sqrt[n]{ax+b}=t,} \int \frac{n}{a}t^{n-1}\cdot R\left(\frac{t^n-b}{a}, t\right)dt.$$

2. $$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx \xlongequal[\text{即}x=\frac{dt^n-b}{a-ct^n}]{\text{令}\sqrt[n]{\frac{ax+b}{cx+d}}=t,} \int \frac{n(ad-bc)t^{n-1}}{(a-ct^n)^2}\cdot R\left(\frac{dt^n-b}{a-ct^n}, t\right)dt$$

3. $$\int R\left(x, \sqrt{1-x^2}\right)dx \xlongequal[t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]]{\text{令}x=\sin t,} \int \cos t\cdot R(\sin t, \cos t)dt$$

4. $$\int R\left(x, \sqrt{x^2+1}\right)dx \xlongequal[t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)]{\text{令}x=\tan t,} \int \frac{1}{\cos^2 t}\cdot R\left(\frac{\sin x}{\cos x}, \frac{1}{\cos t}\right)dt$$

5. $$\int R\left(x, \sqrt{x^2-1}\right)dx \xlongequal[t\in\left(0,\frac{\pi}{2}\right)]{\text{令}x=\sec t,} \int \frac{\sin t}{\cos^2 t}\cdot R\left(\frac{1}{\cos x}, \frac{\sin t}{\cos t}\right)dt.$$

5 定积分的计算

定理1 设$f \in C[a,b]$，$\varphi : [\alpha,\beta] \to [a,b]$可导，且$\varphi' \in R[\alpha,\beta]$，则

$$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du.$$

注1 若$\varphi(\alpha) > \varphi(\beta)$，规定

$$\int_{(\alpha)}^{(\beta)} f(u)du = -\int_{\varphi(\beta)}^{\varphi(\alpha)} f(u)du.$$

证：由$f \in C[a,b]$知f在区间$[a,b]$上有原函数F，于是当$\varphi(\alpha) < \varphi(\beta)$时，

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du &= F(u)|_{\varphi(\alpha)}^{\varphi(\beta)} \quad \text{（Newton-Leibniz公式）}\\ &= F(\varphi(\beta)) - F(\varphi(\alpha));\end{aligned}$$

当$\varphi(\alpha) = \varphi(\beta)$时，

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du &= 0\\ &= F(\varphi(\beta)) - F(\varphi(\alpha));\end{aligned}$$

而当$\varphi(\alpha) > \varphi(\beta)$时，

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du &= -\int_{\varphi(\beta)}^{\varphi(\alpha)} f(u)du\\ &= -F(u)|_{\varphi(\beta)}^{\varphi(\alpha)} \quad \text{（Newton-Leibniz公式）}\\ &= F(\varphi(\beta)) - F(\varphi(\alpha)).\end{aligned}$$

综上，

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du = F(\varphi(\beta)) - F(\varphi(\alpha)).$$

另一方面，由$f \in C[a,b]$，$\varphi : [\alpha,\beta] \to [a,b]$可导知$f \circ \varphi \in C[\alpha,\beta]$；易知函数

$$F \circ \varphi : [\alpha,\beta] \to R$$

是函数$(f \circ \varphi) \cdot \varphi' : [\alpha,\beta] \to R$的一个原函数，故由Newton-Leibniz公式，有

$$\begin{aligned}\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx &= F(\varphi(x))|_{\alpha}^{\beta}\\ &= F(\varphi(\beta)) - F(\varphi(\alpha)),\end{aligned}$$

所以

$$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du.$$

注2 称如下计算定积分$\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx$ 的过程为<u>第一换元法</u>：

$$\begin{aligned}\int_{\alpha}^{\beta} f(\varphi(x))\varphi'(x)dx &\overset{}{=\!=\!=\!=\!=} \int_{\alpha}^{\beta} f(\varphi(x))d\varphi(x) \\ &\underset{\text{令 } \varphi(x)=u}{=\!=\!=\!=\!=} \int_{\varphi(\alpha)}^{\varphi(\beta)} f(u)du.\end{aligned}$$

称如下计算定积分$\int_{g(\alpha)}^{g(\beta)} f(x)dx$ 的过程为<u>第二换元法</u>：

$$\begin{aligned}\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx &\underset{\text{令 } x=\varphi(t)}{=\!=\!=\!=\!=} \int_{\alpha}^{\beta} f(\varphi(t))d\varphi(t) \\ &=\!=\!=\!=\!= \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.\end{aligned}$$

定理2 设$f,g:[a,b]\to R$可导，且$f',g'\in R[a,b]$，则

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx.$$

证：由$f',g'\in R[a,b]$知，$f\cdot g'$，$f'\cdot g\in R[a,b]$，从而$[f\cdot g]'\in R[a,b]$，于是

$$\begin{aligned}\int_a^b f(x)g'(x)dx+\int_a^b f'(x)g(x)dx &=\!=\!=\!=\!= \int_a^b [f(x)g'(x)+f'(x)g(x)]dx \quad \text{（积分的线性性质）} \\ &=\!=\!=\!=\!= \int_a^b [f(x)g(x)]'dx \\ &=\!=\!=\!=\!= f(x)g(x)|_a^b \quad \text{（}[f\cdot g]'\in R[a,b]\text{，}f\cdot g \text{ 是}(f\cdot g)'\text{的一个原函数），}\end{aligned}$$

所以

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx.$$

注3 称如下计算定积分$\int_a^b f(x)g'(x)dx$的过程为<u>分部积分法</u>：

$$\begin{aligned}\int_a^b f(x)g'(x)dx &=\!=\!=\!=\!= \int_a^b f(x)dg(x) \\ &=\!=\!=\!=\!= f(x)g(x)|_a^b - \int_a^b g(x)df(x) \\ &=\!=\!=\!=\!= f(x)g(x)|_a^b - \int_a^b g(x)f'(x)dx.\end{aligned}$$

例1 求定积分$\int_0^{\frac{\pi}{4}}\frac{1}{\cos x}dx$，$\int_0^{\frac{1}{2}}\sqrt{1-x^2}dx$

解：

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\frac{1}{\cos x}dx === \int_0^{\frac{\pi}{4}}\frac{\cos x}{\cos^2 x}dx &===== \int_0^{\frac{\pi}{4}}\frac{1}{1-\sin^2 x}d\sin x\\ &\underset{\text{令 }\sin x=u}{=====} \int_0^{\frac{\sqrt{2}}{2}}\frac{1}{1-u^2}du\\ &===== \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)\Big|_0^{\frac{\sqrt{2}}{2}}\\ &==== \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) == \ln\left(\sqrt{2}+1\right).\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{1}{2}}\sqrt{1-x^2}dx &\underset{\substack{\text{令}x=\sin t,\\ t\in\left[0,\arcsin\frac{1}{2}\right]}}{=====} \int_0^{\arcsin\frac{1}{2}}\cos t d\sin t\\ &===== \int_0^{\arcsin\frac{1}{2}}\cos t\cos t dt\\ &===== \int_0^{\arcsin\frac{1}{2}}\frac{1+\cos 2t}{2}dt\\ &===== \left[\frac{t}{2}+\frac{\sin 2t}{4}\right]\Big|_0^{\arcsin\frac{1}{2}} = \frac{\arcsin\frac{1}{2}}{2}+\frac{\sqrt{3}}{8}.\end{aligned}$$

例2 求定积分 1. $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$; 2. $\int_0^{\frac{\pi}{2}}\frac{\sin^p x}{\sin^p x+\cos^p x}dx$; 3. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\sin^2 x}{1+e^{-x}}dx$.

解：1.

$$\begin{aligned}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx &\underset{\text{令}x=\pi-t}{=====} \int_{\pi}^{0}\frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}d(\pi-t)\\ &===== \int_0^{\pi}\frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}dt\\ &===== \int_0^{\pi}\frac{(\pi-t)\sin t}{1+\cos^2 t}dt,\end{aligned}$$

故

$$\begin{aligned}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx &===== \frac{1}{2}\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}dx\\ &===== -\frac{\pi}{2}\arctan(\cos x)\Big|_0^{\pi} == \frac{\pi^2}{4}.\end{aligned}$$

2.
$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin^p x}{\sin^p x+\cos^p x}dx & \xlongequal[\text{令}x=\frac{\pi}{2}-t]{} \int_{\frac{\pi}{2}}^0 \frac{\sin^p\left(\frac{\pi}{2}-t\right)}{\sin^p\left(\frac{\pi}{2}-t\right)+\cos^p\left(\frac{\pi}{2}-t\right)}d\left(\frac{\pi}{2}-t\right)\\
& ===== \int_0^{\frac{\pi}{2}} \frac{\sin^p\left(\frac{\pi}{2}-t\right)}{\sin^p\left(\frac{\pi}{2}-t\right)+\cos^p\left(\frac{\pi}{2}-t\right)}dt\\
& ===== \int_0^{\frac{\pi}{2}} \frac{\cos^p t}{\cos^p t+\sin^p t}dt,
\end{aligned}
$$
故
$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\sin^p x}{\sin^p x+\cos^p x}dx & ===== \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin^p x}{\sin^p x+\cos^p x}dx+\int_0^{\frac{\pi}{2}} \frac{\cos^p t}{\cos^p t+\sin^p t}dt\right]\\
& ===== \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} dx\right] == \frac{\pi}{4}.
\end{aligned}
$$
3.
$$
\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1+e^{-x}}dx & \xlongequal[\text{令}x=-t]{} \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\sin^2 t}{1+e^t}d(-t)\\
& ===== \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 t}{1+e^t}dt,
\end{aligned}
$$
故
$$
\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1+e^{-x}}dx & ===== \frac{1}{2}\left[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1+e^{-x}}dx+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 t}{1+e^t}dt\right]\\
& ===== \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\left[\frac{1}{1+e^{-x}}+\frac{1}{1+e^x}\right]dx\\
& ===== \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx\\
& ===== \underline{\int_0^{\frac{\pi}{4}} \sin^2 x dx}\\
& ===== \left.\left(\frac{x}{2}-\frac{\sin 2x}{4}\right)\right|_0^{\frac{\pi}{4}} = \frac{\pi}{8}-\frac{1}{4}.
\end{aligned}
$$

例3 求定积分$\int_0^{\frac{\pi}{2}} x\cos x dx$，$\int_0^{\frac{\pi}{2}} \cos^n x dx$， $\int_0^{\frac{\pi}{2}} \sin^n x dx \quad (n=2,3,\cdots)$.

解：
$$
\begin{aligned}
\int_0^{\frac{\pi}{2}} x\cos x dx & ===== \int_0^{\frac{\pi}{2}} x d\sin x\\
& ===== x\sin x|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx = \frac{\pi}{2}+\cos x|_0^{\frac{\pi}{2}} == \frac{\pi}{2}-1.
\end{aligned}
$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\sin^n x dx & \xlongequal[\text{令 } x=\frac{\pi}{2}-t]{} \int_{\frac{\pi}{2}}^{0}\sin^n\left(\frac{\pi}{2}-t\right)d\left(\frac{\pi}{2}-t\right) \\ & \xlongequal{} \int_0^{\frac{\pi}{2}}\sin^n\left(\frac{\pi}{2}-t\right)dt \xlongequal{} \int_0^{\frac{\pi}{2}}\cos^n t dt.\end{aligned}$$

令$I_n=\displaystyle\int_0^{\frac{\pi}{2}}\cos^n x dx \quad (n=1,2,\cdots)$，则当$n\geq 2$ 时，

$$\begin{aligned} I_n & \xlongequal{} \int_0^{\frac{\pi}{2}}\cos^{n-1}x d\sin x \\ & \xlongequal{} \cos^{n-1}x\sin x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin x d\cos^{n-1}x \\ & \xlongequal{} (n-1)\int_0^{\frac{\pi}{2}}\sin x\left[\cos^{n-2}x\sin x\right]dx \\ & \xlongequal{} (n-1)\int_0^{\frac{\pi}{2}}\left[\cos^{n-2}x-\cos^n x\right]dx \\ & \xlongequal{} (n-1)\left[I_{n-2}-I_n\right],\end{aligned}$$

故

$$I_n=\frac{n-1}{n}I_{n-2},$$

所以

$$\begin{aligned} I_{2k} &= \frac{2k-1}{2k}I_{2(k-1)} & I_{2k+1} &= \frac{2k}{2k+1}I_{2k-1} \\ &= \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}I_{2(k-2)} & &= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}I_{2k-3} \\ &= \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdots\cdot\frac{1}{2}I_0 & &= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\cdot\frac{2}{3}I_1 \\ &= \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdots\cdot\frac{1}{2}\cdot\frac{\pi}{2} & &= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\cdot\frac{2}{3} \\ &=: \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}, & &=: \frac{(2k)!!}{(2k+1)!!}.\end{aligned}$$

例4 Taylor公式的积分余项形式：设f在$[a,b]$上有$n+1$阶连续导数，$x_0\in[a,b]$。则$\forall x\in[a,b]$，有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{1}{n!}\int_{x_0}^{x}(x-u)^n f^{(n+1)}(u)du.$$

证：当$n=1$时，根据Newton-Leibnitz公式，

$$f(x)=f(x_0)+\int_{x_0}^{x}f'(u)du.$$

应用分部积分法，

$$\begin{aligned}\int_{x_0}^{x} f'(u)du == \int_{x_0}^{x} f'(u)d(u-x) &===== (u-x)f'(u)|_{x_0}^{x} - \int_{x_0}^{x}(u-x)f''(u)du \\ &===== f'(x_0)(x-x_0) + \int_{x_0}^{x}(x-u)f''(u)du,\end{aligned}$$

结论成立。假定当$n=m$时，结论成立，即

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m + \frac{1}{m!}\int_{x_0}^{x}(x-u)^m f^{(m+1)}(u)du.$$

应用分部积分法，

$$\begin{aligned}\frac{1}{m!}\int_{x_0}^{x}(x-u)^m f^{(m+1)}(u)du &=== -\frac{1}{(m+1)!}\int_{x_0}^{x} f^{(m+1)}(u)d(x-u)^{m+1} \\ &=== -\frac{1}{(m+1)!}(x-u)^{m+1}f^{(m+1)}(u)\Big|_{x_0}^{x} + \frac{1}{(m+1)!}\int_{x_0}^{x}(x-u)^{m+1}f^{(m+2)}(u)du \\ &=== \frac{1}{(m+1)!}f^{(m+1)}(x_0)(x-x_0)^{m+1} + \frac{1}{(m+1)!}\int_{x_0}^{x}(x-u)^{m+1}f^{(m+2)}(u)du.\end{aligned}$$

故

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{1}{(m+1)!}f^{(m+1)}(x_0)(x-x_0)^{m+1} + \frac{1}{(m+1)!}\int_{x_0}^{x}(x-u)^{m+1}f^{(m+2)}(u)du.$$

所以当$n=m+1$时，结论成立。根据数学归纳法，结论对任何自然数n成立。