$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$

期末模拟

1. $z = x f(x, \frac{x}{y})$ 设 $u = x$, $v = \frac{x}{y}$.

$\frac{\partial z}{\partial x} = 1 \cdot f(x, \frac{x}{y}) + x \cdot [f_1 \times 1 + f_2 \times \frac{1}{y}]$

$= f(x, \frac{x}{y}) + x(f_1 + \frac{1}{y} f_2)$

$\therefore \cancel{\frac{\partial^2 z}{\partial x^2} = f_1 \times 1 + f_2 \times \frac{1}{y}}$

$\therefore \frac{\partial^2 z}{\partial x \partial y} = \cancel{f_1 \times 0 +} f_2 \times (\frac{-x}{y^2}) + x \times [\cancel{f_{11} \times 0} + f_{12} \times (\frac{-x}{y^2})$

$+ \frac{1}{y} f_{22} \times (\frac{-x}{y^2}) + (-\frac{1}{y^2}) f_2]$

$= f_2 \times (\frac{-2x}{y^2}) + f_{12}(\frac{-x^2}{y^2}) + f_{22}(\frac{-x}{y^3})$

2. $xyz + \sqrt{x^2+y^2+z^2} = \sqrt{2}$, $z(x,y)$. $x=1, y=0, z=1$.

$\therefore yz + x \cdot y z_x + \frac{1}{2}(x^2+y^2+z)^{-\frac{1}{2}} \cdot (2x + z_x) = (\sqrt{2})' = 0$

$\Rightarrow 0 \quad \frac{1}{2} \times \frac{1}{\sqrt{2}} (2 + z_x) = 0$

$\therefore z_x = -2$

又$\because$ $1 \cdot xz + y \cdot x z_y + \frac{1}{2}(x^2+y^2+z)^{-\frac{1}{2}} \cdot (2y + z_y) = 0$.

$1 + \frac{1}{2\sqrt{2}} z_y = 0$

$\therefore z_y = -2\sqrt{2}$.

$\therefore dz = -2dx - 2\sqrt{2} dy$

3. $d^2 = \dfrac{(x+y-8)^2}{2}$

$\therefore \begin{cases} d^2 \quad \min \\ st. \quad x^2+2xy+3y^2-8y=0. \end{cases}$

令 $L = \dfrac{1}{2}(x+y-8)^2 - \lambda(x^2+2xy+3y^2-8y)$

$Lx = \dfrac{1}{2}\times 2(x+y-8) - \lambda(2x+2y)=0$

$Ly = \dfrac{1}{2}\times 2(x+y-8) - \lambda(2x+6y-8)=0.$

$\therefore \quad y=2, \quad x = -2\pm 2\sqrt{2}.$

$\therefore$ ① $x=-2+2\sqrt{2}, \ y=2$

$x+y-8 = 2\sqrt{2}-8.$

$\therefore d^2\min = \dfrac{(2\sqrt{2}-8)^2}{2} \Rightarrow d\min = 4\sqrt{2}-2$

4. $\because \ x^2+y^2 \le R^2.$ $\qquad \dfrac{1}{2}(1+\cos 2\theta)$

设 $\begin{cases} x = \rho\cos\theta, & \rho\in[0,R], \\ y=\rho\sin\theta & \theta\in[0,2\pi]. \end{cases}$

$\therefore$ 原式 $= \displaystyle\int_0^{2\pi} d\theta \int_0^R \left(\frac{\rho^2\cos^2\theta}{a^2} + \frac{\rho^2\sin^2\theta}{b^2}\right)\rho\, d\rho$

$= \displaystyle\int_0^{2\pi} \left(\frac{R^4}{4a^2}\cos^2\theta + \frac{R^4}{4b^2}\sin^2\theta\right) d\theta$

$= \dfrac{\pi R^4}{4}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}\right)$

在xy的投影为: D: $x^2+y^2=1$.

5. $\therefore \iiint_{\Omega} (x^2+y^2)\,dxdydz$

$$= \iint_D dxdy \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)\,dz.$$

$$= \iint_D (x^2+y^2)[2-2(x^2+y^2)]\,dxdy$$

$$= 2\iint_D (x^2+y^2)[1-x^2-y^2]\,dxdy$$

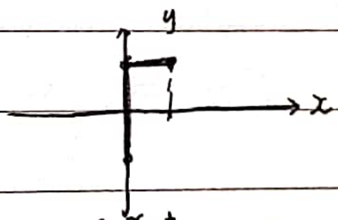
$x=\rho\cos\theta$, $y=\rho\sin\theta$. $\rho\in[0,1]$, $\theta\in[0,2\pi]$

$$= 2\int_0^{2\pi} d\theta \int_0^1 \rho^2(1-\rho^2)\rho\,d\rho$$

$\rho^3-\rho^5$

$$= 4\pi \times \cancel{(\tfrac{1}{3}-\tfrac{1}{6})}\ (\tfrac{1}{4}-\tfrac{1}{6}) = \frac{\pi}{3}$$

6. ☆ $\int_L (x+y)\,dl.$



① $(1,1)\to(0,1). \Rightarrow y=1.$ $\begin{cases} x=t \\ y=1, \end{cases} t\in[1,0].$

$\therefore \int_{L_1} = \int_1^0 (t+1)\times\sqrt{1}\,dt = \frac{1}{2}t^2+t\Big|_1^0 = -\frac{3}{2}.$

② $(0,1)\to(0,-1).$ $\begin{cases} x=0 \\ y=t \end{cases}$ $t\in[1,-1]$

$$\int_1^{-1} t\sqrt{1}\,dt = 0.$$

$$\int_{L^+} (x+y)\,dx + (x+y)\,dy$$

$$= -\frac{3}{2}$$

7. $$\iint_S (2y+z)\,dz\wedge dx + z\,dx\wedge dy$$

$$\vec{F} = (0, 2y+z, z)$$

$$\vec{n} = (\ ,\ ,\ )$$

设 $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$, $\rho\in[0,1]$, $\theta\in[0,2\pi]$

$\therefore$ $\vec{F} = (0, 2\rho\sin\theta+z, z)$

$\vec{r_\theta} = (-\rho\sin\theta, \rho\cos\theta, 0)$

$\vec{r_z} = (0, 0, 1)$

$\therefore$ $\vec{r_\theta}\times\vec{r_z} = (\rho\cos\theta, -\rho\sin\theta, 0)$

$\therefore$ 原式 $= -\int_0^{2\pi} d\theta \int_0^1 dz \int_0^1 2\rho^2\sin^2\theta + \rho\sin\theta z\,\rho\,d\rho$

$= -\frac{1}{2}\pi$

8. $a_0 = \int_0^2 f(x)\,dx = \frac{1}{2}$

$a_n = \int_0^1 x\cos n\pi x\,dx = \frac{(-1)^n-1}{(n\pi)^2}$

$b_n = \int_0^1 x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}$

$\therefore f(x) \sim \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n-1}{(n\pi)^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x \right]$

$$S(x) = \begin{cases} x & 0<x<1 \\ \frac{1}{2} & x=1 \\ 0 & 1<x<2 \end{cases}$$

9\. (1) $S_n(x)=\sum_{n=0}^{+\infty} a_n x^n$

$$S_n'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$S_n''(x)=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

又 $\because$ $S''(x)-2xS'(x)-4S(x)=0.$

$\therefore \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-2x\sum_{n=1}^{\infty} n a_n x^{n-1}-4\sum_{n=0}^{+\infty} a_n x^n=0$

$\nearrow$ 代2. $\quad\quad \nearrow a_0=0$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n-2}-2x\sum_{n=0}^{+\infty}(n+1)a_n x^n$$

$$-4\sum_{n=0}^{\infty} a_n x^n=0.$$

$\therefore$ $(n+2)(n+1)a_{n+2}-2na_n-4a_n=0$

$\Rightarrow$ $a_{n+2}=\frac{2}{n+1}a_n.$

(2) $\because$ $a_0=0$, $a_1=1$

$\therefore$ $a_{2n}=0$, $a_{2n+1}=\frac{1}{n!}$

$\therefore$ $S(x)=\sum_{n=0}^{+\infty} a_{2n+1}x^{2n+1}$

$$=xe^{x^2}.$$