Review

- $f(x)$以$2l$ 为周期,在$[-l,l]$上可积或广义绝对可积,则

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx,\ n = 0,1,2,\cdots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx,\ n = 1,2,\cdots$$

- 定义内积$(f,g) \triangleq \dfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$,则

$$\{\varphi_n\} = \left\{\sqrt{2}/2, \cos x, \sin x, \cdots, \cos nx, \sin nx, \cdots\right\}$$

为$\Re[-\pi,\pi]$中标准正交函数系.

- $$\left\| f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\right)\right\|$$

$$= \min_{\lambda_k,\mu_k \in \mathbb{R}} \left\| f(x) - \left(\frac{\lambda_0}{2} + \sum_{k=1}^{n}(\lambda_k \cos kx + \mu_k \sin kx)\right)\right\|$$

- Bessel不等式

$$f(x) \sim \frac{a_0}{\sqrt{2}}\frac{\sqrt{2}}{2} + \sum_{n=1}^{+\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

$$\frac{a_0^2}{2} + \sum_{n=1}^{+\infty}\left(a_n^2 + b_n^2\right) \le \|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

- $f \in \Re[-\pi,\pi] \Rightarrow \begin{cases} \lim\limits_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0, \\ \lim\limits_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0. \end{cases}$

§2. Fourier级数的收敛性

1. Riemann-Lebesgue引理

设f在$[a,b]$上可积或广义绝对可积, 记为$f \in \Re[a,b]$.

上一节我们证明, 若$f \in \Re[-\pi,\pi]$, 则Fourier系数$a_n \to 0$, $b_n \to 0$, 即

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0,$$

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0.$$

这一结论可以推广为更一般的情形.

Lemma (Riemann-Lebesgue). $f\in\Re[a,b]$, 则

$$\lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x dx=0,\quad \lim_{\lambda\to\infty}\int_a^b f(x)\sin\lambda x dx=0.$$

Proof. 只证第一式, 第二式同理.

Case1. 设f在$[a,b]$上可积, 则f在$[a,b]$上有界, 即

$$\exists M>0, s.t. \left|f(x)\right|\le M, \forall x\in[a,b].$$

任意给定$\lambda>1$, 令$n=\left\lfloor\sqrt{\lambda}\right\rfloor$. n等分$[a,b]$:

$$x_i=a+(b-a)i/n,\quad i=0,1,2,\cdots,n.$$

$$\omega_i(f)=\sup\left\{f(\xi)-f(\eta):\xi,\eta\in[x_{i-1},x_i]\right\},\quad i=1,2,\cdots,n.$$

f在$[a,b]$上可积,则$\lim_{n\to\infty}\sum_{i=1}^{n}\omega_i(f)\Delta x_i=0.$ 于是

$$\left|\int_a^b f(x)\cos\lambda x dx\right|=\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\cos\lambda x dx\right|$$

$$\le\left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}(f(x)-f(x_i))\cos\lambda x dx\right|+\left|\sum_{i=1}^{n}f(x_i)\int_{x_{i-1}}^{x_i}\cos\lambda x dx\right|$$

$$\le\sum_{i=1}^{n}\omega_i(f)\Delta x_i+\sum_{i=1}^{n}|f(x_i)|\left|\int_{x_{i-1}}^{x_i}\cos\lambda x dx\right|$$

$$\le\sum_{i=1}^{n}\omega_i(f)\Delta x_i+\frac{2Mn}{\lambda}=\sum_{i=1}^{\lfloor\sqrt{\lambda}\rfloor}\omega_i(f)\Delta x_i+\frac{2M\lfloor\sqrt{\lambda}\rfloor}{\lambda}$$

$\to 0,$ 当$\lambda\to+\infty$时.

Case2. f在$[a,b]$上广义绝对可积,不妨设a为唯一的瑕点.

则$\forall \varepsilon > 0, \exists \delta > 0$, s.t., f在$[a+\delta, b]$上可积, 且

$$\int_a^{a+\delta} |f(x)|\,dx < \varepsilon/2.$$

从而 $$\left|\int_a^{a+\delta} f(x)\cos\lambda x dx\right| \le \int_a^{a+\delta} |f(x)|\,dx < \varepsilon/2,$$

$$\lim_{\lambda\to+\infty}\int_{a+\delta}^{b} f(x)\cos\lambda x dx = 0.$$

于是$\exists \Lambda > 0$, 当$\lambda > \Lambda$时, $\left|\int_{a+\delta}^{b} f(x)\cos\lambda x dx\right| < \varepsilon/2$, 进而有

$$\left|\int_a^{b} f(x)\cos\lambda x dx\right| \le \left|\int_a^{a+\delta} f(x)\cos\lambda x dx\right| + \left|\int_{a+\delta}^{b} f(x)\cos\lambda x dx\right|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall \lambda > \Lambda. \square$$

例. $\int_0^{+\infty}\frac{\sin t}{t}\mathrm{d}t=\frac{\pi}{2}.$

解: 由广义积分的Dirichlet判别法, $\int_0^{+\infty}\frac{\sin t}{t}\mathrm{d}t$ 收敛. 于是

$$\int_0^{+\infty}\frac{\sin t}{t}\mathrm{d}t=\lim_{\lambda\to+\infty}\int_0^{\lambda\pi}\frac{\sin t}{t}\mathrm{d}t$$

$$=\lim_{\lambda\to+\infty}\int_0^{\pi}\frac{\sin\lambda t}{t}\mathrm{d}t=\lim_{n\to+\infty}\int_0^{\pi}\frac{\sin(n+1/2)t}{t}\mathrm{d}t.$$

恒等式 $\dfrac{\sin\left(n+1/2\right)t}{2\sin\frac{t}{2}}=\dfrac{1}{2}+\sum_{k=1}^{n}\cos kt$ 两边在$[0,\pi]$上积分,得

$$\int_0^{\pi}\frac{\sin\left(n+1/2\right)t}{2\sin\frac{t}{2}}dt=\frac{\pi}{2}.$$

令$g(t)=\dfrac{1}{t}-\dfrac{1}{2\sin\dfrac{t}{2}}$,往证$\lim\limits_{n\to+\infty}\int_0^{\pi}g(t)\sin(n+1/2)t\mathrm{d}t=0.$

$$\lim_{t\to0}g(t)=\lim_{t\to0}\frac{2\sin\dfrac{t}{2}-t}{2t\sin\dfrac{t}{2}}=\lim_{t\to0}\frac{\sin t-t}{2t\sin t}=0.$$

故$t=0$是$g(t)$的可去间断点. 由Riemann-Lebesgue引理,

$$\lim_{n\to+\infty}\int_0^{\pi}g(t)\sin(n+1/2)t\mathrm{d}t=0.\square$$

2. Fourier级数前n项和的积分表示

$f\in\Re[-\pi,\pi]$,即f在$[-\pi,\pi]$上可积或广义绝对可积, 则

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{+\infty}\left(a_n\cos nx+b_n\sin nx\right),$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx\,dx,\ \ k=0,1,2,\cdots$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,dx,\ \ k=1,2,\cdots$$

Question. Fourier级数的逐点收敛性?

着手点: $S_n(x)=\dfrac{a_0}{2}+\displaystyle\sum_{k=1}^{n}\left(a_k\cos kx+b_k\sin kx\right).$

设f以2π为周期，且$f\in\Re[-\pi,\pi]$.

$$S_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}\left(a_k\cos kx+b_k\sin kx\right)$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{2}+\sum_{k=1}^{n}\left(\cos kt\cos kx+\sin kt\sin kx\right)\right]dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k(t-x)\right]dt$$

$$=\frac{1}{\pi}\int_{-\pi-x}^{\pi-x}f(x+\tau)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k\tau\right]d\tau \qquad (\text{令}\ \tau=t-x)$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+\tau)\left[\frac{1}{2}+\sum_{k=1}^{n}\cos k\tau\right]d\tau \quad (2\pi\text{周期性})$$

$$S_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\tau)\frac{\sin\left(n+\frac{1}{2}\right)\tau}{2\sin\frac{\tau}{2}}d\tau$$

$$=\frac{1}{\pi}\left(\int_{-\pi}^{0}+\int_{0}^{\pi}\right)f(x+\tau)\frac{\sin\left(n+\frac{1}{2}\right)\tau}{2\sin\frac{\tau}{2}}d\tau$$

$$=\frac{1}{\pi}\int_{0}^{\pi}\frac{f(x+\tau)+f(x-\tau)}{2\sin\frac{\tau}{2}}\sin\left(n+\frac{1}{2}\right)\tau d\tau$$

$$\forall \delta > 0, g(\tau) \triangleq \frac{f(x_0+\tau)+f(x_0-\tau)}{2\sin\frac{\tau}{2}}, g(\tau) \in \Re[\delta,\pi].$$

$$S_n(x_0) = \frac{1}{\pi}\int_0^{\pi} g(\tau)\sin\left(n+\frac{1}{2}\right)\tau d\tau$$

$$= \frac{1}{\pi}\int_0^{\delta} g(\tau)\sin\left(n+\frac{1}{2}\right)\tau d\tau + \frac{1}{\pi}\int_{\delta}^{\pi} g(\tau)\sin\left(n+\frac{1}{2}\right)\tau d\tau$$

$$\triangleq A_n(x_0) + B_n(x_0).$$

由Riemann-Lebesgue引理, $\lim\limits_{n\to+\infty} B_n(x_0) = 0$. 因此, $S_n(x_0)$ 是否收敛仅仅取决于f在x_0附近的性质.

3. Fourier级数逐点收敛的判别法

Thm (Dini判别法) f以2π为周期，且$f \in \Re[-\pi,\pi]$，$x_0 \in [-\pi,\pi]$，若$\exists s \in \mathbb{R}, \exists \delta > 0, s.t.$

$$\frac{f(x_0+t)+f(x_0-t)-2s}{t} \in \Re[0,\delta],$$

则f的Fourier级数在x_0点收敛于s.

Proof. $S_n(x_0) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \frac{f(x_0+t)+f(x_0-t)}{2\sin\frac{t}{2}} \sin\left(n+\frac{1}{2}\right)t\,dt,$

往证$S_n(x_0) \to s$.

为估计$\left|S_n(x_0)-s\right|$,应将s表示成相应的积分形式.恒等式

$$\frac{1}{2}+\sum_{k=1}^{n}\cos kt=\frac{\sin\left(n+1/2\right)t}{2\sin t/2}$$

两边在区间$[0,\pi]$上积分,得

$$\int_0^{\pi}\frac{\sin\left(n+1/2\right)t}{2\sin t/2}dt=\frac{\pi}{2},$$

$$s=\frac{1}{\pi}\int_0^{\pi}\frac{2s}{2\sin t/2}\cdot\sin\left(n+1/2\right)t\,dt.$$

$$l(t) \triangleq \frac{f(x_0+t)+f(x_0-t)-2s}{2\sin t/2},$$ 则

$$S_n(x_0)-s=\frac{1}{\pi}\int_0^{\pi} l(t)\sin\left(n+1/2\right)tdt \triangleq I_n(x_0).$$

已知$h(t) \triangleq \dfrac{f(x_0+t)+f(x_0-t)-2s}{t} \in \Re[0,\delta], f \in \Re[0,\pi],$

则$h(t) \in \Re[0,\pi],$

$$J_n(x_0) \triangleq \frac{1}{\pi}\int_0^{\pi} h(t)\sin\left(n+1/2\right)tdt,$$

由Riemann-Lebesgue引理, $\lim\limits_{n\to\infty} J_n(x_0)=0.$

欲证$\lim\limits_{n\to\infty} S_n(x_0)=s,$ 只要证$\lim\limits_{n\to\infty} I_n(x_0)=\lim\limits_{n\to\infty} J_n(x_0).$

$$I_n(x_0)-J_n(x_0)=\frac{1}{\pi}\int_0^{\pi}\big(l(t)-h(t)\big)\sin\big(n+1/2\big)t\,dt$$

$$=\frac{1}{\pi}\int_0^{\pi}\left(\frac{1}{2\sin t/2}-\frac{1}{t}\right)\big(f(x_0+t)+f(x_0-t)-2s\big)\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$w(t)\triangleq\frac{1}{2\sin t/2}-\frac{1}{t}=\frac{t-2\sin t/2}{2t\sin t/2},$$

补充定义$w(0)=0$,则$w\in C[0,\pi]$. 从而

$$w(t)\big(f(x_0+t)+f(x_0-t)-2s\big)\in\Re[0,\pi],$$

由Riemann-Lebesgue引理, $\lim\limits_{n\to\infty}\big(I_n(x_0)-J_n(x_0)\big)=0.$ □

Def. 记f在x_0的左、右极限为$f(x_0-0), f(x_0+0)$, 若$\exists\delta>0$, $L>0, \alpha>0, s.t.$

$$\left.\begin{array}{l}\left|f(x_0+t)-f(x_0+0)\right|\le Lt^{\alpha},\\ \left|f(x_0-t)-f(x_0-0)\right|\le Lt^{\alpha},\end{array}\right.\quad \forall t\in(0,\delta),$$

则称f在x_0附近满足广义α阶Lipschitz条件.

Remark. 令$s=\left[f(x_0+0)+f(x_0-0)\right]/2$,

$$g(t)=\frac{f(x_0+t)+f(x_0-t)-2s}{t},\quad \forall t\in(0,\delta).$$

则$\left|g(t)\right|\le\dfrac{\left|f(x_0+t)-f(x_0+0)\right|}{t}+\dfrac{\left|f(x_0-t)-f(x_0-0)\right|}{t}$.

f在x_0附近满足广义α阶Lipschitz条件, 则$\exists\delta>0, L>0, s.t.$

$$|g(t)|\le Lt^{\alpha-1}, \forall t\in(0,\delta).$$

从而 $$\int_0^{\delta}|g(t)|dt\le\int_0^{\delta}Lt^{\alpha-1}dt=\frac{L}{\alpha}t^{\alpha}\Big|_{t=0}^{\delta}<+\infty, \quad (\alpha>0)$$

$g(t)\in\Re[0,\delta]$. 由Dini判别法得以下定理:

Thm (Lipschitz判别法) f以2π为周期, 且$f\in\Re[-\pi,\pi]$, $x_0\in[-\pi,\pi]$, f在x_0附近满足广义α阶Lipschitz条件, 则f的Fourier级数在x_0收敛于$\frac{1}{2}[f(x_0+0)+f(x_0-0)]$.

Corollary. f以2π为周期,且

(1) f在$[-\pi,\pi]$上的不连续点和不可微点至多有限多个;

(2)在每一不连续点ξ处具有第一类间断,即左、右极限

$f(\xi-0)$与$f(\xi+0)$都存在;

(3)在每一不可微点(包括不连续点)η,以下两极限存在

$$\lim_{t\to0+}\frac{f(\eta+t)-f(\eta+0)}{t},\ \lim_{t\to0+}\frac{f(\eta-t)-f(\eta-0)}{-t},$$

则f在每一点x_0的Fourier级数收敛于

$$\frac{1}{2}[f(x_0+0)+f(x_0-0)].$$

Proof. 在所给条件下, $\forall x_0$, 不论x_0是否为f的可微点,以下两极限存在

$$\lim_{t\to 0+}\frac{f(x_0+t)-f(x_0+0)}{t},$$

$$\lim_{t\to 0+}\frac{f(x_0-t)-f(x_0-0)}{-t}.$$

因而f在x_0点满足$\alpha=1$的Lipschitz条件

$$\left|f(x_0\pm t)-f(x_0\pm 0)\right|\le Lt,\forall t\in(0,\delta).$$

由Lipschitz判别法,命题得证.□

Remark. 两极限

$$\lim_{t\to 0+}\frac{f(x_0+t)-f(x_0+0)}{t},$$

$$\lim_{t\to 0+}\frac{f(x_0-t)-f(x_0-0)}{-t}$$

也称为广义单侧导数. 若x_0是f的连续点,则广义单侧导数就是单侧导数.

Def. 称f在$[a,b]$上分段可微,若$\exists[a,b]$的一个分割

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

使得$\forall i = 1, 2, \cdots, n$,

(1) f在(x_{i-1}, x_i)内可微;

(2) f在端点x_{i-1}, x_i处右、左极限分别存在;

(2) f在端点x_{i-1}, x_i处右、左广义导数分别存在.

前面的推论可以等价地描述为:

Corollary. f以2π为周期,且在$[-\pi,\pi]$上分段连续可微,则f在每一点x_0的Fourier级数收敛于

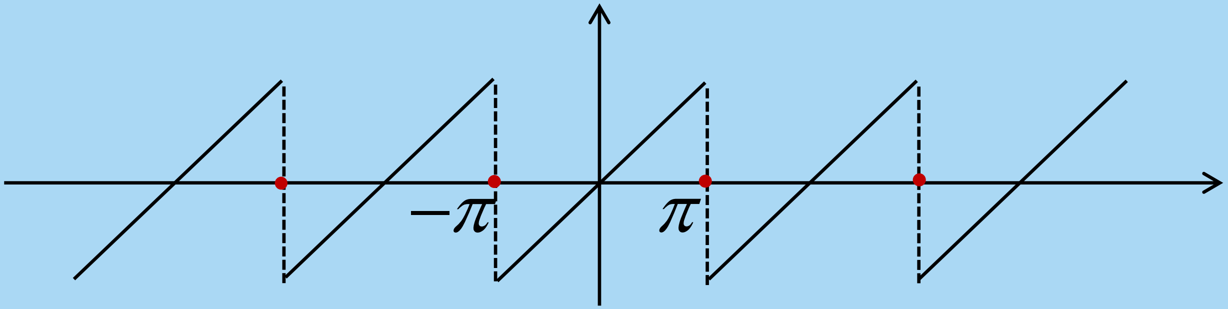
$$\frac{1}{2}\left[f(x_0+0)+f(x_0-0)\right].$$

特别地,若f在x_0连续,则f在x_0的Fourier级数收敛于$f(x_0)$.

例. 将$f(x)=\cos^2 x$展开成2π周期的Fourier级数.

解: $f(x)=\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x,\quad x\in\mathbb{R}.$

例. $f(x)=x$在$(0,\pi)$上的正弦Fourier级数为



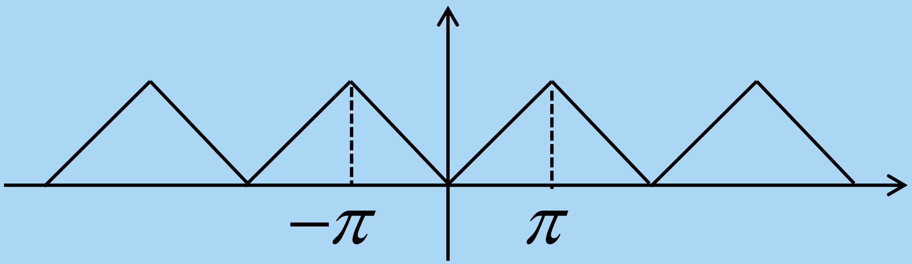
$$f(x)\sim\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{2}{k}\sin kx,$$

$$\sum_{k=1}^{+\infty}(-1)^{k+1}\frac{2}{k}\sin kx=\begin{cases}x, & x\in(-\pi,\pi),\\ 0, & x=\pm\pi.\end{cases}$$

令$x=\dfrac{\pi}{2}$,得

$$1-\frac{1}{3}+\frac{1}{5}+\cdots+(-1)^{n+1}\frac{1}{2n-1}+\cdots=\frac{\pi}{4}.\square$$

例. $f(x)=x$在$(0,\pi)$上的余弦Fourier 级数为



$$f(x)\sim\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{\cos(2k+1)x}{(2k+1)^2}.$$

$$x=f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{\cos(2k+1)x}{(2k+1)^2},\quad \forall x\in[-\pi,\pi].$$

令$x=\pi/4$,得

$$1-\left(\frac{1}{3^2}+\frac{1}{5^2}\right)+\left(\frac{1}{7^2}+\frac{1}{9^2}\right)+\cdots$$

$$+(-1)^n\left(\frac{1}{(4n-1)^2}+\frac{1}{(4n+1)^2}\right)+\cdots=\frac{\sqrt{2}}{16}\pi^2.\square$$

例. $f(x)=x^2, \forall x\in[-1,1]$, f以$T=2$为周期.

$$f(x)\sim\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

$$x^2 = f(x) = \frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos n\pi x,\quad \forall x\in[-1,1].$$

令$x=1$,得

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}.$$

令$x=0$,得

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\frac{(-1)^{n+1}}{n^2}+\cdots=\frac{\pi^2}{12}.\square$$

Thm (Dirichlet判别法) 周期为2π的函数f在$[-\pi,\pi]$上分段单调且有界, 则f的Fourier级数在任意一点x_0收敛于

$$\frac{1}{2}[f(x_0+0)+f(x_0-0)].$$

Proof. $S_n(x_0)=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\frac{f(x_0+t)+f(x_0-t)}{2\sin t/2}\sin\left(n+\frac{1}{2}\right)t\,dt,$

只要证 $\displaystyle\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x_0\pm t)}{2\sin t/2}\sin\left(n+\frac{1}{2}\right)t\,dt=\frac{1}{2}f(x_0\pm 0).$

注意到 $\displaystyle\int_0^{\pi}\frac{\sin(n+1/2)t}{2\sin t/2}dt=\frac{\pi}{2}$,只要证

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{2\sin t/2}\sin\left(n+\frac{1}{2}\right)tdt=0.$$

$$w(t)\triangleq\begin{cases}\dfrac{1}{2\sin t/2}-\dfrac{1}{t}, & t\neq 0\\ 0, & t=0\end{cases}$$, 则$w\in C[0,\pi]$. 而f以2π为周期, 在$[-\pi,\pi]$上分段单调且有界, 则$f\in\Re[x_0-\pi,x_0+\pi]$,

由Riemann-Lebesgue引理,

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}w(t)\big(f(x_0\pm t)-f(x_0\pm 0)\big)\sin\left(n+\frac{1}{2}\right)tdt=0.$$

故往证: $\lim_{n\to\infty}\dfrac{1}{\pi}\displaystyle\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{t}\sin\left(n+\frac{1}{2}\right)tdt=0.$

$\forall x_0, \exists \eta > 0, s.t. f(x_0 \pm t)$在$t \in (0,\eta)$上单调.

$$\int_0^\pi \frac{f(x_0 \pm t) - f(x_0 \pm 0)}{t} \sin\left(n + \frac{1}{2}\right) t dt$$

$$= \left(\int_0^\delta + \int_\delta^\pi\right) \frac{f(x_0 \pm t) - f(x_0 \pm 0)}{t} \sin\left(n + \frac{1}{2}\right) t dt$$

$$\triangleq A_{n,\delta}(x_0) + B_{n,\delta}(x_0)$$

$\delta \in (0,\eta)$时, 由积分第二中值定理, $\exists \xi \in (0,\delta), s.t.$

$$A_{n,\delta}(x_0) = \big(f(x_0 \pm \delta) - f(x_0 \pm 0)\big) \int_\xi^\delta \frac{\sin(n + 1/2)t}{t} dt$$

$$\int_{\xi}^{\delta} \frac{\sin(n+1/2)t}{t} dt = \int_{(n+1/2)\xi}^{(n+1/2)\delta} \frac{\sin\tau}{\tau} d\tau$$

因为$\int_0^{+\infty} \frac{\sin\tau}{\tau} d\tau = \pi/2$,积分收敛,因此$\exists M > 0,\ s.t.$

$$\left|\int_0^{v} \frac{\sin\tau}{\tau} d\tau\right| < M, \forall v \in \mathbb{R}.$$

从而

$$\left|\int_{(n+1/2)\xi}^{(n+1/2)\delta} \frac{\sin\tau}{\tau} d\tau\right| < 2M, \forall \delta, \xi > 0.$$

$$\forall \varepsilon > 0, \exists \delta \in (0,\eta), s.t. \left| f(x_0 \pm \delta) - f(x_0 \pm 0) \right| < \varepsilon/4M,$$

此时,

$$\left|A_{n,\delta}(x_0)\right| < \varepsilon/2.$$

对此ε及δ,由Liemann-Lebesgue引理, $\lim\limits_{n\to\infty} B_{n,\delta}(x_0)=0.$

因此, $\exists N\geq 1, s.t.$

$$\left|B_{n,\delta}(x_0)\right|<\varepsilon/2, \forall n>N.$$

因此

$$\left|\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{t}\sin\left(n+\frac{1}{2}\right)tdt\right|$$

$$\leq\left|A_{n,\delta}(x_0)\right|+\left|B_{n,\delta}(x_0)\right|$$

$$<\varepsilon/2+\varepsilon/2=\varepsilon, \qquad \forall n>N$$

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi}\frac{f(x_0\pm t)-f(x_0\pm 0)}{t}\sin\left(n+\frac{1}{2}\right)tdt=0.\square$$

4. Fourier级数的均方收敛性,Parseval等式

$$(f,g) \triangleq \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx, \quad \forall f,g \in R[-\pi,\pi].$$

$$\varphi_0(x) = 1/\sqrt{2},$$

$$\varphi_{2n-1}(x) = \cos nx, \quad n = 1, 2, \cdots$$

$$\varphi_{2n}(x) = \sin nx, \quad n = 1, 2, \cdots$$

$\{\varphi_k(x)\}$ 为$R[-\pi,\pi]$中标准正交函数系.

Thm. f以2π为周期, 在$[-\pi,\pi]$上可积,$\{\varphi_k\}$如前,

$c_k=(f,\varphi_k)$,则

(1) f的Fourier级数在$[-\pi,\pi]$上均方收敛到$f(x)$,即

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left(f(x)-\sum_{k=0}^{n}c_k\varphi_k(x)\right)^2dx=0$$

(此时称$\{\varphi_k\}$在$R[-\pi,\pi]$中完备);

(2) $\sum_{k=0}^{\infty}c_k^2=\|f\|^2$.(Parseval等式)

这个定理可以翻译为:

Thm. f以2π为周期, 在$[-\pi,\pi]$上可积,

$$f(x)\sim\frac{a_0}{2}+\sum_{n=0}^{\infty}\left(a_n\cos nx+b_n\sin nx\right).$$

(1) f的Fourier级数在$[-\pi,\pi]$上均方收敛到$f(x)$,即

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}\left(f(x)-S_n(x)\right)^2dx=0,$$

其中$S_n(x)=\dfrac{a_0}{2}+\displaystyle\sum_{k=0}^{n}\left(a_k\cos kx+b_k\sin kx\right)$;

(2) $\dfrac{a_0^2}{2}+\displaystyle\sum_{n=1}^{+\infty}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx$.(Parseval等式)

Proof. 证明思路如下：

Step1. Fejer和$\sigma_n(x) \triangleq \frac{1}{n}\sum_{k=0}^{n-1} S_k(x)$.

若$f \in C[-\pi,\pi]$,则$\sigma_n(x)$在$[-\pi,\pi]$上一致收敛到$f(x)$.

Step2. $f \in R[-\pi,\pi], \forall \varepsilon > 0, \exists g \in C[-\pi,\pi], s.t. \| f - g \| < \varepsilon$.

Step3. 由前两步的结论知：任何函数$f \in R[-\pi,\pi]$都可以用三角多项式(即$1, \cos x, \sin x, \cdots, \cos nx, \sin nx, \cdots$的有限线性组合)均方逼近，使得均方误差小于预先给定的正数ε.

Step4. 已知$\{1, \cos x, \sin x, \cdots, \cos nx, \sin nx, \cdots\}$的前$n+1$项线性组合中, f的Fourier级数的部分和$S_n(x)$是f的最佳均方逼近. 于是由Step3的结论知: $\forall f \in R[-\pi, \pi]$,

f的Fourier级数在$[-\pi, \pi]$上均方收敛到$f(x)$,

即$\| f - S_n \| \to 0$.

Step5. 在

$$\left\| f - \sum_{k=0}^{n} c_k \varphi_k \right\|^2 = \| f \|^2 - \sum_{k=0}^{n} c_k^2$$

中令$n \to \infty$, 得Parseval等式.□

例. $x^{1000}=\dfrac{a_0}{2}+\sum_{n=0}^{+\infty}(a_n\cos nx+b_n\sin nx),\ x\in(-\pi,\pi)$. 求$\sum_{n=1}^{+\infty}a_n^2$.

解: 由$Passeval$等式,

$$\frac{a_0^2}{2}+\sum_{n=1}^{+\infty}a_n^2+\sum_{n=1}^{+\infty}b_n^2=\frac{1}{\pi}\int_{-\pi}^{\pi}x^{2000}dx=\frac{2\pi^{2000}}{2001}.$$

x^{1000}为偶函数, 所以$b_n=0,\forall n$.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x^{1000}dx=\frac{2\pi^{1000}}{1001},$$

$$\sum_{n=1}^{+\infty}a_n^2=\frac{2\pi^{2000}}{2001}-\frac{a_0^2}{2}-\sum_{n=1}^{+\infty}b_n^2=2\pi^{2000}\left(\frac{1}{2001}-\frac{1}{1001^2}\right).\ \square$$

作业:习题7.1 No.3.

习题7.2 No.1-4.