Review

曲面的切平面与法线:

曲面方程	点	法向量
$\mathbf{r} = \mathbf{r}(u, v)$	$\mathbf{r}_0 = \mathbf{r}(u_0, v_0)$	$\left.\left(\mathbf{r}'_{u}\times\mathbf{r}'_{v}\right)\right _{\left(u_{0},v_{0}\right)}$
z = f(x, y)	(x_0, y_0, z_0) $z_0 = f(x_0, y_0)$	$(-f_x', -f_y', 1)^{\mathrm{T}}\Big _{(x_0, y_0)}$
F(x, y, z) = 0	$\mathbf{r}_0 = (x_0, y_0, z_0)$	$\operatorname{grad} F(\mathbf{r}_0)$

曲线的切向量:

曲线方程	点	切向量
$\mathbf{r} = \mathbf{r}(t)$	$\mathbf{r}_0 = \mathbf{r}(t_0)$	$\mathbf{r}'(t_0) = \\ \left(x'(t_0), y'(t_0), z'(t_0)\right)$
$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$	$\mathbf{r}_0 = (x_0, y_0, z_0)$	$\operatorname{grad} F(\mathbf{r}_0) \times \operatorname{grad} G(\mathbf{r}_0)$

§ 8. 多元函数的Taylor公式

对充分光滑的一元函数f(x),有

Taylor公式的应用之一是近似计算.

对二元可微函数g(x,y),利用全微分也可以进行近似计算.

$$g(x, y) = g(x_0, y_0) + g'_x(x_0, y_0)(x - x_0)$$

$$+ g'_y(x_0, y_0)(y - y_0)$$

$$+ o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right)$$

$$g(x, y) \approx g(x_0, y_0) + g'_x(x_0, y_0)(x - x_0)$$

$$+ g'_y(x_0, y_0)(y - y_0)$$

Question: 如何进一步提高精度?

Lemma.设A = $(a_{ij})_{n \times n}$ 为n阶实对称矩阵,则

$$\left|x^{\mathrm{T}} A x\right| \leq \left(n \sum_{i=1}^{n} \sum_{j=1}^{n} \left|a_{ij}\right|\right) \cdot \left\|x\right\|^{2}, \quad \forall x \in \mathbb{R}^{n}.$$

Proof. 记
$$\mathbf{A} = (\alpha_1, \dots, \alpha_n), \forall x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$$
,有

$$||Ax|| = ||x_1\alpha_1 + \dots + x_n\alpha_n|| \le |x_1|||\alpha_1|| + \dots + |x_n|||\alpha_n||$$

$$\le (|x_1| + \dots + |x_n|)(||\alpha_1|| + \dots + ||\alpha_n||)$$

$$\leq n \|x\| \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|,$$

于是
$$|x^{T}Ax| \le ||x|| \cdot ||Ax|| \le \left(n \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|\right) \cdot ||x||^{2}$$
.□

Def.设n元函数f在 $B(x_0, \delta)$ 中二阶连续可微,称实对称阵

$$H_{f}(\mathbf{x}_{0}) = \begin{pmatrix} \frac{\partial^{2} f}{\partial x_{1}^{2}} & \frac{\partial^{2} f}{\partial x_{1} \partial x_{2}} & \cdots & \frac{\partial^{2} f}{\partial x_{1} \partial x_{n}} \\ \frac{\partial^{2} f}{\partial x_{2} \partial x_{1}} & \frac{\partial^{2} f}{\partial x_{2}^{2}} & \cdots & \frac{\partial^{2} f}{\partial x_{2} \partial x_{n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^{2} f}{\partial x_{n} \partial x_{1}} & \frac{\partial^{2} f}{\partial x_{n} \partial x_{2}} & \cdots & \frac{\partial^{2} f}{\partial x_{n}^{2}} \end{pmatrix}_{x_{0}}$$

为f在点 x_0 的Hessian矩阵.

Thm.设n元函数f在 $B(x_0,\delta)$ 中二阶连续可微,则

$$\forall x_0 + \Delta x \in B(x_0, \delta), \exists \theta \in (0, 1), s.t.$$
$$f(x_0 + \Delta x) = f(x_0) + J_f(x_0) \Delta x$$
$$+ \frac{1}{2} (\Delta x)^T H_f(x_0 + \theta \Delta x) \Delta x$$

(称为带Lagrange余项的一阶Taylor公式),且

$$f(x_0 + \Delta x) = f(x_0) + J_f(x_0) \Delta x$$
$$+ \frac{1}{2} (\Delta x)^T H_f(x_0) \Delta x + o(\|\Delta x\|^2), \Delta x \to 0 \exists \exists$$

(称为带Peano余项的二阶Taylor公式).

Proof. 构造一元函数

$$g(t) = f(x_0 + t\Delta x) = f(x_0^{(1)} + t\Delta x_1, \cdots, x_0^{(n)} + t\Delta x_n),$$

$$f \in C^2, \text{则} g'(t) = \sum_{i=1}^n f_i''(x_0 + t\Delta x)\Delta x_i = J_f(x_0 + t\Delta x)\Delta x$$

$$g''(t) = \sum_{i=1}^n \sum_{j=1}^n f_{ij}''(x_0 + t\Delta x)\Delta x_i\Delta x_j = (\Delta x)^T H_f(x_0 + t\Delta x)\Delta x$$

$$g(t) = g(0) + g'(0)t + \frac{1}{2}g''(\theta t)t^2, \theta \in (0,1).$$

$$f(x_0 + \Delta x) = g(1) = g(0) + g'(0) + \frac{1}{2}g''(\theta)$$

$$= f(x_0) + J_f(x_0)\Delta x + \frac{1}{2}(\Delta x)^T H_f(x_0 + \theta\Delta x)\Delta x, \theta \in (0,1).$$
带Lagrange条项的一阶Taylor公式得证.

$$\overrightarrow{\mathsf{L}} P(\Delta x) = H_f(x_0 + \theta \Delta x) - H_f(x_0) = (h_{ij})_{n \times n},$$

$$\alpha(\Delta x) = (\Delta x)^T H_f(x_0 + \theta \Delta x) \Delta x - (\Delta x)^T H_f(x_0) \Delta x$$

$$= (\Delta x)^T P(\Delta x) \Delta x.$$

已知 $f \in C^2$,则 $\lim_{\Delta x \to 0} h_{ij} = 0$.由前面的引理得

$$\left|\alpha(\Delta x)\right| \leq \left(n\sum_{i=1}^{n}\sum_{j=1}^{n}\left|h_{ij}\right|\right) \cdot \left\|\Delta x\right\|^{2},$$

因此 $\alpha(\Delta x) = o(\|\Delta x\|^2)$, 当 $\Delta x \to 0$ 时.

故带Peano余项的二阶Taylor公式得证.□

Question. 能否用一元函数带Peano余项的Taylor公式直接证明多元函数带Peano余项的Taylor公式?

Thm. 设函数f(x, y)在区域D中n+1阶连续可微, $M_0(x_0, y_0) \in D, M(x, y) \in D$,且线段 $\overline{M_0M}$ 完全包含在D中. 记

$$h = x - x_0, k = y - y_0,$$

记算子

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{m} \triangleq \sum_{i=0}^{m} C_{m}^{i} h^{i} k^{m-i} \frac{\partial^{m}}{\partial x^{i} \partial y^{m-i}},$$

则f在点 (x_0, y_0) 有

(1)带Lagrange余项的n阶Taylor公式

$$f(x,y) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0)$$

$$+ \dots + \frac{1}{n!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0)$$

$$+ \frac{1}{(n+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0 + \theta h, y_0 + \theta k)$$

$$(0 < \theta < 1)$$

(2)带Peano余项的n+1阶Taylor公式

$$f(x,y) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0)$$

$$+ \dots + \frac{1}{(n+1)!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0, y_0)$$

$$+ o\left(\left(\sqrt{h^2 + k^2}\right)^{n+1}\right).$$

例:写出 $f(x,y) = x^y$ 在点(1,1)的邻域内带Peano余

项的3阶 Taylor公式,并求 $(1.1)^{1.02}$.

解: $f(x, y) = x^y$, $f'_x = yx^{y-1}$, $f'_y = x^y \ln x$,

 $f_{xx}'' = y(y-1)x^{y-2}, f_{yy}'' = x^y \ln^2 x, f_{xy}'' = x^{y-1} + yx^{y-1} \ln x,$

 $f_{xxx}''' = y(y-1)(y-2)x^{y-3}, \quad f_{yyy}''' = x^y \ln^3 x$

 $f_{xxy}''' = (2y-1)x^{y-2} + y(y-1)x^{y-2} \ln x,$

 $f'''_{xyy} = yx^{y-1} \ln^2 x + 2x^{y-1} \ln x.$

 $\Rightarrow x_0 = y_0 = 1, h = x - 1, k = y - 1.$

$$f(1,1) = f'_{x}(1,1) = f''_{xy}(1,1) = f'''_{xxy}(1,1) = 1,$$

$$f'_{y}(1,1) = f''_{xx}(1,1) = f'''_{yy}(1,1)$$

$$= f'''_{xxx}(1,1) = f'''_{yyy}(1,1) = f'''_{xyy}(1,1) = 0$$

$$f(x,y) = f(x_0, y_0) + (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})f(x_0, y_0)$$

$$+ \frac{1}{2!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f(x_0, y_0)$$

$$+ \frac{1}{3!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^3 f(x_0, y_0) + o((\sqrt{h^2 + k^2})^3)$$

$$f(x,y) = 1 + (x-1) + (x-1)(y-1)$$

$$+ \frac{1}{2}(x-1)^{2}(y-1)$$

$$+ o\left(\left(\sqrt{(x-1)^{2} + (y-1)^{2}}\right)^{3}\right)$$

$$(1.1)^{1.02} \approx 1.1 + 0.1 \times 0.02 + \frac{1}{2} \times 0.01 \times 0.02$$

$$= 1.1021.\square$$

Question. 二元函数在一点的Taylor多项式是否唯一?如何证明? 唯一!

例. $\cos(x^2 + y^2)$ 在(0,0)的8阶带Peano余项的Taylor展开式.

解:
$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots + (-1)^n \frac{t^{2n}}{(2n)!} + o(t^{2n}), t \to 0$$
时.

$$\cos(x^2 + y^2) = 1 - \frac{(x^2 + y^2)^2}{2!} + \dots + (-1)^n \frac{(x^2 + y^2)^{2n}}{(2n)!}$$

$$+o((x^2+y^2)^{2n}), x^2+y^2 \to 0$$
 时.

$$\cos(x^{2} + y^{2}) = 1 - \frac{(x^{2} + y^{2})^{2}}{2!} + \frac{(x^{2} + y^{2})^{4}}{4!} + o((x^{2} + y^{2})^{4}),$$

$$x^{2} + y^{2} \to 0 \text{ by.} \Box$$

例. ln(2+x+y+xy)在(0,0)带Peano余项的2阶Taylor展开.

解:
$$x + y + xy \to 0$$
 时,

$$\ln(2 + x + y + xy) = \ln 2 + \ln(1 + \frac{x + y + xy}{2})$$

$$= \ln 2 + \frac{x+y+xy}{2} - \frac{1}{2} \left(\frac{x+y+xy}{2} \right)^2 + o\left((x+y+xy)^2 \right)$$

$$x^2 + y^2 \rightarrow 0$$
时, 必有 $x + y + xy \rightarrow 0$ 时, 因此

$$\frac{o((x+y+xy)^2)}{x^2+y^2} = \frac{o((x+y+xy)^2)}{(x+y+xy)^2} \cdot \frac{(x+y+xy)^2}{x^2+y^2} \to 0,$$

$$\ln(2+x+y+xy) = \ln 2 + \frac{x+y}{2} - \frac{x^2+y^2-2xy}{8} + o(x^2+y^2). \square$$

作业: 习题1.8 No.1,2(2)