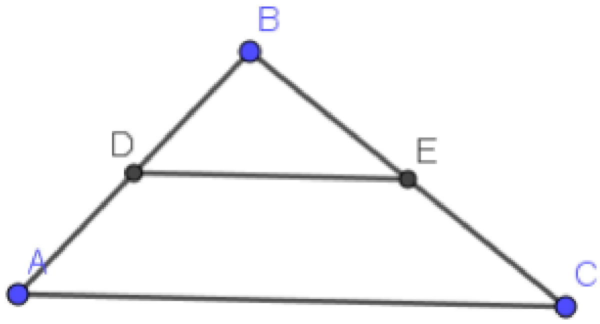
2. Vectors and geometry — Problem Set

Warm-up

1. On triangle ABC, D is the midpoint of AB and E is the midpoint of BC.



   (i) Using vector methods, show that DE is parallel to AC

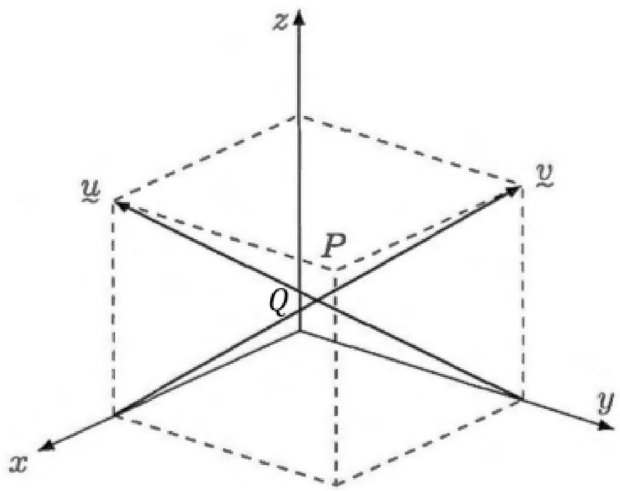
   (ii) Further prove that DE is half the length of AC

Skill-building

1. Prove using vector methods that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of all 4 sides.

Easier Exam Questions

1. (Killara 2022 Q12a) The rectangular prism shown below is produced by the origin and the point P(a, b, c) with all sides either within the coordinate planes or parallel to them.



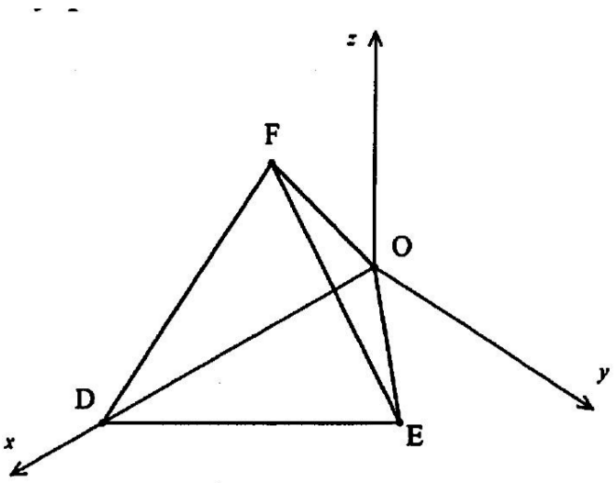
   The main diagonals are represented by the vectors **u** and **v** as shown, intersecting at point Q.

   (i) Find expressions for **u** and **v** in terms of a, b and c. **1**

   (ii) Show that **u** and **v** are perpendicular if and only if $a^2 + b^2 = c^2$. **2**

(iii) Find a point Q such that the main diagonals are perpendicular. **1**

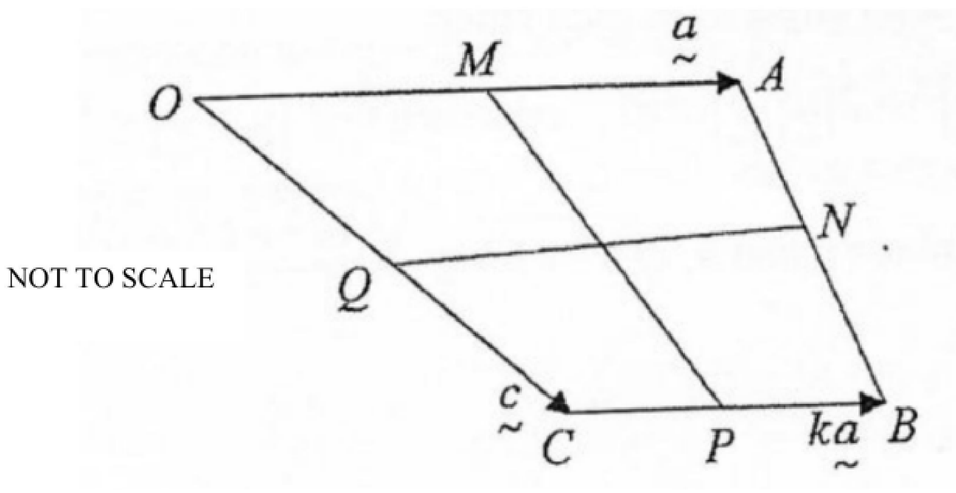
2. (Manly 2020 Q13d) The faces of the tetrahedron ODEF are equilateral triangles of side length 1 unit. Its base ODE lies flat on the xy plane with two vertices at O and D (1,0,0) with F above the xy plane.



i. Show that the coordinates of E are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$ **1**

ii. Using vectors, prove the coordinates of the vertex F are $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right)$. **3**

3. (Manly 2025 Q14d)



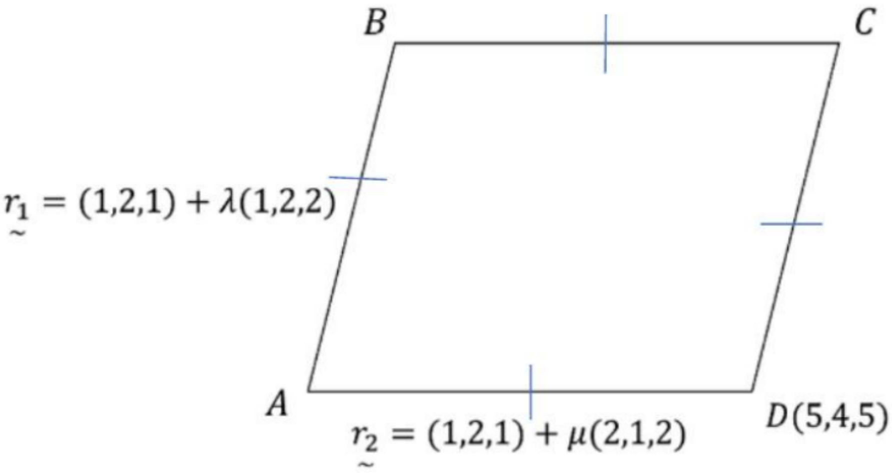
In the diagram $OABC$ is a trapezium in which $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{CB} = k\overrightarrow{OA}$ for some constant $0 < k < 1$.

The points M, N, P, and Q are the midpoints of OA, AB, BC, and CO respectively.

(i) Use vector methods to show that MP and QN bisect each other. **2**

(ii) If $MP \perp QN$ use vector methods to show that $|\overrightarrow{OC}|^2 = |\overrightarrow{AB}|^2$ **3**

4. (Barker 2023 Q13) Two sides of the rhombus ABCD are formed by:
$\mathbf{r_1} = (1, 2, 1) + A(1, 2, 2)$ passing through A and B and
$\mathbf{r_2} = (1, 2, 1) + ţ(2, 1, 2)$ passing through A and D.
The coordinates of D are (5,4,5).

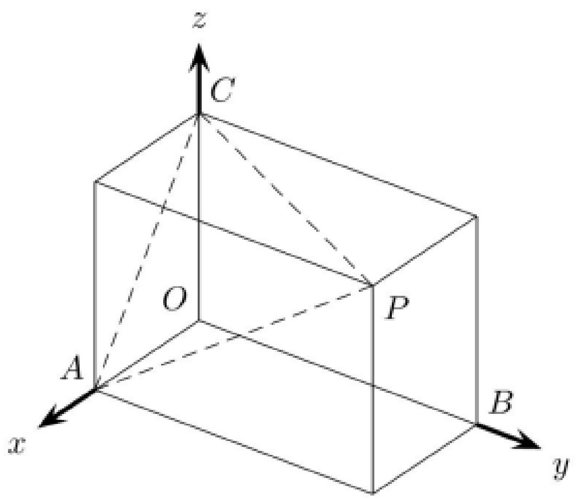


(i) Find the two possible coordinates of B **3**

(ii) Find the exact area of the rhombus. **3**

Harder Exam Questions

1. (Normanhurst Boys 2025 Q14a) Consider the rectangular prism in the diagram.



Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the non-zero position vectors of the points A, B and C.

Using vector methods, prove that triangle ACP **cannot** be a right-angled triangle. **3**

2. (HSC 2021 Q16a)

(i) The point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O. **2**

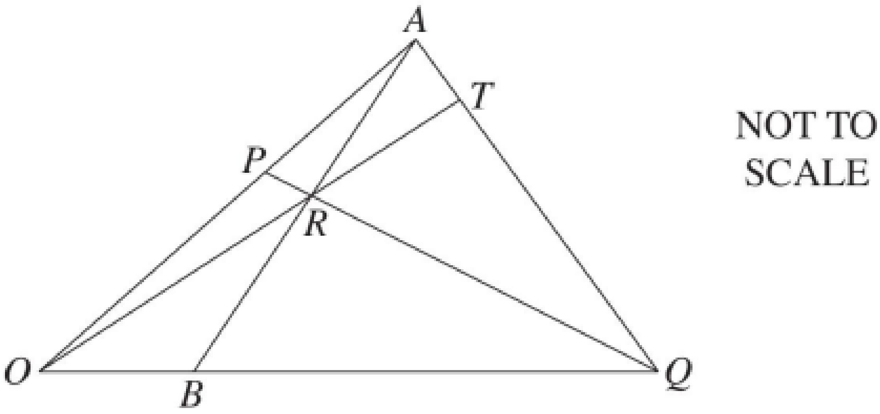
Using the position vector of P, $\overrightarrow{OP} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, and the triangle inequality, or otherwise, show that $|x| + |y| + |z| \geq 1$.

(ii) Given the vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, show that **3**

$|a_1b_1 + a_2b_2 + a_3b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}$.

(iii) As in part (i), the point $P(x, y, z)$ lies on the sphere of radius 1 centred at the origin O. Using part (ii), or otherwise, show that $|x| + |y| + |z| \leq \sqrt{3}$. **2**

3. (HSC 2024 Q14e) The diagram shows triangle OQA. The point P lies on OA so that OP:OA =3:5. The point B lies on OQ so that OB:OQ =1:3. The point R is the intersection of AB and PQ. The point T is chosen on AQ so that O, R and T are collinear.



Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\overrightarrow{PR} = k\overrightarrow{PQ}$ where k is a real number.

(i) Show that $\overrightarrow{OR} = \frac{3}{5}(1-k)\mathbf{a} + 3k\mathbf{b}$. **2**

Writing $\overrightarrow{AR} = h\overrightarrow{AB}$, where h is a real number, it can be shown that $\overrightarrow{OR} = (1-h)\mathbf{a} + h\mathbf{b}$. (Do NOT prove this.)

(ii) Show that $k = \frac{1}{6}$. **2**

(iii) Find $\overrightarrow{OT}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. **2**

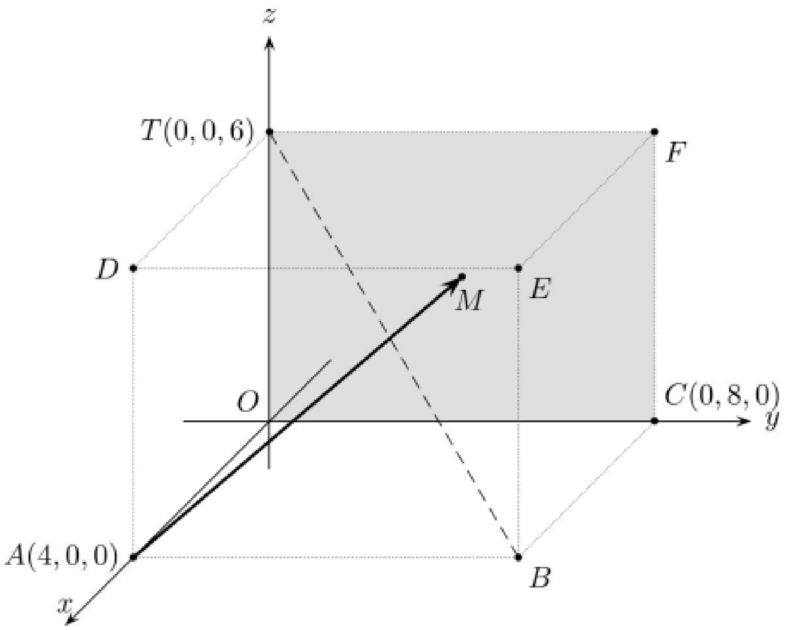
4. (Normanhurst Boys 2024 Q12c) A rectangular prism is defined with vertices A(4,0,0), C(0,8,0) and T(0,0,6). Point M is the centre of the plane with vertices OCFT and M has coordinates (0,4,3). (see next page for diagram)

   i. Determine the vector equation of the line BT, which is one of the diagonals of the prism. **2**

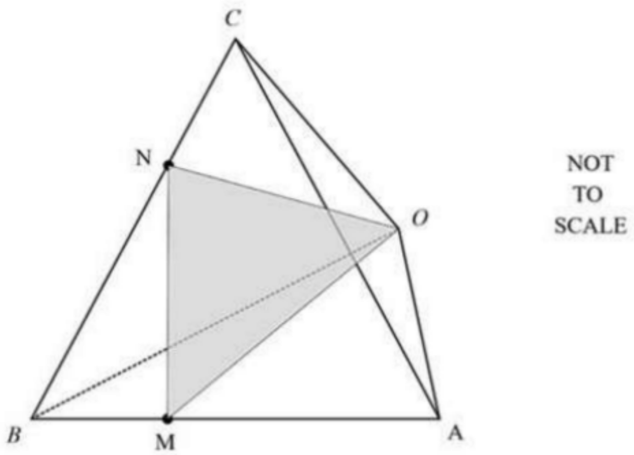
   ii. Find the the vector equation of the sphere that contains all of the vertices of the rectangular prism, in the form $|\mathbf{v} - \mathbf{c}| = r$. **2**

   Hint: Consider the midpoint of the diagonal BT.

   iii. Prove using vector methods, that the lines AM and BT are skew. **3**



5. (Cheltenham Girls 2023 Q13a) A regular tetrahedron OABC consists of four congruent equilateral triangle faces and has sides of length 1, where O is the origin.



Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, and M and N be the points that divides sides AB and BC in the ratio $x : 1 - x$, where $0 < x < 1$.

(i) Show that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} = \frac{1}{2}$. **1**

(ii) Show that $|\overrightarrow{OM}| = \sqrt{1 - x + x^2}$. **2**

(iii) Let $f(x) = \frac{1+x-x^2}{1-x+x^2}$, where $0 < x < 1$.
Without using calculus, prove that $1 < f(x) \leq \frac{5}{3}$. **3**

(iv) Hence, show that $\cos^{-1} \frac{5}{6} \leq \angle MON < \frac{\pi}{3}$. **3**