Arithmetic of Complex Numbers — Problem Set

Warm-up

1. if $z = 2 + 3i$ and $w = 5 + 12i$, find:

    a) $z + w$
    b) $z - w$
    c) zw
    d) $\frac{z}{w}$
    e) z^2
    f) $\sqrt{w}$
    g) $\mathrm{Re}(z)$
    h) $\mathrm{Im}(w)$
    i) $|z|$
    j) $|w|$

2. if $z = 6 - 7i$ and $w = 3 + 4i$, find:

    a) $\bar{z} + \bar{w}$
    b) $z - \bar{w}$
    c) zw
    d) $\frac{\bar{w}}{z}$
    e) w^2
    f) $\sqrt{w}$
    g) $\mathrm{Re}(\bar{z})$
    h) $\mathrm{Im}(\bar{w})$
    i) $|z|$
    j) $|w|$

3. How many distinct roots do the following equations have?

    a) $z^2 + 5z + 1 = 0$
    b) $2z^3 + 6z + 7 = 0$
    c) $(3z^2 + 4z + 5)(z^2 + i) = 0$
    d) $z^5 + 5i = 0$

4. A quadratic polynomial with real coefficients has $1 + i$ as one of its roots. What would be the other root?

Skill-building

1. Solve the following quadratic equations, leaving your answers in the form $a + ib$

   a) $2z^2 - 2z + 5 = 0$
   b) $5z^2 + 2z + 1 = 0$
   c) $z^2 + z + 3 = 0$

2. A complex number, z satisfies $z + \bar{z} = 12$ and $\bar{z} - z = 4$. Find z.

3. Let $z = x + iy$, where $\text{Re}(z) = x$ and $\text{Im}(z) = y$. Prove that:

   a) $\frac{z+\bar{z}}{2} = Re(z)$
   b) $\frac{z-\bar{z}}{2i} = Im(z)$
   c) $|z|^2 = z\bar{z}$

4. Solve the following polynomials:

   a) $z^3 - 4z^2 + 9z - 10 = 0$, given that one root is $1 + 2i$.
   b) $z^3 - 7z^2 + 17z - 15 = 0$, given that one root is $2 - i$.
   c) $z^4 - 6z^3 + 15z^2 - 18z + 10 = 0$, given that one root is $1 + i$.
   d) $z^4 - 4z^3 + 10z^2 - 12z + 5 = 0$, given that one root is $1 - 2i$.

5. Factorise the following polynomials so that all factors have real coefficients.

   a) $z^3 + 3z^2 + 7z + 5 = 0$, given that one root is $-1 + 2i$.
   b) $z^3 - 2z^2 + 2z + 5 = 0$, given that one root is $1 - 2i$.
   c) $z^4 - 6z^3 + 23z^2 - 50z + 50 = 0$, given that one root is $1 - 3i$.
   d) $z^4 + 2z^3 - 2z^2 + 6z - 15 = 0$, given that one root is $i\sqrt{3}$.

Easier Exam Questions

1. (Barker 2020 Q11a) Let $z_1 = -1 + 3i$ and $z_2 = 2 - i$. Find the following, in simplest form:

   (i) $z_1 + z_2$ **1**
   (ii) $z_1 z_2$ **1**
   (iii) $\text{Re}(z_1) - \text{Im}(z_2)$ **1**

2. (Fort St 2023 Q11d) 2-3i is one root of the equation $z^3 + mz + 52 = 0$, where m is real.

   (i) Find the other roots **2**
   (ii) Determine the value of m **2**

3. (Fort St 2025 Q11a) Let z = 3 - i and w = 2+5i.

   (i) Find $|z + w|$ **2**
   (ii) Express $\frac{w}{z}$ in the form $a + ib$, where $a, b \in \mathbb{R}$. **2**

4. (Fort St 2025 Q12a) Find the square roots of $3 + 2i\sqrt{10}$ **3**

5. (HSC 2024 Q14) Let $z = 2 + 3i$ and $w = 1 - 5i$.

   (i) Find $z + \bar{w}$ **1**

   (ii) Find z^2 **1**

6. (Barker 2020 Q8) The number of distinct roots of the equation $(z^2 - 2zi - 1)(z^2 + 2i)(z^2 + 2zi + 2) = 0$ is

   A. 3
   B. 4
   C. 5
   D. 6

7. (Cheltenham Girls 2025 Q5) A cubic polynomial $P(z)$ has all real coefficients. Given that $z = 3$ and $z = 2 + i$ are two roots such that $P(z) = 0$, which equation best describes $P(z)$?

   A. $P(z) = z^3 - 7z^2 - 17z + 15$
   B. $P(z) = z^3 + 7z^2 - 35z + 15$
   C. $P(z) = z^3 - 7z^2 + 17z - 15$
   D. $P(z) = z^3 + 7z^2 + 35z - 15$

Harder Exam Questions

1. (Barker 2024 Q12a)

   (i) Show that $\sqrt{2} - i$ is a root of the equation $z^3 - (\sqrt{2} - i)z^2 + 8z - 8\sqrt{2} + 8i = 0$. **1**

   (ii) Find the other two solutions of the equation. **2**

2. (Fort St 2023 Q11e)

   (i) Find the square roots of $-3 - 4i$ **2**

   (ii) Hence or otherwise, solve the equation $z^2 - 3z + (3 + i) = 0$ **2**

3. (St George Girls 2021 Q12a)

   (i) Find the square root(s) of $7 - 24i$, in the form $x + iy$, where $x, y \in \mathbb{R}$. **3**

   (ii) Hence or otherwise solve $2z^2 + 6z + (1 + 12i) = 0$, for $z \in \mathbb{C}$. **2**

4. (Blacktown Boys 2022 Q12b) Let $P(x) = x^4 + ax^3 - 40x^2 + 41x + b$ where a and b are real numbers.

   It is known that $x = 7$ and $x = \frac{1 - i\sqrt{3}}{2}$ are zeros of $P(x)$.

   i) Explain why $x^2 - x + 1$ must be a factor of $P(x)$. (2)

   ii) Find a and b. (3)

5. (Manly 2024 Q11d) Consider the polynomial given by $P(z) = z^4 - 2z^3 + az^2 + bz + 50$ where $a, b \in \mathbb{R}$ and $P(2 - i) = 0$.

   Fully factorise $P(z)$ such that all factors have real coefficients. (3)

6. (St George Girls 2025 Q14) Factorise the equation $P(z) = z^4 - 6z^3 + 18z^2 - 14z - 39$, given that one of the zeroes of $P(z)$ is $(2 - 3i)$. (3)