Substitution and Trigonometric Integrals— Problem Set

Warm-up

1. a Find $\int \frac{2x}{1+x^2}dx$ using a suitable substitution

   b Recall that $\int \frac{1}{1+x^2} = \tan^{-1} x + C$. Using this and part a), find $\int \frac{x+1}{1+x^2}dx$

2. Find $\int \frac{x}{x+1}dx$

3. Find $\int \frac{x^2}{x^2+x+1}dx$

4. Find $\int_0^{\sqrt{3}} \frac{x+2}{\sqrt{x^2+1}}dx$

5. $\int \frac{x}{(1+x^2)^3}\,dx$

6. $\int e^x \cos(e^x)\,dx$

7. $\int \frac{1}{\sqrt{x}} e^{\sqrt{x}}\,dx$

8. $3\int \sqrt{9-x^2}\,dx \quad (u = 3\sin\theta)$

9. $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx \quad (u = 2\sec\theta)$

10. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,d\theta \quad (x = \tan\frac{\theta}{2})$

11. $\int \frac{1}{\sin x + \cos x}\,dx \quad (u = \tan\frac{x}{2})$

Skill-building

1. $\int_1^3 \frac{\sqrt{x}}{x+1}\,dx$

2. $\int_{1/2}^3 \frac{x-1}{\sqrt{2x+3}}\,dx$

3. $\int \frac{1}{e^x+1}\,dx$

4. $\int_0^1 \frac{x}{x^4+1}\,dx$

5. $\int \frac{e^x + e^{2x}}{1 + e^{2x}}\, dx$

6. $\int x\sqrt{x+1}\, dx$

7. $\int \frac{\sqrt{1-x^2}}{x}\, dx$

8. $\int \sqrt{1-x^2}\, dx$

9. $\int \frac{\sqrt{x}}{1+x}\, dx$

10. $\int \frac{1}{\sqrt{a^2+x^2}}\, dx$

11. $\int cosec(x)dx \quad (x = \tan\frac{\theta}{2})$

12. $\int cot(x)dx \quad (x = \tan\frac{\theta}{2})$

13. $\int_0^{\frac{\pi}{2}} \frac{1}{6 + 7\sin\theta}\, d\theta \quad (x = \tan\frac{\theta}{2})$

Easier Exam Questions

1. (Blacktown Boys 2020 Q6) Using a suitable substitution $\int_a^b x^9 e^{3x^{10}} dx$, where a and b are real constants, can be written as

   A. $\int_a^b e^{3u} du$

   B. $\frac{1}{10}\int_{a^{10}}^{b^{10}} e^{3u} du$

   C. $\frac{1}{30}\int_{30a^9}^{30b^9} e^{u} du$

   D. $\frac{1}{30}\int_{3a^{10}}^{3b^{10}} e^{u} du$

2. (Blacktown Boys 2020 Q12d)

   i) Show that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ **1**

   ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \sin 9x \cos 6x\, dx$ **3**

3. (Hurlstone 2020 Q12a) Evaluate:

   (i) $\int \sin^3 x \cos^2 x\, dx$ **2**

4. (Hurlstone 2021 Q12c,d,and e)

c) By using a suitable substitution, evaluate
$\int_{3\sqrt{2}}^{6} \frac{1}{x^2\sqrt{x^2-9}}\,dx$ **3**

d) Find $\int \cos^5 x\,dx$ **3**

e) Find $\int \frac{dx}{1+\cos x}$ **2**

5. (Hurlstone 2023, Q8)

Which of the following uses a correct substitution for $\int_0^{\sqrt{3}} \frac{\ln(\tan^{-1} x)}{1+x^2}\,dx$?

A. $\int_0^{\frac{\pi}{3}} \ln u\,du$

B. $\int_0^{\frac{\pi}{3}} \frac{\ln u}{1+\tan^2 u}\,du$

C. $\int_0^{\sqrt{3}} \ln u\,du$

D. $\int_0^{\sqrt{3}} \frac{\ln u}{1+\tan^2 u}\,du$

6. (Sydney Boys 2020 Q14d) Consider the integral $I = \int_1^{\sqrt{3}} \frac{dx}{x(1+x^2)}$.

i) Using an appropriate substitution, show that $I = \int_a^b \frac{du}{\tan u}$, stating the values for a and b. **2**

ii) Hence show that $I = \frac{1}{2}(\ln 3 - \ln 2)$ **2**

7. (Sydney Girls 2024 Q14a) Evaluate $\int_0^1 x^2\sqrt{1-x^2}\,dx$. **3**

Harder Exam Questions

1. (Blacktown Boys 2020 Q15b) Use the substitution $t = \tan\frac{x}{2}$ to find the value of a, given that **4**

$$\int_0^{\frac{\pi}{2}} \frac{a}{5+3\sin x+4\cos x}\,dx = 1$$

2. (Hurlstone 2023, Q14d) Use the substitution $u = \cos 2\theta$ to evaluate $\int_{\frac{1}{2}}^{1} \sqrt{\frac{1-u}{1+u}}\,du$. **4**

3. (Sydney Girls 2022 Q11c and d)

(c) Find $\int \frac{1}{1+e^{-x}}\,dx$ **2**

(d) Use an appropriate substitution to find $\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$ **3**

4. (Sydney Girls 2022 Q12a and b)

a) Evaluate $\displaystyle\int_0^1 \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$. **3**

b) Find $\displaystyle\int \sin^4 x \cos^5 x\,dx$. **3**

5. (Sydney Girls 2024 Q11b) Find $\displaystyle\int \frac{\tan^2 x + 2}{x + \tan x}\,dx$. **2**