Vertical and Projectile Resisted Motion — Problem Set

Note: Throughout this worksheet, assume $g = 10ms^{-2}$ unless specified otheriwse.

Warm-up

1. By considering the forces acting on the object, write an equation for vertical acceleration $\ddot{x}$ in terms of g and v for the following scenarios:

   a) An object falling under gravity. Air resistance is negligible.

   b) An object falling under gravity. Air resistance is equal to $2mv^2$, where m is the mass of the object and v is the velocity of the object.

   c) An object is shot upwards at a particular speed (it is not important what this speed is). It experiences air resistance of $2mv^2$ and is pulled down by gravity.

   d) An object falling under gravity. Air resistance is equal to $5mv$, where m is the mass of the object and v is the velocity of the object.

   e) An object is shot upwards at a particular speed (it is not important what this speed is). It experiences air resistance of $5mv$ and is pulled down by gravity.

   f) An object being pushed downwards as it falls with a force of $2m$ Newtons. The object is also being pulled by gravity and experiences air resistance of $8mv$.

2. The terminal velocity is reached when there is no net force acting on the object (i.e when acceleration is 0) as an object falls with air resistance. Calculate the terminal velocity for the following scenarios. Let m be the mass of the object and v be the velocity of the object.

   a) An object falling under gravity. Air resistance is equal to $3mv$.

   b) An object falling under gravity. Air resistance is equal to $4mv^2$.

   c) An object falling under gravity. Air resistance is equal to $5mv$, where m is the mass of the object and v is the velocity of the object.

   d) An object being pushed downwards as it falls with a force of $4m$ Newtons. The object is also being pulled by gravity and experiences air resistance of $8mv$.

3. For the following, find t in terms of v. Initially, the object travelled at 2m/s.

   a) $\ddot{x} = -g - v$. Initially, the object travelled at 1m/s towards the ground.

   b) $\ddot{x} = -g - v^2$. Initially, the object travelled at 3m/s downwards.

   c) $\ddot{x} = -g + v$. Initially, the object travelled at 5m/s upwards.

   d) $\ddot{x} = -g + v^2$. Initially, the object travelled at 7m/s upwards.

   (Hint: $\ddot{x} = \frac{dv}{dt}$)

4. For the solutions in Question 3, rearrange to find $v(t)$ (v in terms of t).

5. For the following, find x in terms of v.

   a) $\ddot{x} = -g - v$. Initially, the object travelled at 1m/s towards the ground at a height of 10m.

   b) $\ddot{x} = -g - v^2$. Initially, the object travelled at 3m/s downwards at a height of 100m.

   c) $\ddot{x} = -g + v$. Initially, the object travelled at 5m/s upwards from the ground.

d) $\ddot{x} = -g + v^2$. Initially, the object travelled at 7m/s upwards from a 49m high cliff. Take the base of the cliff to be $x = 0$.

(Hint: $\ddot{x} = v\frac{dv}{dx}$)

6. For the solutions in Question 5, rearrange to find $v(x)$ (v in terms of x).

7. A projectile is being launched at an angle of 30 degrees from the ground with a speed of 10m/s. The air resistance experienced is $2mv$, so the equations of motion are

$$\ddot{x} = -2v, \quad \ddot{y} = -g - 2v$$

Solve the above to find $\dot{x}$ and $\dot{y}$ in terms of t.

Skill-building

1. A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2 where v is the speed and k is a positive constant.

   a) Show that $v^2 = \frac{g}{k}(1 - e^{-2kx})$ where x is the distance fallen

   b) As the distance it has fallen increases from d_1 to $2d_1$ the speed increases from v_1 to $\frac{5}{4}v_1$. Express the greatest possible speed of the particle in terms of v_1

2. A particle of mass m is projected from the ground with initial velocity U m/s. During the journey it experiences an air resistance which is numerically equal to mv^2

   a) find the time taken for it to reach maximum height

   b) find the maximum height

   c) Find the velocity when it reaches the ground. Is this more or less than U?

   d) Find the time taken to reach the ground

3. A particle is moving vertically downward in a medium which exerts a resistance to the motion which is proportional to the square of the speed of the particle. The particle is released from rest at O and its terminal velocity is V. Find the distance it has fallen below O and the time taken when its velocity reaches half the terminal velocity.

4. A particle of unit mass is projected from the origin at a speed of 1 m/s at an angle of 30 degrees and experiences resistance that is equal to half the particle's velocity.

   a) Solve for the equations of motion on both upwards and downwards motion separately

   b) Find the maximum height reached by the particle

   c) Calculate how far the particle travelled from the launch point (horizontally) when it reaches maximum height.

Easier Exam Questions

1. (Sydney Boys 2024 Q8) A particle of mass m falls vertically from rest under gravity in a medium in which the resistance to the motion has magnitude

$$\frac{1}{40}mv^2,$$

Where v m/s is the speed of the particle and $g = 9.8$ m/s^2 is the acceleration due to gravity. What is the terminal velocity of the particle?

   (A) 400 m/s
   (B) 392 m/s
   (C) 20 m/s
   (D) 19.8 m/s

2. (Sydney Boys 2025 Q9) A particle is moving vertically in a resistive medium under the influence of gravity. The resistive force is proportional to the velocity of the particle.

   Which of the following is false?

   (A) If the particle is initially moving downwards, then its speed could either increase or decrease.
   (B) If the particle is initially moving upwards, then its speed could either increase or decrease.
   (C) If the particle is initially moving upwards, it will eventually reach a terminal velocity.
   (D) If the particle is initially moving downwards, it will eventually reach a terminal velocity.

3. (Sydney Boys 2020 Q4) A mass of m kilograms falls from a stationary balloon at height h metres above the ground. It experiences air resistance during its fall equal to mkv^2 where v m/s is its speed and k is a positive constant. The distance, in metres, of the mass to the ground as it falls is x. The acceleration due to gravity is given by g and the positive direction is taken to be upwards. What is the equation of motion?

   (A) $\ddot{x} = g - kv^2$
   (B) $\ddot{x} = g + kv^2$
   (C) $\ddot{x} = -g + kv^2$
   (D) $\ddot{x} = -g - kv^2$

4. (Caringbah 2021 Q16b) An object of mass 50 kg falls from a height under a gravitational acceleration of g meters per second squared and air resistance of $0.05v$ newtons.

   (i) Show that $t = 1000 \ln\left(\frac{g}{g-0.001v}\right)$. **3**
   (ii) Find the velocity of the object after time t. **1**
   (iii) What is the limiting (terminal) velocity of the object? **1**
   (iv) How far has the object fallen after time t. **2**

Harder Exam Questions

1. (Barker 2023 Q13c) A unit mass is projected vertically in a medium where the resistance to motion is $0.1v^2$ N and $g = 10$ m/s^2.

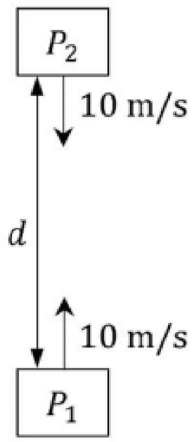
   (i) In the upward direction, show that the equation of motion is:

   $$\ddot{x} = -\frac{100 + v^2}{10}.$$

   1

   (ii) In the downward direction, show that the terminal velocity is 10 m/s. **1**

   (iii) A unit mass, P_2, is projected vertically downward at 10 m/s, and at the same instant a second identical unit mass, P_1, is projected vertically upwards at 10 m/s. Initially they are d metres apart.

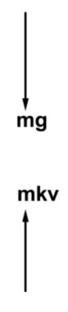
   

   If the two unit masses collide at the instant P_1 reaches its maximum height, show that:

   $$d = 5\left(\ln 2 + \frac{\pi}{2}\right) \text{ metres.}$$

   4

2. (Girraween 2021 Q14b) A particle with mass m is dropped from a stationary balloon. If the force of gravity on the particle is mg (downwards obviously!) and the particle experiences air resistance proportional to its speed (mkv) in the opposite direction to its motion (see diagram)

   (i) Find an expression for the particle's acceleration in terms of velocity and find the velocity it can't exceed as it falls (terminal velocity) if $g = 10$ m/s^2 and $k = \frac{1}{6}$ (*Note: Down is positive in this question!!!*) **2**

(ii) Show $t = -6\ln\left(1 - \frac{v}{60}\right)$ and find the time at which it hits the ground if it hits at half the terminal velocity. **2**

(iii) Show that $x = 60t + 360(e^{-\frac{t}{6}} - 1)$ and find the height from which the particle was dropped. **2**

3. (Girraween 2024 Q15b) A relief parcel is dropped vertically from a stationary helicopter 50 m above the ground. It experiences acceleration due to gravity of 10 m/s^2 and acceleration due to air resistance against its direction of motion of $\frac{v^2}{40}$ m/s^2.

   (i) Taking down as positive, express acceleration in terms of velocity and find the parcel's terminal velocity. **1**

   (ii) Show that the distance the parcel has dropped from the helicopter is given by $x = -20\ln\left(\frac{400-v^2}{400}\right)$ and find the speed at which the parcel hits the ground. **4**

4. (Hsc 2021 Q15c) An object of mass 1 kg is projected vertically upwards with an initial velocity of u m/s. It experiences air resistance of magnitude kv^2 newtons where v is the velocity of the object, in m/s, and k is a positive constant. The height of the object above its starting point is x metres. The time since projection is t seconds and acceleration due to gravity is g m/s^2.

   (i) Show that the time for the object to reach its maximum height is

   $$\frac{1}{\sqrt{gk}}\arctan\left(u\sqrt{\frac{k}{g}}\right) \text{ seconds.}$$

   3

   (ii) Find an expression for the maximum height reached by the object, in terms of k, g and u. **3**

5. (Girraween 2022 Q14a) A 20 kg projectile is launched upwards from the ground at 500 m/s. It experiences force from gravity of 200 Newtons and air resistance in the opposite direction to its motion of $\frac{v^2}{18}$ Newtons.

   Letting the ground be $x = 0$,

   (i) Show that the weight's acceleration is given by $\ddot{x} = -10 - \frac{v^2}{360}$ **1**

   (ii) Show that $x = 180\ln\left(\frac{253\,600}{3600+v^2}\right)$ and find the maximum height the projectile reaches. **3**

   (iii) Show that the time taken for the projectile to reach a velocity of v m/s is given by $t = 6\tan^{-1}\left(\frac{25}{3}\right) - 6\tan^{-1}\frac{v}{60}$ and find the time taken for the projectile to reach its maximum height. **2**

6. (Girraween 2025 Q16c) A projectile with mass m kg is launched vertically from the ground with a velocity of v_0 metres per second where $v_0 < 200$ metres per second. It experiences gravity of $10m$ Newtons and air resistance of $0.05mv$ Newtons against its direction of motion, where v is the current velocity. If the projectile lands in the same place 5 seconds after launch, find its initial velocity v_0. **6**

Harder Exam Questions - Resisted Projectile Motion

1. (Barker 2020 Q15a) The trajectory of a projectile fired with speed u m s^{-1} at an angle θ to the horizontal, in a medium whose resistance to the projectile's motion is proportional to the projectile's velocity, is represented by the parametric equations

$$x = \frac{u\cos\theta}{k}(1 - e^{-kt}) \quad \text{and} \quad y = \frac{(10 + ku\sin\theta)}{k^2}(1 - e^{-kt}) - \frac{10t}{k},$$

where k is the constant of proportionality of the resistance.

   (i) Show the greatest height is reached when $t = \frac{1}{k}\log_e\left(\frac{10+ku\sin\theta}{10}\right)$. **2**

   (ii) If $k = 0.5, u = 40$ and $\theta = 30°$, show that the greatest height is reached when the projectile is at the point $(20\sqrt{3}, 40(1 - \log_e 2))$. **2**

2. (Barker 2020 Q14a) A particle is projected from a fixed origin, O. There are two forces acting on the particle: gravity and air resistance. The particle's displacement is measured in metres, and time is measured in seconds.

   (i) The initial horizontal component of the velocity is u. Its subsequent horizontal motion is modelled by $\frac{d\dot{x}}{dt} = -k\dot{x}$. Show that the particle's horizontal displacement is given by $x = \frac{u}{k}(1 - e^{-kt})$. **3**

   (ii) The initial vertical component of the velocity is w. Its subsequent vertical motion is modelled by $\frac{d\dot{y}}{dt} = -k\dot{y} - g$ where g is the acceleration due to gravity. Show that the particle's vertical displacement is given by $y = \frac{kw+g}{k^2}(1 - e^{-kt}) - \frac{gt}{k}$. **3**

   (iii) Show that the Cartesian equation of motion is given by $y = \left(\frac{kw+g}{ku}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{u}\right)$. **2**

   (iv) Suppose that $u = 9$ m/s$, w = 6$ m/s$, k = 0.2$ and $g = 10$ m/s^2. Determine whether the particle will pass over a wall of height 0.15m at a horizontal distance of 9 m from the origin. **1**

3. (Caringbah 2025 Q15c) A particle of mass m is thrown from the top, O, of a very tall building with an initial velocity u at an angle α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \text{ and } \ddot{y} = -k\dot{y} - g$$

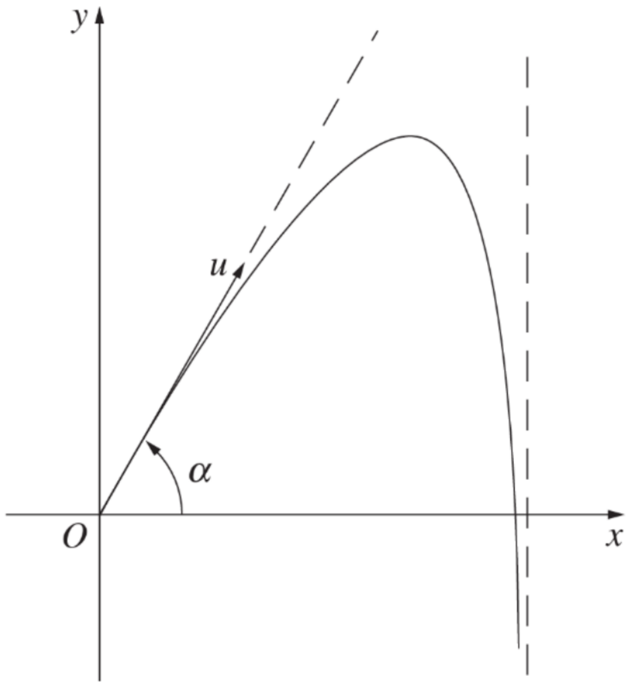
where k is a constant and the acceleration due to gravity is g. (You are not required to show these.) (i) Derive the result $\dot{x} = ue^{-kt}\cos\alpha$ from the relevant equation of motion. **2**

(ii) Verify that $\dot{y} = \dfrac{1}{k}((ku\sin\alpha + g)e^{-kt} - g)$ satisfies the appropriate equation of motion. **2**

(iii) Find the value of t when the particle reaches its maximum height. **1**

(iv) What is the limiting value of the horizontal displacement of the particle? **2**

4. (HSC 2022 Q16b) A projectile of mass M kg is launched vertically upwards from a horizontal plane with initial speed $v_0\, ms^{-1}$ which is less than 100 ms^{-1}. The projectile experiences a resistive force which has magnitude $0.1Mv$ newtons, where $v\, ms^{-1}$ is the speed of the projectile. The acceleration due to gravity is 10 ms^{-2}. The projectile lands on the horizontal plane 7 seconds after launch. Find the value of v_0, correct to 1 decimal place. **4**



5. (HSC 2023 Q13c) A particle of mass 1 kg is projected from the origin with speed $40\,\text{m s}^{-1}$ at an angle $30°$ to the horizontal plane.

   (i) Use the information above to show that the initial velocity of the particle is **1**

$$\mathbf{v}(0) = \begin{pmatrix} 20\sqrt{3} \\ 20 \end{pmatrix}.$$

   The forces acting on the particle are gravity and air resistance. The air resistance is proportional to the velocity vector with a constant of proportionality 4. Let the acceleration due to gravity be $10\,\text{m s}^{-2}$.

   The position vector of the particle, at time t seconds after the particle is projected, is $\mathbf{r}(t)$ and the velocity vector is $\mathbf{v}(t)$.

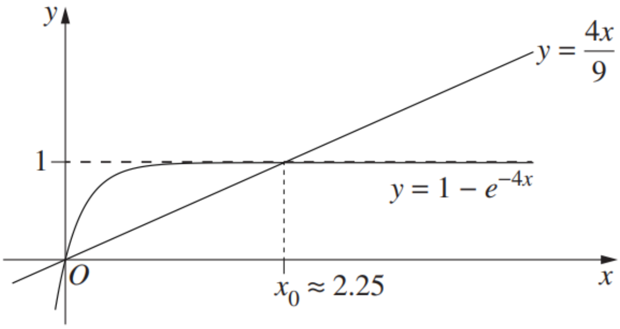
   (ii) Show that **3**

$$\mathbf{v}(t) = \begin{pmatrix} 20\sqrt{3}\,e^{-4t} \\ \dfrac{45}{2}e^{-4t} - \dfrac{5}{2} \end{pmatrix}.$$

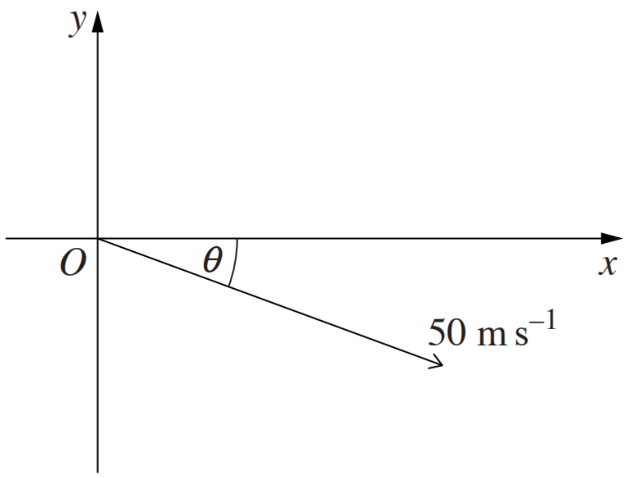
   (iii) Show that **2**

$$\mathbf{r}(t) = \begin{pmatrix} 5\sqrt{3}\,(1 - e^{-4t}) \\ \dfrac{45}{8}(1 - e^{-4t}) - \dfrac{5}{2}t \end{pmatrix}.$$

   (iv) The graphs $y = 1 - e^{-4x}$ and $y = \dfrac{4x}{9}$ are given in the diagram below (next page). **2**
   Using the diagram, find the horizontal range of the particle, giving your answer rounded to one decimal place.



6. (HSC 2025 Q16b) A particle of mass 1 kg is projected from the origin with a speed of 50 m s^{-1}, at an angle θ below the horizontal into a resistive medium. **5**



The position of the particle t seconds after projection is (x, y), and the velocity of the particle at that time is

$$\mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

The resistive force, $\mathbf{R}$, is proportional to the velocity of the particle, so that

$$\mathbf{R} = -k\mathbf{v},$$

where k is a positive constant.

Taking the acceleration due to gravity to be 10 m s^{-2}, and the upwards vertical direction to be positive, the acceleration of the particle at time t is given by

$$\mathbf{a} = \begin{pmatrix} -k\dot{x} \\ -k\dot{y} - 10 \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

Derive the Cartesian equation of the motion of the particle, given $\sin\theta = \frac{3}{5}$.