Powers and roots of complex numbers — Problem Set

Warm-up

1. In your own words, state De Moivre's theorem.

2. If $z^5 = 32\text{cis}(\frac{\pi}{3})$, there are 5 solutions for z.

    a) State one solution for z by applying De Moivre's theorem in reverse.

    b) Let us rotate z anticlockwise by $\frac{2\pi}{5}$. Show that this also satisfies $z^5 = 32\text{cis}(\frac{\pi}{3})$.

    c) By repeatedly rotating your solution by $\frac{2\pi}{5}$, find the 5 solutions to $z^5 = 32\text{cis}(\frac{\pi}{3})$.

    d) In your own words, why does rotating the solution by $\frac{2\pi}{5}$ produce new solutions? What happens if we produce a sixth solution by rotating further?

    e) Notice that some of your solutions do not use the principal argument. How might we change our method so that all solutions have arguments between $-\pi$ and π?

3. Find the 5 fifth roots of $8cis(\frac{5\pi}{4})$ in modulus-argument form.

4. Find the 3 cube roots of -1 in the form x+iy.

5. Let us consider finding the seventh roots of unity (i.e what real or complex number z satisfies $z^7 = 1$). A trivial (obvious) solution is that $z = 1$, but is not the only solution.

    a) Considering $1 = 1cis(2\pi)$, apply De Moivre's theorem in reverse to find one non-real solution to $z^7 = 1$.

    b) Let us call the solution you found in part (a) as ω. Show that ω^2 also satisfies $z^7 = 1$

    c) In the same way, we can also show $\omega^3, \omega^4, \omega^5$and ω^6 also satisfy $z^7 = 1$ (you don't need to show this for this question).

       Plot all 7 solutions to $z^7 = 1$. What shape do they make?

    d) If we added up all 7 roots, what do we get?

6. From the last question (part (d)), we can guess that if ω is a non-real solution to $\omega^3 = 1$, then $1 + \omega + \omega^2 = 0$.

   Using this, and considering $\omega^3 = 1$, simplify $(1 - \omega^8)(1 - \omega^4)(1 - \omega^2)(1 - \omega)$

7. If w is a non-real cube root of 1, simplify $(1 + 2w - 3w^2)(1 + 3w - 2w^2)$

Skill-building

1. Find the 3 ciube roots of $8i$ in modulus-argument form.

2. Solve the equation $z^4 + 16 = 0$ and plot the solutions in the complex plane.

3. If $z = \cos(\theta) + i\sin(\theta)$,

    a) Write z^3 in modulus-argument form using De-moivre's theorem.

b) Write z^3 in the form x+iy by expanding $(\cos(\theta)+i\sin(\theta))^3$ and collecting real and imaginary parts

c) If two complex numbers are equal, their real parts must be equal and their imaginary parts must be equal. Use this to show that $\sin(3\theta) = 3sin(\theta) - 4\sin^3(\theta)$

d) Letting $\sin(3\theta) = 1$, and substituting $x = \sin\theta$ results in the cubic equation

$$-4x^3 + 3x - 1 = 0$$

. By solving $\sin(3\theta) = 1$, and substituting θ into $x = \sin(\theta)$, solve the cubic equation.

4. Considering Question 3(d), show that $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$, and by letting $\cos(3\theta) = 1$ and by making a suitable substitution, solve

$$4x^3 - 3x - 1 = 0$$

5. a) Let $z = cis(\theta)$, show that $z - \frac{1}{z} = 2i\sin\theta$

b) Further show that $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

c) Let us consider $(z - \frac{1}{z})^5$. Expand this via binomial expansion.

d) From part (a), we already know

$$(z - \frac{1}{z})^5 = (2i\sin(\theta))^5 = 32i\sin^5(\theta)$$

By pairing terms with the same coefficient in your expansion in part (c) and using the results in part (b), show that

$$2i\sin(5\theta) - 10i\sin(3\theta) + 20i\sin(\theta) = 32i\sin^5(\theta)$$

e) Simplify this to show that

$$\sin^5(\theta) = \frac{1}{16}(\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta))$$

Easier Exam Questions

1. (Gosford 2023 Q16a) Find the fourth roots of $2 + 2\sqrt{3}i$ **4**

2. (HSC 2021 Q14c) Using De Moivre's theorem and the binomial expansion of $(\cos\theta + i\sin\theta)^5$, or otherwise, show that: **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

3. (Hornsby Girls 2025 Q5) One of the complex solutions to $z^5 = -a$, where a is a positive real constant, is $a^{\frac{1}{5}}\operatorname{cis}\left(\frac{\pi}{5}\right)$. One of the other solutions is a real number and is equal to:

(A) $a^{\frac{1}{5}}\operatorname{cis}\left(\frac{3\pi}{5}\right)$

(B) $-a^{\frac{1}{5}}$

(C) $a^{\frac{1}{5}}$

(D) $a^{\frac{1}{5}}\operatorname{cis}\left(\frac{9\pi}{5}\right)$

4. (Hornsby Girls 2025 Q7) If ω is the complex cube root of unity, then the value of $(1-3\omega+\omega^2)(1+\omega-8\omega^2)$ is:

   (A) $36\omega^3$
   (B) -36
   (C) $-36\omega^3$
   (D) 36

5. (Manly 2020 Q6) Given that $w^5 = 1$ and w is a complex number, what is the value of $1+w+w^2+w^3+w^4+w^5$?

   (A) 1
   (B) 0
   (C) w
   (D) $-w$

Harder Exam Questions

1. (Blacktown Boys 2022 Q11a) Given that $z = \sqrt{3} - i$

   (i) Express z in modulus-argument form. **2**
   (ii) Use De Moivre's theorem to evaluate z^7, and leave your answer in the form $x+iy$. **2**
   (iii) Use De Moivre's theorem to evaluate $\frac{z^7}{(\bar{z})^7}$, and leave your answer in the form $x+iy$. **2**

2. (Caringbah 2020 Q14a) Consider the equation $z^5+1=0$.

   (i) Draw a sketch of the roots of $z^5+1=0$ on an Argand Diagram. **1**
   (ii) Factor z^5+1 into irreducible factors with real coefficients. **2**
   (iii) Deduce that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$. **2**
   (iv) Write a quadratic equation with integer coefficients which has roots $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$. Hence find the value of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ as surds. **2**

3. (Cheltenham Girls 2023 Q5) Let $\omega = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$. Which of the following polynomials has $\omega, \omega^3, \omega^7$ and ω^9 as its zeros?

   (A) $z^4+z^3+z^2+z+1$
   (B) $z^4+z^3-z^2-z+1$
   (C) $z^4-z^3-z^2+z+1$
   (D) $z^4-z^3+z^2-z+1$

4. (Cheltenham Girls 2024 Q14a) Given that $z^n+\frac{1}{z^n}=2\cos n\theta$, where $z=\cos\theta+i\sin\theta$. **3**
   Express $\cos^6\theta$ in terms of $\cos n\theta$.

5. (Fort St 2025 Q14a) Consider the expansion of z^5 for $z=\cos\theta+i\sin\theta$.

(i) Using De Moivre's theorem to show that:

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

3

(ii) Hence show that $x = \pm\cos\frac{\pi}{10}$ and $x = \pm\cos\frac{3\pi}{10}$ are the solutions to the equation $16x^4 - 20x^2 + 5 = 0$. **3**

(iii) Deduce the exact value of $\cos\frac{\pi}{10}$. **3**

6. (Killara 2020 Q12d)

(i) Show that if $1, \omega_1, \omega_2$ are the cube roots of 1,

$$1 + \omega_1 + \omega_1^2 = 0$$

and

$$1 + \omega_2 + \omega_2^2 = 0$$

1

(ii) If n is not a multiple of 3, prove that

$$x^{2n} + x^n + 1 \text{ is divisible by } x^2 + x + 1$$

3