Integration by Parts and Recurrence Relations — Problem Set

Warm-up

1. $\int x \ln x \, dx$

2. $\int x \sin x \, dx$

3. $\int \ln x \, dx$

4. $\int_0^{\pi} e^x \sin x \, dx$

5. If $I_n = \int_0^1 x(1-x^3)^n \, dx$ for $n \geq 0$, show that $I_n = \dfrac{3n}{3n+2} I_{n-1}$ for $n \geq 1$.
   Hence find an expression for I_n in terms of n for $n \geq 0$.

Skill-building

1. $\int \tan^{-1} x \, dx$

2. Show that
$$\int_0^{\pi} x^2 \sin\left(\frac{1}{2}x\right) \, dx = 8\pi - 16$$

3. If $I_n = \int \tan^n x \, dx$ for $n \geq 0$, show that $I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$ for $n \geq 2$.

4. If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$,
   (i) show that $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$.
   (ii) Hence, evaluate $I_4 = \int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$.

Easier Exam Questions

1. (HSC 2024 Q11a) Find $\int xe^x \, dx$. **2**

2. (Hurlstone 2025 Q15b) Evaluate $\int_0^2 x^2 \ln x \, dx$. **3**

3. (Girraween 2025 Q14a) If $I_N = \int_0^{\frac{\pi}{2}} \sin^N x \, dx$
   (i) Show that $I_N = \dfrac{N-1}{N} I_{N-2}$ **2**
   (ii) Find the exact value of I_6 **2**

4. (Killara 2025 Q13b) Consider $y = \ln(x), x \in [1, e]$.

   (i) Find the exact value of **2**

$$\int_1^e [\ln(x)]^2 \, dx$$

   (ii) $y = \ln(x)$ on the interval $[1, e]$, is rotated about the x-axis. Find the exact volume of the solid generated. **2**

5. (Sydney Tech 2025 Q16b) Given $I_n = \int_{-3}^{0} x^n \sqrt{x+3} \, dx$

   i) Show that $I_n = -\frac{6n}{2n+3} I_{n-1}$. **3**

   ii) Hence, find I_3. **2**

Harder Exam Questions

1. (Cheltenham 2025 Q16c) Consider the following definite integral.

$$I = \int_2^3 \tan^{-1}\left(\frac{1}{x^2 - 5x + 7}\right) dx$$

   (i) Show that $\tan^{-1}\left(\frac{1}{x^2 - 5x + 7}\right) = \tan^{-1}(x-2) - \tan^{-1}(x-3)$. **2**

   (ii) Hence, evaluate I, leaving your answer in exact form. **2**

2. (HSC 2024 Q14d) The following argument attempts to prove that $0 = 1$.

   We evaluate $\int \frac{1}{x} \, dx$ using the method of integration by parts.

$$\begin{aligned} \int \frac{1}{x} \, dx &= \int \frac{1}{x} \times 1 \, dx \\ &= \frac{1}{x} \times x - \int -\frac{1}{x^2} x \, dx \\ &= 1 + \int \frac{1}{x} \, dx \end{aligned}$$

   So we have

$$\int \frac{1}{x} \, dx = 1 + \int \frac{1}{x} \, dx$$

   We may now subtract $\int \frac{1}{x} \, dx$ from both sides to show that $0 = 1$.

   Explain what is wrong with this argument. **2**

3. (Hornsby Girls 2025) Find $\int sec^3(x) dx$ **3**

4. (Normanhurst 2025 Q15c) Let $I_{3n+2} = \int_0^1 x^{3n+2} e^{x^3} \, dx$, where $n \in \mathbb{Z}$ and $n \geq 0$.

   i. Prove that $I_{3n+2} = \frac{e}{3} - n I_{3n-1}$. **2**

   ii. Hence find I_8 as an exact value. **2**

   iii. Using part (i), prove that **3**

$$3 \int_0^1 (1 + x^3) x^{3n-1} e^{x^3} \, dx \leq e$$

5. (Manly 2025 Q15c) Let $I_n = \int_0^{\frac{\pi}{3}} \tan^{n+1} x \sec x \, dx$. Given that $\int \tan x \sec x \, dx = \sec x$,

(i) Show that $I_n = \dfrac{2 \times 3^{\frac{n}{2}}}{n+1} - \dfrac{n}{n+1} I_{n-2}$ **3**

(ii) Hence find I_2 **2**

6. (St George Girls 2025 Q15b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, where n is an integer, and $n \geq 3$.

(i) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$. **3**

(ii) Hence evaluate I_7. **3**