Mathematics in the World Around Us
Student Workbook

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Contents

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Chapter 0

Introductory Challenges

0.0 Mathematical Outcome

The mathematical process is deeper than simply arriving at the correct answer. The goal of this chapter is to introduce the students and the instructor to the format of the text. The authors' hope is that the activities provided will foster an interactive and cooperative classroom environment. This chapter is a set of logic problems which require little or no prerequisite knowledge and establish a foundation of thinking which will be used in activities throughout the text.

Instructors will find that students will begin developing the basic problem solving skills throughout this chapter's activities. They will encounter such questioning as; "Is this the optimal answer?" "Can we show there is not a better way?" "Can we explain why our technique will solve the problem?" etc. In answering these questions students are engaged in the mathematical method of developing reasoning through concrete examples.

To further this extension from the concrete to the abstract we begin asking students to change the variable of the problem to see if the previous solution technique continues to apply. In some cases a technique can be developed which will work for all possible arrangements, in which case it is natural to ask "how do we know the method will work for all possibilities?" However, students are also exposed to solution methods that are not valid when a variable is changed. The contrast in the problems will hopefully develop the necessary curiosity that initiates the investigation process.

As a framework for the entire course, Polya's four step method for problem solving should be introduced, emphasized, and reemphasized throughout. Reference to these steps will be made from time to time within the instructor's notes, but they should be kept in the forefront of students' minds whether the prompts are there or not.

Polya's Problem Solving Method

1. Understand the problem (This may include reading the question several times, asking "what if" questions, tinkering with parts of the question, and so on.)

2. Devise a plan (Though this is often a completely separate step from understanding the problem, the tinkering used to understand may lead to an idea for a plan of action.)

3. Carry out the plan (When the plan can be completed, great! However, one may find they are unable to complete their plan of action due to its difficulty or faultiness. It is at this point, the method becomes iterative. The problem solver will need to revisit understand the problem or devising a plan to refine their approach.)

4. Review (Even when the plan can be carried out successfully, a review of the result is needed. Do the results answer the question asked? What evidence is there that the results are correct? Is that answer actually correct? If not, here is another place where the method becomes iterative. The problem solver will have to revisit one of the first three steps depending on the flaw identified.)

It is important to emphasize that a natural part of problem solving is temporary failure. Noticing that there is a flaw in the solution to a problem (during the review) is a critical step in solving difficult problems. It should not be expected that anyone will devise correct, complete answers on their first try, and that this should not be considered failure. It is a temporary setback that should launch the problem solver down a new path. One only fails when one gives up trying new ideas.

0.1 Activity: Manager's Schedule

Let's assume you are the manager of some business. This could be any business, such as an advertising firm, a theater troop, or IT department. You have 9 projects that must be completed. You predict that each project will take a day to complete and have assigned the employees needed for each project in the table below. No person can work on two projects in a single day but must be present the entire day the project is being worked on. (Maybe they are in different locations, national safety procedure require this, etc.)

Project Number	Employees assigned
1	Anna, Beth, Chris, Doug
2	Eve, Chris
3	Doug,Eve,Fran,Ian
4	Anna, Eve, Fran
5	Beth, Chris, Fran, Greg, Hugh
6	Anna, Hugh, Ian
7	Doug, Greg, Hugh
8	Beth, Greg
9	Ian

1. Make a first attempt at creating a project schedule.

2. Double check the schedule you made for Question 1 to make sure no person is required to work on two projects in the same day. Describe or illustrate how you know your schedule has met that criterion.

Unfortunately, you have to stay until all projects are completed and the holidays are quickly approaching. So you really want to create a schedule that allows yourself to go on vacation ASAP!

3. Create a better way to organize the interactions between the projects that can help you to make your schedule. How will you use this organization to make a project schedule with the fewest number of days?

4. Using the information you have gathered, can you provide a number of days you know you will need more than to schedule the projects? What is your reasoning you cannot finish in that many days?

5. Now create a project schedule that uses the fewest number of days.

6. Does your schedule get you on break as early as your answer from Question 4? If not, explain why there is no schedule that finishes faster or make a better schedule.

0.1.1 Exit Slip: The Manager's Process

It is likely you will need to do this again with next year's schedule, but there may be a different number of projects or employees at that time. To help you remember what you learned for next year or possibly pass this along to the next manager, write down the process of finding an optimal schedule making sure to highlight at least three things you learned are important about the process.

0.2 Activity: Making $20

How many different ways can you have $20 in US bills?

Sometimes a simple question can have many layers, so we first must unpack and organize our thoughts before answering! We will use Polya's 4-step problem solving method to guide the process.

1. **Understand the problem.**

   (a) Can you write down three different ways to make $20 in US bills? If yes, write them down—this is good progress toward understanding the problem! If no, a discussion with your peers is in order (What are "US bills"? What is meant by "make $20"? etc.)

   (b) What makes your three ways different?

   (c) Are there other ways to make $20? How many? (It's ok to make a wild guess at this point!)

   (d) What denominations of bills are possible to use in a collection that sums to $20?

   (e) Do you think you are being asked for an exact number for the total, or is an estimate good enough?

2. **Devise a plan.**

   (a) Provide an example of a collection of bills that sums to $20 that contains exactly one $1 bill?

   (b) Is it possible to have a collection of bills that sums to $20 that contains exactly one $1 bill and no $5 bills? If so provided an example or explain why none exists?

(c) **One of the best ways to start problem solving is to solve a simpler similar problem.** One way to do that is to work on a lower total. List all the ways that U.S. bills can make a total of 5 dollars. HINT: There are more than two ways.

(d) Let's continue our exploration of simpler problems. Write down all the collections of bills that make $10.

(e) Did you use the result from 2(c) in solving 2(d)? If no, what is one way you could have? HINT: See if you can find a connection between your answer to part 2(c) and your answer to part 2(d).

(f) Another way to solve a similar simpler problem is to make the total \$20 but restrict the types of bills that may be used. List all the ways to make \$20 where the highest denomination is a \$1 bill (meaning you must use at least one \$1 bill and no higher denominations).

(g) List all the ways to make \$20 where the highest denomination is a \$2 bill (meaning you must use at least one \$2 bill and no higher denominations).

(h) How do you know there are no repeats in your answers to questions 2(f) and 2(g)?

At this point we have explored two ways to create simpler similar problems (restrict the total dollar amount and restrict the types of bills we can use), and we've seen that one simpler solution may be used to solve another (using the \$5 totals to help make the list of \$10 totals). Using one or both of these ideas, devise a plan for finding all collections of US bills that make \$20. Summarize your plan here. DO NOT carry out the plan here. Just describe a plan that you think will work.

3. **Carry out the plan.** Try to do what you just described to find out how many different ways there are to make \$20 using US bills. If you are struggling to make your plan work or you have determined that it just won't work, go back and revise your plan!

How many ways did you come up with?

4. **Review the solution.**

   (a) How do you know you have not counted any collection more than once? In other words, how do you know each of your collections is different?

   (b) How do you know you have all collections that make \$20? How do you know you have not missed any?

If you can't answer these questions or you have doubts that you have the correct answer you may need to revise your plan or how you carried it out.

0.2.1 Exit Slip

1. How many different ways can you have \$15 in US bills? Provide a tree diagram, an organized list, or written justification for your answer.

2. At Panera Bread, you are presented with the following lunch choices: a starter of soup or salad; 3 different sandwiches; and a side of bread, chips, or an apple. How many different lunches consisting of one starter, one sandwich, and one side can be made?

0.3 Project Choices

1. As the manager of a chemical distribution company, your job is to make sure that all of the chemicals your company manufactures are safely stored in warehouses. But your job is not as easy as you may think. Some of the chemicals cannot be in close proximity to some others, so some chemicals cannot be stored together in the same warehouse. Due to the complex chemical names for each of the 7 chemicals produced, your company just refers to the chemicals as Chemical 1 through Chemical 7. To reduce expenses, we want to use as few warehouses as possible. Table 1 shows the chemicals that cannot be stored together. Provide a storage arrangement for the chemicals in to the fewest number of warehouses and provide evidence on why your company cannot safely store the chemicals in fewer warehouses than your solution provides.

Chemical X	Chemicals that cannot be stored with X
1	2, 3, and 7
2	1, 3, and 4
3	1, 2, and 4
4	2, 3, 5, and 6
5	4, 6, and 7
6	4, 5, and 7
7	1, 5, and 6

Table 1: List of the chemicals that cannot be stored together

2. How many different ways can you make $30 in US bills? Provide a tree diagram to support your claim and write a paragaraph about how you used results from the $20 class activity to find your answer.

3. At Domino's Pizza, the following unusual vegetable toppings are available: jalapeno peppers, roasted red peppers, spinach, and basil. How many different unusual veggie pizzas can you make using these toppings? Support your answer with a tree diagram. Provide written explanation of how you know you have counted them all and haven't counted anything extra. This may be an explanation of how you know your tree diagram works.

4. The division championship in major league baseball goes to the winner of a best-of-5 series. The first team to win 3 games wins the division, so the series will never go more than 5 games. For example, if the teams are Boston and Tampa Bay (as they were in the 2021 American League Division Series), one way it could have unfolded was B-T-B-T-T, meaning Boston won the first, Tampa Bay won second, Boston won the third, and then Tampa Bay won the fourth and fifth games. It also could have

unfolded T-B-B-B (and this is what actually happened in 2021). How many different ways could the division series have unfolded (including the two already mentioned)? Support your answer with a tree diagram. Provide written explanation of how you know you have counted them all and haven't counted anything extra. This may be an explanation of how you know your tree diagram works.

5. The curator at a natural history museum is designing a penguin display. He would like to put penguins of 4 different species in the display—Emperor, Northern Rockhopper, Macaroni, and African—arranged one next to the other in a row. However, he wants to be sure the Emperor Penguin is not next to the Macaroni Penguin. How many ways can he arrange the penguins in the display? Support your answer with a tree diagram. Provide written explanation of how you know you have counted them all and haven't counted anything extra. This may be an explanation of how you know your tree diagram works.

0.4 Additional Exercises

1. How many different ways can you make \$50 in US bills? Provide a tree diagram to support your claim and write a statement about how you used results from the \$20 class activity to find your answer.

2. You're at a meeting where there is a make-your-own-sandwich platter. There is only one kind of bread, but there is a choice of what to put on it. There are three different lunch meats, two different cheeses, and extras—mustard, mayonnaise, and lettuce—to choose from. How many sandwiches can you make using one lunch meat, one cheese and any combination of extras?

Chapter 1

Logic

1.0 Mathematical Outcome

One fundamental concept of mathematics is logic. Sound mathematics comes from the ability to state definitively whether something is true or false. In order to explore mathematics from a theoretical standpoint one must first understand the basics of logic. This chapter uses students previous and intuitive knowledge to formally develop the concepts and terminology which will be used in developing conjectures and theorems throughout the text.

Definition 1.1. A declarative statement (either verbally or written) which can be judged as either true or false (but never both) is known as a **statement**. Determining a statement to be true or false is called the statement's truth value.

Generally there are two types of statements. The first is called a **simple statement** which only contains one idea, which we will denote using letters throughout this chapter. The second is the idea of combining a number of simple statements with conjunctions which form **compound statements**. You will see compound statements in written and symbolic form which use connections such as "and" and "or" along with others. The concepts of logic follow from being able to consider the truth value of the simple statements and conclude the truth value of a compound statement.

The two most common and basic conjunctions used to form compound statements are "and" and "or." Both of these conjunctions are explored in Activity 1.1 from an intuitive approach.

The "and" conjunction is usually the most intuitive and students easily realize the only time when the statement constructed by adjoining two statements (either simple or compound) with an "and" is true is if both of the statements being adjoined are true, otherwise the compound statement is false.

Conversely, the "or" conjunction is often used in common vernacular to mean only one of the adjoined statements is true. For example when you ask a child "Would you like chocolate **or** vanilla ice cream?" you are not intending for them to choose both; you are simply giving them an option of choosing either. In logic however the conjunction "or" is evaluated as determining if at least one of the adjoining statements is true. Therefore we will determine that the compound statement adjoined by an "or" is true whenever one or both of the adjoined statements is true, hence it will only be false if both of the adjoined statements are false.

Definition 1.2. A tabular format for displaying all possible outcome combinations of a compound statement is known as a **truth table**.

Truth tables are useful in comparing and contrasting compound statements which are made up of the same simple statements. We begin expelling the basics of truth tables in Activity 1.0.1 and continue to explore the more advanced concepts throughout the chapter.

We often wish a statement to be positive if it is false. We see this in our everyday lives when getting tested for diseases, it is a good thing if the test proves negative or false. Therefore we may want to consider the truth value of a particular statement to be the opposite. This concept is known as **negation**. If we place a "not" in front of any statement we have changed its truth value at each instance. We explore the use of negation in Activity 1.1.4.

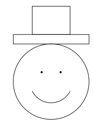
One of the more complex conjunctions is the "if...then..." statement. While there are other conjunctions such as "if and only if," "else," "while," and others, we will finish our exploration in this text here. We have chosen to explore the "if...then..." statement since it is the basics of formulating conjectures and theorems in mathematics.

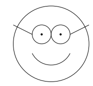
The "if ... then..." statement is one which determines if one statement implies the other. We can only definitively conclude that a statement does not imply another if the first statement is true yet the second is false, because we can see that the first statements truth does not imply that the others will also be true. However when the first statement is false there is no conclusion about its implication on the second statement, therefore we must conclude that the conjunction is true.

An example of this is the combination of the statements "A person is pregnant" and "a person is a woman". We will look at the compound statement "If a person is pregnant, then they are a woman." In this case knowing that a person is pregnant allows us to conclude definitively that they are also a woman, however knowing the the person being pregnant is false we can make no conclusion about the gender of that person.

1.0.1 Entrance Activity: Top Hat and Glasses

Consider the following statements P and Q:

P: I am wearing a top hat. 

Q: I am wearing glasses. 

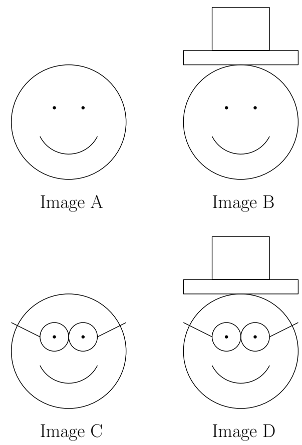


Figure 1.1: Top Hats and Glasses

1. For which of the images in Figure 1.1 is P true?

2. For which of the images in Figure 1.1 is Q true?

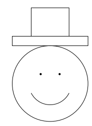
3. For which of the images in Figure 1.1 is "P and Q" true?

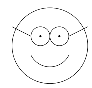
4. For which of the images in Figure 1.1 is "P or Q" true?

1.1 Making a Statement

1.1.1 Activity: Exploring AND and OR Statements

Consider the following statements P and Q:

P: I am wearing a top hat. 

Q: I am wearing glasses. 

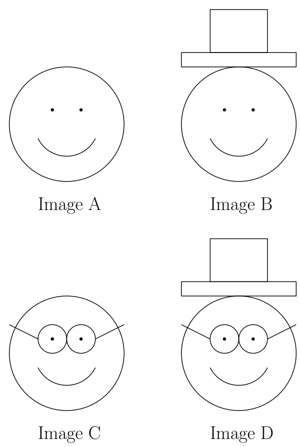


Figure 1.2: Top Hats and Glasses

1. What does it mean when the statement P and the statement Q are both true? Explain what it means if both P and Q are false?

2. If I am wearing a top hat and no glasses, would the statement "P and Q" be true or false? Explain your answer.

3. What would it mean if we knew that the statement "P or Q" is true? Write out all the possible answers, describing the images in Figure 1.2 that make "P or Q" true.

4. What would it mean if we knew that the statement "P or Q" is false? Write out all the possible answers.

5. If I am wearing a top hat and no glasses, what would you say about the statement "P or Q"; is it true or false?

6. List all the different ways I can look depending on whether statement P or statement Q is true or false?

1.1.2 Activity: Top Hats, Overalls, and Rain boots

Let $P \wedge Q$ be the single statement "P and Q" and let $P \vee Q$ be the single statement "P or Q." Consider the statements below:

P: I am wearing a top hat.
Q: I am wearing overalls.
R: I am wearing rain boots.

7. What do I look like if $P \wedge Q$ is true? Am I wearing the same thing if $Q \wedge P$ is true?

8. What do I look like if $P \wedge Q$ is false? Is there more than one answer? If so, list out all the possible ways I could be dressed.

9. What am I wearing if $P \vee Q$ is true? Is there any difference in my outfit if $Q \vee P$ is true? List all possible outfits that make these statements true.

10. Fill in the table below. Each cell in the table should contain either "True" or "False." First write in all the possible truth values for the pair of statements P and Q in the first two columns (see Question 6 for some help). Then in the third and fourth columns describe how $P \wedge Q$ and $P \vee Q$ will be affected by your choice of P and Q in the corresponding row. This table is called a truth table.

P	Q	$P \wedge Q$	$P \vee Q$

11. What do I look like if $P \wedge Q \wedge R$ is true?

12. Does my outfit change if we reorder P, Q, and R in Question 11? Explain your answer.

13. Describe my clothing if $(P \vee Q) \wedge R$ is true. List out all the possible clothing options.

14. Describe my clothing if $P \vee (Q \wedge R)$ is true. List out all the possible clothing options.

15. Fill in the table below. Each cell in the table should contain either "True" or "False." As in Question 10 first fill in the first three columns so that no two rows have exactly the same three truth values for P, Q and R. The your choice of P, Q, and R in each row will determine the truth value in the next three columns.

P	Q	R	$(P \vee Q) \wedge R$	$P \vee (Q \wedge R)$

16. Using the truth table in Question 15 determine if the statement $P \vee (Q \wedge R)$ the same as the statement $(P \vee Q) \wedge R$. Explain yourself thoroughly.

17. Suppose I am wearing a top hat and rain boots, and I am not wearing overalls. Is the statement $(P \wedge Q) \wedge R$ true or false? Is the statement $(P \wedge R) \vee Q$ true or false?

1.1.3 Exit Slip

Fill in the table below. Each cell in the table should contain either "True" or "False." As in Question 10, begin by completing the first three columns.

P	Q	R	$(P \wedge Q) \wedge R$	$(P \vee Q) \vee R$	$P \vee (Q \wedge R)$

1.1.4 Activity: Negation

Let $\neg P$ represent the statement “not P”. For example, if P represents the statement "It is raining." then $\neg P$ represents the statement "It is not raining." $\neg P$ is called the negation of P. Again we will consider the statements below.

P: I am wearing a top hat.
Q: I am wearing overalls.
R: I am wearing rain boots.

27. If $\neg P$ is true, is anything on my head?

28. If $\neg P$ is false, is anything on my head?

29. Fill in the truth table below. Each cell in the table should contain either “True” or “False.” As in Question 10 begin by completing the first column.

P	$\neg P$

30. Suppose you know the statement $P \wedge R$ is false. Describe what I’m wearing. Is there a statement that says the same thing when it is true?

31. Fill in the truth table below so that each cell contains either "True" or "False." As in Question 10, begin by completing the first two columns so that all ways truth values can be assigned to P and Q are represented in the four rows.

P	Q	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$

32. Using the truth table in Question 31 determine if the statement $\neg(P \wedge R)$ is the same as the statement $\neg P \vee \neg R$. Explain yourself thoroughly.

33. Fill in the table below so that each cell contains either "True" and "False." As in Question 10, begin by completing the first three columns.

P	Q	R	$(P \wedge Q) \wedge \neg R$	$(\neg P \vee \neg Q) \vee R$	$(P \vee \neg Q) \wedge \neg R$

1.1.5 Activity: The Implications of Your Actions

The last activity introduced the basic form of an argument used by mathematicians. They assume that something is true and then see if that means whether you know other things are true. This step is called an **implication**. We say that "P implies Q", and write $P \Rightarrow Q$, if whenever P is true Q is also true. So the only way that the statement $P \Rightarrow Q$ can be false is when P is true and Q is false.

1. Consider the statement "If it's Monday, my day will suck." This is an implication where P = "today is Monday" and Q = "I will have a sucky day today." Discuss with your group members: What are the 4 possible options for T/F for P and Q. In which of those scenarios is the person who claims "If it's Monday, my day will suck" saying something true? In which scenario are they wrong?

2. Fill in the following table so that each cell contains either "True" or "False." First write in all the possible truth values for the pair of statements P and Q in the first two columns. Then in the third column indicate how $P \Rightarrow Q$ will be affected by your choice of P and Q in the corresponding row.

P	Q	$P \Rightarrow Q$

3. Notice that $P \Rightarrow Q$ is always true whenever P is false. Why do you think that is the case?

4. Suppose the statement "If it is Tuesday then it must be raining all day" is true. What do you know if:

   (a) it is not raining outside.

   (b) it is Tuesday today.

   (c) it is Wednesday today.

Consider the statements below for Questions 5–6.

P: It is raining today.
Q: I bring an umbrella with me.
R: I play basketball during recess.
S: I took a test in English class.

5. Write the statement
   `"If it is raining today then I will bring an umbrella with me."`
   using the symbolic statements P, Q, R, and S. Explain why you know your answer is correct.

6. Write the statement
   `"If it is raining today then I will play basketball and take a test in English class."`
   using the symbolic statements P, Q, R, and S. Explain why you know your answer is correct.

7. Fill in Table 1.1 so that each cell contains either "True" or "False." First write in all the possible truth values for the pair of statements P and Q in the first two columns. Then in the remaining columns indicate how the other statements will be affected by your choice of P and Q in the corresponding row.

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$

Table 1.1: Table of logical statements using the $\Rightarrow$ symbol

8. If $P \Rightarrow Q$ is true, does that mean that $Q \Rightarrow P$ is true. Give a justification for your answer.

9. Which symbolic statements in the Table 1.1 are equivalent?

1.2 Logic in Games

1.2.1 Entrance Activity: Statements in Minesweeper

The game of Minesweeper is a popular game that requires players to use logic to solve the puzzle. The game begins with a board of squares, some of which contain hidden mines. Game play can be summarized in one sentence: **the number on a block shows the number of mines adjacent to it and you have to flag all the mines**. In more detail:

- Start by clicking any random place since it is your first move. You do not have any clues to the locations of the mines yet!
- When you identify a square hiding a mine, flag it by right-clicking. This will place a flag on that spot.
- When you identify a square that is not hiding a mine, clear it by clicking it. This will reveal the number of mines in the adjacent squares, or more if there are no adjacent mines.
- If you accidentally clear a mine you lose.
- If you flag all the mines and clear all other locations, you win!

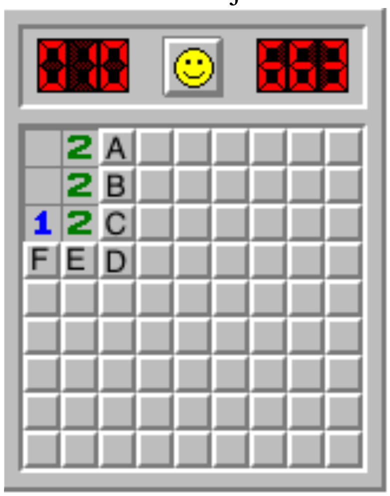
Before reading any further go to https://freeminesweeper.org/ and play a few games! If it is your first time playing, you might want to select Beginner from the Game menu.

Here is a sample of the type of question you will be asked and how you are expected to answer. Shown is a game in progress where the player has indicated the locations of 4 mines (marked by the 4 flags). The 6 above the game board and on the left indicates there are 6 remaining mines. In this example, we have labelled one location with the letter "A" to indicate a space we would like to discuss. You will not see letters on the squares while playing the game online.

Sample Question: Determine whether the following statement is true or false and explain. Square A is a mine.

Sample Answer: FALSE. The square directly below the flag next to the A contains a 1, indicating that one and only one square adjacent to it is a mine. Since the square above the 1 (the flag) is a mine, square A must not be a mine. We can conclude that the statement "Square A is a mine." is false.

Consider the in-progress game below and determine if each statement below is true or false. Provide a brief justification for your claim.



1. Squares A and B are mines.

2. Square C is a mine

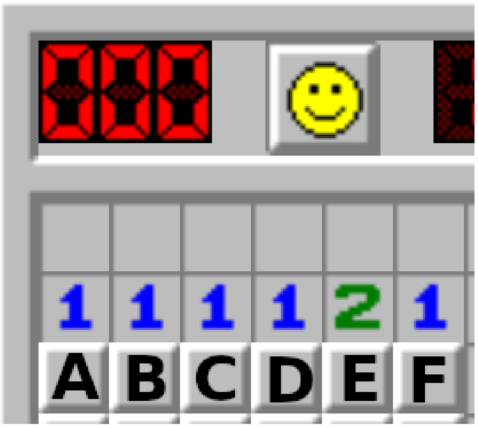
3. Squares F and E are mines

4. Square D is a mine.

1.2.2 Implications in Minesweeper

In the game Minesweeper sometimes we cannot determine whether a square is a mine or not at first glance. If we can determine the state of a nearby square, however, that may help us figure it out. Here we wish to explore these scenarios and how logic plays a roll in beating the game.

1. Consider this portion of an in-progress game. Are the statements below true, false, or indeterminate (need more information)? Explain.



   (a) A is a mine.

   (b) B is a mine.

   (c) A is a mine or B is a mine.

   (d) A is a mine and B is a mine.

2. What can we conclude from the exploration in question 1? Circle one and explain.

   (a) Neither A nor B is a mine.
   (b) Both A and B are mines.
   (c) Either A is a mine or B is a mine, but not both.

3. So which is it? Is A the mine? or is B? To figure this out logically, we make A a mine and see what happens. So suppose A is a mine (as if we knew this to be a fact) and determine whether the following statements are true or false. You should be able to explain each one. Use the boards below, which show the same board without the letters, for reference.

   (a) If A is a mine, then B is clear.

   (b) If A is a mine, then B is clear and C is clear.

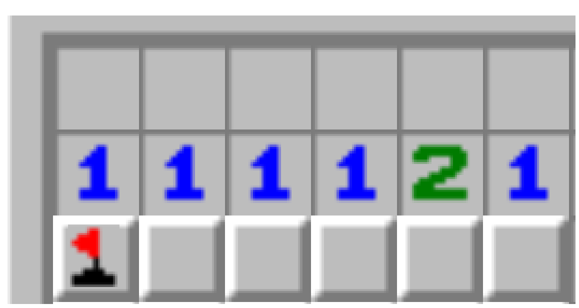
   (c) If A is a mine, then B is clear and C is clear and D is a mine.

   (d) If A is a mine, then B is clear and C is clear and D is a mine and E is a mine and F is a mine.

   (e) If A is a mine, then B is clear and C is clear and D is a mine and E is a mine and F is clear.

   (f) If A is a mine, then B is clear and C is clear and D is a mine and E is clear and F is a mine.

   (g) If A is a mine, then B is clear and C is clear and D is a mine and E is clear and F is clear.



Now suppose B is a mine (as if we knew this to be a fact) and start over, assuming we don't know anything about any other square to begin. Determine whether the following statements are true or false. You should be able to explain each one. Use the boards below, which show the same board without the letters, for reference.

(h) If B is a mine, then A is clear.

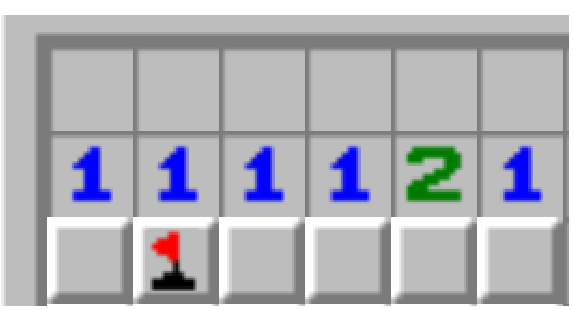
(i) If B is a mine, then A is clear and C is clear.

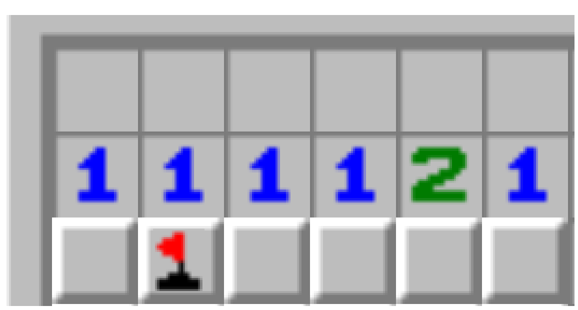
(j) If B is a mine, then A is clear and C is clear and D is clear.

(k) If B is a mine, then A is clear and C is clear and D is clear and E is a mine and F is a mine.

(l) If B is a mine, then A is clear and C is clear and D is clear and E is a mine and F is clear.

(m) If B is a mine, then A is clear and C is clear and D is clear and E is clear and F is a mine.





4. According to your answers in question 3, which do you think is the mine? A or B? Explain.

5. Below is more of the board from question 1. Use this new information to determine whether each square A through J is a mine or is clear. A board without the letters is supplied for you to work on. You may want to use a pencil!

Were you right about squares A and B?

6. Consider the in-progress game below. Are the following statements true or false? Explain.

(a) Square Q is a mine.

(b) Square N is a mine.

(c) Square I is a mine and square J is a mine.

(d) Square H is a mine.

(e) Square K is a mine or square L is a mine.

(f) If square K is a mine, then square J is not a mine.

(g) If square L is a mine, then square J is not a mine.

(h) Square J is not a mine.

7. Below is an argument table showing that the statement "Square A is a mine" is true for the game shown in question 6. Create an argument table for each statement below.

True Statement	Reason
O is a mine	O is the only uncleared square adjacent to the 1 on its left.
G is not a mine.	G is adjacent to the 1 above the O (which is a mine).
A is a mine or G is a mine, but not both	A and G are the only uncleared squares adjacent to the 1 next to A.
A is a mine.	The second and third rows (of this table) prove A is a mine.

Table 1.2: Argument

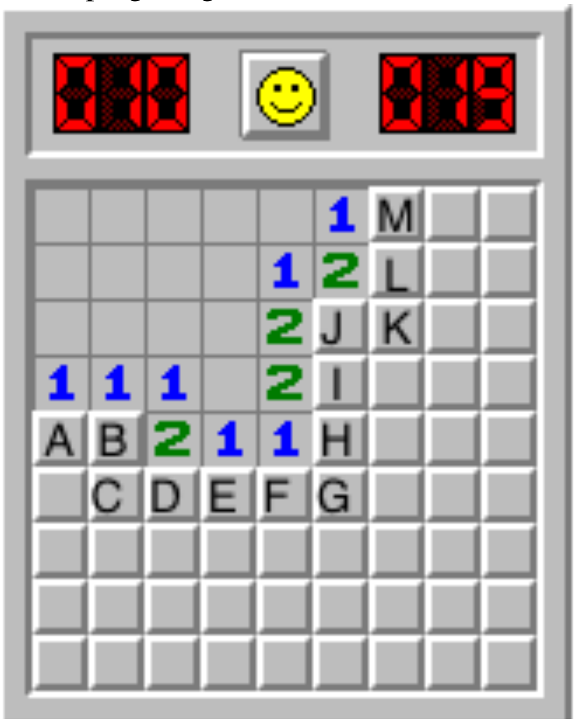
You may use the statements in question 6, and you may bring in new statements as needed.

(a) Square P is not a mine.

(b) Square I is a mine.

1.2.3 Exit Slip

For the in-progress game below, create three true statements, one for each type listed below.



1. Statement containing an “and”
2. Statement containing an “or”
3. An implication statement in the form “ If . . . , then . . . .”

1.3 Extension Activity: Arguments

1.3.1 Activity: Nancy Out On The Town

Nancy's friends are trying to find her based on whether the following statements are true or false. They know that she is at exactly one of: home, the movie theater, the grocery store, or a clothing store.

P: Nancy is at home.
Q: Nancy is at the movie theater.
R: Nancy is at the grocery store.
S: Nancy is at a clothing store.
T: Nancy is trying on clothes.
U: Nancy is buying food.

1. If P is true is Q true or false? Explain your reasoning.
2. What statements could be true if T is true? Explain your reasoning.
3. If P is true, is $T \vee U$ true or false? Explain your answer.
4. If P is true, is $\neg S$ true or false? Explain your answer.
5. What statements are false if $Q \vee R$ is true?
6. If U is true, is $Q \wedge R$ true or false? Explain your conclusion.
7. If U is false, what do you know must be false? Explain your answer.
8. If T is true and U is false, what single true statement can you create concerning Nancy's location? Explain your results.
9. If U is true, what single true statement can you create concerning Nancy's locations? Explain your results.
10. If P is true and Q is false, is $R \vee (T \wedge U)$ true or false? Explain your results.
11. If T is true, is $\neg Q \wedge (S \vee P)$ true or false? Why?

1.4 Project Choices

1. **Project Question:** You are given an in-progress Minesweeper game. On the given board:

   (a) Mark each box adjacent to a numbered square with an O if it can be determined to be a mine or X if it can be determined that it is not a mine. * *There are at least 3 squares that are mines and at least 3 squares that are not mines. There may also be squares adjacent to the numbers that cannot be determined. Leave them blank.* *

   (b) Write an argument table for one of the squares marked with an O. * *Do not pick a square located in a "corner" adjacent to a square with a* 1. *

   (c) Write an argument table for one of the squares marked with an X.

   You may want to visit https://freeminesweeper.org/ to learn or practice the game.

2. **Project Question:** You are given an empty nonogram board. Your project is to do the following.

   (a) Fill all squares correctly, making note of your logic.

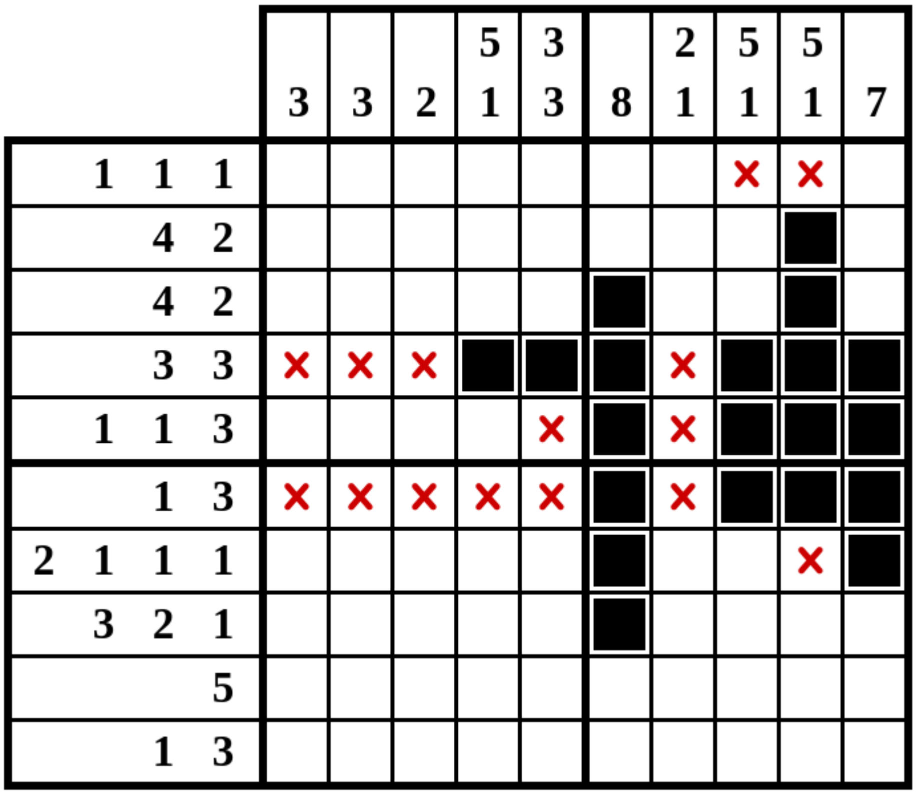
   (b) Write one true (compound) logical statement that includes an "and" and helped you complete the puzzle.

   (c) Write one true (compound) logical statement that includes an "or" and helped you complete the puzzle.

   The rules for filling the board are as follows.

   - Each box must be filled in black or marked with X.
   - Beside each row of the grid are listed the lengths of the runs of consecutive black squares on that row.
   - Above each column are listed the lengths of the runs of consecutive black squares in that column.
   - Boxes that are not part of a run of black squares must be marked with X.

Here is an example of a puzzle in progress.



The four squares in the rightmost column were filled in first. 7 consecutive squares in that column must be filled in and there are only 10 squares total, so you can only skip three of them from the top **or** three of them from the bottom. The black squares and X's immediately to the left of those four were filled in next. For example, in row 4 the tenth square in that row is filled in **and** the last run of squares must be length 3, so the two squares next to the last one must also be filled in black. That completes the run so the square next to that (the 6th from the left in that row) must not be filled (so is marked with an X). See if you can figure out why the rest of the marked squares are the way they are.

You may want to visit https://www.puzzle-nonograms.com/ to try some nonogram puzzles at a variety of difficulty levels online.

3. **Project Question:** You are given two KenKen puzzles. Your project is to do the following.

   (a) Fill all squares correctly in each puzzle, making note of your logic.

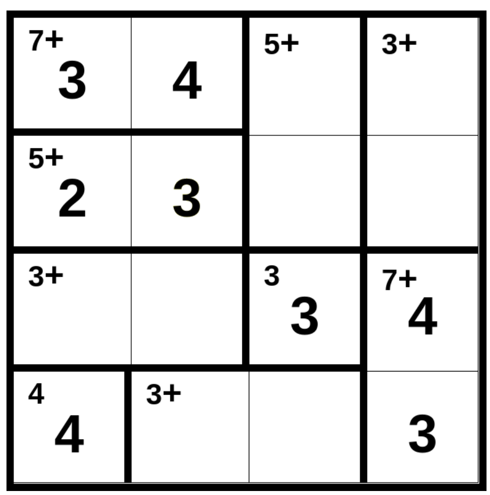
   (b) Write one true (compound) logical statement that includes an "and" and helped you complete the puzzle.

(c) Write one true (compound) logical statement that includes an "or" and helped you complete the puzzle.

The rules for solving a KenKen are as follows.

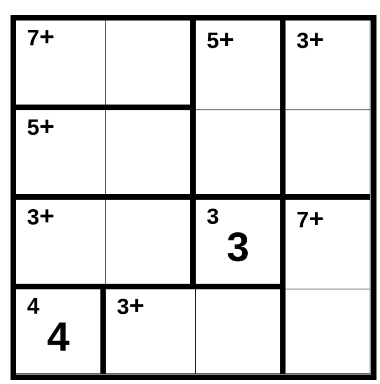
- The only numbers you may write are $1, 2, 3, \ldots$ up to the number of rows and columns. In a 4×4 puzzle you will use the numbers 1 through 4. In a 6×6 puzzle you will use the numbers 1 through 6, and so on.
- Every allowable number must appear in every row and every column. No number may be repeated in any row or any column.
- Each region bounded by a heavy border, called a **cage**, contains a target number and usually an arithmetic operation. You must fill that cage with numbers that reach the target using the specified arithmetic operation. Numbers may repeat within a cage, if needed, as long as they do not repeat within a single row or column.

Here is an example of a puzzle in progress.

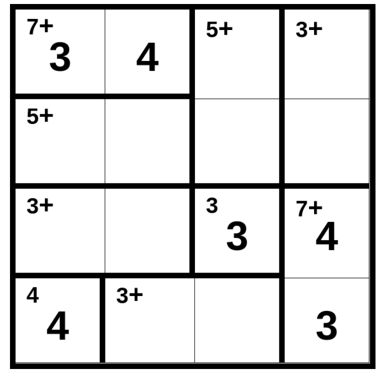


Here is how the numbers were filled in to this point.

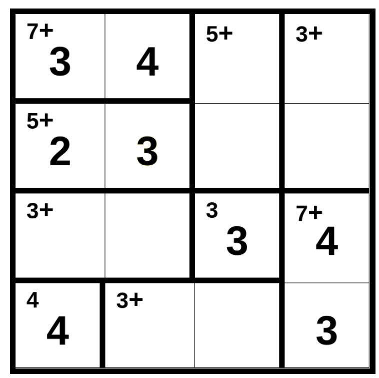
- The 4 in the bottom left corner and the 3 in the third row, third column were filled in first. Their cages are only one block so there is no operation to consider. The target number is the number that needs to be filled in.



- The two cages with $7+$ in them were filled in next. In each cage, the numbers 3 and 4 must be placed since the only way for two **different** numbers between 1 and 4 to add up to seven is to use 3 and 4. For the cage in the bottom right, the 4 must be in the third row since there is already a 3 in that row. You can come to the same conclusion by noticing there is a 4 in the fourth row (bottom left corner) already, so the 4 for this cage cannot be put in the fourth row. See if you can understand why the 3 and 4 in the $7+$ cage in the top left of the board were placed the way they were. They cannot be placed the other way around.



- The $5+$ cage with the 2 and 3 in it was filled in next. There are two ways to use different numbers between 1 and 4 to sum to five—1 and 4 or 2 and 3. However, there are already 4's in columns one and two, so a 4 cannot be used in this cage. Hence the cage will be filled with 2 and 3. Since there is already a 3 in column one, the 3 for this cage must be in column two and therefore the 2 must be in column one.



See if you can complete the puzzle. HINT: The first column already has the numbers $2, 3, 4$ in it. There is only one number that can be put in the empty box of that column!

You may want to visit https://www.kenkenpuzzle.com/ to try some KenKen puzzles at a variety of difficulty levels online. Begin by clicking the "Create Custom Puzzle" button.

4. **Project Question:** You are given a Sudoku Pair Puzzle. This isn't your regular Sudoku though!

   (2,3) - Sudoku Pair Puzzle Rules: Each completed puzzle is a 6 by 6 grid that contains exactly one integer from 1 to 6 in each cell. Furthermore, there are regions that must contain every integer exactly once. The regions are described below and outlined in the puzzle:

   (a) Each row is its own region.

   (b) Each column is its own region.

(c) Each 2 by 3 rectangle outlined by bold lines is its own region.

(d) Each 3 by 2 rectangle outlined by dashed lines is its own region.

Here is an example of a completed puzzle.

(2,3)-Sudoku Pair Puzzle

5	**3**	6	1	2	4
1	4	**2**	3	6	**5**
2	6	4	5	1	3
3	5	1	**6**	**4**	2
6	**2**	3	4	5	1
4	1	**5**	2	3	6

We've been describing $(2,3)$ Sudoku pair puzzles. A slightly harder example would be $(2,5)$ Sudoku pair puzzles. There are many variations of these puzzles that were originally designed by Dr. James Hammer in 2012. We have these puzzles on our webpage (https://open-math-book.github.io/MWAU/23spp/index.html for the $(2,3)$ Sudoku pair puzzles) that will allow you to generate and practice on multiple games.

1.5 Additional Exercises

1. Construct a truth table for the following statements.

   (a) $\neg(\neg P \wedge \neg Q)$

   (b) $\neg(\neg P \vee \neg Q)$

   (c) $\neg(P \Rightarrow \neg Q)$

   (d) $(P \vee Q) \Rightarrow P$

2. The $\Leftrightarrow$ operator is known as the "if and only if" operator and is defined to be the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Its truth table is given in Table 1.5. Find a way to write the statement $P \Leftrightarrow Q$ using the statements P and Q, and the operators $\neg$, $\vee$, or $\wedge$. (*Hint*: Construct a truth table first.)

P	Q	$P \Leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

Table 1.3: Truth table for $\Leftrightarrow$ operator

3. Construct a truth table for the following statements.

   (a) $\neg P \Leftrightarrow P$

   (b) $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

   (c) $\neg(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow P)$

   (d) $((P \Leftrightarrow Q) \wedge (P \Rightarrow R)) \vee (\neg P \Rightarrow (Q \wedge R))$

   (e) $\neg(\neg P \Rightarrow (Q \vee R)) \Rightarrow (\neg P \wedge Q)$

4. Write a single statement using P, Q, and R along with the symbols $\neg$, $\vee$, $\wedge$, or $\Rightarrow$ that is always true.

5. Consider the following statements about Jake.

   `"If today is Wednesday, then Jake's class has library day. If Jake's class has library day, then Jake will bring home a new book. Jake did not bring any new books home today."`

   What can you conclude based on the given information? Explain your answer.

6. Show that the two statements $((P \wedge Q) \Rightarrow R)$ and $(\neg P) \vee (\neg Q) \vee R$ are equivalent. Give an example in which P, Q, and R each stand for some statement (like in Activity 1.3.1, then explain what the statements $((P \wedge Q) \Rightarrow R)$ and $(\neg P) \vee (\neg Q) \vee R$ mean in your given context.

7. Write out a statement using P and Q with the operators $\neg$, $\wedge$, or $\vee$ that has the indicated truth table.

(a)

P	Q	
True	True	True
True	False	True
False	True	False
False	False	True

(b)

P	Q	
True	True	False
True	False	False
False	True	True
False	False	False

(c)

P	Q	
True	True	True
True	False	True
False	True	True
False	False	True

(d)

P	Q	
True	True	True
True	False	True
False	True	True
False	False	False

(e)

P	Q	
True	True	False
True	False	False
False	True	True
False	False	True

8. The $\oplus$ symbol is known as XOR and is the statement that is true if exactly one of P and Q is true. Its truth table is given in Table 1.6. Write a statement that is equivalent to $P \oplus Q$ using the symbols $\neg$, $\vee$, or $\wedge$.

P	Q	$P \oplus Q$
True	True	False
True	False	True
False	True	True
False	False	False

Table 1.4: Truth table for $\oplus$ symbol

9. Show that the pair of statements are equivalent using truth tables.

   (a) $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$

   (b) $((P \vee Q) \wedge (\neg P)) \wedge Q$ and $\neg P \wedge Q$

10. During your voyage across the 7 seas you come across an island that is populated by two tribes: monks and peasants (or so they call themselves). From your interaction with them you realize that the monks always tell the truth while the peasants tell nothing but lies. Unfortunately, you cannot tell the inhabitants apart from each other. One day, two of the inhabitants, let us call them Amber and Betty, came up to you to tell you something:

    Amber says "Exactly one of us is a monk."

    Betty says "Only a peasant would say that Amber is a peasant."

    Can you determine to which tribe each of Amber and Betty belong?

11. You meet another inhabitant of island described in Question 10. You ask an inhabitant if there is gold on the island. He tells you "There is gold on this island if and only if I am a monk." Translate this statement into a logical symbol. Is this person a monk or peasant? Is there gold on the island. (*Hint*: The "if and only if" operator ($\Leftrightarrow$) is discussed in Question 2.)

12. Construct a new symbol $*$ that gives true or false responses depending on whether the statements are true or false (just like $\vee$, $\wedge$, and $\Rightarrow$). Explain how it works. Fill in the first three columns of the following table to make it clear what $P * Q$ means. Then fill in the remainder of the truth table.

P	Q	$P * Q$	$((P * Q) \wedge P) \Rightarrow Q$	$((P \Rightarrow \neg Q) * Q) \Rightarrow \neg P$	$P * Q \Rightarrow P$

Chapter 2

Proportions, Ratios, and Scaling

2.0 Mathematical Outcome

The activities of this section are designed to strengthen understanding of fractions and the related concepts of ratios and similarity through applications of proportion, ratio, and scaling. For unit conversion it is assumed the student has a working knowledge of fractions including multiplication and division thereof. The sections on ratios and similarity are presented assuming little to no prior knowledge of the topics.

Through a series of activities mostly focused on things in the world around us such as the moon, peoples' eyes, and the buildings in which we teach and learn, students will

1. learn to convert units through multiplying fractions
2. work with ratios
3. see the connection between equivalent fractions and equivalent ratios
4. connect proportionality and scale with the world around them
5. calculate heights and weights of large objects by gathering data about smaller ones and using mathematics to extrapolate
6. explore the concept of self-similarity

2.1 Unit Conversion

2.1.1 Entrance Activity: One Billion Seconds

1. Write one billion as a numeral (writing its digits, not words).

2. Convert 5 minutes into seconds. (How many seconds are there in five minutes?)

3. Convert 240 seconds into minutes. (How many minutes are in 240 seconds?)

4. Convert 2,022 minutes into seconds.

5. Convert 2,022 seconds into minutes.

6. What mathematical operation(s) do you need to do these conversions? ($+$, $-$, $\times$, $\div$, $\sqrt{}$, etc.)

2.1.2 Activity: One Billion Seconds

On what day will you (or did you) become one billion seconds old?

1. As we have done before, let's solve a simpler problem first by reducing the billion seconds to one million seconds. Write one million as a numeral (writing its digits, not words).

2. One plan of action is to convert one million seconds into days, and count that many days from your birthday. Not many people know how to convert seconds to days directly, though. We can figure it out!

   (a) Convert one day into hours.

   (b) Convert the number of hours you got in part 2(a) to minutes. This is how many minutes in one day.

   (c) Convert the number of minutes you got in part 2(b) to seconds. This is how many seconds in one day.

3. Use your work from question 2 to convert one million seconds to days.

4. Write down your birthday, including the year.

5. Use your answers to parts 3 and 4 to determine on what day you will become (or became) one million seconds old.
   To think about: Does it matter what time you were born?

6. Use what you learned from questions 1 through 5 to determine on what day you will become (or became) one billion seconds old.
   To think about: Do all months have the same number of days? Do all years have the same number of days?

2.1.3 Exit Slip

1. How many seconds are there in one year? Does this change from year to year? If so, give all possible answers and explain why there are multiple answers.

2.1.4 Activity: Carats and Carrots

When you are told the carat weight of a gem stone or piece of jewelry, you are actually being given its mass in carats. A unit of mass you are probably more familiar with is the gram. In fact, 1 gram $= 5$ carats, and we call the number 5 the conversion factor between carats and grams.

1. Convert 26 grams to carats.

2. If the mass of an object is written in both carats and grams, which number will be numerically larger—the number of grams or the number of carats?

3. Should you multiply by 5 or divide by 5 to convert a mass in carats to a mass in grams? Why?

4. What is the mass of a $\frac{1}{4}$-carat diamond in grams?

5. Given that 1 pound $= 453.6$ grams, how many pounds does a 1-carat diamond weigh?

6. If you buy a half pound of carrots, how many carats of carrots do you have? Convert first from pounds to grams, and then from grams to carats.

7. If you ordered $2,000$ carats of cheese at the deli, how many pounds would you expect to get (assuming the deli-worker didn't just stand there looking at you funny!)?

2.1.5 Activity: To the Moon!

NASA's Artemis project is intended to bring people to the surface of the moon as early as 2025. The first Artemis mission was scrubbed on August 29, 2022 and then again on September 3, 2022. The first launch was then rescheduled for mid-October 2022. "With Artemis missions, NASA will land the first woman and first person of color on the Moon, using innovative technologies to explore more of the lunar surface than ever before...We're going back to the Moon for scientific discovery, economic benefits, and inspiration for a new generation of explorers." (Artemis)

The Activity: How far is the moon from Earth?

I bet you expected to have to work for that answer. Not this time! On average, the moon is $.00000004063$ light years away . OK, OK, so you'll have to work a little.

1. Will the number of miles from Earth to the moon be greater or less than the number of light years?

2. 1,000,000,000,000 miles=0.170107795023 light years so dividing one by the other equals a sort of "one". Multiplying by either fraction, therefore, leaves distances unchanged! Which fraction should you multiply $.00000004063$ light years by to convert to miles, and why?

$$\frac{1,000,000,000,000 \text{ miles}}{0.170107795023 \text{ light years}} \quad \text{or} \quad \frac{0.170107795023 \text{ light years}}{1,000,000,000,000 \text{ miles}}$$

3. Carry out the product as planned.

   $.00000004063$ light years $\times$ __________________ $=$

4. Now write that product without the numbers, retaining only the units.

   light years $\times$ __________________ $=$

   Do the light years cancel?

5. Redo the calculation using the equation 1 light year=5,878,625,000,000 miles.

   $.00000004063$ light years $\times$ __________________ $=$

   Did you get the same answer?

2.1.6 Exit Slip

1. Convert the average distance between the sun and Earth, 93 million miles, into light years.

2. A typical tire pressure for automobiles is 220 kPa (kilopascals). Use the fact that 1.25 kPa equals 0.1813 psi (pounds per square inch) to convert typical tire pressure to psi. Source: Pirelli.

3. Prior to 1896, the length of the Kentucky Derby was 12 furlongs. Source: KentuckyDerby.com. Use the fact that one furlong equals one eighth of a mile to convert the original length of the Kentucky Derby to miles.

4. Using the facts that (a) one furlong equals one eighth of a mile and (b) there are 5280 feet in one mile, what is the conversion factor between feet and furlongs?

2.2 Proportions and Ratios

2.2.1 Entrance Activity: Rice

You've been following the directions on the bag of rice you recently purchased at the grocery store.

Every time you make the rice, you do it according to the directions—2 cups of water, 1 cup of rice—but it just makes too much! About one third of it is always left over. Learning from this experience, you decide you will use only 2/3 cups of rice. But how much water should you use? Still 2 cups? No. The directions on the bag have you using twice as much water as rice, so you should continue to use twice as much water as rice! For 2/3 cups of rice, you should use $2 \cdot (2/3) = 4/3$, or $1\frac{1}{3}$, cups of water. If you used 2 cups of water with 2/3 cups of rice, the water and rice would be out of proportion. You would be using 3 times as much water as rice!

1. You're having guests and need to cook two cups of rice. How much water should you use?

2. A recipe for rice pudding asks that you start by cooking 3/4 cups of (uncooked) white rice according to package directions. You still have that much rice left in your package. How much water will you need to use?

3. How much rice can you cook with 3 cups of water?

2.2.2 Activity: Ratios and the Colon Notation

1. The proper way to cook any amount of rice from the bag in the entrance activity is to use twice as much water as rice. The volumes of water and rice used in cooking should stay in proportion. We might also say that to cook the rice we maintain a 2 to 1 water-to-rice **ratio**. The symbol used for ratios is the colon. Fill in the blanks to see the water-to-rice ratio written using the colon notation.

   $$\underset{\text{amount of water}}{\underline{\hspace{3cm}}} : \underset{\text{amount of rice}}{\underline{\hspace{3cm}}}$$

2. In a baseball game where the score is 9 to 6, the announcer might say the winning team is outscoring their opponent 3-to-2, meaning that the winning team is scoring 3 points for every 2 points scored by the losing team. Write the ratio 3-to-2 using colon notation.

3. In a large high school auditorium, the girls outnumber the boys 5-to-4, meaning for every 5 girls there are only 4 boys. Write the ratio 5-to-4 using colon notation.

4. Fill in the blanks in the table where the winning team is outscoring their opponent by a 3:2 ratio.

winning team score	3	6	9	12	15
losing team score			6		

5. Fill in the blanks in the table where girls outnumber boys 5:4.

girls	100		195	
boys		124		200

2.2.3 Activity: Body Mass Index

In a 2012 MedicalNewsToday article, research suggests that waist-to-height ratio is a better predictor of health problems such as high blood pressure, diabetes, heart attacks, and strokes than is BMI (Body Mass Index).

1. A $5\frac{1}{2}$ foot tall person with a 32 inch waist has what waist-to-height ratio?

2. According to the article, keeping your waist-to-height ratio at 0.5:1 or lower can increase your life expectancy. A 6 foot tall person should keep their waistline how small to increase their life expectancy?

2.2.4 Activity: One Blue, Seven Brown

1. What is the ratio between the number of students in this class with brown eyes to that of students with other colored eyes?

2. If the University has 9,127 students, and the ratio of brown-eyed students to non-brown-eyed students matches that in this classroom, how many brown-eyed students are there in total?

3. Repeat the exercise, but now catalog students as having blue eyes, brown eyes, or neither (somewhere in between). The person with the bluest eyes in the classroom has blue eyes.

2.2.5 Activity: Equivalent Ratios

1. There are 2.54 centimeters in an inch. That means there is a 2.54:1 ratio between its length measured in centimeters and its length measured in inches. How many centimeters are there in one foot (12 inches)?

2. It is also true that the ratio between a length measured in centimeters and a length measured in inches is 127:50. Use this ratio to calculate the number of centimeters in one foot. Did you get the same answer?

3. Is it true that the ratio between a length measured in centimeters and a length measured in inches is 254:100? Explain.

4. Show that $\frac{2.54}{1}$, $\frac{127}{50}$, $\frac{254}{100}$ are equivalent fractions by rewriting them all with the same denominator.

5. Show that 2.54:1, 127:50, and 254:100 are equivalent ratios in the same way. Rewrite them by multiplying or dividing each number in a given ratio by the same nonzero quantity so they all have the same second number.

6. For this question, the ratio between two quantities is 4:7.

    (a) If the smaller quantity is 36 fathoms, what is the larger quantity?

    (b) If the larger quantity is 28 minutes, what is the smaller quantity?

    (c) A Lego set has 50 blue blocks and 28 brown ones. Are these quantities in a 4:7 ratio? Explain.

    (d) A fountain has two tiers, one with a volume of 3,936 cm^3 and another with a volume of 6,888 cm^3. Are the volumes of the two tiers in a 4:7 ratio? Explain.

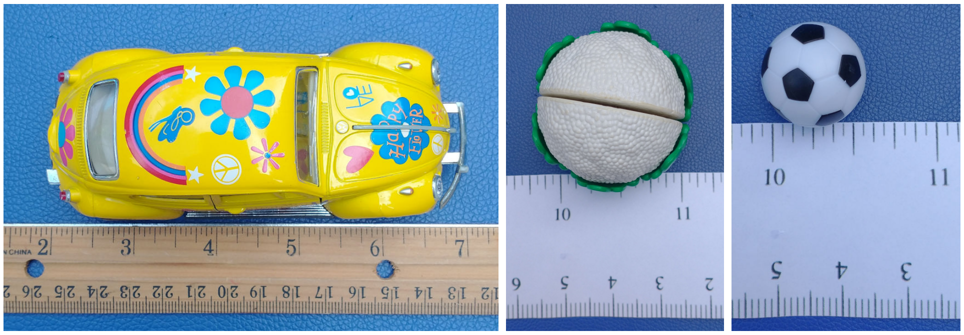
2.2.6 Activity: Out of Proportion

The following photos have been digitally edited so the images look weird. Using the word "ratio" and ratios as appropriate, explain why they look weird and/or what editing has been done to them.

1. Cantaloupe:

2. Hand:

2.2.7 Exit Slip: Automobile, Cauliflower, and a Soccer Ball



1. Use the pictures to measure the size of the model automobile, cauliflower, and soccer ball.

   (a) model automobile:

   (b) model cauliflower:

   (c) model soccer ball:

2. Approximately what is the true (real-life) size of an automobile, head of cauliflower, and soccer ball?

   (a) real automobile:

   (b) real cauliflower:

   (c) real soccer ball:

3. The scale of a miniature (like the automobile, caulilflower, and soccer ball, or a model train set) is often reported as the ratio between the model size and the real life size of that object. A common scale for model trains is 1:87, for example (meaning something that measures 1 inch on the model would measure 87 inches on the full size train). Based on your answers above, what are the scales of the following objects?

   (a) model automobile?

   (b) model cauliflower?

   (c) model soccer ball?

4. Would it make sense to pair any of these objects in the same farmhouse diorama? Would the cauliflower and soccer ball look "right" next to each other? The cauliflower and automobile? The soccer ball and automobile? Explain in terms of their scales.

2.2.8 Entrance Activity: The Odditorium

You're on a family vacation, and your parents get the idea to bring everyone to the nearby odditorium. You're not sure it will be any fun, but your parents are insistent and your sibling thinks it's a great idea. It turns out the things at the odditorium are truly odd. There's the world's smallest automobile, authentic shrunken heads, and sculptures made from recyclables. There's also this mannequin of the world's tallest person next to a young lady clearly enjoying the place. You spy the same young lady just outside the odditorium with this humongous marble.

Curiously, she is spinning the marble all by herself! The sign nearby notes that the sphere "is floating on 1/254 inches of water and can easily be stopped and spun in another direction—try it!" You think to yourself, "that ball looks really heavy. Just how much weight is she pushing around?"

1. Gather several stones, each one the size of a walnut or smaller. Put them somewhere so you won't forget to bring them to class!

2. Make a guess as to the weight of the giant marble.

Before starting this activity take a good look at the pictures from the entrance slip. Use them to estimate the diameter of the humongous marble.

2.2.9 Activity: Marble Weight Using Density

The density of an object is the ratio of its mass to its volume. In this activity, we will determine the densities of the stones you brought to class and use that information to predict the weight of the giant marble at the odditorium.

1. Choose one of the stones you brought to class and
   (a) use the scale to find its mass

   (b) use the graduated cylinder to find its volume

   (c) write down the density of the stone as a ratio. Use colon notation and integers on each side of the colon.

2. Repeat for two of the other stones.
3. Do any of the densities you calculated seem wrong (out of line with the others), or did they all come out about the same?

4. Pick one of the densities from above as an estimate of the density of the stones you brought in. Choose the density you think best matches the density of the marble in the picture.

5. Before starting this activity you should have estimated the diameter of the giant marble from the entrance slip (see the first sentence on this page!) Use your estimate to calculate the volume of the giant marble. The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

6. Calculate the mass of the giant marble based on your answers to questions 4 and 5.

2.3 Project Choices

1. Water Displacement A 12"$\times$10" rectangular prismic foot bath is filled 3" deep using water from a 10' diameter circular pool (see photo). Knowing that water has been removed from the pool for use in the foot bath, logic would suggest that the level of water in the pool must decrease. The pool contains less water after all! However, the pool looks just as full after removing water as it did beforehand. What gives?

   (a) Calculate the volume of water removed from the pool in cubic inches.

   (b) A cylinder with diameter 10' and the same volume as that calculated in part 1(a) has what height (in inches)? NOTE: The volume of a cylinder is $V = \pi r^2 h$ where r is the radius of the cylinder and h is its height.

   (c) Explain why the height calculated in part 1(b) is a good approximation of the amount the level in the pool must have decreased.

   (d) How does this calculation resolve the paradox?

2. Lake Mead Formed by the Hoover dam, Lake Mead is the largest reservoir in the United States. Much of the lake straddles the border between Nevada and Arizona, states that have been in a drought for over two decades. Due to record low levels in the lake in 2022, the Bureau of Reclamation mandated water usage cutbacks in both Nevada and Arizona. In Nevada, the cutback is $21,000$ acre $\cdot$ feet. (As reported in The Acronym May 22, 2022)

   (a) Convert $21,000$ acre $\cdot$ feet to cubic feet using the fact that one acre is $43,560$ square feet.

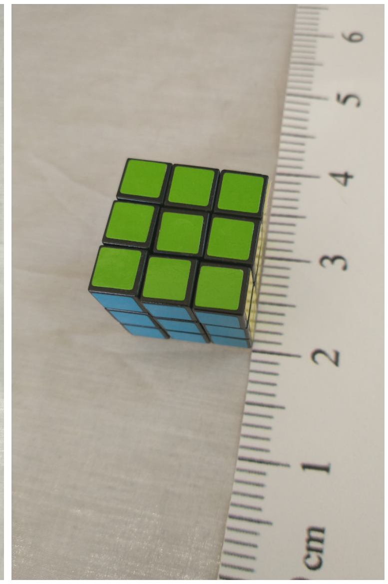
   (b) Convert the volume from part 2(a) into gallons using the estimate that 25 cubic feet equals 187 gallons.

   (c) Does this seem like a large cutback? Explain.

3. Ratios

(a) Verbalize the similarities, differences, and connections between the concepts of ratios, fractions, and unit conversion.

(b) Would this Rubik's cube look abnormally small, abnormally large, or just about right in this doll's hands? Explain in terms of ratios.



(c) In mixing pumpkin pie spice from individual spices, the ratio of cinnamon to ginger to cloves to nutmeg is 4:2:1:0.5. This means, for example, that you can mix 4 teaspoons cinnamon with 2 teaspoons ginger and 1 teaspoon cloves and a half teaspoon nutmeg to make 7.5 teaspoons of pumpkin pie spice.

i. How many tablespoons are 7.5 teaspoons? If you don't know the conversion between teaspoons and tablespoons, look it up!

ii. How much of each ingredient should you use to make 6 tablespoons of pumpkin pie spice?

iii. Is it practical to make 6 tablespoons of pumpkin pie spice using this recipe?

iv. What would be practical amounts of each spice to mix instead given that you need 6 tablespoons for your fall baking?

4. Model Trains

(a) The ratio between the size of any feature of an O scale model train and the size of the same feature on the actual train is 1:48. If you have an O scale model train that measures 17 inches long, how long is the train it models?

(b) An actual CSX hi-roof boxcar has an inside height of 13 feet (CSX.com). How high should the inside of an O scale model of this boxcar be?

(c) The ratio between the size of the features of a G scale model train and the actual train is 2:45. If the wheel of a train measures 36 inches in diameter, what would be the diameter of a G scale model of that train?

(d) Would a G scale model of a railroad car be larger or smaller than an O scale model of the same railcar? Explain.

5. Tile New Haven The Area of New Haven The land area of New Haven, CT is 18.69 square miles (U.S. Census)

   (a) Convert the area of New Haven into square feet. Note that square miles is written miles^2 (or miles $\times$ miles) as a unit. Use the fact that 1 mile $= 5280$ feet. HINT: It's more than a million.

   (b) A ceramic tile for a traditional subway tiling pattern is 2 inches tall and 4 inches wide. How many tiles does it take to cover one square foot?

   (c) How many tiles would it take to cover the entire area of New Haven?

   (d) Convert the area covered by one letter size sheet of paper (which is 8.5 inches wide and 11 inches tall) into square feet.

   (e) How many sheets of letter size paper would it take to cover New Haven? In other words, convert the area of New Haven from square feet into sheets of letter size paper.

Chapter 3

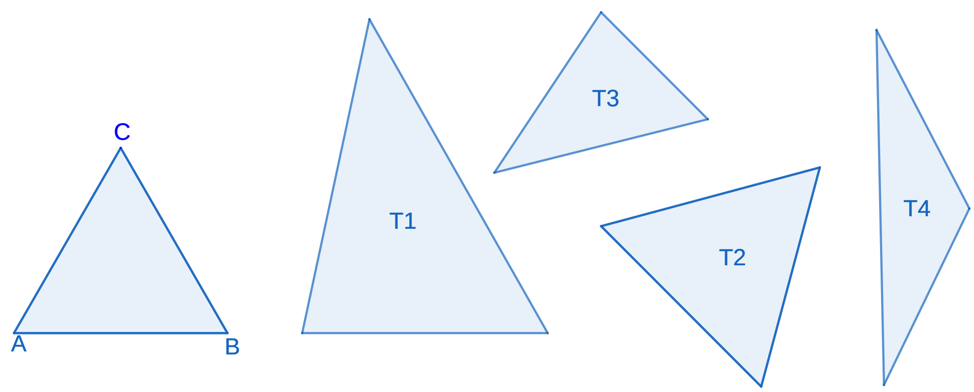
Geometry

3.1 Dilation and similarity

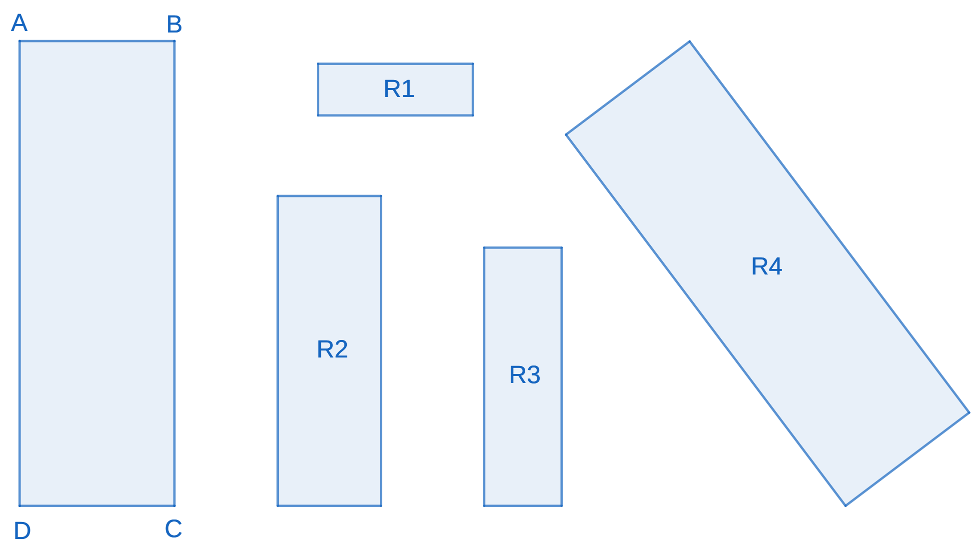
3.1.1 Activity: Dilation

Dilation is the enlarging or shrinking of a mathematical object (a point on a coordinate grid, polygon, line segment, sphere, etc.) using a specific scale factor. Dilation does not change the shape of the object. The size of a figure can change, but not the shape. The preimage—the object before scaling—is enlarged, inverted, or shrunken to form the image. The preimage and image are dilations of one another. You can think of the **preimage** as the original figure, and the **image** as the new figure. The **scale factor** of a dilation is the amount by which all original lengths are enlarged or shrunken. In the language of the previous section, everything about the dilated shape is in proportion.

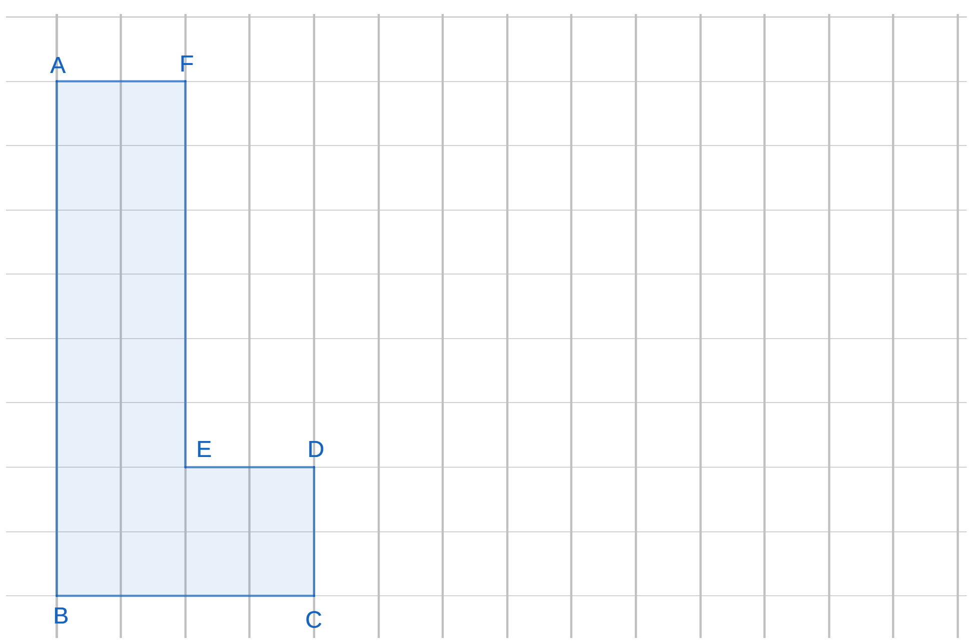
1. Which triangle is a dilation of ΔABC?



2. Rectangles R1, R2, R3, and R4 are all dilations of rectangle $ABCD$. Which one is a dilation with scale factor $\frac{1}{2}$?



3. Draw two separate dilations of hexagon $ABCDEF$—one with scale factor $\frac{1}{2}$ and the other with scale factor $\frac{2}{3}$.



4. A circle with radius 4 is dilated with scale factor 3.

   (a) What is the radius of its image?

   (b) What is the area of its image?

   (c) What is the ratio of the area of the image to the area of the preimage? Careful! It is not 3:1.

5. A cube with side length 2 cm is dilated with scale factor 10. What is the ratio of the volume of the image to the volume of the preimage?

3.1.2 Activity: Marble Weight Using Dilation

A typical shooter marble has a 5/8 inch diameter (source MarbleCollecting.com),

weighs about 5.4 grams, and has about the same density as the humongous marble. We will use this information to calculate the weight of the humongous marble from the odditorium (see 2.2.8)!

1. Make sure you understand the problem.

2. The plan for solving the problem comprises parts 3-7. You will carry out that plan by answering those questions.

3. Convert the weight of the shooter marble (5.4 grams) into pounds. Recall 1 pound $=$ 453.6 grams.

4. Estimate the diameter of the humongous marble in inches.

5. Since a typical shooter marble is a sphere and the humongous marble is also a sphere, the humongous marble is a dilation of the tiny shooter marble. What is the scale factor?

6. What is the ratio of the volume of the humongous marble to the volume of the shooter marble?

7. The ratio of the weight of the humongous marble to the weight of the shooter marble is the same as the ratio in part 6 (because weight and volume are proportional). Use this information to calculate the weight of the humongous marble.

8. Review (problem solving step 4): Did this calculation give a weight similar to that from the calculation using density? Do the calculated weights seem reasonable? How do they compare to your initial guess? If you note any discrepancies or problems, where did the calculation(s) go wrong? See if you can spot the error(s) in this calculation or the calculation based on density.

3.1.3 Activity: How tall is Engleman Hall?

When the sun casts shadows onto level ground the triangle formed by a vertical object, its shadow, and the imaginary line from the top of the object to the tip of the shadow is similar to any other. This fact allows us to calculate the height of tall objects without the dangers of climbing them or unrolling a tape measure from their top to bottom.

1. Draw a schematic diagram of an object, its shadow, and the imaginary line from the top of the object to the tip of its shadow. You may include a schematic of the sun, the ground, and anything else that you find helpful.

2. Go outside and measure

   (a) Your height

   (b) The length of your shadow

3. Find 3 tall objects whose heights you would like to calculate. Pick objects where their shadow, such as that of Engleman Hall, other building, tree, lamppost, flagpole, sculpture, etc. is cast on level ground where you can measure it. Write down the objects, their shadow lengths, and make a guess as to how tall each one is. Record your measurements and your guesses here. Then return to your classroom.

Object	Shadow length	Guess of height

4. Use the fact that all triangles defined by a vertical object, its shadow, and the imaginary line from the top of the object to the tip of its shadow are dilations of one another to calculate the height of Engleman Hall (or other object whose shadow you measured).

3.1.4 Entrance Activity: Sharpie!

When we say that two things are similar in English, we mean that they resemble one another in some way, but are not exactly the same. Just how closely they resemble one another and in what way is entirely up to the speaker. However, when we say two things are similar in mathematics, we mean they are the exact same shape but perhaps different sizes. The way in which they resemble one another is precisely defined. In terms of ratios, two objects are similar if corresponding lengths between the two objects maintain a constant ratio.

Say you have two photos of the same clown, one is a 4x6 and the other 8x10. In the 4x6 photo, the distance between the clown's pupils is one inch and in the 8x10 photo, the distance between the clown's pupils in 2 inches. If the two photos are similar in the mathematical sense, then all corresponding distances between the two photos will have a 2:1 ratio. Say the distance from the center of his big red nose to the furthest point on his big right ear is 1.3 inches on the 4x6 photo. Then the distance from the center of his big red nose to the furthest point on his big right ear on the 8x10 photo must be 2.6 inches. The conversion factor from distance on the 4x6 photo to distance of the 8x10 photo is 2. If this ratio holds for all distances in the photo, then the photos (clowns) are similar in the mathematical sense. However, if there is even one distance (say the width of the clown's nose) for which this ratio does not hold (perhaps it is 1/3 inches in the 4x6 and only 1/2 inches in the 8x10), then the photos are not similar. In the language of the previous sections, we would say the width of the clown's nose is out of proportion or the 8x10 photo is not a dilation of the 4x6 photo. For two shapes to be similar all of their features must be in proportion. They must be dilations of one another.

1. Are the markers similar in the English sense?

2. Are the markers similar in the mathematical sense?

3.1.5 Entrance Activity: Self-similarity

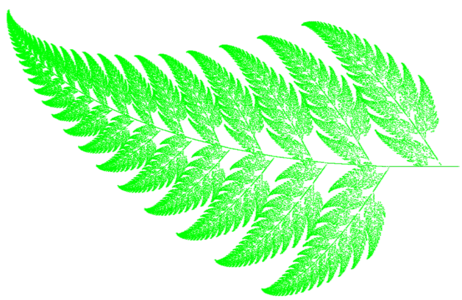
Watching movies is fun. Surfing the web is fun. Thumbing through a magazine is fun. Everywhere we look we see images. Behind each image is a great deal of mathematics. Some digital images are processed. They may have had their colors changed or their subjects touched up. Some digital images are compressed. They have been encoded to use less computer memory. To make storing, processing, compressing, and reproducing digital images happen a great deal of mathematics is used. One technique used is fractal image compression (see this IJSR article, for example). Can you tell which of these images are fractals and which are not?

Artwork or fractal?

Maple leaf or fractal?

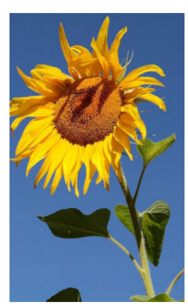
Photo or fractal?

They are all fractals! The basic idea behind all of these fractals is self-similarity. An object or shape is self-similar if parts of it look like the whole. Broccoli is a great example of a self-similar shape from nature.



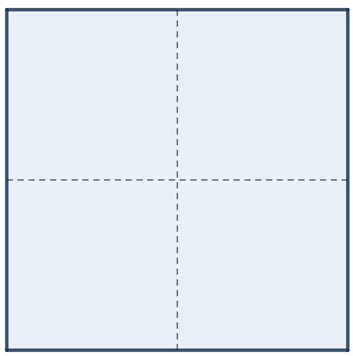
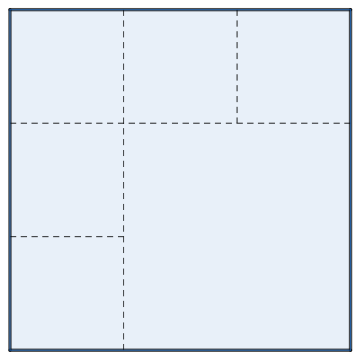
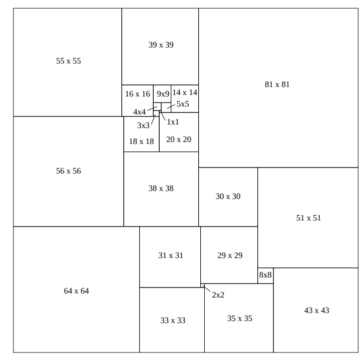
The small piece of the broccoli head resembles the whole head! The fronds of the fern resemble the whole fern!

Which of the following natural shapes are self-similar (have parts that resemble the whole)? Circle them.

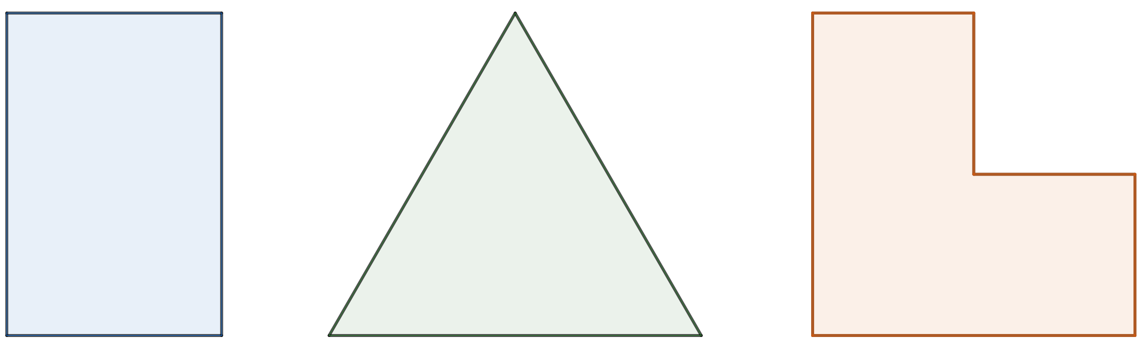


3.1.6 Activity: Rep-tiles

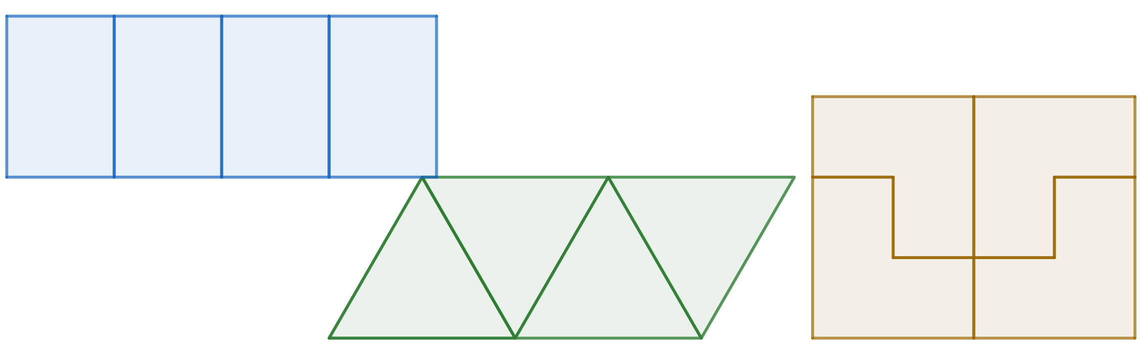
Rep-tiles are perfectly self-similar shapes. A rep-tile can be replicated by fitting together copies of itself without overlap. For example, a square is a rep-tile because four congruent squares can be fitted together to form a square. There are other ways to fit squares together to form a square too! As seen below, five congruent squares plus a sixth square of double their size (dilation by a factor of 2) can be fitted together to form a square. Even though it may not be clear from the diagram, the rightmost image shows 31 squares of different sizes fitting together to form a square.

Can you fit four congruent copies of the following three rep-tiles together to create a replica of the whole? The rectangles must be put together to form a rectangle. The equilateral triangles must be put together to form an equilateral triangle. The L-shaped hexagons must be fitted together to form an L-shaped hexagon.



Cut out the copies below and see if you can fit them together on the shapes above.



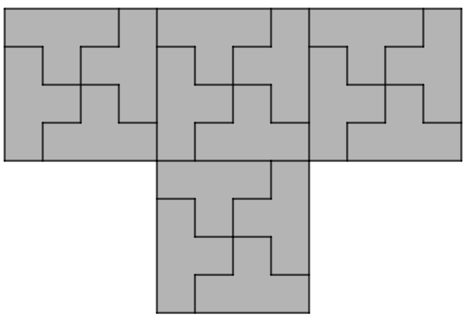
3.1.7 Activity: Similarity and Self-similarity

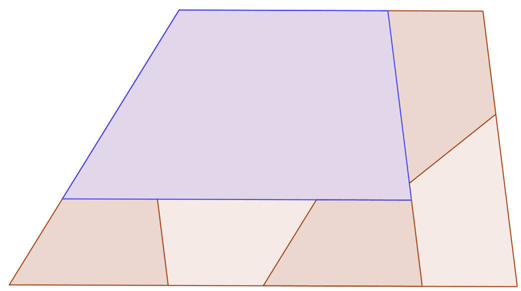
1. Consider this photo of three pencils.

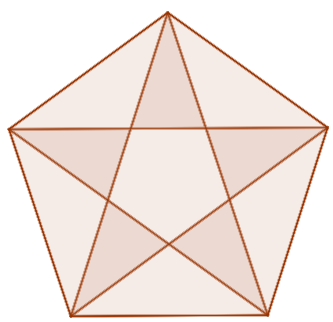
   (a) Are the pencils similar in the English sense?

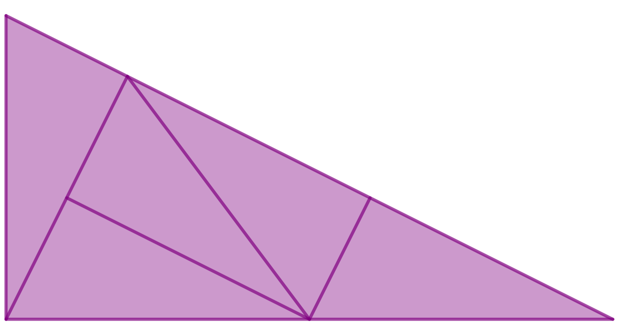
   (b) Are the pencils similar in the mathematical sense? Consider only the shapes, not the patterns printed on them.

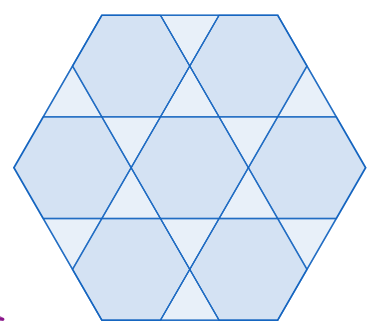
2. Which illustrations are rep-tiles (perfectly self-similar shapes)? Circle them.



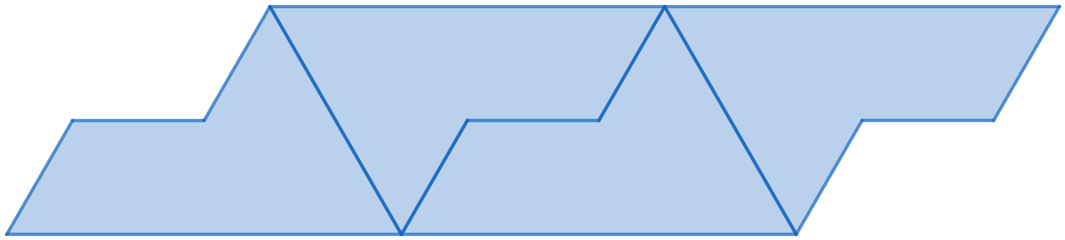


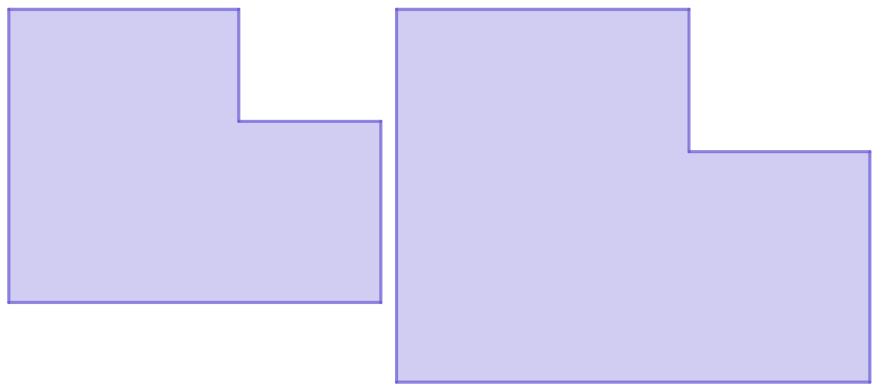


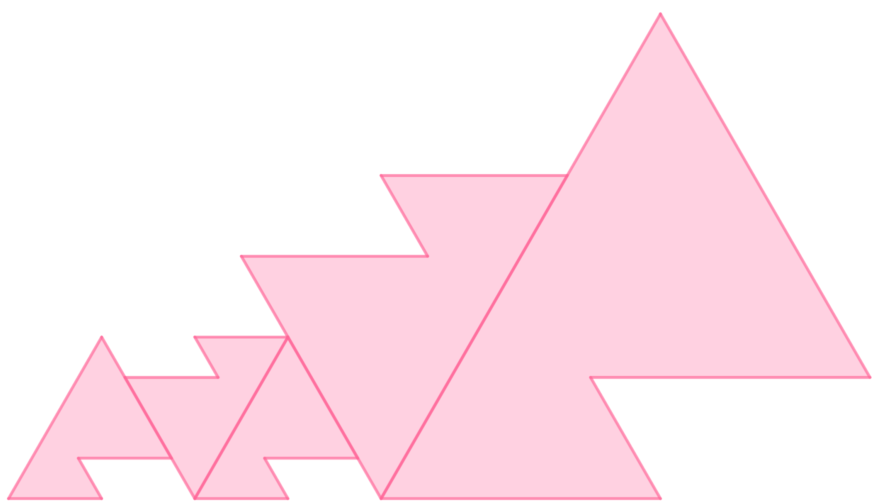


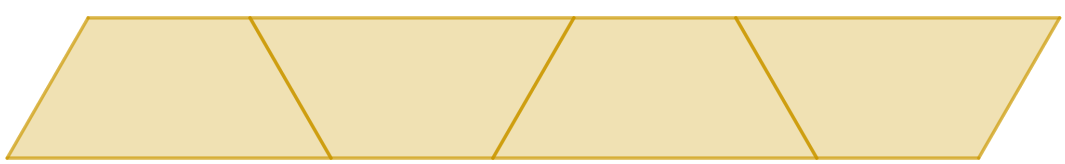


3. Pieces to assemble four different rep-tiles appear below. Cut out the pieces and see how many of the four rep-tiles you can fit together.



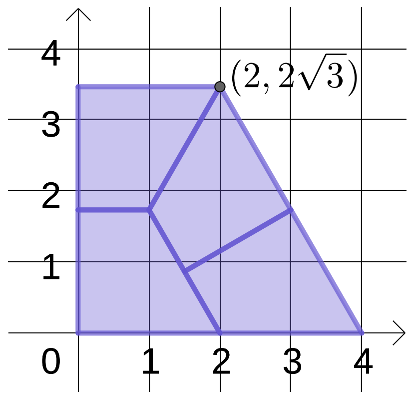




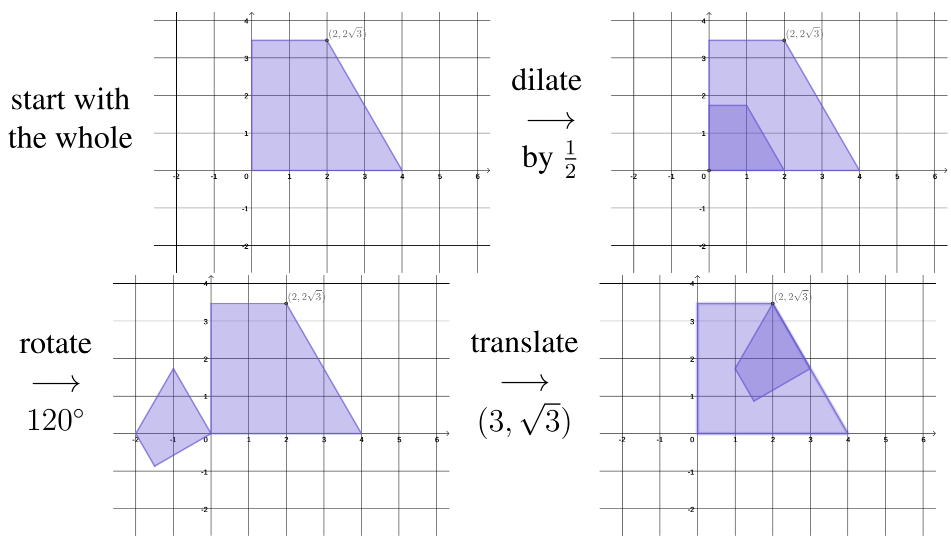


3.1.8 Activity: Defining a rep-tile

You've probably heard the phrases "two points define a line" or "a circle is defined by its center and radius". One way to understand what we mean by "define" in these two phrases is that the information describes exactly one line or one circle. For any two points, there is exactly one line that passes through both. For every center-radius pair there is exactly one circle with that center and radius. In the same way, a rep-tile is defined by the relationships between its parts and the whole. If we can precisely describe these relationships, there will be exactly one rep-tile with that description. Take the rep-tile below.

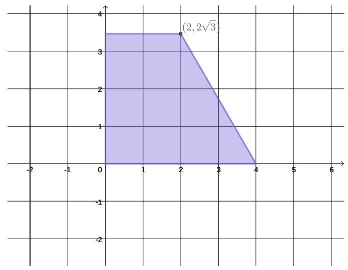


It is a trapezoid formed by fitting together four similar trapezoids. With the axes superimposed on the rep-tile, we can mathematically describe the relationships between the parts and the whole in terms of dilations, rotations, reflections, and translations. For example, if we take the whole trapezoid and follow the steps below, it turns into one of the parts of the assembly.

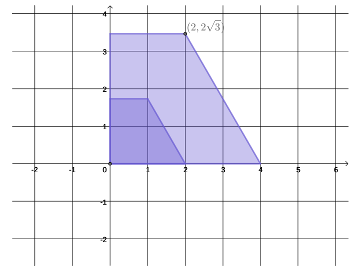


So ONE of the parts of the description that defines this rep-tile is *dilate by* $\frac{1}{2}$*; rotate* $120°$*; translate* $(3, \sqrt{3})$. A second one is much simpler:

start with the whole



dilate $\longrightarrow$ by $\frac{1}{2}$



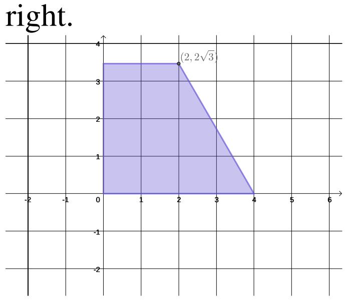
So ANOTHER part of the description that defines this rep-tile is *dilate by* $\frac{1}{2}$.

See if you can find descriptions that turn the whole trapezoid into the remaining parts. You may use

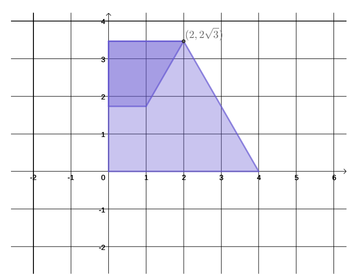
- **Reflection**. Reflection is done about the x-axis or about the y-axis only.
- **Dilation**. Dilation is done "toward" the origin, $(0, 0)$, as seen above. That means the point $(0, 0)$ does not move under dilation.
- **Rotation**. Rotation is done *counterclockwise* about the origin, $(0, 0)$, as seen above. That means the point $(0, 0)$ does not move under rotation and positive angles mean counterclockwise rotation. Use negative angles for clockwise rotation.
- **Translation**. Translation means changing location horizontally and/or vertically and is indicated by an ordered pair, (horizontal move, vertical move) as above.

in that order.

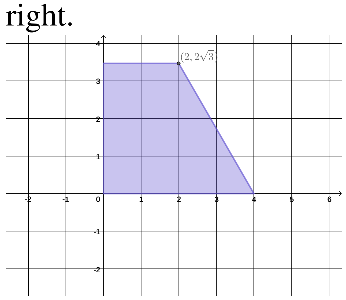
1. Write a description that turns the figure on the left (the whole) into the part on the right.



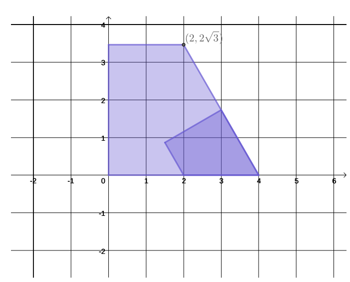
$\longrightarrow$



2. Write a description that turns the figure on the left (the whole) into the part on the right.



$\longrightarrow$



3. To check that you have the descriptions right

(a) Write the FOUR parts of the description of the trapezoid from the introduction and the previous exercises in the table below.

	Description 1	Description 2	Description 3	Description 4
Reflect about				
Dilate by				
Rotate				
Translate				

(b) Enter them into the rep-tile designer at https://lqbrin.github.io/tea-time-linear/rep-tile-designer.html.

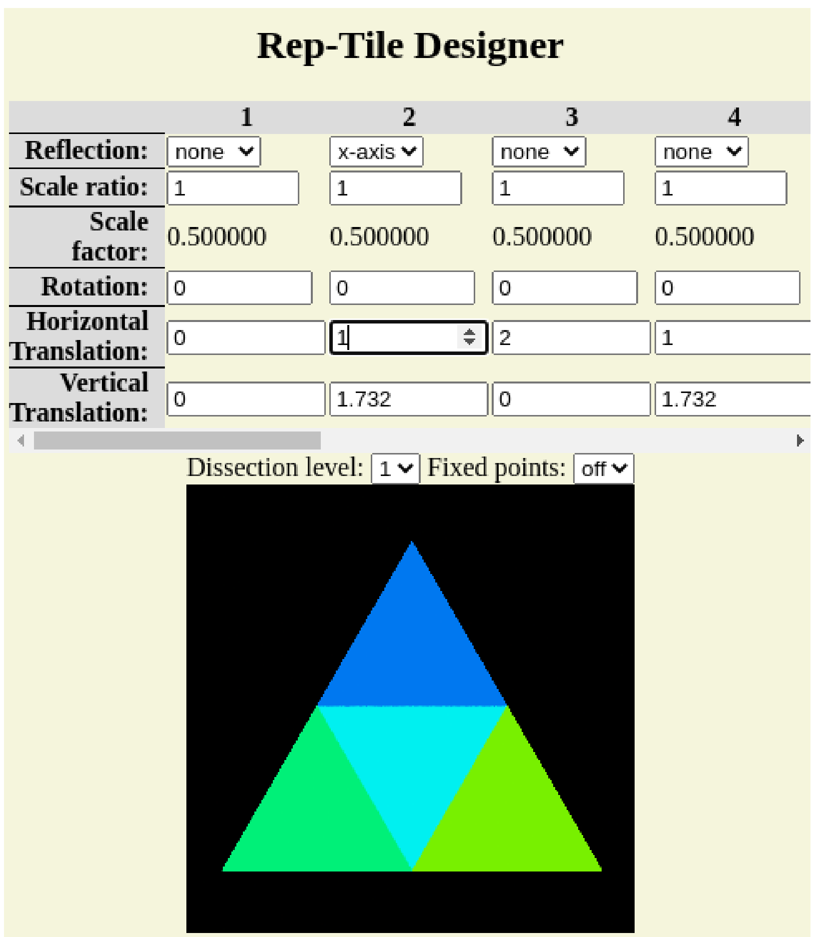
As an example, shown below is the rep-tile designer with descriptions for the parts of the equilateral triangle:

dilate by $\frac{1}{2}$

reflect about x*-axis; dilate by* $\frac{1}{2}$*; translate* $(1, \sqrt{3})$

dilate by $\frac{1}{2}$*; translate* $(2, 0)$

dilate by $\frac{1}{2}$*; translate* $(1, \sqrt{3})$

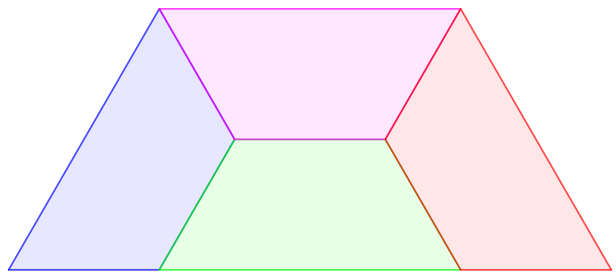


Notes:

- You only need to put in the ratio between the sizes of the parts. Rep-tile designer will caclulate the dilation (scale) factors for you.
- Select dissection level 1 to see the parts.

3.1.9 Exit Slip: Create your own

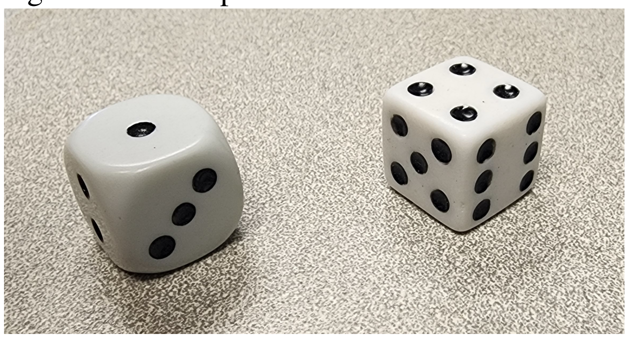
Below is an image of an isosceles trapezoid rep-tile from our previous activity. There is no coordinate system attached to this, but the "bottom" angles measure $60°$ and the "bottom" is twice as long as the "top" and "sides." Find a way to create this image in the Rep-tile designer and write down the description for each of the four parts.



3.2 Funny Dice

3.2.1 Entrance Activity: Dice

Dice are certainly familiar items from the world around us. There are games that can be played with dice alone, and there are many games that use dice to add an element of randomness – for instance in Monopoly, each player rolls the dice when it's their turn and that determines how many positions their token moves. A single die is a random number generator with possible values from 1 to 6.



Often (when numbers larger than 6 are required), we use two die. That gets us numbers from 2 to 12, but there's a funny consequence – some numbers are more likely to come up than others.

There are 6 different ways that you can roll a 7 with two die. Can you list them?

die 1	die 2

How many ways can you roll an 11?

So in a game played with rolling two dice, do you expect to see more 7's or more 11's throughout the game?

3.2.2 Activity: Why Only Five?

If we want "dice-like things" (random number generators that give equal weight to the options) we'll need solid geometric objects that are nice and regular, but that have different numbers of sides. Here are a few examples:

There are other options, but a good place to start is shapes that have all the same regular polygon as their sides. These are called Platonic solids, named after the greek philosopher Plato. The smallest platonic solid is known as a tetrahedron – gamers call it D4. The pyramids that can be found in Egypt and a few other places around the world have triangular sides, but square bases. Can you imagine what it would look like if a pyramid's base was also a triangle?

(a) Notice that there are 3 edges on each triangle (duh!) and since there are 4 triangles that makes a total of $3 \cdot 4 = 12$ edges. But there are only 6 edges on the D4 we just made. Can you explain why?

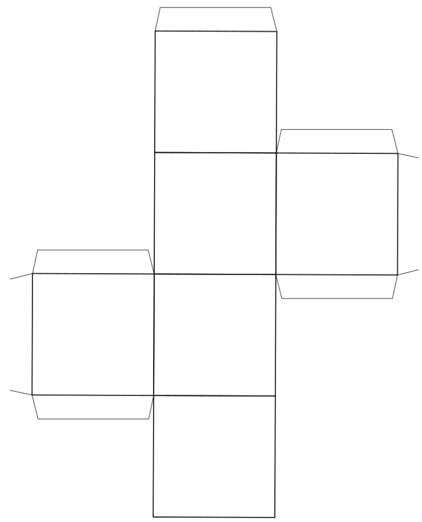
(b) Let's start by thinking a bit about why cubes are the perfect 3-d shape to use for dice. Cubes are made up of squares, but you could make an imperfect cube out of rectangles – think of a brick. Explain why tossing a brick would be a poor choice for generating random numbers.

It seems we should build our "fancy" dice using faces that are **regular** like a square. The regular polygons that can be used are:

(a) equilateral triangles
(b) squares
(c) regular pentagons

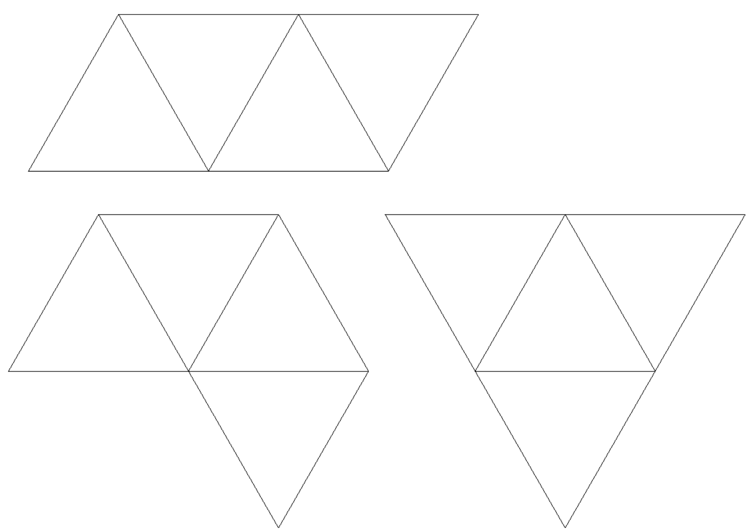
Of course, triangles are the smallest possible polygons, but what about shapes with more sides? Can you come up with a reason why we couldn't make a 3-d object with regular hexagons, heptagons, octagons? (If you have physical objects like these, try to build a 3-d object, what happens?)

A tool we can use to think carefully about polyhedra (that's the generic term for a solid with many sides) is the so-called *net* of the polyhedron. There are generally many different nets for a given polyhedron. Here's one for a cube:



Officially, a net wouldn't include the little tabs that are meant for gluing the model together. But, as you can probably tell from the example, the net of a solid is a flat template that can be folded up to cover the exterior of the solid.

Which of the following is a net for the tetrahedron? (Cut them out and see!)



A quicker way to rule out a potential net is to count how many polygons will meet at each corner. In a tetrahedron, exactly 3 triangles meet at each corner. Now do you see which one of the above is impossible?

Let's recap for a minute. We're trying to figure out how to make "fancy" dice that will have a different number of sides than a cube does. We know that the faces of our dice will have 3, 4, or 5 sides. Finally, it seems that the number of faces that meet at a corner is important. Some of you may know about D8, D12 and D20, but the only solids we've discovered together so far are D4 and D6 – the tetrahedron and the cube.

Let's collect what we've found so far in a table.

Also, let's (please!) agree to use the following abbreviations:

$F =$ the number of faces on the object.

$C =$ the number of corners on the object.

$F/C =$ the number of faces that meet at a corner.

$C/F =$ the number of corners on each face.

$E =$ the number of edges.

name	F	C	F/C	C/F	E
tetrahedron					
cube					

Near the beginning of this activity we asked, "Can anyone come up with a reason why we couldn't make a 3-d object with hexagons, heptagons, octagons, etc. as the sides?" There are two facts that explain the situation.

Fact 1: You have to have 3 or more faces meeting at a corner.
(What would it look like if only two faces (of whatever type (sorry about the nested parentheses)) met at a corner?)

Fact 2: The sum of the angles on the faces that meet at a corner must be less than $360°$

Fact 2 might best be illustrated by an example. What would it look like, if instead of having 3 squares meet at a corner, we tried to get 4 squares to meet at a corner. (Hint: look down.)

The angle on the corner of a hexagon is $120°$ and if 3 of them met at a corner, we'd get a tiling of the plane – not a polyhedron of any sort. Here's a hard question: what is the angle found on the corner of a regular pentagon, and how many regular pentagons could meet at the corners of a polyhedron?

Alright then! If a polyhedron has squares as its faces there must be _______ of them meeting at a corner, and that gets us a cube! If a polyhedron has regular pentagons as its faces there must also be _______ of them meeting at a corner, and that beast is called the dodecahedron. We'll come back to that!

What are the possible values of F/C if the faces are triangles?

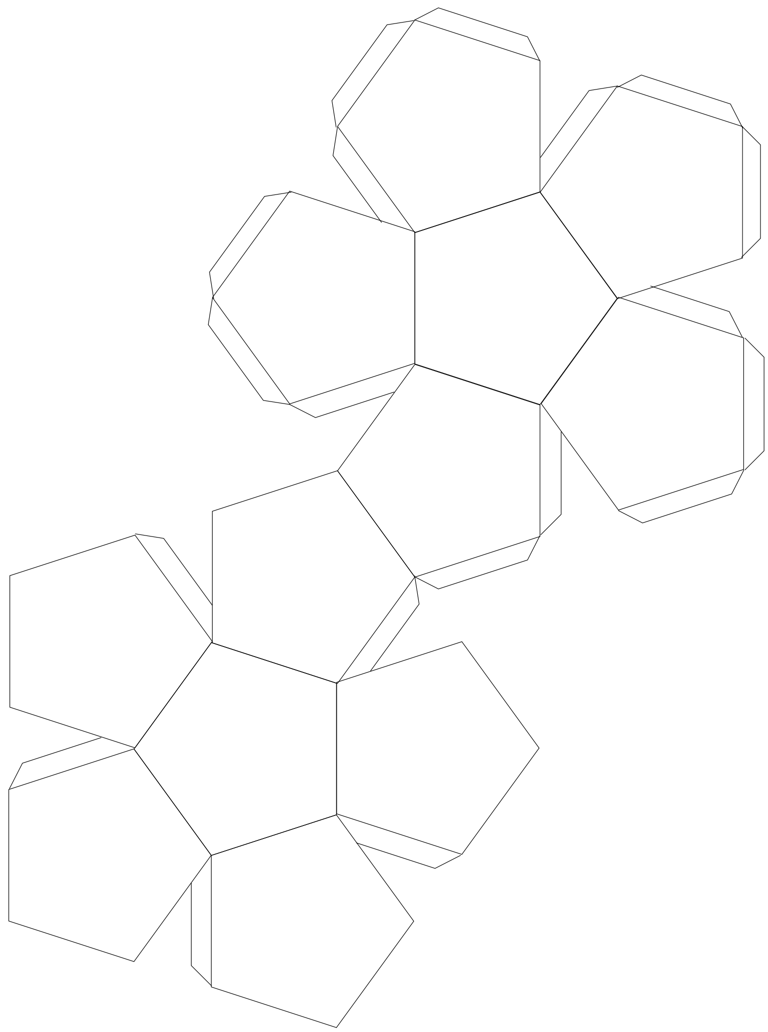
Here's a bit more of the table we started previously. The new object in the row that's been added is called the *octahedron* and the entries that are pre-filled tell us that it's made of triangles with 4 of them meeting at a corner. A little thinking about etymology will probably let you guess how many faces it has. Rather than building a model, we'll just try to get you to visualize this solid in your mind's eye. Imagine taking two (Egyptian-style, square based) pyramids and gluing them together along their square bases – the resulting thing would only have triangles (8 of them!) visible on the outside and there would be 4 meeting at each corner.

With that visualization you should now be able to fill out the remaining entries in the octahedron row.

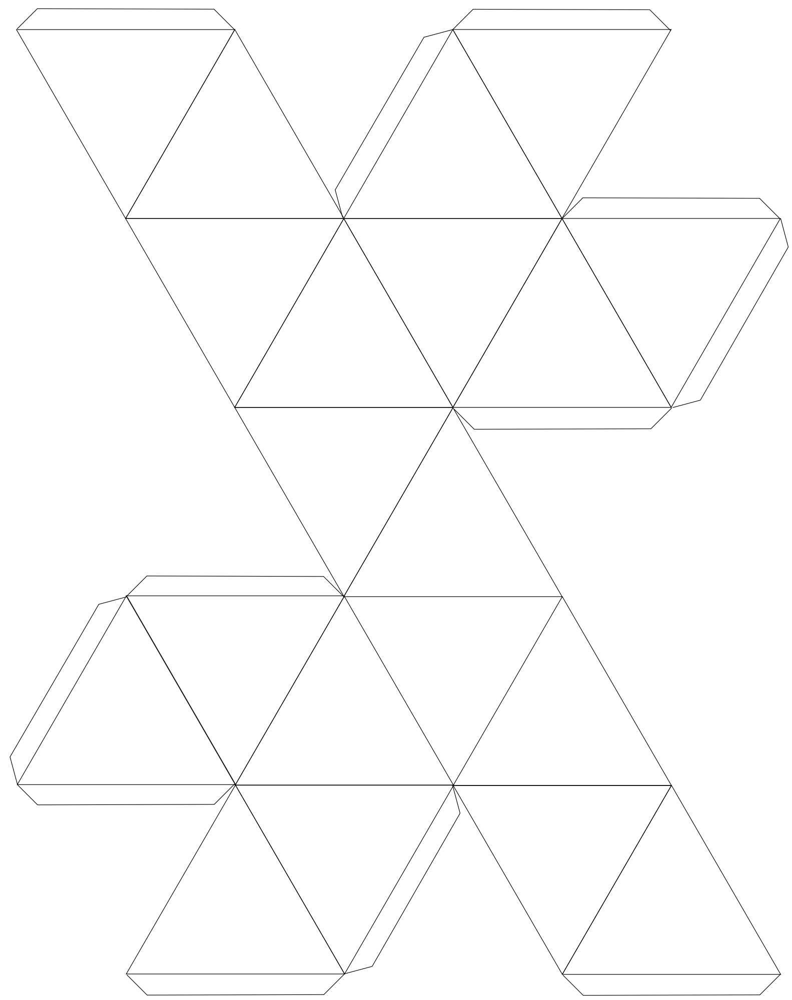
name	F	C	F/C	C/F	E
tetrahedron	4	4	3	3	6
cube	6	8	3	4	12
octahedron			4	3	

There are two remaining objects in this list of the 5 Platonic solids we're trying to build. The *dodecahedron* (aka D12) and the *icosahedron* (aka D20). One of them is built of 5-sided polygons meeting 3 at a corner. The other is made of 3-sided polygons meeting 5 at a corner. Isn't that weird switching-around of 3's and 5's interesting?

Here's a net for the dodecahedron:



This is a net for the icosahedron:



You guessed it! It's time to completely finish the table!

name	F	C	F/C	C/F	E
tetrahedron	4	4	3	3	6
cube	6	8	3	4	12
octahedron	8	6	4	3	12
dodecahedron					
icosahedron					

Look carefully at your completed table. Search for patterns. Remember that weird switching around of 3's and 5's? Mathematicians say that "the tetrahedron is self-dual and the other platonic solids can be divided into pairs that are dual to one another." Any idea what they're talking about?

3.2.3 Exit Slip

(a) What would you guess the Greek prefix *icosa* means?

(b) The word “dozen” literally means $2 + 10$. How is this silly fact relevant to the naming of Platonic solids?

(c) Let’s try some multi-counting. Every one of the pentagons on a dodecahedron has 5 corners, and since there are 12 pentagonal faces that makes for $12 \cdot 5 = 60$ corners. By what factor did we over-count the corners? _______

How many corners are actually on a dodecahedron? _______

(d) Every face of a cube has 4 corners . Since there are 6 faces on a cube, we get $6 \cdot 4 = 24$ corners. By what factor did we over-count the corners? _______

How many corners are actually on a cube? _______

3.2.4 Duals

Look at the final table in the previous activity, and notice that there are some weird coincidences in the numbers. For example, the dodecahedron and the icosahedron have the number of faces and the number of corners interchanged. These coincidences lead to the idea of *duality*.

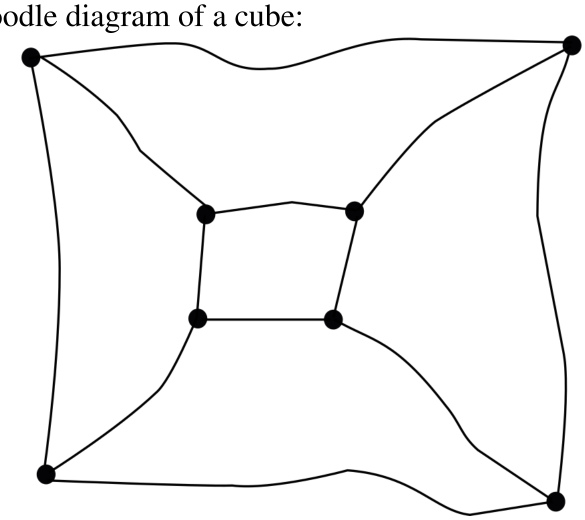
Which pairs of Platonic solids are dual to one another?

Why is the tetrahedron called self-dual?

You can make drawings of the platonic solids that lie flat in the plane, using some ideas from the mathematical area known as Topology. (Topology means the study of shape. Topologists study the things that remain the same even when an object is deformed somewhat.)

Imagine making a Platonic solid out of spaghetti noodles (maybe with meatballs holding the corners together). Then (very carefully) cook your solid until the noodles get noodly. You should now be able to lay your "solid" out flat in the plane in such a way that no noodles cross each other.

Here's a wet noodle diagram of a cube:

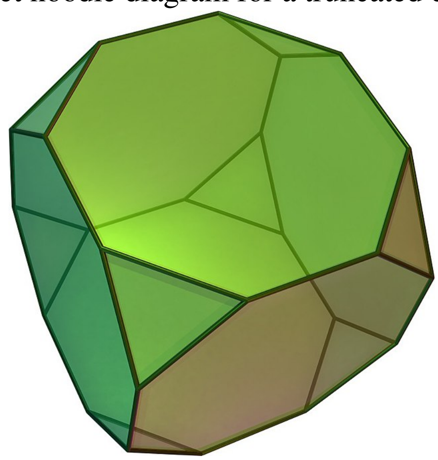


(a) Make wet noodle diagrams of all 5 platonic solids.

(b) The corners of your Platonic solid are the points (meatballs) in the wet noodle diagram. The edges of the solid are the noodles. The faces of the Platonic solid have become regions in the plane surrounded/bordered by noodles. Count the regions in each of your 5 noodle diagrams.

(c) What happened to the missing faces?

(d) There is a process known as *truncation* that essentially means "cut off the corners." Draw a wet noodle diagram for a truncated cube:



(e) Now try making a wet noodle diagram for a truncated tetrahedron.

(f) Try a wet noodle diagram of a 5-sided pyramid.

(g) Let's do some counting! How many regions, edges and vertices (the fancy way to say 'corners') are there in each of your diagrams?

Count the region outside your diagrams too when figuring out R.

name	R	E	V
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			
truncated cube			
truncated tetrahedron			
pentagonal pyramid			

(h) Make a fairly complicated, but random wet noodle diagram – it doesn't need to actually come from something solid. Find R, E, and V for your diagram. What does $R - E + V$ equal?

(i) For each of your Platonic solid noodle diagrams, draw a point somewhere in each region (don't forget to put a point on the outside!) Connect points with a noodle if the corresponding regions are separated by a noodle. It would be a good idea to make the original and the new diagram you're making be in different colors. (BTW, this process is called *dualizing*).

3.2.5 Exit Slip

(a) Draw the WND and the dualized WND for a four-sided pyramid.

(b) Fill in the blanks:

The dualized wet noodle diagram for the tetrahedron is a wet noodle diagram for a ________________________

The dualized wet noodle diagram for the cube is a wet noodle diagram for a(n) ________________________.

The dualized wet noodle diagram for the dodecahedron is a wet noodle diagram for a(n) ________________________

(c) Verify the identity $R - E + V = 2$ for a soccer ball.