Mathematics in the World Around Us
Student Workbook

Leon Brin, Braxton Carrigan, Joseph Fields

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Contents

Acknowledgements

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Chapter 0

Introductory Challenges

0.0 Mathematical Outcome

The mathematical process is deeper than simply arriving at the correct answer. The goal of this chapter is to introduce the students and the instructor to the format of the text. The authors' hope is that the activities provided throughout the text will foster an interactive and cooperative classroom environment. This chapter is a set of logic problems which require little or no prerequisite knowledge and establish a foundation of thinking which will be used in activities throughout the text.

You will find that the students will begin developing the basic problem solving skills throughout this chapters activities. They will encounter such questioning as; "Is this the optimal answer?" "Can we show there is not a better way?" "Can we explain why our technique will solve the problem?" etc. In answering these questions students are engaged in the mathematical method of developing reasoning through concrete examples.

To further this extension from the concrete to the abstract we begin asking students to change the variable of the problem to see if the previous solution technique continues to apply. In some cases a technique can be developed which will work for all possible arrangements of which it is natural to ask "how do we know the method will work for all possibilities?" However, the students are also exposed to solution methods which are not valid when a variable is changed. The contrast in the problems will hopefully develop the necessary curiosity which initiates the investigation process.

As a framework for the entire course, Polya's four step method for problem solving should be introduced, emphasized, and reemphasized throughout. Reference to these steps will be made from time to time within the instructor's notes, but they should be kept in the forefront of students' minds whether the prompts are there or not.

Polya's Problem Solving Method

1. Understand the problem (This may include reading the question several times, asking "what if" questions, tinkering with parts of the question, and so on.)

2. Devise a plan (Though this is often a completely separate step from understanding the problem, the tinkering used to understand may lead to an idea for a plan of action.)

3. Carry out the plan (When the plan can be completed, great! However, one may find they are unable to complete their plan of action due to its difficulty or faultiness. It is at this point, the method becomes iterative. The problem solver will need to revisit understand the problem or devising a plan to refine their approach.)

4. Review (Even when the plan can be carried out successfully, a review of the result is needed. Do the results answer the question asked? What evidence is there that the results are correct? Is that answer actually correct? If not, here is another place where the method becomes iterative. The problem solver will have to revisit one of the first three steps depending on the flaw identified.)

It is important to emphasize that a natural part of problem solving is temporary failure. Noticing that there is a flaw in the solution to a problem (during the review) is a critical step in solving difficult problems. It should not be expected that anyone will devise correct, complete answers on their first try, and that this should not be considered failure. It is a temporary setback that should launch the problem solver down a new path. One only fails when one gives up trying new ideas.

0.1 Activity: Making $20

How many different ways can you have $20 in US bills?

Sometimes a simple question can have many layers, so we first must unpack and organize our thoughts before answering! We will use Polya's 4-step problem solving method to guide the process.

1. **Understand the problem.**

   (a) Can you write down three different ways to make $20 in US bills? If yes, write them down—this is good progress toward understanding the problem! If no, a discussion with your peers is in order (What are "US bills"? What is meant by "make $20"? etc.)

   (b) What makes your three ways different?

   (c) Are there other ways to make $20? How many? (It's ok to make a wild guess at this point!)

   (d) What denominations of bills are possible to use in a collection that sums to $20?

   (e) Do you think you are being asked for an exact number for the total, or is an estimate good enough?

2. **Devise a plan.**

   (a) Provide an example of a collection of bills that sums to $20 that contains exactly one $1 bill?

   (b) Is it possible to have a collection of bills that sums to $20 that contains exactly one $1 bill and no $5 bills? If so provided an example or explain why none exists?

(c) **One of the best ways to start problem solving is to solve a simpler similar problem.** One way to do that is to work on a lower total. List all the ways that U.S. bills can make a total of 5 dollars. HINT: There are more than two ways.

(d) Let's continue our exploration of simpler problems. Write down all the collections of bills that make $10.

(e) Did you use the result from 2(c) in solving 2(d)? If no, what is one way you could have? HINT: See if you can find a connection between your answer to part 2(c) and your answer to part 2(d).

(f) Another way to solve a similar simpler problem is to make the total \$20 but restrict the types of bills that may be used. List all the ways to make \$20 where the highest denomination is a \$1 bill (meaning you must use at least one \$1 bill and no higher denominations).

(g) List all the ways to make \$20 where the highest denomination is a \$2 bill (meaning you must use at least one \$2 bill and no higher denominations).

(h) How do you know there are no repeats in your answers to questions 2(f) and 2(g)?

At this point we have explored two ways to create simpler similar problems (restrict the total dollar amount and restrict the types of bills we can use), and we've seen that one simpler solution may be used to solve another (using the \$5 totals to help make the list of \$10 totals). Using one or both of these ideas, devise a plan for finding all collections of US bills that make \$20. Summarize your plan here. DO NOT carry out the plan here. Just describe a plan that you think will work.

3. **Carry out the plan.** Try to do what you just described to find out how many different ways there are to make \$20 using US bills. If you are struggling to make your plan work or you have determined that it just won't work, go back and revise your plan!

How many ways did you come up with?

4. **Review the solution.**

   (a) How do you know you have not counted any collection more than once? In other words, how do you know each of your collections is different?

   (b) How do you know you have all collections that make $20? How do you know you have not missed any?

If you can't answer these questions or you have doubts that you have the correct answer you may need to revise your plan or how you carried it out.

0.1.1 Exit Slip

1. How many different ways can you have $15 in US bills? Provide a tree diagram, an organized list, or written justification for your answer.

For the remaining two questions, your solution must include (i) the tree diagram, (ii) the answer, and (iii) a few sentences describing why you believe your tree diagram gives the correct answer (shows all the answers and no repeats).

2. Solve **ONE** of the following problems using a tree diagram.

   (a) At Panera Bread, you are presented with the following lunch choices: a starter of soup or salad; 3 different sandwiches; and a side of bread, chips, or an apple. How many different lunches consisting of one starter, one sandwich, and one side can be made?

   (b) At Domino's Pizza, the following unusual vegetable toppings are available: jalapeno peppers, roasted red peppers, spinach, and basil. How many different unusual veggie pizzas can you make using these toppings?

3. Solve **ONE** of the following problems using a tree diagram.

   (a) The division championship in major league baseball goes to the winner of a best-of-5 series. The first team to win 3 games wins the division, so the series will never go more than 5 games. For example, if the teams are Boston and Tampa Bay (as they were in the 2021 American League Division Series), one way it could have unfolded was B-T-B-T-T, meaning Boston won the first, Tampa Bay won second, Boston won the third, and then Tampa Bay won the fourth and fifth games. It also could have unfolded T-B-B-B (and this is what actually happened in 2021). How many different ways could the division series have unfolded (including the two already mentioned)?

   (b) The curator at a natural history museum is designing a penguin display. He would like to put penguins of 4 different species in the display—Emperor, Northern Rockhopper, Macaroni, and African—arranged one next to the other in a row. However, he wants to be sure the Emperor Penguin is not next to the Macaroni Penguin. How many ways can he arrange the penguins in the display?

0.2 Activity: Crossing the Bridge

Alfred, **B**ruce, **C**athy, and **D**oug were coming back to their village in the middle of the night. The tiny bridge to the town is too small for more than 2 people to cross it at the same time, and it is only safe to cross if someone is holding a flashlight. Since they only brought one flashlight, someone will have to return with the flashlight before two more people can cross. Doug can walk across the bridge in 10 minutes, Cathy takes 5 minutes to cross, Bruce can skip across in 2 minutes, and Alfred can cross the bridge in 1 minute. Unfortunately, when you walk across the bridge with someone else, you can only cross the bridge at the speed of the slower person. What is the least amount of time needed to get everyone across the bridge?

1. **Understand the Problem:** Act out the situation. Members of your group will play the roles of Alfred, Bruce, Cathy, and Doug. If you are a group of 3, you will only be able to play Alfred, Bruce, and Cathy. Use a prop for the flashlight and define a physical space in the classroom that will stand in as the bridge. Play act until you have successfully gotten everyone across the bridge. Are the rules clear?

2. **Devise a plan:** Solve a simpler problem first. Imagine Doug is ill and therefore not with the group. What is the quickest way for Alfred, Bruce, and Cathy to get across the bridge?

3. How do you know this is the quickest way? Construct a tree diagram of all the ways they can cross the bridge without any obvious waste of time (repeating a position—going from one position to another and then back to the position you were before—is an obvious waste of time).

4. For each way of crossing the bridge, calculate the total time. What is the least? How can it be done? Are there more ways than one?

5. **Carry out the plan.** Now return to the original problem and use the same process as you used for the simpler problem. What is the least amount of time needed to get everyone across the bridge? How can it be done?

6. **Review.** Why is there no quicker way to get everyone across? Thoroughly explain your answer.

7. It turns out that Bruce stubbed his toe on the way to the bridge and will actually take 4 minutes to cross the bridge. Now how long would it take for everyone to get across the bridge?

8. Why is there no quicker way to get everyone across in Question 7? Thoroughly explain your answer.

0.3 Project Choices

1. **Project Question:** How many different ways can you make $50 in US bills? Provide a tree diagram to support your claim and write a statement about how you used results from the $20 class activity to find your answer.

2. You need to cross a river in a small rowboat and take your possessions with you. The boat is only big enough for you and one possession at a time. Unfortunately, some of your belongings can't be left alone on the river bank while you transport the other. Your belongings are: a wolf, a goat, and a big bag of cabbage. Problems: the wolf will eat the goat and the goat will eat the cabbages if they are able to do so. How do you get everything across?

0.4 Additional Exercises

1. You're at a meeting where there is a make-your-own-sandwich platter. There is only one kind of bread, but there is a choice of what to put on it. There are three different lunch meats, two different cheeses, and extras—mustard, mayonnaise, and lettuce—to choose from. How many sandwiches can you make using one lunch meat, one cheese and any combination of extras?

2. How many different ways can you have \$30 in US bills? Make sure to utilize the techniques and strategies you have learned above and pay close attention to work from above that you may be able to re-use.

Chapter 1

Logic

1.0 Mathematical Outcome

One fundamental concept of mathematics is logic. Sound mathematics comes from the ability to state definitively whether something is true or false. In order to explore mathematics from a theoretical standpoint one must first understand the basics of logic. This chapter uses students previous and intuitive knowledge to formally develop the concepts and terminology which will be used in developing conjectures and theorems throughout the text.

Definition 1.1. A declarative statement (either verbally or written) which can be judged as either true or false (but never both) is known as a **statement**. Determining a statement to be true or false is called the statement's truth value.

Generally there are two types of statements. The first is called a **simple statement** which only contains one idea, which we will denote using letters throughout this chapter. The second is the idea of combining a number of simple statements with conjunctions which form **compound statements**. You will see compound statements in written and symbolic form which use connections such as "and" and "or" along with others. The concepts of logic follow from being able to consider the truth value of the simple statements and conclude the truth value of a compound statement.

The two most common and basic conjunctions used to form compound statements are "and" and "or." Both of these conjunctions are explored in Activity 1.1 from an intuitive approach.

The "and" conjunction is usually the most intuitive and students easily realize the only time when the statement constructed by adjoining two statements (either simple or compound) with an "and" is true is if both of the statements being adjoined are true, otherwise the compound statement is false.

Conversely, the "or" conjunction is often used in common vernacular to mean only one of the adjoined statements is true. For example when you ask a child "Would you like chocolate **or** vanilla ice cream?" you are not intending for them to choose both; you are simply giving them an option of choosing either. In logic however the conjunction "or" is evaluated as determining if at least one of the adjoining statements is true. Therefore we will determine that the compound statement adjoined by an “or” is true whenever one or both of the adjoined statements is true, hence it will only be false if both of the adjoined statements are false.

Definition 1.2. A tabular format for displaying all possible outcome combinations of a compound statement is known as a **truth table**.

Truth tables are useful in comparing and contrasting compound statements which are made up of the same simple statements. We begin expelling the basics of truth tables in Activity 1.0.1 and continue to explore the more advanced concepts throughout the chapter.

We often wish a statement to be positive if it is false. We see this in our everyday lives when getting tested for diseases, it is a good thing if the test proves negative or false. Therefore we may want to consider the truth value of a particular statement to be the opposite. This concept is known as **negation**. If we place a “not” in front of any statement we have changed its truth value at each instance. We explore the use of negation in Activity 1.1.4.

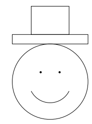
One of the more complex conjunctions is the “if...then...” statement. While there are other conjunctions such as “if and only if,” “else,” “while,” and others, we will finish our exploration in this text here. We have chosen to explore the “if...then...” statement since it is the basics of formulating conjectures and theorems in mathematics.

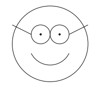
The “if ... then...” statement is one which determines if one statement implies the other. We can only definitively conclude that a statement does not imply another if the first statement is true yet the second is false, because we can see that the first statements truth does not imply that the others will also be true. However when the first statement is false there is no conclusion about its implication on the second statement, therefore we must conclude that the conjunction is true.

An example of this is the combination of the statements “A person is pregnant” and “a person is a woman”. We will look at the compound statement “If a person is pregnant, then they are a woman.” In this case knowing that a person is pregnant allows us to conclude definitively that they are also a woman, however knowing the the person being pregnant is false we can make no conclusion about the gender of that person.

1.0.1 Entrance Activity: Top Hat and Glasses

Consider the following statements P and Q:

P: I am wearing a top hat. 

Q: I am wearing glasses. 

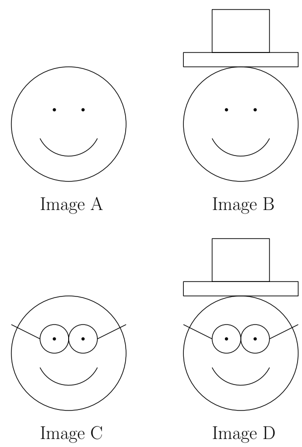


Figure 1.1: Top Hats and Glasses

1. For which of the images in Figure 1.1 is P true?

2. For which of the images in Figure 1.1 is Q true?

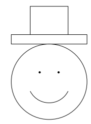
3. For which of the images in Figure 1.1 is "P and Q" true?

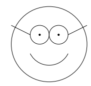
4. For which of the images in Figure 1.1 is "P or Q" true?

1.1 Making a Statement

1.1.1 Activity: Exploring AND and OR Statements

Consider the following statements P and Q:

P: I am wearing a top hat. 

Q: I am wearing glasses. 

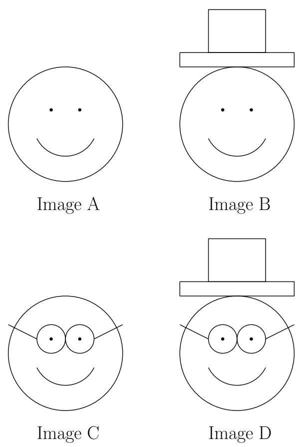


Figure 1.2: Top Hats and Glasses

1. What does it mean when the statement P and the statement Q are both true? Explain what it means if both P and Q are false?

2. If I am wearing a top hat and no glasses, would the statement "P and Q" be true or false? Explain your answer.

3. What would it mean if we knew that the statement "P or Q" is true? Write out all the possible answers, describing the images in Figure 1.2 that make "P or Q" true.

4. What would it mean if we knew that the statement "P or Q" is false? Write out all the possible answers.

5. If I am wearing a top hat and no glasses, what would you say about the statement "P or Q"; is it true or false?

6. List all the different ways I can look depending on whether statement P or statement Q is true or false?

1.1.2 Activity: Top Hats, Overalls, and Rain boots

Let $P \wedge Q$ be the single statement "P and Q" and let $P \vee Q$ be the single statement "P or Q." Consider the statements below:

P: I am wearing a top hat.
Q: I am wearing overalls.
R: I am wearing rain boots.

7. What do I look like if $P \wedge Q$ is true? Am I wearing the same thing if $Q \wedge P$ is true?

8. What do I look like if $P \wedge Q$ is false? Is there more than one answer? If so, list out all the possible ways I could be dressed.

9. What am I wearing if $P \vee Q$ is true? Is there any difference in my outfit if $Q \vee P$ is true? List all possible outfits that make these statements true.

10. Fill in the table below. Each cell in the table should contain either "True" or "False." First write in all the possible truth values for the pair of statements P and Q in the first two columns (see Question 6 for some help). Then in the third and fourth columns describe how $P \wedge Q$ and $P \vee Q$ will be affected by your choice of P and Q in the corresponding row. This table is called a truth table.

P	Q	$P \wedge Q$	$P \vee Q$

11. What do I look like if $P \wedge Q \wedge R$ is true?

12. Does my outfit change if we reorder P, Q, and R in Question 11? Explain your answer.

13. Describe my clothing if $(P \vee Q) \wedge R$ is true. List out all the possible clothing options.

14. Describe my clothing if $P \vee (Q \wedge R)$ is true. List out all the possible clothing options.

15. Fill in the table below. Each cell in the table should contain either "True" or "False." As in Question 10 first fill in the first three columns so that no two rows have exactly the same three truth values for P, Q and R. The your choice of P, Q, and R in each row will determine the truth value in the next three columns.

P	Q	R	$(P \vee Q) \wedge R$	$P \vee (Q \wedge R)$

16. Using the truth table in Question 15 determine if the statement $P \vee (Q \wedge R)$ the same as the statement $(P \vee Q) \wedge R$. Explain yourself thoroughly.

17. Suppose I am wearing a top hat and rain boots, and I am not wearing overalls. Is the statement $(P \wedge Q) \wedge R$ true or false? Is the statement $(P \wedge R) \vee Q$ true or false?

1.1.3 Exit Slip

Fill in the table below. Each cell in the table should contain either “True” or “False.” As in Question 10, begin by completing the first three columns.

P	Q	R	$(P \wedge Q) \wedge R$	$(P \vee Q) \vee R$	$P \vee (Q \wedge R)$

1.1.4 Activity: Negation

Let $\neg P$ represent the statement "P is false". ($\neg P$ is called the negation of P.) Again we will consider the statements below.

P: I am wearing a top hat.
Q: I am wearing overalls.
R: I am wearing rain boots.

27. If $\neg P$ is true, is anything on my head?

28. If $\neg P$ is false, is anything on my head?

29. Fill in the truth table below. Each cell in the table should contain either "True" or "False." As in Question 10 begin by completing the first column.

P	$\neg P$

30. Suppose you know the statement $P \wedge R$ is false. Describe what I'm wearing. Is there a statement that says the same thing when it is true?

31. Fill in the truth table below so that each cell contains either "True" or "False." As in Question 10, begin by completing the first two columns so that all ways truth values can be assigned to P and Q are represented in the four rows.

P	Q	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$

32. Using the truth table in Question 31 determine if the statement $\neg(P \wedge R)$ is the same as the statement $\neg P \vee \neg R$. Explain yourself thoroughly.

33. Fill in the table below so that each cell contains either "True" and "False." As in Question 10, begin by completing the first three columns.

P	Q	R	$(P \wedge Q) \wedge \neg R$	$(\neg P \vee \neg Q) \vee R$	$(P \vee \neg Q) \wedge \neg R$

1.1.5 Activity: The Implications of Your Actions

The last activity introduced the basic form of an argument used by mathematicians. They assume that something is true and then see if that means whether you know other things are true. This step is called an **implication**. We say that "P implies Q", and write $P \Rightarrow Q$, if whenever P is true Q is also true. So the only way that the statement $P \Rightarrow Q$ can be false is when P is true and Q is false.

1. Consider the statement "If it's Monday, my day will suck." This is an implication where $P =$ "today is Monday" and $Q =$ "I will have a sucky day today." Discuss with your group members: What are the 4 possible options for T/F for P and Q. In which of those scenarios is the person who claims "If it's Monday, my day will suck" saying something true? In which scenario are they wrong?

2. Fill in the following table so that each cell contains either "True" or "False." First write in all the possible truth values for the pair of statements P and Q in the first two columns. Then in the third column indicate how $P \Rightarrow Q$ will be affected by your choice of P and Q in the corresponding row.

P	Q	$P \Rightarrow Q$

3. Notice that $P \Rightarrow Q$ is always true whenever P is false. Why do you think that is the case?

4. Suppose the statement “If it is Tuesday then it must be raining all day” is true. What do you know if:

   (a) it is not raining outside.

   (b) it is Tuesday today.

   (c) it is Wednesday today.

Consider the statements below for Questions 4–5.

P: It is raining today.
Q: I bring an umbrella with me.
R: I play basketball during recess.
S: I took a test in English class.

4. Write the statement
   "If it is raining today then I will bring an umbrella with me."
   using the symbolic statements P, Q, R, and S. Explain why you know your answer is correct.

5. Write the statement
   "If it is raining today then I will play basketball and take a test in English class."
   using the symbolic statements P, Q, R, and S. Explain why you know your answer is correct.

6. Fill in Table 1.1 so that each cell contains either "True" or "False." First write in all the possible truth values for the pair of statements P and Q in the first two columns. Then in the remaining columns indicate how the other statements will be affected by your choice of P and Q in the corresponding row.

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P \Rightarrow \neg Q$	$\neg Q \Rightarrow \neg P$

Table 1.1: Table of logical statements using the $\Rightarrow$ symbol

7. If $P \Rightarrow Q$ is true, does that mean that $Q \Rightarrow P$ is true. Give a justification for your answer.

8. Which symbolic statements in the Table 1.1 are equivalent?

1.2 Logic in Games

1.2.1 Entrance Activity: Statements in Minesweeper

The game of Minesweeper is a popular game that requires players to use logic to solve the puzzle. The game begins with a board of squares, some of which contain hidden mines. Game play can be summarized in one sentence: **the number on a block shows the number of mines adjacent to it and you have to flag all the mines**. In more detail:

- Start by clicking any random place since it is your first move. You do not have any clues to the locations of the mines yet!
- When you identify a square hiding a mine, flag it by right-clicking. This will place a flag on that spot.
- When you identify a square that is not hiding a mine, clear it by clicking it. This will reveal the number of mines in the adjacent squares, or more if there are no adjacent mines.
- If you accidentally clear a mine you lose.
- If you flag all the mines and clear all other locations, you win!

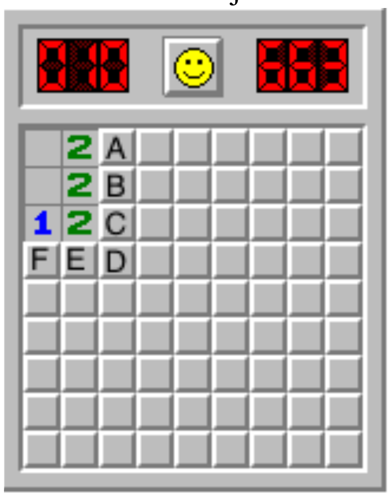
Before reading any further go to https://freeminesweeper.org/ and play a few games! If it is your first time playing, you might want to select Beginner from the Game menu.

Here is a sample of the type of question you will be asked and how you are expected to answer. Shown is a game in progress where the player has indicated the locations of 4 mines (marked by the 4 flags). The 6 above the game board and on the left indicates there are 6 remaining mines. In this example, we have labelled one location with the letter "A" to indicate a space we would like to discuss. You will not see letters on the squares while playing the game online.

Sample Question: Determine whether the following statement is true or false and explain. Square A is a mine.

Sample Answer: FALSE. The square directly below the flag next to the A contains a 1, indicating that one and only one square adjacent to it is a mine. Since the square above the 1 (the flag) is a mine, square A must not be a mine. We can conclude that the statement "Square A is a mine." is false.

Consider the in-progress game below and determine if each statement below is true or false. Provide a brief justification for your claim.



1. Squares A and B are mines.

2. Square C is a mine

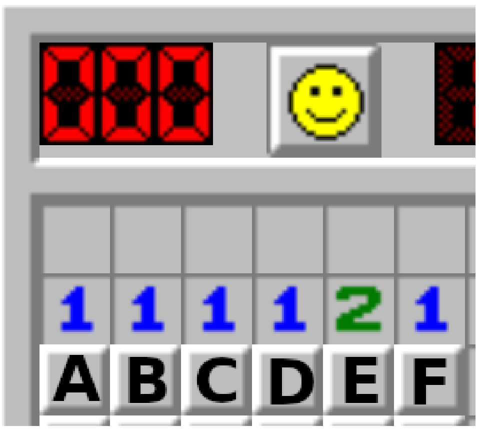
3. Squares F and E are mines

4. Square D is a mine.

1.2.2 Implications in Minesweeper

In the game Minesweeper sometimes we cannot determine whether a square is a mine or not at first glance. If we can determine the state of a nearby square, however, that may help us figure it out. Here we wish to explore these scenarios and how logic plays a roll in beating the game.

1. Consider this portion of an in-progress game. Are the statements below true, false, or indeterminate (need more information)? Explain.



   (a) A is a mine.

   (b) B is a mine.

   (c) A is a mine or B is a mine.

   (d) A is a mine and B is a mine.

2. What can we conclude from the exploration in question 1? Circle one and explain.

   (a) Neither A nor B is a mine.
   (b) Both A and B are mines.
   (c) Either A is a mine or B is a mine, but not both.

3. So which is it? Is A the mine? or is B? To figure this out logically, we make A a mine and see what happens. So suppose A is a mine (as if we knew this to be a fact) and determine whether the following statements are true or false.

   (a) If A is a mine, then B is clear.

   (b) If A is a mine, then B is clear and C is clear.

   (c) If A is a mine, then B is clear and C is clear and D is a mine.

   (d) If A is a mine, then B is clear and C is clear and D is a mine and E is a mine and F is a mine.

   (e) If A is a mine, then B is clear and C is clear and D is a mine and E is a mine and F is clear.

   (f) If A is a mine, then B is clear and C is clear and D is a mine and E is clear and F is a mine.

   (g) If A is a mine, then B is clear and C is clear and D is a mine and E is clear and F is clear.

   Now suppose B is a mine (as if we knew this to be a fact) and start over, assuming we don't know anything about any other square to begin. Determine whether the following statements are true or false.

   (a) If B is a mine, then A is clear.

   (b) If B is a mine, then A is clear and C is clear.

   (c) If B is a mine, then A is clear and C is clear and D is clear.

   (d) If B is a mine, then A is clear and C is clear and D is clear and E is a mine and F is a mine.

   (e) If B is a mine, then A is clear and C is clear and D is clear and E is a mine and F is clear.

   (f) If B is a mine, then A is clear and C is clear and D is clear and E is clear and F is a mine.

4. According to your answers in question 3, which do you think is the mine? A or B? Explain.

5. Below is more of the board from question 1. Use this new information to determine whether each square A through J is a mine or is clear. A board without the letters is supplied for you to work on. You may want to use a pencil!

Were you right about squares A and B?

6. Consider the in-progress game below. Are the following statements true or false? Explain.

(a) Square Q is a mine.

(b) Square N is a mine.

(c) Square I is a mine and square J is a mine.

(d) Square H is a mine.

(e) Square K is a mine or square L is a mine.

(f) If square K is a mine, then square J is not a mine.

(g) If square L is a mine, then square J is not a mine.

(h) Square J is not a mine.

7. Below is an argument table showing that the statement "Square A is a mine" is true for the game shown in question 6. Create an argument table for each statement below.

True Statement	Reason
O is a mine	O is the only uncleared square adjacent to the 1 on its left.
G is not a mine.	G is adjacent to the 1 above the O (which is a mine).
A is a mine or G is a mine	A and G are the only uncleared squares adjacent to the 1 next to A.
A is a mine.	The second and third rows (of this table) prove A is a mine.

Table 1.2: Argument

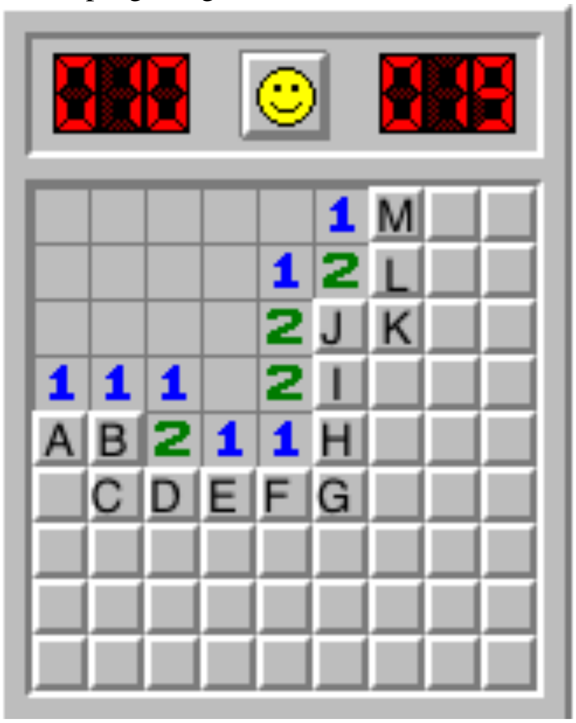
You may use the statements in question 6, and you may bring in new statements as needed.

(a) Square P is not a mine.

(b) Square I is a mine.

1.2.3 Exit Slip

For the in-progress game below, create three true statements, one for each type listed below.



1. Statement containing an "and"
2. Statement containing an "or"
3. An implication statement in the form " If . . . , then . . . ."

1.3 Extension Activity: Arguments

1.3.1 Activity: Nancy Out On The Town

Nancy's friends are trying to find her based on whether the following statements are true or false. They know that she is at exactly one of: home, the movie theater, the grocery store, or a clothing store.

P: Nancy is at home.
Q: Nancy is at the movie theater.
R: Nancy is at the grocery store.
S: Nancy is at a clothing store.
T: Nancy is trying on clothes.
U: Nancy is buying food.

1. If P is true is Q true or false? Explain your reasoning.
2. What statements could be true if T is true? Explain your reasoning.
3. If P is true, is $T \vee U$ true or false? Explain your answer.
4. If P is true, is $\neg S$ true or false? Explain your answer.
5. What statements are false if $Q \vee R$ is true?
6. If U is true, is $Q \wedge R$ true or false? Explain your conclusion.
7. If U is false, what do you know must be false? Explain your answer.
8. If T is true and U is false, what single true statement can you create concerning Nancy's location? Explain your results.
9. If U is true, what single true statement can you create concerning Nancy's locations? Explain your results.
10. If P is true and Q is false, is $R \vee (T \wedge U)$ true or false? Explain your results.
11. If T is true, is $\neg Q \wedge (S \vee P)$ true or false? Why?

1.4 Project Choices

1. **Project Question:** You are given an in-progress Minesweeper game. On the given board:

   (a) Mark each box adjacent to a numbered square with an O if it can be determined to be a mine or X if it can be determined that it is not a mine. * *There are at least 3 squares that are mines and at least 3 squares that are not mines. There may also be squares adjacent to the numbers that cannot be determined. Leave them blank.* *

   (b) Write an argument table for one of the squares marked with an O. * *Do not pick a square located in a "corner" adjacent to a square with a* 1. *

   (c) Write an argument table for one of the squares marked with an X.

2. **Project Question:** You are given an empty nonogram board. Your project is to do the following.

   (a) Fill all squares correctly, making note of your logic.

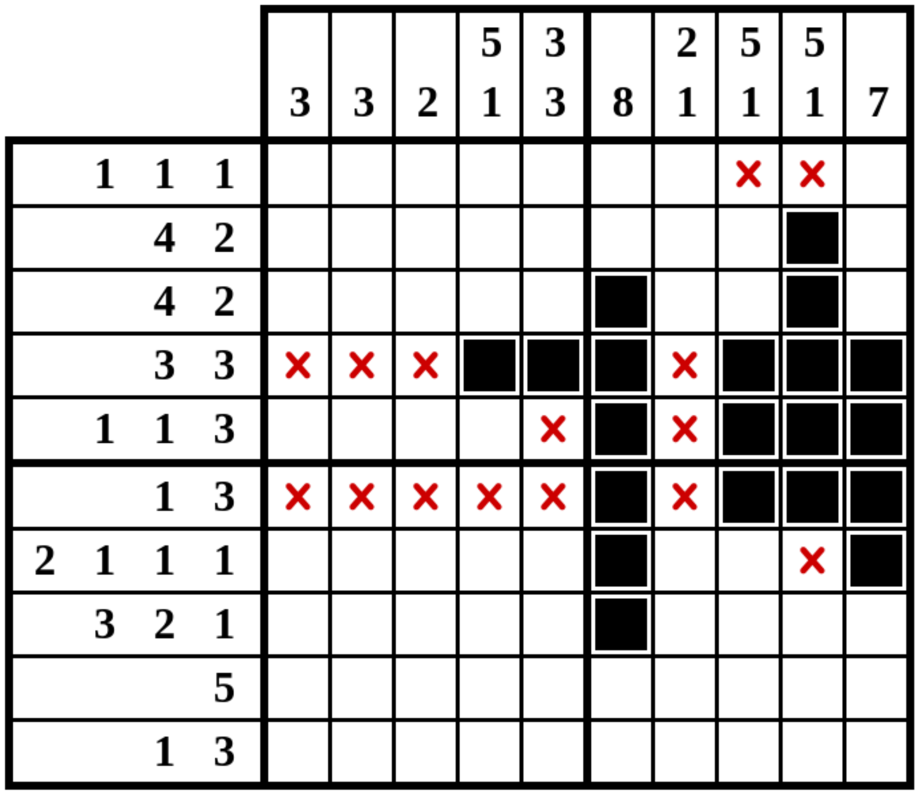
   (b) Write one true statement using an "and" that helped you complete the puzzle.

   (c) Write one true statement using an "or" that helped you complete the puzzle.

   The rules for filling the board are as follows.

   - Each box must be filled in black or marked with X.
   - Beside each row of the grid are listed the lengths of the runs of consecutive black squares on that row.
   - Above each column are listed the lengths of the runs of consecutive black squares in that column.
   - Boxes that are not part of a run of black squares must be marked with X.

Here is an example of a puzzle in progress.



The four squares in the rightmost column were filled in first. 7 consecutive squares in that column must be filled in and there are only 10 squares total, so you can only skip three of them from the top **or** three of them from the bottom. The black squares and X's immediately to the left of those four were filled in next. For example, in row 4 the tenth square in that row is filled in **and** the last run of squares must be length 3, so the two squares next to the last one must also be filled in black. That completes the run so the square next to that (the 6th from the left in that row) must not be filled (so is marked with an X). See if you can figure out why the rest of the marked squares are the way they are.

You may want to visit https://www.puzzle-nonograms.com/ to try some nonogram puzzles at a variety of difficulty levels online.

3. **Project Question:** You are given two KenKen puzzles. Your project is to do the following.

   (a) Fill all squares correctly in each puzzle, making note of your logic.

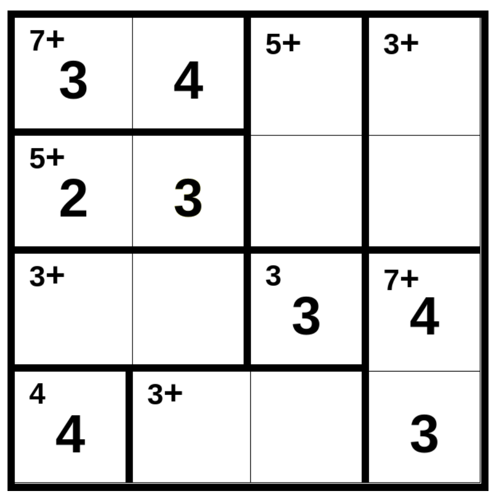
   (b) Write one true statement using an "and" that helped you complete one of the puzzles.

(c) Write one true statement using an "or" that helped you complete one of the puzzles.

The rules for solving a KenKen are as follows.

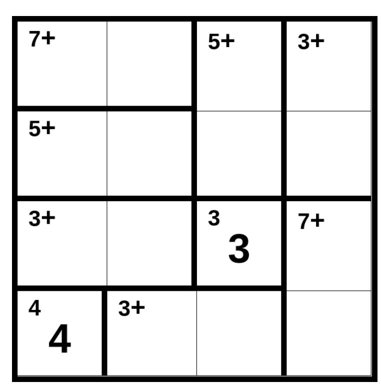
- The only numbers you may write are $1, 2, 3, \ldots$ up to the number of rows and columns. In a 4×4 puzzle you will use the numbers 1 through 4. In a 6×6 puzzle you will use the numbers 1 through 6, and so on.
- Every allowable number must appear in every row and every column. No number may be repeated in any row or any column.
- Each region bounded by a heavy border, called a **cage**, contains a target number and usually an arithmetic operation. You must fill that cage with numbers that reach the target using the specified arithmetic operation. Numbers may repeat within a cage, if needed, as long as they do not repeat within a single row or column.

Here is an example of a puzzle in progress.

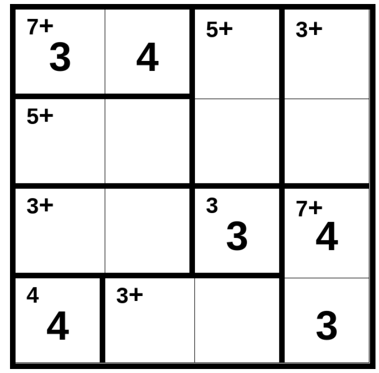


Here is how the numbers were filled in to this point.

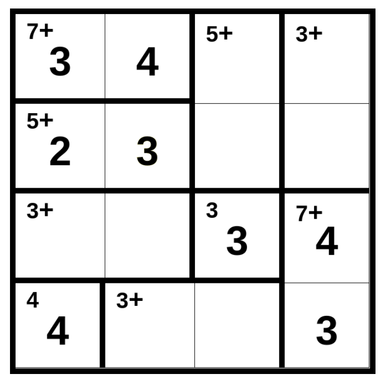
- The 4 in the bottom left corner and the 3 in the third row, third column were filled in first. Their cages are only one block so there is no operation to consider. The target number is the number that needs to be filled in.



- The two cages with $7+$ in them were filled in next. In each cage, the numbers 3 and 4 must be placed since the only way for two **different** numbers between 1 and 4 to add up to seven is to use 3 and 4. For the cage in the bottom right, the 4 must be in the third row since there is already a 3 in that row. You can come to the same conclusion by noticing there is a 4 in the fourth row (bottom left corner) already, so the 4 for this cage cannot be put in the fourth row. See if you can understand why the 3 and 4 in the $7+$ cage in the top left of the board were placed the way they were. They cannot be placed the other way around.



- The $5+$ cage with the 2 and 3 in it was filled in next. There are two ways to use different numbers between 1 and 4 to sum to five—1 and 4 or 2 and 3. However, there are already 4's in columns one and two, so a 4 cannot be used in this cage. Hence the cage will be filled with 2 and 3. Since there is already a 3 in column one, the 3 for this cage must be in column two and therefore the 2 must be in column one.



See if you can complete the puzzle. HINT: The first column already has the numbers $2, 3, 4$ in it. There is only one number that can be put in the empty box of that column!

You may want to visit https://www.kenkenpuzzle.com/ to try some KenKen puzzles at a variety of difficulty levels online. Begin by clicking the "Create Custom Puzzle" button.

4. **Project Question:** You are given a Sudoku Pair Puzzle. This isn't your regular Sudoku though! Fully complete the puzzle and provide a paragraph description of how you used logic to complete the puzzle, using examples from your specific game.

   (2,5) - Sudoku Pair Puzzle Rules: Each completed puzzle is a 10 by 10 grid that contains exactly one integer from 1 to 10 in each cell. Furthermore, there are regions that must contain every integer exactly once. The regions are described below and outlined in the puzzle:

   (a) Each row is its own region.

(b) Each column is its own region.

(c) Each 2 by 5 rectangle outlined by bold lines is its own region.

(d) Each 5 by 2 rectangle outlined by dashed lines is its own region.

Here is an example of a completed puzzle.

(2,5)-Sudoku Pair Puzzle

5	9	3	8	10	4	7	1	**6**	2
6	**1**	7	4	2	3	5	9	8	**10**
7	3	**6**	10	9	**8**	2	4	1	**5**
8	4	2	**5**	**1**	7	10	6	9	3
2	10	9	1	6	5	**3**	8	4	7
3	8	5	**7**	4	9	1	10	2	6
10	2	4	**3**	8	**6**	**9**	7	5	1
9	**5**	1	6	**7**	2	4	3	10	8
4	6	**10**	2	3	1	8	**5**	7	9
1	7	8	**9**	**5**	10	**6**	**2**	**3**	4

We've been describing $(2,5)$ Sudoku pair puzzles. A slightly simpler example would be $(2,3)$ Sudoku pair puzzles. It might be helpful to warm up on some of these! There is a webpage which will let you practice online:

http://www2.cedarcrest.edu/academic/math/jmhammer/SudokuWebapp/23SPLS/

1.5 Additional Exercises

1. Construct a truth table for the following statements.

   (a) $\neg(\neg P \wedge \neg Q)$

   (b) $\neg(\neg P \vee \neg Q)$

   (c) $\neg(P \Rightarrow \neg Q)$

   (d) $(P \vee Q) \Rightarrow P$

2. The $\Leftrightarrow$ operator is known as the "if and only if" operator and is defined to be the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Its truth table is given in Table 1.3. Find a way to write the statement $P \Leftrightarrow Q$ using the statements P and Q, and the operators $\neg$, $\vee$, or $\wedge$. (*Hint*: Construct a truth table first.)

P	Q	$P \Leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

Table 1.3: Truth table for $\Leftrightarrow$ operator

3. Construct a truth table for the following statements.

   (a) $\neg P \Leftrightarrow P$

   (b) $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$

   (c) $\neg(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow P)$

   (d) $((P \Leftrightarrow Q) \wedge (P \Rightarrow R)) \vee (\neg P \Rightarrow (Q \wedge R))$

   (e) $\neg(\neg P \Rightarrow (Q \vee R)) \Rightarrow (\neg P \wedge Q)$

4. Write a single statement using P, Q, and R along with the symbols $\neg$, $\vee$, $\wedge$, or $\Rightarrow$ that is always true.

5. Consider the following statements about Jake.

   `"If today is Wednesday, then Jake's class has library day. If Jake's class has library day, then Jake will bring home a new book. Jake did not bring any new books home today."`

   What can you conclude based on the given information? Explain your answer.

6. Show that the two statements $((P \wedge Q) \Rightarrow R)$ and $(\neg P) \vee (\neg Q) \vee R$ are equivalent. Give an example in which P, Q, and R each stand for some statement (like in Activity 1.3.1, then explain what the statements $((P \wedge Q) \Rightarrow R)$ and $(\neg P) \vee (\neg Q) \vee R$ mean in your given context.

7. Write out a statement using P and Q with the operators $\neg$, $\wedge$, or $\vee$ that has the indicated truth table.

(a)

P	Q	
True	True	True
True	False	True
False	True	False
False	False	True

(b)

P	Q	
True	True	False
True	False	False
False	True	True
False	False	False

(c)

P	Q	
True	True	True
True	False	True
False	True	True
False	False	True

(d)

P	Q	
True	True	True
True	False	True
False	True	True
False	False	False

(e)

P	Q	
True	True	False
True	False	False
False	True	True
False	False	True

8. The $\oplus$ symbol is known as XOR and is the statement that is true if exactly one of P and Q is true. Its truth table is given in Table 1.4. Write a statement that is equivalent to $P \oplus Q$ using the symbols $\neg$, $\vee$, or $\wedge$.

P	Q	$P \oplus Q$
True	True	False
True	False	True
False	True	True
False	False	False

Table 1.4: Truth table for $\oplus$ symbol

9. Show that the pair of statements are equivalent using truth tables.

   (a) $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$

   (b) $((P \vee Q) \wedge (\neg P)) \wedge Q$ and $\neg P \wedge Q$

10. During your voyage across the 7 seas you come across an island that is populated by two tribes: monks and peasants (or so they call themselves). From your interaction with them you realize that the monks always tell the truth while the peasants tell nothing but lies. Unfortunately, you cannot tell the inhabitants apart from each other. One day, two of the inhabitants, let us call them Amber and Betty, came up to you to tell you something:

    Amber says "Exactly one of us is a monk."

    Betty says "Only a peasant would say that Amber is a peasant."

    Can you determine to which tribe each of Amber and Betty belong?

11. You meet another inhabitant of island described in Question 10. You ask an inhabitant if there is gold on the island. He tells you "There is gold on this island if and only if I am a monk." Translate this statement into a logical symbol. Is this person a monk or peasant? Is there gold on the island. (*Hint*: The "if and only if" operator ($\Leftrightarrow$) is discussed in Question 2.)

12. Construct a new symbol $*$ that gives true or false responses depending on whether the statements are true or false (just like $\vee$, $\wedge$, and $\Rightarrow$). Explain how it works. Fill in the first three columns of the following table to make it clear what $P * Q$ means. Then fill in the remainder of the truth table.

P	Q	$P * Q$	$((P * Q) \wedge P) \Rightarrow Q$	$((P \Rightarrow \neg Q) * Q) \Rightarrow \neg P$	$P * Q \Rightarrow P$

Chapter 2

Proportions, Ratios, and Scaling

2.0 Mathematical Outcome

The activities of this section are designed to strengthen understanding of fractions and the related concepts of ratios and similarity through applications of proportion, ratio, and scaling. For unit conversion it is assumed the student has a working knowledge of fractions including multiplication and division thereof. The sections on ratios and similarity are presented assuming little to no prior knowledge of the topics.

Through a series of activities mostly focused on things in the world around us such as the moon, peoples' eyes, and the buildings in which we teach and learn, students will

1. learn to convert units through multiplying fractions
2. work with ratios
3. see the connection between equivalent fractions and equivalent ratios
4. connect proportionality and scale with the world around them
5. calculate heights and weights of large objects by gathering data about smaller ones and using mathematics to extrapolate
6. explore the concept of self-similarity

2.1 Unit Conversion

2.1.1 Entrance Activity: One Billion Seconds

1. Write one billion as a numeral (writing its digits, not words).

2. Convert 5 minutes into seconds. (How many seconds are there in five minutes?)

3. Convert 240 seconds into minutes. (How many minutes are in 240 seconds?)

4. Convert 2,022 minutes into seconds.

5. Convert 2,022 seconds into minutes.

6. What mathematical operation(s) do you need to do these conversions? ($+$, $-$, $\times$, $\div$, $\sqrt{\ }$, etc.)

2.1.2 Activity: One Billion Seconds

On what day will you (or did you) become one billion seconds old?

1. As we have done before, let's solve a simpler problem first. On what day will you (or did you) become one million seconds old? Write one million as a numeral (writing its digits, not words).

2. One plan of action is to convert one million seconds into days, and count that many days from your birthday. Not many people know how to convert seconds to days directly, though. We can figure it out!

   (a) Convert one day into hours.

   (b) Convert the number of hours you got in part 2(a) to minutes. This is how many minutes in one day.

   (c) Convert the number of minutes you got in part 2(b) to seconds. This is how many seconds in one day.

3. Use your work from question 2 to convert one million seconds to days.

4. Write down your birthday, including the year.

5. Use your answers to parts 3 and 4 to determine on what day you will become (or became) one million seconds old.
   To think about: Does it matter what time you were born?

6. Use what you learned from questions 1 through 5 to determine on what day you will become (or became) one billion seconds old.
   To think about: Do all months have the same number of days? Do all years have the same number of days?

2.1.3 Exit Slip

1. How many seconds are there in one year? Does this change from year to year? If so, give all possible answers and explain why there are multiple answers.

2.1.4 Activity: Carats and Carrots

When you are told the carat weight of a gem stone or piece of jewelry, you are actually being given its mass in carats. A unit of mass you are probably more familiar with is the gram. In fact, 1 gram $= 5$ carats, and we call the number 5 the conversion factor between carats and grams.

1. Convert 26 grams to carats.

2. If the mass of an object is written in both carats and grams, which number will be numerically larger—the number of grams or the number of carats?

3. Should you multiply by 5 or divide by 5 to convert a mass in carats to a mass in grams? Why?

4. What is the mass of a $\frac{1}{4}$-carat diamond in grams?

5. Given that 1 pound $= 453.6$ grams, how many pounds does a 1-carat diamond weigh?

6. If you buy a half pound of carrots, how many carats of carrots do you have? Convert first from pounds to grams, and then from grams to carats.

7. If you ordered $2,000$ carats of cheese at the deli, how many pounds would you expect to get (assuming the deli-worker didn't just stand there looking at you funny!)?

2.1.5 Activity: To the Moon!

NASA's Artemis project is intended to bring people to the surface of the moon as early as 2025. The first Artemis mission was scrubbed on August 29, 2022 and then again on September 3, 2022. The first launch was then rescheduled for mid-October 2022. "With Artemis missions, NASA will land the first woman and first person of color on the Moon, using innovative technologies to explore more of the lunar surface than ever before...We're going back to the Moon for scientific discovery, economic benefits, and inspiration for a new generation of explorers." (Artemis)

The Activity: How far is the moon from Earth?

I bet you expected to have to work for that answer. Not this time! On average, the moon is $.00000004063$ light years away . OK, OK, so you'll have to work a little.

1. Will the number of miles from Earth to the moon be greater or less than the number of light years?

2. 1,000,000,000,000 miles=0.170107795023 light years so dividing one by the other equals a sort of "one". Multiplying by either fraction, therefore, leaves distances unchanged! Which fraction should you multiply $.00000004063$ light years by to convert to miles, and why?

$$\frac{1,000,000,000,000 \text{ miles}}{0.170107795023 \text{ light years}} \quad \text{or} \quad \frac{0.170107795023 \text{ light years}}{1,000,000,000,000 \text{ miles}}$$

3. Carry out the product as planned.

   $.00000004063 \text{ light years} \times \rule{4cm}{0.4pt} =$

4. Now write that product without the numbers, retaining only the units.

   $\text{light years} \times \rule{4cm}{0.4pt} =$

   Do the light years cancel?

5. Redo the calculation using the equation 1 light year=5,878,625,000,000 miles.

   $.00000004063 \text{ light years} \times \rule{4cm}{0.4pt} =$

   Did you get the same answer?

2.1.6 Exit Slip

1. Convert the average distance between the sun and Earth, 93 million miles, into light years.

2. A typical tire pressure for automobiles is 220 kPa (kilopascals). Use the fact that 1.25 kPa equals 0.1813 psi (pounds per square inch) to convert typical tire pressure to psi. Source: Pirelli.

3. Prior to 1896, the length of the Kentucky Derby was 12 furlongs. Source: KentuckyDerby.com. Use the fact that one furlong equals one eighth of a mile to convert the original length of the Kentucky Derby to miles.

4. Using the facts that (a) one furlong equals one eighth of a mile and (b) there are 5280 feet in one mile, what is the conversion factor between feet and furlongs?

2.1.7 Activity: Tile New Haven

The Area of New Haven The land area of New Haven, CT is 18.69 square miles (U.S. Census)

1. Convert the area of New Haven into square feet. Note that square miles is written miles^2 (or miles $\times$ miles) as a unit. Use the fact that 1 mile $= 5280$ feet. HINT: It's more than a million.

2. A ceramic tile for a traditional subway tiling pattern is 2 inches tall and 4 inches wide. How many tiles does it take to cover one square foot?

3. How many tiles would it take to cover the entire area of New Haven?

4. Convert the area covered by one letter size sheet of paper (which is 8.5 inches wide and 11 inches tall) into square feet.

5. How many sheets of letter size paper would it take to cover New Haven? In other words, convert the area of New Haven from square feet into sheets of letter size paper.

2.2 Proportions and Ratios

2.2.1 Entrance Activity: Rice

You've been following the directions on the bag of rice you recently purchased at the grocery store.

Every time you make the rice, you do it according to the directions—2 cups of water, 1 cup of rice—but it just makes too much! About one third of it is always left over. Learning from this experience, you decide you will use only 2/3 cups of rice. But how much water should you use? Still 2 cups? No. The directions on the bag have you using twice as much water as rice, so you should continue to use twice as much water as rice! For 2/3 cups of rice, you should use $2 \cdot (2/3) = 4/3$, or $1\frac{1}{3}$, cups of water. If you used 2 cups of water with 2/3 cups of rice, the water and rice would be out of proportion. You would be using 3 times as much water as rice!

1. You're having guests and need to cook two cups of rice. How much water should you use?

2. A recipe for rice pudding asks that you start by cooking 3/4 cups of (uncooked) white rice according to package directions. You still have that much rice left in your package. How much water will you need to use?

3. How much rice can you cook with 3 cups of water?

2.2.2 Activity: Ratios and the Colon Notation

1. The proper way to cook any amount of rice from the bag in the entrance activity is to use twice as much water as rice. The volumes of water and rice used in cooking should stay in proportion. We might also say that to cook the rice we maintain a 2 to 1 water-to-rice **ratio**. The symbol used for ratios is the colon. Fill in the blanks to see the water-to-rice ratio written using the colon notation.

   ______________ : ______________
   amount of water   amount of rice

2. In a baseball game where the score is 9 to 6, the announcer might say the winning team is outscoring their opponent 3-to-2, meaning that the winning team is scoring 3 points for every 2 points scored by the losing team. Write the ratio 3-to-2 using colon notation.

3. In a large high school auditorium, the girls outnumber the boys 5-to-4, meaning for every 5 girls there are only 4 boys. Write the ratio 5-to-4 using colon notation.

4. Fill in the blanks in the table where the winning team is outscoring their opponent by a 3:2 ratio.

winning team score	3	6	9	12	15
losing team score			6		

5. Fill in the blanks in the table where girls outnumber boys 5:4.

girls	100		195	
boys		124		200

2.2.3 Activity: Body Mass Index

In a 2012 MedicalNewsToday article, research suggests that waist-to-height ratio is a better predictor of health problems such as high blood pressure, diabetes, heart attacks, and strokes than is BMI (Body Mass Index).

1. A $5\frac{1}{2}$ foot tall person with a 32 inch waist has what waist-to-height ratio?

2. According to the article, keeping your waist-to-height ratio at 0.5:1 or lower can increase your life expectancy. A 6 foot tall person should keep their waistline how small to increase their life expectancy?

2.2.4 Activity: One Blue, Seven Brown

1. What is the ratio between the number of students in this class with brown eyes to that of students with other colored eyes?

2. If the University has 9,127 students, and the ratio of brown-eyed students to non-brown-eyed students matches that in this classroom, how many brown-eyed students are there in total?

3. Repeat the exercise, but now catalog students as having blue eyes, brown eyes, or neither (somewhere in between). The person with the bluest eyes in the classroom has blue eyes.

2.2.5 Activity: Equivalent Ratios

1. There are 2.54 centimeters in an inch. That means there is a 2.54:1 ratio between its length measured in centimeters and its length measured in inches. How many centimeters are there in one foot (12 inches)?

2. It is also true that the ratio between a length measured in centimeters and a length measured in inches is 127:50. Use this ratio to calculate the number of centimeters in one foot. Did you get the same answer?

3. Is it true that the ratio between a length measured in centimeters and a length measured in inches is 254:100? Explain.

4. Show that $\frac{2.54}{1}$, $\frac{127}{50}$, $\frac{254}{100}$ are equivalent fractions by rewriting them all with the same denominator.

5. Show that 2.54:1, 127:50, and 254:100 are equivalent ratios in the same way. Rewrite them by multiplying or dividing each number in a given ratio by the same nonzero quantity so they all have the same second number.

6. For this question, the ratio between two quantities is 4:7.

   (a) If the smaller quantity is 36 fathoms, what is the larger quantity?

   (b) If the larger quantity is 28 minutes, what is the smaller quantity?

   (c) A Lego set has 50 blue blocks and 28 brown ones. Are these quantities in a 4:7 ratio? Explain.

   (d) A fountain has two tiers, one with a volume of 3,936 cm^3 and another with a volume of 6,888 cm^3. Are the volumes of the two tiers in a 4:7 ratio? Explain.

2.2.6 Activity: Model Trains

1. The ratio between the size of any feature of an O scale model train and the size of the same feature on the actual train is 1:48. If you have an O scale model train that measures 17 inches long, how long is the train it models?

2. An actual CSX hi-roof boxcar has an inside height of 13 feet (CSX.com). How high should the inside of an O scale model of this boxcar be?

3. The ratio between the size of the features of a G scale model train and the actual train is 2:45. If the wheel of a train measures 36 inches in diameter, what would be the diameter of a G scale model of that train?

4. Would a G scale model of a railroad car be larger or smaller than an O scale model of the same railcar? Explain.

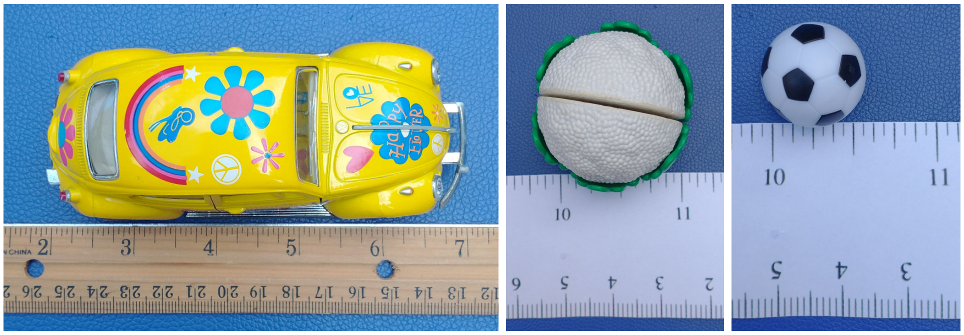
2.2.7 Activity: Out of Proportion

The following photos have been digitally edited so the images look weird. Using ratios, explain why they look weird and/or what editing has been done to them.

1. Cantaloupe:

2. Hand:

2.2.8 Exit Slip: Automobile, Cauliflower, and a Soccer Ball



1. Use the picture to measure the size of the model automobile, cauliflower, and soccer ball.

2. Approximately what is the true (real-life) size of an automobile, head of cauliflower, and soccer ball?

3. What are the scales of the

   (a) model automobile?

   (b) model cauliflower?

   (c) model soccer ball?

4. Would it make sense to pair any of these objects in the same farmhouse diorama? Would the cauliflower and soccer ball look "right" next to each other? The cauliflower and automobile? The soccer ball and automobile? Explain.

2.3 Dilation and similarity

2.3.1 Entrance Activity: The Odditorium

You're on a family vacation, and your parents get the idea to bring everyone to the nearby odditorium. You're not sure it will be any fun, but your parents are insistent and your sibling thinks it's a great idea. It turns out the things at the odditorium are truly odd. There's the world's smallest automobile, authentic shrunken heads, and sculptures made from recyclables. There's also this mannequin of the world's tallest person next to a young lady clearly enjoying the place. You spy the same young lady just outside the odditorium with this humongous marble.

Curiously, she is spinning the marble all by herself! The sign nearby notes that the sphere "is floating on 1/254 inches of water and can easily be stopped and spun in another direction—try it!" You think to yourself, "that ball looks really heavy. Just how much weight is she pushing around?"

1. Gather several stones, each one the size of a walnut or smaller. Put them somewhere so you won't forget to bring them to class!

2. Make a guess as to the weight of the giant marble.

Before starting this activity take a good look at the pictures from the entrance slip. Use them to estimate the diameter of the humongous marble.

2.3.2 Activity: Marble Weight Using Density

The density of an object is the ratio of its mass to its volume. In this activity, we will determine the densities of the stones you brought to class and use that information to predict the weight of the giant marble at the odditorium.

1. Choose one of the stones you brought to class and

   (a) use the scale to find its mass

   (b) use the graduated cylinder to find its volume

   (c) write down the density of the stone as a ratio. Use colon notation and integers on each side of the colon.

2. Repeat for two of the other stones.

3. Do any of the densities you calculated seem wrong, or did they all come out about the same?

4. Pick one of the densities as an estimate of the density of the stones you brought in. Choose the density you think best matches the density of the marble in the picture.

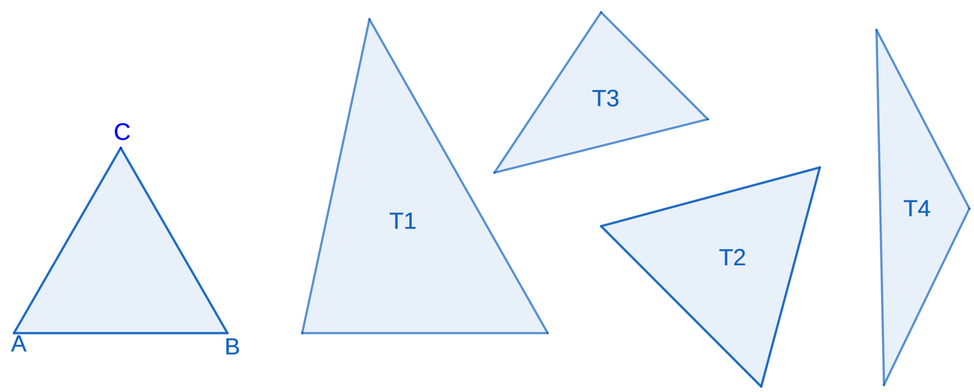
5. Using your estimate of the diameter of the giant marble, calculate its volume. The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

6. Calculate the mass of the giant marble based on your answers to questions 4 and 5.

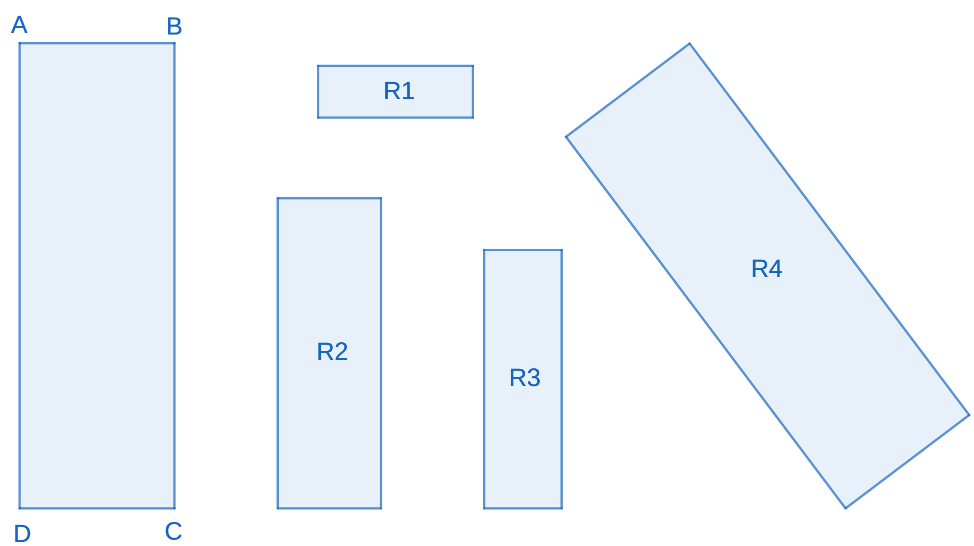
2.3.3 Activity: Dilation

Dilation is the enlarging or shrinking of a mathematical object (a point on a coordinate grid, polygon, line segment, sphere, etc.) using a specific scale factor. Dilation does not change the shape of the object. The size of a figure can change, but not the shape. The preimage—the object before scaling—is enlarged, inverted, or shrunken to form the image. The preimage and image are dilations of one another. You can think of the **preimage** as the original figure, and the **image** as the new figure. The **scale factor** of a dilation is the amount by which all original lengths are enlarged or shrunken. In the language of the previous section, everything about the dilated shape is in proportion.

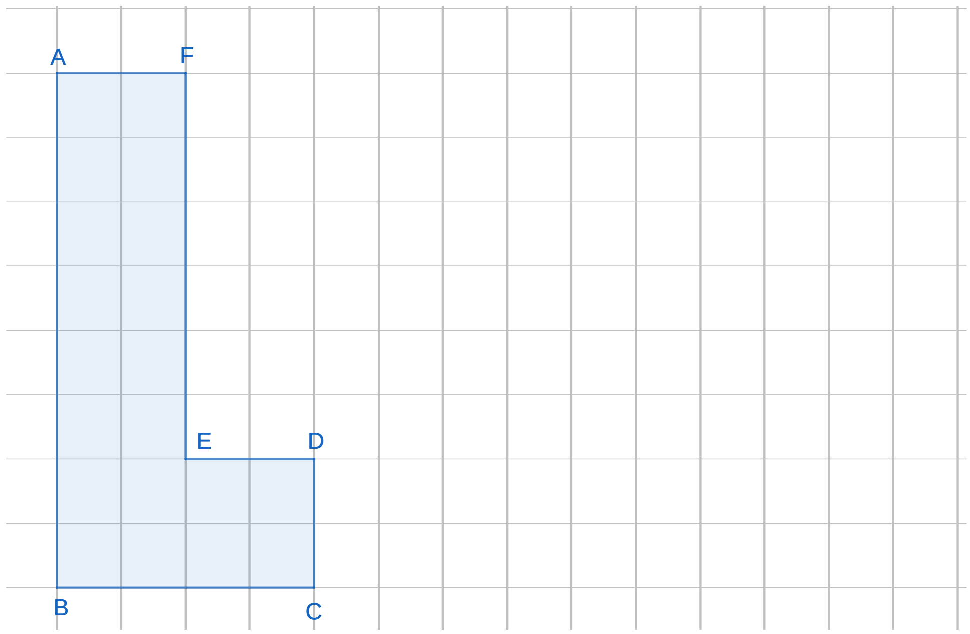
1. Which triangle is a dilation of ΔABC?



2. Rectangles R1, R2, R3, and R4 are all dilations of rectangle $ABCD$. Which one is a dilation with scale factor $\dfrac{1}{2}$?



3. Draw two separate dilations of hexagon $ABCDEF$—one with scale factor $\frac{1}{2}$ and the other with scale factor $\frac{2}{3}$.



4. A circle with radius 4 is dilated with scale factor 3.

   (a) What is the radius of its image?

   (b) What is the area of its image?

   (c) What is the ratio of the area of the image to the area of the preimage? Careful! It is not 3:1.

5. A cube with side length 2 cm is dilated with scale factor 10. What is the ratio of the volume of the image to the volume of the preimage?

2.3.4 Activity: Marble Weight Using Dilation

A typical shooter marble has a 5/8 inch diameter (source MarbleCollecting.com),

weighs about 5.4 grams, and has about the same density as the humongous marble. We will use this information to calculate the weight of the humongous marble!

1. Make sure you understand the problem.
2. Parts 3-7 comprise the plan for solving the problem. You will carry out that plan by answering those questions.
3. Convert the weight of the shooter marble (5.4 grams) into pounds. Recall 1 pound $=$ 453.6 grams.
4. Estimate the diameter of the humongous marble in inches.
5. Since a typical shooter marble is a sphere and the humongous marble is also a sphere, the humongous marble is a dilation of the tiny shooter marble. What is the scale factor?
6. What is the ratio of the volume of the humongous marble to the volume of the shooter marble?
7. The ratio of the weight of the humongous marble to the weight of the shooter marble is the same as the ratio in part 6 (because weight and volume are proportional). Use this information to calculate the weight of the humongous marble.
8. Review (problem solving step 4): Did this calculation give a weight similar to that from the calculation using density? Do the calculated weights seem reasonable? How do they compare to your initial guess? If you note any discrepancies or problems, where did the calculation(s) go wrong? See if you can spot the error(s) in this calculation or the calculation based on density.

2.3.5 Activity: How tall is Engleman Hall?

When the sun casts shadows onto level ground the triangle formed by a vertical object, its shadow, and the imaginary line from the top of the object to the tip of the shadow is similar to any other. This fact allows us to calculate the height of tall objects without the dangers of climbing them or unrolling a tape measure from their top to bottom.

1. Draw a schematic diagram of an object, its shadow, and the imaginary line from the top of the object to the tip of its shadow. You may include a schematic of the sun, the ground, and anything else that you find helpful.

2. Go outside and measure

   (a) Your height

   (b) The length of your shadow

   (c) The length of the shadow of Engleman Hall (or other building), a tree, lamppost, flagpole, sculpture or other tall object whose shadow is cast on level ground where you can measure it, for a total of 3 objects.

3. Write down the objects, their shadow lengths, and your estimates of how tall each one is.

Object	Shadow length	Estimated height

4. Use the fact that all triangles defined by a vertical object, its shadow, and the imaginary line from the top of the object to the tip of its shadow are dilations of one another to calculate the height of Engleman Hall (or other object whose shadow you measured).

2.3.6 Entrance Activity: Sharpie!

When we say that two things are similar in English, we mean that they resemble one another in some way, but are not exactly the same. Just how closely they resemble one another and in what way is entirely up to the speaker. However, when we say two things are similar in mathematics, we mean they are the exact same shape but perhaps different sizes. The way in which they resemble one another is precisely defined. In terms of ratios, two objects are similar if corresponding lengths between the two objects maintain a constant ratio.

Say you have two photos of the same clown, one is a 4x6 and the other 8x10. In the 4x6 photo, the distance between the clown's pupils is one inch and in the 8x10 photo, the distance between the clown's pupils in 2 inches. If the two photos are similar in the mathematical sense, then all corresponding distances between the two photos will have a 2:1 ratio. Say the distance from the center of his big red nose to the furthest point on his big right ear is 1.3 inches on the 4x6 photo. Then the distance from the center of his big red nose to the furthest point on his big right ear on the 8x10 photo must be 2.6 inches. The conversion factor from distance on the 4x6 photo to distance of the 8x10 photo is 2. If this ratio holds for all distances in the photo, then the photos (clowns) are similar in the mathematical sense. However, if there is even one distance (say the width of the clown's nose) for which this ratio does not hold (perhaps it is 1/3 inches in the 4x6 and only 1/2 inches in the 8x10), then the photos are not similar. In the language of the previous sections, we would say the width of the clown's nose is out of proportion or the 8x10 photo is not a dilation of the 4x6 photo. For two shapes to be similar all of their features must be in proportion. They must be dilations of one another.

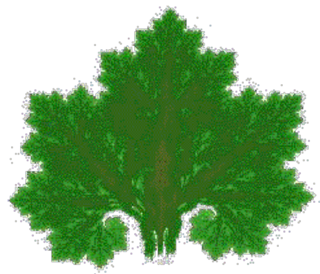
1. Are the markers similar in the English sense?

2. Are the markers similar in the mathematical sense?

2.3.7 Entrance Activity: Self-similarity

Watching movies is fun. Surfing the web is fun. Thumbing through a magazine is fun. Everywhere we look we see images. Behind each image is a great deal of mathematics. Some digital images are processed. They may have had their colors changed or their subjects touched up. Some digital images are compressed. They have been encoded to use less computer memory. To make storing, processing, compressing, and reproducing digital images happen a great deal of mathematics is used. One technique used is fractal image compression (see this IJSR article, for example). Can you tell which of these images are fractals and which are not?

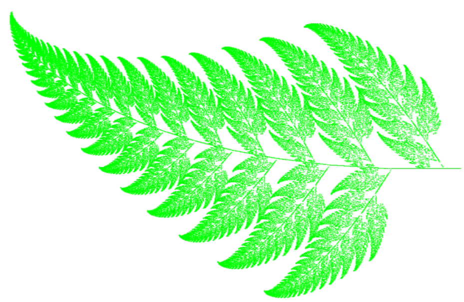
Artwork or fractal?



Maple leaf or fractal?

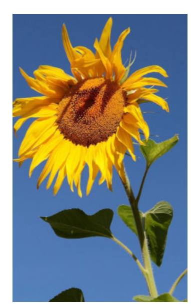
Photo or fractal?

They are all fractals! The basic idea behind all of these fractals is self-similarity. An object or shape is self-similar if parts of it look like the whole. Broccoli is a great example of a self-similar shape from nature.



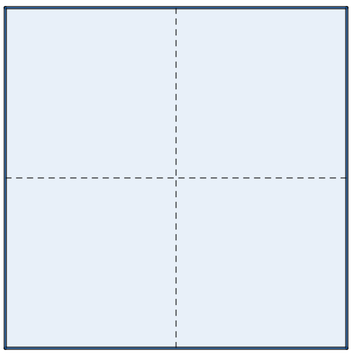
The small piece of the broccoli head resembles the whole head! The fronds of the fern resemble the whole fern!

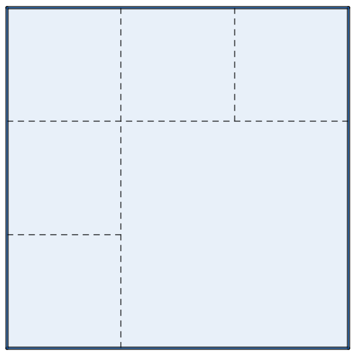
Which of the following natural shapes are self-similar (have parts that resemble the whole)? Circle them.

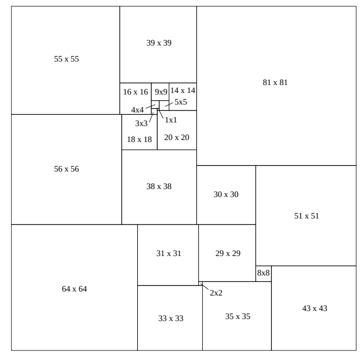


2.3.8 Activity: Rep-tiles

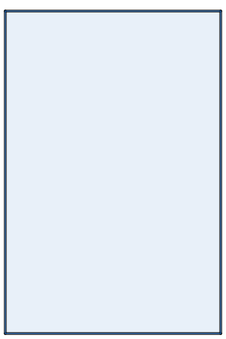
Rep-tiles are perfectly self-similar shapes. A rep-tile can be replicated by fitting together copies of itself without overlap. For example, a square is a rep-tile because four congruent squares can be fitted together to form a square. There are other ways to fit squares together to form a square too! As seen below, five congruent squares plus a sixth square of double their size (dilation by a factor of 2) can be fitted together to form a square. Even though it may not be clear from the diagram, the rightmost image shows 31 squares of different sizes fitting together to form a square.

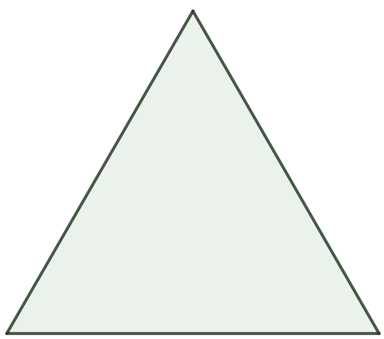


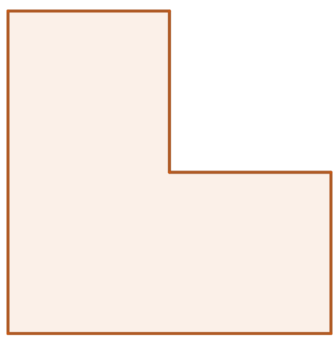




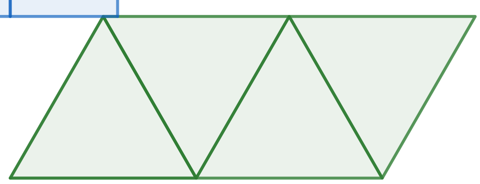
Can you fit four congruent copies of the following three rep-tiles together to create a replica of the whole? The rectangles must be put together to form a rectangle. The equilateral triangles must be put together to form an equilateral triangle. The L-shaped hexagons must be fitted together to form an L-shaped hexagon.

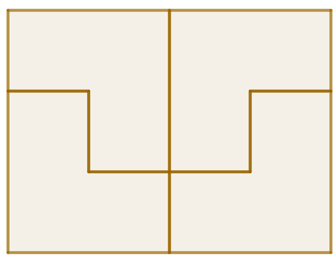






Cut out the copies below and see if you can fit them together on the shapes above.





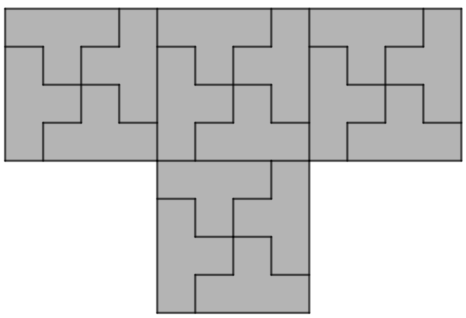
2.3.9 Activity: Similarity and Self-similarity

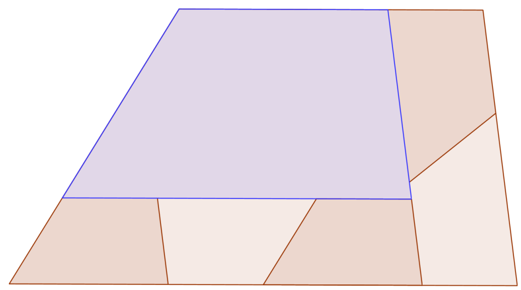
1. Consider this photo of three pencils.

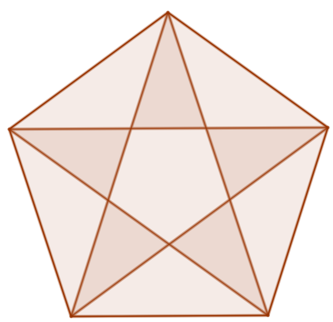
   (a) Are the pencils similar in the English sense?

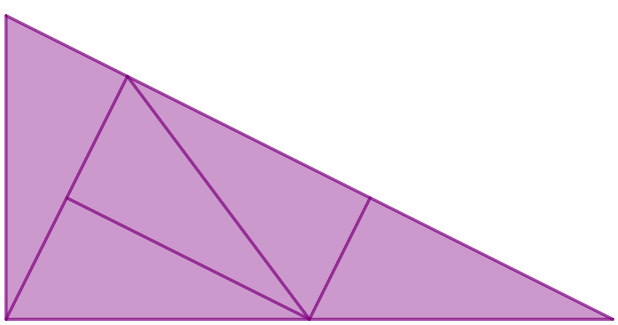
   (b) Are the pencils similar in the mathematical sense? Consider only the shapes, not the patterns printed on them.

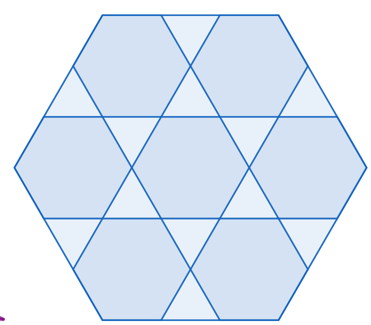
2. Which illustrations are rep-tiles (perfectly self-similar shapes)? Circle them.



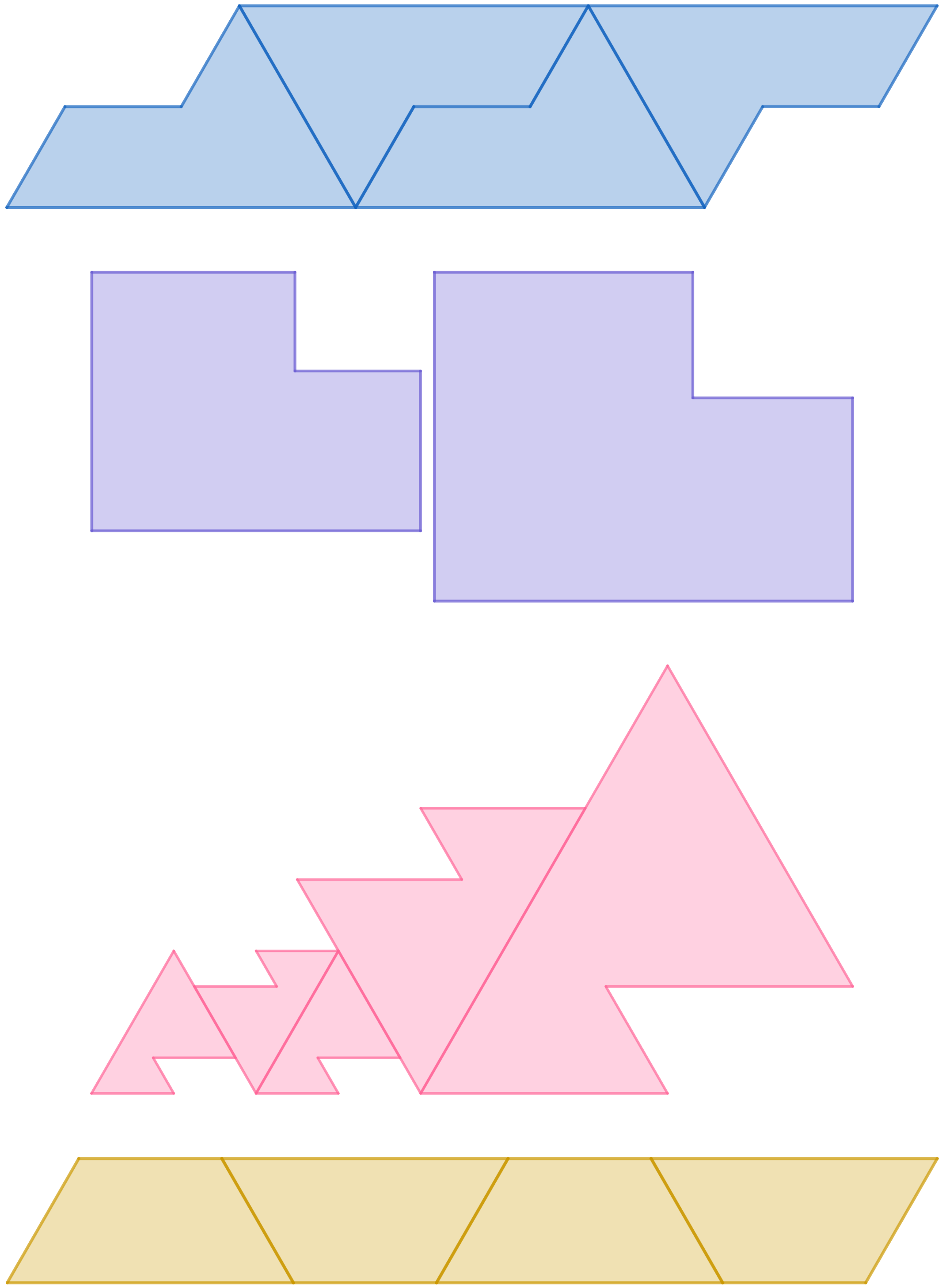






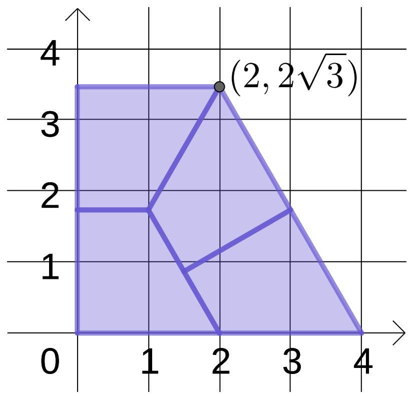


3. Pieces to assemble four different rep-tiles appear below. Cut out the pieces and see how many of the four rep-tiles you can fit together.

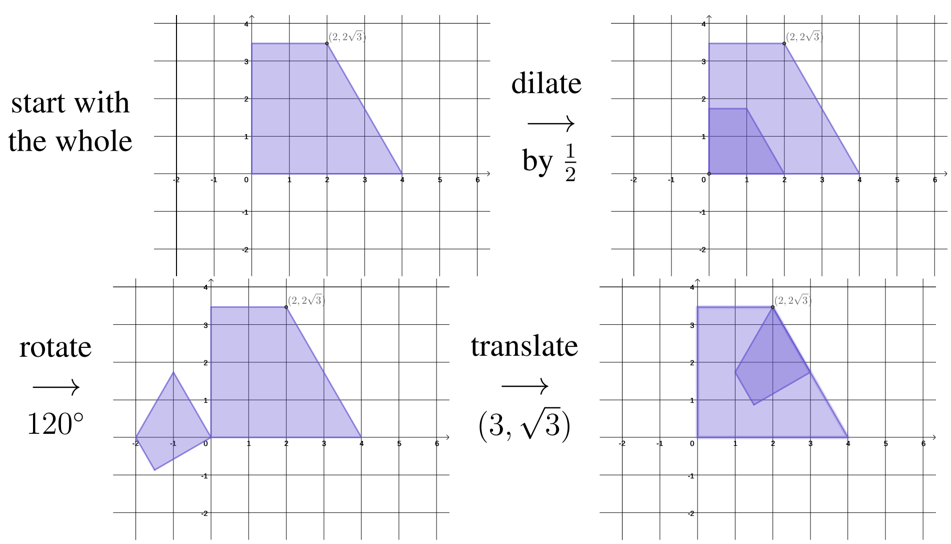


2.3.10 Activity: Defining a rep-tile

You've probably heard the phrases "two points define a line" or "a circle is defined by its center and radius". One way to understand what we mean by "define" in these two phrases is that the information describes exactly one line or one circle. For any two points, there is exactly one line that passes through both. For every center-radius pair there is exactly one circle with that center and radius. In the same way, a rep-tile is defined by the relationships between its parts and the whole. If we can precisely describe these relationships, there will be exactly one rep-tile with that description. Take the rep-tile below.

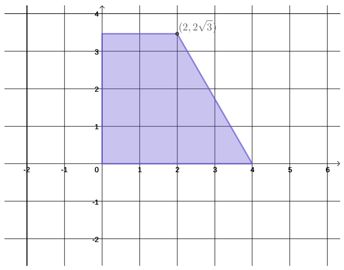


It is a trapezoid formed by fitting together four similar trapezoids. With the axes superimposed on the rep-tile, we can mathematically describe the relationships between the parts and the whole in terms of dilations, rotations, reflections, and translations. For example, if we take the whole trapezoid and follow the steps below, it turns into one of the parts of the assembly.

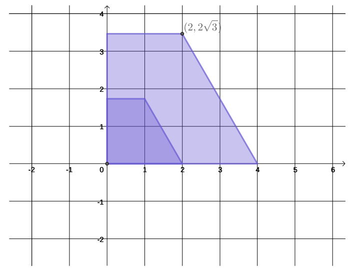


So ONE of the parts of the description that defines this rep-tile is *dilate by* $\frac{1}{2}$*; rotate* $120°$*; translate* $(3, \sqrt{3})$. A second one is much simpler:

start with
the whole



dilate
$\longrightarrow$
by $\frac{1}{2}$



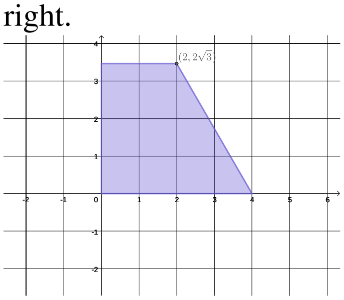
So ANOTHER part of the description that defines this rep-tile is *dilate by* $\frac{1}{2}$.

See if you can find descriptions that turn the whole trapezoid into the remaining parts. You may use

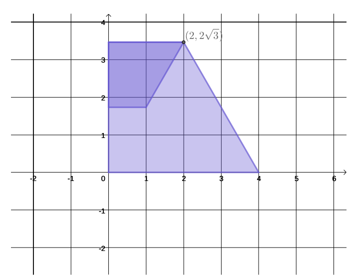
- **Reflection**. Reflection is done about the x-axis or about the y-axis only.
- **Dilation**. Dilation is done "toward" the origin, $(0, 0)$, as seen above. That means the point $(0, 0)$ does not move under dilation.
- **Rotation**. Rotation is done *counterclockwise* about the origin, $(0, 0)$, as seen above. That means the point $(0, 0)$ does not move under rotation and positive angles mean counterclockwise rotation. Use negative angles for clockwise rotation.
- **Translation**. Translation means changing location horizontally and/or vertically and is indicated by an ordered pair, $($horizontal move, vertical move$)$ as above.

in that order.

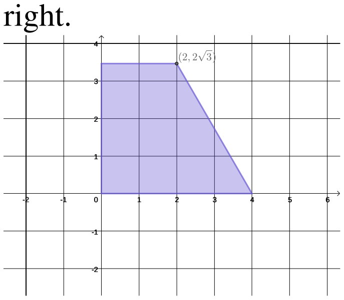
1. Write a description that turns the figure on the left (the whole) into the part on the right.



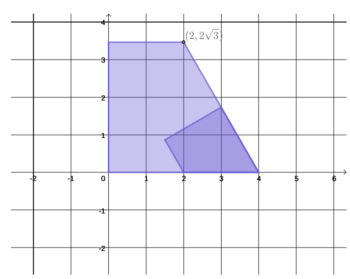
$\longrightarrow$



2. Write a description that turns the figure on the left (the whole) into the part on the right.



$\longrightarrow$



3. To check that you have the descriptions right, enter them into the rep-tile designer at https://lqbrin.github.io/tea-time-linear/rep-tile-designer.html.

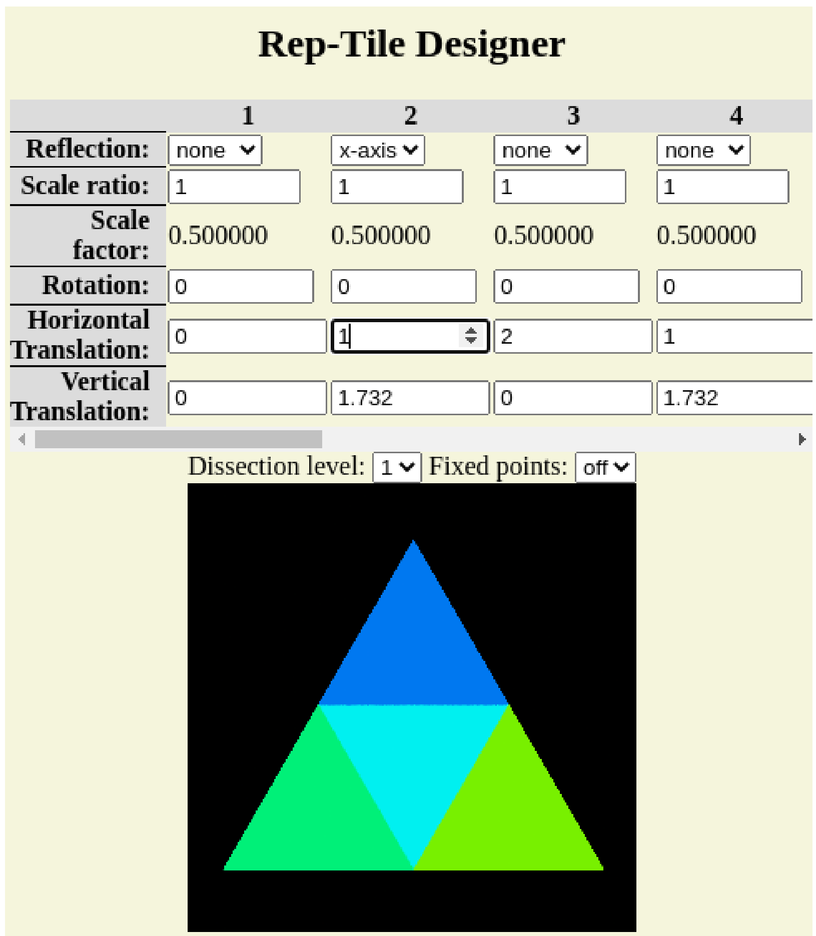
   As an example, shown below is the rep-tile designer with descriptions for the parts of the equilateral triangle:

   dilate by $\frac{1}{2}$

   reflect about x*-axis; dilate by* $\frac{1}{2}$*; translate* $(1, \sqrt{3})$

   dilate by $\frac{1}{2}$*; translate* $(2, 0)$

   dilate by $\frac{1}{2}$*; translate* $(1, \sqrt{3})$

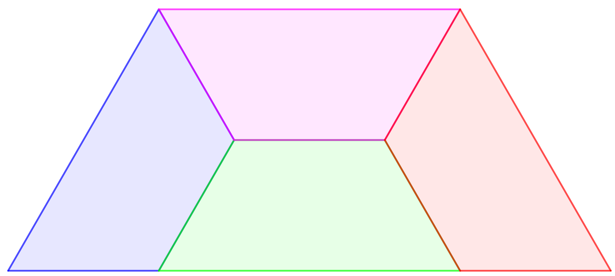


   Notes:

   - You only need to put in the ratio between the sizes of the parts. Rep-tile designer will caclulate the dilation (scale) factors for you.
   - Select dissection level 1 to see the parts.

2.3.11 Exit Slip: Create your own

Below is an image of an isosceles trapezoid rep-tile from our previous activity. There is no coordinate system attached to this, but the "bottom" angles measure $60°$ and the "bottom" is twice as long as the "top" and "sides." Find a way to create this image in the Rep-tile designer and write down the description for each of the four parts.



2.4 Project Choices

1. Unit Conversion

   (a) **Paradox resolved** A 12"×10" rectangular prismic foot bath is filled 3" deep using water from a 10' diameter circular pool (see photo). Knowing that water has been removed from the pool for use in the foot bath, logic would suggest that the level of water in the pool must decrease. The pool contains less water after all! However, the pool looks just as full after removing water as it did beforehand. What gives?

   i. Calculate the volume of water removed from the pool in cubic inches.
   ii. A cylinder with diameter 10' and the same volume as that calculated in part 1(a)i has what height (in inches)? NOTE: The volume of a cylinder is $V = \pi r^2 h$ where r is the radius of the cylinder and h is its height.
   iii. Explain why the height calculated in part 1(a)ii is a good approximation of the amount the level in the pool must have decreased.
   iv. How does this calculation resolve the paradox?

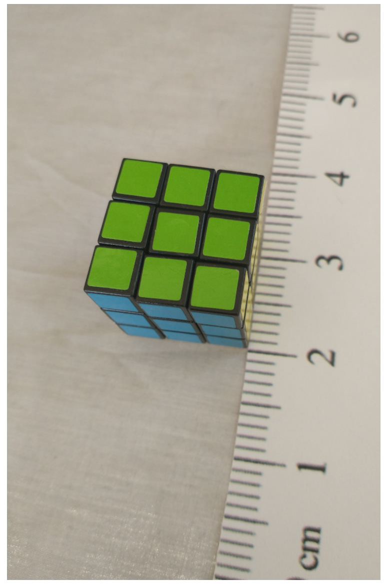
   (b) **Lake Mead** Formed by the Hoover dam, Lake Mead is the largest reservoir in the United States. Much of the lake straddles the border between Nevada and Arizona, states that have been in a drought for over two decades. Due to record low levels in the lake in 2022, the Bureau of Reclamation mandated water usage cutbacks in both Nevada and Arizona. In Nevada, the cutback is $21,000$ acre $\cdot$ feet. (As reported in The Acronym May 22, 2022)

   i. Convert $21,000$ acre$\cdot$feet to cubic feet using the fact that one acre is $43,560$ square feet.
   ii. Convert the volume from part 1(b)i into gallons using the estimate that 25 cubic feet equals 187 gallons.
   iii. Does this seem like a large cutback? Explain.

2. Ratios

(a) Verbalize the similarities, differences, and connections between the concepts of ratios, fractions, and unit conversion.

(b) Would this Rubik's cube look abnormally small, abnormally large, or just about right in this doll's hands? Explain in terms of ratios.



(c) In mixing pumpkin pie spice from individual spices, the ratio of cinnamon to ginger to cloves to nutmeg is 4:2:1:0.5. This means, for example, that you can mix 4 teaspoons cinnamon with 2 teaspoons ginger and 1 teaspoon cloves and a half teaspoon nutmeg to make 7.5 teaspoons of pumpkin pie spice.

   i. How many tablespoons are 7.5 teaspoons? If you don't know the conversion between teaspoons and tablespoons, look it up!

   ii. How much of each ingredient should you use to make 6 tablespoons of pumpkin pie spice?

   iii. Is it practical to make 6 tablespoons of pumpkin pie spice using this recipe?

   iv. What would be practical amounts of each spice to mix instead given that you need 6 tablespoons for your fall baking?

3. Similarity—chose ONE of the following

(a) How big is the SCSU pond? With some mathematics and some safe, dry, on-ground measurements, you can calculate it!

i. Bring some friends, a camera, tape measure, and markers out to the field. Measure the distances marked $\mathbf{w}$, $\mathbf{x}$, and $\mathbf{y}$ as in the diagram above. Notice the spot where the hypotenuse crosses the line labeled with $\mathbf{x}$ and $\mathbf{y}$ is in the woods! Be creative getting those measurements, but be safe!

ii. Take pictures of yourself doing the activity.

iii. Use your measurements and the fact that the two triangles are similar to calculate the distance across the pond, thereby measuring the distance across the pond without ever crossing the pond or laying a tape measure over the water.

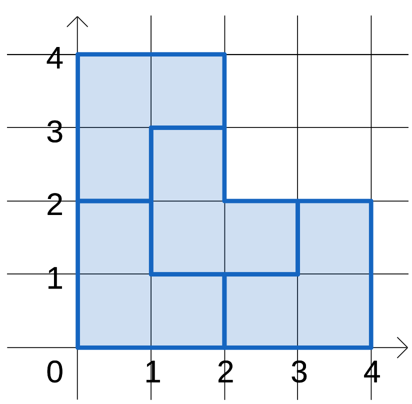
iv. Write a one or two page report that includes

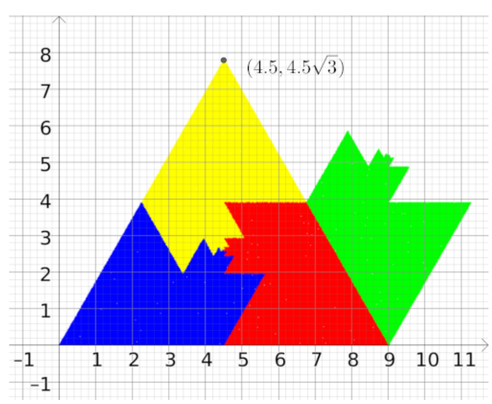
   A. a description of your adventure taking the measurements.

   B. at least one photo of you taking measurements.

   C. your computation of the distance across the pond with an explanation.

(b) Dissecting a rep-tile. Choose one of the following rep-tiles for dissection.





i. For each of the four parts of the rep-tile, find a description that transforms the whole into the part.
ii. Enter your descriptions into the rep-tile designer.
iii. Take a screenshot of the rep-tile designer with your descriptions entered and the dissection level set to 1.

Chapter 3

Foundations of Counting

3.0 Mathematical Outcome

Activity 3.1.1 focuses on developing a method to quickly count large numbers of related objects. This method of counting is called the Fundamental Counting Principle. Before we describe this principle we must first define necessary terminology.

Definition 3.1. Two events are **independent** if the occurrence of a particular outcome for the first event does not make the outcomes in the second event any more or less likely to occur.

For example, if the number of chairs in a house is one event and the number of houses in the street is the other event, then these events are independent. In other words, as long as the number of possible outcomes of event 2 does not change after you choose an outcome from event 1, your events are independent. When making a choice in one event changes the number of possible outcomes in another event, we say these events are dependent.

Definition 3.2. The **Fundamental Counting Principle** is the method of counting the number of ways to sequentially choose x outcomes, each one coming from one of a sequence (a list that has an order) of s independent events. If event i has n_i possible outcomes then the number of different choices of x outcomes is the product $n_1 \times n_2 \times \cdots \times n_x$.

Through the students' work in Activity 3.1.3, they should notice that there are situations where the Fundamental Counting Principle is not applicable. As is suggested in the definition, the Fundamental Counting Principle will work only if the events are independent.

We will refer to the occurrence of choosing a single outcome from each event as an option. There are at least two distinct ways of counting the number of outcomes in each question from Activity 3.1.1. One may most naturally use either multiplication or repeated addition to accomplish this task. Both are acceptable methods. Say one has 2 independent events where event 1 has n_1 options and event 2 has n_2 options. Then there are $n_1 \times n_2$ different outcomes. The other way to find the number of different outcomes is to say you have n_1 options in event 1 for each of the n_2 options in event 2. So there are

$$\underbrace{n_1 + n_1 + \cdots + n_1}_{n_2 \text{ times}}$$

total outcomes. These counting methods are useful for calculating recursions as well as for pattern recognition.

There are many tools that can be used to systematically count the number of options in Activity 3.1. One such tool is the tree diagram seen in Questions 2 and 6 from Activity 3.1.1.

Definition 3.3. A **tree diagram** is a tool used to systematically list all outcomes when choices are being made sequentially.

In this chapter, our trees start with "Customer Arrives" so as to be more understandable to the problem at hand. Then at each subsequent step in the sequence, branches are drawn from each vertex in the previous level, each such branch listing one of the possible options based on the choices made so far.

Figure **??** is one such example of a tree diagram. Trees are a useful tool to form a systematic list of all the outcomes of an event, making it simpler to avoid missing any of the outcomes. Counting using the tree structure can be applied to both independent or dependent events, so we can use trees in all discussions on counting. The only downside is that if you have a lot of outcomes for the first event and a lot of outcomes for the second event, then already the tree may be too big to manage. Even so, trees are excellent method to further students' understanding of the Fundamental Counting Principle.

When counting the number of options in Activity 3.1.3, the Fundamental Counting Principle cannot be used directly because these are dependent events. However, one may use the Fundamental Counting Principle developed in Activity 3.1.1 and then subtract (or add) the unwanted (the missing) options.

While we will not formally develop the idea of permutations and combinations, the arrangement of all the combination numbers into a triangular array such that the n^{th} row (starting with $n = 0$) contains all the choose numbers $\binom{n}{i}$ for all i between 0 and n is accredited to the French mathematician Blaise Pascal (although other mathematicians were using this arrangement well before Pascal). Hence the name Pascal's triangle. This arrangement of numbers allows us to recognize many relationships between the combination numbers.

$$
\begin{array}{c c}
\text{Row number} & \\
0 & 1 \\
1 & 1 \quad 1 \\
2 & 1 \quad 2 \quad 1 \\
3 & 1 \quad 3 \quad 3 \quad 1 \\
4 & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
5 & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

Figure 3.0(a): First six rows of Pascal's triangle

We begin with the first row as the single number $\binom{0}{0} = 0! = 1$. One can simply regard it as a definition; it turns out to be useful in that it fits the purpose in formulas in which it plays a role. Now the second row consists of the two numbers: $\binom{1}{0} = 1$ (there is only one way to choose no objects) and $\binom{1}{1} = 1$. We continue this indefinitely placing in the n^{th} row the numbers $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ from left to right as in Figure 3.0(a).

In this arrangement it is easy to see the symmetry of the triangle. If we draw a vertical line down the center of the triangle we see that each number to the left of the line also appears on the right of the line. To explain why this is true we asks the students to think about $\binom{n}{r}$ versus $\binom{n}{n-r}$ in Question 4 of Activity 3.2.3. There are many other patterns mathematicians study inside Pascal's triangle but the depth of that study is beyond the scope of this book.

3.1 Activity: Ice Cream Parlor

3.1.1 Entrance Activity: Jack's Ice Cream Parlor

Jack's famous ice cream shack serves only two flavors (vanilla and chocolate) and offers only one cone type (sugar). Jack's ice cream shack was suddenly inundated with customers, so to save time a customer must choose either one scoop or two scoops of a single flavor of ice cream. (So no mixing of flavors is allowed. This restriction applies throughout all of Activity 3.1.1 and 3.1.2.)

1. A family of five enters Jack's shop. All five hope to place different orders. Is this possible? Find a systematic way to list all the different possible orders.

2. Jack suddenly becomes generous and allows customers to choose up to three scoops of a single flavor of ice cream. How many different orders can be made? Find a systematic way to list all the different orders.

3.1.2 Activity: No Mixing Flavors Allowed

Jack's famous ice cream shack served only two flavors (vanilla and chocolate) and offers only one cone type (sugar). To expand costumer choice, he add a third flavor of strawberry ice cream. Customer must choose either one scoop or two scoops of a single flavor of ice cream. (So no mixing of flavors is allowed. This restriction applies throughout all of Activity 3.1.1 and 3.1.2.)

1. If Jack allows up to two scoops of one of the three flavors (no mixing of flavors on a cone is allowed!), how many different orders can be made?

2. Jack decided to always have up to 2 scoops and just one type of cone, but made more flavors available. Complete the table below by finding the number of different possible orders in each case. Mixing of flavors on the cone is still not allowed. (You already know the answer for the first two rows.)

Maximum number of scoops	Number of flavors available	Number of types of cones	Number of different orders
2	2	1	
2	3	1	
2	4	1	
2	5	1	
2	f	1	

3. Now suppose Jack allows up to 3 scoops but still just one type of cone. Complete the table below by finding the number of different orders for the various numbers of flavors; mixing of flavors on the cone is not allowed. (You already know the answer for the first row.)

Maximum number of scoops	Number of flavors available	Number of types of cones	Number of different orders
3	2	1	
3	3	1	
3	4	1	
3	5	1	
3	f	1	

4. Jack allows up to 4 scoops of ice cream of one of the f available flavors. How many different orders can be made?

5. In general, how many different orders are there when Jack allows up to s scoops and f flavors with sugar cones being the only receptacle available to put the ice cream in?

6. Jack opens his emergency pack of waffle cones. If customers are allowed to choose from 3 flavors (no mixing of flavors on a cone is allowed!), up to 2 scoops, and 2 cone types available, how many different orders can be placed? Develop a systematic list of all your ice cream orders.

7. Complete the following table as in Question 2.

Maximum number of scoops	Number of flavors available	Number of types of cones	Number of different orders
2	5	2	
3	5	2	
3	10	2	
3	f	2	
4	f	2	
s	f	2	
s	f	3	
s	f	4	
s	f	c	

8. Suddenly, a delivery truck arrives with a surplus of ice cream toppings. There are f flavors, up to s scoops, c types of cones, and t toppings available. How many different orders are available at Jack's shack?

With 2 flavors (chocolate and vanilla), up to 5 scoops, and 2 cone types (sugar and waffle) available, Jack realizes that a sugar cone cannot handle the weight of the 4^{th} or 5^{th} scoops.

9. So he insists that if you order 4 or 5 scoops then you must choose the waffle cone. How many different possible ice cream orders are there? Systematically list them all. How do you know that you have them all?

10. Even with 3 scoops, the sugar cone is unstable. Now if you ask for 3, 4, or 5 scoops then you must choose a waffle cone. What are all the possible ice cream orders now?

11. How are the answers to Questions 9 and 10 different to the entry of the first row in Question 7? How are they the same?

12. Jack brings out a 3^{rd} cone choice: cake cones. Cake cones can hold up to 3 scoops, sugar cones can hold up to 2 scoops and waffle cones can hold up to 5 scoops. With 2 flavors available, how many possible orders are there?

3.1.3 Activity: Mixing Flavors Is Allowed

Customers are tired of all the rules that Jack is enforcing, so they head toward Jill's ice cream parlor. Jill allows customers to have multiple scoops where each scoop can be any flavor no matter whether it is the same or different from the previous one. Jill also places no restriction on the number of scoops allowed in each type of cone.

13. With 2 flavor choices, up to 3 scoops, and 2 cone types available, how many different ice cream orders could you possibly have?

14. Jill now allows up to 4 scoops with 2 flavors and 2 cone types. How many different orders are there?

15. Jill now allows up to 5 scoops with 2 flavors and 2 cone types. Find how many orders are possible. Can you describe how you would find the number of orders possible with n scoops?

16. Suppose that Jill considers an order of chocolate on vanilla to be different from an order with vanilla on chocolate. Find the number of possible orders you can make when Jill allows up to 2 cone types, up to 4 scoops, and 3 different flavors available.

17. Suppose that Jill considers an order of chocolate on vanilla to be different from an order with vanilla on chocolate. Can you describe a method for finding the number of possible orders with up to s scoops, f flavors, and 2 cone types available?

Jack and Jill join together to set up the ice cream emporium on top of a near by hill. Each sees that some of the other's rules have merit. They decide that customers are now allowed to have a different flavor for each scoop, sugar cones can hold up to 2 scoops, cake cones can hold up to 3 scoops, and waffle cones can hold up to 5 scoops. They are willing to let you decide whether or not chocolate on vanilla should be considered different from vanilla on chocolate. (Keen young minds may want to consider both options!)

18. How many possible orders can be made with 2 flavors, up to 5 scoops, and only waffle and sugar cones allowed.

19. Suppose there are 2 flavors, up to 5 scoops, and we can choose a waffle cone, sugar cone, or cake cone for our cone type. How many possible orders are there?

20. Jack and Jill go up the hill to their ice cream emporium and find a delivery of a third flavor: strawberry ice cream. With the 3 flavors, up to 5 scoops, and 3 cone types available, how many possible orders are there?

3.1.4 Activity: Which ice cream is different?

Jill allows customers to have multiple scoops where each scoop can be the same or different from the other scoops but she considers an order with chocolate on top of vanilla to be the same as an order with vanilla on chocolate.

21. If Jill has only 1 cone type, up to two scoops, and 3 flavors are available to her customers, how many possible orders are there?

22. If Jill has only 1 cone type, up to 3 scoops, and 3 flavors are available to her customers, how many possible order are there?

3.1.5 Exit Activity: Area Codes

Originally area codes were made up of three numbers (0 through 9) where the middle digit was either 0 or 1 and the first digit was non-zero.

1. In this way, how many different area codes were possible?

2. To avoid emergency contact numbers such as 911 and 411 no area code is allowed to end with consecutive 1's. How many different area codes were there if all area codes that end in 11 were forbidden?

3.1.6 Extension Activity: A Chance of Something Tasty

In discrete settings in which all outcomes are equally likely to occur, probability calculations rely on being able to count well. The probability of a certain event E occurring is simply the number of outcomes that satisfy the event E divided by the total number of possible outcomes. So our efforts to consolidate counting techniques will stand us in good stead as the following questions are addressed.

Jill's ice cream parlor allows customers to have multiple scoops where each scoop can be the same or different to the previous flavor choices. Jill also places no restriction on the number of scoops allowed in each type of cone.

Any order can have:

(i) up to 3 scoops, with 2 flavor choices (vanilla and chocolate), and 2 choices of cone type (waffle and sugar).

Also, it is important to know that:

(ii) at this time, Jill considers that having chosen the cone type, putting vanilla on top of chocolate in the cone is the **same** as putting chocolate on top of vanilla (so it is just the number of scoops of each flavor that matters).

Late one night, Jill wonders what the probability is for a particular type of order to be requested, assuming that:

(iii) each order of ice cream has equal probability of occurring.

1. What is the probability that a customer orders exactly 2 scoops, both being vanilla, on a sugar cone?
2. What is the probability that a customer orders precisely 2 scoops, both being vanilla, on any cone?
3. What is the probability that the customers' order includes 2 scoops of vanilla ice cream on a sugar cone?
4. What is the probability that a customer orders exactly 2 scoops of any flavors of ice cream on any cone type?
5. What is the probability that a customer orders at most 2 scoops of any flavors of ice cream on any cone type?

3.2 Activity: Mario's Charity Raffle

3.2.1 Entrance Activity

A local charity fund raiser has decided to run a raffle. Five balls are placed in an opaque box (the five balls are marked 1, 2, 3, 4, and 5 respectively). When the raffle begins, one or more balls will be chosen and announced. Currently the raffle consists of choosing one ball and then choosing a second ball *without* replacing the first ball in the box. Each participant donates to the charity, then writes down which ball they think will be drawn first and which will be drawn second. The judge of the raffle, Mario, informs everyone that the order in which the balls are drawn matters. As an example, if you write "$(4, 1)$" or "4 will be drawn first and 1 will be drawn second", then you win if that is what happens (but, for example, you would **not** win if the first ball drawn is a 1 and the second ball is a 4).

1. How many different ways are there for two balls to be chosen in the manner described above?

2. Lots of people started winning at the raffle, so Mario decides to make it more difficult for people to win. He starts the raffle over and increases the number of balls to eight (the eight balls are marked 1, 2, 3, 4, 5, 6, 7, and 8). Mario still draws two balls to determine the winners. Is it harder for participants to win? Explain your answer.

3.2.2 Activity: Raffles

You want to be prepared in case Mario pulls a fast one on the participants again. So you and your buddies decide to determine how likely you are to win, no matter what number of balls are in the opaque box.

1. If there are 15 balls to choose from, how many possible balls are you able to choose during the first draw? How many balls remain after you remove the first chosen ball? So how many different ways are there to choose two balls (where the order in which they are chosen is important)?

2. Say we have 20 balls. So how many different ways are there to choose two balls (where the order in which they are chosen is important)?

3. How many balls are available to be chosen for the first draw given n balls? How many balls remain? Explain your answer.

4. How many different ways can 2 balls be chosen if Mario starts with n balls? Explain your answer.

5. If you would like there to be at least 1000 different ways of choosing two balls, what would be the fewest number of balls you would need in the opaque box? Explain how you reached your answer.

6. If you would like there to be at least 1000 different ways of drawing n balls, what would be the fewest number of draws you would need if there were 7 balls in the opaque box?

7. How would your answer change in Questions 4 and 5 if the first ball drawn is replaced before the second is drawn?

Mario decides to be far sneakier than before. He will draw more than 2 balls during the raffle. Each time he draws a ball, he will set it aside before drawing the next ball.

8. If Mario starts with 10 balls, how many balls remain after the 2^{nd} draw? So how many different ways are there to draw three balls (where the order in which they are chosen is important)?

9. If Mario only has 10 balls, how many will he need to draw to make sure there are at least 1000 ways to choose the balls?

10. If Mario starts with 10 balls, how many balls do you have to choose from on the 10^{th} draw?

11. If Mario starts with 10 balls, how many balls do you have to choose from on the 11^{th} draw? So how many ways are there to have 11 draws with 10 balls?

12. You are given a box in which the number of balls is given in the first column of the following table. The second column lists how many balls you have to draw, the order of which they are drawn being important. In column 3, list the number of different ways the balls can be drawn.

Number of balls in the box	Number of balls to be drawn	Number of different ways
5	3	
6	3	
n	3	
5	4	
5	5	
5	6	
n	m (with $m \leq n$)	

3.2.3 Extension Activity: Selecting a Jury

Entrance Activity

Imagine you are working in a small business with four people (say Alice, Bob, Charlie, and you) and many of the projects your business needs to complete require partners to work together.

1. How many different partners could you possible work with on a project? List all the partnerships you can be apart of.

2. How many different partners could Alice work with on a project? List all the pairs of people containing Alice?

3. Do you see any pairs that show up in questions 1 and 2?

4. List all the pairs of people that exist at your work.

3.2.4 Jury Duty

The small town of Spoonerville has experienced its first felony. The deputies' office has apprehended a suspect and Mayor Catie wants him to stand trial in their very own courtroom. She has gathered 7 citizens for jury selection: **A**lice, **B**ob, **C**harlie, **D**oug, **E**rika, **F**red, and **G**eorge. Of those 7, the mayor needs to pick 3 to sit on the jury. To make the selection process fair, for each possible combination of 3 people selected from these 7 citizens in turn, she writes the names of the 3 people on a sheet of paper. She places all the sheets of paper in a box and then selects one piece of paper. The people with names on the selected piece of paper become the jury. Your job is to determine how many pieces of paper she will need so that all possible juries are placed into the box.

1. Describe several methods to make a list of the possible juries. Make sure your methods do not miss nor duplicate any juries!

2. Which of your lists from Question 1 is organized in a way that is easy to count? If none, come up with a way to organize the list so it is easy to count.

3. Is each jury represented through the organization process given in Question 2? Explain your results. List all the possibilities, choosing an approach that makes it clear that each 3-person jury is included exactly once. How many juries did you find?

4. If the mayor had to choose 4 people for the jury from the 7 people, how many juries could be formed? Is there some relationship between your answer to this question and the number of possibilities you gave in Question 3?

5. For the future, we would like to know how many possible juries there are when picking any number of jurors from 7 people. So if we have a pool of 7 people from which to choose, fill in the table below.

Number of people in jury	Number of different juries
1	
2	
3	
4	
5	
6	
7	

Mayor Catie was thinking about the number of 3-person juries she could select from the pool of 7 people: **A**lice, **B**ob, **C**harlie, **D**oug, **E**rika, **F**red, and **G**eorge. She really wanted to make use of her preference for using permutations whenever possible. After all, sitting next to the window in the jury box might be a desirable seat!

6. What would happen to the number of different juries if the jury selected is to be seated in a certain order? For example, if Alice sits in the left seat, Bob sits in the middle seat and Charlie sits in the right seat, it is a different jury to if Bob sits in the left seat, Alice sits in the middle seat and Charlie sits in the right seat. How many different *ordered* jury seatings are there if 3 jurors out of 7 people are to be seated?

7. Suppose Alice, Bob, and Charlie are selected to sit on the jury. How many ways are there to seat these 3 jurors if the order in which they are seated matters?

8. Does the number of seated juries found in Question 7 change if we have a different set of three jurors? Explain your answer.

9. Return to Question 3 where we found the number of juries of 3 people that Mayor Catie can choose from 7 people when the order in which they sit does *not* matter. How can you use the answers to Questions 6, 7, and 8 above to find this number? (*Hint*: Think about how many times each unordered jury is counted in your answer to Question 6.)

Let us try to determine what could happen in any situation. So assume that the mayor needs to select a jury of r people from a population of n people. The mayor does not care where they sit, but at first we will! So assume that we have r seats in a row, numbered 1 to r from left to right. We will start by seeing how many seating arrangements we can have, then group them by which seating arrangements have selected the same group of jurors.

10. For the first seat how many people are available to be chosen to sit in this seat?

11. After we have chosen our first person, how many people are available to be chosen for our second seat?

12. So for the i^{th} chair, how many people are available to be chosen?

13. Can you use the approach used in Questions 7 and 8 to count the total number of options to seat r jurors in precisely r seats where the order in which they sit does matter?

14. Can you use the answers to Questions 12 and 13 to count the number of juries with r people that can be chosen from a group of n people where the order in which they sit is irrelevant?

15. Can you devise a way which accurately gives the number of possible juries without listing all combinations? Test your hypothesis on finding the number of combinations selecting 3 jurors from 8 people.

3.3 Shipping Candy

3.3.1 Entrance Activity: It's time for a break!

A KitKat package comes with four individual wafer bars as shown in the image above.

1. How many wafer bars are there in 4 KitKat packages?

2. If you have 97 wafer bars, how many **complete** KitKat packages could you have?

3.3.2 Activity: How many can we pack in a Shipment?

You are charged with filling and invoicing store shipment orders of KitKat to individual convenience stores across the country. For some reason the website for purchasing is set up to request the number of wafers a store wishes to purchase.

1. You cannot ship individual wafers as each must be sealed into a package, so which of the following stores are you able to fill an order for?
   (a) Store A has ordered 368 wafers
   (b) Store B has ordered 528 wafers
   (c) store C has ordered $12,346$ wafers
2. Can you describe an easy way to determine if the order can be filled?

To invoice the stores for shipping you need to figure out the total cost to your company. Obviously buying in bulk will be helpful to stores but packaging and shipping varies depending on the exact order as described below.

- A single package of KitKats cost \$0.72 to package and ship.
- A sleeve of KitKats consists of four packages and costs \$1.54 to package and ship.
- A carton of KitKats consists of four sleeves and costs \$3.18 to package and ship.
- A box of KitKats consists of four cartons and costs \$6.46 to package and ship.
- A crate of KitKats consists of four boxes and costs \$13.02 to package and ship.

For example, if a store ordered 20 wafer bars, that would be packaged and shipped in the containers of one sleeve and one package to minimize cost. The total cost for shipping this way is \$1.54 + \$.72 = \$2.26. Your job is to make sure your orders are shipping for the lowest cost.

3. How many wafers are in 2 cartons and 3 sleeves?

4. How many wafers are in 1 crate?

5. How many wafers are in 2 boxes, 3 cartons, 1 sleeve, and 1 package? How much does it cost to ship these wafers?

6. In what packaging containers would you ship 56 wafer bars? How much would that cost? Remember, you are trying to ship them at minimum cost.

7. In what packaging containers would you ship 156 wafer bars? How much would that cost?

8. Al owns three convenience stores and realizes he can save money if he has all three of his orders combined into one. He calls to ask this after you have calculated each of his shipments, which are each listed below. Can you figure out his combined order without counting the number of individual wafers ordered?

   - Store 1: 1 box, 2 cartons, 3 sleeves and 1 package.
   - Store 2: 3 cartons and 3 packages.
   - Store 3: 2 boxes, 2 cartons, and 2 sleeves.

You have obviously impressed your bosses with your money saving acumen. You have been promoted to manager in charge of all candy distribution!

Hershey bars are packaged in groups of 12 and shipped by the following requirements:

- A single package of Hershey cost $0.12 to package and ship.
- A box of Hershey consists of twelve packages and costs $1.30 to package and ship.
- A carton of Hershey consists of twelve boxes and costs $12.50 to package and ship.
- A crate of Hershey consists of twelve cartons and costs $140.75 to package and ship.

9. How many Hershey pieces are in 1 crate?

10. How many Hershey pieces are in 2 cartons, 11 boxes, and 5 packages? How much does it cost to ship these pieces?

11. In what packaging containers would you ship 1,020 Hershey pieces? How much would that cost?

12. In what packaging containers would you ship 3,744 Hershey pieces? How much would that cost?

13. Al calls to combine his three orders. Without calculating the total number of individual pieces or packages ordered what is the total shipment packaging and what does it cost?

    - Store 1: 10 cartons, 8 boxes and 11 packages.
    - Store 2: 5 cartons and 9 packages.
    - Store 3: 2 crates, 7 cartons, and 2 boxes.

3.3.3 Exit Activity: Caramello Bars

As you can see Caramellos are packaged in groups of 6. They are shipped by the following requirements:

- A single package of Caramellos cost \$0.15 to package and ship.
- A sleeve of Caramellos consists of six packages and costs \$0.85 to package and ship.
- A box of Caramellos consists of six sleeves and costs \$4.50 to package and ship.
- A crate of Caramellos consists of six boxes and costs \$24.75 to package and ship.

1. How many Caramellos are there in 5 boxes, 4 sleeves, and 3 packages?

2. In what packaging containers would you ship 270 Caramellos? How much would that cost?

For sake of efficiency, the candy shipping company uses an abbreviation for each shipment. They simply list the number of crates, boxes, sleeves, packages, and individual Caramellos to be sent using a sequence of digits, in that order. For example, a shipment of 5 crates, 3 boxes, and 4 packages would be abbreviated as 53040. Notice the third digit is 0 since zero sleeves are being sent, and there will always be a 0 at the end since, at this time, individual Caramellos cannot be shipped.

3. How many individual Caramellos, packages, sleeves, boxes, and crates are sent in the order abbreviated by 20450?

4. How many Caramellos, in total, are sent in the order abbreviated by 20450?

5. What would be the abbreviation for the least expensive shipment of 1000 Caramellos?

3.4 Counting Rhythm

3.4.1 Entrance Activity: Writing Rhythm

Music is the beautiful embodiment of patterns in sound. While there are enough connections between mathematics and music to create an entire course, we will explore just one connection by looking at rhythm.

Listen to the clip at https://www.youtube.com/watch?v=SnjqxgtLJlA. In the background of the audio you hear a small ticking that is keeping track of “time" or “counting time." The first four ticks are just meant as introduction as the music starts on the fifth tick. Find a way to describe on which ticks the musician plays his instrument and which he does not. Use this method to describe the entire sequence of rhythm played in the video.

3.4.2 Activity: Exploration of Rhythm

Begin by discussing how each member of your group denoted the rhythm played in the video.

1. Come to a consensus in your group for how you will write down a rhythm and re-write the rhythm from the video here. What symbols will you use?

2. Does the rhythm repeat itself, and if so how long does it take to begin repeating? Write down the shortest length of rhythm that is then continuously repeated? (Hint: professionals would say this rhythm is an 8-count (has 8 beats).

We want to know how many different rhythms exist, but first we have to understand when two rhythms will be different.

3. Can you list a different rhythm with 5 strikes (and 3 rests) within an 8-count?

4. John plays three "rounds" of your rhythm. So write the rhythm as 24 counts by repeating the eight count rhythm twice.

5. Imagine someone didn't hear the first three beats of John playing your rhythm. What might they say was the eight count rhythm? Is this the same rhythm or different?

3.4.3 Activity: Finding Your Rhythm

Let's see if we can't find all the 8-count rhythms that exist. This may be a bit difficult, but we can at least try! As we have done before, let's start by finding all the rhythms in the world on a small number of counts and build up from there.

1. List all the rhythms with 3 strikes (and 1 rest) on four counts. Think about how you can organize this list. You may find using a tree diagram helpful.

2. List all the rhythms with 2 strikes (and 2 rests) on four counts.

3. Imagine you have listened to the first four counts of a five count rhythm.
   (a) What are the possibilities for the last count?
   (b) If you knew how many strikes were in the rhythm before beginning, would you be able to guess what the last count would be after listening to the first four counts?

4. How can the answers from questions 1-3 help you find all the rhythms on five counts?

In exercise 4 you should have outlined a plan for finding all the rhythms on some number of counts based on a list of the rhythms on one less count. If your thought process/plan was not that general, you may wish to reconsider your answer. We will put your plan into action in the next few exercises.

5. Below is a list of all three count rhythms. Notice rhythms with zero, one, two, and three strikes are all included. All other three count rhythms will sound like one of these. How can you build on these to create the rhythms you listed in question 2 (2 strikes on 4 counts)?

   - - -   ♪ - -   ♪ ♪ -   ♪ ♪ ♪

6. And now back to five count rhythms...

   (a) Make a list of all the different four count rhythms with 1 strike or 2 strikes. You can collect some of this information from previous exercises.

   (b) Use your plan to extend these four count rhythms to list all the five count rhythms with 2 strikes. Remember, we are only looking for rhythms that sound different.

7. Can you use what you have learned so far to fill out the entire table below? Describe the process you are using. Be careful with the six count rhythms. It may take some time to figure all of them out.

	1 strike	2 strikes	3 strikes	4 strikes	5 strikes	6 strikes
1 count						
2 counts						
3 counts						
4 counts						
5 counts						
6 counts						

3.5 Project Choices

1. **Fundamental Counting Principle & Dependent Choices:** While we generally say that the fundamental counting principle can only be used when working with independent "choices" provide an example of a situation (not used in class!) where one can use the fundamental counting principle to count the number of outcomes in a situation that has dependant choices. (Hint: Mario's Charity Raffle is an example.)

2. **License Plates:** Connecticut passenger car license plates come in three standard forms as shown below. The first form contains 3 digits followed by 3 letters; the second contains 2 letters followed by 5 digits; and the third contains a digit followed by 4 letters followed by a second digit.

   (a) Which form provides the most unique license plates?

   (b) Given that all three types of license plates are in use, what is the greatest number of passenger cars that can be registered in Connecticut using standard plates where no two cars have the same plate?

   (c) What is the population of Connecticut according to the 2020 census?

   (d) Are there enough standard plates that everyone in CT can have a different one?

   (e) A standard New York passenger car license plate is formatted with 3 letters followed by 4 digits. Are there more standard passenger car license plates available in NY or in CT?

   (f) Research another state's standard passenger car license plate formats. How many formats are there? What are the formats? How many standard passenger car plates can be made in that state?

3. **When Order Does Not Matter:** Your local ice cream shop offers two types of cones: sugar and waffle. Sugar cones can hold up to two scoops of ice cream; waffle cones can hold up to three scoops of ice cream. Four flavors are available: chocolate, vanilla, strawberry, and rocky road. Assume that the order the scoops that come on the ice cream DOES NOT matter (e.g. two scoops with vanilla on top of chocolate is the same as two scoops with chocolate on top of vanilla).

   (a) Explain why you cannot use the fundamental counting principle to solve this problem.

   (b) Create an effective and organized way to list all the possible orders, and use it to list them.

(c) How many possible orders are there?

4. **Rhythms:**

A music box like the one shown above contains a wheel that rotates to play the music, so after the wheel has made a full revolution it repeats the same music over. Also since the crank determines where the music begins and ends, the music could start and stop at any point along the wheel. Just as we did with the counting rhythm exercises, we are only interested in rhythms that sound different, this time in a music box.

(a) In class we created a list of all four count rhythms with two strikes that sound different. There were two fo them. Whether you did it while completing the table of rhythms or not, list here all the different four count rhythms. Include rhythms with 0, 1, 2, 3, and 4 strikes. The zero-strike music box represents one that is broken!

(b) Find how many different bracelets you can make with four beads if you only have two different colors of beads.

(c) Explain the connection between the four bead bracelets and the four count music boxes. Why are there the same number of each?

(d) Pick a rhythm on 8 counts that you like, and record yourself playing it on your instrument of choice (bells, guitar, drum, clapping, stomping your feet, etc.) three times over (three rounds of it). You can use your phone's sound recorder. Write down the rhythm using the same symbols you used in class and say why you like it.

(e) Extend the table from the "Finding your Rhythm" section to contain rhythms up to 8 counts and 8 strikes. Correct answers for up to 6 counts have already been filled in.

	Strikes									Total
	0	1	2	3	4	5	6	7	8	Rhythms
1 count	1	1								2
2 counts	1	1	1							3
3 counts	1	1	1	1						4
4 counts	1	1	2	1	1					6
5 counts	1	1	2	2	1	1				8
6 counts	1	1	3	4	3	1	1			14
7 counts										
8 counts										

3.6 Additional Exercises

1. The Cuddle Toy factory makes 3 different types of fluffy animals, each with its choices of add-ons: the turtle, the bunny, and the kitten.

   (a) The turtle and bunny can each come with any number of the following: a top hat, a cane, and a monocle. The kitten and turtle have a choice of any number of the following: overalls, scarf, a shirt, and shoes. All three animals are available in one of the following three colors: blue, red, or green. How many different types of fluffy animals are there?

   (b) The color options are still the same for the three fluffy animals. The turtle and bunny still can each come with any number of the following: a top hat, a cane, and a monocle. The kitten and turtle are now only allowed a choice of any number of the following: overalls, bell-bottoms, a shirt, and shoes. How many different types of fluffy animals are there?

2. The Bebop manufacturing company produces license plates consisting of the following in the order given: one lower-case letter, one upper-case letter, two digits (0 through 9), and finally three letters that may be upper- or lower-case. How many different license plates can Bebop make? If I am assigned a plate at random, what is the probability that it starts with "aA1"?

3. Originally area codes were made up of three numbers (0 through 9) where the middle digit was either 0 or 1 and the first digit was non-zero.

   (a) In this way, how many different area codes were possible?

   (b) To avoid emergency contact numbers such as 911 and 411 no area code is allowed to end with consecutive 1's. How many different area codes were there if all area codes that end in 11 were forbidden?

4. Steve rolls three differently colored 6-sided dice. How many different ways can the dice land?

5. Using the three dice in Question 4, you want to know the number of ways the dice can be rolled so that exactly two of the dice show the same number. How many different ways can this happen?

6. A restaurant has 12 appetizers, 13 main courses, and 11 desserts.

   (a) How many ways can you order a three-course meal (you must choose 1 appetizer, 1 main course, and 1 dessert for your meal)?

   (b) What is the probability that my desert is tiramisu (one of the 11 choices for dessert) if I choose my 3 courses randomly?

7. A special license plate consists of 4 characters. Each character is either a letter or a digit (there are 26 letters and 10 digits. How many 4-character license plates can be made where characters can be repeated on each plate?

8. Using the description of a license plate from Question 7, how many 5-character license plates can be made if repetition is allowed?

9. A car company makes cars of 2 different body styles (compact and station wagon), 6 different colors (blue, orange, green, black, red, and white), and 3 model types (standard, all wheel drive, and luxury package).

   (a) How many different types of cars can be built?

   (b) If I choose my car at random from all the possibilities, what is the probability that I get a black station wagon?

   (c) How many different types of cars can be built if the station wagon cannot be made in black?

10. There are 20 true or false questions on a quiz given in your History class. In how many different ways can a student answer the test?

11. There are 10 true or false questions on a quiz given in a History class.

    (a) Consider two students to have answered the test differently if they disagree in at least one of their 10 answers. How many different ways are there to answer the 10 questions on the test?

    (b) If the student just rolls a die to randomly answer each question, what is the probability the student will get 100% on the test (i.e., get all questions correct)?

12. How many 4-digit even numbers use only the numbers 1, 2, 3, 4, and 5?

13. How many 4-digit odd numbers use only the numbers 1, 2, 3, 4, and 5?

14. There are 15 people in a line to get their license at the Department of Motor Vehicles. Alice and Bob are two of the people in this line. In how many ways can the line be formed such that Alice is ahead of Bob?

15. Your local ice cream shop offers 2 types of cones: sugar and waffle. Sugar cones can hold up to 2 scoops of ice cream; waffle cones can hold up to 3 scoops of ice cream. Two flavors are available: chocolate and vanilla. Assume that the order the scoops come on the ice cream DOES NOT matter.

    (a) Draw a tree that lists all the possible orders.

    (b) How many possible orders are there altogether?

    (c) If I select an ice cream order at random, what is the probability that my ice cream is on a waffle cone?

16. Out of 10 ice cream flavors, how many different ways can you choose 9 of them?

17. Out of 200 ice cream flavors, how many different ways can you choose 5 of them?

18. In a student council meeting, there are 10 people and you need to decide who is the chairperson, vice-chairperson, treasurer, and secretary. In how many different ways can this be done?

19. In a student council meeting, there are 10 people and you need to choose 4 people to be on a committee. In how many different ways can this be done?

20. King Arthur and his 9 knights sit at the round table but one of the chairs is reserved for King Arthur. In how many ways can the 9 knights seat themselves at the table?

21. In how many different ways can a deck of 52 cards be arranged so that cards of the same suit are together? (Each of the cards falls into one of 4 suits, and there are 13 cards in each suit.)

22. In how many different ways can you be dealt 5 cards from a deck of 52 cards?

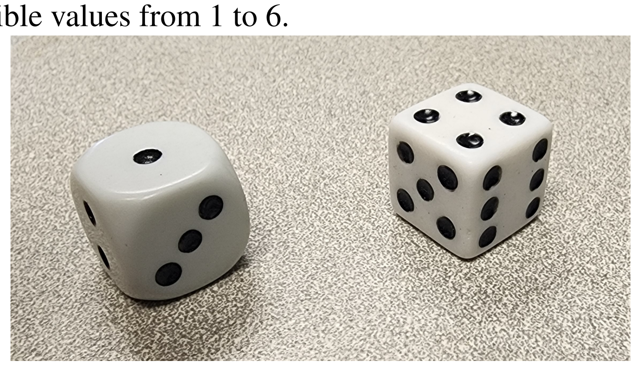
Chapter 4

Polyhedra

4.1 Funny Dice

4.1.1 Entrance Activity: Dice

Dice are certainly familiar items from the world around us. There are games that can be played with dice alone, and there are many games that use dice to add an element of randomness – for instance in Monopoly, each player rolls the dice when it's their turn and that determines how many positions their token moves. A single die is a random number generator with possible values from 1 to 6.



Often (when numbers larger than 6 are required), we use two die. That gets us numbers from 2 to 12, but there's a funny consequence – some numbers are more likely to come up than others.

There are 6 different ways that you can roll a 7 with two die. Can you list them?

die 1	die 2

How many ways can you roll an 11?

So, on average, how many 7's would you see for each 11 if you were rolling the dice a bunch of times?

If we want "dice-like things" (random number generators that give equal weight to the options) we'll need solid geometric objects that are nice and regular, but that have different numbers of sides. Here are a few examples:

There are other options, but a good place to start is with the five Platonic solids. Apparently, Plato (an ancient Greek philosopher) knew about these five, extremely regular, solid objects. The smallest platonic solid is known as a tetrahedron – gamers call it D4.

The pyramids that can be found in Egypt and a few other places around the world have triangular sides, but square bases. Can you imagine what it would look like if a pyramid's base was also a triangle?

You're going to make your own D4 today. The style of construction method we'll use is called *plaiting*. Carefully cut out the two strips from the handout. Pre-crease the edges between triangles, then weave the two strips together. It may seem hard at first, but the pattern is just over-under-over-under. Be sure that the markings are on the outside when you are finished!

Notice that there are 3 edges on each triangle (duh!) and since there are 4 triangles that makes a total of $3 \cdot 4 = 12$ edges. But there are only 6 edges on the D4 we just made. Can you explain why?

4.1.2 Activity: Why Only Five?

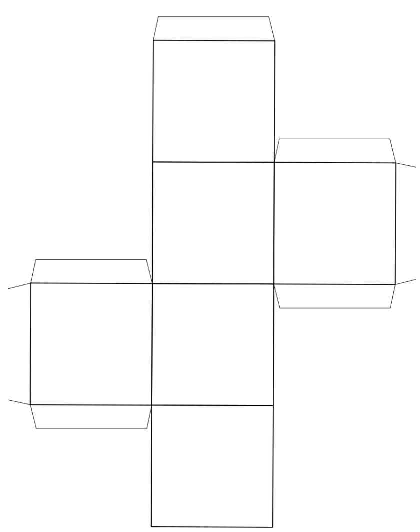
Let's start by thinking a bit about why cubes are the perfect 3-d shape to use for dice. Cubes are made up of squares, but you could make an imperfect cube out of rectangles – think of a brick. Explain why tossing a brick would be a poor choice for generating random numbers.

It seems we should build our "fancy" dice using faces that are **regular** like a square. The regular polygons that can be used are:

1. equilateral triangles
2. squares
3. regular pentagons

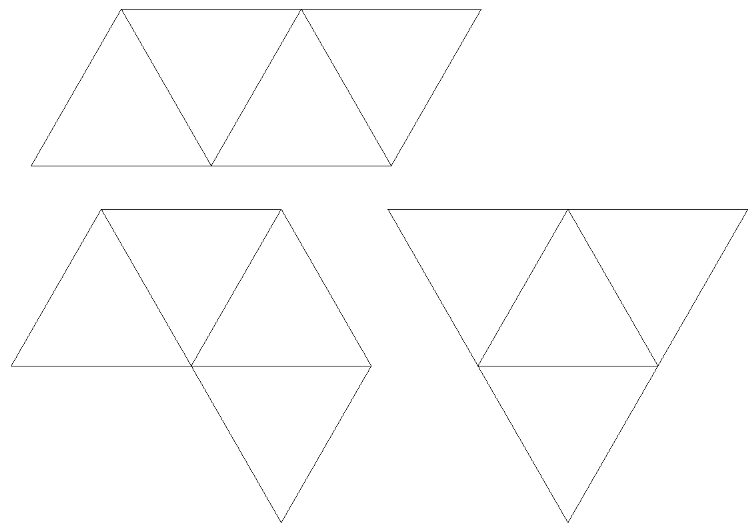
Of course, triangles are the smallest possible polygons, but what about on the big side? Can anyone come up with a reason why we couldn't make a 3-d object with hexagons, heptagons, octagons, etc. as the sides?

A tool we can use to think carefully about polyhedra (that's the generic term for a solid with many sides) is the so-called *net* of the polyhedron. There are generally many different nets for a given polyhedron. Here's one for a cube:



Officially, a net wouldn't include the little tabs that are meant for gluing the model together. But, as you can probably tell from the example, the net of a solid is a flat template that can be folded up to cover the exterior of the solid.

Which of the following is a net for the tetrahedron?



A quick way to rule out a potential net is to count how many polygons will meet at each corner. In a tetrahedron, exactly 3 triangles meet at each corner. Now do you see which one of the above is impossible?

Let's recap for a minute. We're trying to figure out how to make "fancy" dice that will have a different number of sides than a cube does. We know that the faces of our dice will have 3, 4, or 5 sides. Finally, it seems that the number of faces that meet at a corner is important. Some of you may know about D8, D12 and D20, but the only solids we've discovered together so far are D4 and D6 – the tetrahedron and the cube.

Let's collect what we've found so far in a table.

Also, let's (please!) agree to use the following abbreviations:

F = the number of faces on the object.

C = the number of corners on the object.

F/C = the number of faces that meet at a corner.

C/F = the number of corners on each face.

E = the number of edges.

name	F	C	F/C	C/F	E
tetrahedron					
cube					

Near the beginning of this activity we asked, "Can anyone come up with a reason why we couldn't make a 3-d object with hexagons, heptagons, octagons, etc. as the sides?" There are two facts that explain the situation.

Fact 1: You have to have 3 or more faces meeting at a corner.
(What would it look like if only two faces (of whatever type (sorry about the nested parentheses)) met at a corner?)

Fact 2: The sum of the angles on the faces that meet at a corner must be less than $360°$

Fact 2 might best be illustrated by an example. What would it look like, if instead of having 3 squares meet at a corner, we tried to get 4 squares to meet at a corner. (Hint: look down.)

The angle on the corner of a hexagon is $120°$ and if 3 of them met at a corner, we'd get a tiling of the plane – not a polyhedron of any sort. Here's a hard question: what is the angle found on the corner of a regular pentagon, and how many regular pentagons could meet at the corners of a polyhedron?

Alright then! If a polyhedron has squares as its faces there must be _______ of them meeting at a corner, and that gets us a cube! If a polyhedron has regular pentagons as its faces there must also be _______ of them meeting at a corner, and that beast is called the dodecahedron. We'll come back to that!

What are the possible values of F/C if the faces are triangles?

Here's a bit more of the table we started previously.

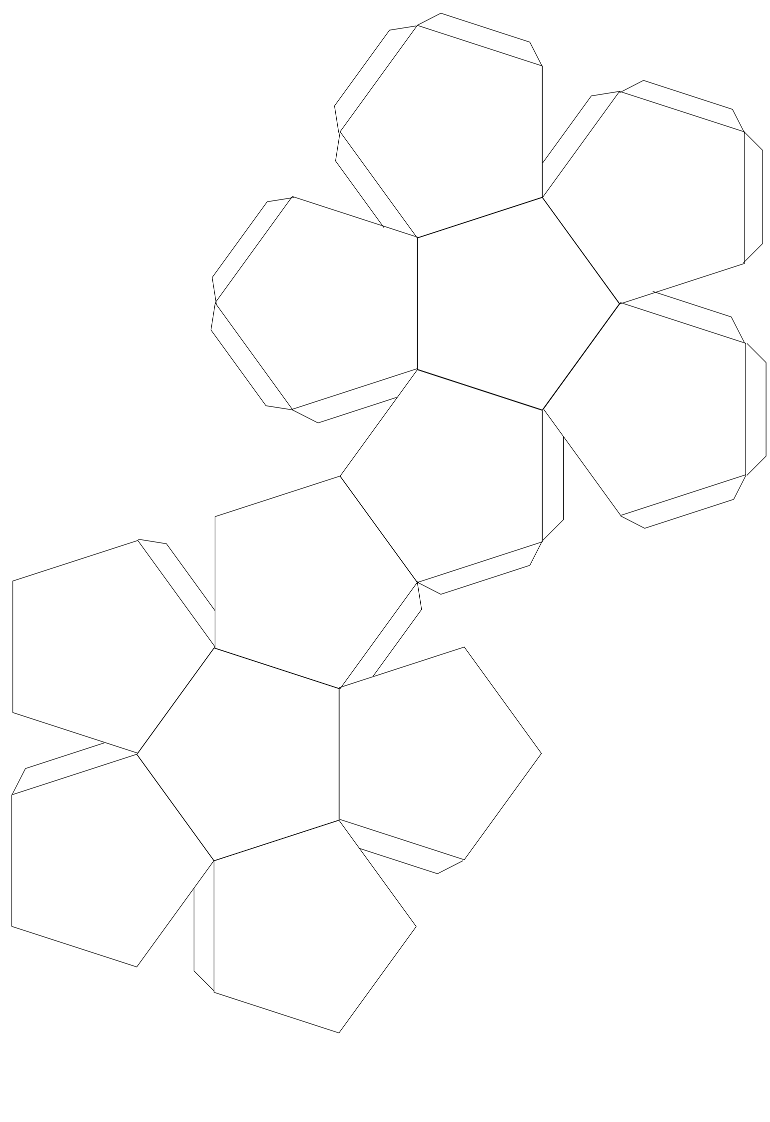
name	F	C	F/C	C/F	E
tetrahedron	4	4	3	3	6
cube	6	8	3	4	12
octahedron			4	3	

The new object in the row that's been added is called the *octahedron* and the entries that are pre-filled tell us that it's made of triangles with 4 of them meeting at a corner. A little thinking about etymology will probably let you guess how many faces it has. Rather than building a model, we'll just try to get you to visualize this solid in your mind's eye. Imagine taking two (Egyptian-style, square based) pyramids and gluing them together along their square bases – the resulting thing would only have triangles (8 of them!) visible on the outside and there would be 4 meeting at each corner.

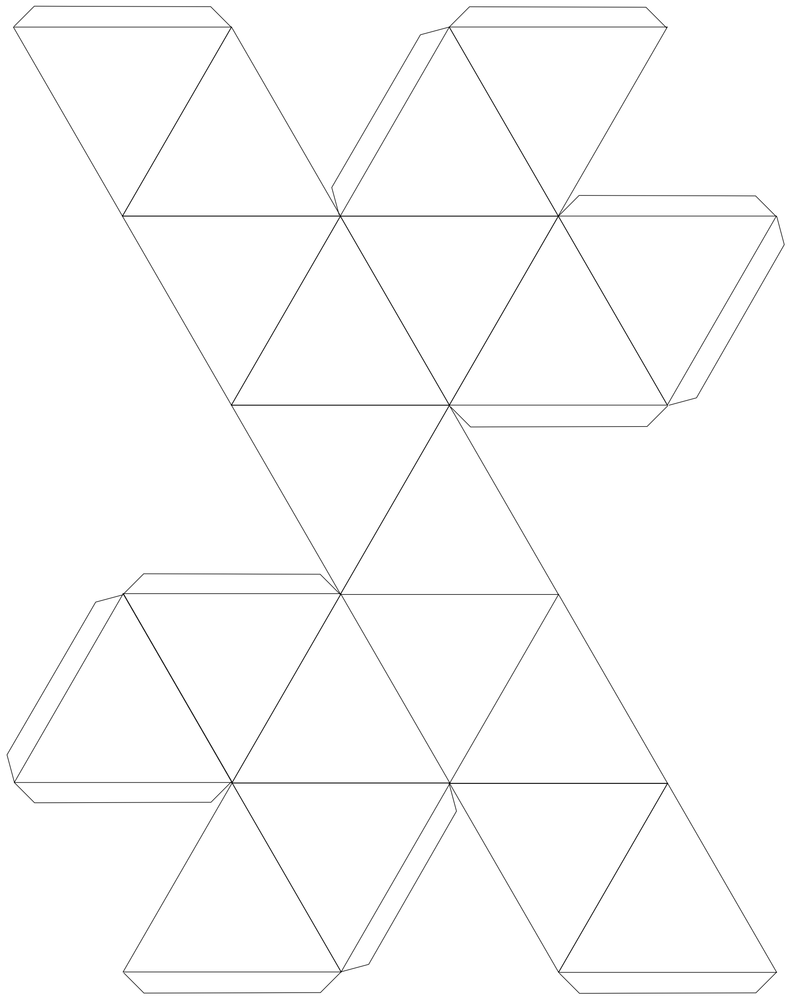
With that visualization you should now be able to fill out the remaining entries in the octahedron row.

There are two remaining objects in this list of the 5 Platonic solids we're trying to build. The *dodecahedron* (aka D12) and the *icosahedron* (aka D20). One of them is built of 5-sided polygons meeting 3 at a corner. The other is made of 3-sided polygons meeting 5 at a corner. Isn't that weird switching-around of 3's and 5's interesting?

Here's a net for the dodecahedron:



This is a net for the icosahedron:



You guessed it! It's time to completely finish the table!

name	F	C	F/C	C/F	E
tetrahedron	4	4	3	3	6
cube	6	8	3	4	12
octahedron	8	6	4	3	12
dodecahedron					
icosahedron					

Look carefully at your completed table. Search for patterns. Remember that weird switching around of 3's and 5's? Mathematicians say that "the tetrahedron is self-dual and the other platonic solids can be divided into pairs that are dual to one another." Any idea what they're talking about?

4.1.3 Exit Slip

1. What would you guess the Greek prefix *icosa* means?

2. The word "dozen" literally means $2+10$. How is this silly fact relevant to the naming of Platonic solids?

3. Let's try some multi-counting. Every one of the pentagons on a dodecahedron has 5 corners, and since there are 12 pentagonal faces that makes for $12 \cdot 5 = 60$ corners. By what factor did we over-count the corners? _______

   How many corners are actually on a dodecahedron? _______

4. Every face of a cube has 4 corners . Since there are 6 faces on a cube, we get $6 \cdot 4 = 24$ corners. By what factor did we over-count the corners? _______

   How many corners are actually on a cube? _______

4.1.4 Duals

Look at the final table in the previous activity, and notice that there are some weird co-incidences in the numbers. For example, the dodecahedron and the icosahedron have the number of faces and the number of corners interchanged. These coincidences lead to the idea of *duality*.

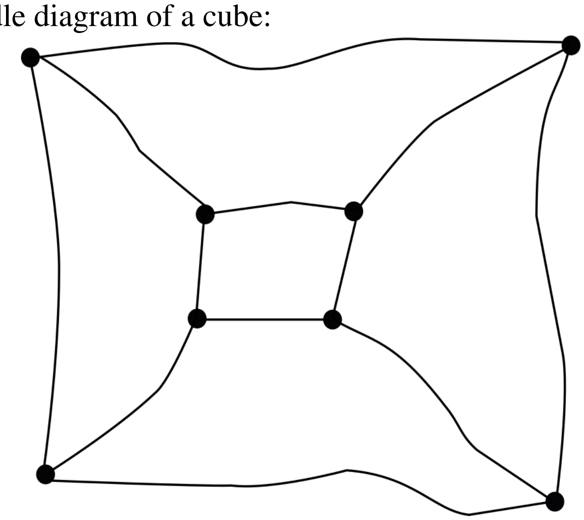
Which pairs of Platonic solids are dual to one another?

Why is the tetrahedron called self-dual?

You can make drawings of the platonic solids that lie flat in the plane, using some ideas from the mathematical area known as Topology. (Topology means the study of shape. Topologists study the things that remain the same even when an object is deformed somewhat.)

Imagine making a Platonic solid out of spaghetti noodles (maybe with meatballs holding the corners together). Then (very carefully) cook your solid until the noodles get noodly. You should now be able to lay your "solid" out flat in the plane in such a way that no noodles cross each other.

Here's a wet noodle diagram of a cube:

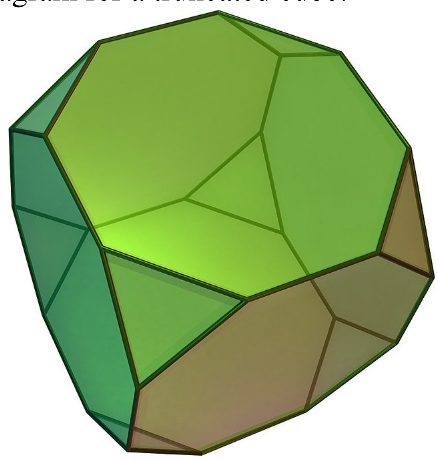


1. Make wet noodle diagrams of all 5 platonic solids.

2. The corners of your Platonic solid are the points (meatballs) in the wet noodle diagram. The edges of the solid are the noodles. The faces of the Platonic solid have become regions in the plane surrounded/bordered by noodles. Count the regions in each of your 5 noodle diagrams.

3. What happened to the missing faces?

4. There is a process known as *truncation* that essentially means "cut off the corners." Draw a wet noodle diagram for a truncated cube:



5. Now try making a wet noodle diagram for a truncated tetrahedron.

6. Try a wet noodle diagram of a 5-sided pyramid.

7. Let's do some counting! How many regions, edges and vertices (the fancy way to say 'corners') are there in each of your diagrams?

   Count the region outside your diagrams too when figuring out R.

name	R	E	V
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			
truncated cube			
truncated tetrahedron			
pentagonal pyramid			

8. Make a fairly complicated, but random wet noodle diagram – it doesn't need to actually come from something solid. Find R, E, and V for your diagram. What does $R - E + V$ equal?

9. For each of your Platonic solid noodle diagrams, draw a point somewhere in each region (don't forget to put a point on the outside!) Connect points with a noodle if the corresponding regions are separated by a noodle. It would be a good idea to make the original and the new diagram you're making be in different colors. (BTW, this process is called *dualizing*).

4.1.5 Exit Slip

1. Draw the WND and the dualized WND for a four-sided pyramid.

2. Fill in the blanks:

   The dualized wet noodle diagram for the tetrahedron is a wet noodle diagram for a __________________________

   The dualized wet noodle diagram for the cube is a wet noodle diagram for a(n) __________________________.

   The dualized wet noodle diagram for the dodecahedron is a wet noodle diagram for a(n) __________________________

3. Verify the identity $R - E + V = 2$ for a soccer ball.

4.1.6 Project Choices

1. Make all five Platonic solids.

   You can make these models in many different ways: 3-d printing, plaited models (like the D4 we started this section with), paper nets can be assembled with glue and tabs, wire and wire nuts, pipecleaners, clay (or Play Doh), straws and paper clips, wood, solid gold or platinum, etc.

2. Make 5 models that show the stages of a cube being transformed into an octahedron (or vice versa) via truncation.

3. Make a calendar - a plaited model of a rhombic dodecahedron marked with a calendar for 2023 is available.

4. The "wet noodle diagrams" we talked about are actually known as Planar Graphs. Do some research and give a presentation on the Königsberg Bridge Problem and how this strange recreational problem lead to the area now known as Graph Theory.

5. A flat plane and a sphere are hard to tell apart (ask any member of the Flat Earth Society). The number we calculated R-E+V (which is usually rendered as V-E+F) is always 2 when your drawing is done on a plane or a sphere. This is known as the Euler characteristic. Demonstrate to your classmates that V-E+F need not be 2 if you make a drawing on the surface of a doughnut!