Lab: 9
Finding Solutions in Sage

Solving equations

(Note: follow along with these initial computations then try the problems.)

We will investigate functions of the form $f(x) = ax^2 + bx + c$. To begin with, we will consider when $a = 1$, $b = 0$, and $c = 0$, i.e. $f(x) = x^2$.

Start by defining $f(x) = x^2$ in Sage.

```
f(x)=x^2
```

To solve the equation $f(x) = 1$ (i.e. $x^2 = 1$) in Sage, we can use the **solve** command:

```
solve(f(x)==1, x)
```

This will solve the equation $f(x) = 1$, solving for the variable x. Note that when solving, we need to put two equals signs to denote equality. This is because Sage will interpret a single equals sign as an assignment (i.e. we would be defining $f(x)$ to be the function that is always equal to 1).

The output we obtain is

```
[x == -1, x == 1]
```

These are the two solutions that we expect of $x = -1$ and $x = 1$.

Exercises

1. Try to solve the equation $f(x) = 4$ using Sage.
2. Solve the equation $f(x) = 3$ using Sage.
3. Solve the equation $f(x) = 0$ using Sage.
4. Do you expect for there to be any solutions for $f(x) = -9$? Try it in Sage and see what happens.
5. For what values of h does $f(x) = h$ have two real solutions? Only 1 solution? Two imaginary solutions?
6. Define a new function $g(x) = x^2 + 10x + 21$. Let's try two things:

(a) Use the `factor()` member function to see how $g(x)$ factors.

(b) Use the `solve()` command to see when $g(x)$ is equal to zero.

Can you explain the minus signs?

7. Do the things `factor()` and `solve()` in the previous problem for

   (a) $g(x) = x^2 + 10x + 22$

   (b) $g(x) = x^2 + 10x + 23$

   (c) $g(x) = x^2 + 10x + 24$

   (d) $g(x) = x^2 + 10x + 25$

   (e) $g(x) = x^2 + 10x + 26$

   Sometimes Sage factors the polynomial into linear factors.
   Sometimes Sage is refusing to factor because the zeros are not nice numbers.
   Once, Sage refuses to factor because things have gotten truly weird.

   Identify which is which? What do you suppose `I - 5` means?

Graphing functions

(Again, follow along with the first few computations, then try the problems.)

Another useful way to analyze things is by visualizing functions using plots of their graphs. Let's begin by plotting the function $f(x) = x^2$:

```
f(x)=x^2
plot(f(x))
```

The first line defines the function $f(x) = x^2$, then the second line will plot the function. Note that since we did not specify the range for the axes, the plot will automatically pick a range. If we want to see more of the graph, we can try adjusting the **xmin**, **xmax**, **ymin**, and **ymax** values, which correspond to the minimum and maximum values of the x- and y- axes that we desire. For example, changing the plot command to

```
plot(f(x),xmin=-3,xmax=3,ymin=-2,ymax=9)
```

will gives us an x-axis that ranges from -3 to 3 and a y-axis that ranges from -2 to 9.

We can also plot multiple functions on the same plot, in case we want to compare them together. Let's try overlaying the graph of the function $g(x) = 1$ onto the same set of axes:

```
f(x)=x^2
g(x)=1
plot([f(x), g(x)],xmin=-3,xmax=3,ymin=-2,ymax=9)
```

We have defined two functions now, $f(x) = x^2$ and $g(x) = 1$. To plot both, we put them both into a list that is enclosed in square brackets, with the two functions separated by a comma. This produces graphs of both $f(x)$ and $g(x)$, in two different colors.

How many times do the graphs of $f(x)$ and $g(x)$ intersect?

Exercises

1. Graph the functions $f(x) = x^2$ and $g(x) = 4$ together on the same set of axes. How many times do the graphs of $f(x)$ and $g(x)$ intersect?

2. Repeat with $f(x) = x^2$ and $g(x) = 3$. How many times do the graphs of $f(x)$ and $g(x)$ intersect?

3. Repeat with $f(x) = x^2$ and $g(x) = 0$. How many times do the graphs of $f(x)$ and $g(x)$ intersect?

4. Repeat with $f(x) = x^2$ and $g(x) = -3$. How many times do the graphs of $f(x)$ and $g(x)$ intersect?

5. What is the relationship between the number of times that the graphs of $f(x) = x^2$ and $g(x) = k$ intersect and the number of solutions to $x^2 = k$?

6. Are there any intersections of the graphs $f(x) = x^2 + 1$ and $g(x) = x + 2$?
   Use a sage `plot()` command to visualize the situation, then use the `solve()` command to find the points of intersection.

7. Notice that in the last problem the `solve()` command only gives us the x coordinates of the points of intersection. How can we find the y coordinates?

8. Recall that the quadratic formula is

   $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   What are the a, b and c in this case? Careful! You need to rearrange so that you have a quadratic polynomial set equal to zero.

   Does the answer from using Sage's `solve()` command agree with what the quadratic formula tells us?

Numerical Approximations to Solutions of Equations

(There are no problems in this section, so just read along and do the computations as you encounter them.)

Some equations are difficult to solve exactly even with the assistant of a computer and computer software, and Sage is no exception to this.

Use the **solve** function to have Sage solve the equation $x^5 - 3x^4 + x^3 + 2 = 0$.

The output that we get is

```
[0 == x^5 - 3*x^4 + x^3 + 2]
```

which is another way of Sage telling us that it could not find a solution. However, plotting the function $x^5 - 3x^4 + x^3 + 2$ tells us another story. Plot the graph of $x^5 - 3x^4 + x^3 + 2$ in Sage to see whether it has any roots. How many are there, and what are the approximate values from the graph? You may want to play around with the range of the x-axis (using xmin and xmax) to get a clearer picture.

(3 roots, roughly -0.8, 1.2, and 2.5)

We see that there are 3 roots, at roughly $x = -0.8$, 1.2, and 2.5. Although Sage cannot get the exact values for them by solving the equation $x^5 - 3x^4 + x^3 + 2 = 0$, it can get approximate decimal values for the roots by using the **find_root** function. In short, a computer algebra system such as Sage uses a sophisticated version of "find where the graph crosses the x-axis, and zoom in repeatedly around that point to get a more precise estimate of the x-coordinate of where the graph crosses the x-axis." To find a root x where $-2 < x < 0$, we would use the command

```
find_root(x^5-3*x^4+x^3+2==0, -2, 0)
```

The output of this is the root

```
-0.7931397744702121
```

If we want to find a different root, we can change the interval on which we instruct Sage to look for a root. For example, if we want to find the root near 1.2, we could try something like

```
find_root(x^5-3*x^4+x^3+2==0, 1, 2)
```

which gives us the value

```
1.199258801379252
```

Try to find the approximate value of the root of $x^5 - 3x^4 + x^3 + 2$ near $x = 2.5$.

(value is 2.563623765649018)

Note that we said that these values are approximate. Let's verify what we mean by that. Recall that a root of a function $f(x)$ is a value of x that makes $f(x)$ equal to exactly 0, i.e. $f(x) = 0$. Use Sage to define the function $f(x) = x^5 - 3x^4 + x^3 + 2$, then plug in the values of the approximate roots from above, e.g. find $f(-0.7931397744702121)$. If -0.7931397744702121 is truly a solution to $x^5 - 3x^4 + x^3 + 2 = 0$, then $f(-0.7931397744702121)$ should equal exactly 0.

However, the value that we get from Sage is

```
1.35419453428653e-13
```

The e here is used for <u>e</u>ngineering notation, where 1.35419453428653e-13 means

$$1.35419453428653 \times 10^{-13}$$

In other words, the part before the "e" is a decimal number, but then the part after the "e" is the exponent that we should raise 10 to then multiply the decimal. Another way to think of this is that the "e-13" means we should start with the 1.35419453428653 then move the decimal to the left 13 times, making our number smaller. In contrast, if we had seen

```
1.35419453428653e5
```

that would mean move the decimal to the right 5 places, resulting in the number 135419.453428653.

The number $1.35419453428653 \times 10^{-13}$ is a really small number, but it's not exactly equal to 0. This demonstrates that the decimal "solution" we obtained using **find_root** is only approximate and not an exact solution.

Another thing to be cautious about when using **find_root** is that it will quit as soon as it finds one approximate root. In other words, even if there is another root in the interval that you specify, **find_root** will only tell you the value of one of them. If we try

```
find_root(x^5-3*x^4+x^3+2==0, 1, 3)
```

in hopes of finding both roots that are between 1 and 3, it will not give us both. Try this out and see what you get!

(only the solution 2.563623765649033 is found)