Modeling Loss Severity II

A short course authored by the Actuarial Community

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Risk Retention – Deductibles and Limits

Risk Retention Framework

- Now consider the following framework:
  - Policyholder or insured suffers a loss of amount X
  - Under an insurance contract, the insurer is obligated to cover a portion of X, denoted as Y
  - Y represents the insurer's claim payment
- We introduce standard mechanisms that insurers use to reduce, or mitigate, their risk, including deductibles and policy limits
- Further, we examine how the distribution of the insurer's obligations depends on these mechanisms

Risk Retention Function

- Known risk retention function $g(\cdot)$ maps the amount insured to the amount retained by the insurer, that is, $Y = g(X)$
- **Special Case 1. Deductible (d)**

$$g(x) = (x - d)_+ = \begin{cases} 0 & x \leq d \\ x - d & x > d. \end{cases}$$

Notation "$(\cdot)_+$" means "Take the positive part of"

- **Special Case 2. Limit (u)**

$$g(x) = x \wedge u = \begin{cases} x & x \leq u \\ u & x > u. \end{cases}$$

Notation "$\wedge$" means "take the minimum of"

- **Special Case 3. Coinsurance**. Define $Y = g(X) = cX$. Typically, $0 < c < 1$, and so represents the proportion of claims retained by the insurer

Information Set for Deductibles

- Specify what type of information is available to the insurer
- **Special Case 4. Policyholder Deductible**. Define:

$$g_P(x) = \begin{cases} \text{undefined/not observed} & x \leq d \\ x - d & x > d \end{cases}$$

- Insurance only pays amounts in excess of the deductible d. If a loss is less than the deductible, the insurer does not observe the loss.
  - Random variable $Y^P = g_P(X)$ is the claim that an insurer observes
  - "P" subscript indicates that the retained loss is on a per payment basis
  - For case where a claim of zero is observed for losses $X \leq d$, terminology per loss is used.
  - Notation is $Y^L = (X - d)_+$

Distributions of Retained Risks - Deductible

- Consider two types of ordinary deductible:
- Cost (amount of payment) per loss event

$$Y^L = (X - d)_+ = \begin{cases} 0 & X \le d \\ X - d & X > d \end{cases}$$

- Cost (amount of payment) per payment event

$$Y^P = \begin{cases} \textit{undefined} & X \le d \\ X - d & X > d \end{cases}$$

Example. Exponential Distribution. Suppose that the loss X has cumulative distribution function $F(x) = 1 - \exp(-x/1000)$. Compute the cdf and pdf for Y^L and Y^P with $d = 250$

Expectations of Retained Risks

Limited Expected Value

- Use a generic "u" for the upper limit. Expected value of limited loss variable $(X \wedge u)$ is

$$E(X \wedge u) = \int_0^u (1 - F(x))\, dx = \int_0^u S(x) dx.$$

Pareto Policy Limit. Recall

$$1 - F(x) = S(x) = \Pr(X > x) = \left(\frac{\theta}{x+\theta}\right)^{\alpha}$$

with mean $\mathrm{E}\,(X) = \frac{\theta}{\alpha-1}$. Thus, the limited expected value is

$$\begin{aligned}
E(X \wedge u) &= \theta^{\alpha} \int_0^u (x+\theta)^{-\alpha} dx = \theta^{\alpha} \left.\frac{(x+\theta)^{-\alpha+1}}{-\alpha+1}\right|_0^u \\
&= \theta^{\alpha} \left(\frac{\theta^{-\alpha+1} - (u+\theta)^{-\alpha+1}}{\alpha - 1}\right) \\
&= \frac{\theta}{\alpha-1} \left\{1 - \left(\frac{\theta}{u+\theta}\right)^{\alpha-1}\right\}.
\end{aligned}$$

Pareto Deductible

- Claim amount on a per loss basis is $Y^L = (X-d)_+$ for a deductible d
- To calculate $E(X-d)_+$, use $X \wedge d + (X-d)_+ = X$
- For the Pareto distribution, recall $E(X) = \frac{\theta}{\alpha-1}$ and

$$E(X \wedge d) = \frac{\theta}{\alpha-1}\left\{1-\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right\}.$$

Thus,

$$\begin{aligned} E(X-d)_+ &= E(X)-E(X\wedge d) = \frac{\theta}{\alpha-1} - \frac{\theta}{\alpha-1}\left\{1-\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right. \\ &= \frac{\theta}{\alpha-1}\left\{\left(\frac{\theta}{d+\theta}\right)^{\alpha-1}\right\}. \end{aligned}$$

Mean Excess Loss

- For the per payment random variable associated with the policyholder deductible case,

$$g_P(x) = \begin{cases} \text{undefined/not observed} & x \leq d \\ x - d & x > d \end{cases}$$

we can calculate the expectation as

$$e_X(d) = e(d) = \mathrm{E}\left(X - d | X > d\right)$$

- $e_X(d)$ is the mean excess loss that We can write this as

$$\begin{aligned} e(d) &= E(X - d|X > d) \\ &= \frac{\int_d^\infty (x-d)f(x)dx}{S(d)} \\ &= \frac{E(X-d)_+}{S(d)} \\ &= \frac{\int_d^\infty S(x)dx}{S(d)} \end{aligned}$$

Example. Exam M Fall 2005, Exercise 26

For an insurance:

- Losses have the density function

$$f_X(x) = \begin{cases} 0.02x & 0 < x < 10 \\ 0 & \text{elsewhere} \end{cases}$$

- Insurance has an ordinary deductible of 4 per loss
- Y^P is the claim payment per payment random variable

Calculate $E[Y^P]$

Summary of Limited Loss Variables

Random Variable	**Expectation**
Excess loss random variable $Y = X - d$ if $X > d$ left truncated	$e_X(d) = E(Y) = \mathrm{E}(X - d \vert X > d)$ mean excess loss function mean residual life function complete expectation of life $e_X^k(d) = E\left[(X-d)^k \vert X > d\right]$
$(X-d)_+ = \begin{cases} 0 & X \le d \\ X - d & X > d \end{cases}$ left-censored and shifted variable	$E\,(X-d)_+ = e(d)S(d)$ $E\,(X-d)_+^k = e^k(d)S(d)$
$min(X,d) = X \wedge d = \begin{cases} X & X \le d \\ d & X > d \end{cases}$ limited loss variable	$E\,(X \wedge d)$ – limited expected value right censored

Note that $(X-d)_+ + (X \wedge d) = X$. Thus, $E\,(X-d)_+ + E\,(X \wedge d) = E(X)$

For nonnegative, continuous random variables,

$$E\,(X \wedge d) = \int_0^d S\,(x)\,dx \quad \text{and} \quad E(X-d)_+ = \int_d^\infty S\,(x)\,dx$$

LER and More Risk Retention

Loss Elimination Ratio (LER)

- Consider an ordinary deductible, cost (amount of payment) per loss event
- Loss elimination ratio at deductible d is

$$
\begin{aligned}
LER &= \frac{\mathrm{E}\left(X \wedge d\right)}{E(X)} \\
&= \frac{\text{limited exp value}}{\text{exp value}}
\end{aligned}
$$

  What fraction of the losses have been eliminated by introducing the deductible?

Example. Losses have a lognormal distribution with $\mu = 6$ and $\sigma = 2$. There is a deductible of 2,000

Determine the loss elimination ratio

Risk Retention Function II

- Combining three special cases of coverage modifications (deductible, limit, coinsurance) results in

$$g(x) = \begin{cases} 0 & x \leq d \\ c(x-d) & d < x \leq u \\ c(u-d) & x > u. \end{cases}$$

- Think about these as parameters in a contract between a policyholder and an insurer and so represent modifications of the underlying contract
- Note: $g(X) = c\left((X \wedge u) - (X \wedge d)\right)$