Aggregate Loss Models

A short course authored by the Actuarial Community

19 Jan 2021

Introduction

Basic Terminology

- Claim - indemnification upon the occurrence of an insured event
- Loss - some authors use claim and loss interchangeably, others think of loss as the amount suffered by the insured whereas claim is the amount paid by the insurer
- Frequency - how often an insured event occurs (claim count) in one period (typically six months or one year)
- Severity - Amount, or size, of each payment for an insured event
- Aggregate Loss - Total amount paid for a defined set of insureds in one period (typically six months or one year)

Motivation

- Goal: Build a model for the total payments by an insurance system
  - Aggregate loss for a single policy, a group insurance contract, a business line, an entire company, etc
  - Frequency and severity models are building blocks
- Applications
  - Ratemaking
  - Capital management
  - Risk financing

Models

- Two ways to build a model for aggregate losses
  - in a fixed time period, and
  - on a defined set of insurance contracts
- Collective Risk Model (a.k.a. compound model): record losses as claims are made and then add them up
- Individual Risk Model: record losses for each contract and then add them up

Individual Risk Model

Individual Risk Model

- The **individual risk model** represents the aggregate loss as a sum of a fixed number of insurance contracts

$$S = X_1 + \ldots + X_n,$$

  where

- S denotes the aggregate loss for n (a fixed number) contracts
- X_i denotes the loss cost for the ith contract for $i = 1, \ldots, n$
- X_i are assumed to independent but are not necessarily identically distributed, due to different coverage or exposure
- X_i usually has a probability mass at zero

Moments and Generating Functions

- The **individual risk model** represents the aggregate loss as a sum of a fixed number of insurance contracts

$$S = X_1 + \ldots + X_n,$$

- It is straightforward to show

$$\begin{array}{llll} \mathrm{E}(S) & = \sum_{i=1}^{n} \mathrm{E}(X_i), & \mathrm{Var}(S) = \sum_{i=1}^{n} \mathrm{Var}(X_i) \\ P_S(z) & = \prod_{i=1}^{n} P_{X_i}(z), & M_S(z) = \prod_{i=1}^{n} M_{X_i}(z) \end{array}$$

Applications

- Originally developed for life insurance
  - Probability of death within a year is q_i;
  - Fixed benefit paid for the death of the ith person is b_i.
- The distribution of the loss to the insurer for the ith policy is

$$f_{X_i}(x) = \begin{cases} 1 - q_i, & x = 0 \\ q_i, & x = b_i \end{cases}$$

- Calculate $\mathrm{E}(S)$, $\mathrm{Var}(S)$, $P_S(z)$, and $M_S(z)$
- In P&C, we often use the Tweedie distribution, which has a probability mass at zero, to model the loss cost X_i.

A General Approach

- Two-part framework

$$X_i = I_i \times B_i = \begin{cases} 0, & I_i = 0 \\ B_i, & I_i = 1 \end{cases}$$

- $I_1, \ldots, I_n$, $B_1, \ldots, B_n$ are independent.
- I_i is an indicator (Bernoulli) that is 1 with probability q_i and 0 with probability 1-q_i. It indicates whether the ith policy has a claim.
- B_i, a r.v. with nonnegative support, represents the amount of losses of policy i, given that a claim is made. It can follow a degenerate distribution such as the life insurance example.

A General Approach

Denote $\mu_i = \mathrm{E}(B_i)$ and $\sigma_i^2 = \mathrm{Var}(B_i)$, we show

$$
\begin{aligned}
\mathrm{E}(X_i) \quad & = \mathrm{E}[\mathrm{E}(X_i|I_i)] = \mathrm{E}[\mathrm{E}(I_i \times B_i|I_i)] = \mathrm{E}[I_i \times \mathrm{E}(B_i|I_i)] \\
& = \mathrm{E}[I_i \times \mathrm{E}(B_i)] = \mathrm{E}(I_i) \times \mathrm{E}(B_i) = q_i \mu_i \\
\mathrm{Var}(X_i) \quad & = \mathrm{E}(X_i^2) - \mathrm{E}(X_i)^2 = q_i(1-q_i)\mu_i^2 + q_i \sigma_i^2 \\
P_{X_i}(z) \quad & = \mathrm{E}(z^{X_i}) = \mathrm{E}(z^{I_i \times B_i}) = \mathrm{E}[\mathrm{E}(z^{I_i \times B_i}|I_i)] \\
& = \mathrm{E}(z^{I_i \times B_i}|I_i = 0) \times \mathrm{Pr}(I_i = 0) \\
& \quad + \mathrm{E}(z^{I_i \times B_i}|I_i = 1) \times \mathrm{Pr}(I_i = 1) \\
& = 1 - q_i + q_i P_{B_i}(z) \\
M_{X_i}(z) \quad & = 1 - q_i + q_i M_{B_i}(z)
\end{aligned}
$$

Computing the Aggregate Loss Distribution

- Parametric approximations: use moment matching to estimate parameters
- Analytic results: some special cases
- Exact calculation: involves numerous convolutions (not covered)

Computing Aggregate Loss Distribution

- Parametric Approximations
- When n is large, use normal distribution

$$F_S(s) = \Pr(S \leq s) = \Phi\left(\frac{s - \mu_S}{\sigma_S}\right)$$

where $\mu_S = \mathrm{E}(S)$ and $\sigma_S = \sqrt{\mathrm{Var}(S)}$. - When n is small, use skewed distribution, e.g. lognormal

$$\Pr(S \leq s) = \Pr(\ln S \leq \ln s) = \Phi\left(\frac{\ln s - \mu_S}{\sigma_S}\right)$$

where μ_S and σ_S are found from

$$\begin{cases} \exp\{\mu_S + \sigma_S^2/2\} = \mathrm{E}(S) \\ \exp\{2\mu_S + 2\sigma_S^2\} = \mathrm{E}(S^2) = \mathrm{E}(S)^2 + \mathrm{Var}(S) \end{cases}$$

Computing Aggregate Loss Distribution

Analytic Results Summary

- Severity Distributions
  - If $X_i \sim N(\mu_i, \sigma_i^2)$, then $S \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$
  - If $X_i \sim \mathit{Exponential}(\theta)$, then $S \sim \mathit{Gamma}(n, \theta)$
  - If $X_i \sim \mathit{Gamma}(\alpha_i, \theta)$, then $S \sim \mathit{Gamma}(\sum_{i=1}^n \alpha_i, \theta)$
- Frequency Distributions
  - If $X_i \sim \mathit{Poisson}(\lambda_i)$, then $S \sim \mathit{Poisson}(\sum_{i=1}^n \lambda_i)$
  - If $X_i \sim \mathit{Bin}(m_i, q)$, then $S \sim \mathit{Bin}(\sum_{i=1}^n m_i, q)$
  - If $X_i \sim \mathit{Geometric}(\beta)$, then $S \sim \mathit{NegBin}(n, \beta)$
  - If $X_i \sim \mathit{NegBin}(r_i, \beta)$, then $S \sim \mathit{NegBin}(\sum_{i=1}^n r_i, \beta)$

Collective Risk Model

Collective Risk Model

The **collective risk model** has representation

$$S = X_1 + \ldots + X_N,$$

with S the aggregate loss of N (a random number) individual claims $(X_1, \ldots, X_N)$

Two building blocks: frequency N and severity X

- Key assumptions
  - Conditional on $N = n$, $X_1, \ldots, X_n$ are i.i.d. random variables
  - Frequency N and severity X are independent

Model

The assumptions suggest that we can build an aggregate loss model, the compound model, in three steps:

1. Develop a model for the frequency distribution of N, the primary distribution, based on data
2. Develop a model for the severity distribution of X, the secondary distribution, based on data
3. Using these two models, carry out the necessary calculations to obtain the distribution of S

Moments

$$S = X_1 + \ldots + X_N$$

Using the law of iterated expectations to calculate the mean

$$\mathrm{E}(S) = \mathrm{E}[\mathrm{E}(S|N)] = \mathrm{E}[N\mathrm{E}(X)] = \mathrm{E}(N)\mathrm{E}(X)$$

Using the law of total variation to calculate the variance

$$\begin{aligned}
\mathrm{Var}(S) &= \mathrm{E}[\mathrm{Var}(S|N)] + \mathrm{Var}[\mathrm{E}(S|N)] \\
&= \mathrm{E}[N\mathrm{Var}(X)] + \mathrm{Var}[N\mathrm{E}(X)] \\
&= \mathrm{E}(N)\mathrm{Var}(X) + \mathrm{Var}(N)\mathrm{E}(X)^2
\end{aligned}$$

Generating Functions

The moment generating function (mgf) of S is

$$\begin{aligned}
M_S(z) &= E\left[e^{zS}\right] = \mathrm{E}\left[\mathrm{E}\left(e^{zS}|N\right)\right] \\
&= \mathrm{E}\left[e^{0}\right]\mathrm{Pr}(N=0) \\
&\qquad + \textstyle\sum_{n=1}^{\infty}\mathrm{E}\left[e^{z(X_1+\ldots+zX_n)}|N=n\right]\mathrm{Pr}(N=n) \\
&= \mathrm{Pr}(N=0) + \textstyle\sum_{n=1}^{\infty}\prod_{i=1}^{n}\mathrm{E}\left[e^{zX_i}\right]\mathrm{Pr}(N=n) \\
&= \textstyle\sum_{n=0}^{\infty}\mathrm{Pr}(N=n)[M_X(z)]^n \\
&= \mathrm{E}\left[M_X(z)^N\right] = P_N[M_X(z)]
\end{aligned}$$

Similarly, the probability generating function (pgf) of S is

$$P_S(z) = P_N[P_X(z)]$$

Compound Distribution

The cumulative distribution function (cdf) is $F_S(s)$. The probability density function (pdf) of S or probability mass function (pmf) is $f_S(s)$. Specifically,

$$\begin{aligned} F_S(s) &= \Pr(S \leq s) \\ &= \sum_{n=0}^{\infty} \Pr(N=n) \cdot \Pr(S \leq s | N=n) \\ &= \sum_{n=0}^{\infty} \Pr(N=n) \cdot F_X^{*n}(s) \end{aligned}$$

$$f_S(s) = \sum_{n=0}^{\infty} \Pr(N=n) \cdot f_X^{*n}(s)$$

Examples

The frequency and severity distributions are summarized by:

N	0	1	2	3	4	5	6	7	8	
$f_N(n)$	0.05	0.10	0.15	0.20	0.25	0.15	0.06	0.03	0.01	
X	1	2	3	4	5	6	7	8	9	10
$f_X(x)$	0.150	0.200	0.250	0.125	0.075	0.050	0.050	0.050	0.025	0.025

`R` demo

Stop-Loss Insurance

Definition Insurance on the aggregate losses, subject to a deductible, is called *stop-loss insurance*.

- The expected cost of this insurance is called the *net stop-loss premium*.
- It can be computed as $\mathrm{E}[(S-d)_+]$, where S is the aggregate loss and d is the deductible.
- For any aggregate distribution:

$$\begin{aligned} E[(S-d)_+] &= E(S) - E(S \wedge d) \\ &= \int_0^\infty [1-F_S(s)]ds - \int_0^d [1-F_S(s)]ds \\ &= \int_d^\infty [1-F_S(s)]ds \end{aligned}$$

Stop-loss Insurance

For a continuous distribution, the net stop-loss premium is

$$\mathrm{E}[(S-d)_+] = \int_d^\infty (s-d) f_S(s) ds$$

For a discrete distribution, the net stop-loss premium is

$$\mathrm{E}[(S-d)_+] = \sum_{s>d} (s-d) f_S(s) ds$$

Computing the Aggregate Loss Distribution

The computation of the compound distribution function of the aggregate loss S is not an easy task

$$F_S(s) = \sum_{n=0}^{\infty} \mathrm{Pr}(N = n) \cdot F_X^{*n}(s)$$

Several strategies:

- **direct calculation**: difficult part is the evaluation of n-fold convolutions
- **approximating the distribution**: to avoid direct calculation
- **recursive method**: considerable savings in computer time
- **analytic results**: available only for select combinations of frequency and severity
- **Monte Carlo simulation**

Computing the Aggregate Loss Distribution

Use parametric distributions to approximate the compound distribution

- When $\mathrm{E}(N)$ is large, the normal distribution provides a good approximation.
  - This is supported by central limit theorem as parameter λ (Poisson), or m (Bin), or r (NegBin) goes to infinity
- When $\mathrm{E}(N)$ is small, the distribution of S is likely to be skewed.
  - This suggests using the lognormal or other skewed distributions, though no theoretical support

Computing the Aggregate Loss Distribution

Use the recursive method, assuming

- severity distribution, $f_X(x)$, is defined on $0, 1, 2, \ldots, m$, representing a multiple of some convenient monetary unit
- frequency distribution, $f_N(n)$, is a member of the $(a, b, 1)$ class

$$f_S(s) = \frac{[p_1 - (a+b)p_0]f_X(s) + \sum_{y=1}^{\min(s,m)} (a + by/s) f_X(y) f_S(s-y)}{1 - a f_X(0)}$$

Analytic Results

For most choices of distributions of N and X, the compound distributional values can only be obtained numerically

For certain combinations of choices, simple analytic results are available

- compound geometric-exponential
- sum of independent compound Poisson

Analytic Results: Geometric-Exponential

Example 1. Determine the distribution of S when the frequency distribution is geometric with parameter β and the severity distribution is exponential with parameter θ. The mgf of S is

$$\begin{aligned}
M_S(z) &= P_N[M_X(z)] = P_N[(1-\theta z)^{-1}] \\
&= \left(1-\beta[(1-\theta z)^{-1}-1]\right)^{-1} \\
&= 1+\frac{\beta}{1+\beta}\left([1-\theta(1+\beta)z]^{-1}-1\right)
\end{aligned}$$

Analytic Results: Geometric-Exponential

Note that

$$\begin{aligned} M_{X^*}(z) &= [1 - \theta(1+\beta)z]^{-1} \\ P_{N^*}(z) &= 1 + \frac{\beta}{1+\beta}(z-1) \end{aligned}$$

Therefore, the mgf of S can also be represented as

$$M_S(z) = P_N[M_X(z)] = P_{N^*}[M_{X^*}(z)]$$

i.e., $S = X_1^* + \cdots + X_{N^*}^*$ where

- $N^* \sim Bin(1, \beta/(1+\beta))$ and
- $X^* \sim Exponential(\theta(1+\beta))$

Analytic Results: Geometric-Exponential

It is a two-component mixture distribution:

$$S = \begin{cases} 0 & \text{with probability } \Pr(N^* = 0) \\ X^* & \text{with probability } \Pr(N^* = 1) \end{cases}$$

$$F_S(s) = \frac{1}{1+\beta} F_0(s) + \frac{\beta}{1+\beta} F_{X^*}(s)$$

$$= \frac{1}{1+\beta}(1) + \frac{\beta}{1+\beta}[1 - \exp[-s/(\theta(1+\beta))]]$$

$$= 1 - \frac{\beta}{1+\beta} \exp[-s/(\theta(1+\beta))]$$

Analytic Results: Compound Poisson

Example 2. Suppose that S_i has compound Poisson distribution with

- frequency parameter λ_i and
- severity distribution F_{X_i} for $i = 1, \cdots, m$.

Given $S_1, \cdots, S_m$ are independent, then $S = \sum_{i=1}^m S_i$ has a compound Poisson distribution with

- frequency parameter $\lambda = \sum_{i=1}^m \lambda_i$ and
- severity distribution

$$F_X(x) = \sum_{i=1}^m \frac{\lambda_i}{\lambda} F_{X_i}(x).$$

The model can be useful for analyzing aggregate claims from

- group insurance contracts
- multiple lines of business

Analytic Results: Compound Poisson

Proof. The mgf of S_i is

$$M_{S_i}(z) = P_{N_i}(M_{X_i}(z)) = \exp\left\{\lambda_i[M_{X_i}(z) - 1]\right\},$$

and, by independence, the mgf of S is

$$\begin{aligned} M_S(z) &= \textstyle\prod_{i=1}^m M_{S_i}(z) = \prod_{i=1}^m \exp\left\{\lambda_i[M_{X_i}(z) - 1]\right\} \\ &= \exp\left\{\textstyle\sum_{i=1}^m \lambda_i M_{X_i}(z) - \lambda\right\} \\ &= \exp\left\{\lambda\left[\textstyle\sum_{i=1}^m \dfrac{\lambda_i}{\lambda} M_{X_i}(z) - 1\right]\right\} \end{aligned}$$

Analytic Results: Compound Poisson

Proof. Let $M_X(z) = \sum_{i=1}^m \frac{\lambda_i}{\lambda} M_{X_i}(z)$, then

$$M_S(z) = \exp\{\lambda[M_X(z) - 1]\} = P_N(M_X(z))$$

where

- N is Poisson with parameter λ
- X has cdf $F_X(x) = \sum_{i=1}^m \frac{\lambda_i}{\lambda} F_{X_i}(x)$.

Compound Frequency Distributions

Compound Frequency Distributions

- We consider a special case of the compound distribution

$$S = M_1 + \cdots + M_N$$

where

- Both N and M are count random variables (r.v.)
- General relations apply:

$$\begin{aligned} &P_S(z) = P_N[P_M(z)] \qquad && M_S(z) = P_N[M_M(z)], \\ &E(S) = E(N)E(M) \qquad && Var(S) = E(N)Var(M) + E(M)^2 Var(N) \end{aligned}$$

Compound Poisson (Frequency) Distribution

- Consider the primary distribution with Poisson parameter λ and the secondary distribution as the following mixture

$$M = \begin{cases} 0 & \text{with probability } 1-q \\ M^* & \text{with probability } q \end{cases}$$

- The resulting compound distribution is equivalent to a compound Poisson model with Poisson parameter λq and the secondary distribution M^*

Compound Poisson (Frequency) Distribution

- Thinning of Poisson: Let N have a Poisson distribution with mean λ.

Assume that each outcome can be categorized in to a specific type.

The probability of an event being of type i is p_i.

Then the number of type i events, denoted by N_i, follows a Poisson distribution with mean $p_i\lambda$.

Compound Poisson (Frequency) Distribution

- Suppose that S_i has compound Poisson distribution with primary parameter λ_i and secondary distribution probability mass function (pmf) f_{M_i} for $i = 1, \cdots, m$.

Given $S_1, \cdots, S_m$ are independent, then $S = \sum_{i=1}^{m} S_i$ has a compound Poisson distribution with primary distribution parameter $\lambda = \sum_{i=1}^{m} \lambda_i$ and secondary distribution

$$f_M(x) = \sum_{i=1}^{m} \frac{\lambda_i}{\lambda} f_{M_i}(x)$$

Tweedie Distribution

Tweedie Distribution

The Tweedie distribution is defined as a Poisson sum of gamma random variables

$$S = (X_1 + \cdots + X_N)/\omega$$

where - ω is exposure - $N \sim \mathit{Poisson}(\omega\lambda)$ and $X \sim \mathit{Gamma}(\alpha, \theta)$

Tweedie Distribution

The cdf of S is

$$\begin{aligned} F_S(s) &= \sum_{n=0}^{\infty} \Pr(N=n) \cdot \Pr(S \le s | N=n) \\ &= \Pr(N=0) + \sum_{n=1}^{\infty} \Pr(N=n) \cdot \Pr(S \le s | N=n) \\ &= e^{-\omega\lambda} + \sum_{n=1}^{\infty} e^{-\omega\lambda} \frac{(\omega\lambda)^n}{n!} \Gamma\left(n\alpha; \frac{s}{\theta/\omega}\right) \end{aligned}$$

Note that

$$S|(N=n) = X_1 + \cdots + X_n \sim Gamma(n\alpha, \theta/\omega)$$

Tweedie Distribution

- The Tweedie variable has a mass probability at zero

$$f_S(0) = \Pr(S = 0) = \Pr(N = 0) = e^{-\omega\lambda}$$

- The density of Tweedie at $s > 0$ is

$$f_S(s) = \sum_{n=1}^{\infty} e^{-\omega\lambda} \frac{(\omega\lambda)^n}{n!} \frac{1}{\Gamma(n\alpha)} \frac{s^{n\alpha-1}}{(\theta/\omega)^{n\alpha}} e^{-\frac{s}{\theta/\omega}}$$

Tweedie Distribution

Definition The distribution of *linear exponential family* is

$$f_Y(y;\theta,\phi/\omega)=\exp\left\{\frac{y\theta-b(\theta)}{\phi/\omega}+c(y;\phi/\omega)\right\}$$

- θ is the parameter of interest
- ϕ is scale parameter and ω (known) is exposure
- $b(\theta)$ depends only on θ not y
- $c(y;\phi/\omega)$ does not depend on θ
- $\mathrm{E}(y)=b'(\theta)$ and $\mathrm{Var}(y)=\frac{\phi}{\omega}b''(\theta)$

Members include binomial, normal, Poisson, gamma, and inverse-Gaussian distributions

Tweedie Distribution

Consider reparameterizations

$$\lambda = \frac{\mu^{2-p}}{\phi(2-p)}, \quad \alpha = \frac{2-p}{p-1}, \quad \theta = \phi(p-1)\mu^{p-1}$$

For $p \in (1,2)$, the Tweedie distribution can be presented as:

$$f_S(s) = \exp\left\{\tfrac{\omega}{\phi}\left(\tfrac{-s}{(p-1)\mu^{p-1}} - \tfrac{\mu^{2-p}}{2-p}\right) + c(s; p, \phi/\omega)\right\}$$

and

$$\mathrm{E}(S) = \mu, \quad \mathrm{Var}(S) = \frac{\phi}{\omega}\mu^p$$

Tweedie Distribution

- Two limiting distributions
- $p \to 1 \implies$ Over-dispersed Poisson
- $p \to 2 \implies$ Gamma
- Examples in `R`

Effects of Coverage Modifications

Effects of Coverage Modifications

- Effect of exposure on frequency
- Impact of deductibles on claim frequency
- Impact of individual policy modifications on aggregate claims

Exposure and Frequency

- Consider the number of claims from a portfolio of n policies

$$N = Y_1 + \cdots + Y_n$$

where we assume Y_i are i.i.d.

- Use a policy as the exposure base, the exposure is n
- The pgf of N is $P_N(z) = [P_Y(z)]^n$
- Exposure changes from n to n^*, the pgf of N^* is $P_N(z) = [P_Y(z)]^{n^*} = [P_N(z)]^{n^*/n}$

Exposure and Frequency

- To account for exposure n in frequency models
  - if $Y \sim Poisson(\lambda)$, $N \sim Poisson(n\lambda)$
  - if $Y \sim NegBin(r, \beta)$, $N \sim NegBin(nr, \beta)$
  - in this case, exposure is a large positive number
- Further, the units of exposure may be fractions - Using policy year as the exposure base, exposure represents the fraction of the year that a policyholder had insurance coverage
- Effect of exposure on MLE
- Exposure change

Deductibles and Frequency Distribution

- Consider an insurance contract with deductible d
- Denote N^L the number of losses or accidents, and given $N^L = n^L$, Y_i is the ground-up loss for $i = 1, \cdots, n^L$
- We are interested in N^P, the number of payments or claims
- Assume
  - Given $N^L = n^L$, Y_i are i.i.d. with common distribution Y
  - Changing the deductible does not change policyholder behavior

Deductibles and Frequency Distribution

- Represent N^P as a compound frequency distribution

$$N^P = I_1 + \cdots + I_{N^L}$$

where

$$I_i = \begin{cases} 1 & \text{with probability } \Pr(Y_i > d) \\ 0 & \text{with probability } \Pr(Y_i \leq d) \end{cases}$$

- Let $v = \Pr(Y > d)$, the pgf of N^P is

$$\begin{aligned} P_{N^P}(z) &= P_{N^L}[P_I(z)] \\ &= P_{N^L}[1 + v(z-1)] \end{aligned}$$

Deductibles and Frequency Distribution

- **Example:** $N^L \sim Poisson(\lambda)$
- The pgf of N^P is

$$\begin{aligned} P_{N^P}(z) &= \exp\{\lambda[1 + v(z-1) - 1]\} \\ &= \exp\{v\lambda(z-1)\} \end{aligned}$$

- N^L and N^P are from the same family and $N^P \sim Poisson(\lambda^* = v\lambda)$
- Similar results for binomial and negative binomial distributions in the $(a, b, 0)$ class

Deductibles and Frequency Distribution

- Now we consider changing the deductible from d_1 to d_2. Let θ be the parameter in the distribution of group-up loss. Recall that $\theta^* = v\theta$ where $v = \Pr(Y > d)$
- Define $v_1 = \Pr(Y > d_1)$ and $v_2 = \Pr(Y > d_2)$, then

$$\theta_1^* = v_1 \cdot \theta \text{ and } \theta_2^* = v_2 \cdot \theta = \left(\frac{v_2}{v_1}\right)\theta_1^*$$

- Two special cases:
  - $d_1 = 0,\ d_2 = d \to \dfrac{v_2}{v_1} = v$
  - $d_1 = d,\ d_2 = 0 \to \dfrac{v_2}{v_1} = v^{-1}$

Impact on Aggregate Claims

Recall the following notation:

- Frequency
  - $N^L \equiv$ number of losses
  - $N^P \equiv$ number of payments
- Severity
  - $Y \equiv$ ground-up loss
  - $Y^L \equiv$ amount of payment on per-loss basis
  - $Y^P \equiv$ amount of payment on per-payment basis
- Coverage Modifications
  - $d \equiv$ amount of deductible (ordinary or franchise)
  - $r \equiv$ uniform rate of inflation
  - $u \equiv$ policy limit
  - $\alpha \equiv$ coinsurance percentage

Impact on Aggregate Claims

Recall:

$$
\begin{aligned}
Y^L &= \begin{cases} 0 & Y < \dfrac{d}{1+r} \\ \alpha[(1+r)Y - d] & \dfrac{d}{1+r} \leq Y < \dfrac{u}{1+r} \\ \alpha(u-d) & Y \geq \dfrac{u}{1+r} \end{cases} \\
&= \alpha(1+r)\left(\left[Y \wedge \left(\tfrac{u}{1+r}\right)\right] - \left[Y \wedge \left(\tfrac{d}{1+r}\right)\right]\right)
\end{aligned}
$$

Impact on Aggregate Claims

Recall:

- Frequency

$$\begin{aligned} N^P &= I_1 + \cdots + I_{N^L} \\ P_{N^P} &= P_{N^L}[1 - v + vz] \end{aligned}$$

- Severity

$$\begin{aligned} Y^P &= Y^L | (Y^L > 0) \\ F_{Y^L}(y) &= 1 - v + vF_{Y^P}(y) \\ M_{Y^L}(z) &= 1 - v + vM_{Y^P}(z) \end{aligned}$$

Impact on Aggregate Claims

- The aggregate claim can be represented on either a per-loss or per-payment basis
  - per-loss basis: $S = Y_1^L + \cdots + Y_{N^L}^L$
  - per-payment basis: $S = Y_1^P + \cdots + Y_{N^P}^P$
- The two representations are equivalent:

$$M_S(z) = P_{N^L}\left[M_{Y^L}(z)\right] = P_{N^P}\left[M_{Y^P}(z)\right]$$