

Quantitative Models for Options Pricing

Fabian Farestam, typeset by Om Gupta

September 2022

1 Introduction

This document is an overview of quantitative models for option pricing to be implemented in OpenBBTerminal. When learning about options pricing, I recommend you to start with learning the Cox-Ross-Rubinstein Binomial model, then Black-Scholes model and later on advance to more complicated models.

Note that none of this is my (Fabian Farestam's) research. It's just a reformulation and a collection of material from multiple sources.

2 Cox-Ross-Rubinstein Binomial Model

The binomial pricing model [1] has a simple concept, namely that it builds a tree of possible pricing with a certain amount of steps until the time of the options maturity and calculates the arbitrage free price based on this tree. It's easy to implement (can be $O(2^n)$) and also very practical while clearly demonstrating the no arbitrage argument. In special limiting cases of this we can get both the Black-Scholes model as well as the Cox-Ross jump process model (similar to the hybrid jump diffusion model by Merton).

The model assumes that:

- The stock price follows a multiplicative binominal process over discrete periods
- The interest rate is constant and one might borrow and lend as much as wished at this interest rate. (For convenience we assume that $r > 1$)

2.1 Notation

- u : Asset return with the probability of q
- d : Asset return with the probability of $1 - q$
- S : Stock Price
- q : Probability of tree path taken
- r : Riskless interest rate
- C : Value of a call option
- K : Strike Price

2.2 One Step Tree

The setup is simple, the stock returns over each period is either $u - 1$ with the probability of q , or $d - 1$ with the probability $1 - q$. Thus the price at the end of the period will be uS or dS . Note that the relation $u > r > d$ must always be held. An illustration of this is:

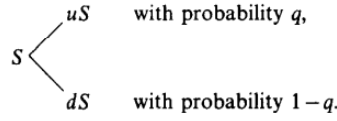


Figure 1: Source: Cox, Ross, Rubinstein

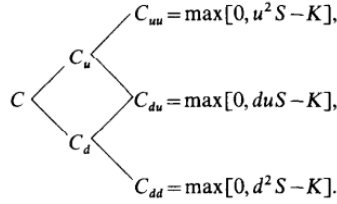


Figure 2: Source: Cox, Ross, Rubinstein

To calculate the value of this simple setup, where there's only one step, we use:

$$C_u = \max(0, uS - K)$$

$$C_d = \max(0, dS - K)$$

When using a number of shares, g (some use δ instead of g) and dollar invested in riskless bonds, B . Thus:

$$C_u = guS + rB$$

$$C_d = gdS + rB$$

After solving for g and B ,

$$g = \frac{C_u - C_d}{(u - d)S}$$

$$B = \frac{uC_d - dC_u}{(u - d)r}$$
(1)

If no arbitrage exists then:

$$C = gS + B$$

Which then implies

$$C = \frac{pC_u + (1 - p)C_d}{r}$$
(2)

where $p = \frac{r-d}{u-d}$ and $1 - p = \frac{u-r}{u-d}$

From 2 we can make three observations:

1. q is not used, thus even if investors have different opinions about the probabilities they can still agree to this relationship! (p is q if investors were to be risk neutral)
2. Investor's risk attitude doesn't matter.
3. The stock price is the only random variable.

2.3 Two Step Tree

For a two step tree we get the following structure:

To this new tree we adapt 2

$$\begin{aligned} C_u &= \frac{pC_{uu} + (1-p)C_{ud}}{r} \\ C_d &= \frac{pC_{du} + (1-p)C_{dd}}{r} \end{aligned} \quad (3)$$

We don't need to change 1 but just we need to adjust our hedge portfolio with the new values gotten from 3

So to get the value of the call we substitute 3 into 2, which results in:

$$\begin{aligned} C &= \frac{p^2 C_{uu} + 2p(1-p)C_{ud} + (1-p)^2 C_{dd}}{r} \\ &= \frac{p^2 * \max(0, u^2 S - K) + 2p(1-p) * \max(0, duS - K)}{r} \end{aligned} \quad (4)$$

From this we conclude that all assumptions made from 2 also hold for 4. Now the list of variables that determine C is S, K, n, u, d and r .

2.4 Binomial tree with any amount of steps

We can find the call value recursively, by starting at the expiration and working backwards, for any amount of steps:

$$C = \frac{\sum_{j=0}^n \binom{n}{j} p^j (1-p)^{n-j} \max(0, u^j d^{n-j} S - K)}{r^n} \quad (5)$$

We're not yet done since we can express this in a way nicer form by some transformations:

$$C = \frac{\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} (u^j d^{n-j} S - K)}{r^n} \quad (6)$$

where a (represents minimum stock upwards moves for finishing in the money) is the smallest non negative larger greater than $\frac{\log(K/Sd^n)}{\log(u/d)}$

We'll break up the above formula into two terms:

$$C = S * \frac{\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j} (u^j d^{n-j})}{r^n} - K * \frac{\sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j}}{r^n} \quad (7)$$

In formula 7 we have two brackets, when taking a closer look at them one realises that we can express them as complementary binomial distribution function ϕ . We can sum this up as the final simple Binomial Pricing formula:

$$C = S * \phi(a; n, p') - Kr^{-n} * \phi(a; n, p) \quad (8)$$

where $p = \frac{r-d}{u-d}$ and $p' = \frac{u}{r}p$

Natural values for u and d are $u = e^{\sigma\Delta t}$ and $d = \frac{1}{u} = e^{-\sigma\Delta t}$ where σ represents the volatility of the asset under consideration

Note that the formula above is only valid for American non-dividend paying call options.

2.5 Dividends

In this part in order to account for the dividend we assume that the stock has a constant dividend yield, δ on the ex-dividend date and a total of v ex-dividend days. To account for dividend we need to define ρ as one plus the interest rate over a period of $h = \frac{t}{n}$. The total return is $\rho^n = r^t$, and we can express $\rho = r^{\frac{t}{n}}$.

A major difference in this model "extension" is that early exercise is optimal in some cases. Early exercise is in fact optimal when $S > S^*$, where S^* is the "critical" price of the stock where the following equation holds:

$$S^* - K = \frac{p * C(n, i-1, j+1) + (1-p) * C(n, i-1, j)}{\rho} \quad (9)$$

for $j = 0, 1, 2, \dots, n-i$ where $C(n, i, j)$ = call value $n-i$ periods from now, $S^* = u^j d^{n-i-j} (1-\delta)^{v(n,1)} * S$, and $v(n, i) = \sum_{k=1}^{n-1} v_k$

From 9, we get

$$C(n, i, j) = \max \left(u^j d^{n-i-j} (1-\delta)^{v(n,1)} * S - K, \frac{p * C(n, i-1, j+1) + (1-p) * C(n, i-1, j)}{\rho} \right) \quad (10)$$

Important to note is that here we traverse the price the binomial tree recursively, thus starting with $i = 0$ and ending with $i = n$. We apply the previous results into the calculations of the previous ones.

The hedge ratio of the portfolio is:

$$g = \frac{C(n, n-1, 1) - C(n, n-1, 0)}{(u-d) * S}$$

2.6 Limiting Cases: Black-Scholes + Cox-Ross jump process

We'll start exploring these limiting cases by splitting our steps into small pieces. By setting h , the time elapsed between each step, as $h = \frac{t}{n}$. Now to get to a continuous tree we simply set $n \rightarrow \infty$. Again, we have $\rho^n = r^t$ so that $\rho = r^{\frac{t}{n}}$

From here we need to express u and d in terms of n . There's two ways, one results in the Black Scholes model and the other in the Cox-Ross jump process model.

2.6.1 Black-Scholes

We'll express the returns u and d from here on as $\log u$ and $\log d$, so over multiple periods we get that the final stock return is:

$$\log\left(\frac{S^*}{S}\right) = j \log u + (n-j) * \log d = j \log\left(\frac{u}{d}\right) + n \log d \quad (11)$$

j is the random number of upwards moves, which occurred during n periods. We can now express the expected return and the return variance:

$$\begin{aligned} E\left[\log\left(\frac{S^*}{S}\right)\right] &= \log\left(\frac{u}{d}\right) * E[j] + n \log d \\ \text{Var}\left[\log\left(\frac{S^*}{S}\right)\right] &= \left(\log\left(\frac{u}{d}\right)\right)^2 * \text{Var}[j] \end{aligned}$$

Since $E[j] = nq$; and $\text{Var}[j] = nq(1-q)$; (due to the variance of one period being: $q(1-q)^2 + (1-q)(0-q)^2 = q(1-q)$). Now we simplify the formulas above:

$$\begin{aligned} E\left[\log\left(\frac{S^*}{S}\right)\right] &= \left(\log\left(\frac{u}{d}\right) q + \log d\right) * n = \mu n \\ \text{Var}\left[\log\left(\frac{S^*}{S}\right)\right] &= \left(\log\left(\frac{u}{d}\right)\right)^2 q(1-q) * n = \sigma n \end{aligned} \quad (12)$$

Now we need to adjust u and d so that we get realistic results. Below $\bar{\mu}$ and $\bar{\sigma}$ are the empirical values of μ and σ so we get the empirical values after adjusting u and d .

As $n \rightarrow \infty$:

$$\begin{aligned} \left(\log\left(\frac{u}{d}\right) q + \log d\right) * n &\rightarrow \bar{\mu} n \\ \left(\log\left(\frac{u}{d}\right)\right)^2 q(1-q) * n &\rightarrow \bar{\sigma} n \end{aligned}$$

Thus,

$$\begin{aligned} u &= \exp\left(\bar{\sigma}h^{\frac{1}{2}}\right) \\ d &= \exp\left(-\bar{\sigma}h^{\frac{1}{2}}\right) \\ q &= \frac{1 + \frac{\bar{\sigma}}{\mu}\left(\frac{n}{t}\right)^{\frac{1}{2}}}{2} \end{aligned} \tag{13}$$

This makes that every probability and possible outcome get changed. Now we use the central limit theorem, when applied it to our problem, says that as $n \rightarrow \infty$, if 14 below holds then 15 applies, where $N(z)$ is the standard normal distribution:

$$\frac{q|\log u - \mu|^3 + (1-q)|\log d - \mu|^3}{\sigma^3 n^{\frac{1}{2}}} \rightarrow 0 \tag{14}$$

$$\text{Prob}\left[\frac{\log\left(\frac{S^*}{S}\right) - \mu n}{\sigma n^{\frac{1}{2}}}\right] \rightarrow N(z) \tag{15}$$

When we rewrite the Black Scholes theorem in terms of the the previously used notation, we get:

$$C = SN(x) - Kr^{-t}N(x - \sigma t^{\frac{1}{2}}) \tag{16}$$

where $x = \frac{\log\left(\frac{S}{Kr^{-t}}\right)}{\sigma t^{\frac{1}{2}}} + \frac{\sigma t^{\frac{1}{2}}}{2}$

2.6.2 Cox Ross Jump Process

To get the Cox-Ross jump process model, we set u , d , and q to the following values:

$$\begin{aligned} u &= u \\ d &= \exp\left(\zeta\left(\frac{t}{n}\right)\right) \\ q &= \lambda\left(\frac{t}{n}\right) \end{aligned}$$

where λ is the intensity function (or rate) of the underlying Poisson process and ζ is related to the intensity measure of the distribution.

The central limit theorem no longer holds for these values and the stock price converges to a log-Poisson distribution as $n \rightarrow \infty$. This complementary Poisson distribution can be expressed in the following form:

$$\psi(x, y) = \sum_{i=x}^{\infty} \frac{e^{-y} y^i}{i!} \tag{17}$$

This allows the Cox-Ross jump process model to be written as:

$$C = S\psi(x, y) - Kr^{-t}\psi\left(x, \frac{y}{u}\right) \tag{18}$$

where $y = \frac{(\log r - \xi)ut}{u-1}$ and x is the the smallest non-negative integer greater than $\frac{\log\left(\frac{K}{S}\right)}{\log u}$

3 Black-Scholes Model

3.1 Notation

- S_t : Price of underlying asset at time t
- μ : Annualized drift rate of S

- σ : Standard deviation of the asset's log returns
- $\phi(m, s)$: Normal distribution with mean m and standard deviation of s
- η : Continuously compounded rate of return
- U : Annualized annual expected return in period Δt
- V : Annualized annual expected return expressed in a compounding frequency of Δt over a longer period of time
- ϵ : random from the standard normal distribution $\phi(0, 1)$
- r : Annual risk-free rate of interest
- f : Price of derivative

3.2 Assumptions

These assumptions are used to derive the Black-Scholes-Merton [3] differential equation and the equation only holds in this theoretical environment. Note that some of these assumptions can and will be accounted for at a later stage of the document. For instance interest rates can be stochastic, thus relaxing the conditions assumed.

- Random walk - Asset prices follow geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz$$

- Lognormal distributions - this results in that the price at time T corresponds to:

$$\ln S_T \sim \phi \left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma T^{\frac{1}{2}} \right) \quad (19)$$

- Non dividend (will later show dividend extension)
- Market:
 - No arbitrage
 - Lend and borrow
 - Buy and short
 - No transaction fees
 - Continuous security trading
- μ , r , and σ are constant
- r is the same for all maturities

3.3 Calculations

If we start with the normal distribution of returns we get that the expected value of S_T , $E[S_T]$:

$$E[S_T] = S_0 e^{\mu T}$$

and variance of S_T :

$$\text{Var}[S_T] = S_0^2 e^{2\mu T} [e^{\sigma^2 T} - 1]$$

3.3.1 Rate of Return

Further using the log normal properties of the price at time T we get:

$$S_T = S_0 e^{\eta T}$$

and it follows that

$$\eta = \frac{1}{T} \ln \left(\frac{S_T}{S_0} \right) \quad (20)$$

For correctly understanding 20 we need to get $\ln \left(\frac{S_T}{S_0} \right)$, we get this when transforming 19:

$$\ln \left(\frac{S_T}{S_0} \right) \sim \phi \left(\left(\mu - \frac{\sigma^2}{2} \right) T, \sigma T^{\frac{1}{2}} \right) \quad (21)$$

So it then follows that

$$\eta = \left(\mu - \frac{\sigma^2}{2}, \frac{\sigma}{T^{\frac{1}{2}}} \right) \quad (22)$$

Now we can define V (expected return over a longer period of time) from 22, which results in:

$$V = \mu - \frac{\sigma^2}{2} \quad (23)$$

In order to compare it to U and properly understand it, we express geometric Brownian motion in discrete-time:

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon (\Delta t)^{\frac{1}{2}} \quad (24)$$

U is then naturally:

$$U = \mu \Delta t \quad (25)$$

3.3.2 Volatility

Further drawing from 24 we get that as Δt approaches zero, we can see that $\sigma(\Delta t)^{\frac{1}{2}}$ is approximately equal to the standard deviation of proportional change in asset price in Δt

In equation 20 “the volatility of a stock price can be defined as the standard deviation of the return provided by the stock in one year when the return is expressed using continuous compounding” (citation from John C. Hull see sources).

While in 19 we see that vol (volatility) is also the stdev (standard deviation) of the natural logarithmic asset price at the end of one year.

3.3.3 Ito's Lemma

The price of any derivate is a function of stochastic variables of the derivative underlying and time, thus we need to understand some of the behaviour of functions of stochastic variables, known as Ito's lemma. The value of a variable x follows an Ito process:

$$dx = a(x, t)dt + b(x, t)dz \quad (26)$$

Here a and b are functions of x and t and dz is a Wiener process. The drift rate of x is thus a and the variance of x is b^2 .

What Ito's lemma says is essentially that a function, G , of x and t , where dz is the same Wiener process as above in 26, follows an Ito process:

$$dG = \left[\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} * b^2 \right] dt + \frac{\partial G}{\partial x} b dz \quad (27)$$

Thus, the drift rate is

$$\text{drift}(G) = \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} * b^2 \quad (28)$$

and the variance

$$\text{Var}[G] = \left(\frac{\partial G}{\partial x}b\right)^2 \quad (29)$$

Now from Ito's lemma we can set $a = \mu S$, $b = \sigma S$ and $x = S$ into 27:

$$dG = \left[\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} * \sigma^2 S^2 \right] dt + \frac{\partial G}{\partial S} \sigma S dz \quad (30)$$

(Note that the source of uncertainty dz affects both S and G)

3.3.4 Derivation of the Black-Scholes-Merton differential equation

Since f is the price of the derivate on S , f must be a function of S and t . Now we'll use formula 30 with respect to f rather than G :

$$df = \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} * \sigma^2 S^2 \right] dt + \frac{\partial f}{\partial S} \sigma S dz \quad (31)$$

or in discrete form (note 24)

$$\Delta f = \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} * \sigma^2 S^2 \right] \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z \quad (32)$$

In the discrete form of the Brownian process,

$$\Delta S = \mu S \Delta t + \sigma S \Delta z \quad (33)$$

As mentioned in the section about Ito's lemma, 26 and 27 have the same Wiener process, thus we get that f and S follow the same Wiener process! We can therefore say $\Delta z = \sigma \epsilon (\Delta t)^{\frac{1}{2}}$ see 24 in the formulas 32 and 33 are the same!

Hull describes it the best here:

It follows that by choosing a portfolio of stock and the derivate, the Wiener process can be eliminated. The appropriate portfolio is:

-1: derivate
 $\frac{\partial f}{\partial S}$: Shares

The holder of this portfolio is short one derivate and long an amount of $\frac{\partial f}{\partial S}$ of shares.

So we define Π as the portfolio value, thus:

$$\begin{aligned} \Pi &= -f + \frac{\partial f}{\partial S} S \\ \Delta \Pi &= -\Delta f + \frac{\partial f}{\partial S} \Delta S \end{aligned} \quad (34)$$

Since we assume that there is no arbitrage Δz must be 0, since the portfolio must be riskless during Δt and earn the risk-free rate (otherwise there would be arbitrage). It follows that:

$$\Delta \Pi = r \Pi \Delta t \quad (35)$$

When we plug 32 and 33 in 34, we get

$$\Delta \Pi = - \left[\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} * \sigma^2 S^2 \right] \Delta t \quad (36)$$

Using 35 and 36, we get

$$\frac{\partial f}{\partial t} + r S \frac{\partial f}{\partial S} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} * \sigma^2 S^2 - r f = 0 \quad (37)$$

Finally we have arrived to the Black-Scholes-Merton differential equation!

The BSM differential equation is riskless for a infinitesimally small time period ($\lim_{\Delta t \rightarrow 0}$). Since S and t changes, $\frac{\partial f}{\partial S}$ changes as well, so the ratio between stocks and derivatives kept in the portfolio gets rebalanced.

The particular derivative depends on the boundary conditions that are used. Different boundary conditions result in different derivatives. For instance for European call options you need to use the following boundary conditions at $t = T$:

$$f = \max(S - K, 0)$$

or for an European put option, it is:

$$f = \max(K - S, 0)$$

3.4 Pricing Formulas

3.4.1 European options

We get the formulas for European options by solving the BSM differential equation for the previously mentioned boundaries (where c for call and p for put is used):

$$\begin{aligned} c &= S_0 N(d_1) - X e^{-rT} N(d_2) \\ p &= X e^{-rT} N(-d_2) - S_0 N(-d_1) \end{aligned} \quad (38)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r^2 + \frac{\sigma^2}{2}\right) T}{\sigma T^{\frac{1}{2}}} \\ d_2 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r^2 - \frac{\sigma^2}{2}\right) T}{\sigma T^{\frac{1}{2}}} = d_1 - \sigma T^{\frac{1}{2}} \end{aligned}$$

The input for dividend paying European options has to be adjusted, which is covered in the Appendix

3.4.2 American call options with non-dividend paying stocks

American call option on a stock that pays no dividends during the life of the option will not be exercised early and, hence, can be valued as a European option with the standard Black-Scholes formula.

3.4.3 European warrants issued by firm on its own stocks

$$W = \left[\left(S_0 + \frac{M}{N} W \right) N(d_1) - K e^{-rT} N(d_2) \right] \frac{N\gamma}{N + M\gamma} \quad (39)$$

where

- γ : The amount of shares per warrant
- N : Outstanding shares
- M : Outstanding European warrants

3.5 Modelling Input

3.5.1 Estimating volatility

If we define

- $n + 1$: Observations
- τ : Time in years
- u_i : Continuously compounded return in i th interval

Due to $S_i = S_{i-1}e^{u_i}$,

$$u_i = \ln \left(\frac{S_i}{S_{i-1}} \right) \quad (40)$$

To then calculate the deviations from the mean we use a similar calculation as with the standard deviation:

$$s = \left(\frac{1}{n-1} \sum_{i=1}^n u_i^2 - \frac{1}{n(n-1)} \left(\sum_{i=1}^n u_i \right)^2 \right)^{\frac{1}{2}}$$

Now, we see from 19 that $\sigma = \frac{s}{\tau^{\frac{1}{2}}}$ but this has some flaws so the standard error can be shown to be

$$\alpha = \frac{\sigma}{(2n)^{0.5}} \quad (41)$$

Note that the volatility has to be annualised since the time used is in years. Choosing n is hard, so here is the general recommendation from Hull:

Choosing an appropriate value for n is not easy. Ceteris paribus, more data generally leads to more accuracy. However, σ does change over time and data that are too old may not be relevant for predicting the future. A compromise that seems to work reasonably well is to use closing prices from daily data over the most recent 90 to 180 days. An often used rule of thumb is to set the time period over which the volatility is measured equal to time period over which it is applied.

GARCH can also be used, see the GARCH Models in volatility models.

If the stock pays dividend, we just need to change u_i during a time interval that includes an ex-dividend day, given by:

$$u_i = \ln \left(\frac{S_i + D}{S_{i-1}} \right) \quad (42)$$

where D is the dividend amount

4 Merton's Jump-Diffusion Model

The Jump-Diffusion model [4] [5] assumes that underlying asset follows a jump-diffusion process. The downside from this is that it requires two more input parameters, which are “hard” to observe in the market. The parameters are the following:

- λ : Expected number of jumps per year
- γ : Percentage of total volatility explained by the jumps

The JmpDif (Jump-Diffusion model) solves the problem with the “Mes Mile-effect” that comes with the BSM model. It also doesn't assume the log normal distribution. Prices of assets vary in “normal” vibrations in price, for example due to temporary supply and demand imbalances, and in “abnormal” vibrations, caused by asset/sector specific news (for example quarterly reports). The “normal” vibrations can be modelled by geometric Brownian motion and the “abnormal” by a “jump” process.

4.1 Additional Notation

- a : Instantaneous expected asset return
- ϵ : Expectations operator
- k : Percent change of asset price if event occurs
- W : Options price
- P : Portfolio value
- τ : Time to expiry
- E : Exercise price

4.2 Calculations

4.2.1 Poisson-driven Process

To start to describe the above mentioned “jump“ process, we’ll use a “Poisson-driven“ process. We now look at the probability of an event occurring in this Poisson distributed process:

1. Prob[event doesn’t occurs in time interval $(t, t + h)$] = $1 - \lambda h + O(h)$
2. Prob[event occurs once in time interval $(t, t + h)$] = $\lambda h + O(h)$
3. Prob[event occurs multiple times in time interval $(t, t + h)$] = $O(h)$

Above, we can set h to whatever we want. $O(h)$ is the asymptotic order symbol defined by $\psi(h) = O(h)$ when $\lim_{h \rightarrow 0} \left[\frac{\psi(h)}{O(h)} \right] = 0$. λ is the mean number of arrivals per time unit.

If this event occurs, then we draw Y from a distribution, so if we ignore the continuity we would get something like $S(t + h) = S(t)Y$

The stock price can be expressed stochastically as:

$$\frac{dS}{S} = (a - \lambda k)dt + \sigma dz + dq \quad (43)$$

In the formula above σ^2 is the variance at the moment (if no Poisson event occurs), dz is the standard Gauss-Wiener process and $q(t)$ is the Poisson process of the event described above. k is defined as $\epsilon(Y - 1)$, where $(Y - 1)$ is the random variable percentage change in the stock price if the Poisson event occurs.

To make it clear, 43 can be rewritten as:

$$\frac{dS}{S} = (a - \lambda k)dt + \sigma dz$$

if event doesn’t occur and

$$\frac{dS}{S} = (a - \lambda k)dt + \sigma dz + (Y - 1)$$

if event does occur

Setting a , λ , k and σ to constants, then the random ratio of stock price at time 0 and time t is:

$$\frac{S_t}{S_0} = \exp \left(a - \frac{\sigma^2}{2} - \lambda kt + \sigma z(t) \right) Y(n) \quad (44)$$

$z(t)$ is a Gaussian random variable, which has a mean of 0 and variance of t . $Y(n)$ is defined as:

$$Y(n) = \begin{cases} 1, & \text{if } n = 0 \\ \prod_{j=1}^n Y_j, & \text{if } n \geq 1 \end{cases} \quad (45)$$

Y_j are independently and identically distributed and n is Poisson distributed with parameter λt . Now, we apply 43 to the options price, W , where $W(t) = F(S, t)$

$$\frac{dW}{W} = (a_W - \lambda k_W)dt + \sigma_W dz + dq_W \quad (46)$$

It has the same terminology as 43 but everything is for the option

When applying Ito’s lemma to the continuous part and a similar to the Poisson process we get:

$$\begin{aligned} a_W &= \frac{\frac{\sigma^2}{2} S^2 F''_{SS}(S, t) + (a - \lambda k) S F'_S + F'_t + \lambda \epsilon(F(S, Y, t) - F(S, t))}{F(S, t)} \\ \sigma_W &= \frac{F'_S(S, t) \sigma S}{F(S, t)} \end{aligned} \quad (47)$$

Since $q_W(t)$ is dependent on $q(t)$ (so only if $q(t)$ occurs $q_W(t)$ occurs). If the event manifests $Y = y$ and $Y_W = \frac{F(SY, t)}{F(S, t)}$. This is how we partly get the result in 47.

Let's try to model portfolio value similarly to 43 and 4.2.1, where w_1 , w_2 , and w_3 will be the proportions of stock, option and riskless asset:

$$\frac{dP}{P} = (a_P - \lambda k_P)dt + \sigma_P dz + dq_P \quad (48)$$

Based on 43 and 4.2.1 we get ($w_3 = 1 - w_1 - w_2$ has been substituted) formulas similar to 47, but notice the differences and similarities:

$$\begin{aligned} a_P &= w_1(a - r) + w_2(a_W - r) + r \\ \sigma_P &= w_1\sigma + w_2\sigma_W \\ Y_P - 1 &= w_1(Y - 1) + w_2 \frac{F(SY, t) - F(S, t)}{F(S, t)} \end{aligned} \quad (49)$$

Due to portfolio mixing being a linear operation and the option price being a non-linear function of the stock price, we can't use the simple method with assumptions we used to get the BSM differential equation (see appendix for greater detail).

4.2.2 Getting a options pricing formula

When the Black Scholes "no arbitrage" method cannot be used, we can still derive a options formula if we can figure of the options expected return as a function of the stock price and time to expiry. We define the function of $g(S, \tau)$ to be the equilibrium momentary expected return. We'll use this and 47, we get that F (a function of τ this time) must hold the following equation (which is a bit similar to the BSM):

$$\frac{1}{2}\sigma^2 S^2 F''_{SS} + (a - \lambda k)SF'_S - F'_\tau - g(S, \tau)F + \lambda\epsilon(F(S, Y, t) - F(S, t)) = 0 \quad (50)$$

subject to the boundary conditions $F(0, \tau) = 0$ and $F(S, 0) = \max(0, S - E)$

The equation of 50 can be solved, but then you have to know a and $g(S, \tau)$. However this is a bit hard and a not so fruitful approach.

Now we'll try price it with CAPM (Capital Asset Pricing Model), which BSM also assumed to be true. Here the jumps will represent non-systematic risks and this is the only source of uncertainty. Thus the β of the portfolio must be 0 and in CAPM the expected return of such portfolios (with 0 β) must be r ($a_{P^*} = r$). Here follows from 49 $w_1^*(a - r) + w_2^*(a_W - r) = 0$ and we substitute w_1^* w_2^* :

$$\frac{a - r}{\sigma} = \frac{a_W - r}{\sigma_W} \quad (51)$$

When using 51, 47 we get that F must hold the following equation:

$$\frac{1}{2}\sigma^2 S^2 F''_{SS} + (r - \lambda k)SF'_S - F'_\tau - rF + \lambda\epsilon(F(S, Y, t) - F(S, t)) = 0 \quad (52)$$

52 still depends on the boundaries specified before and the same equation, except that we have replaced $ag(S, \tau)$ with r .

4.2.3 Solution to the equation

To solve the equation 52 and obtain a solution, we have to further specify the distribution of Y , a partial solution. We'll start by defining $W(S, \tau; E, r, \sigma^2)$ as the no jump Black Scholes formula:

$$W(S, \tau; E, r, \sigma^2) = S\phi(d_1) - Ee^{-r\tau}\phi(d_2) \quad (53)$$

where $\phi(y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^y \exp\left(-\frac{s^2}{2}\right) ds$ i.e. the cumulative normal distribution function and

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\tau^{\frac{1}{2}}} \\ d_2 &= d_1 - \sigma\tau^{\frac{1}{2}} \end{aligned}$$

Defining the random variable X_n to have the same distribution as the product of n independently and identically distributed random variables, with the same distribution of Y used in 43, where $X_0 = 1$. ϵ_n is the expectation operator over the distribution of X_n . The solution to 52 for the options price when the current stock price is S can be written as:

$$F(S, \tau) = \sum_{n=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} [\epsilon_n W(SX_n e^{-\lambda\tau}, \tau; E, \sigma^2, r)] \quad (54)$$

4.3 Special cases of the pricing formula

In these special cases formula 54 can be vastly simplified. Case number 1 is when the Poisson event occurs, the stock price goes to zero and case number 2 is when Y has a log-normal distribution.

4.3.1 Case 1

We need to set $Y = 0$ with a probability to one if the event occurs, $X_n = 0$ if $n \neq 0$ and $k = -1$. Then 54 can be written as:

$$F(S, \tau) = e^{-\lambda\tau} [W(S e^{\lambda\tau}, \tau; E, \sigma^2, r)] = W(S, \tau; E, \sigma^2, r + \lambda) \quad (55)$$

It is identical to the standard Black Scholes, but the interest rate is $r' = r + \lambda$

4.3.2 Case 2

If Y is log-normally distributed, $\gamma = \log(k + 1)$ and σ^2 is the variance of the logarithm of Y . X_n has a log-normal distribution of which $\sigma^2 n$ is the variance of $\log X_n$ and $\epsilon_0(X_0) = e^{n\gamma}$

$$f_n(S, \tau) = W(S, \tau; E, v_n^2, r_n) \quad (56)$$

where $v_n^2 = \sigma^2 + \frac{n\sigma^2}{\tau}$ and $r_n = r - \lambda k + \frac{n\gamma}{\tau}$

It is the standard Black Scholes option, where v_n^2 is the “formal” stocks variance per unit time and r_n is the “formal” instantaneous rate of interest. Finally we have:

$$F(S, \tau) = \sum_{n=1}^{\infty} e^{-\lambda\tau} \frac{(\lambda(1+k)\tau)^n}{n!} f_n(S, \tau) \quad (57)$$

What happens in these “magic” formulas is that f_n is the value of the options when we know how many Poisson jumps will occur during the options life time. $F(S, \tau)$, the actual value of the option, is the weighted sum of these prices, where the weight responds to the probability that a Poisson random variable (with characteristic parameter $\lambda(1+k)\tau$) will take the value of n .

5 Heston Model

The Heston model [2] is a stochastic volatility model, thus it has one Stochastic Differential Equation (SDE) for the asset and another SDE for the stochastic volatility.

5.1 Calculations

5.1.1 Volatility Modelling

In the Heston model we assume that the stock price follows:

$$dS = \mu S dt + v^{\frac{1}{2}} S dz_1 \quad (58)$$

If the volatility, v , follows an Ornstein-Uhlenbeck process:

$$dv^{\frac{1}{2}} = -\beta v^{\frac{1}{2}} dt + \sigma dz_2 \quad (59)$$

Then by using Ito's lemma we see that the variance v follows:

$$dv = (\sigma^2 - 2\beta v)dt + 2\sigma v^{\frac{1}{2}}dz_2 \quad (60)$$

60 can be rewritten as the square-root process and as in CIR:

$$dv = K(\theta - v)dt + \sigma v^{\frac{1}{2}}dz_2 \quad (61)$$

z_1 and z_2 have the correlation of ρ . We assume a constant interest rate r . The price of the asset $f(S, v, t)$ must hold 62, to follow the no arbitrage argument:

$$\frac{1}{2}vS^2\frac{\partial^2 f}{\partial S^2} + \rho\sigma vS\frac{\partial^2 f}{\partial S\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 f}{\partial v^2} + rS\frac{\partial f}{\partial S} + [K(\theta - v) - \lambda]\frac{\partial f}{\partial v} - rf + \frac{\partial f}{\partial t} = 0 \quad (62)$$

In 62 λ represents the price of volatility risk and is asset independent. Research suggests that λ is non zero for equity options; there's approaches to model it for consumption assets but we won't consider them here. We use the following boundary conditions for a European call option with the strike X and with T as maturity in 62:

$$\begin{aligned} f(S, v, T) &= \max(S - X, 0) \\ U(0, v, t) &= 0 \\ \frac{\partial f}{\partial S}(\infty, v, t) &= 1 \\ f(S, \infty, t) &= S \\ rS\frac{\partial f}{\partial S}(S, 0, t) + K\theta\frac{\partial f}{\partial v}(S, 0, t) - rf(S, 0, t) + \frac{\partial f}{\partial t}(S, 0, t) &= 0 \end{aligned} \quad (63)$$

From similarity with the Black Scholes formula, we guess that our solution will be along the lines of:

$$C(S, v, t) = SP_1 - XP(t, T)P_2 \quad (64)$$

where SP_1 is the present value of spot price in optimal exercise and $XP(t, T)P_2$ is the present value of strike price payment We plug 64 into 63 and P_1 and P_2 must satisfy 65 below

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2} + \rho\sigma v\frac{\partial^2 P_j}{\partial x\partial v} + \frac{1}{2}v\sigma^2\frac{\partial^2 P_j}{\partial v^2} + (r + u_j v)\frac{\partial P_j}{\partial x} + [a - b_j v]\frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0 \quad (65)$$

for $j = 1, 2$ and where we have

$$\begin{aligned} x &= \ln S \\ u_1 &= \frac{1}{2} \\ u_2 &= -\frac{1}{2} \\ a &= K\theta \\ b_1 &= K + \lambda - \rho\sigma \\ b_2 &= K + \lambda \end{aligned}$$

Since 65 must follow 62, we have the terminal condition

$$P_j(x, v, T; \ln(X)) = 1_{(x \geq \ln(X))} \quad (66)$$

From this formula we get that the probabilities are risk-neutralized. The conditional probability, P_j , of the option expiring ITM is

$$P_j(x, v, T; \ln(X)) = \text{Prob}[x(T) \geq \ln(X) \mid x(t) = x, v(t) = v] \quad (67)$$

67 holds when x follows

$$\begin{aligned} dx(t) &= (r + u_j v)dt + v(t)^{\frac{1}{2}}dz_1(t) \\ dv &= (a_j - b_j v)dt + \sigma v(t)^{\frac{1}{2}}dz_2(t) \end{aligned} \quad (68)$$

5.2 Characteristics

Since there's no closed form solution to the probability from 67, we have to calculate the characteristic functions. At time T in the future, $f(x, v, t)$ is the expectation of a function $g(x(T), v(T))$, where $x(t)$ and $v(t)$ follow the risk neutral process from 66:

$$f(x, v, t) = E[g(x(T), v(T)) | x(t) = x, v(t) = v] \quad (69)$$

We apply Ito's lemma to this:

$$df = \left[\frac{1}{2}v \frac{\partial^2 f}{\partial x^2} + \rho\sigma v \frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2}v\sigma^2 \frac{\partial^2 f}{\partial v^2} + (r + u_j v) \frac{\partial f}{\partial x} + [a - b_j v] \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} \right] dt + [r + u_j v] \frac{\partial f}{\partial x} dz_1 + [a - b_j v] \frac{\partial f}{\partial v} dz_2 \quad (70)$$

$E[df] = 0$ and when we apply this knowledge to 70, we get

$$\frac{1}{2}v \frac{\partial^2 f}{\partial x^2} + \rho\sigma v \frac{\partial^2 f}{\partial x \partial v} + \frac{1}{2}v\sigma^2 \frac{\partial^2 f}{\partial v^2} + (r + u_j v) \frac{\partial f}{\partial x} + [a - b_j v] \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t} = 0 \quad (71)$$

71 is called the Fokker-Planck forward equation and is very useful, since we know that:

$$f(x, v, T) = g(x, v) \quad (72)$$

Due to this we can set $g(x, v)$ equal to some function and then solve $f(x, v, T)$ from it. Let's take a look at some examples:

- $g(x, v) = \delta(x - x_0) \rightarrow$ Conditional probability density that $x(T) = x_0$ at time t .
- $g(x, v) = 1_{(x \geq \ln(X))} \rightarrow$ Conditional probability that $x(T) > \ln(K)$ at time t .
- $g(x, v) = e^{i\phi x} \rightarrow$ Characteristic functions

To solve for the characteristic functions (which are the solution to the last example of $g(v, x)$ above) we have to have an idea on the functional form. By using the linearity of 70 coefficients, we guess:

$$f(x, v, t) = \exp[C(T - t) + D(T - t)v + i\phi x] \quad (73)$$

In order to reduce the functional form to two ordinary differential equations, we plug 73 into 70:

$$-\frac{1}{2}\sigma^2\phi^2 + \rho\sigma\phi iD + \frac{1}{2}D^2 + u_j\phi i - b_jD + \frac{\partial D}{\partial t} = 0 \quad (74)$$

$$r\phi i + aD + \frac{\partial C}{\partial t} = 0 \quad (75)$$

subject to $C(0) = 0$ and $D(0) = 0$

5.3 Result

To summarize we previously get that the characteristic functions of 66 which hold the terminal condition $f_j(x, v, T; \phi) = e^{i\phi x}$ satisfy 62. The final solution of the characteristic function is:

$$f_j(x, v, t; \phi) = \exp(C(T - t; \phi) + D(T - t; \phi)v + i\phi x); \quad (76)$$

where

$$\begin{aligned} C(\tau, \phi) &= r\phi i\tau + \frac{a}{\sigma^2} * (b_j - \rho\sigma\phi i + d)\tau - 2 \ln \left[\frac{1 - ge^{dr}}{1 - g} \right] \\ D(\tau, \phi) &= \frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2} \frac{1 - e^{dr}}{1 - ge^{dr}} \\ g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} \\ d &= (\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i + \phi^2) \end{aligned} \quad (77)$$

If we want the probabilities, we can invert the characteristic functions:

$$P_j(x, v, T; \ln(X)) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[\frac{e^{-i\phi \ln(X)} f_j(x, v, T; \phi)}{i\phi} \right] d\phi \quad (78)$$

If we want to get the price of a European call option, then we have to get the solution from 64, 76 and 78.

Parameter	Value
Mean reversion	$K^* = 2$
Long-run variance	$\theta^* = 0.01$
Current variance	$v(t) = 0.01$
Correlation of $z_1(t)$ and $z_2(t)$	$\rho = 0.0$
Volatility	$\sigma = 0.1$
Option maturity	$\tau = 0.5$ years
Interest rate	$r = 0$
Strike Price	$X = 100$

Table 1: Default parameters for use with the Heston Model

5.4 Modelling Input

Heston recommends the following default parameter values for the simulation of option prices: Note that you need to adjust (at least) some of these to get realistic results, for instance the strike price, X and the options maturity.

References

- [1] John C. Cox, Stephen A. Ross, and Mark Rubinstein. “Option pricing: A simplified approach”. In: *Journal of Financial Economics* 7.3 (Sept. 1979), pp. 229–263. DOI: [10.1016/0304-405x\(79\)90015-1](https://doi.org/10.1016/0304-405x(79)90015-1). URL: <https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.379.7582&rep=rep1&type=pdf>.
- [2] Steven L. Heston. “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”. In: *Review of Financial Studies* 6.2 (Apr. 1993), pp. 327–343. DOI: [10.1093/rfs/6.2.327](https://doi.org/10.1093/rfs/6.2.327). URL: <https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.379.7582&rep=rep1&type=pdf>.
- [3] John C Hull. *Options, Futures, and Other Derivatives (9th Edition)*. Pearson, 2014. ISBN: 978-0133456318.
- [4] S.G. Kou. “Chapter 2 Jump-Diffusion Models for Asset Pricing in Financial Engineering”. In: *Financial Engineering*. Elsevier, 2007, pp. 73–116. DOI: [10.1016/S0927-0507\(07\)15002-7](https://doi.org/10.1016/S0927-0507(07)15002-7). URL: [https://doi.org/10.1016/S0927-0507\(07\)15002-7](https://doi.org/10.1016/S0927-0507(07)15002-7).
- [5] Robert Merton. “Option Prices When Underlying Stock Returns Are Discontinuous”. In: *Journal of Financial Economics* 3 (Jan. 1976), pp. 125–144. DOI: [10.1016/0304-405X\(76\)90022-2](https://doi.org/10.1016/0304-405X(76)90022-2).