Chapter 1. Preliminaries: Set theory and categories

3 Categories

Problem 3.1. $\triangleright$ Let C be a category. Consider a structure C^{op} with

1. $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$
2. for A, B objects of C^{op} (hence objects of C), $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A)$.

Show how to make this into a category (that is, define composition of morphisms in C^{op} and verify the properties listed in §3.1). Intuitively, the 'opposite' category C^{op} is simply obtained by 'reversing all the arrows' in C.

Solution. Remember that by definition, a category must have i) a composition morphism that satisfies associativity for any pairs of morphisms and ii) an identity morphism that is unital for all objects in the category (e.g. for $f : X \to Y, f1_X = 1_Y f = f$).

i) Composition morphism: Let's define composition of morphisms as follows: for $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, define $f \circ' g \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, C)$ such that $f \circ' g = g \circ f \in \mathrm{Hom}_{\mathsf{C}}(C, A)$. We know $g \circ f$ exists because C is a category and thus satisfies the condition of having a composition morphism for all pairs of morphisms.

To show that the composition morphism is associative, let $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B), g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$, and $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(C, D)$. Then $(f \circ' g) \circ' h = (g \circ f) \circ' h = h \circ (g \circ f) = (h \circ g) \circ f = (g \circ' h) \circ f = f \circ' (g \circ' h)$. Where the 1st, 2nd, 4th, and 5th equalities are due to the definition we chose for the composition of morphisms, and the 3rd equality is true because C is a category and thus its morphisms are associative.

ii) Identity morphism: If A is an object in C^{op}, it also exists in C (by definition of $\mathrm{Obj}(\mathsf{C}^{op})$). Since C is a category, it satisfies the property of having an identity morphism for each object. Let's define 1_A for any A in C^{op} to be the same as 1_A for the same A in C.

To show that the identity morphism is unital, let $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$, then:

1. $f \circ' 1_A = 1_A \circ f = f$
2. $1_B \circ' f = f \circ 1_B = f$

Where for both statements, the first equality is due to our definition of composition and the second equality is true because C is a category and thus its identity morphisms are unital. $\square$

Problem 3.2. If A is a finite set, how large is $\mathrm{End}_{\mathsf{Set}}(A)$?

Solution. By definition $\mathrm{End}_{\mathsf{Set}}(A) = \mathrm{Hom}_{\mathsf{Set}}(A, A)$, and since we are working with the category of sets, we can think of morphisms as set-functions (§3.2). In other words, $\mathrm{End}_{\mathsf{Set}}(A)$ is the set of all functions $f : A \to A$, otherwise denoted as A^A (§2.1, 3.2), and we are asked to find the count of all possible functions $f : A \to A$.

Since A is finite, the count of all possible functions $|\mathrm{End}_{\mathsf{Set}}(A)| = |A|^{|A|}$ (where $|A|$ is the number of elements in A). $\square$

Problem 3.3. ▷ Formulate precisely what it means to say that 1_a is an identity with respect to composition in Example 3.3, and prove this assertion. [§3.2]

Solution. 1_a is an identity if for any three objects $z, a, b \in S$, where $e \in \text{Hom}(z, a)$ and $f \in \text{Hom}(a, b)$, $1_a e = e$ and $f 1_a = f$.

As described in Example 3.3, we only have one choice for $1_a \in \text{Hom}(a, a)$ where $1_a = (a, a)$. By our definition of morphism, we also have $e = (z, a)$ and $f = (a, b)$. Using our definition of composition, $1_a e \in \text{Hom}(z, a)$ and $f 1_a \in \text{Hom}(a, b)$. It follows that $1_a e = (z, a) = e$ and $f 1_a = (a, b) = f$.

□

Problem 3.4. Can we define a category in the style of Example 3.3 using the relation $<$ on the set $\mathbf{Z}$?

Solution. No, since $<$ is not reflexive, if follows that the set $\text{Hom}(A, A)$ is empty, and therefore we cannot define an identity morphism. □

Problem 3.5. ▷ Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3. [§3.2]

Solution. For the sake of clarity, let S' represent the set in Example 3.3 and S represent the set in Example 3.4. We can think of $S' = \mathscr{P}(S)$ where each element $a, b \in S'$ represents $A, B \subseteq S$. Both $\sim$ and $\subseteq$ are reflexive and transitive. For both categories, a morphism between objects is either a pair, (a, b) if $a \sim b$ or (A, B) if $A \subseteq B$, or $\varnothing$ otherwise. □

Problem 3.6. ▷ (Assuming some familiarity with linear algebra.) Define a category V by taking $\text{Obj}(\mathsf{V}) = \mathbf{N}$ and letting $\text{Hom}_{\mathsf{V}}(n, m) =$ the set of $m \times n$ matrices with real entries, for all $n, m \in \mathbf{N}$. (We will leave the reader the task of making sense of a matrix with 0 rows or columns.) Use product of matrices to define composition. Does this category 'feel' familiar? [§VI.2.1, §VIII.1.3]

Solution. Reminder: Remember that if V is n-dimensional and W is m-dimensional, then $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are isomorphic [LADR 3.60], where $\mathcal{L}(V, W)$ is the set of all linear maps from V to W [LADR 3.3] and $\mathbf{F}^{m,n}$ is the set of all matrices with m rows and n columns [LADR 3.39]. Additionally, two finite-dimensional vector spaces over $\mathbf{F}$ (and in this case $\mathbf{R}$ which is a subset of $\mathbf{F}$) are isomorphic if and only if they have the same dimension [LADR 3.59].

The question gives us a category theoretic way of describing linear maps. In linear algebra we deal exclusively with finite-dimensional vector spaces. Since every linear map $T \in \mathcal{L}(V, W)$, where V is n-dimensional and W is m-dimensional, can be represented as an m-by-n matrix, it follows that we can represent every linear map as a morphism from n to m where $n, m \in \text{Obj}(\mathsf{V})$.

If a matrix has 0 columns, we can try to interpret it as a linear map from a zero-dimensional vector space to a non-zero dimensional vector space. However this contradicts the definition of linear maps which must satisfy additivity [LADR 3.11]. Thus, we can say that if $m \neq 0$, then $\mathrm{Hom}_{\mathsf{V}}(m, 0) = \varnothing$.

If a matrix has 0 rows, we can interpret it as a linear map to a zero-dimensional vector space. In other words, for all $v \in V$, $Tv = 0v = 0$. We can represent this as any m-by-n matrix with every element in the matrix $= 0$. □

Problem 3.7. ▷ Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition [§3.2].

Solution. Let C be a category and A be an object in C. We define the category $\overline{\mathsf{C}}_A$ as follows:

1. $\mathrm{Obj}(\overline{\mathsf{C}}_A) := \mathrm{Hom}_{\mathsf{C}}(A, X)$ where X is any object in C. Pictorially, an object of $\overline{\mathsf{C}}_A$ is an arrow:

$$\begin{tikzcd} A \arrow[d, "f_1"] \\ X_1 \end{tikzcd}$$

2. Morphisms in $\overline{\mathsf{C}}_A$ can defined as commutative diagrams. Let f_1 and f_2 be objects of $\overline{\mathsf{C}}_A$, that is two arrows:

$$\begin{tikzcd} A \arrow[d, "f_1"] & A \arrow[d, "f_2"] \\ X_1 & X_2 \end{tikzcd}$$

The morphism $f_1 \to f_2$ can be defined as the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "f_1"] & A \arrow[d, "f_2"] \\ X_1 \arrow[r, "\sigma"] & X_2 \end{tikzcd}$$

However, since we are always dealing with A as the domain, we can simplify as follows:

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 \arrow[rr, "\sigma"] & & X_2 \end{tikzcd}$$

3. We define composition of morphisms as putting two commutative diagrams side-by-side:

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f_2"] \arrow[dr, "f_3"] & \\ X_1 \arrow[r, "\sigma"] & X_2 \arrow[r, "\tau"] & X_3 \end{tikzcd}$$

Because C is commutative and $f_1, f_2, f_3, \sigma, \tau$ are morphisms in C, we can remove the central arrow:

$$\begin{tikzcd}
& A \arrow[dl, "f_1"'] \arrow[dr, "f_3"] & \\
X_1 \arrow[rr, "\tau\sigma"] & & X_3
\end{tikzcd}$$

4. We define the identity morphism using the identities in C. For $f_1 : A \to X$ in $\overline{\mathsf{C}}_A$, the identity 1_f corresponds to:

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "f"] & \\
X \arrow[rr, "1_f"] & & X
\end{tikzcd}$$

Given our choice of composition and identities, we need to now check whether the morphisms are associative and unital.

a) Associativity: Let $f_1, f_2, f_3, f_4 \in \overline{\mathsf{C}}_A$ such that:

$$\begin{tikzcd}
& & A \arrow[dll, bend right, "f_1"'] \arrow[dl, "f_2"'] \arrow[dr, "f_3"] \arrow[drr, bend left, "f_4"] & & \\
X_1 \arrow[r, "\sigma"] & X_2 \arrow[rr, "\tau"] & & X_3 \arrow[r, "\rho"] & X_4
\end{tikzcd}$$

Since C is commutative, we can remove either of the two central arrows to get the following diagrams:

$$\begin{tikzcd}
& & A \arrow[dll, bend right, "f_1"'] \arrow[dr, "f_3"] \arrow[drr, bend left, "f_4"] & & \\
X_1 \arrow[rrr, "\tau\sigma"] & & & X_3 \arrow[r, "\rho"] & X_4
\end{tikzcd}$$

$$\begin{tikzcd}
& & A \arrow[dll, bend right, "f_1"'] \arrow[dl, "f_2"'] \arrow[drr, bend left, "f_4"] & & \\
X_1 \arrow[r, "\sigma"] & X_2 \arrow[rrr, "\rho\tau"] & & & X_4
\end{tikzcd}$$

Since C is commutative, both diagrams are equivalent. Thus $(\rho\tau)\sigma = \rho(\tau\sigma)$.

b) Unital: Let $f_0, f_1, f_2 \in \overline{\mathsf{C}}_A$ such that:

$$\begin{tikzcd}
A \arrow[d, "f_0"] & A \arrow[d, "f_1"] & A \arrow[d, "f_2"] \\
X_0 & X_1 & X_2
\end{tikzcd}$$

Then given our definition of identity and composition, we get:

$$\begin{tikzcd} & A \arrow[dl, "f_0"'] \arrow[d, "f_1"] \arrow[dr, "f_1"] & \\ X_0 \arrow[r, "\phi"'] & X_1 \arrow[r, "1_{f_1}"'] & X_1 \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "f_1"] \arrow[dr, "f_2"] & \\ X_1 \arrow[r, "1_{f_1}"'] & X_1 \arrow[r, "\sigma"'] & X_2 \end{tikzcd}$$

And by removing the central arrows, we get:

$$\begin{tikzcd} & A \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\ X_0 \arrow[rr, "\phi=1_{f_1}\phi"'] & & X_1 \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ X_1 \arrow[rr, "\sigma=\sigma 1_{f_1}"'] & & X_2 \end{tikzcd}$$

It is clear from the diagrams that $\phi = 1_{f_1}\phi$ and $\sigma = \sigma 1_{f_1}$, thus proving that our identity morphisms are unital.

$\square$

Problem 3.8. $\triangleright$ A *subcategory* C' of a category C consists of a collection of objects of C, with morphisms $\mathrm{Hom}_{\mathsf{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects A, B in $\mathrm{Obj}(\mathsf{C}')$, such that identities and compositions in C make C' into a category. A subcategory C' is *full* if $\mathrm{Hom}_{\mathsf{C}'}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all A, B in $\mathrm{Obj}(\mathsf{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of Set. [4.4, §VI.1.1, §VIII.1.3]

Solution. Let $\mathsf{Set}_{\mathrm{inf}}$ be the category whose objects are the infinite sets in Set and whose morphisms are all the set-functions between infinite sets in Set. In other words, if A, B are infinite sets, $\mathrm{Hom}_{\mathsf{Set}_{\mathrm{inf}}}(A, B) = \mathrm{Hom}_{\mathsf{Set}}(A, B)$. We also inherit composition and identity from Set. It suffices to show that $\mathsf{Set}_{\mathrm{inf}}$ satisfies associativity and unity (i.e. is a category), which is a trivial exercise (since we inherited composition, identity, set-functions, and sets from Set which is a category). $\square$

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Problem 3.9.

Problem 3.10.

Problem 3.11. $\triangleright$ Draw the relevant diagrams and define composition and identities for the category $\mathsf{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10 [§5.5, 5.12].

Solution. [Example 3.9] Given two objects A, B of C, we define a new category $\mathsf{C}^{A,B}$ as follows

1. $\mathrm{Obj}(\mathsf{C}_{A,B})$ = diagrams in C, where $Z \in \mathrm{Obj}(\mathsf{C})$;

$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
& Z \\
B \arrow[ur, "g"] &
\end{tikzcd}$$

2. morphisms correspond *commutative* diagrams;

$$\begin{tikzcd}
A \arrow[dr, "f_1"] & & & A \arrow[dr, "f_2"] & \\
& Z_1 & \xrightarrow{\sigma} & & Z_2 \\
B \arrow[ur, "g_1"] & & & B \arrow[ur, "g_2"] &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}$$

3. compositions are obtained by placing commutative diagrams side-by-side;

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] \arrow[drrr, bend left, "f_3"] & & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3 \\
B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_2"'] \arrow[urrr, bend right, "g_3"'] & & &
\end{tikzcd}$$

we can then remove the center diagram;

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_3"] & & \\
& Z_1 \arrow[r, "\tau\sigma"] & Z_3 \\
B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_3"'] & &
\end{tikzcd}$$

4. identity morphisms are inherited from identities in C;

$$
\begin{tikzcd}
A \arrow[dr, "f"] \arrow[drr, bend left, "f"] & & \\
& Z \arrow[r, "1_Z"] & Z \\
B \arrow[ur, "g"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}
$$

Since C is commutative, associative, and unital, $\mathsf{C}^{A,B}$ is also commutative, associate, and unital.

[Example 3.10] To define $\mathsf{C}^{\alpha,\beta}$, we choose two fixed morphisms $\alpha : C \to A, \beta : C \to B$ in C. We then consider the data of $\mathsf{C}^{\alpha,\beta}$ as follows:

1. $\text{Obj}(\mathsf{C}^{\alpha,\beta})$ = commutative diagrams in C, where $Z \in \text{Obj}(\mathsf{C})$;

$$
\begin{tikzcd}
& A \arrow[dr, "f"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \\
& B \arrow[ur, "g"'] &
\end{tikzcd}
$$

2. morphisms correspond to *commutative* diagrams;

$$
\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \\
& B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}
$$

3. composition correspond to placing commutative diagrams side-by-side;

$$
\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] \arrow[drrr, bend left, "f_3"] & & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3 \\
& B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_2"'] \arrow[urrr, bend right, "g_3"'] & & &
\end{tikzcd}
$$

after which one can remove the center diagram, resulting in a diagram that also commutes;

$$\begin{tikzcd}
 & A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_3"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\tau\sigma"] & Z_3 \\
 & B \arrow[ur, "g_1"] \arrow[urr, bend right, "g_3"'] & &
\end{tikzcd}$$

4. identity morphisms are inherited from C, resulting in the following diagram;

$$\begin{tikzcd}
 & A \arrow[dr, "f"] \arrow[drr, bend left, "f"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \arrow[r, "1_Z"] & Z \\
 & B \arrow[ur, "g"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Since C is commutative, associative, and unital, $\mathsf{C}^{\alpha,\beta}$ is also commutative, associate, and unital. $\square$

4 Morphisms

Problem 4.1.

Problem 4.2.

Problem 4.3. Let A, B be objects of a category C, and let $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ be a morphisms.

1. Prove that if f has a right-inverse, then f is an epimorphism.
2. Show that the converse does not hold, by giving an explicit example of a category and an epimorphisms without a right-inverse.

Solution. Let $f' : B \to A$ be a right-inverse of f and let $B', B'' \in \mathrm{Hom}_{\mathsf{C}}(B, A)$. If $B'f = B''f \implies B'ff' = B''ff' \implies B'1_B = B''1_B \implies B' = B''$.

To show the converse doesn't hold, we use Example 4.5 (and Example 3.3), where the objects of C are the integers and $\mathrm{Hom}_{\mathsf{C}}(a, b) =$ the pair (a, b) if $a \leq b$ or $\emptyset$ otherwise, where $a, b \in \mathrm{Obj}(\mathsf{C})$. Because there is at most one morphism between two objects, the fact that each morphism is epimorphic is vacuously true. Moreover, the pair (a, b) generally does not have a right-inverse unless $a = b$. $\square$

5 Morphisms

Problem 5.1.

Problem 5.2.

Problem 5.3.

Problem 5.4.

Problem 5.5.

Problem 5.6.

Problem 5.7.

Problem 5.8. Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic, if they exist. (Hint: Observe that they both satisfy the universal property for the product of A and B; then use Proposition 5.4.)

Solution. We use a similar proof to the one provided in §I.5.4. But first, we define $B \times A$ as the *product of sets* such that $B \times A = \{(b, a) | b \in B, a \in A\}$ and the *natural projections* $\pi'_A((b, a)) := a$ and $\pi'_B((b, a)) := b$. If $A \times B$ exists, then $B \times A$ also exists and vice versa. Let Z be an object in C.

For any object
$$\begin{tikzcd} & A \\ Z \arrow[ur, "f_A"] \arrow[dr, "f_B"'] & \\ & B \end{tikzcd}$$
, there exists a unique $\tau : Z \to B \times A$ such that the following diagram commutes:
$$\begin{tikzcd} & & A \\ Z \arrow[r, "\tau"] \arrow[urr, bend left, "f_A"] \arrow[drr, bend right, "f_B"'] & B \times A \arrow[ur, "\pi'_A"] \arrow[dr, "\pi'_B"] & \\ & & B \end{tikzcd}$$
. For all $z \in Z$, we define

$$
\begin{aligned}
\tau(z) &= (f_B(z), f_A(z)) \text{ where} \\
\pi'_B\tau(z) &= \pi'_B(f_B(z), f_A(z)) = f_B(z) \\
\pi'_A\tau(z) &= \pi'_A(f_B(z), f_A(z)) = f_A(z),
\end{aligned}
$$

showing that the diagram commutes. Since τ is defined by the object [?? since f_B, f_A are unique ??], it is thus a unique morphism from that object to $B \times A$, thereby proving that $B \times A$ is a terminal object.

Now that we've proved that $B \times A$ is also a terminal object, we use Proposition 5.4 to show that is it isomorphic to $A \times B$ which is also a terminal object, thereby completing the proof $\square$

Problem 5.9. Let C be a category of products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of *three* objects of C ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Solution. It is a terminal object for any object Z $\square$