

Problem Set 1: Sets and Categories

July 12, 2021

1. Define a relation \sim on the set \mathbb{R} of real numbers by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R} \setminus \sim$. Do the same for relation \approx on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$.

Solution

Let $a, b, c \in \mathbb{R}$. We have $a - a = 0 \in \mathbb{Z}$, so $a \sim a$ and \sim is reflexive. If $a \sim b$ then $a - b \in \mathbb{Z}$. As negating an integer yields an integer we have that $-(a - b) = b - a \in \mathbb{Z}$, which implies $b \sim a$. Hence \sim is symmetric. Suppose that both $a \sim b$ and $b \sim c$ are true. Then $a - b$ and $b - c$ are integers. Since the sum of two integers is an integer we have that $(a - b) + (b - c) = a - c \in \mathbb{Z}$; hence $a \sim c$ and \sim is transitive. We have shown that \sim is an equivalence relation.

We can identify $\mathbb{R} \setminus \sim$ with the interval $[0, 1)$ very naturally. Let $[r]_{\sim} \in \mathbb{R} \setminus \sim$. The *fractional part* of r , denoted $\{r\}$, is the difference between r and its integer part, i.e. $r - [r]$, where $[r]$ denotes the floor of r . Almost by definition we have $r \sim \{r\}$, i.e. $[r]_{\sim} = [\{r\}]_{\sim}$, and further $\{r\} \in [0, 1)$. Conversely, elements of $[0, 1)$ correspond to distinct equivalence classes: if $a, b \in [0, 1)$ then $a - b$ is not an integer.

The proof that \approx is an equivalence relation is similar to the above. We can identify $(\mathbb{R} \times \mathbb{R}) \setminus \approx$ with the set $[0, 1) \times [0, 1)$.

2. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

Solution

For some sets A, B let $f: A \rightarrow B$ be a bijection, and let $f^{-1}: B \rightarrow A$ be its two-sided inverse. This means $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$. But these same equations show that f is a two-sided inverse of f^{-1} . Hence f^{-1} is also a bijection.

Suppose C is another set and let $g: B \rightarrow C$ be a bijection with $g^{-1}: C \rightarrow B$ as its inverse. We claim that $g \circ f$ has $f^{-1} \circ g^{-1}$ as an inverse. Indeed,

since

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^1) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\ &= (g \circ 1_B) \circ g^{-1} \\ &= g \circ g^{-1} = 1_C,\end{aligned}$$

we see that $f^{-1} \circ g^1$ is a right inverse. Proving that it's a left inverse is entirely analogous. So, $g \circ f$ is bijective.

3. Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \sqcup B$ is well-defined up to isomorphism.

Solution

Let $f: A' \rightarrow A''$ and $g: B' \rightarrow B''$ be bijections (these exist by definition of isomorphism). Define a function $h: A' \cup B' \rightarrow A'' \cup B''$ by the following rule. For all $x \in A' \cup B'$ we set

$$h(x) := \begin{cases} f(x) & \text{if } x \in A' \\ g(x) & \text{if } x \in B'. \end{cases}$$

This function is well defined since $A' \cap B' = \emptyset$, so, for every $x \in A' \cup B'$, we have either $x \in A'$ or $x \in B'$ but not both. We will show that h is a bijection, finishing the proof.

Suppose that $h(x) = h(y)$ for some $x, y \in A' \cup B'$. Then either $x, y \in A'$ or $x, y \in B'$ but not both; this is because if, for example, $x \in A'$ and $y \in B'$ then $h(x) = f(x) \in A''$ and $h(y) = g(y) \in B''$, which contradicts $h(x) = h(y)$ as $A'' \cap B'' = \emptyset$. If $x, y \in A'$ then $f(x) = h(x) = h(y) = f(y)$; since f is injective it follows that $x = y$. Similarly, if $x, y \in B'$ then $g(x) = g(y)$ and so $x = y$. Hence, h is injective.

Let $z \in A'' \cup B''$. If $z \in A''$ then there is some $x \in A'$ such that $z = f(x) = h(x)$, given that f is surjective. On the other hand, if $z \in B''$ then there is some $y \in B'$ such that $z = g(y) = h(y)$, since g is surjective. Then h is surjective, finishing the proof.

3. Let \mathbf{C} be a category. Consider the structure \mathbf{C}^{op} with

- $\text{Obj}(\mathbf{C}^{op}) := \text{Obj}(\mathbf{C})$;
- for A, B objects of \mathbf{C}^{op} (hence objects of \mathbf{C}), $\text{Hom}_{\mathbf{C}^{op}}(A, B) := \text{Hom}_{\mathbf{C}}(B, A)$.

Show how to make this into a category.

Solution

We are already given objects and morphisms of \mathcal{C}^{op} , so all we need to check is that they satisfy the axioms.

For any object A of \mathcal{C}^{op} (or, equivalently, \mathcal{C}), we have that $\text{Hom}_{\mathcal{C}}(A, A) = \text{Hom}_{\mathcal{C}^{op}}(A, A)$; hence 1_A , the identity morphism of A in \mathcal{C} , also belongs to $\text{Hom}_{\mathcal{C}^{op}}(A, A)$. So, each object has an endomorphism.

Let A, B, C be objects of \mathcal{C}^{op} , so that there are some morphisms $f \in \text{Hom}_{\mathcal{C}^{op}}(A, B)$ and $g \in \text{Hom}_{\mathcal{C}^{op}}(B, C)$. Then, $f \in \text{Hom}_{\mathcal{C}}(B, A)$ and $g \in \text{Hom}_{\mathcal{C}}(C, B)$. By composition in \mathcal{C} , there is a morphism $fg \in \text{Hom}_{\mathcal{C}}(C, A)$. But then $fg \in \text{Hom}_{\mathcal{C}^{op}}(A, C)$; this proves that one can compose morphisms in \mathcal{C}^{op} .

By the above paragraph, composition in \mathcal{C}^{op} is just composition in \mathcal{C} , so associativity is preserved in this new category. Similarly, identity morphisms are identities with respect to composition and morphisms preserve information about the source and target.

4. Define a category \mathbf{V} by taking $\text{Obj}(\mathbf{V}) := \mathbb{N}$ and letting $\text{Hom}_{\mathbf{V}}(n, m) :=$ the set of $m \times n$ matrices with real entries, for all $m, n \in \mathbb{N}$. Use matrix multiplication to define composition. Does this category “feel” familiar?

Solution

This category trivially satisfies all of the properties morphisms ought to have. One can think of \mathbf{V} as the category of real finite-dimensional vector spaces, where morphisms are linear maps.

5. Draw the relevant diagrams and define composition and identities for category $\mathcal{C}^{A,B}$, mentioned in Example 3.9. Do the same for category $\mathcal{C}^{\alpha,\beta}$ mentioned in Example 3.10.

Solution

6. Let A, B be objects of a category \mathcal{C} , and let $f \in \text{Hom}_{\mathcal{C}}(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

Solution

Suppose f has a right-inverse, $g \in \text{Hom}_{\mathcal{C}}(B, A)$. Let Z be an object of \mathcal{C} and let $\beta', \beta'' \in \text{Hom}_{\mathcal{C}}(B, Z)$. Furthermore, assume that $\beta' \circ f = \beta'' \circ f$. Then we can apply g to both sides:

$$\beta' \circ 1_B = \beta' \circ (fg) = \beta'' \circ (fg) = \beta'' \circ 1_B,$$

where we have used associativity. It follows that $\beta' = \beta''$ and hence f is an epimorphism.

In the category defined by \leq on \mathbb{Z} , there is exactly one epimorphism from 3 to 5. Yet this morphism does not have a right inverse; indeed, there isn't even a morphism from 5 to 3.

7. Show that in every category \mathbf{C} the products $A \times B$ and $B \times A$ are isomorphic, if they exist.

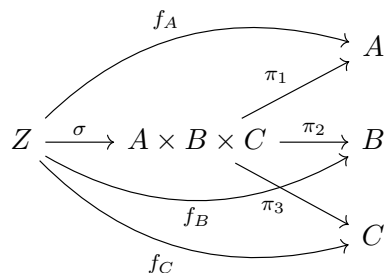
Solution

Note that $\mathbf{C}^{A,B}$ and $\mathbf{C}^{B,A}$ are the same category. Hence both $A \times B$ and $B \times A$ are final objects of the same category, thus isomorphic.

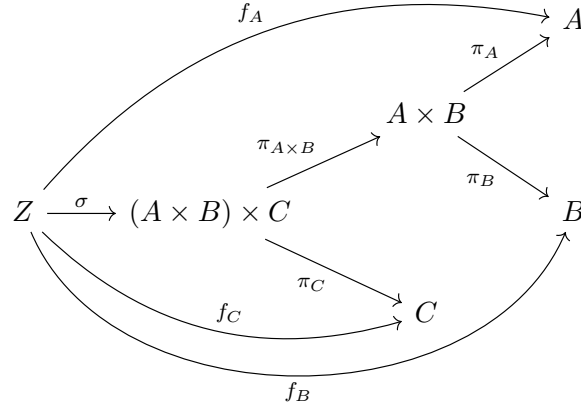
8. Let \mathbf{C} be a category with products. Find a reasonable candidate for the universal property that the universal product $A \times B \times C$ of *three* objects of \mathbf{C} ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Solution

For objects A, B, C the triple product $A \times B \times C$ must satisfy the following. There must be three morphisms π_1, π_2, π_3 such that for any object Z and morphisms f_A, f_B, f_C there exists a unique morphism σ such that the following diagram commutes.

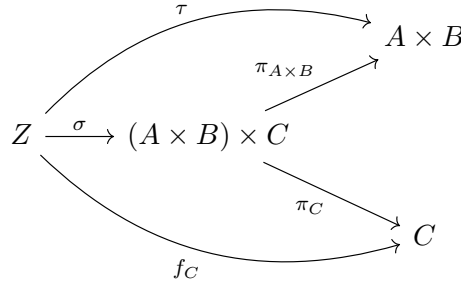


We show that $(A \times B) \times C$ satisfies this property. Since $A \times B$ is a product, there are two associated projection morphisms, π_A and π_B , to this product. Similarly, since $(A \times B) \times C$ is a product, there are two associated projection morphisms, $\pi_{A \times B}$ and π_C , to this product. We claim that there is a unique σ such that the following diagram commutes.



If we can show this, then we have shown that $(A \times B) \times C$ is a triple product, since we can take $\pi_1 := \pi_A \circ \pi_{A \times B}$, and $\pi_2 := \pi_B \circ \pi_{A \times B}$, and $\pi_3 := \pi_C$ in the first diagram.

As $A \times B$ is a product, there is a unique morphism τ from Z to $A \times B$ such that $\pi_A \circ \tau = f_A$ and $\pi_B \circ \tau = f_B$. In addition, we have the following sub-diagram.



Which commutes for a unique σ since $(A \times B) \times C$ is a product. For completeness, we can check that this σ indeed makes the whole diagram commute. We have three equalities to check, one of which is already given by the sub-diagram, namely $\pi_C \circ \sigma = f_C$. Next, consider $\pi_A \circ \pi_{A \times B} \circ \sigma$, which equals $\pi_A \circ \tau$ by commutativity of the sub-diagram, and this in turn equals f_A (we remarked this when we defined τ). Similarly, one can check that $\pi_B \circ \pi_{A \times B} \circ \sigma = f_B$.

So, we have shown the existence of a σ that makes the diagram commute, but we haven't yet shown that it is the unique morphism with this property. Let σ' be a morphism that also makes the diagram commute. Then

$$\begin{aligned} f_A &= \pi_A \circ (\pi_{A \times B} \circ \sigma'), \\ f_B &= \pi_B \circ (\pi_{A \times B} \circ \sigma'). \end{aligned}$$

We conclude that $\pi_{A \times B} \circ \sigma' = \tau$ since τ is the unique morphism that

satisfies the above identities. Then we have

$$\begin{aligned}\tau &= \pi_{A \times B} \circ \sigma' \\ f_C &= \pi_C \circ \sigma.\end{aligned}$$

Hence $\sigma' = \sigma$ since σ is the only morphism that makes the sub-diagram commute. We have shown that $(A \times B) \times C$ is a triple product, and in a similar fashion one can prove that $A \times (B \times C)$ is a triple product. One can define a category $\mathbf{C}^{A,B,C}$ such that triple products are terminal in that category; hence all triple products are isomorphic.