Problem Set 1: Sets and Categories

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1. Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}\backslash \sim$. Do the same for relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$.

Solution

Let $a, b, c \in \mathbb{R}$. We have $a - a = 0 \in \mathbb{Z}$, so $a \sim a$ and $\sim$ is reflexive. If $a \sim b$ then $a - b \in \mathbb{Z}$. As negating an integer yields an integer we have that $-(a - b) = b - a \in \mathbb{Z}$, which implies $b \sim a$. Hence $\sim$ is symmetric. Suppose that both $a \sim b$ and $b \sim c$ are true. Then $a - b$ and $b - c$ are integers. Since the sum of two integers is an integer we have that $(a-b)+(b-c) = a-c \in \mathbb{Z}$; hence $a \sim c$ and $\sim$ is transitive. We have shown that $\sim$ is an equivalence relation.

We can identify $\mathbb{R}\backslash \sim$ with the interval $[0, 1)$ very naturally. Let $[r]_\sim \in \mathbb{R}\backslash \sim$. The *fractional part* of r, denoted $\{r\}$, is the difference between r and its integer part, i.e. $r - \lfloor r \rfloor$, where $\lfloor r \rfloor$ denotes the floor of r. Almost by definition we have $r \sim \{r\}$, i.e. $[r]_\sim = [\{r\}]_\sim$, and further $\{r\} \in [0, 1)$. Conversely, elements of $[0, 1)$ correspond to distinct equivalence classes: if $a, b \in [0, 1)$ then $a - b$ is not an integer.

The proof that $\approx$ is an equivalence relation is similar to the above. We can identify $(\mathbb{R} \times \mathbb{R})\backslash \approx$ with the set $[0, 1) \times [0, 1)$.

2. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

Solution

For some sets A, B let $f : A \to B$ be a bijection, and let $f^{-1} : B \to A$ be its two-sided inverse. This means $f \circ f^{-1} = 1_B$ and $f^{-1} \circ f = 1_A$. But these same equations show that f is a two-sided inverse of f^{-1}. Hence f^{-1} is also a bijection.

Suppose C is another set and let $g : B \to C$ be a bijection with $g^{-1} : C \to B$ as its inverse. We claim that $g \circ f$ has $f^{-1} \circ g^1$ as an inverse. Indeed, since

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\ &= (g \circ 1_B) \circ g^{-1} \\ &= g \circ g^{-1} = 1_C,\end{aligned}$$

we see that $f^{-1} \circ g^{1}$ is a right inverse. Proving that it's a left inverse is entirely analogous. So, $g \circ f$ is bijective.

3. Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \sqcup B$ is well-defined *up to isomorphism*.

Solution

Let $f\colon A' \to A''$ and $g\colon B' \to B''$ be bijections (these exist by definition of isomorphism). Define a function $h\colon A' \cup B' \to A'' \cup B''$ by the following rule. For all $x \in A' \cup B'$ we set

$$h(x) := \begin{cases} f(x) & \text{if } x \in A' \\ g(x) & \text{if } x \in B'. \end{cases}$$

This function is well defined since $A' \cap B' = \emptyset$, so, for every $x \in A' \cup B'$, we have either $x \in A'$ or $x \in B'$ but not both. We will show that h is a bijection, finishing the proof.

Suppose that $h(x) = h(y)$ for some $x, y \in A' \cup B'$. Then either $x, y \in A'$ or $x, y \in B'$ but not both; this is because if, for example, $x \in A'$ and $y \in B'$ then $h(x) = f(x) \in A''$ and $h(y) = g(y) \in B''$, which contradicts $h(x) = h(y)$ as $A'' \cap B'' = \emptyset$. If $x, y \in A'$ then $f(x) = h(x) = h(y) = f(y)$; since f is injective it follows that $x = y$. Similarly, if $x, y \in B'$ then $g(x) = g(y)$ and so $x = y$. Hence, h is injective.

Let $z \in A'' \cup B''$. If $z \in A''$ then there is some $x \in A'$ such that $z = f(x) = h(x)$, given that f is surjective. On the other hand, if $z \in B''$ then there is some $y \in B'$ such that $z = g(y) = h(y)$, since g is surjective. Then h is surjective, finishing the proof.

3. Let C be a category. Consider the structure C^{op} with

- $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$;
- for A, B objects of C^{op} (hence objects of C), $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A)$.

Show how to make this into a category.

Solution

We are already given objects and morphisms of C^{op}, so all we need to check is that they satisfy the axioms.

For any object A of C^{op} (or, equivalently, C), we have that $\mathrm{Hom}_{\mathsf{C}}(A, A) = \mathrm{Hom}_{\mathsf{C}^{op}}(A, A)$; hence 1_A, the identity morphism of A in C, also belongs to $\mathrm{Hom}_{\mathsf{C}^{op}}(A, A)$. So, each object has an endomorphism.

Let A, B, C be objects of C^{op}, so that there are some morphisms $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, C)$. Then, $f \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(C, B)$. By composition in C, there is a morphism $fg \in \mathrm{Hom}_{\mathsf{C}}(C, A)$. But then $fg \in \mathrm{Hom}_{\mathsf{C}^{op}}(A, C)$; this proves that one can compose morphisms in C^{op}.

By the above paragraph, composition in C^{op} is just composition in C, so associativity is preserved in this new category. Similarly, identity morphisms are identities with respect to composition and morphisms preserve information about the source and target.

4. Define a category V by taking $\mathrm{Obj}(\mathsf{V}) := \mathbb{N}$ an letting $\mathrm{Hom}_{\mathsf{V}}(n, m) :=$ the set of $m \times n$ matrices with real entries, for all $m, n \in \mathbb{N}$. Use matrix multiplication to define composition. Does this category "feel" familiar?

Solution

This category trivially satisfies all of the properties morphisms ought to have. One can think of V as the category of real finite-dimensional vector spaces, where morphisms are linear maps.

5. Draw the relevant diagrams and define composition and identities for category $\mathsf{C}^{A,B}$, mentioned in Example 3.9. Do the same for category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10.

Solution

6. Let A, B be objects of a category C, and let $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

Solution

Suppose f has a right-inverse, $g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$. Let Z be an object of C and let $\beta', \beta'' \in \mathrm{Hom}_{\mathsf{C}}(B, Z)$. Furthermore, assume that $\beta' \circ f = \beta'' \circ f$. Then we can apply g to both sides:

$$\beta' \circ 1_B = \beta' \circ (fg) = \beta'' \circ (fg) = \beta'' \circ 1_B,$$

where we have used associativity. It follows that $\beta' = \beta''$ and hence f is an epimorphism.

In the category defined by $\leq$ on $\mathbb{Z}$, there is exactly one epimorphism from 3 to 5. Yet this morphism does nor have a right inverse; indeed, there isn't even a morphism from 5 to 3.

7. Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic, if they exist.

Solution

Note that $\mathsf{C}^{A,B}$ and $\mathsf{C}^{B,A}$ are the same category. Hence both $A \times B$ and $B \times A$ are final objects of the same category, thus isomorphic.

8. Let C be a category with products. Find a reasonable candidate for the universal property that the universal product $A \times B \times C$ of *three* objects of C ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Solution

For objects A, B, C the triple product $A \times B \times C$ must satisfy the following. There must be three morphisms π_1, π_2, π_3 such that for any object Z and morphisms f_A, f_B, f_C there exists a unique morphism σ such that the following diagram commutes.

$$
\begin{tikzcd}
 & & A \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "f_A"] \arrow[rr, bend right, "f_B"'] \arrow[drr, bend right, "f_C"'] & A \times B \times C \arrow[ur, "\pi_1"] \arrow[r, "\pi_2"] \arrow[dr, "\pi_3"] & B \\
 & & C
\end{tikzcd}
$$

We show that $(A \times B) \times C$ satisfies this property. Since $A \times B$ is a product, there are two associated projection morphisms, π_A and π_B, to this product. Similarly, since $(A \times B) \times C$ is a product, there are two associated projection morphisms, $\pi_{A \times B}$ and π_C, to this product. We claim that there is a unique σ such that the following diagram commutes.

$$
\begin{tikzcd}
& & & A \\
& & A \times B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"] & \\
Z \arrow[r, "\sigma"] \arrow[uurrr, bend left, "f_A"] \arrow[drr, bend right, "f_C"'] \arrow[rrr, bend right=70, "f_B"'] & (A \times B) \times C \arrow[uur, "\pi_{A \times B}"] \arrow[dr, "\pi_C"] & & B \\
& & C &
\end{tikzcd}
$$

If we can show this, then we have shown that $(A \times B) \times C$ is a triple product, since we can take $\pi_1 \coloneqq \pi_A \circ \pi_{A\times B}$, and $\pi_2 \coloneqq \pi_B \circ \pi_{A\times B}$, and $\pi_3 \coloneqq \pi_C$ in the first diagram.

As $A \times B$ is a product, there is a unique morphism τ from Z to $A \times B$ such that $\pi_A \circ \tau = f_A$ and $\pi_B \circ \tau = f_B$. In addition, we have the following sub-diagram.

$$
\begin{tikzcd}
& & A \times B \\
Z \arrow[r, "\sigma"] \arrow[urr, bend left, "\tau"] \arrow[drr, bend right, "f_C"'] & (A \times B) \times C \arrow[ur, "\pi_{A \times B}"] \arrow[dr, "\pi_C"] & \\
& & C
\end{tikzcd}
$$

Which commutes for a unique σ since $(A \times B) \times C$ is a product. For completeness, we can check that this σ indeed makes the whole diagram commute. We have three equalities to check, one of which is already given by the sub-diagram, namely $\pi_C \circ \sigma = f_C$. Next, consider $\pi_A \circ \pi_{A\times B} \circ \sigma$, which equals $\pi_A \circ \tau$ by commutativity of the sub-diagram, and this in turn equals f_A (we remarked this when we defined τ). Similarly, one can check that $\pi_B \circ \pi_{A\times B} \circ \sigma = f_B$.

So, we have shown the existence of a σ that makes the diagram commute, but we haven't yet shown that it is the unique morphism with this property. Let σ' be a morphism that also makes the diagram commute. Then

$$
\begin{aligned}
f_A &= \pi_A \circ (\pi_{A\times B} \circ \sigma'), \\
f_B &= \pi_B \circ (\pi_{A\times B} \circ \sigma').
\end{aligned}
$$

We conclude that $\pi_{A\times B} \circ \sigma' = \tau$ since τ is the unique morphism that satisfies the above identities. Then we have

$$\tau = \pi_{A\times B} \circ \sigma'$$
$$f_C = \pi_C \circ \sigma.$$

Hence $\sigma' = \sigma$ since σ is the only morphism that makes the sub-diagram commute. We have shown that $(A \times B) \times C$ is a triple product, and in a similar fashion one can prove that $A \times (B \times C)$ is a triple product. One can define a category $\mathsf{C}^{A,B,C}$ such that triple products are terminal in that category; hence all triple products are isomorphic.