Problem 1

Let x and y be two elements of a group G. Assume that each of the elements x, y, xy has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of G, and it has order 4.

Proof.

- We already have the identity $1 \in H$.
- As x, y, xy are defined to be of order 2, thus they happen to be their own inverses. As $x^2 = xx = 1$, $y^2 = yy = 1$ and $(xy)^2 = xyxy = 1$. And 1 is also the inverse of itself but it is *not* of order 2.
- Now we have to check that the multiplication is closed in H. One can make a Cayley multiplication table for this, but trivially,

$$
\begin{aligned}
1x = x1 = x &\in H \\
1y = y1 = y &\in H \\
xx = 1 &\in H \\
yy = 1 &\in H \\
xyy = x &\in H \\
xxy = y &\in H \\
1xy = xy1 = xy &\in H
\end{aligned}
$$

  Thus, indeed the multiplication is closed in H.
- And this is satisfied if we can show that all the elements of H are distinct. Assume $x = 1$ or $y = 1$, then $xx = 1x$ must be 1 as x has an order of 2, but that's not the case since $1x = x$. Asssume $x = y$, then $yx = yy$ is contradictory since $yy = 1$ and x already is the inverse of itself. Now, for the last case, assume $x = xy$ or $y = xy$, now this is again impossible because if you multiply both the equations by y and x respectively, you get $xy = xyy$ whch is contradictory beause $xyy = x1 = x$ and 1 is the unique identity. Similarly it's impossible for $xy = xxy$. Thus all the elements of the set are unique. Making $H \subset G$ a well-defined subgroup and $|H| = 4$.

$\square$

Problem 2 continued on next page...

Problem 2

Let G be a group and let $a \in G$ have order k. If p is a prime divisor of k, and if there is $x \in G$ with $x^p = a$, prove that x has order pk.

Proof. □

Problem 3

Let G be a group in which $x^2 = 1$ for every $x \in G$, prove that G must be abelian.

Proof. As $x^2 = 1 \quad \forall x \in G$, thus x is the inverse of itself. So for two distinct elements $x, y \in G \quad x \neq y$ we have,

$$(xy)^{-1} = y^{-1}x^{-1} = yx \implies xyyx = 1$$

So, yx is the inverse of xy. But also as we have assosciativity under this group, $xyxy = xxyy = 1$, making xy to be the inverse of itself, but as we know that in a well-defined groupo, the inverse of each element has to be unique so $yx = xy \quad \forall x, y \in G$, proving G to be abelian. □

Problem 4

If G is a group with an even number of elements, prove that the number of elements in G of order 2 is odd. In particular, G must contain an element of order 2.

Proof. □

Problem 5

Let H be a subgroup of G and $C(H) = \{g \in G : gh = hg \quad \forall h \in H\}$ Prove that $C(H)$ is a subgroup of G. This is known as the *centralizer* of H in G.

Problem 5 continued on next page...

Proof. • Let $g_1, g_2 \in C(H)$ with $g_2h_1 = h_2g_1$ and $g_2h_2 = h_2g_2$ for some $h_1, h_2 \in H$. Multiplying these two equations we get, $g_1g_2h_1h_2 = h_1h_2g_1g_2 \implies g_1g_2 \in C(H)$. Thus, the closure property is satisfied.

- The identity in G would be the identity in H, thus for $1 \in H \quad 1h = h1 \quad \forall h \in H.$. Thus $1 \in G$. So the identity is well-defined.

- The inverses also exist, for $g \in C(H)$ we have $gh = hg$, for some $h \in H$. We have the inverse $g^{-1} \in H \subset G$. We first multiply this inverse on the left side, so we have $ghg^{-1} = hgg^{-1}$. Now we again multiply the inverse on the right side, so $g^{-1}ghg^{-1} = g^{-1}hgg^{-1} \implies hg^{-1} = g^{-1}h$. Thus $g^{-1} \in C(H)$.

□

Problem 6

Let G b a group, let X be a set, and let $f : G \to X$ be a bijection. Show that there is a unique operation on X, so that X is a group and f is an isomorphism.

Proof. As G is a group so we can define an operation on it such as : $m_G : G \times G \to G$. And similarly for X, $m_X : X \times X \to X$. And as we have $f : G \to X$ is a bijection thus it induces another bijection, namely : $f \times f : G \times G \to X \times X$. Thus,

$$\begin{array}{ccc} G \times G & \xrightarrow{f \times f} & X \times X \\ {\scriptstyle m_G}\downarrow & & \downarrow{\scriptstyle m_X} \\ G & \xrightarrow[f]{} & X \end{array}$$

Thus, for X to be a group, m_G has to be unique and f would be a bijective group homomorphism, i.e an isomorphism. □

Problem 7

Determine the center of $GL_n(\mathbb{R})$.

Proof. □

Problem 8

Suppose given on E an (associative and commutative) addition under which all the elements of E are invertible and a multiplication which is non-associative but commutative and doubly distributive with respect to additoin. Suppose further that $n \in \mathbb{Z}, n \neq 0$ and $nx = 0 \implies x = 0$ in E. Show hat if, writing $[xy, y, z] = (xy)z - z(yz)$, the identity

$$[xy, u, z] + [yz, u, z] + [zx, u, y] = 0$$

holds, then $x^{m+n} = x^m x^n$ for all x (show, by induction on p, the identity $[x^q, y, x^{p-q}] = 0$ holds for $1 \leq q < p$).

Proof. □