

Chapter 1. Preliminaries: Set theory and categories

3 Categories

Problem 3.1. \triangleright Let \mathbf{C} be a category. Consider a structure \mathbf{C}^{op} with

1. $\text{Obj}(\mathbf{C}^{op}) := \text{Obj}(\mathbf{C})$
2. for A, B objects of \mathbf{C}^{op} (hence objects of \mathbf{C}), $\text{Hom}_{\mathbf{C}^{op}}(A, B) := \text{Hom}_{\mathbf{C}}(B, A)$.

Show how to make this into a category (that is, define composition of morphisms in \mathbf{C}^{op} and verify the properties listed in §3.1). Intuitively, the ‘opposite’ category \mathbf{C}^{op} is simply obtained by ‘reversing all the arrows’ in \mathbf{C} .

Solution. Remember that by definition, a category must have i) a composition morphism that satisfies associativity for any pairs of morphisms and ii) an identity morphism that is unital for all objects in the category (e.g. for $f : X \rightarrow Y$, $f1_X = 1_Y f = f$).

i) Composition morphism: Let’s define composition of morphisms as follows: for $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B)$ and $g \in \text{Hom}_{\mathbf{C}^{op}}(B, C)$, define $f \circ' g \in \text{Hom}_{\mathbf{C}^{op}}(A, C)$ such that $f \circ' g = g \circ f \in \text{Hom}_{\mathbf{C}}(C, A)$. We know $g \circ f$ exists because \mathbf{C} is a category and thus satisfies the condition of having a composition morphism for all pairs of morphisms.

To show that the composition morphism is associative, let $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B)$, $g \in \text{Hom}_{\mathbf{C}^{op}}(B, C)$, and $h \in \text{Hom}_{\mathbf{C}^{op}}(C, D)$. Then $(f \circ' g) \circ' h = (g \circ f) \circ' h = h \circ (g \circ f) = (h \circ g) \circ f = (g \circ' h) \circ f = f \circ' (g \circ' h)$. Where the 1st, 2nd, 4th, and 5th equalities are due to the definition we chose for the composition of morphisms, and the 3rd equality is true because \mathbf{C} is a category and thus its morphisms are associative.

ii) Identity morphism: If A is an object in \mathbf{C}^{op} , it also exists in \mathbf{C} (by definition of $\text{Obj}(\mathbf{C}^{op})$). Since \mathbf{C} is a category, it satisfies the property of having an identity morphism for each object. Let’s define 1_A for any A in \mathbf{C}^{op} to be the same as 1_A for the same A in \mathbf{C} .

To show that the identity morphism is unital, let $f \in \text{Hom}_{\mathbf{C}^{op}}(A, B)$, then:

1. $f \circ' 1_A = 1_A \circ f = f$
2. $1_B \circ' f = f \circ 1_B = f$

Where for both statements, the first equality is due to our definition of composition and the second equality is true because \mathbf{C} is a category and thus its identity morphisms are unital. \square

Problem 3.2. If A is a finite set, how large is $\text{End}_{\text{Set}}(A)$?

Solution. By definition $\text{End}_{\text{Set}}(A) = \text{Hom}_{\text{Set}}(A, A)$, and since we are working with the category of sets, we can think of morphisms as set-functions (§3.2). In other words, $\text{End}_{\text{Set}}(A)$ is the set of all functions $f : A \rightarrow A$, otherwise denoted as A^A (§2.1, 3.2), and we are asked to find the count of all possible functions $f : A \rightarrow A$.

Since A is finite, the count of all possible functions $|\text{End}_{\text{Set}}(A)| = |A|^{|A|}$ (where $|A|$ is the number of elements in A). \square

Problem 3.3. ▷ Formulate precisely what it means to say that 1_a is an identity with respect to composition in Example 3.3, and prove this assertion. [§3.2]

Solution. 1_a is an identity if for any three objects $z, a, b \in S$, where $e \in \text{Hom}(z, a)$ and $f \in \text{Hom}(a, b)$, $1_a e = e$ and $f 1_a = f$.

As described in Example 3.3, we only have one choice for $1_a \in \text{Hom}(a, a)$ where $1_a = (a, a)$. By our definition of morphism, we also have $e = (z, a)$ and $f = (a, b)$. Using our definition of composition, $1_a e \in \text{Hom}(z, a)$ and $f 1_a \in \text{Hom}(a, b)$. It follows that $1_a e = (z, a) = e$ and $f 1_a = (a, b) = f$.

□

Problem 3.4. Can we define a category in the style of Example 3.3 using the relation $<$ on the set \mathbf{Z} ?

Solution. No, since $<$ is not reflexive, it follows that the set $\text{Hom}(A, A)$ is empty, and therefore we cannot define an identity morphism. □

Problem 3.5. ▷ Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3. [§3.2]

Solution. For the sake of clarity, let S' represent the set in Example 3.3 and S represent the set in Example 3.4. We can think of $S' = \mathcal{P}(S)$ where each element $a, b \in S'$ represents $A, B \subseteq S$. Both \sim and \subseteq are reflexive and transitive. For both categories, a morphism between objects is either a pair, (a, b) if $a \sim b$ or (A, B) if $A \subseteq B$, or \emptyset otherwise. □

Problem 3.6. ▷ (Assuming some familiarity with linear algebra.) Define a category \mathbf{V} by taking $\text{Obj}(\mathbf{V}) = \mathbf{N}$ and letting $\text{Hom}_{\mathbf{V}}(n, m) =$ the set of $m \times n$ matrices with real entries, for all $n, m \in \mathbf{N}$. (We will leave the reader the task of making sense of a matrix with 0 rows or columns.) Use product of matrices to define composition. Does this category ‘feel’ familiar? [§VI.2.1, §VIII.1.3]

Solution. Reminder: Remember that if V is n -dimensional and W is m -dimensional, then $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$ are isomorphic [LADR 3.60], where $\mathcal{L}(V, W)$ is the set of all linear maps from V to W [LADR 3.3] and $\mathbf{F}^{m,n}$ is the set of all matrices with m rows and n columns [LADR 3.39]. Additionally, two finite-dimensional vector spaces over \mathbf{F} (and in this case \mathbf{R} which is a subset of \mathbf{F}) are isomorphic if and only if they have the same dimension [LADR 3.59].

The question gives us a category theoretic way of describing linear maps. In linear algebra we deal exclusively with finite-dimensional vector spaces. Since every linear map $T \in \mathcal{L}(V, W)$, where V is n -dimensional and W is m -dimensional, can be represented as an m -by- n matrix, it follows that we can represent every linear map as a morphism from n to m where $n, m \in \text{Obj}(\mathbf{V})$.

If a matrix has 0 columns, we can try to interpret it as a linear map from a zero-dimensional vector space to a non-zero dimensional vector space. However this contradicts the definition of linear maps which must satisfy additivity [LADR 3.11]. Thus, we can say that if $m \neq 0$, then $\text{Hom}_V(m, 0) = \emptyset$.

If a matrix has 0 rows, we can interpret it as a linear map to a zero-dimensional vector space. In other words, for all $v \in V$, $Tv = 0v = 0$. We can represent this as any m -by- n matrix with every element in the matrix = 0. \square

Problem 3.7. \triangleright Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition [§3.2].

Solution. Let \mathcal{C} be a category and A be an object in \mathcal{C} . We define the category $\overline{\mathcal{C}}_A$ as follows:

1. $\text{Obj}(\overline{\mathcal{C}}_A) := \text{Hom}_{\mathcal{C}}(A, X)$ where X is any object in \mathcal{C} . Pictorially, an object of $\overline{\mathcal{C}}_A$ is an arrow:

$$\begin{array}{c} A \\ \downarrow f_1 \\ X_1 \end{array}$$

2. Morphisms in $\overline{\mathcal{C}}_A$ can be defined as commutative diagrams. Let f_1 and f_2 be objects of $\overline{\mathcal{C}}_A$, that is two arrows:

$$\begin{array}{ccc} A & & A \\ \downarrow f_1 & & \downarrow f_2 \\ X_1 & & X_2 \end{array}$$

The morphism $f_1 \rightarrow f_2$ can be defined as the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow f_1 & & \downarrow f_2 \\ X_1 & \xrightarrow{\sigma} & X_2 \end{array}$$

However, since we are always dealing with A as the domain, we can simplify as follows:

$$\begin{array}{ccc} & A & \\ f_1 \swarrow & & \searrow f_2 \\ X_1 & \xrightarrow{\sigma} & X_2 \end{array}$$

3. We define composition of morphisms as putting two commutative diagrams side-by-side:

$$\begin{array}{ccccc} & & A & & \\ & f_1 \swarrow & \downarrow f_2 & \searrow f_3 & \\ X_1 & \xrightarrow{\sigma} & X_2 & \xrightarrow{\tau} & X_3 \end{array}$$

Because \mathbf{C} is commutative and $f_1, f_2, f_3, \sigma, \tau$ are morphisms in \mathbf{C} , we can remove the central arrow:

$$\begin{array}{ccc} & A & \\ f_1 \swarrow & & \searrow f_3 \\ X_1 & \xrightarrow{\tau\sigma} & X_3 \end{array}$$

4. We define the identity morphism using the identities in \mathbf{C} . For $f_1 : A \rightarrow X$ in $\overline{\mathbf{C}}_A$, the identity 1_f corresponds to:

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow f \\ X & \xrightarrow{1_f} & X \end{array}$$

Given our choice of composition and identities, we need to now check whether the morphisms are associative and unital.

- a) Associativity: Let $f_1, f_2, f_3, f_4 \in \overline{\mathbf{C}}_A$ such that:

$$\begin{array}{ccccccc} & & & A & & & \\ & \swarrow f_1 & & & \searrow f_3 & & \swarrow f_4 \\ X_1 & \xrightarrow{\sigma} & X_2 & \xrightarrow{\tau} & X_3 & \xrightarrow{\rho} & X_4 \end{array}$$

Since \mathbf{C} is commutative, we can remove either of the two central arrows to get the following diagrams:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow f_1 & & \searrow f_3 & \swarrow f_4 \\ X_1 & \xrightarrow{\tau\sigma} & & X_3 & \xrightarrow{\rho} & X_4 \end{array}$$

$$\begin{array}{ccccc} & & A & & \\ & \swarrow f_1 & & \searrow f_2 & \swarrow f_4 \\ X_1 & \xrightarrow{\sigma} & X_2 & \xrightarrow{\rho\tau} & X_4 \end{array}$$

Since \mathbf{C} is commutative, both diagrams are equivalent. Thus $(\rho\tau)\sigma = \rho(\tau\sigma)$.

- b) Unital: Let $f_0, f_1, f_2 \in \overline{\mathbf{C}}_A$ such that:

$$\begin{array}{ccc} A & A & A \\ \downarrow f_0 & \downarrow f_1 & \downarrow f_2 \\ X_0 & X_1 & X_2 \end{array}$$

Then given our definition of identity and composition, we get:

$$\begin{array}{ccc}
& A & \\
f_0 \swarrow & \downarrow f_1 & \searrow f_1 \\
X_0 & \xrightarrow{\phi} & X_1 \xrightarrow{1_{f_1}} X_1
\end{array}
\qquad
\begin{array}{ccc}
& A & \\
f_1 \swarrow & \downarrow f_1 & \searrow f_2 \\
X_1 & \xrightarrow{1_{f_1}} & X_1 \xrightarrow{\sigma} X_2
\end{array}$$

And by removing the central arrows, we get:

$$\begin{array}{ccc}
& A & \\
f_0 \swarrow & & \searrow f_1 \\
X_0 & \xrightarrow{\phi=1_{f_1}\phi} & X_1
\end{array}
\qquad
\begin{array}{ccc}
& A & \\
f_1 \swarrow & & \searrow f_2 \\
X_1 & \xrightarrow{\sigma=\sigma 1_{f_1}} & X_2
\end{array}$$

It is clear from the diagrams that $\phi = 1_{f_1}\phi$ and $\sigma = \sigma 1_{f_1}$, thus proving that our identity morphisms are unital. □

Problem 3.8. \triangleright A *subcategory* \mathcal{C}' of a category \mathcal{C} consists of a collection of objects of \mathcal{C} , with morphisms $\text{Hom}_{\mathcal{C}'}(A, B) \subseteq \text{Hom}_{\mathcal{C}}(A, B)$ for all objects A, B in $\text{Obj}(\mathcal{C}')$, such that identities and compositions in \mathcal{C} make \mathcal{C}' into a category. A subcategory \mathcal{C}' is *full* if $\text{Hom}_{\mathcal{C}'}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$ for all A, B in $\text{Obj}(\mathcal{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of \mathbf{Set} . [4.4, §VI.1.1, §VIII.1.3]

Solution. Let $\mathbf{Set}_{\text{inf}}$ be the category whose objects are the infinite sets in \mathbf{Set} and whose morphisms are all the set-functions between infinite sets in \mathbf{Set} . In other words, if A, B are infinite sets, $\text{Hom}_{\mathbf{Set}_{\text{inf}}}(A, B) = \text{Hom}_{\mathbf{Set}}(A, B)$. We also inherit composition and identity from \mathbf{Set} . It suffices to show that $\mathbf{Set}_{\text{inf}}$ satisfies associativity and unity (i.e. is a category), which is a trivial exercise (since we inherited composition, identity, set-functions, and sets from \mathbf{Set} which is a category). □

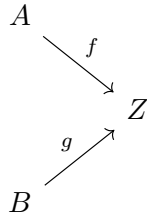
Problem 3.9.

Problem 3.10.

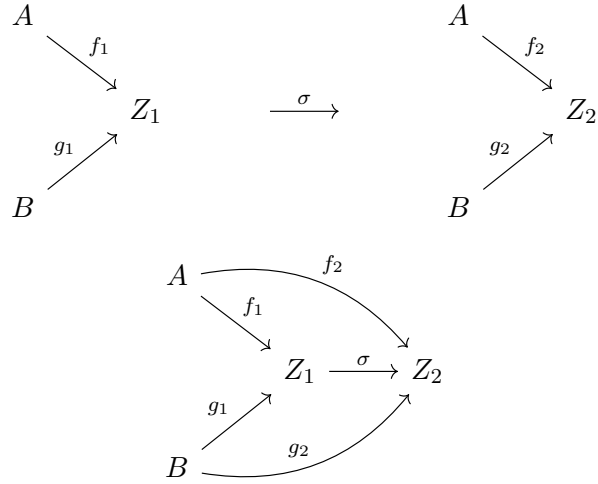
Problem 3.11. \triangleright Draw the relevant diagrams and define composition and identities for the category $\mathcal{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathcal{C}^{\alpha,\beta}$ mentioned in Example 3.10 [§5.5, 5.12].

Solution. [Example 3.9] Given two objects A, B of \mathcal{C} , we define a new category $\mathcal{C}^{A,B}$ as follows

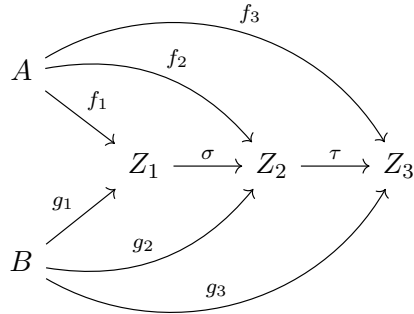
1. $\text{Obj}(\mathcal{C}_{A,B}) = \text{diagrams in } \mathcal{C}, \text{ where } Z \in \text{Obj}(\mathcal{C});$



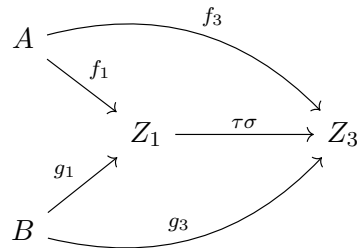
2. morphisms correspond *commutative* diagrams;



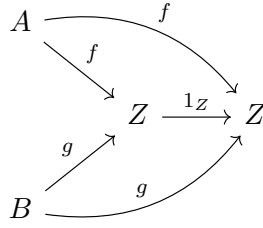
3. compositions are obtained by placing commutative diagrams side-by-side;



we can then remove the center diagram;



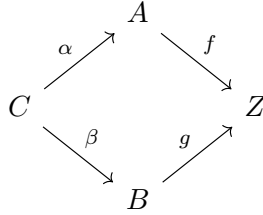
4. identity morphisms are inherited from identities in C;



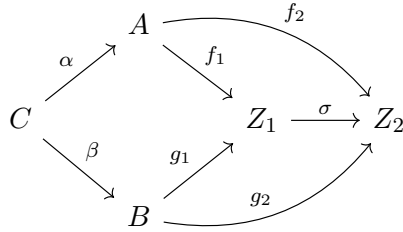
Since \mathbf{C} is commutative, associative, and unital, $\mathbf{C}^{A,B}$ is also commutative, associative, and unital.

[Example 3.10] To define $\mathbf{C}^{\alpha,\beta}$, we choose two fixed morphisms $\alpha : C \rightarrow A, \beta : C \rightarrow B$ in \mathbf{C} . We then consider the data of $\mathbf{C}^{\alpha,\beta}$ as follows:

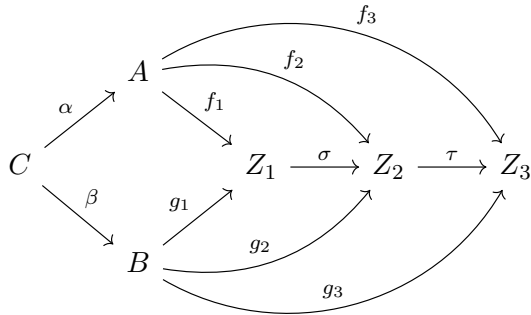
1. $\text{Obj}(\mathbf{C}^{\alpha,\beta}) = \text{commutative diagrams in } \mathbf{C}, \text{ where } Z \in \text{Obj}(\mathbf{C});$



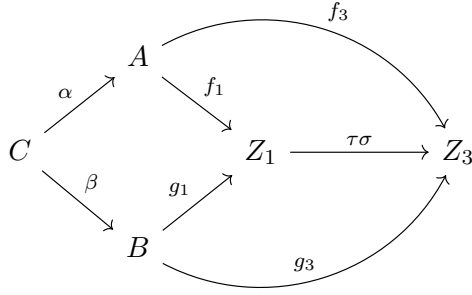
2. morphisms correspond to *commutative* diagrams;



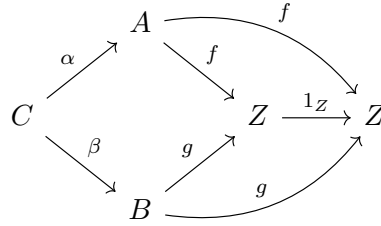
3. composition correspond to placing commutative diagrams side-by-side;



after which one can remove the center diagram, resulting in a diagram that also commutes;



4. identity morphisms are inherited from \mathbf{C} , resulting in the following diagram;



Since \mathbf{C} is commutative, associative, and unital, $\mathbf{C}^{\alpha, \beta}$ is also commutative, associate, and unital. \square

4 Morphisms

Problem 4.1.

Problem 4.2.

Problem 4.3. Let A, B be objects of a category \mathbf{C} , and let $f \in \text{Hom}_{\mathbf{C}}(A, B)$ be a morphisms.

1. Prove that if f has a right-inverse, then f is an epimorphism.
2. Show that the converse does not hold, by giving an explicit example of a category and an epimorphisms without a right-inverse.

Solution. Let $f' : B \rightarrow A$ be a right-inverse of f and let $B', B'' \in \text{Hom}_{\mathbf{C}}(B, A)$. If $B'f = B''f \implies B'ff' = B''ff' \implies B'1_B = B''1_B \implies B' = B''$.

To show the converse doesn't hold, we use Example 4.5 (and Example 3.3), where the objects of \mathbf{C} are the integers and $\text{Hom}_{\mathbf{C}}(a, b) =$ the pair (a, b) if $a \leq b$ or \emptyset otherwise, where $a, b \in \text{Obj}(\mathbf{C})$. Because there is at most one morphism between two objects, the fact that each morphism is epimorphic is vacuously true. Moreover, the pair (a, b) generally does not have a right-inverse unless $a = b$. \square

5 Morphisms

Problem 5.1.

Problem 5.2.

Problem 5.3.

Problem 5.4.

Problem 5.5.

Problem 5.6.

Problem 5.7.

Problem 5.8. Show that in every category \mathbf{C} the products $A \times B$ and $B \times A$ are isomorphic, if they exist. (Hint: Observe that they both satisfy the universal property for the product of A and B ; then use Proposition 5.4.)

Solution. We use a similar proof to the one provided in §I.5.4. But first, we define $B \times A$ as the *product of sets* such that $B \times A = \{(b, a) | b \in B, a \in A\}$ and the *natural projections* $\pi'_A((b, a)) := a$ and $\pi'_B((b, a)) := b$. If $A \times B$ exists, then $B \times A$ also exists and vice versa. Let Z be an object in \mathbf{C} .

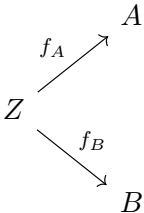
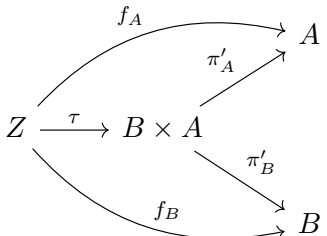
For any object Z  , there exists a unique $\tau : Z \rightarrow B \times A$ such that the following

diagram commutes:  . For all $z \in Z$, we define

$$\begin{aligned}\tau(z) &= (f_B(z), f_A(z)) \text{ where} \\ \pi'_B \tau(z) &= \pi'_B(f_B(z), f_A(z)) = f_B(z) \\ \pi'_A \tau(z) &= \pi'_A(f_B(z), f_A(z)) = f_A(z),\end{aligned}$$

showing that the diagram commutes. Since τ is defined by the object $[\text{?}]$ since f_B, f_A are unique $[\text{?}]$, it is thus a unique morphism from that object to $B \times A$, thereby proving that $B \times A$ is a terminal object.

Now that we've proved that $B \times A$ is also a terminal object, we use Proposition 5.4 to show that it is isomorphic to $A \times B$ which is also a terminal object, thereby completing the proof \square

Problem 5.9. Let \mathbf{C} be a category of products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of *three* objects of \mathbf{C} ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Solution. It is a terminal object for any object Z \square