1. Define a relation ~ on the real numbers by setting $a \sim b$ if $b - a \in \mathbb{Z}$.

This defines an equivalence relation on $\mathbb{R}$:
Reflexivity
For every $a \in \mathbb{R}$, $a - a = 0$. Since $0 \in \mathbb{Z}$, $a \sim a$.
Symmetry
Suppose $a, b \in \mathbb{R}$ and $a \sim b$. Then $b - a = n$ for some $n \in \mathbb{Z}$. But then $a - b = -n$. Since $-n$ is also an integer, $b \sim a$.
Transitivity
Suppose $a, b, c \in \mathbb{R}$ with $a \sim b$ and $b \sim c$. Then there are integers m, n such that $b - a = m$ and $c - b = n$. But then $c - a = (c - b) + (b - a) = n + m$. Since the sum of two integers is an integer, $a \sim c$.

Notice that each equivalence class will have exactly one representative in the interval $[0,1)$, and that $0 \sim 1$. Thus we may identify $\mathbb{R}/\sim$ with an interval with the two endpoints joined. Topologically, this is a circle, S^1. We can make an explicit identification of $\mathbb{R}/\sim$ with the unit circle in the complex plane by sending $[x]_\sim$ to $e^{2\pi i x}$. (Since $e^{2\pi i n} = 1$ for $n \in \mathbb{Z}$, this mapping is well-defined.)

The second part of this problem is very similar.
Here, we define a relation ≈ on $\mathbb{R} \times \mathbb{R}$ by $(a_1, a_2) \approx (b_1, b_2)$ if $(b_1 - a_1, b_2 - a_2) \in \mathbb{Z} \times \mathbb{Z}$.
I'll run through the argument a little faster this time.
Reflexivity
$(a_1 - a_1, a_2 - a_2) = (0,0) \in \mathbb{Z} \times \mathbb{Z}$, hence $(a_1, a_2) \approx (a_1, a_2)$
Symmetry
If $(a_1, a_2) \approx (b_1, b_2)$, then $(b_1 - a_1, b_2 - a_2) = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$. Thus $(a_1 - b_1, a_2 - b_2) = (-m_1, -m_2) \in \mathbb{Z} \times \mathbb{Z}$, showing $(b_1, b_2) \approx (a_1, a_2)$.
Transitivity
If $(a_1, a_2) \approx (b_1, b_2)$ and $(b_1, b_2) \approx (c_1, c_2)$, then $(b_1 - a_1, b_2 - a_2) = (m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$ and $(c_1 - b_1, c_2 - b_2) = (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}$. Thus
$(c_1 - a_1, c_2 - a_2) = (c_1 - b_1, c_2 - b_2) + (b_1 - a_1, b_2 - a_2) = (n_1 + m_1, n_2 + m_2) \in \mathbb{Z} \times \mathbb{Z}$.

Here, $\mathbb{R} \times \mathbb{R}/\approx$ can be identified with the torus $S^1 \times S^1$. An explicit map is given by sending $[(x, y)]_\approx$ to $(e^{2\pi i x}, e^{2\pi i y}) \in \mathbb{C}^2$. ■

2. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

Both of these follow quickly from Corollary 2.2 in Aluffi: A function $f : A \to B$ is a bijection if and only if it has a (two-sided) inverse.

If $f : A \to B$ is a bijection, then it has an inverse $f^{-1} : B \to A$, with $f^{-1} \circ f = id_A$ and $f \circ f^{-1} = id_B$. But these equations show that f^{-1} has a two-sided inverse, namely f. Thus f^{-1} is a bijection.

In a similar way, if $f : A \to B$ and $g : B \to C$ are bijections, they have inverses f^{-1}, g^{-1}, respectively. Associativity of composition shows that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = id_A$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = id_C$, showing that $g \circ f$ has a two-sided inverse, hence is bijective.

Notes: (1) This proof elides the fact that Corollary 2.2 uses both parts of Proposition 2.1, and only the first part of Prop. 2.1 was proved in the text.
(2) For those working through the ideas of injection and surjection the first time, it is probably more instructive to prove these results directly, along the way showing that the composition of injective functions is injective and the composition of surjective functions is surjective. ■

---

3. Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \amalg B$ is well-defined, up to isomorphism.

Let $f_1 : A' \to A''$ and $f_2 : B' \to B''$ be isomorphisms, and define $g : A' \cup B' \to A'' \cup B''$ by

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in A' \\ f_2(x) & \text{if } x \in B' \end{cases}.$$

Notice that g is well-defined precisely because $A' \cap B' = \emptyset$.

Similarly, define $h : A'' \cup B'' \to A' \cup B'$ by

$$h(x) = \begin{cases} f_1^{-1}(x) & \text{if } x \in A'' \\ f_2^{-1}(x) & \text{if } x \in B'' \end{cases}.$$

By considering cases, it's easy to see that h is both a left- and right-inverse of g, hence g is an isomorphism between $A' \cup B'$ and $A'' \cup B''$.

Aluffi defines the disjoint union of $A \amalg B$ by this process:

Find disjoint sets A' and B' such that $A \cong A'$ and $B \cong B'$.
(This can always be done, using $A' = \{0\} \times A$ and $B' = \{1\} \times B$.)
Define $A \amalg B := A' \cup B'$.

Note that under a different choice of disjoint sets, say A'', B'', it is generally not true that $A' \cup B' = A'' \cup B''$; however, by the work above, it is true that $A' \cup B' \cong A'' \cup B''$. Thus $A \amalg B$ is (only) well-defined up to isomorphism. ■

3'. Let C be a category. Consider a structure C^{op} with

- Obj(C^{op}) := Obj(C);
- for A, B objects of C^{op} (hence objects of C), $Hom_{C^{op}}(A, B) := Hom_C(B, A)$.

Show how to make this into a category.

First, note that $Hom_{C^{op}}(A, A) = Hom_C(A, A)$ contains 1_A.
Next, let's define a law of composition (apologies in advance for unattractive notation):
Suppose $f \in Hom_{C^{op}}(A, B) = Hom_C(B, A)$ and $g \in Hom_{C^{op}}(B, C) = Hom_C(C, B)$.
We define $(gf)_{op} \in Hom_{C^{op}}(A, C)$ by $(gf)_{op} = fg$, where this composition is done in the category C.

Let's verify that this law of composition is associative and that the morphisms 1_A have the desired properties under composition.
Let $f \in Hom_{C^{op}}(A, B)$ and $g \in Hom_{C^{op}}(B, C)$ as above, and suppose $h \in Hom_{C^{op}}(C, D)$.
Then $((hg)_{op}f)_{op} = ((gh)f)_{op} = f(gh) = (fg)h = (h(fg))_{op} = (h(gf)_{op})_{op}$.
Here, the middle equality is because C is a category. The other equalities are from the definition of the law of composition in C^{op}.
Next, observe that $(f1_A)_{op} = 1_A f = f$, by the property of the morphism 1_A in C. Similarly, $(1_B f)_{op} = f 1_B = f$.

Finally, note that since $Hom_{C^{op}}(A, B) = Hom_C(B, A)$ and $Hom_{C^{op}}(C, D) = Hom_C(D, C)$, we have $Hom_{C^{op}}(A, B) \cap Hom_{C^{op}}(C, D) = \varnothing \iff Hom_C(B, A) \cap Hom_C(D, C) = \varnothing$ $\iff A = B$ and $C = D$. ■

---

4. Define a category V by taking $Obj(V) = \mathbb{N}$ and letting $Hom_V(n, m) =$ the set of $m \times n$ matrices with real entries, for all $n, m \in \mathbb{N}$. Use product of matrices to define composition. Does this category "feel" familiar?

I'm finding it tricky to extend the definition of $Hom_V(n, m)$ to deal with the special case when m or $n = 0$. The main difficulty arises in distinguishing the single point in the set $\mathbb{R}^0$ from the number $0 \in \mathbb{R}^1$. To distinguish the two, let's say $\mathbb{R}^0 = \{ * \}$. Now, for $n \geq 0$ we can define $Hom_V(n,0) = *_n$, thinking of this as the constant function taking all of $\mathbb{R}^n$ to the point $*$. For $m > 0$, we'll define $Hom_V(0,m) = \mathbf{0}_m$, the mapping taking $*$ to the $m \times 1$ zero matrix. (We can think of this as a mapping taking $*$ to the zero vector in $\mathbb{R}^m$.)

We now need to add the composition rules for compositions involving morphisms in $Hom_V(n, m)$ when m or $n = 0$.
First, a couple of special cases:
We define $*_n \mathbf{0}_n = *_0$ (the identity mapping from $\mathbb{R}^0$ to itself), and
set $\mathbf{0}_m *_n = \mathbf{0}_{mn}$, the $m \times n$ zero matrix.
For $A \in Hom_V(m, n)$, with $m, n \neq 0$,
$A\mathbf{0}_m = \mathbf{0}_n$, and we'll define $*_n A = *_m$.

If I've done all of this correctly, under these definitions we can think of V as representing the collection of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, $n, m \geq 0$, with the usual composition of functions. ■

---

5. Draw the relevant diagrams and define composition and identities for category $C^{A,B}$ mentioned in Example 3.9. Do the same for category $C^{\alpha,\beta}$ mentioned in Example 3.10.

Ugh. Typing this is gross.
An object in $Obj(C^{A,B})$ will consist of a pair of morphisms $f : A \to Z$ and $g : B \to Z$ for some object $Z \in Obj(C)$.

$$\begin{tikzcd} A \arrow[dr, "f"] & \\ & Z \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

A morphism

$$
\begin{tikzcd}
A \arrow[dr, "f_1"] & & & A \arrow[dr, "f_2"] & \\
& Z_1 & {} \arrow[r] & {} & Z_2 \\
B \arrow[ur, "g_1"'] & & & B \arrow[ur, "g_2"'] &
\end{tikzcd}
$$

Is a commutative diagram

$$
\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, "f_2"] & & \\
& Z_1 \arrow[r, "\sigma_{12}"] & Z_2 \\
B \arrow[ur, "g_1"'] \arrow[urr, "g_2"'] & &
\end{tikzcd}
$$

The identity

$$\begin{tikzcd}
A \arrow[dr, "f_1"] & & & A \arrow[dr, "f_1"] & \\
& Z_1 \arrow[rrr] & & & Z_1 \\
B \arrow[ur, "g_1"'] & & & B \arrow[ur, "g_1"'] &
\end{tikzcd}$$

Is given by $\sigma = 1_{Z_1}$.

The composition of

$$\begin{tikzcd}
A \arrow[dr, "f_1"] & & & A \arrow[dr, "f_2"] & \\
& Z_1 \arrow[rrr, "\sigma_{12}"] & & & Z_2 \\
B \arrow[ur, "g_1"'] & & & B \arrow[ur, "g_2"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[dr, "f_2"] & & & A \arrow[dr, "f_3"] & \\
& Z_2 \arrow[rrr, "\sigma_{23}"] & & & Z_3 \\
B \arrow[ur, "g_2"'] & & & B \arrow[ur, "g_3"'] &
\end{tikzcd}$$

is given by the composition $\sigma_{23} \circ \sigma_{12}$.

Sorry, I can't bear to try to draw the diagrams for $C^{\alpha,\beta}$. Ugh.
😐

---

6. Let A, B be objects of a category C, and let $f \in Hom_C(A, B)$ be a morphism.

- Prove that if f has a right inverse, then f is an epimorphism.
- Show that the converse doesn't hold, by giving an explicit example of a category and an epimorphism without a right inverse.

Suppose that $f \in Hom_C(A, B)$ has a right inverse $g \in Hom_C(B, A)$. Let Z be any object of C and $\beta', \beta'' \in Hom_C(B, Z)$ such that $\beta' f = \beta'' f$.
If we compose with g, we have $(\beta' f)g = (\beta'' f)g$, which is the central equality in this chain: $\beta' = \beta' 1_B = \beta'(fg) = (\beta' f)g = (\beta'' f)g = \beta''(fg) = \beta'' 1_b = \beta''$, showing that $\beta' = \beta''$.

To show that the converse doesn't hold, consider the category $\mathbb{Z}, \leq$ discussed in the text. As noted on page 30, every morphism in this category is both a monomorphism and an endomorphism. However, only the identities will have left- or right-inverses. For example, $1 \leq 2$ is a morphism in $Hom(1,2)$, but $Hom(2,1)$ is empty, so there is no candidate to be a right-inverse. ■

---

7. Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic, if they exist.

First, let me flesh out the categorical definition of $A \times B$. On page 36, Aluffi seems to define $A \times B$ as a final object in the category $C_{A,B}$ *if this category has final objects*. I doubt that this is literally what he means, for then it would not be possible to compare $A \times B$ and $B \times A$ directly, since they live in different categories, $C_{A,B}$ and $C_{B,A}$, respectively. Here's what I think he intends:

Let A, B be objects in category C. If the category $C_{A,B}$ has final objects, then each final object is a triple of the form (Z_T, f_A, f_B), where $Z_T \in Obj(C)$, $f_A \in Hom_C(Z_T, A)$, and $f_B \in Hom_C(Z, B)$. We define $A \times B := Z_T$. Note that this depends on our choice of Z_T, so the product is only defined up to isomorphism, since terminal objects in a category are isomorphic.

Using this as our definition, here's what I'll prove:
If (Z_T, f_A, f_B) is a final object in $C_{A,B}$ (so $A \times B$ is Z_T), then (Z_T, f_B, f_A) is a final object in $C_{B,A}$ (so $B \times A$ is also Z_T).

So let's say (Z_T, f_A, f_B) is a terminal object in $C_{A,B}$.

Consider the object (Z_T, f_B, f_A) in $C_{B,A}$:

$$\begin{tikzcd} & B \\ Z_T \arrow[ur, "f_B"] \arrow[dr, "f_A"'] & \\ & A \end{tikzcd}$$

Suppose (Z, g_B, g_A) is any other object in $C_{B,A}$:

$$\begin{tikzcd} & B \\ Z \arrow[ur, "g_B"] \arrow[dr, "g_A"'] & \\ & A \end{tikzcd}$$

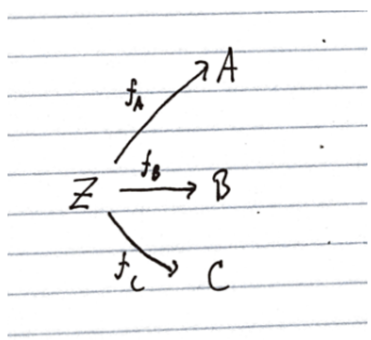
Since (Z_T, f_A, f_B) is final in $C_{A,B}$, we know that there is exactly one morphism $\sigma \in Hom_C(Z, Z_T)$ such that $g_B = f_B\sigma$ and $g_A = f_A\sigma$. But this is all we need to show that (Z_T, f_A, f_B) is final in $C_{B,A}$. ■

---

8. Let C be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of *three* objects of C ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Let's define the category $C_{A,B,C}$ in this way:

$Obj_{C_{A,B,C}} = \{(Z, f_A, f_B, f_C) : Z \in Obj(C), f_A \in Hom_C(A), f_B \in Hom_C(B), f_C \in Hom_C(C)\}$



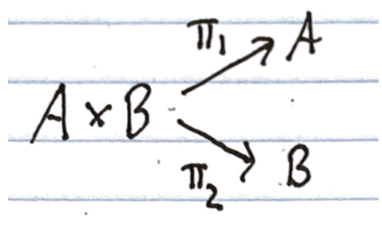
$Hom_{C_{A,B,C}}((Z, f_A, f_B, f_C), (Z', f'_A, f'_B, f'_C)) = \{\sigma \in Hom_C(Z, Z') : f_A = \sigma f'_A, f_B = \sigma f'_B, f_C = \sigma f'_C\}$
(In other words, homomorphisms are commutative diagrams between objects like the one above.)

If (Z_T, f_A, f_B, f_C) is a final object in $C_{A,B,C}$, we'll say that $Z_T \in Obj(C)$ is our desired product $A \times B \times C$ (up to isomorphism).

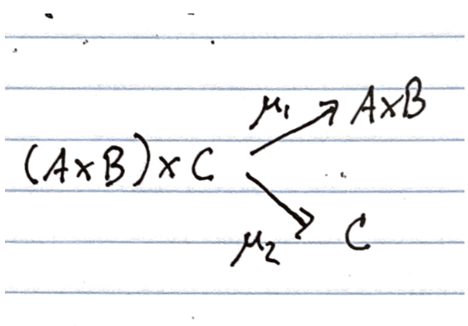
This is equivalent to saying that if we have any object Z in C and morphisms g_A, g_B, g_C from Z to A, B, C, respectively, then there is exactly one morphism $\sigma \in Hom_C(Z, Z_T)$ such that $g_A = \sigma f_A$, etc.

Let's show that $(A \times B) \times C$ satisfies this universal property…but we must be careful. We must specify the associated "projections" from $(A \times B) \times C$ to A, B, C.

$A \times B$ is defined by the property that $(A \times B, \pi_1, \pi_2)$ is a final object in $C_{A,B}$ for some morphisms π_1, π_2.



Similarly, $(A \times B) \times C$ is defined by the property that $((A \times B) \times C, \mu_1, \mu_2)$ is a final object in $C_{A \times B, C}$ for some morphisms μ_1, μ_2.



I'll show that $((A \times B) \times C, \pi_1\mu_1, \pi_2\mu_1, \mu_2)$ is a final object in $C_{A,B,C}$.

To this end, suppose (Z, g_A, g_B, g_C) is an object in $C_{A,B,C}$.
I must show that there is a unique $\sigma \in Hom_C(Z, (A \times B) \times C)$ such that
$g_A = \pi_1\mu_1\sigma, \quad g_B = \pi_2\mu_1\sigma, \quad$ and $g_C = \mu_2\sigma$.

Since $(A \times B, \pi_1, \pi_2)$ is final in $C_{A,B}$, there is a unique morphism $\sigma_1 \in Hom_C(Z, A \times B)$ that makes the following diagram commute:

$$\begin{tikzcd}
 & & A \\
Z \arrow[r, dashed, "\sigma_1"] \arrow[urr, bend left, "g_A"] \arrow[drr, bend right, "g_B"'] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & B
\end{tikzcd}$$

Next, because $((A \times B) \times C, \mu_1, \mu_2)$ is final in $C_{A \times B, C}$, there is a unique $\sigma \in Hom_C(Z, (A \times B) \times C)$ that makes this diagram commute. (It's labeled σ_2 in this picture.)

$$\begin{tikzcd}
 & & A \times B \\
Z \arrow[r, dashed, "\sigma_2"] \arrow[urr, bend left, "\sigma_1"] \arrow[drr, bend right, "g_c"'] & (A \times B) \times C \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \\
 & & C
\end{tikzcd}$$

Notice that if we read from these last two diagrams,
$g_A = \pi_1 \sigma_1 = \pi_1 \mu_1 \sigma$,
$g_B = \pi_2 \sigma_1 = \pi_2 \mu_1 \sigma$, and
$g_C = \pi_2 \sigma$,
as desired.

The argument that that $A \times (B \times C)$ satisfies the same universal property follows along the same lines. Finally, since $A \times B \times C$, $(A \times B) \times C$, and $A \times (B \times C)$ (together with their respective projections) are all final in $C_{A,B,C}$, they are all isomorphic to one another. ■