Problem Set 2 : Group Homomorphisms and Isomorphisms

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Referenced Texts :
Artin's *Algebra*: Chapter 2; Section 1-6.
Rotman's *Advanced Modern Algebra* : Chapter 2; Section 1-3.
Rotman's *Introduction to Theory of Groups*: Chapter 1.
Thomas Judson's *Abstract Algebra: Theory and Applications*: Chapter 3; Section 1-4
Nicholas Bourbaki's *Algebre, Vol.I*: Chapter 1-3

1. Let x and y be two elements of a group G. Assume that each of the elements x, y and xy has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of G, and it has order 4. [**Artin Exercise 4.7**]

2. Let G be a group and let $a \in G$ have order k. If p is a prime divisor of k, and if there is $x \in G$ with $x^p = a$, prove that x has order pk. [**Rotman, *Advanced Modern Algebra* Exercise 2.24**]

3. If G is a group in which $x^2 = 1$ for every $x \in G$, prove that G must be abelian. [**Rotman's *Advanced Modern Algebra* Exercise 2.26**]

4. If G is a group with an even number of elements, prove that the number of elements in G of order 2 is odd. In particular, G must contain an element of order 2. [**Rotman, *Advanced Modern Algebra*: Exercise 2.27**]

5. Let H be a subgroup of G and

$$C(H) = \{g \in G : gh = hg \quad \forall h \in H\}$$

Prove that $C(H)$ is a subgroup of G. This is known as the *centralizer* of H in G. [**Judson Exercise 3.4.53**]

6. Let G be a group, let X be a set, and let $f : G \to X$ be a bijection. Show that there is a unique operation on X, so that X is a group and f is an isomorphism. [**Rotman, *Intro to Theory of Groups*: Exercise 1.44**]

7. Determine the center of $GL_n(\mathbb{R})$. [**Artin, Exercise 5.6**]

8. [Optional] Suppose given on E an (assosciative and commutative) addition under which all the elements of E are invertive and a multiplication which is *non-assosciative*, but commutative and doubly distributive with respect to addition. Suppose further that $n \in \mathbb{Z}, n \neq 0$ and $nx = 0 \implies x = 0$ in E. Show that if, writing $[xy, y, z] = (xy)z - x(yz)$, the identity

$$[xy, u, z] + [yz, u, z] + [zx, u, y] = 0$$

holds, then $x^{m+n} = x^m x^n$ for all x (show, by induction on p, the identity $[x^q, y, x^{p-q}] = 0$ holds for $1 \leq q < p$). **[N. Bourbaki, *Algebre, Vol.I*: Exercise 9, §3.]**