Abstract Algebra Problem Set 1

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Problem 1: Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting $a \sim b \Leftrightarrow b-a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \Leftrightarrow b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$.

[Aluffi Exercise 1.6]

Proof. We need to check the three conditions for equivalence relations:
1) Symmetry: $a - a = 0 \in \mathbb{Z} \Leftrightarrow a \sim a$
2) Reflexivity: $a \sim b \Leftrightarrow a - b \in \mathbb{Z} \Leftrightarrow b - a = -(a-b) \in \mathbb{Z} \Leftrightarrow b \sim a$
3) Transitivity: $(a \sim b) \wedge (b \sim c) \Leftrightarrow (a - b \in \mathbb{Z}) \wedge (b - c \in \mathbb{Z}) \Leftrightarrow a - c = (a-b) + (b-c) \in \mathbb{Z} \Leftrightarrow a \sim c$
Thus $\sim$ on $\mathbb{R}$ as it is defined is an equivalence relation. $\mathbb{R}/\sim$ is basically a collection of all possible displaced copies of the standard integer lattice on $\mathbb{R}$. Now perform the same checks for the second relation:
1) Symmetry: $(a_1 - a_1 = 0 \in \mathbb{Z}) \wedge (a_2 - a_2 = 0 \in \mathbb{Z}) \Leftrightarrow (a_1, a_2) \approx (a_1, a_2)$
2) Reflexivity: $(a_1, a_2) \approx (b_1, b_2) \Leftrightarrow (a_1 - b_1 \in \mathbb{Z}) \wedge (a_2 - b_2 \in \mathbb{Z}) \Leftrightarrow (b_1 - a_1 = -(a_1 - b_1) \in \mathbb{Z}) \wedge (b_2 - a_2 = -(a_2 - b_2) \in \mathbb{Z}) \Leftrightarrow (a_1, a_2) \approx (b_1, b_2)$
3) Transitivity: $((a_1, a_2) \approx (b_1, b_2)) \wedge ((b_1, b_2) \approx (c_1, c_2)) \Leftrightarrow (a_1 - b_1 \in \mathbb{Z}) \wedge (a_2 - b_2 \in \mathbb{Z}) \wedge (b_1 - c_1 \in \mathbb{Z}) \wedge (b_2 - c_2) \in \mathbb{Z}) \Leftrightarrow (a_1 - c_1 = (a_1 - b_1) + (b_1 - c_1) \in \mathbb{Z}) \wedge (a_2 - c_2 = (a_2 - b_2) + (b_2 - c_2) \in \mathbb{Z}) \Leftrightarrow (a_1, a_2) \approx (c_1, c_2)$
as required. Similarly to the 1D case, $\mathbb{R}/\approx$ would look like multiple copies of the integer lattice on $\mathbb{R}^2$. □

Problem 2: Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection. [Aluffi Exercise 2.3]

Proof. Let $f : A \to B$ be a bijective function and $g : B \to A$ be its inverse. Suppose $a_1, a_2 \in A$, so $f(a_1), f(a_2) \in B$. Then by the injectivity of f we know that whenever $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$. The surjectivity of f then requires this condition to go both ways since there is no element in B that cannot be reached (and this is the only way the opposite direction can be violated). So $a_1 \neq a_2 \Leftrightarrow f(a_1) \neq f(a_2)$ and furthermore all elements from A can be reached starting from elements in B which proves the bijectivity of the inverse (defined as $g(f(a)) = a \ \ \forall a \in A$ where there is no ambiguity in taking $f(a)$ as the argument since we know all elements in B are reached by f anyway)
Suppose $f : A \to B$ and $g : B \to C$ are both bijective and consider the composition gf:
1) For some $a_1, a_2 \in A$, suppose $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ due to bijectivity of $f \Rightarrow g(f(a_1)) = (gf)(a_1) \neq (gf)(a_2) = g(f(a_2))$ due to bijectivity of g. Thus gf must be injective.
2) Due to bijectivity of f and g we know that $Im_f(A) = f(A) = B$ and $Im_g(B) = g(B) = C$. Then $Im_{gf}(A) = (gf)(A) = g(f(A)) = g(B) = C$ as required for surjectivity of gf.
$\Rightarrow gf$ is bijective. □

Problem 3: Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \varnothing$ and $A'' \cup B'' = \varnothing$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the disjoint union of A and B is well-defined up to isomorphism. [Aluffi Exercise 2.9]

Proof. Let f, g be the bijections for $A' \cong A''$ and $B' \cong B''$ respectively. We define a new function $\gamma : A' \cup B' \to A'' \cup B''$ by setting $\gamma(x) = f(x)$ whenever $x \in A'$ and $\gamma(x) = g(x)$ whenever $x \in B'$.

Suppose $y \in A'' \cup B'' \Rightarrow (y \in A'') \vee (y \in B'')$ since $A'' \cap B'' = \varnothing$. If $y \in A''$, by the bijectivity of f we know that there is $x \in A' \subset A' \cup B' : \ \gamma(x) = f(x) = y$. Likewise, if $y \in B''$, by the bijectivity of g we know that there is $x \in B' \subset A' \cup B' : \ \gamma(x) = g(x) = y$. Thus γ is surjective.

Now suppose $x_1, x_2 \in A' \cup B'$ with $x_1 \neq x_2$. Using the bijectivity of f and g, we have a couple of different options:
1) $x_1, x_2 \in A' \Rightarrow \gamma(x_1) = f(x_1) \neq f(x_2) = \gamma(x_2)$ proving the injectivity of γ for this case.
2) $x_1, x_2 \in B' \Rightarrow \gamma(x_1) = g(x_1) \neq g(x_2) = \gamma(x_2)$ proving the injectivity of γ for this case.
3) $(x_1 \in A' \wedge x_2 \in B') \vee (x_1 \in B' \wedge x_2 \in A')$. W.l.o.g. we consider the first case: $\gamma(x_1) = f(x_1) \neq g(x_2) = \gamma(x_2)$ due to the fact that $f(x_1) \in A''$ and $g(x_2) \in B''$ whereas $A'' \cap B'' = \varnothing$.

Since γ is both injective and surjective, then it must be bijective.

The constructions of A'' to A' and B'' to B' correspond to the method used to form the disjoint union of A' and B'. Since $\cong$ is an equivalence relation, we can consider the disjoint union to be well-defined up to isomorphisms. □

Problem 4: Let $\mathbb{C}$ be a category. Consider a structure $\mathbb{C}^{op}$ with:
1) $Obj(\mathbb{C}^{op}) = Obj(\mathbb{C})$
2) For A, B objects of $\mathbb{C}^{op}$, $Hom_{\mathbb{C}^{op}}(A, B) = Hom_{\mathbb{C}}(B, A)$
Show how to make this into a category. [Aluffi Exercise 3.1]

Proof. Let $A, B, C \in Obj(\mathbb{C}^{op}) = Obj(\mathbb{C})$. Let $f \in Hom_{\mathbb{C}^{op}}(A, B) = Hom_{\mathbb{C}}(B, A)$ and $g \in Hom_{\mathbb{C}^{op}}(B, C) = Hom_{\mathbb{C}}(C, B)$. We need to define a composition, so let that be $(gf)_{\mathbb{C}^{op}} \in Hom_{\mathbb{C}^{op}}(A, C) = Hom_{\mathbb{C}}(C, A)$. If one stares at this long enough, then $(gf)_{\mathbb{C}^{op}} = (fg)_{\mathbb{C}}$.
1) Identity: we know that $\mathbb{C}$ is a category so we must have an identity morphism $\mathbb{1}_A \in Hom_{\mathbb{C}}(A, A) = Hom_{\mathbb{C}^{op}}(A, A)$ for any $A \in Obj(\mathbb{C}) = Obj(\mathbb{C}^{op})$ so we know we have the same morphism in $\mathbb{C}^{op}$.
2) Compositions: our definition of $(gf)_{\mathbb{C}^{op}}$ is basically the standard composition $(fg)_{\mathbb{C}}$.
3) Associativity: let $A, B, C, D \in Obj(\mathbb{C})$, $f \in Hom_{\mathbb{C}^{op}}(A, B)$, $g \in Hom_{\mathbb{C}^{op}}(B, C)$, $h \in Hom_{\mathbb{C}^{op}}(C, D)$:
$\Rightarrow ((hg)_{\mathbb{C}^{op}} f)_{\mathbb{C}^{op}} = (f(gh)_{\mathbb{C}})_{\mathbb{C}} = ((fg)_{\mathbb{C}} h)_{\mathbb{C}} = (h(gf)_{\mathbb{C}^{op}})_{\mathbb{C}^{op}}$ as required.
4) Identities w.r.t. composition: for $f \in Hom_{\mathbb{C}^{op}}(A, B) = Hom_{\mathbb{C}}(B, A)$:
$\Rightarrow (f\mathbb{1}_A)_{\mathbb{C}^{op}} = (\mathbb{1}_A f)_{\mathbb{C}} = f$ and $(\mathbb{1}_B f)_{\mathbb{C}^{op}} = (f\mathbb{1}_B)_{\mathbb{C}} = f$ as required.
5) Let $A, B, C, D \in ObjC = Obj\mathbb{C}$. Due to $\mathbb{C}$ being a category, if $A \neq C$ and $B \neq D$ then $Hom_{\mathbb{C}}(B, A) = Hom_{\mathbb{C}^{op}}(A, B)$ and $Hom_{\mathbb{C}}(D, C) = Hom_{\mathbb{C}op}(C, D)$ are disjoint which is the final requirement.
$\Rightarrow \mathbb{C}^{op}$ is a category. □

Problem 5: Let A, B be objects of a category $\mathbb{C}$, and let $f \in Hom_{\mathbb{C}}(A, B)$ be a morphism.
1) Prove that if f has a right-inverse, then f is an epimorphism.
2) Show that the converse doesn't hold, by giving an explicit example of a category and an epimorphism without a right inverse.
[Aluffi Exercise 4.3]

Proof. Let $g : B \to A$ be the right inverse of some $f : A \to B$. Then $1_B = fg : B \to B$. Let $Z \in Obj(\mathbb{C})$ and $\beta', \beta'' : A \to Z$. Now suppose $\beta' f = \beta'' f$ and apply g to the RHS:
$(\beta' f)g = (\beta'' f)g \Rightarrow \beta'(fg) = \beta''(fg) \Rightarrow \beta' 1_B = \beta'' 1_B \Rightarrow \beta' = \beta''$ thus f is an epimorphism.

Now let $\mathbb{Z} = Obj(\mathbb{C})$ and define $Hom_{\mathbb{C}}(a, b)$ to be $\{(a, b)\}$ whenever $a \leq b$ and $\varnothing$ otherwise. Furthermore, if a, b, c are objects and $f \in Hom_{\mathbb{C}}(a, b), g \in Hom_{\mathbb{C}}(b, c)$ define the composition $(gf)_{\mathbb{C}} = \{(a, c)\}$. We know that every morphism in $\mathbb{C}$ is an epimorphism, however not every epimorphism can have a right inverse due to the $\varnothing$ in the definition. □