Problem Set 2 : Group Homomorphisms and Isomorphisms

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1. Each element has order 2, $xy = xyxx = xyxex = xyxyyx = (xy)(xy)yx = yx$, thus G is abelian. Then we can calculate this table. $H = \{1, x, y, xy\}$ is a subset of G, closure, identify, inverse are obvious.

	1	x	y	xy
1	1	x	y	xy
x	x	1	xy	y
y	y	xy	1	x
xy	xy	y	x	1

2. $x^{pk} = 1$ so $|x|$ divides pk. Assume $|x| \neq pk$. Suppose $|x| = np$ for some $n < k$, then $a^n = x^{np} = 1$, contradiction. So $|x|$ can't be np for some $n < k$, then $|x|$ must divides k. Since $p \mid k$, $|x|$ can't be k. We must have $|x| < k$, then $a^{|x|} = x^{p|x|} = 1$, contradiction again, so $|x|$ must be pk.

3. Proved in question 1.

4. Suppose G has even number of elements of order 2, then odd number of elements have order neither 1 or 2. These odd number of elements must find its exclusive inverse pair, which is impossible for odd number of elements.

Suppose there's no element of order 2. Then there are odd number of elements have order neither 1 or 2. Proof are similar as above.

5. Verify three requirements:

- Identity: $\forall h \in H, eh = he \implies i \in C(H)$
- Closure: $\forall a, b \in C(H), \forall h \in H, (ab)h = abh = ahb = hab = h(ab) \implies ab \in C(H)$
- Inverse. $\forall a \in C(H), \forall h \in H, a^{-1}h = a^{-1}haa^{-1} = a^{-1}aha^{-1} = ha^{-1} \implies a^{-1} \in C(H)$

6. $x_1x_2 \mapsto f(f^{-1}(x_1)f^{-1}(x_2))$ is the operation.

X **is group**. $\forall x_1, x_2 \in X$, $f^{-1}(x_1) \in G$ and $f^{-1}(x_2) \in G$, by the law of composition under group G we get another element $g \in G$, and f sends it back to X. This operation is closed. Associativity is obvious because this operation is just compose of other functions. $f(1_G)$ is the identity element. For $x \in X$, its inverse is $f((f^{-1}(x))^{-1})$. So X is group under this operation.

f **is isomorphism**. $\forall x_1, x_2 \in X, f^{-1}(x_1x_2) = f^{-1}(f(f^{-1}(x_1)f^{-1}(x_2))) = f^{-1}(x_1)f^{-1}(x_2)$, then f^{-1} is an isomorphism, so is f.

Uniqueness. If there exists another operation noted by $*$, then $\forall x_1, x_2 \in X, x_1 * x_2 = f(f^{-1}(x_1)) * f(f^{-1}(x_2)) = f(f^{-1}(x_1)f^{-1}(x_2)) = f(f^{-1}(x_1x_2)) = x_1x_2$, so this operation is unique.

7. $\{aI \mid a \in \mathbb{R}\backslash 0\}$ is center of $GL_n(\mathbb{R})$. Let E_{ij} denotes matrix that has 0 entry except for $e_{ij} = 1$. Consider:

$$A(I + E_{ij}) = AI + AE_{ij} = A + \begin{pmatrix} & & & a_{1i} & & & \\ 0 & \cdots & 0 & \vdots & 0 & \cdots & 0 \\ & & & a_{ni} & & & \end{pmatrix}$$

$$(I+E_{ij})A = IA + E_{ij}A = A + \begin{pmatrix} 0 & \cdots & 0 \\ & \vdots & \\ a_{j1} & \cdots & a_{jn} \\ & \vdots & \\ 0 & \cdots & 0 \end{pmatrix}$$

If A is in center, necessarily, A's off-diagonal entries are zeros.

There's another constraint: Elementary matrices generate $GL_n(\mathbb{R})$, and it's straightforward that matrices commutes with elementary matrices are center. Let a diagonal matrix left and right multiply with the row switching matrix(column switching for right multiply), we get two matrices:

$$\begin{pmatrix} a_{11} & & & & & \\ & \ddots & & & & \\ & & 0 & & a_{jj} & \\ & & & \ddots & & \\ & & a_{ii} & & 0 & \\ & & & & & \ddots & \\ & & & & & & a_{nn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & & & & & \\ & \ddots & & & & \\ & & 0 & & a_{ii} & \\ & & & \ddots & & \\ & & a_{jj} & & 0 & \\ & & & & & \ddots & \\ & & & & & & a_{nn} \end{pmatrix}$$

And they must be equal, that is $a_{ii} = a_{jj}$ for A. Other elementary matrices didn't bring new constaints.

Under all the constraints, $\{aI \mid a \in \mathbb{R}\backslash 0\}$ is the center.

8. Let $x = y = z$, the identify formula becomes:

$$[xy, u, z] + [yz, u, z] + [zv, u, y] = 3[xx, u, x] = 0$$

which implies $[xx, u, x] = 0$. This is the base case of induction for $[x^q, y, x^{p,q}] = 0$ holds for $1 \le q < p$. Suppose this holds for $p = n, n > 1$, that is $[x^q, y, x^{n-q}] = (x^q y)x^{n-q} - x^q(yx^{n-q}) = 0$ holds for $1 \le q < n$. For $p = n+1$:

$$\begin{aligned}
\left[x^q, y, x^{n+1-q}\right] &= (x^q y)x^{n+1-q} - x^q(yx^{n+1-q}) \\
&= (x^q y)x^{n+1-q} - x^q(yx^{n+1-q}) - ((x^q y)x^{n-q} - x^q(yx^{n-q}))x \\
&= (x^q y)(x^{n-q}x) - x^q(y(x^{n-q}x)) - ((x^q y)x^{n-q} - x^q(yx^{n-q}))x \\
&= (x^q y)(x^{n-q}(x-1)) - x^q(y(x^{n-q}(x-1))) - ((x^q y)x^{n-q} - x^q(yx^{n-q}))(x-1) \\
&= \dots \\
&= (x^q y)(x^{n-q}(x-x)) - x^q(y(x^{n-q}(x-x))) - ((x^q y)x^{n-q} - x^q(yx^{n-q}))(x-x) \\
&= 0
\end{aligned}$$

Induction done.

Let $y = x$, then $[x^q, y, x^{p-q}] = [x^q, x, x^{p-q}] = x^{q+1}x^{p-q} - x^q x^{p-q+1} = 0$, that is $x^{q+1}x^{p-q} = x^q x^{p-q+1}$ Thus,

$$x^m x^n = x^{m-1}x^{n+1} = \dots = x^{m-m}x^{n+m} = x^{n+m} = x^{m+n}$$

for all x.