Problem Set 1 : Sets and Categories

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1 Naive Set Theory

Proof. Let's consider three requirements for $\sim$:

- reflexivity: $\forall a \in \mathbb{R}$, we have $a - a = 0 \in \mathbb{Z}$, then $a \sim a$.
- symmetry: $\forall a, b \in \mathbb{R}$, if $a \sim b$, that it $a - b \in \mathbb{Z}$, then $b - a \in \mathbb{Z}$, that is $b \sim a$.
- transitivity: $\forall a, b, c \in \mathbb{R}$, if $a \sim b$ and $b \sim c$,we have $a - c = (a - b) - (b - c) \in \mathbb{Z}$, that is $a \sim c$.

$\square$

$\mathbb{R}/\sim$ contains equivalence classes which has same fraction.

Proof. Let's consider $\approx$:

- reflexivity: $\forall (a_1, a_2) \in \mathbb{R} \times \mathbb{R}$, we have $a_1 - a_1 = 0 \in \mathbb{Z}$ and $a_2 - a_2 = 0 \in \mathbb{Z}$, then $a \approx a$.
- symmetry: $\forall (a_1, a_2), (b_1, b_2) \in \mathbb{R} \times \mathbb{R}$, if $(a_1, a_2) \approx (b_1, b_2)$, that it $b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$, then $a_1 - b_1 \in \mathbb{Z}$ and $a_2 - b_2 \in \mathbb{Z}$, that is $b \approx a$.
- transitivity: $\forall (a_1, a_2), (b_1, b_2), (c_1, c_2) \in \mathbb{R} \times \mathbb{R}$, if $(a_1, a_2) \approx (b_1, b_2)$ and $(b_1, b_2) \approx (c_1, c_2)$, we can show that $c1 - a1 = (c1 - b1) - (b1 - a1) \in \mathbb{Z}$ and $c2 - a2 = (c2 - b2) - (b2 - a2) \in \mathbb{Z}$, that is $a \approx c$.

$\square$

if we split a 2D plane into consecutive 1×1 squares that align with x and y axis, $\mathbb{R} \times \mathbb{R}/\approx$ contains equivalence classes which has same relative position within the square.

2 Functions on Sets

2.1 By corollary 2.2, function f is a bijection iff it has a two sided inverse, that is $f \circ g = 1$ and $g \circ f = 1$, and this tells us g has a two sided inverse at the same time, thus g is a bijection by this corollary. Suppose we can compose two bijection as $f \circ g$, we know that $(f \circ g) \circ (g^{-1} \circ f^{-1}) = 1$ and $(g^{-1} \circ f^{-1}) \circ (f \circ g) = 1$, so their composition is also a bijection.

2.2 Support we have bijection $f : A' \mapsto A''$ and $g : B' \mapsto B''$, and consider $h : A' \cup B' \mapsto A'' \cup B''$:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A' \\ g(x) & \text{if } x \in B' \end{cases}$$

It has a two sided inverse:

$$h^{-1}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in A'' \\ g^{-1}(y) & \text{if } y \in B'' \end{cases}$$

So $A' \cup B' \cong A'' \cup B''$.

We choose copys of original set such that they are disjoint, which fulfills the condition, so disjoint unions are isomorphic.

3 Categories

3.1

- Identity: $Hom_{C^{op}}(A, A) = Hom_C(A, A)$
- Compose:

  $Hom_{C^{op}}(A, B) \circ Hom_{C^{op}}(B, C) := Hom_C(C, B) \circ Hom_C(B, A) = Hom_C(C, A) = Hom_{C^{op}}(A, C)$
- Associativity:

  $(Hom_{C^{op}}(A, B) \circ Hom_{C^{op}}(B, C)) \circ Hom_{C^{op}}(C, D) = Hom_{C^{op}}(A, C) \circ Hom_{C^{op}}(C, D) = Hom_{C^{op}}(A, D)$

  $Hom_{C^{op}}(A, B) \circ (Hom_{C^{op}}(B, C) \circ Hom_{C^{op}}(C, D)) = Hom_{C^{op}}(A, B) \circ Hom_{C^{op}}(B, D) = Hom_{C^{op}}(A, D)$
- Identify in compose:

  $Hom_{C^{op}}(A, A) \circ Hom_{C^{op}}(A, B) = Hom_C(B, A) \circ Hom_C(A, A) = Hom_C(B, A) = Hom_{C^{op}}(A, B)$

  $Hom_{C^{op}}(A, B) \circ Hom_{C^{op}}(B, B) = Hom_C(B, B) \circ Hom_C(B, A) = Hom_C(B, A) = Hom_{C^{op}}(A, B)$
- Different morphism for difference objects.

3.2

- Identity: Identify Matrix $N \times N$
- Compose: Matrix multiplication.
- Associativity: Linear tranformation is associative.
- Identify in compose: Multiply by identity matrix.
- Different morphism for difference objects.

But can't see how to deal with 0-order...

3.3 Example 3.9

- Obj($\mathtt{C}^{A,B}$) = diagrams

$$\begin{tikzcd} A \arrow[dr, "f"] & \\ & Z \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

- morphisms

$$\begin{tikzcd} A \arrow[dr, "f"] & & & A \arrow[dr, "f"] & \\ & Z_1 & \longrightarrow & & Z_2 \\ B \arrow[ur, "g"'] & & & B \arrow[ur, "g"'] & \end{tikzcd}$$

  are commutative diagrams

$$\begin{tikzcd} A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\ & Z_1 \arrow[r, "\sigma"] & Z_2 \\ B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & & \end{tikzcd}$$

Example 3.10

- $\mathrm{Obj}(\mathtt{C}^{\alpha,\beta})$ = commutative diagrams

$$\begin{tikzcd}
& A \arrow[dr, "f"] & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \\
& B \arrow[ur, "g"'] &
\end{tikzcd}$$

- morphisms

$$\begin{tikzcd}
& A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z_1 \arrow[r, "\sigma"] & Z_2 \\
& B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}$$

4 Morphisms

Proof. $\forall$ morphisms b' and b'', if $b' \circ f = b'' \circ \circ f$, compose both side with its right-inverse, that is $b' \circ f \circ f_{right}^{-1} = b'' \circ f \circ f_{right}^{-1}$, then $b' = b''$, so f is an epimorphism. $\square$

In the category defined by $\leq$ on $\mathbb{Z}$, every morphism is an epimorphism, but it doesn't have any right-inverse, they just don't exist.

5 Universal Properties

5.1 Products are universal properties, they are initial objects of category thus should be isomorphic.

5.2

$$\begin{tikzcd}
& A \\
A \times B \times C \arrow[ur, "\pi_A"] \arrow[r, "\pi_B"] \arrow[dr, "\pi_C"] & B \\
& C
\end{tikzcd}$$

For every set Z and morphisms

$$\begin{tikzcd}
& A \\
Z \arrow[ur, "f_A"] \arrow[r, "f_B"] \arrow[dr, "f_C"] & B \\
& C
\end{tikzcd}$$

there exist a unique morphism $\sigma : Z \longrightarrow A \times B \times C$ such that the diagram

$$\begin{tikzcd}
& A & \\
Z \arrow[ur, "f_A"] \arrow[r, "f_B"] \arrow[dr, "f_C"'] \arrow[rr, bend left=70, "\sigma"] & B & A \times B \times C \arrow[ul, "\pi_A"'] \arrow[l, "\pi_B"] \arrow[dl, "\pi_C"] \\
& C &
\end{tikzcd}$$

commutes.

This is 5.4 Products from Aluffi.

$$\begin{tikzcd}
 & & A \\
Z \arrow[r, "\sigma"] \arrow[rru, bend left, "f_A"] \arrow[rrd, bend right, "f_B"'] & A \times B \arrow[ru, "\pi_A"] \arrow[rd, "\pi_B"] & \\
 & & B
\end{tikzcd}$$

We do it again and get what we want:

$$\begin{tikzcd}
 & & A \times B \\
Z \arrow[r, "\sigma'"] \arrow[rru, bend left, "\sigma"] \arrow[rrd, bend right, "f_C"'] & (A \times B) \times C \arrow[ru, "\pi_{AB}"] \arrow[rd, "\pi_C"] & \\
 & & C
\end{tikzcd}$$

σ' for $A \times (B \times C)$ can be constructed in a similiar way.

$(A \times B) \times C$ and $A \times (B \times C)$ are both final objects, thus they are isomorphic.