Problem Set 1 : Sets and Categories

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0.1 Naive Set Theory

1. Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$. [Aluffi Exercise 1.6]

0.2 Functions on Sets

2. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection. [Aluffi Exercise 2.3]

3. Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \varnothing$ and $A'' \cap B'' = \varnothing, then$A' $\cup B' \cong A'' \cup B''$. Conclude that the operation $A \sqcup B$ is well-defined *up to isomorphism* [Aluffi Exercise 2.9]

0.3 Categories

3. Let C be a category. Consider a structure C^{op} with

- Obj(C^{op}) := Obj(C);
- for A,B objects of C^{op}(hence objects of C), $Hom_{C^{op}}(A, B) := Hom_C(B, A)$.

Show how to make this into a category. [Aluffi Exercise 3.1]

4. Define a category V by taking Obj(V) $= \mathbb{N}$ and letting $Hom_V(n.m) =$ the set of $m \times n$ matrices with real entires, for all $m.n \in \mathbb{N}$. Use matrix multiplication to define composition. Does this category "feel" familiar? [Aluffi Exercise 3.6]

5. Draw the relevant diagrams and define composition and identities for category $C^{A,B}$, mentioned in Example 3.9. Do the same for category $C^{\alpha,\beta}$ mentioned in Example 3.10. [Aluffi Exercise 3.11]

0.4 Morphisms

6. Let A, B be objects of a category C, and let $f \in Hom_C(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse doesn't hold, by giving an explicit example of a category and an epimorphism without a right inverse.

[Aluffi Exercise 4.3]

0.5 Universal Properties

7. Show that in every category C the produts $A \times B$ and $B \times A$ are isomorphic, if they exist. [Aluffi Exercise 5.8]

8. Let C be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of *three* objects of C ought to satisfy,and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic. [Aluffi Exercise 5.9]