Problem Set 3 : Equivalence Relations, Products and Quotients

27th July, 2021

1 Refernced Texts :

Artin's *Algebra* : Chapter 2; Section 7-12
Aluffi's *Chapter 0 : Algebra* : Chapter 7
Hungerford's *Algebra*: Chapter 5, Section 5
Birkhoff, MacLane's *Algebra*: Chapter 2, Section 9 & 10
Nicholas Bourbaki's *Algebre* : Chapter 1, §3 - §5
Serge Lang's *Algebra* : Chapter 1

2 Problems

1. Let H and K be subgroups of a group G. Prove that the product set HK is a subgroup of G iff $HK = KH$. [**Artin Exercise 11.9**]

2. Let G_1 and G_2 be groups and let Z_i be the center of G_i. Prove that the center of the product group $G_i \times G_2$ is $Z_i \times Z_2$. [**Artin Exercise 11.5**]

3. In the general linear group $GL_3(\mathbb{R})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $*$ represents an arbitrary real number. Show that H is a subgroup of GL_3, that K is a normal subgroup of H, and identify the quotient group H/K. Determine the center of H. [**Artin Exercise 12.2**]

4. Let P be a partition of a group G with the property that for any pair of elements A, B of the partition, the product set AB contained entirely within another element C of the partition. Let N be the element of P that contains 1. Prove that N is a normal subgroup of G and that P is the set of its cosets. [**Artin Exercises 12.3**]

5. let G be a group, and let n be a positive integer. Consider the relation

$$a \sim b \iff (\exists g \in G) ab^{-1} = g^n$$

- Show that in general $\sim$ is *not* an equivalence relation.
- Prove that $\sim$ is an equivalence relation if G is commutative, and determine the corresponding subgroup of G.

[**Aluffi Exercise 7.6**]

6. Let G be a group, and let $[G, G]$ be the subgroup of G generated by all elements of the form $aba^{-1}b^1$. (This is the *commutator* subgroup) Prove that $[G, G]$ is normal in G. And prove that $G/[G, G]$ is commutative. [**Aluffi Exercise 7.11**]

7. If $f : G \to H$ is a homomorphism, H is abelian and N is a subgroup of G containing the kernel of f, then N is normal in G. [**Hungerford, Section 5 Exercise 16.**]

8. For subgroups $N \subset M$ of G, both normal in G, prove that $(G/N)/(M/N) \cong G/M$. [**MacLane, Ch.II; Section 10, Exercise 9**]

9. [**Optional**] Let (G_α) be a family of normal subgroups of a group G such that $\bigcap_\alpha G_\alpha = \{\text{e}\}$; show that G is isomorphic to a subgroup of the product group $\Pi_\alpha(G/G_\alpha)$. [**N.Bourbaki, §4 Exercise 6.**]

10. [**Optional**] Let H be a subgroup of a finite abelian group G. Show that G has a subgroup that is isomorphic to G/H. [**Lang, Ch.I, Exercise 43**]