On the Average Value of Correlated Time Series, with Applications in Dendroclimatology and Hydrometeorology

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ABSTRACT

In a number of areas of applied climatology, time series are either averaged to enhance a common underlying signal or combined to produce area averages. How well, then, does the average of a finite number (N) of time series represent the population average, and how well will a subset of series represent the N-series average? We have answered these questions by deriving formulas for 1) the correlation coefficient between the average of N time series and the average of n such series (where n is an arbitrary subset of N) and 2) the correlation between the N-series average and the population. We refer to these mean correlations as the subsample signal strength (SSS) and the expressed population signal (EPS). They may be expressed in terms of the mean inter-series correlation coefficient $\bar{r}$ as

$$\mathrm{SSS} \equiv (\bar{R}_{n,N})^2 \approx \frac{n(1 + (N - 1)\bar{r})}{N(1 + (n - 1)\bar{r})},$$

$$\mathrm{EPS} \equiv (\bar{R}_N)^2 \approx \frac{N\bar{r}}{1 + (N - 1)\bar{r}}.$$

Similar formulas are given relating these mean correlations to the fractional common variance which arises as a parameter in analysis of variance. These results are applied to determine the increased uncertainty in a tree-ring chronology which results when the number of cores used to produce the chronology is reduced. Such uncertainty will accrue to any climate reconstruction equation that is calibrated using the most recent part of the chronology. The method presented can be used to define the useful length of tree-ring chronologies for climate reconstruction work. A second application considers the accuracy of area-average precipitation estimates derived from a limited network of raingage sites. The uncertainty is given in absolute terms as the standard error of estimate of the area-average expressed as a function of the number of gage sites and the mean inter-site correlation.

1. Introduction

This paper is concerned with estimating the statistical uncertainty in the average value of a set of correlated time series. This type of problem occurs in areas of applied climatology where sets of time series are averaged to enhance some common underlying signal, or are combined to produce spatial averages. For example, in dendroclimatology time series of indexed ring widths are averaged to produce "chronologies" which reflect variations in ring widths that are common to all trees at a particular site. A chronology based on, say, N ring-width series is an estimate of a hypothetical population chronology which may, in turn, be regarded as the potential climate signal. It is useful to know how well a given N-series chronology estimates the population chronology, and, also, how well a chronology based on a subset of n series estimates the larger N-series chronology.

A second example is the estimation of area-averages of meteorological variables, such as temperature and precipitation. This application is less direct since it usually involves some weighting of the time series before averaging, and the weights may be time-dependent. Although the temperature case is of considerable interest, we will concentrate here on the hydrologically-oriented example, the estimation of area-average precipitation. In an ideally instrumented river catchment area, the areal average precipitation will be the average value of all the individual precipitation values from a regularly spaced network of raingages. How few gages are required to yield a good estimate of the true area average?

In mathematical terms we pose the following questions. Suppose that W_{ij} is the jth observation of a parameter W_i ($i = 1, N; j = 1, J$) and that each W_i reflects some controlling process μ which is obscured by noise which depends on both i and j (i.e., $W_{ij} = \mu_j + e_{ij}$). If the e_{ij} have similar variances the best estimate of the jth value of the controlling process, which is common to all i, is

$$\bar{W}_{\cdot j}(N) = \frac{1}{N} \sum_{i=1}^{N} W_{ij}, \tag{1}$$

where (N) indicates that the average is over N time series. How good is $\bar{W}_{\cdot j}(N)$ as an estimate of the population value μ_j, and how good is $\bar{W}_{\cdot j}(n)$ as an estimate of $\bar{W}_{\cdot j}(N)$ for $n < N$? The answers depend on how strongly the time series W_{ij} are correlated. If they are highly intercorrelated, then they will have most of their variability in common and μ_j can be well estimated even if N is small. If they are poorly intercorrelated (i.e. subject to a large error or noise element) then N must be large in order to reliably estimate μ_j. We seek to quantify these qualitative results.

2. Theory

a. Introduction

To determine the accuracy of $\bar{W}_{\cdot j}(n)$ as an estimate of $\bar{W}_{\cdot j}(N)$ we will use the average value of the correlations between $\bar{W}_{\cdot j}(N)$ and all possible subsets n of N. For a particular subset, the correlation between $\bar{W}_{\cdot j}(n)$ and $\bar{W}_{\cdot j}(N)$ is

$$R_{n,N} = \frac{\sum_{j=1}^{J} (\bar{W}_{\cdot j}(n) - \bar{W}(n))(\bar{W}_{\cdot j}(N) - \bar{W}(N))}{(J-1)s(n)s(N)}, \quad (2)$$

where

$$[s(N)]^2 = \frac{1}{J-1} \sum_{j=1}^{J} [\bar{W}_{\cdot j}(N) - \bar{W}(N)]^2 \quad (3)$$

with $\bar{W}_{\cdot j}(N)$ defined by Eq. (1) and

$$\bar{W}(N)\ [\equiv \bar{W}_{\cdot\cdot}(N)] = \frac{1}{J} \sum_{j=1}^{J} \bar{W}_{\cdot j}(N),$$

[similarly for $s(n)$ and $\bar{W}(n)$]. A measure of the accuracy of $\bar{W}_{\cdot j}(n)$ as an estimate of $\bar{W}_{\cdot j}(N)$ is provided by the average value of $R_{n,N}$ (viz. $\bar{R}_{n,N}$) over all possible subsets of size n. By taking the limit as $N \to \infty$ of $\bar{R}_{n,N}$ and replacing n by N we obtain a measure of the accuracy of $\bar{W}_{\cdot j}(N)$ as an estimate of μ_j, viz. $\bar{R}_N$. Determination of $\bar{R}_{n,N}$ (and $\bar{R}_N$) is the main objective of this paper (See Appendix A for nomenclature).

To determine $\bar{R}_{n,N}$ we need a number of intermediate results relating to the statistical properties of W_{ij} and $\bar{W}_{\cdot j}$ and to the correlations between individual time series and between any particular series and the average series. If the basic time series are strongly correlated then, for given N, the average series will be a better estimate of the hypothetical population average, so the mean inter-series correlation is obviously a key parameter.

b. Analysis of variance results

All of the expressions used in our analyses involve sums of squares which appear naturally in a two-way analysis of variance (ANOVA) of the time series data portrayed, for example, as a matrix with the time series (variables) as rows and the observations as columns. It is convenient, therefore, (although not strictly necessary) to begin with an ANOVA Table. This is a useful step because it helps to relate our results to other published work in dendroclimatology which makes extensive use of ANOVA (Fritts, 1976); such comparisons will be made in a later section of this paper.

Parameter	Degrees of freedom	Mean square	Expected value
Sum of squares between observations = SSY	$J - 1$	MSY = SSY/ $(J - 1)$	$E\{\text{MSY}\} = N\sigma_y^2 + \sigma_e^2$
Sum of squares between series = SSC	$N - 1$	MSC = SSC/ $(N - 1)$	$E\{\text{MSC}\} = J\sigma_c^2 + \sigma_e^2$
Error sum of squares = SSE	$(N - 1)(J - 1)$	MSE = SSE/ $(N - 1)(J - 1)$	$E\{\text{MSE}\} = \sigma_e^2$
Total sum of squares = SST	$NJ - 1$	MST = SST/ $(NJ - 1)$	

Here σ_y^2 is the population between-observation (or within-series) variance, σ_c^2 is the population between-series variance and σ_e^2 the population error variance. If $\bar{W}_{\cdot j} \equiv \bar{W}_{\cdot j}(N)$ is the average value over all N series of observation j [Eq. (1)] and $\bar{W}_{i\cdot} \equiv \bar{W}_{i\cdot}(J)$ is the average value over all J observations for series i, the sums of squares terms are defined by

$$\text{SSY} = N \sum_{j=1}^{J} (\bar{W}_{\cdot j} - \bar{W})^2, \quad (4)$$

$$\text{SSC} = J \sum_{i=1}^{N} (\bar{W}_{i\cdot} - \bar{W})^2, \quad (5)$$

$$\text{SST} = \sum_{i=1}^{N} \sum_{j=1}^{J} (W_{ij} - \bar{W})^2, \quad (6)$$

$$\text{SSE} = \text{SST} - \text{SSY} - \text{SSC}. \quad (7)$$

We need also to define three variances, the variance for the ith series (s_i^2), the variance across series for observation j (s_j^2), and the variance for the average series $\{s^2 \equiv [s(N)]^2\}$. The last of these has already been defined [Eq. (3)]. From Eq. (4) we have

$$s^2 \equiv [s(N)]^2 = \frac{\text{SSY}}{N(J-1)}. \quad (8)$$

We also have

$$(J-1)s_i^2 = \sum_{j=1}^{J} (W_{ij} - \bar{W}_{i\cdot})^2, \quad (9)$$

$$(N-1)s_j^2 = \sum_{i=1}^{N} (W_{ij} - \bar{W}_{\cdot j})^2. \quad (10)$$

Note that s^2 is not an unbiased estimator of the population between-observation (or '*common*') variance σ_y^2. Since

$$\sigma_y^2 = (E\{\text{MSY}\} - \sigma_e^2)/N = (E\{\text{MSY} - \text{MSE}\})/N$$
$$= \frac{E\{\text{SSY} - \text{SSE}/(N-1)\}}{N(J-1)}, \quad (11)$$

an unbiased estimator of the common variance is

$$\hat{s}^2 \equiv [\hat{s}(N)]^2 = \frac{1}{N(J-1)}\left(\text{SSY} - \frac{\text{SSE}}{N-1}\right). \quad (12)$$

Finally, two fractional variance terms a and $\hat{a}$ will be useful in later derivations. These terms are measures of the strength of the common signal which our time series averaging aims to detect and isolate. As will be shown later, these fractional common variances are closely related to the average inter-series correlation which in turn is a direct measure of how closely the various time series are related. One would naturally expect closely related series to have a strong common signal. Here a and $\hat{a}$ are defined by

$$a = \frac{\text{SSY}}{\text{SSY} + \text{SSE}}, \quad (13)$$

$$\hat{a} = \frac{\text{SSY} - \text{SSE}/(N-1)}{\text{SSY} + \text{SSE}}. \quad (14)$$

The notation $\hat{a}$ is used to indicate that this parameter involves unbiased estimates of corresponding population parameters, viz.

$$\frac{\sigma_y^2}{\sigma_y^2 + \sigma_e^2} = \frac{E\{\text{SSY} - \text{SSE}/(N-1)\}/N(J-1)}{E\{\text{SSY} + \text{SSE}\}/N(J-1)}.$$

Therefore, $\hat{a}$ has no systematic dependence on sample size. The parameters a and $\hat{a}$ are related by

$$a = \hat{a} + \frac{1-\hat{a}}{N} \quad (15)$$

which shows how $a \to \hat{a}$ as $N \to \infty$.

c. *Mean inter-series correlation*

This is a key parameter in our derivation. We begin by considering the correlation between two particular time series, series i and series I: viz.

$$r_{iI} = \frac{\sum_{j=1}^{J} (W_{ij} - \bar{W}_{i\cdot})(W_{Ij} - \bar{W}_{I\cdot})}{(J-1)s_i s_I}. \quad (16)$$

As a measure of the overall similarity of the whole set of N time series we use the average value of r_{iI}, averaged over all i and I. This average can be estimated approximately using the averages of the separate elements in Eq. (16) (see Appendix B for details). We obtain

$$\bar{r}^* \approx \frac{\text{SSY}}{\text{SSY} + \text{SSE}} \equiv a \quad (17)$$

for the average correlation over all N^2 possible pairs (i, I) including the N pairs $i = I$ for which $r_{iI} \equiv 1$. Excluding these gives

$$\bar{r} \approx \frac{\text{SSY} - \text{SSE}/(N-1)}{\text{SSY} + \text{SSE}} \equiv \hat{a}. \quad (18)$$

Although (17) and (18) are only approximate results we can show empirically that they are excellent approximations (see below).

The inter-series correlation, $\bar{r}^*$, is also closely related to the average correlation between series i and the average series. This is defined by

$$r_{i,N} = \frac{\sum_{j=1}^{J} (W_{ij} - \bar{W}_{i\cdot})(\bar{W}_{\cdot j} - \bar{W})}{(J-1)s_i s}. \quad (19)$$

The average value of $r_{i,N}$ (over $i = 1, N$), viz. $\bar{r}_N$, can be shown to be given by (cf. the derivation of $\bar{r}$)

$$(\bar{r}_N)^2 \approx \frac{\text{SSY}}{\text{SSY} + \text{SSE}} \quad (20)$$

so that

$$(\bar{r}_N)^2 \approx \bar{r}^*. \quad (21)$$

These results can also be verified empirically.

d. *Derivation of $\bar{R}_{n,N}$*

Equation (2) defined the correlation between the n-series average and the N-series average. We are now in a position to evaluate the mean value of this correlation, viz. $\bar{R}_{n,N}$. The details are given in Appendix B. We give three alternative results

$$(\bar{R}_{n,N})^2 \approx \frac{n\bar{r}^*}{1 + (n-1)\bar{r}} = \frac{n[1 + (N-1)\bar{r}]}{N[1 + (n-1)\bar{r}]}, \quad (22)$$

$$(\bar{R}_{n,N})^2 \approx \frac{n\text{SSY}}{n\text{SSY} + \left(\frac{N-n}{N-1}\right)\text{SSE}}, \quad (23)$$

$$(\bar{R}_{n,N})^2 \approx \frac{na}{1 + (n-1)\hat{a}} = \frac{n[1 + (N-1)\hat{a}]}{N[1 + (n-1)\hat{a}]}, \quad (24)$$

where a, $\hat{a}$, $\bar{r}$, r^* and the sums of squares must be determined using all N series.

To find $\bar{R}_N$, the expected correlation between an N-series average, $\bar{W}_{\cdot j}(N)$, and the population average μ_j we simply take the limit of Eqs. (22) or (24) as $N \to \infty$ and replace n by N to give

$$(\bar{R}_N)^2 \approx \frac{N\bar{r}}{1 + (N-1)\bar{r}} \quad (25)$$

TABLE 1. Empirical verification of Eq. (18) and Eqs. (22–24).

Number	Site/data type	N	J	SST	SSY	SSC	SSE	$\bar{r}$ Observed	$\bar{r}$ Eq. (18) $\bar{r} = \hat{a}$	$(\bar{R}_{1,N})^2 \equiv \bar{r}_N$ Observed	$(\bar{R}_{1,N})^2 \equiv \bar{r}_N$ Eq. (22) $(\bar{R}_{1,N})^2 = \bar{r}^*$	$(\bar{R}_{1,N})^2 \equiv \bar{r}_N$ Eq. (24) $(\bar{R}_{1,N})^2 = a$
1	Radley/tree rings*	18	100	1133.94	625.00	1.22	507.72	0.5297	0.5254	0.5556	0.5558	0.5518
2	Radley/tree rings (normalized)	18	100	1800.00	1000.51	0.00	799.49	0.5297	0.5297	0.5558	0.5558	0.5558
3	Hesleyside/tree rings*	23	100	1916.15	680.29	13.70	1222.16	0.3461	0.3284	0.3729	0.3745	0.3576
4	Hesleyside/tree rings (normalized)	23	100	2300.00	861.42	0.00	1438.58	0.3461	0.3461	0.3745	0.3745	0.3745
5	Clovelly/tree rings*	13	100	1360.61	307.95	9.56	1043.10	0.1779	0.1636	0.2390	0.2411	0.2279
6	Clovelly/tree rings (normalized)	13	100	1300.00	313.48	0.00	986.52	0.1779	0.1779	0.2411	0.2411	0.2411
7	River Wye/precipitation†	17	300	1418.00	931.88	259.62	226.50	0.8524	0.7922	0.8582	0.8611	0.8045
8	River Wye/precipitation (weighted)†	17	300	1162.28	997.20	1.36	163.73	0.8524	0.8502	0.8610	0.8611	0.8590

* SST, SSY, SSC, SSE values divided by 10^5 to facilitate comparison.
† SST, SSY, SSC, SSE values divided by 10^4 to facilitate comparison.

or

$$(\bar{R}_N)^2 \approx \frac{N\hat{a}}{1 + (N-1)\hat{a}}. \tag{26}$$

In terms of the sums of squares, Eq. (26) becomes

$$(\bar{R}_N)^2 \approx \frac{\mathrm{SSY} - \dfrac{\mathrm{SSE}}{N-1}}{\mathrm{SSY}}. \tag{27}$$

(It is not valid simply to let $N \rightarrow \infty$ in (23) since SSY and SSE depend on N.) It is easy to show that, as one would expect,

$$(\bar{R}_n)^2 = (\bar{R}_{n,N})^2(\bar{R}_N)^2. \tag{28}$$

Although we hesitate to introduce new "jargon," we have found a need for descriptive terms for $(\bar{R}_{n,N})^2$ and $(\bar{R}_N)^2$, viz. the subsample signal strength (SSS) and the expressed population signal (EPS).

e. Empirical verification

The results for $(\bar{R}_{n,N})^2$ and $(\bar{R}_N)^2$ are approximate and require empirical verification. To do this we used oak tree-ring width data from three sites in England [Radley (Oxfordshire), Hesleyside (Northumberland) and Clovelly (Devon); unpublished tree-ring data collected by K. R. Briffa and P. D. Jones] and precipitation data from the River Wye catchment (southeast Wales; area 4040 km^2). The tree-ring data span a range of fractional common variances $\hat{a}$ from ~0.16 to 0.53 while the precipitation data have $\hat{a} = 0.79$ (0.85 for the weighted precipitation data). Our verifications therefore span a wide range of $\hat{a}$ values. Readers willing to accept the above results can move directly to the next section.

The tree-ring data require some preliminary explanation. The basic data are time series of "indexed" ring widths (W_{ij}) which are averaged to produce tree-ring chronologies [$\bar{W}._j(N)$]. In building chronologies for dendroclimatological purposes indexing is used to remove long-term growth trends (see, e.g., Fritts, 1976; Baillie, 1982). The usual technique is to fit a curve (in this case, a low-order polynomial) to each individual ring-width series and to divide this into the raw data to give the indexed series. Indexing ensures that the means, $\bar{W}_i.$, are approximately equal so that SSC ≈ 0. For the empirical verification analysis we used data spanning 1880–1979 (i.e., $J = 100$ years), although all of the series actually extended back before 1880. The precipitation data used were monthly values for 1951–75 from a network of 17 sites ($J = 300$ months). Further details are given in Table 1.

The results which are most easily verified are those for the mean inter-series correlation, viz. $\bar{r} \approx \hat{a}$ [Eq. (18)] and for the mean correlation between a single series and the N-series average, viz. $(\bar{R}_{1,N})^2 \approx \bar{r}^*$ [Eq. (22) with $n = 1$] and $(\bar{R}_{1,N})^2 \approx a$ [Eq. (24) with $n = 1$]. Values for $\bar{r}$, $\bar{r}^*$ and $\bar{R}_{1,N}$ were calculated directly from the time series, and a and $\hat{a}$ were calculated from the ANOVA results using Eqs. (13) and (14). Note that $\bar{R}_{1,N} \equiv \bar{r}_N$ so that verification of Eq. (22) for $n = 1$ also verifies Eq. (21).

It is of interest also to examine the effect of SSC on the results; the various formulas should be more accurate for small SSC, but their accuracy for large SSC is difficult to estimate *a priori*. In the tree-ring case, the indexed series have SSC approximately zero. To examine the case where SSC is identically zero we produced a second set of tree-ring series for each site by normalizing the original series (although subtracting the means would have been sufficient). For the precipitation case we also produced a second set of series by weighting the original precipitation data using weights appropriate for calculating the area-averaged precipitation (see later). The raw precipitation data had a relatively large SSC contribution to the total sum of squares, while SSC for the weighted data was negligibly small. Overall, then, we tested Eqs. (18), (22) and (24) with eight sets of data.

The results are shown in Table 1. The first thing to note is that the observed mean correlations $\bar{r}$ and $\bar{r}^*$ are independent of any normalizing or weighting. Since this type of massaging only changes the origin and/or scale it cannot affect the mean inter-series correlations, and can only affect $\bar{R}_{1,N}$ slightly. Secondly, we note that the formulas work exactly for the normalized tree-ring data (numbers 2, 4 and 6 in Table 1). In this case all series variances are the same and the formula derivations are exact. When SSC is small (numbers 1, 3, 5 and 8 in Table 1) the variability in variance from series to series is relatively small, so that approximations are excellent and all formulas work extremely well. In these cases, however, the expression relating $(\bar{R}_{1,N})^2$ to the mean inter-series correlation [Eq. (22)] works noticeably better than that relating $(\bar{R}_{1,N})^2$ to the fractional common variance [Eq. (24)], or the equivalent ANOVA sums of squares form [Eq. (23)]. This tendency is much more pronounced in the case where SSC is comparable in size to SSE (number 7 in Table 1). Here both Eq. (18) for $\bar{r}$ and Eqs. (23) and (24) for $(\bar{R}_{1,N})^2$ are in error by 6–7%, but Eq. (22), which relates $(\bar{R}_{1,N})^2$ to $\bar{r}$, is in error by only 0.3%.

The validity of Eqs. (22–24) for general n was confirmed by Monte Carlo simulation in the following way. For each set of N tree-ring time series (corresponding to the data in rows 1, 3 and 5 of Table 1) we took a large number of randomly chosen subsets of n series ($n = 1, 2, 3, \ldots, N$), calculated the n-series average and then calculated $R_{n,N}$. For each value of n we then averaged the $R_{n,N}$ over all subsets to obtain an estimate of $\bar{R}_{n,N}$. The actual value of $R_{n,N}$ depends on the particular subset of n series chosen out of the available N, so we also calculated the standard deviations of these $R_{n,N}$ values as a measure of the scatter of individual realizations. We also estimated $\bar{R}_{n,N}$ using the average inter-series correlation and Eq. (22). For small SSC this equation is equivalent to Eqs. (23) and (24), but it is noticeably better for larger SSC. The results are compared in Fig. 1. The agreement between the simulated $\bar{R}_{n,N}$ and the formula estimates is excellent confirming that Eqs. (22–24) provide a correct representation of the n-dependence of $\bar{R}_{n,N}$. Further confirmation is provided by a similar analysis of the precipitation data (see Fig. 4 in section 3.b).

The relationship between $\bar{W}_{\cdot j}(N)$ and μ_j cannot be tested directly since μ_j is unknown. However, since $\bar{R}_N$ is a limiting form for $\bar{R}_{n,N}$, verification of the latter allows us to place some confidence also in Eqs. (25–27).

3. Applications

a. *Dendroclimatology*

Some of the terms introduced before are already used in dendroclimatology. The fractional common variance concept ($\hat{a}$) is equivalent to the percent common variance which Fritts (1976) defines as

$$\%Y = 100\,\frac{\mathrm{SSY} - \mathrm{SSE}/(N-1)}{\mathrm{SSY} + \mathrm{SSE}}. \tag{29}$$

Fritts (1976, p. 294) has noted, empirically, that $\%Y/100$ is almost identical to the mean inter-series correlation, a result which we have derived theoretically. Fritts' observation adds further support to the validity of our Eq. (18).

The expressed population signal $(\bar{R}_N)^2$ is closely related to the concept of chronology signal-to-noise ratio (SNR) as used in dendroclimatology and defined (e.g., DeWitt and Ames, 1978; Graybill, 1982; Cropper, 1982) as

$$\mathrm{SNR} = \frac{N(\%Y)}{100 - \%Y}. \tag{30}$$

Alternative forms for SNR are

$$\mathrm{SNR} = \frac{(N-1)\mathrm{SSY} - \mathrm{SSE}}{\mathrm{SSE}}, \tag{31}$$

$$\mathrm{SNR} = \frac{N\hat{a}}{1-\hat{a}}. \tag{32}$$

Chronology signal-to-noise ratio has been used to evaluate the relative strength of the common variance signal in tree-ring chronologies from different regions. For example (following DeWitt and Ames, 1978) individual chronologies from southwestern USA have maximum SNR ~ 15 with $N \sim 10$, whereas eastern USA chronologies (which tend to have only about half the fractional common variance) require, since SNR $= N\hat{a}/(1-\hat{a})$, a substantially larger number of cores to attain the same SNR.

SNR and $(\bar{R}_N)^2$ are related [e.g., using Eqs. (27) and (31)] by

$$(\bar{R}_N)^2 \approx \frac{\mathrm{SNR}}{1 + \mathrm{SNR}}. \tag{33}$$

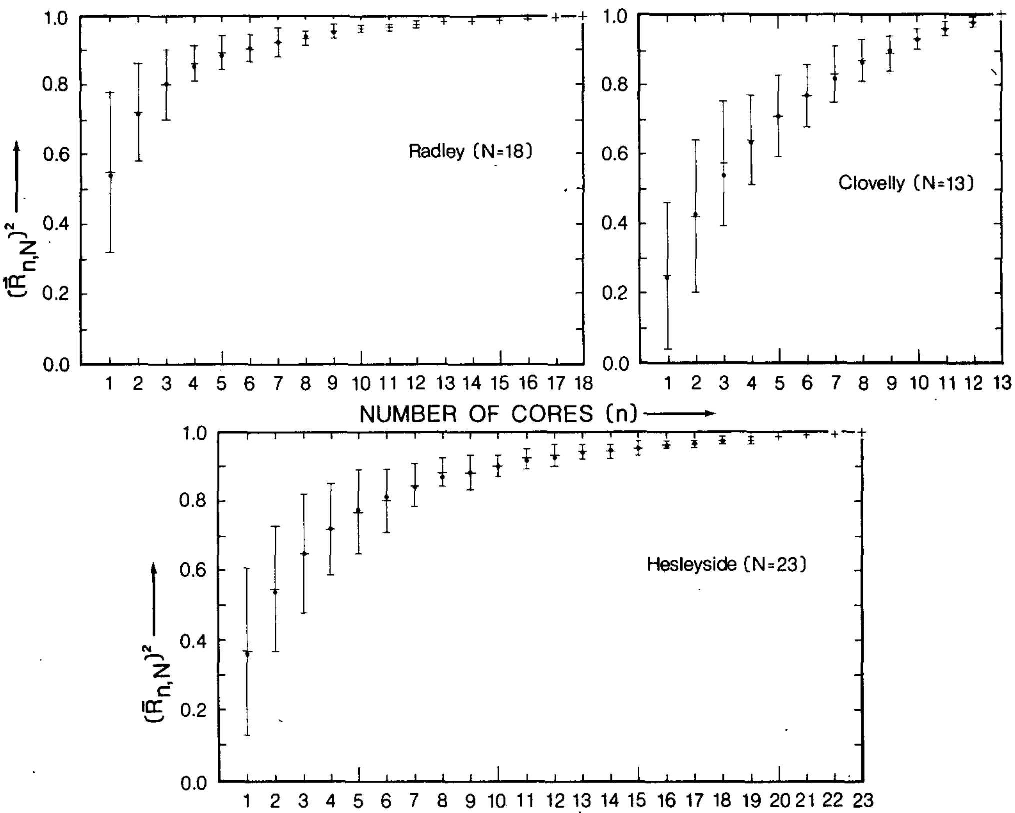

FIG. 1. Empirical verification of the n-dependence of the subsample signal strength $(\bar{R}_{n,N})^2$ using three sets of tree-ring data. The solid circles show $(\bar{R}_{n,N})^2$ values based on Eq. (22) for $n = 1, 2, \ldots, N$. These are compared with empirically determined values obtained by averaging individual values of $R_{n,N}$ over a large number of randomly chosen subsets of n ring-width time series, shown by crosses (mean values) and vertical bars (two-sigma limits). Limits less than ± 0.01 are not shown, and $(\bar{R}_{n,N})^2$ values for $n > 10$ (Radley) and $n > 19$ (Hesleyside) based on Eq. (22) are omitted since they are identical to the simulated means. Agreement is generally excellent, hence confirming the validity of Eq. (22).

The ratio [SNR/(1 + SNR)] when expressed as a percentage, has been called the "percent common signal" by Cropper (1982, p. 66). Eq. (33) identifies this ratio as the amount of population chronology variance which is explained by an N-core chronology. SNR is an indicator of the strength of the signal which is common to a set of tree-ring data, but it behaves in a markedly non-linear fashion. For large SNR, large increases in SNR lead to only minimal changes in $(\bar{R}_N)^2$. Although SNR and $(\bar{R}_N)^2$ give the same information, the latter term (or Cropper's percent common signal) should be easier to interpret.

Even so, the significance of these terms in dendroclimatology is a little obscure. Although common climate forcing may well be a major contributing cause of common variance, the strength of the common signal [as determined by SNR or $(\bar{R}_N)^2$] cannot be interpreted solely in climatic terms, since common variance may also arise from other factors—management, pests and disease, pollution, etc. Furthermore, there is no obvious way to determine how high SNR or $(\bar{R}_N)^2$ should be to ensure that a particular chronology is suitable for climate reconstruction purposes. Nevertheless, when carefully interpreted, SNR can be a useful concept in dendroclimatology, and the present work may be viewed as a contribution towards its clarification through $(\bar{R}_N)^2$ and its generalization through $(\bar{R}_{n,N})^2$.

The parameter $(\bar{R}_{n,N})^2$ should be of considerable value in dendroclimatology. In climate reconstruction work a statistical link between climate and a chronology (or set of chronologies) is usually established using a recent part of the chronology generally made up of a maximum or near-maximum number of cores (N). Reconstructions are then based on earlier sections of the chronology which tend to contain smaller and smaller numbers of cores (n) as one goes back in time (see Fig. 2). As n becomes smaller, the n-core and N-core chronologies become more disparate, so the quality of the reconstruction must diminish. Since $(\bar{R}_{n,N})^2$ is a measure of the uncertainty in a chronology due to a reduction from N cores to n cores, this parameter must also be an indicator of the parallel loss of reconstruction quality. At some point $(\bar{R}_{n,N})^2$ will

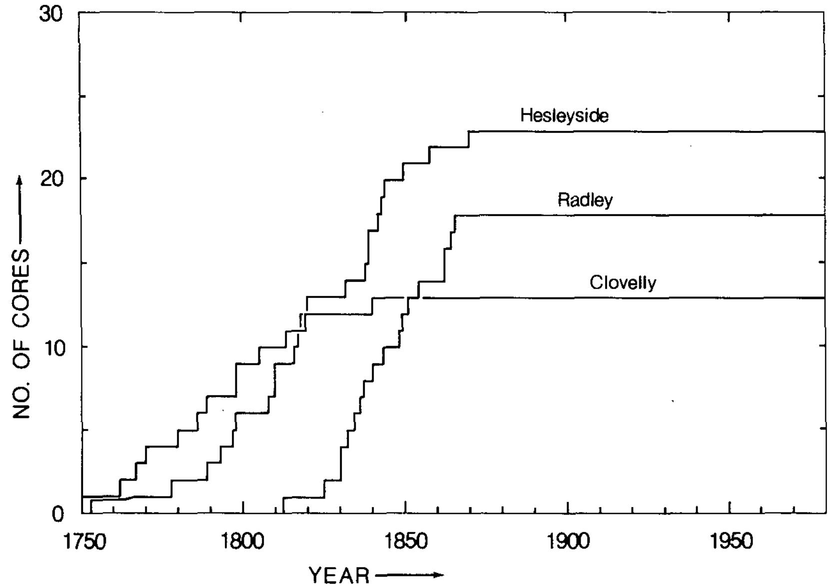

FIG. 2. Changes in the number of cores with time for the three chronologies. A few additional cores exist after 1850 for Clovelly but these have been omitted from the present study because they terminate well before 1980.

become sufficiently low that we may judge the n-core chronology to be an inadequate representation of the N-core chronology. This point can be defined by some pre-specified threshold value of $(\bar{R}_{n,N})^2$ or $(\bar{R}_{n,N})$.

Any such threshold value must be a subjective choice based on the particular application and on the user's experience. Since it is desirable for any reconstruction uncertainty resulting from chronology uncertainty to be considerably less than the uncertainty inherent in the statistical climate-chronology link, the threshold value should be substantially higher than the explained climate variance in the reconstruction. This would ensure that there was, in absolute terms, only a small loss of explained climate variance due to n-core versus N-core chronology uncertainty. In the best climate reconstructions based on tree-ring widths, the climate variance explained is of order 50% (e.g. Fritts *et al.*, 1981; Duvick and Blasing, 1981; Briffa *et al.*, 1983; and various papers in Hughes *et al.*, 1982). As a rough guide, let us assume that chronology uncertainty of order 15% due to a reduction in the number of cores is acceptable [corresponding to a threshold $(\bar{R}_{n,N})^2$ of 0.85]. For explained climate variance of 50% based on an N-core chronology, reduction in the number of cores to the point where $(\bar{R}_{n,N})^2 \approx 0.85$ would only reduce the explained climate variance to around 43% ($0.85 \times 50\%$), a noticeable reduction, but, in most cases, probably not a statistically significant one. The absolute size of the additional loss of explained variance due to $R_{n,N}$ effects is even smaller if the climate variance explained is less than 50%.

The value 0.85 for a threshold $(\bar{R}_{n,N})^2$ is given here only as a guide; in any particular case the chosen threshold will depend on the user's subjective evaluation of accuracy needs. However, if we accept 0.85 as reasonable, then we can easily determine the corresponding minimum number of cores for any given N and $\hat{a}$. Fig. 3 shows the minimum n as a function of N and $\hat{a}$, with the three chronologies used earlier plotted as examples. For Clovelly, with the lowest fractional common variance, at least 8 of the original 13 cores are required; for Hesleyside, with a somewhat higher $\hat{a}$, 7 out of the original 23 are required; while for Radley, with the highest $\hat{a}$, as few as 4 cores are judged to give a satisfactory chronology. Instead of using Fig. 3 the same results can be obtained from the plots of $(\bar{R}_{n,N})^2$ as a function of n, such as in Fig. 1. From Fig. 2 we can see that the three chronologies are satisfactory back to 1798 (Clovelly), 1808 (Hesleyside) and 1832 (Radley).

In practice, the application of this technique for evaluating the maximum satisfactory length of a chronology is straightforward. In the process of chronology construction, it is standard procedure to perform an analysis of variance on the maximum-core-number part of the chronology. Fractional common variance ($\hat{a}$ or Fritts' $\%Y$) is also determined routinely. Theoretical plots (cf. Fig. 1) of $(\bar{R}_{n,N})^2$ as a function of n can easily be constructed using Eq. (23) or (24), or Fig. 3 can be used as an immediate guide.

b. *Hydrometeorology*

One of the concerns of hydrometeorology is the calculation of area-average precipitation, either to estimate total precipitation over river basin catchment

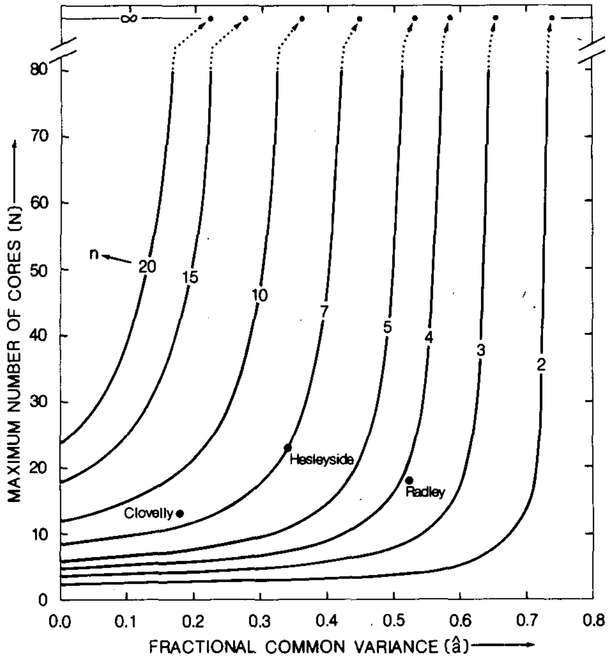

FIG. 3. The minimum number of cores required for an n-core subset of an N-core chronology to maintain climate reconstruction accuracy, based on a critical $(\bar{R}_{n,N})^2$ value of 0.85. The curves are based on Eq. (24). The uppermost points for each curve show the limiting value as $N \to \infty$.

areas for reservoir planning or hydrological experiments or as an indicator of regional-scale climate and climatic change. In a typical situation, with N raingage sites distributed over an area, we may wish to know how well the area-average based on the N-site network represents that for an ideal, fully instrumented area, or how well the N-site area-average is approximated by a smaller set of n raingage sites. The parameters $(\bar{R}_N)^2$ and $(\bar{R}_{n,N})^2$ are measures of the strength of these relationships.

This particular application is more complex than the tree-ring case since area-average precipitation is not a simple average of the individual site values: instead

$$\bar{W}_{\cdot j}(N) = \frac{1}{N}\sum_{i=1}^{N} c_i X_{ij} \equiv \frac{1}{N}\sum_{i=1}^{N} W_{ij}, \tag{34}$$

where X_{ij} is the precipitation at site i at time j, c_i is the weight for site i and W_{ij} are the weighted precipitations. (The X_{ij} may be hourly, daily, monthly, etc. values; our example uses monthly data.) Depending on the method, the weights c_i may differ according to how many sites are used. (This would be the case if the area-average were calculated using Thiessen polygons, for example.) If so, then adding or removing a time series from the N-series set will change all of the W_{ij} series and the theory we have developed will not be immediately applicable. For a rigorous application a weighting system which is independent of N is required, as provided by the method in which

$$c_i = \frac{\langle \bar{W}_{\cdot j}\rangle}{\langle X_{ij}\rangle}, \tag{35}$$

where angle brackets denote the average over some chosen reference period for which the area-average is known accurately.

As an example we have used monthly data from the River Wye catchment upstream of Redbrook (5 km south of Monmouth). These data have already been discussed in reference to Table 1. For this catchment we have compiled a basic data set using a 17-site network, which extends back to 1899 (Jones, 1983). Annual precipitation varies considerably over the catchment, from around 690 mm to 1600 mm. For the period 1916–50 an accurate area-average value has been calculated based on isohyetal analyses using all available data (more sites than our basic 17-site network). This value and the corresponding site values were used to determine the weights c_i defined by Eq. (35). We are interested in knowing how few gage sites are needed to produce a reasonable area-average (required, in this case, to estimate river flows). The appropriate parameter is $(\bar{R}_n)^2$.

To calculate $(\bar{R}_N)^2$ and $(\bar{R}_{n,N})^2$ we need to know either the mean inter-series correlation $\bar{r}$ or the fractional common variance $\hat{a}$. We have already determined these using data for the period 1951–75; the results are given in Table 1. Note that $\bar{r}$ gives the better estimate of $(\bar{R}_N)^2$ and $(\bar{R}_{n,N})^2$ (although the gain over using $\hat{a}$ is only marginal in this case), and that $\bar{r}$ can be calculated using the raw, unweighted data. Using Eqs. (22) and (25) we have

$$(\bar{R}_{n,N})^2 = \frac{n(1 + 16(0.8524))}{17(1 + (n-1)0.8524} = \frac{1.0102n}{n + 0.1732}, \tag{36}$$

$$(\bar{R}_n)^2 = \frac{0.8524n}{1 + (n-1)0.8524} = \frac{n}{n + 0.1732}, \tag{37}$$

$$(\bar{R}_N)^2 \equiv (\bar{R}_{17})^2 = 0.9899. \tag{38}$$

The $(\bar{R}_{n,N})^2$ and $(\bar{R}_n)^2$ based on these equations are shown in Fig. 4 together with estimates of $(\bar{R}_{n,N})^2$ based on Monte Carlo simulations and standard deviations of these estimates (cf. Fig. 1). The agreement between theory and simulation is excellent.

We also show in Fig. 4 the observed values of $(R_{n,N})^2$ for the actual n-site network which existed at various times prior to 1899 (see Fig. 5). These individual realizations differ somewhat from the expected values but they are generally within the 95% (i.e., two-sigma) confidence limits. The fourth site included in the network is an upland site for which precipitation is highly correlated with the 17-site area-average (based on 1951–

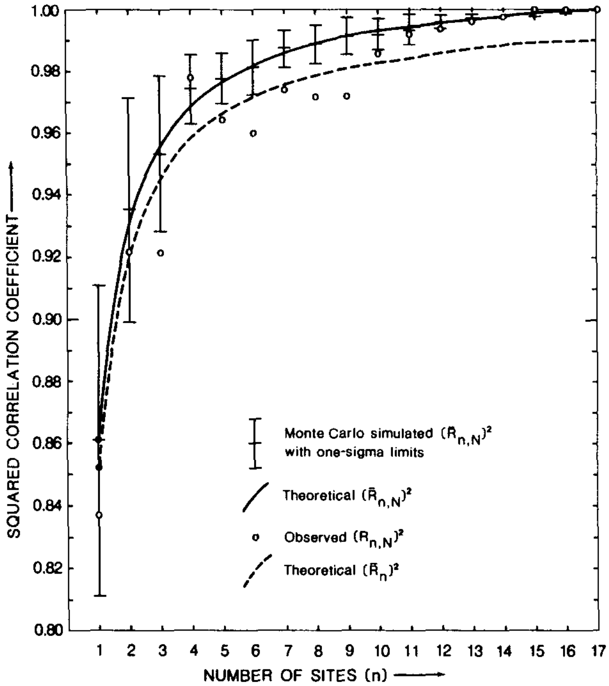

FIG. 4. Variations in $(\bar{R}_{n,N})^2$ for weighted precipitation time series from the River Wye catchment area. Values for increasing n based on Eq. (22) are shown by the solid curve and values based on Monte Carlo simulations are shown by crosses (mean values) and vertical bars (one sigma limits). The open circles show observed values of $(R_{n,N})^2$ for the particular raingage configurations which existed at different times (see Fig. 5). The dashed curve shows $(\bar{R}_n)^2$ based on Eq. (25).

75 data) and so gives an unusually large improvement in $(R_{n,N})^2$. The fifth and sixth sites, however, are relatively poor in their correlations with the area-average and these cause a noticeable depression in $(R_{n,N})^2$ below $(\bar{R}_{n,N})^2$. It is clear from Fig. 4 that a number of the individual networks are noticeably inferior in that their $(R_{n,N})^2$ values are well below the corresponding $(\bar{R}_{n,N})^2$ value. A part of this departure must be due to sampling uncertainty in $(R_{n,N})^2$; in Fig. 4 only the n = 6, 7, 8 and 9 cases are significantly different from $(\bar{R}_{n,N})^2$ at the 0.05 level. Conversely, for any given n some networks will be superior, with $(R_{n,N})^2$ noticeably larger than $(\bar{R}_{n,N})^2$.

Our theoretical results can be used to determine the number of gages required to give an accurate estimate of area-average precipitation (according to some predetermined criterion for accuracy) for the case of randomly located measuring sites. (Note that a fewer number may be satisfactory if the gages could be optimally located and provided the corresponding $R_{n,N}$ value was significantly above $\bar{R}_{n,N}$). For randomly located gages the number required for given accuracy in area-averaging can be determined using $(\bar{R}_n)^2$, the squared-correlation between an n-site area-average and the population (i.e., the true) area-average. Since the true area-average is not known, $(\bar{R}_n)^2$ can only be determined theoretically. Applying this theoretical result to real-world situations depends on the assumption that the real site network is uniform, in the sense that all sites are equally important in determining the area-average. This is a common assumption (see, e.g., Sutcliffe, 1966; Herbst and Shaw, 1969; Clarke and Edwards, 1972). Fig. 4 shows that noticeable differences can occur between the theoretical uniform network and a particular network, but these differences tend to become smaller as the number of sites increases. Relatively few differences are statistically significant.

We now use the River Wye data as an example and determine the minimum acceptable number of gages for calculating area-average precipitation. We could specify a critical $\bar{R}_n$ value for acceptable accuracy, but it is more convenient to consider accuracy in terms of the standard error (SE_n) of the area-average estimate. Unfortunately, this standard error cannot be uniquely defined, nor specified uniquely in terms of sample statistics. If we make use of the relationship

$$(SE_n)^2 = S^2(1 - (\bar{R}_n)^2), \qquad (39)$$

where S^2 is the variance of the area-average precipitation, then we have alternative results depending on which expressions are used for S and $\bar{R}_n$. For example, using the $\hat{a}$ form for $\bar{R}_n$ [Eq. (26)] we have either

$$(SE_n)^2 = \frac{SSY}{N(J-1)} \frac{1-\hat{a}}{1+(n-1)\hat{a}}, \qquad (40a)$$

using the sample variance for S^2 [Eq. (8)], or

$$(SE_n)^2 = \frac{(N-1)SSY - SSE}{N(N-1)(J-1)} \frac{1-\hat{a}}{1+(n-1)\hat{a}}, \qquad (40b)$$

using the best estimate of $\sigma_y{}^2$ for S^2 [Eq. (12)]. Note that we cannot use the sum of squares form for $(\bar{R}_n)^2$ since the sums of squares are functions of n and are

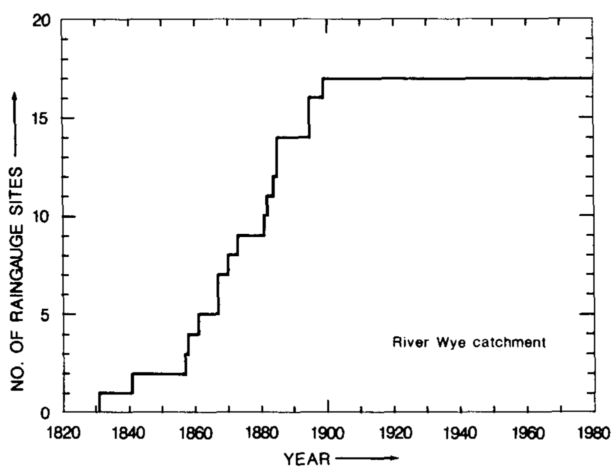

FIG. 5. Changes in the number of raingage sites with time for the River Wye catchment.

only known for $n = N$. Eq. (40a) can be expanded [using Eq. (14)] to give

$$(\mathrm{SE}_n)^2 = \frac{\mathrm{SSE}}{n(N-1)(J-1)}\left[1 + \frac{N-n}{n(N-1)}\frac{\mathrm{SSE}}{\mathrm{SSY}}\right]^{-1}.$$

(Note that here, and elsewhere, SSE and SSY are determined from the N-site network.) Hence, provided n is not too small,

$$(\mathrm{SE}_n)^2 \approx \frac{\mathrm{SSE}}{n(N-1)(J-1)}. \qquad (40\mathrm{c})$$

Eq. (40c) is similar to the result given by Clarke and Edwards (1972).

A fourth alternative obtains if we use Eq. (39) with the $\bar{r}$ form for $(\bar{R}_n)^2$ [Eq. (25)] and Eq. (12) for S^2. This gives

$$(\mathrm{SE}_n)^2 = \frac{(N-1)\mathrm{SSY} - \mathrm{SSE}}{N(N-1)(J-1)}\,\frac{1-\bar{r}}{1+(n-1)\bar{r}}. \qquad (40\mathrm{d})$$

The differences between these expressions for SE_n are small and, in most cases, within the other uncertainties inherent in this type of analysis of precipitation data. Eq. (40d) is preferred since the $\bar{r}$ forms of our results have been shown, empirically, to be superior to the $\hat{a}$ forms.

The $\bar{r}$ form for $(\mathrm{SE}_n)^2$ is one which can be exploited in order to apply the result to arbitrary catchment areas. In general, the value of $\bar{r}$ will depend on catchment size, topography and location (i.e., climatic zone), and also on the precipitation time unit being considered ($\bar{r}$ is smaller for hourly than for monthly data, for example). For any given catchment, $\bar{r}$ can be expressed in terms of a spatial-correlation decay length and Eq. (40d) can usefully be rewritten in terms of this decay length. This approach will be developed further elsewhere.

For the River Wye catchment, using the unweighted precipitation data (number 7 in Table 1), Eqs. (40c) and (40d) become

$$\mathrm{SE}_n = \frac{21.76}{n^{1/2}}, \qquad (40\mathrm{c}')$$

$$\mathrm{SE}_n = \frac{17.68}{(n+0.1732)^{1/2}}. \qquad (40\mathrm{d}')$$

These results are compared in Fig. 6. Eq. (40d′) implies a smaller error than (40c′) for all n. On the basis of (40d′) and assuming a uniform gage network, 4 gages would be required to achieve a standard error of less

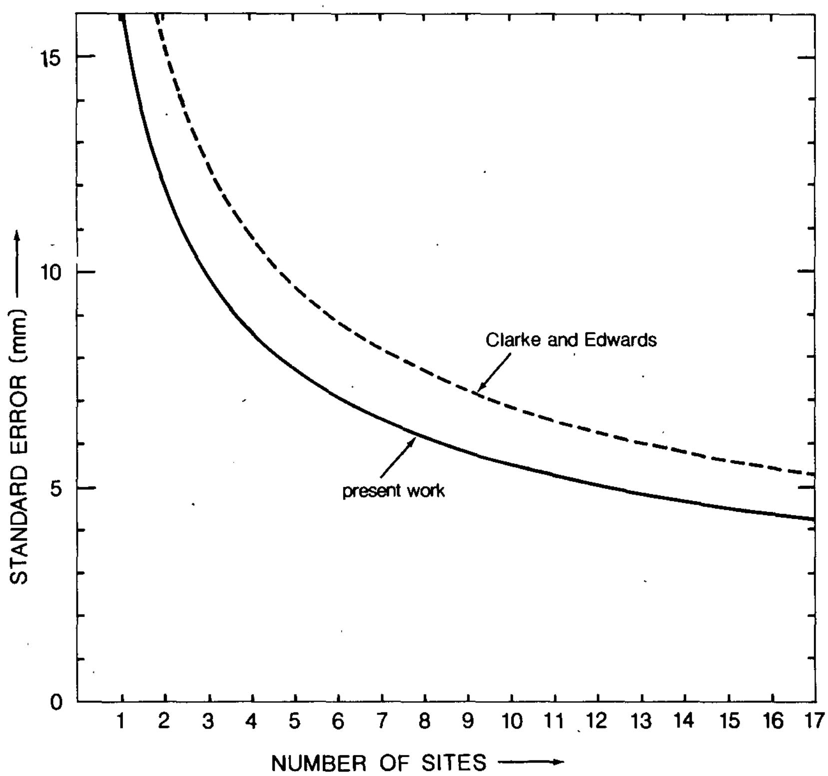

FIG. 6. Standard error of the estimate of area-average precipitation for the River Wye catchment as a function of the number of raingage sites based on the result given by Clarke and Edwards [Eqs. (40c) and (40c′), upper curve] and on the preferred result derived here [Eqs. (40d) and (40d′), lower curve].

than 10 mm [5 gages using (40c′)], while 13 gages would give a standard error below 5 mm [19 gages using (40c′)]. The minimum acceptable SE will depend on the use to be made of the area-average precipitation, but it is unlikely to be much less than 5 mm. For riverflow reconstruction, for example, an SE of $\sim$10 mm is usually adequate (see Jones, 1984). For minimum SE less than $\sim$5 mm any substantial gain in accuracy requires a large number of additional raingage sites.

Since $\bar{r}$ is the same regardless of whether the series are weighted or not, Eq. (40d) can be used independently of the procedure used to calculate the area-average. For any given network it is, of course, possible to optimize the combination of data to give a best estimate of area-average precipitation (e.g. by multiple regression analysis, or using trend surfaces, Thiessen polygons or isohyetal analysis). This may allow one to satisfactorily estimate the area-average precipitation in particular cases with less than the minimum number of gages determined from our uniform network theory if one assumes that the optimized result is stable in time. Alternatively, for a given number of sites their locations can be optimized to maximize the amount of information obtained. Since our method takes no account of the particular gage configuration, it provides a base-line result against which optimized networks can be judged.

4. Summary and conclusions

We have developed a theory for estimating uncertainties in the average of a set of correlated time series. The determining parameters are the mean inter-series correlation ($\bar{r}$) and/or the fractional common variance ($\hat{a}$). Approximate formulas have been derived for the mean correlation between the average of N time series and the average for a subset n of N ($\bar{R}_{n,N}$), and for the mean correlation between the N-series average and the population average ($\bar{R}_N$). Empirical verification has shown the formulas involving $\bar{r}$ to be more accurate than those involving $\hat{a}$. All formulas are extremely accurate when the between-series sum of squares (SSC) is small.

We have applied these results to tree-ring time series and to precipitation time series. In the former case we have shown that $\bar{r} \approx \hat{a}$, a result previously demonstrated empirically by Fritts (1976). We have also shown that $(\bar{R}_N)^2$ is closely related to the signal-to-noise ratio as used in dendroclimatology. Here $(\bar{R}_{n,N})^2$ is a measure of the loss of reconstruction accuracy which occurs when an n-core chronology is used to reconstruct past climate with a transfer function which is derived from an N-core chronology. The n-dependence of $(\bar{R}_{n,N})^2$ allows one to estimate the minimum number of cores required to reduce this loss of reconstruction accuracy to below any chosen threshold level and hence to estimate the maximum useful length of a tree-ring chronology. The minimum number of cores depends on N and $\hat{a}$ (or $\bar{r}$) and on the acceptable accuracy level. Chronology sections based on as few as 4 cores were shown to be potentially useful for climate reconstruction, but the number of cores required varies widely with N and $\hat{a}$. Our analysis only considers the case of one core per tree but it can be adapted to more general cases. For example, multiple cores may be averaged to produce a single time series for each tree, or $\bar{r}$ (or $\hat{a}$) can be determined using only a single core per tree if the early parts of a chronology suggest that such a strategy is more appropriate.

We have also applied our theoretical results to the problem of determining the accuracy of area-average precipitation estimates. While $\hat{a}$ is the more useful parameter in the dendroclimatological case (since it is already calculated as a routine procedure), the mean inter-series correlation is more useful in the precipitation case. Although the area-average may be based on a weighted average of precipitation series from different raingage sites, $\bar{r}$ can be calculated from the raw, unweighted data. The important accuracy-determining parameter is $(\bar{R}_n)^2$. This can be used to estimate the number of raingage sites required to achieve any desired level of accuracy. [This is done more conveniently when $(\bar{R}_n)^2$ is expressed in terms of the standard error of estimate for the area-average.] Although our results here apply strictly to uniformly distributed raingage networks, they provide a baseline for evaluating the results of network optimizations. We considered an example using monthly data from a river basin catchment area. In more general applications there are some practical considerations in dealing with zeros (in the case of short precipitation time units) and in applying the method to cases where the data show strong seasonal cycles. These problems are more appropriately discussed in the hydrological literature.

In both the dendroclimatological and hydrometeorological applications our analysis makes the tacit assumption that $\hat{a}$ and $\bar{r}$ are well-defined quantities which show no systematic variation with N. This assumption may not be correct for the hydrometeorological case where $\bar{r}$ could increase as N increases because this will increase the relative number of closer-spaced, more-highly-correlated sites. For $N \geqslant 10$, however, we have found $\bar{r}$ variations to be slight, but this is an aspect which requires further study.

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APPENDIX A

Terminology

a	Fractional common variance [Eq. (13)]
$\hat{a}$	Unbiased fractional common variance [Eq. (14)]
i	Subscript for i-th time series
j	Subscript for j-th observation
J	Total number of observations
n, N	Total number of series
r_{iI}	Correlation between series i and series I [Eq. (16)]: $r(W_{ij}, W_{Ij})$
$\bar{r}^*$	Average value of r_{iI} over all i, I
$\bar{r}$	Average value of r_{iI}, $i \neq I$
$r_{i,N}$	Correlation between series i and the series obtained by averaging N time series [Eq. (19)]: $r[W_{ij}, \bar{W}._j(N)]$
$\bar{r}_N$	Average value of $r_{i,N}$ over $i = 1, N$
$R_{n,N}$	Correlation between the n-series average and the N-series average where n is a subset of N [Eq. (2)]: $r[\bar{W}._j(n), \bar{W}._j(N)]$
$\bar{R}_{n,N}$	Average value of $R_{n,N}$ for all subsets of size n
$\bar{R}_n(\bar{R}_N)$	Correlation between the $n(N)$-series average and the population (limiting form of $\bar{R}_{n,N}$)
s_i^2	Variance of the i-th series [Eq. (9)]
s_j^2	Variance across series of observation j [Eq. (10)]
$s^2 \equiv s^2(N)$	Variance for the N-series average [Eqs. (3) and (8)]
$\hat{s}^2 \equiv \hat{s}^2(N)$	Unbiased estimate of the population common variance, σ_y^2 [Eq. (12)]
S^2	Variance of area-average precipitation
SE_n	Standard error of area-average estimate [Eq. (39)]
SNR	Signal-to-noise ratio as used in dendroclimatology [Eq. (30)]
SSC	Between-series sum of squares [Eq. (5)]
SSE	Error sum of squares [Eq. (7)]
SSY	Between-observations (i.e. within-series) sum of squares [Eq. (4)]
SST	Total sum of squares [Eq. (6)]
W_{ij}	j-th observation of parameter W_i
$\bar{W}_i. \equiv \bar{W}_i.(J)$	Average value of W_{ij} over $j = 1, J$
$\bar{W}._j \equiv \bar{W}._j(N)$	Average value of W_{ij} over $i = 1, N$ [Eq. (1)]
$\bar{W} \equiv \bar{W}..(N)$	Average value of W_{ij} over $i = 1, N$ and $j = 1, J$ [strictly, $\bar{W} = \bar{W}..(N, J)$]
%Y	Percent common variance as used in dendroclimatology [Eq. (29)]. %$Y \equiv 100\hat{a}$
μ_j	Population value of $\bar{W}._j$
σ_c^2	Population between-series variance
σ_e^2	Population error variance
σ_y^2	Population between-observation variance $\equiv$ common variance

APPENDIX B

Derivation of Expressions for $\bar{r}^*$ and $(\bar{R}_{n,N})^2$

The relationship $\bar{r}^* \approx a$ [Eq. (17)] comes directly from the definition of r_{iI} [Eq. (16)]. We use the fact that, if $A = B/(C^2D^2)^{1/2}$ then $\bar{A} \approx \bar{B}/(\overline{C^2}\,\overline{D^2})^{1/2}$ provided the constituent terms do not have large coefficients of variation. The numerator average for Eq. (16) is simply

$$\sum_{j=1}^{J} (\bar{W}._j - \bar{W})^2 = \mathrm{SSY}/N,$$

while the denominator, using $\overline{s_j^2} = (\mathrm{SST} - \mathrm{SSC})/N(J - 1)$, becomes $(\mathrm{SST} - \mathrm{SSC})/N$. Hence

$$\bar{r}^* \approx \frac{\mathrm{SSY}}{\mathrm{SST} - \mathrm{SSC}} \equiv a. \tag{B1}$$

To obtain the expressions for $(\bar{R}_{n,N})^2$ [viz. Eqs. (22–24)] we begin with the definition of $R_{n,N}$ [Eq. (2)]. This becomes

$$R_{n,N} = \frac{1}{(J-1)s(n)s(N)} \sum_{j=1}^{J} \left\{[\bar{W}._j(N) - \bar{W}(N)] \times \left(\frac{1}{n}\sum_{i=1}^{n} W_{ij} - \frac{1}{J}\sum_{j=1}^{J}\frac{1}{n}\sum_{i=1}^{n} W_{ij}\right)\right\}$$

$$= \frac{1}{n(J-1)s(n)s(N)} \sum_{j=1}^{J} \left\{[\bar{W}._j(N) - \bar{W}(N)] \times \left(\sum_{i=1}^{n}\left\{W_{ij} - \frac{1}{J}\sum_{j=1}^{J} W_{ij}\right\}\right)\right\}.$$

The first two summations can be commuted to give

$$R_{n,N} = \frac{1}{ns(n)} \sum_{i=1}^{n} \left\{\frac{1}{(J-1)s(N)} \times \sum_{j=1}^{J} [\bar{W}._j(N) - \bar{W}(N)](W_{ij} - \bar{W}_i.)\right\}.$$

Using Eq. (19) this may be written

$$R_{n,N} = \frac{1}{ns(n)} \sum_{i=1}^{n} s_i r_{i,N}. \tag{B2}$$

Averaging this expression over all subsets n of N gives

$$\bar{R}_{n,N} \approx \frac{(\overline{s_i^2})^{1/2}(\overline{\bar{r}_N^2})^{1/2}}{[\overline{s^2(n)}]^{1/2}}. \tag{B3}$$

To find $\overline{s^2(n)}$ recall that $s^2(n)$ is defined by

$$s^2(n) = \frac{1}{(J-1)} \sum_{j=1}^{J} [\bar{W}._j(n) - \bar{W}(n)]^2$$

which becomes

$$s^2(n) = \frac{1}{J-1}\sum_{j=1}^{J}\left[\frac{1}{n}\sum_{i=1}^{n}(W_{ij} - \bar{W}_{i}.)\right]^2$$

or

$$s^2(n) = \frac{1}{n^2(J-1)}\sum_{j=1}^{J}[\sum_{i=1}^{n}(W_{ij} - \bar{W}_{i}.)^2 + 2\sum_{i\neq I}(W_{ij} - \bar{W}_{i}.)(W_{Ij} - \bar{W}_{I}.)].$$

The summation signs can be interchanged and the result simplified to

$$s^2(n) = \frac{1}{n^2}[\sum_{i=1}^{n} s_i^2 + 2\sum_{i\neq I} s_i s_I r_{iI}]. \tag{B4}$$

Averaging over all subsets n of N gives

$$\overline{s^2(n)} \approx \frac{1}{n}\overline{s_i^2}(1 + (n-1)\bar{r}). \tag{B5}$$

As a check on this result we note that, in the case $n = 1$, $\overline{s^2(n)}$ and $\overline{s_i^2}$ are the same thing, and that Eq. (B5) reduces to this equality when $n = 1$. Substituting (B5) into (B3) and using the expression for $(\bar{r}_N)^2$ derived earlier [Eq. (21)] gives

$$(\bar{R}_{n,N})^2 \approx \frac{n\bar{r}^*}{1 + (n-1)\bar{r}} = \frac{n[1 + (N-1)\bar{r}]}{N[1 + (n-1)\bar{r}]}. \tag{B6}$$

Using Eq. (18) to express $\bar{r}$ in terms of the sums of squares gives

$$(\bar{R}_{n,N})^2 \approx \frac{n\mathrm{SSY}}{n\mathrm{SSY} + \left(\frac{N-n}{N-1}\right)\mathrm{SSE}} \tag{B7}$$

or, in terms of the fractional common variances

$$(\bar{R}_{n,N})^2 \approx \frac{na}{1 + (n-1)\hat{a}} = \frac{n[\hat{a} + (1-\hat{a})/N]}{1 + (n-1)\hat{a}}, \tag{B8}$$

where a, $\hat{a}$, $\bar{r}$, $\bar{r}^*$ and all sums of squares are determined using all N series. Note that, while (B7) and (B8) are formally equivalent, these results involve additional approximations when derived from (B6).

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