$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k \\ y(t) &= C(t)x(t) + D(t)u(t) \quad y \in \mathbb{R}^m \end{aligned} \tag{1}$$

Prove the following impulse response theorem for (1).

Theorem 1 (Impulse Response). *Consider a discrete-time lienar system with k inputs and m outputs. There exists a matrix-valued signal $G(t,\tau) \in \mathbb{R}^{m\times k}$ such that for every input u, a corresponding output is given by*

$$u \rightsquigarrow y(t) = \sum_{\tau=0}^{\infty} G(t,\tau)u(\tau) \qquad \forall t \geq 0. \tag{3}$$

The matrix-valued signal $G(t,\tau) \in \mathbb{R}^{m\times k}$ is called a discrete-time impulse response. Its entry $g_{ij}(t,\tau)$can be viewed as the ith entry of an output at time t, corresponding to a unit discrete-time pulse applied at the jth input at time τ.

Proof: As for discrete-time inputs, we can use $\delta(t,\tau) := \begin{cases} 1, & \text{if } t=\tau \\ 0, & \text{otherwise} \end{cases}$ to approximate input signal u. To simplify the analysis, just prove the theorem for SISO system, which is: $u \rightsquigarrow y(t) = \sum_{\tau=0}^{\infty} g(t,\tau)u(\tau)$ $(\forall t \geq 0)$, where $g(t,\tau)$, $t,\tau \geq 0$, is an output corresponding to the input $\delta(t,\tau)$. Since

$$u(t) = \sum_{\tau}^{\infty} u(\tau)\delta(t,\tau) \; (\forall t \geq 0) \xrightarrow{\text{linearity}} u \rightsquigarrow y(t) := \sum_{\tau=0}^{\infty} u(\tau) g(t,\tau), \; \forall t \geq 0.$$

1. For causal systems, one can choose the impulse response to satisfy $G(t,\tau) = 0, \forall \tau > t$.
2. For time-invariant systems, one can choose the impulse response to satisfy $G(t+T, \tau+T) = G(t,\tau)$, $\forall t,\tau,T \geq 0$. In particular for $\tau = 0, t_1 = T, t_2 = t+T$, $G(t_2,t_2) = G(t_2-t_1, 0)$, $\forall t_2 \geq t_1 \geq 0$ which shows that $G(t_2,t_1)$ is just a function of $t_2 - t_1$.
3. For causal, time-invariant systems, we can write (3) as

$$u \rightsquigarrow y(t) = \sum_{\tau=0}^{t} G(t-\tau)u(\tau) := (G \star u)(t) \tag{4}$$

where $\star$ denotes the convolution operator on discrete signals.

Proof: 1. Consider for two outputs of u: $\begin{cases} u \rightsquigarrow y_1 \\ u \rightsquigarrow y_2 \end{cases}$, let $u=0$, we have $u+u = 0+0 = 0$, because of the linearity, we have $y_1+y_2 = y_1 = y_2$, hence $y_1 = y_2 = 0$. Since the impulse $\delta(t,\tau)$ at time τ is equal to the zero input u on $[0,\tau-1]$, it must have an output that is zero on $[0,\tau-1]$; i.e., $\delta(t,\tau) = 0, \forall 0 \leq t \leq \tau-1 \Rightarrow \exists \bar{y}: \delta(t-\tau) \rightsquigarrow \bar{y}$ and $\bar{y}(t) = 0, \forall \leq t \leq \tau-1 < \tau$. choose this input to construct the impulse response, we can obtain $\exists G(t,\tau) = 0, \forall \tau > t$.

2. By the defination of impulse response: $\delta(t-\tau-T) \rightsquigarrow y(t) := G(t, \tau+T)$, where $\delta(t-\tau-T)$ is an impulse at time $\tau+T$. By time invariance, the impulse $\delta(t-\tau)$ at time τ must have an output $\bar{y}$ such that: $\delta(t-\tau) \rightsquigarrow \bar{y}(t) = y(t+T) = G(t+T, \tau+T)$. By using this output to construct the impulse response, we obtain $G(t+T, \tau+T) = G(t,\tau)$, $\forall t, T \geq 0$.

3. From "2." above we know that $G(t_2, t_1)$ is just a function of $t_2 - t_1$, hence we can replace $G(t,\tau)$ in (3) with $G(t-\tau)$: $u \rightsquigarrow y(t) = \sum_{\tau=0}^{\infty} G(t-\tau)u(\tau), \forall t \geq 0$. From "1" above we know $G(t,\tau) = 0$, $\forall \tau > t$, i.e., $\forall t < \tau$, hence we can further replace the ∞ in the upper summation limit by t since $G(t,\tau) = G(t-\tau) = 0, \forall \tau > t$. Hence we prove that $u \rightsquigarrow y(t) = \sum_{\tau=0}^{t} G(t-\tau)u(\tau) := (G * u)(t)$

For discrete-time systems the $\mathcal{Z}$-transform plays a role analogous to the Laplace transform in defining a transfer function, essentially by replacing all the integrals by summations. Given a discrete-time signal $y(\cdot)$, its $\mathcal{Z}$-transform is given by

$$\hat{y}(z) = \mathcal{Z}[y(t)] := \sum_{t=0}^{\infty} z^{-t} y(t), \; z \in \mathbb{C}.$$

The transfer function for a discrete-time LTI system is defined as:

Definition 1 (Transfer function)**.** The transfer function of a discrete-time LTI system is the $\mathcal{Z}$-transform

$$\hat{G}(z) = \mathcal{Z}[G(t)] := \sum_{t=0}^{\infty} z^{-t} G(t), \; z \in \mathbb{C}$$

of an impulse response given in (4).

1. Prove the following theorem:

   Theorem 2. *For every input u, the $\mathcal{Z}$-transform of a corresponding output y is given by $\hat{y}(z) = \hat{G}(z)\hat{u}(z)$.*

2. Prove the follwoing theorem:

   Theorem 3. *The impulse response and transfer function of the DLTI system (2) are given by*

   $$G(t) = \mathcal{Z}^{-1}\left[C(zI - A)^{-1}B + D\right] \quad \textit{and} \quad \hat{G}(z) = C(zI - A)^{-1}B + D$$

   respectively. Moreover, the output given by (3) corresponds to the zero initial condition $x(0) = 0$.

Proof: 1. Because of Theorem 1, the discrete-time linear system has an output: $y(t) = \sum_{\tau=0}^{\infty} G(t-\tau)u(\tau), \forall t \geq 0$. Taking its

Z-transform, one obtains: $\hat{y}(z) = \sum_{t=0}^{\infty} z^{-t} \sum_{\tau=0}^{\infty} G(t-\tau)u(\tau), \forall t \geq 0.$

Changing the other of summation:

$\hat{y}(z) = \sum_{t=0}^{\infty} \left(\sum_{\tau=0}^{\infty} z^{-(t-\tau)} G(t-\tau) dt \right) z^{-\tau} u(\tau) d\tau$ ①. But because of causality, $\sum_{t=0}^{\infty} z^{-(t-\tau)} G(t-\tau) dt = \sum_{\bar{t}=-\tau}^{\infty} z^{-\bar{t}} G(\bar{t}) d\bar{t}$

$$= \sum_{\bar{t}=0}^{\infty} z^{-\bar{t}} G(\bar{t}) d\bar{t} =: \hat{G}(z)$$ ②.

Substituting ② into ① and remove $\hat{G}(z)$ from the summation, we conclude that: $\hat{y}(z) = \sum_{\tau=0}^{\infty} \hat{G}(z) e^{-\tau} u(\tau) d\tau$

$$= \hat{G}(z) \sum_{\tau=0}^{\infty} e^{-\tau} u(\tau) d\tau = \hat{G}(z) \hat{u}(z)$$

2, consider the discrete-time LTI system:

$$x^{+} = Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^{n}, u \in \mathbb{R}^{k}, y \in \mathbb{R}^{m}.$$

Since $Z(x^{+}) = Z(x(t+1)) = \sum_{t=0}^{\infty} x(t+1) z^{-t} = z \sum_{t=1}^{\infty} x(t) z^{-t} = z\left(\sum_{t=0}^{\infty} x(t) z^{-t} - x(0) z^{0}\right) = z\hat{G}(z) - x(0) z$, we can take the z-transform of both sides of the two equations:

$$z\hat{x}(z) - zx(0) = A\hat{x}(z) + B\hat{u}(z), \quad \hat{y}(z) = C\hat{x}(z) + D\hat{u}(z).$$

Solving for $\hat{x}(z)$, we obtain: $(zI - A)\hat{x}(z) = zx(0) + B\hat{u}(z)$

$\Rightarrow \hat{x}(z) = (zI-A)^{-1} B \hat{u}(z) + (zI-A)^{-1} z x(0)$, from which we conclude that $\hat{y}(z) = \hat{\Psi}(z) x(0) + \hat{G}(z) \hat{u}(s)$, where:

$\hat{\Psi}(z) := C(zI-A)^{-1}$, $\hat{G}(z) := C(SI-A)^{-1}B+D$. Comming back to the time domain by applying invers z-transform, we obtain $y(t) = \Psi(t)x(0) + (G*u)(t) = \Psi(t)x(0) + \sum_{\tau=0}^{t} G(t-\tau)u(\tau)$. ① Comparing ① with equation (4), we know the output given by (3) corresponds to the zero initial condition $x(0)=0$.

1. Find a realization for the proper rational matrix (i.e., find the system matrices (A, B, C, D) of a continuous linear time invariant (CLTI) system whose transfer function is given by the following proper rational matrix,)

$$\hat{G}(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{s}{s+2} \end{bmatrix}. \tag{5}$$

2. Find a realization for each column of $\hat{G}(s)$ above, and then connect them as shown in Figure 1(a) to obtain a realization of $\hat{G}(s)$. What is the dimension of this realization (i.e. dimension of matrix A)? Compare this dimension with the one in 1.

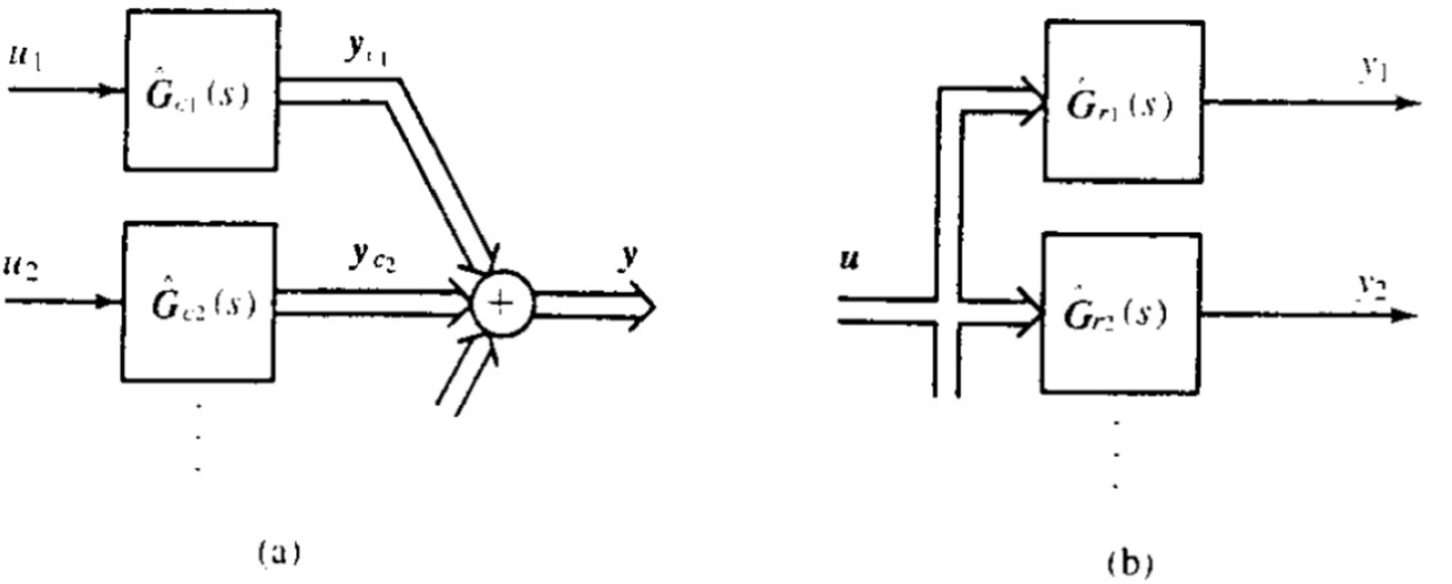

Figure 1: Connection

3. Find a realization for each row of $\hat{G}(s)$ above, and then connect them as shown in Figure 1(b) to obtain a realization of $\hat{G}(s)$. What is the dimension of this realization (i.e. dimension of matrix A)? Compare this dimension with the one in part 1. and the one in part 2.

Solution: 1. $\lim_{s\to\infty}\hat{G}(s) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = D$.

$$\hat{G}_{sp}(s) = \hat{G}(s) - D = \begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{-3}{s+1} & \frac{-2}{s+2} \end{bmatrix}.$$

$$d(s) = (s+1)(s+2) = s^2+3s+2 \Rightarrow \alpha_1 = 3, \alpha_2 = 2$$

$$\hat{G}_{sp}(s) = \begin{bmatrix} 2(s+2) & 2s-3 \\ -3(s+2) & -2(s+1) \end{bmatrix} / d(s) = \frac{1}{d(s)}\left(\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix} s + \begin{bmatrix} 4 & -3 \\ -6 & -2 \end{bmatrix}\right)$$

$\Rightarrow N_1 = \begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$, $N_2 = \begin{bmatrix} 4 & -3 \\ -6 & -2 \end{bmatrix}$. Hence we have:

$$A = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -3 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & 2 & 4 & -3 \\ -3 & -2 & -6 & -2 \end{bmatrix}$$

2. $$\begin{cases} \hat{G}_{c1}(s) = \begin{bmatrix} \frac{2}{s+1} \\ \frac{s-2}{s+1} \end{bmatrix} \\ \hat{G}_{c2}(s) = \begin{bmatrix} \frac{2s-3}{(s+1)(s+2)} \\ \frac{s}{s+2} \end{bmatrix} \end{cases} \Rightarrow \begin{cases} D_{c1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \hat{G}_{spc_1} = \frac{1}{s+1}\begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ D_{c2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \hat{G}_{spc_2} = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 2s-3 \\ -2(s+1) \end{bmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} d_{c1}(s) = s+1 \ (\alpha_1 = 1), \ N_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \ (n=1) \\ d_{c2}(s) = s^2+3s+2 \ (\alpha_1 = 3, \alpha_2 = 2), \ N_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, N_2 = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \ (n=2) \end{cases}$$

Hence we have: $(k=1)$

$A_{c1} = -1$, $B_{c1} = 1$, $C_{c1} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $D_{c1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A_{c2} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$, $B_{c2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C_{c2} = \begin{bmatrix} 2 & -3 \\ -2 & -2 \end{bmatrix}$, $D_{c2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

As for connection (a), $u' = \begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix}$, $y' = y_{c1} + y_{c2}$, we have:

$$\begin{bmatrix} \dot{x}_{c1} \\ \dot{x}_{c2} \end{bmatrix} = \begin{bmatrix} A_{c1} & 0 \\ 0 & A_{c2} \end{bmatrix} \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix} + \begin{bmatrix} B_{c1} & 0 \\ 0 & B_{c2} \end{bmatrix} \begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix} \Rightarrow \dot{x}' = A'x' + B'u'$$

$$y' = C_{c1}x_{c1} + D_{c1}u_{c1} + C_{c2}x_{c2} + D_{c2}u_{c2}$$

$$= [C_{c1}, C_{c2}] \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix} + [D_{c1}, D_{c2}] \begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix} = C'x' + D'u'.$$

Hence the dimension of this realization is $1+2=3$, which is larger than that of 1.

3、$\begin{cases} \hat{G}_{r1}(s) = \left[\frac{2}{s+1}, \frac{2s-3}{(s+1)(s+2)} \right] \\ \hat{G}_{r2}(s) = \left[\frac{s-2}{s+1}, \frac{s}{s+2} \right] \end{cases} \Longrightarrow \begin{cases} D_{r1} = [0, 0] \\ D_{r2} = [1, 1] \end{cases}$. Let $d(s) = (s+1)(s+2) = s^2 + 3s + 2$,

$$\begin{cases} \hat{G}_{spr1} = \frac{1}{d(s)} [2(s+2) \quad 2s-3] = \frac{1}{d(s)} ([2 \ 2]s + [4 \ -3]) \\ \hat{G}_{spr2} = \frac{1}{d(s)} [-3(s+2) \quad -2(s+1)] = \frac{1}{d(s)} ([-3 \ -2]s + [-6 \ -2]) \end{cases}$$

Hence we have: $A_{r1} = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -3 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = A_{r2} = A$

$$B_{r1} = B_{r2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{r1} = [2\ \ 2\ \ 4\ \ -3], \quad D_{r1} = [0, 0]$$

$$C_{r2} = [-3\ \ -2\ \ -6\ \ -2], \quad D_{r2} = [1, 1]$$

As for connection (b), $y' = \begin{bmatrix} y_{r1} \\ y_{r2} \end{bmatrix}$, we have:

$$\begin{bmatrix} \dot{x}_{r1} \\ \dot{x}_{r2} \end{bmatrix} = \begin{bmatrix} A_{r1} & 0 \\ 0 & A_{r2} \end{bmatrix} \begin{bmatrix} x_{r2} \\ x_{r2} \end{bmatrix} + \begin{bmatrix} B_{r1} \\ B_{r2} \end{bmatrix} u \Rightarrow \dot{x}' = A'x' + B'u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_{r1} & 0 \\ 0 & C_{r2} \end{bmatrix} \begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix} + \begin{bmatrix} D_{r1} \\ D_{r2} \end{bmatrix} u \Rightarrow y' = C'x' + D'u$$

Hence the dimension of this realization is $4+4=8$, which is much larger than both 1 and 2.