1. The solution to HDTLV system is given by $x(t)=\phi(t,t_0)x_0$, $x_0\in R^n$, $t\geq t_0$

where $\phi(t,t_0):=\begin{cases} I, & t=t_0 \\ A(t-1)A(t-2)\cdots A(t_0+1)A(t_0), & t>t_0 \end{cases}$, because $\phi(t+1,t_0)=A(t)\phi(t,t_0)$.

And for DLTI system the state transition matrix is simply given by:

$\phi(t,t_0)=A^{t-t_0}$ since $A(t):=A$.

Going back to the nonhomogeneous case, to verify this conclusion, note that at $t=t_0$ we get $x(t_0)=x_0$. Taking $t=t+1$ of each side with respect to time, we obtain: $x(t+1)=\phi(t+1,t_0)x_0+\sum_{\tau=t_0}^{t+1-1}\phi(t+1,\tau+1)B(\tau)u(\tau)$

$$=A(t)\phi(t,t_0)x_0+\phi(t+1,t+1)B(t)u(t)+A(t)\sum_{\tau=t_0}^{t-1}\phi(t,\tau+1)B(\tau)u(\tau)$$

$=A(t)x(t)+B(t)u(t)$, which verifies the conclusion in the question. And the expression for $y(t)$ is obtained by direct substitution of $x(t)$ in $y(t)=Cx(t)+Du(t)$.

2. $J=PAP^{-1}$ be the jordan normal form of $A \Longrightarrow A=P^{-1}JP$,

$A^t=P^{-1}\begin{bmatrix} J_1^t & & & \\ & J_2^t & & \\ & & \ddots & \\ & & & J_l^t \end{bmatrix}P$, where J_i are the Jordan blocks of A and

$$J_i=\begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix} \Longrightarrow J_i^t=\begin{bmatrix} \lambda_i^t & t\lambda_i^{t-1} & \frac{t!\lambda_i^{t-2}}{(t-2)!2!} & \frac{t!\lambda_i^{t-3}}{(t-3)!3!} & \cdots & \frac{t!\lambda_i^{t-n_i+1}}{(t-n_i+1)!(n_i-1)!} \\ 0 & \lambda_i^t & t\lambda_i^{t-1} & \frac{t!\lambda_i^{t-2}}{(t-2)!2!} & \cdots & \frac{t!\lambda_i^{t-n_i+2}}{(t-n_i+2)!(n_i-2)!} \\ 0 & 0 & \lambda_i^t & t\lambda_i^t & \cdots & \frac{t!\lambda_i}{(t-n_i+3)!(n_i-3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & & t\lambda_i^{t-1} \\ 0 & 0 & 0 & 0 & \cdots & \lambda_i^t \end{bmatrix}$$

Hence:

① When all the eigenvalues of A have magnitude strictly smaller than 1, then $J_i^t \to 0$ as $t \to \infty$, and therefore $A^t \to 0$ as $t \to \infty$, i.e. $x(t) = A^{t-t_0} x_0 \to 0$ as $t \to \infty$, hence the H-DLTI system is asymptotically stable.

② When all the eigenvalues of A have magnitude smaller or equal to 1 and all the Jordan blocks corresponding to eigenvalues with magnitude equal to 1 are 1×1, then all the J_i^t remain bounded as $t \to \infty$, and consequently, A^t remains bounded as $t \to \infty$, i.e. $x(t) = A^{t-t_0} x_0$ is uniformly bounded as $t \to \infty$, hence the H-DLTI system is marginally stable.

③ Since when "2" A^{t-t_0} converge to zero exponentially fast and therefore $\|A^t\|$ converges to zero exponentially fast (for every matrix norm); i.e. there exist constants $c, \lambda > 0$ such that $\|A^t\| \le c\lambda^t$, $\forall t \in \mathbb{R}$. In this case, for a submultiplicative norm, we have:

$\|x(t)\| = \|A^{t-t_0} x_0\| \le \|A^{t-t_0}\| \, \|x_0\| \le c\lambda^{t-t_0} \|x_0\|$, $\forall t \ge t_0$, this means that asymptotic stability and exponential stability are equivalent concepts for H-DLTI systems and "2" := "3".

④ When at least one eigenvalue of A has magnitude larger than 1 or magnitude equal to 1, but the corresponding Jordan block is larger than 1×1, then A^t is unbounded as $t \to \infty$, hence $x(t) = A^{t-t_0} x_0$ is unbounded and the system is unstable.

3. The equivalence between condition 1, 2, and 3 has already been proved in "2". We prove that condition 2 $\Rightarrow$ condition 4 by showing that the unique solution is given by $P := \sum_{t=0}^{\infty} A^{Tt} Q A^t$. To verify this is so, four steps are needed.

① The sum is well defined (i.e., it is finite). This is a consequence of the fact that the system is exponentially stable, and therefore $\|A^{Tt} Q A^t\|$ converges to zero exponentially fast as $t \to \infty$.

② The matrix P solves the equation. To verify this, we compute

$$A^T P A - P = \sum_{t=0}^{\infty} (A^T A^{Tt} Q A^t A - A^{Tt} Q A^t).$$

$$= \sum_{t=0}^{\infty} (A^{T(t+1)} Q A^{t+1} - A^{Tt} Q A^t)$$

$$= A^{Tt} Q A^t \Big|_0^{\infty} = \lim_{t\to\infty} A^{Tt} Q A^t - A^{T0} Q A^0.$$ Since $\lim_{t\to\infty} A^t = 0$ because of asymptotic stability and that $A^0 = I \Rightarrow A^T P A - P = -Q$.

③ The matrix P is symmetric and positive-define. Symmetry comes from the fact that $P^T = \sum_{t=0}^{\infty} (A^{Tt} Q A^t)^T = \sum_{t=0}^{\infty} A^{Tt} Q^T A^t = \sum_{t=0}^{\infty} A^{Tt} Q A^t = P$. To check that P is positive-define, we pick an arbitrary (constant) vector $z \in R^n$ and compute $z^T P z = \sum_{t=0}^{\infty} z^T A^{Tt} Q A^t z = \sum_{t=0}^{\infty} w(t)^T Q w(t)$, where $w(t) := A^t z$, $\forall t \geq 0$. Since Q is positive-define, we conclude that $z^T P z \geq 0$. Moreover, $z^T P z = 0 \Rightarrow \sum_{t=0}^{\infty} w(t)^T Q w(t) = 0$, which can only happen if $w(t) = A^t z = 0$, $\forall t \geq 0$, from which one concludes that $z = 0$, because A^t is nonsingular. Therefore P is positive-definite.

④ No other matrix solves this equation. To prove this by contradiction, assume that there exists another solution $\bar{P}$; i.e., $A^T \bar{P} A - \bar{P} = -Q$ and

$A^T \bar{P} A - \bar{P} = -Q$. Then $A^T(P-\bar{P})A - (P-\bar{P}) = 0$. Multiplying this equation on the left and right by A^{Tt} and A^t, respectively, we conclude that

$$A^{Tt}A^T(P-\bar{P})AA^t - A^{Tt}(P-\bar{P})A^t = 0, \forall t \geq 0.$$ On the other hand,

$$A^{T(t+1)}(P-\bar{P})A^{t+1} - A^{Tt}(P-\bar{P})A^t = \Delta(A^{Tt}(P-\bar{P})A^t) = 0,$$ and therefore

$A^{Tt}(P-\bar{P})A^t$ must remain constant for all times. But, because of stability this quantity must converge to zero as $t \to \infty$, so it must be always zero. Since A^t is nonsingular, this is possible only if $P = \bar{P}$.

The implication that condition $4 \Rightarrow$ condition 5 follows immediately, because if we select $Q = I$ in condition 4, then the matrix P that solves "(3)" also satisfies "(4)".

To prove that condition $5 \Rightarrow$ condition 2, let P be a symmetric positive-define matrix for which (4) holds and let $Q := -(A^T P A - P) > 0$. Consider the evolution of the signal $v(t) = x'(t) P x(t), \forall t \geq t_0$. In this case, along solutions to the system, we have $v(t+1) = x'(t+1) P x(t+1) = x'(t) A' P A x(t)$, and (3) guarantees that $v(t+1) = x'(t)(P-Q)x(t) = v(t) - x'(t) Q x(t), \forall t \geq 0$. From this we conclude that $v(t)$ is nonincreasing and, with a little more effort, that it actually decreases to zero exponentially fast.

4. Since $$\phi(0,t) = \phi^{-1}(t,0) = \frac{1}{e^{-t}\cos^2 2t + e^{-t}\sin^2 2t}\begin{bmatrix} e^{-2t}\cos 2t & -e^{-2t}\sin 2t \\ e^{t}\sin 2t & e^{t}\cos 2t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\cos 2t & -e^{-t}\sin 2t \\ e^{2t}\sin 2t & e^{2t}\cos 2t \end{bmatrix}$$

Hence $\phi(t,t_0)=\phi(t,0)\phi(0,t_0)=\begin{bmatrix} e^{t}\cos 2t & e^{-2t}\sin 2t \\ -e^{t}\sin 2t & e^{-2t}\cos 2t \end{bmatrix}\begin{bmatrix} e^{-t_0}\cos 2t_0 & -e^{-t_0}\sin 2t_0 \\ e^{2t_0}\sin 2t_0 & e^{2t_0}\cos 2t_0 \end{bmatrix}$

② $\frac{d}{dt}\phi(t,t_0)=A(t)\phi(t,t_0) \Rightarrow A(t)=\frac{d}{dt}\phi(t,t_0)\,\phi(t_0,t)$

$$=\frac{d}{dt}\phi(t,t_0)\,\phi(t_0,0)\,\phi(0,t)$$

$$=\begin{bmatrix} e^{t}\cos 2t-2e^{t}\sin 2t & 2e^{-2t}\sin 2t+2e^{-2t}\cos 2t \\ -e^{t}\sin 2t-2e^{t}\cos 2t & -2e^{-2t}\cos 2t-2e^{-2t}\sin 2t \end{bmatrix}\begin{bmatrix} e^{-t}\cos 2t & -e^{-t}\sin 2t \\ e^{2t}\sin 2t & e^{2t}\cos 2t \end{bmatrix}$$

$$=\begin{bmatrix} 1-3\sin^2 2t & 2-3\sin 2t\cos 2t \\ -(2+3\sin 2t\cos 2t) & 1-3\cos^2 2t \end{bmatrix}$$

③ $|\lambda I-A|=0 \Rightarrow (\lambda-1+3\sin^2 2t)(\lambda-1+3\cos^2 2t)+(2-3\sin 2t\cos 2t)(2+3\sin 2t\cos 2t)=0$

$$\Rightarrow (\lambda-1)^2+3(\lambda-1)+(3\sin 2t\cos 2t)^2+4-(3\sin 2t\cos 2t)^2=0$$

$$\Rightarrow (\lambda-1)^2+3(\lambda-1)+4=0 \Rightarrow \lambda-1=\frac{-3\pm i\sqrt{7}}{2}$$

$$\Rightarrow \lambda=\frac{-1\pm i\sqrt{7}}{2}$$

④ Since all the eigenvalues of $A(t)$ have strictly negative real parts, the system is exponentially (asymptotically) stable.

5. (a) $e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A)^k$. Since $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \Rightarrow e^{At} b = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

(b) $W_R[0,1] = \int_0^1 e^{At} b \, (e^{At} b)^T dt = \int_0^1 \begin{bmatrix} t \\ 1 \end{bmatrix} [t \;\, 1] dt = \int_0^1 \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} dt$

$$= \begin{bmatrix} \frac{1}{3} t^3 \big|_0^1 & \frac{1}{2} t^2 \big|_0^1 \\ \frac{1}{2} t^2 \big|_0^1 & t \big|_0^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}, \quad x_f = W_R[0,1] \alpha \Rightarrow \alpha = W_R[0,1]^{-1} x_f$$

$$\Rightarrow \alpha = 12 \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} = \begin{bmatrix} 12 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} 6\sqrt{2} = 12\sqrt{2} \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

(c) $u(t) = b^T e^{A(1-t)^T} \alpha = [0, 1] \begin{bmatrix} 1 & 0 \\ 1-t & 1 \end{bmatrix} \begin{bmatrix} 9 \\ -5 \end{bmatrix} 12\sqrt{2}$

$$= [1-t, 1] \begin{bmatrix} 9 \\ -5 \end{bmatrix} 12\sqrt{2} = 12\sqrt{2}(9 - 9t - 5)$$

$$= 12\sqrt{2}(4 - 9t)$$

$u(t) = b^T e^{A(1-t)^T} \alpha \Rightarrow \int_0^1 u^2(t) dt = \int_0^1 \| b^T e^{A(1-t)^T} \alpha \|^2 dt = \alpha^T W_R[0,1] \alpha$, which is the energy required for the minimum-energy control. Hence $u(t)$ is the minimum-energy control.

6. $C = [B \;\, AB \;\, A^2B \;\, A^3B]$

①

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2w \\ 0 & 1 \\ -2w & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 3w^2 & 0 & 0 & 2w \\ 0 & -w^2 & 0 & 2w \\ 0 & -2w & 0 & 1 \\ -6w^3 & -2w & 0 & -4w^2+1 \end{bmatrix}$, $A^2B = \begin{bmatrix} 0 & 2w \\ -w^2 & 2w \\ -2w & 1 \\ -2w & -4w^2+1 \end{bmatrix}$,

$$A^3=\begin{bmatrix} 0 & -w^2 & 0 & 2w \\ -3w^4 & -4w^2 & 0 & -2w^3+2w \\ -6w^3 & -2w & 0 & 1-4w^2 \\ 6w^3 & 2w^3-2w & 0 & 1-3w^2 \end{bmatrix},\quad A^3B=\Big[$$

Hence $C=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2w & -w^2 & 2w \\ 1 & 0 & 0 & 2w & -w^2 & 2w & -4w^2 & -2w^3+2w \\ 0 & 0 & 0 & 1 & -2w & 1 & -2w & 1-4w^2 \\ -2w & 1 & 0 & 1 & -2w & -4w^3+1 & 2w^3-2w & 1-8w^2 \end{bmatrix}$, $\text{rank}(C)=4=n$, hence the system is controllable.

② when the radial thruster fails, then $B=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$,

$C=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2w & 0 & 2w \\ 0 & 0 & 0 & 2w & 0 & 2w & 0 & -2w^3+2w \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1-4w^2 \\ 0 & 1 & 0 & 1 & 0 & 1-4w^2 & 0 & 1-8w^2 \end{bmatrix}$, $\text{rank}(C)=4=n$, hence the system is still controllable.

when the tangential thruster fails, then $B=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$C=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & -w^2 & 0 \\ 1 & 0 & 0 & 0 & -w^2 & 0 & -4w^2 & 0 \\ 0 & 0 & 0 & 0 & -2w & 0 & -2w & 0 \\ 0 & 0 & -2w & 0 & -2w & 0 & 2w^3-2w & 0 \end{bmatrix}$, $\text{rank}(C)=3<4=n$, hence the system is not controllable.

3. Since $\text{rank}(A=3)$, hence $\lambda=0$ is one eigenvalue of A, which means the system can't be asymptotically stable.

7. We should prove that $\text{rank}[A-\lambda I \;\; B]=nk, \forall\lambda\in\mathbb{C}$.

$$[A-\lambda I \;\; B] = \begin{bmatrix} -(\lambda+\alpha_1)I_{k\times k} & -\alpha_2 I_{k\times k} & \cdots & -\alpha_{n-1}I_{k\times k} & -\alpha_n I_{k\times k} & I_{k\times k} \\ I_{k\times k} & -\lambda I_{k\times k} & \cdots & 0_{k\times k} & 0_{k\times k} & 0_{k\times k} \\ 0_{k\times k} & I_{k\times k} & \cdots & 0_{k\times k} & 0_{k\times k} & 0_{k\times k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{k\times k} & 0_{k\times k} & \cdots & I_{k\times k} & -\lambda I_{k\times k} & 0_{k\times k} \end{bmatrix}_{nk\times(n+1)k}$$

By using Gaussion Eimination, we have:

$$\begin{bmatrix} 0_{k\times k} & 0_{k\times k} & \cdots & 0_{k\times k} & 0_{k\times k} & I_{k\times k} \\ I_{k\times k} & 0_{k\times k} & \cdots & 0_{k\times k} & 0_{k\times k} & 0_{k\times k} \\ 0_{k\times k} & I_{k\times k} & \cdots & 0_{k\times k} & 0_{k\times k} & 0_{k\times k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0_{k\times k} & 0_{k\times k} & \cdots & I_{k\times k} & 0_{k\times k} & 0_{k\times k} \end{bmatrix}_{nk\times(n+1)k}$$

, it's obvious that the matrix haw k zero columns and nk linear independent columns, which means the rank is nk.

Hence $\operatorname{rank}[A-\lambda I \;\; B] = nk$, $\forall \lambda \in \mathbb{C}$, such a system is always controllable.