Estimation and Control of Dynamical Systems
Fall 2023
Professor: Xuan Zhang

Homework 4

Due: Jan. 2nd, 2024, Submit to TA

Exercise 1. Observability

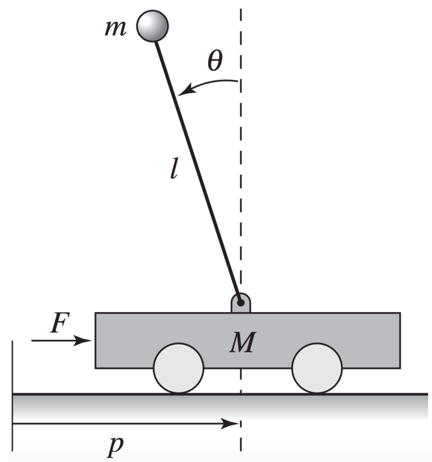

Figure 1: Cart-pendulum system

The dynamics of the cart-pendulum system shown in Figure 1 can be computed using Newtonian mechanics and have the form

$$\begin{bmatrix} (M+m) & -ml\cos\theta \\ -ml\cos\theta & (J+ml^2) \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{p} + ml\sin\theta\dot{\theta}^2 \\ \gamma\dot{\theta} - mgl\sin\theta \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{1}$$

where $p, \dot{p}$ represent the position and velocity of the base of the system, $\theta, \dot{\theta}$ are the angle and angular rate of the structure above the base, F represents the force applied at the base of the system, assumed to be in the horizontal direction (aligned with p), M is the mass of the base, m and J are the mass and moment of inertia of the system to be balanced, l is the distance from the base to the center of mass of the balanced body, c and γ are coefficients of viscous friction, and g is the acceleration due to gravity.

1. Define the state vector as $x = (p, \theta, \dot{p}, \dot{\theta})^T$, the input as $u = F$, and the output vector as $y = (p, \theta)^T$. Also, let $M_t = M + m$ and $J_t = J + ml^2$. Write down the system model in the form $\dot{x} = f(x, u), y = g(x, u)$.

2. In many cases, the angle θ will be very close to 0, and hence we can use the approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Furthermore, if $\dot{\theta}$ is small, we can ignore quadratic and higher terms in $\dot{\theta}$. Under these approximations, write down a linear state space equation of the model obtained in 1.

3. Let us further simplify the linear state space model obtained in 2., by setting $c = \gamma = 0$. For this simplified system, can we determine the state x from measuring y? How about the cases when we can only measure the position p, or the angle θ?

solusion:

1. $$\begin{cases} M_t\ddot{p}-(ml\cos\theta)\ddot{\theta}+c\dot{p}+(ml\sin\theta)\dot{\theta}^2=F=u \\ (-ml\cos\theta)\ddot{p}+J_t\ddot{\theta}+\gamma\dot{\theta}-mgl\sin\theta=0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{p}=\frac{1}{M_t}\left[(ml\cos\theta)\ddot{\theta}-c\dot{p}-(ml\sin\theta)\dot{\theta}^2+u\right] \\ \ddot{p}=\frac{1}{ml\cos\theta}\left[J_t\ddot{\theta}+\gamma\dot{\theta}-mgl\sin\theta\right] \end{cases}$$

$$\Downarrow$$

$$M_t\left[J_t\ddot{\theta}+\gamma\dot{\theta}-mgl\sin\theta\right]=(ml\cos\theta)^2\ddot{\theta}-ml\cos\theta c\dot{p}-(ml)^2\sin\theta\cos\theta\,\dot{\theta}^2+u(ml\cos\theta)$$

$$\Downarrow$$

$$\ddot{\theta}=\left[-ml\cos\theta c\dot{p}-(ml)^2\sin\theta\cos\theta\,\dot{\theta}^2+u(ml\cos\theta)-M_t\gamma\dot{\theta}+M_t(mgl\sin\theta)\right]\cdot\left(M_tJ_t-(ml\cos\theta)^2\right)^{-1}$$

$$=g(x,u)$$

$$\begin{cases} \ddot{\theta}=\frac{1}{ml\cos\theta}\left[M_t\ddot{p}+c\dot{p}+(ml\sin\theta)\dot{\theta}^2-u\right] \\ \ddot{\theta}=\frac{1}{J_t}\left[ml\cos\theta\ddot{p}-\gamma\dot{\theta}+mgl\sin\theta\right] \end{cases}$$

$$\Downarrow$$

$$J_t\left[M_t\ddot{p}+c\dot{p}+(ml\sin\theta)\dot{\theta}^2-u\right]=ml\cos\theta\left[ml\cos\theta\ddot{p}-\gamma\dot{\theta}+mgl\sin\theta\right]$$

$$\Downarrow$$

$$\ddot{p}=\left[-J_tc\dot{p}-J_t(ml\sin\theta)\dot{\theta}^2+J_tu-(ml\cos\theta)\gamma\dot{\theta}+g(ml)^2\cos\theta\sin\theta\right]\cdot\left(J_tM_t-(ml\cos\theta)^2\right)^{-1}$$

$$=h(x,u)$$

Hence $\dot{x}=\begin{pmatrix}\dot{p}\\ \dot{\theta}\\ \ddot{p}\\ \ddot{\theta}\end{pmatrix}=f(x,u)=\begin{pmatrix}\dot{p}\\ \dot{\theta}\\ g(x,u)\\ h(x,u)\end{pmatrix}$, $y=\begin{pmatrix}p\\ \theta\end{pmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}p\\ \theta\\ \dot{p}\\ \dot{\theta}\end{bmatrix}$

2.

$$\begin{bmatrix} M_t & -ml \\ -ml & J_t \end{bmatrix}\begin{bmatrix} \ddot{p} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{p} \\ \gamma\dot{\theta} - mgl\theta \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} M_t\ddot{p} - ml\ddot{\theta} + c\dot{p} = u \\ -ml\ddot{p} + J_t\ddot{\theta} + \gamma\dot{\theta} - mgl\theta = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \frac{M_t}{ml}(J_t\ddot{\theta} + \gamma\dot{\theta} - mgl\theta) - ml\ddot{\theta} + c\dot{p} = u \\ M_tJ_t\ddot{\theta} + M_t\gamma\dot{\theta} - M_tmgl\theta - (ml)^2\ddot{\theta} + mlc\dot{p} = mlu \\ \ddot{\theta}(M_tJ_t - (ml)^2) = M_tmgl\theta - M_t\gamma\dot{\theta} - mlc\dot{p} + mlu \\ \ddot{\theta} = (M_tJ_t - (ml)^2)^{-1}\left(\begin{bmatrix} 0 & M_tmgl & -mlc & -M_t\gamma \end{bmatrix}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + mlu\right) \end{cases}$$

$$\begin{cases} M_t\ddot{p} - \frac{ml}{J_t}(ml\ddot{p} - \gamma\dot{\theta} + mgl\theta) + c\dot{p} = u \\ M_tJ_t\ddot{p} - (ml)^2\ddot{p} + ml\gamma\dot{\theta} - g(ml)^2\theta + J_tc\dot{p} = J_tu \\ \ddot{p}(M_tJ_t - (ml)^2) = g(ml)^2\theta - ml\gamma\dot{\theta} - J_tc\dot{p} + J_tu \\ \ddot{p} = (M_tJ_t - (ml)^2)^{-1}\left(\begin{bmatrix} 0 & g(ml)^2 & -J_tc & -ml\gamma \end{bmatrix}\cdot\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + J_tu\right) \end{cases}$$

Hence $\dot{x} = \begin{pmatrix} \dot{p} \\ \dot{\theta} \\ \ddot{p} \\ \ddot{\theta} \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g(ml)^2 & -J_tc & -ml\gamma \\ 0 & M_tmgl & -mlc & -M_t\gamma \end{pmatrix}\begin{pmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ J_t \\ ml \end{pmatrix}u\right](M_tJ_t - (ml)^2)^{-1}$

3. $$\dot{X} = \begin{pmatrix}\dot{p}\\ \dot{\theta}\\ \ddot{p}\\ \ddot{\theta}\end{pmatrix} = \left[\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&g(ml)^2&0&0\\0&M_t mgl&0&0\end{pmatrix}\begin{pmatrix}p\\ \theta\\ \dot{p}\\ \dot{\theta}\end{pmatrix} + \begin{pmatrix}0\\0\\J_t\\ml\end{pmatrix}u\right](M_tJ_t-(ml)^2)^{-1}$$

$$C=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix},\quad A=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&\frac{g(ml)^2}{M_tJ_t-(ml)^2}&0&0\\0&\frac{M_tmgl}{M_tJ_t-(ml)^2}&0&0\end{bmatrix}.$$

Let $\begin{cases}\frac{g(ml)^2}{M_tJ_t-(ml^2)}=a\\ \frac{M_tmgl}{M_tJ_t-(ml)^2}=b\end{cases}$

$$O=\begin{bmatrix}C\\CA\\CA^2\\CA^3\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&a&0&0\\0&b&0&0\\0&0&0&a\\0&0&0&b\end{bmatrix}$$

$\operatorname{rank} O=4 \Rightarrow$ observable, i.e. we can determine the state x from measuring y.

When only measure P, then $C=[1\,0\,0\,0]$, we have:

$$O=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&a&0&0\\0&0&0&a\end{bmatrix},$$ $\operatorname{rank} O=4 \Rightarrow$ observable, i.e. we can determine the state x from just measuring P.

When only measure θ, then $C=[0\,1\,0\,0]$, we have:

$$O=\begin{bmatrix}0&1&0&0\\0&0&0&1\\0&b&0&0\\0&0&0&b\end{bmatrix},$$ $\operatorname{rank} O=2 \Rightarrow$ not observable, i.e. we can't determine the state x from only measuring θ.