Tsinghua-Berkeley Shenzhen Institute
OPERATIONS RESEARCH
Fall 2023

Homework 7

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- **Acknowledgments:** This assignment refers to the textbook: Introduction to Operations Research(10th).
- **Collaborators:** I finish this assignment by myself.

The answers below are arranged according to the order of the following questions in the textbook: 13.2-6(a)(b)(c)(d)(e), 13.2-9, 13.3-4, 13.4-1, 13.6-4.

7.1. 13.2-6 SOLUTION

(a) $f^{''}(x) = -2 < 0$, so f(x) is convex.
(b) $f^{''}(x) = 12x^2 + 12 > 0$, so f(x) is concave.
(c) $f^{''}(x) = 12x - 6$, so f(x) is neither convex or concave.
(d) $f^{''}(x) = 12x^2 + 2 > 0$, so f(x) is concave.
(e) $f^{''}(x) = 6x + 12x^2 = 6x(1 + 2x)$, so f(x) is neither convex or concave.

7.2. 13.2-9 SOLUTION

(a) Since $\frac{\partial f(x)}{\partial x_1^2} = \frac{\partial f(x)}{\partial x_2^2} = \frac{\partial f(x)}{\partial x_1 \partial x_2} = 0$, and $\frac{\partial f(x)}{\partial x_1^2}\frac{\partial f(x)}{\partial x_2^2} - \left[\frac{\partial f(x)}{\partial x_1 \partial x_2}\right]^2 = 0$, f(x) is convex. And $\frac{\partial g(x)}{\partial x_1^2} = \frac{\partial g(x)}{\partial x_2^2} = \frac{\partial g(x)}{\partial x_1 \partial x_2} = 2 > 0$, and $\frac{\partial g(x)}{\partial x_1^2}\frac{\partial g(x)}{\partial x_2^2} - \left[\frac{\partial g(x)}{\partial x_1 \partial x_2}\right]^2 = 4 > 0$, g(x) is convex, too. Hence the problem is a convex programming problem.
(b) The graph is shown as Figure 1.

7.3. 13.3-4 SOLUTION

(a) SOLUTION
Let $x_1 = e^{y_1}$ and $x_2 = e^{y_2}$, the problem becomes:
Objective:

$$Minimize \quad f(y) = 2e^{-2y_1 - y_2} + e^{-y_1 - 2y_2}, \tag{1}$$

subject to:

$$g(y) = 4 * e^{y_1 + y_2} + e^{2y_1 + 2y_2} - 12 \leq 0. \tag{2}$$

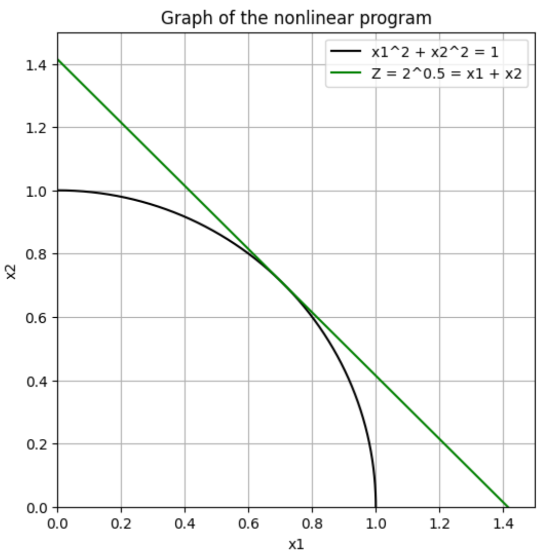

Figure 1: Graph of the nonlinear program

(b) Since:
$\frac{\partial f(y)}{\partial y_1^2} = 8e^{-2y_1-y_2} + e^{-y_1-y_2} \geq 0$, $\frac{\partial f(y)}{\partial y_2^2} = 2e^{-2y_1-y_2} + 4e^{-y_1-y_2} \geq 0$,
$\frac{\partial f(y)}{\partial y_1^2}\frac{\partial f(y)}{\partial y_2^2} - \left[\frac{\partial f(y)}{\partial y_1 \partial y_2}\right]^2 = 18e^{-3y_1-3y_2} \geq 0$.
$\frac{\partial g(y)}{\partial y_1^2} = \frac{\partial g(y)}{\partial y_2^2} = \left[\frac{\partial g(y)}{\partial y_1 y_2}\right]^2 = 4e^{y_1+y_2} + 4e^{y_1+y_2}$,
$\frac{\partial g(y)}{\partial y_1^2}\frac{\partial g(y)}{\partial y_2^2} - \left[\frac{\partial g(y)}{\partial y_1 \partial y_2}\right]^2 = 0$.
Hence f(y) and g(y) are both convex, which means the model formulated in part (a) is indeed a convex programming problem.

7.4. 13.4-1 SOLUTION

(a) $f^{'}(x) = 3x^2 + 2 - 4x - x^3$ and the answer is shown in Figure 2.

Itertation	f'(x)	lower x	upper x	New x'	f(x')
0		0	2.4	1.2	0.7296
1	-0.208	0	1.2	0.6	0.6636
2	0.464	0.6	1.2	0.9	0.745
3	0.101	0.9	1.2	1.05	0.7487
4	-0.05	0.9	1.05	0.975	0.7497
5	0.025	0.975	1.05	1.0125	0.7499
Stop Solution: x = 1.0125					

Figure 2: Bisection method

(b) $f^{''}(x) = 6x - 4 - 3x^2$ and the answer is shown in Figure 3.

Itertation i	xi	f(xi)	f'(xi)	f''(xi)	xi+1	|xi-xi+1|
1	1.2000	0.7296	-0.208	-1.12	1.0143	0.1857
2	1.0143	0.7499	-0.014	-1.0006	1.0000	0.0143
3	1.0000	0.75	0	-1	1.0000	0.0000
Stop Solution: x = 1.000006 (more accurate)						

Figure 3: Newton's method

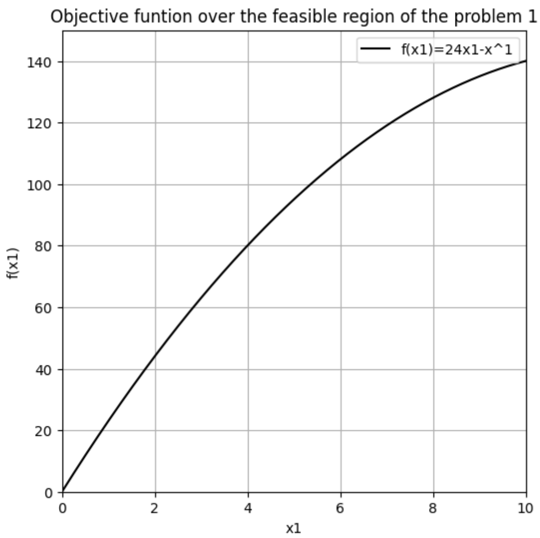

Figure 4: Objective function over the feasible region of subproblem 1

7.5. 13.6-4

(a) SOLUTION:
The KKT conditions are as followed:

$$
\begin{aligned}
1(j=1) &: 24 - 2x_1 - u_1 \leq 0 \\
2(j=1) &: x_1(24 - 2x_1 - u_1) = 0 \\
1(j=2) &: 10 - 2x_2 - u_2 \leq 0 \\
2(j=2) &: x_2(10 - 2x_2 - u_2) = 0 \\
3 &: x_1 - 10 \leq 0, x_2 - 15 \leq 0 \\
4 &: u_1(x_1 - 10) = 0, u_2(x_2 - 15) = 0 \\
5 &: x_1 \geq 0, x_2 \geq 0 \\
6 &: u_1 \geq 0, u_2 \geq 0
\end{aligned}
\tag{3}
$$

(b) SOLUTION:
The subproblem 1 is:
Objective:

$$Maximize \quad f_1(x_1) = 24x_1 - x_1^2, \tag{4}$$

subject to: $0 \leq x_1 \leq 10$.

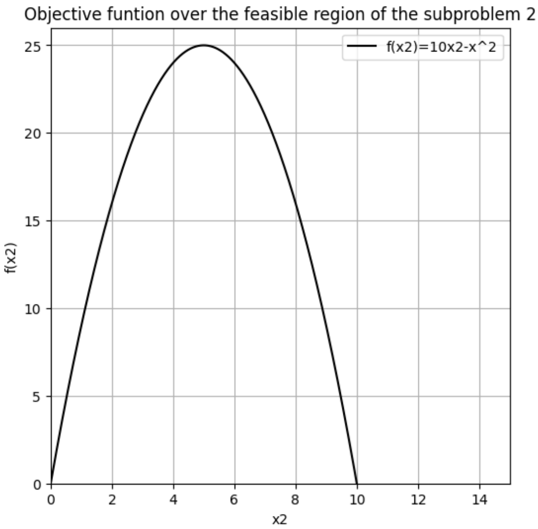

Figure 5: Objective function over the feasible region of subproblem 2

The subproblem 2 is:
Objective:

$$Maximize \quad f_2(x_2) = 10x_2 - x_2^2, \tag{5}$$

subject to: $0 \leq x_2 \leq 15$.
The objective funtion over the feasible region of the two problems are shown in Figure 4 and Figure 5.
Since $\frac{\partial f(x1)}{\partial x_1} = 24 - 2x_1 > 0$ when $0 \leq x_1 \leq 10$, hence $x_1 = 10$ is a global optimal solution over the feasible region.
Since $\frac{\partial f(x2)}{\partial x_2} = 10 - 2x_2 = 0$ when $x_2 = 5 < 15$ and $\frac{\partial f(x2)}{\partial x_2^2} = -2 < 0$, hence $x_2 = 5$ is a global optimal solution.

A Relevant Files

Relevant files can be found in my GitHub repository:
`https://github.com/OpenGHz/TBSI-MyHomework.git`.