Tsinghua-Berkeley Shenzhen Institute
OPERATIONS RESEARCH
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**Homework 2**

Haizhou Ge 2023312829 October 17, 2023

- **Acknowledgments:** This assignment refers to the textbook: Introduction to Operations Research(10th).
- **Collaborators:** I finish this assignment by myself.

The answers below are arranged according to the order of the following questions in the textbook: 4.4-5 (a) (b), 4.5-2 (a) (b) (c) (d) (e), 4.6-9 (a) (b), 3.4-11(a) (b) (c).

2.1. 4.4-5

(a) SOLUTION Augmented form of the model is as follows.
Objective:

$$Maximize \quad z = 2x_1 + 4x_2 + 3x_3, \tag{1}$$

subject to

$$3x_1 + 4x_2 + 2x_3 + x_4 = 60 \tag{2}$$

$$2x_1 + x_2 + 2x_3 + x_5 = 40 \tag{3}$$

$$x_1 + 3x_2 + 2x_3 + x_6 = 80 \tag{4}$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$.
Then choose $x_1, x_2, x_3$ to be the nonbasic variables for the initial BF solution (0,0,0,60,40,80), which is not optimal. Increase $x_2$ and stop when the first basic variable $(x_4, x_5, x_6)$ drops to zero($x_4$). With $x_2$ now a basic variable and $x_4$ now a nonbasic variable, solve the system equation: $(x_2 = 15, x_5 = 25, x_6 = 65)$, so (0,0,15,0,25,65) is the new BF solution, which is not optimal. Increase $x_3$, and stop when the first basic variable $(x_2, x_5, x_6)$ drops to zero($x_5$), so $x_5$ now a nonbasic variable and $x_3$ now a basic variable, solve the system equation:$(x_2 = \frac{20}{3}, x_3 = \frac{50}{3}, x_6 = \frac{80}{3})$, so $(0, \frac{20}{3}, \frac{50}{3}, \frac{80}{3})$ is now the new BF solution and is finally optimal because increasing any nonbasic variable $(x_1, x_4, x_5)$ decreases z and now $z_{max} = \frac{230}{3}$.

(b) Solution: The tabular form is shown in Figure 1.

2.2. 4.5-2

(a) As Figure 2 shows, the feasible region is unbounded when x1 and x2 increase from 0 to $+\infty$.

(b) Yes. In order to maximize $z = -x_1 + x_2$, draw the line $x_2 = x_1 + z$ in Figure 3. We can see that with z increasing, the line moves up and the last intersection point is $x_1 = 0, x_2 = 10$, where $z_{max} = 10$.

| Iteration | Basic Variable | Eq. | Coefficient of: z | x1 | x2 | x3 | x4 | x5 | x6 | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | z | 0 | 1 | -2 | -4 | -3 | 0 | 0 | 0 | 0 |
| | x4 | 1 | 0 | 3 | 4 | 2 | 1 | 0 | 0 | 60 |
| | x5 | 2 | 0 | 2 | 1 | 2 | 0 | 1 | 0 | 40 |
| | x6 | 3 | 0 | 1 | 3 | 2 | 0 | 0 | 1 | 80 |
| 1 | z | 0 | 1 | 1 | 0 | -1 | 1 | 0 | 0 | 60 |
| | x2 | 1 | 0 | 3/4 | 1 | 1/2 | 1/4 | 0 | 0 | 15 |
| | x5 | 2 | 0 | 5/4 | 0 | 3/2 | - 1/4 | 1 | 0 | 25 |
| | x6 | 3 | 0 | -5/4 | 0 | 1/2 | - 3/4 | 0 | 1 | 35 |
| 2 | z | 0 | 1 | 11/6 | 0 | 0 | 5/6 | 2/3 | 0 | 230/3 |
| | x2 | 1 | 0 | 1/3 | 1 | 0 | 1/3 | -1/3 | 0 | 20/3 |
| | x3 | 2 | 0 | 5/6 | 0 | 1 | -1/6 | 2/3 | 0 | 50/3 |
| | x6 | 3 | 0 | -5/3 | 0 | 0 | -2/3 | -1/3 | 1 | 80/3 |

Figure 1: Simplex method in tabular form to Question 4.5-2(made in Excel)

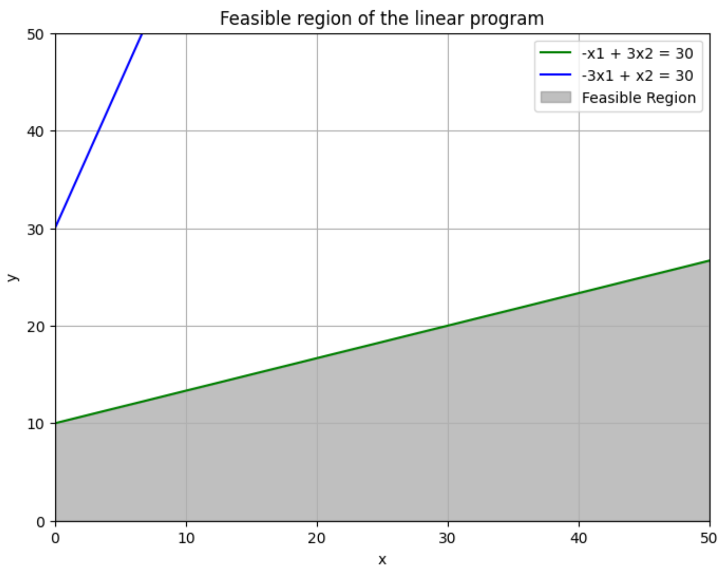


Figure 2: Feasible region to Question 4.5-2

(c) No. Similar to (b), shown in Figure 4, we can see that with z increasing to $+\infty$ the line moves down and always intersects with the feasible region. So the model has no optimal solution.

(d) No, it does not necessarily mean that there are no good solutions according to the model. An objective function of a linear program can have no optimal solution for two reasons below:

- The feasible region is unbounded. This means that there are feasible solutions that can make the objective function value infinitely large or small. In this case, there are still good solutions, but they are not optimal. Actually we can choose the solutions according to some other experience, such as choosing small-value decision variables and so on, to make the solutions better in a sense.
- The feasible region is empty. This means that there are no feasible solutions at all, and so that there are no good solutions, too.

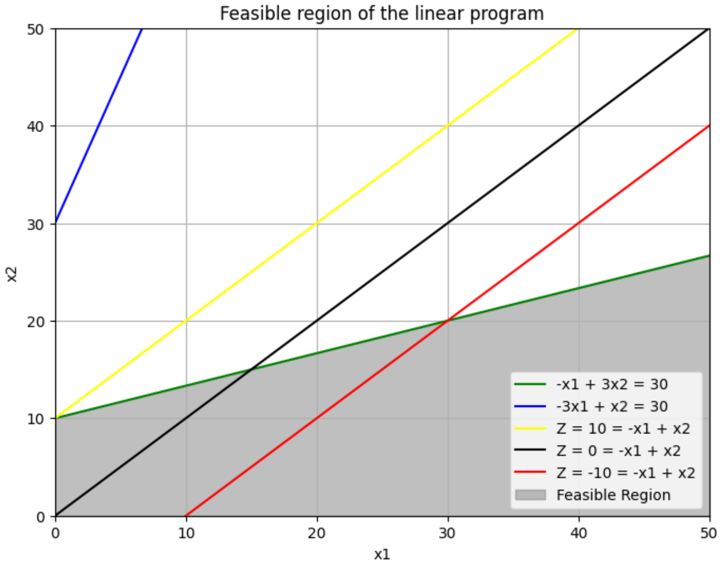


Figure 3: Feasible region and objective to Question 4.5-2(b)

Regarding the question of "What might have gone wrong when formulating the model?", the first reason is due to the stringent constraints that are unrealistic and self-contradictory. The solution to this situation is to examine the inconsistencies in the constraints, modify them to relax certain restrictions, and thus, to make the optimization problem meaningful; for the second reason, it is because the constraints for the objective are so loose. But obviously in reality, resources are often limited and the decision variables cannot increase to $+\infty$. Therefore, this may be due to an oversight of certain necessary conditions in the model.
In general, it is important to note that linear programs are a simplified model of reality. There may be other factors that are not considered in the model that can affect the quality of the solution. Therefore, even if a linear program has an optimal solution, it is important to evaluate the solution carefully to ensure that it is realistic and achievable.

(e) Choose $z = x_1 - x_2$, and the augmented form of the problem is as follows.
Objective:

$$Maximize \quad z = x_1 - x_2, \tag{5}$$

subject to

$$-x_1 + 3x_2 + x_3 = 30 \tag{6}$$
$$-3x_1 + x_2 + x_4 = 30 \tag{7}$$

and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

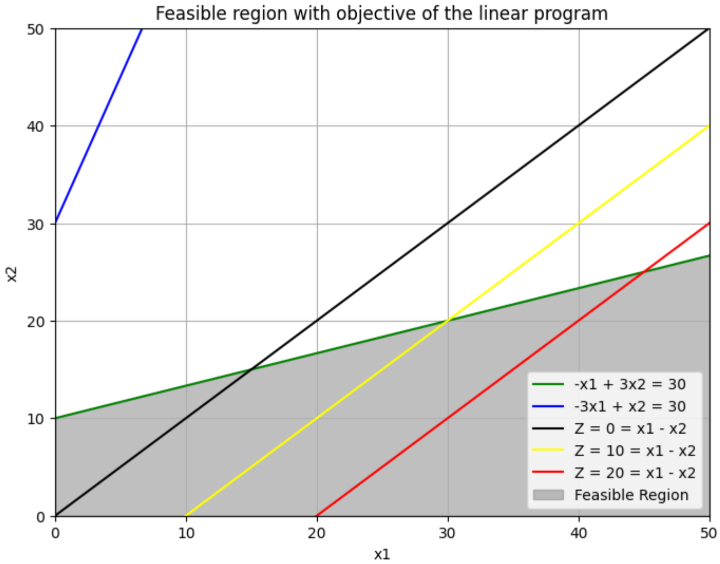


Figure 4: Feasible region and objective to Question 4.5-2(c)

Then choose $x_1, x_2$ to be the nonbasic variables for the initial BF solution (0,0,30,30), which is not optimal. Increase $x_1$, and because the coefficient of x1 is negative, the value of $x_3$ and $x_4$ will increase with $x_1$ increasing to satisfy the equation, so that neither $x_3$ and $x_4$ decreases to zero. As a result, $x_1$ can be as bigger as possible to maximize z, indicating that z is unbounded.

2.3. 4.6-9

(a) SOLUTION The artificial problem is:

$$Maximize \quad -z = -3x_1 - 2x_2 - 4x_3 - Mx_5 - Mx_6 \tag{8}$$

subject to

$$3x_1 + 3x_2 + 5x_3 - x_4 + x_5 = 120 \tag{9}$$
$$2x_1 + x_2 + 3x_3 + x_6 = 60 \tag{10}$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$.
Then the big M method is shown in Figure 5.

(b) Solution
The phase 1 problem is:

$$Minimize \quad z = x_5 + x_6 \tag{11}$$

subject to

$$3x_1 + 3x_2 + 5x_3 - x_4 + x_5 = 120 \tag{12}$$
$$2x_1 + x_2 3x_3 + x_6 = 60 \tag{13}$$

| Iteration | Basic Variable | Eq. | Coefficient of: z | x1 | x2 | x3 | x4 | x5 | x6 | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | z | 0 | -1 | 3-5M | 2-4M | 4-8M | M | 0 | 0 | -180M |
| | x5 | 1 | 0 | 3 | 3 | 5 | -1 | 1 | 0 | 120 |
| | x6 | 2 | 0 | 2 | 1 | 3 | 0 | 0 | 1 | 60 |
| 1 | z | 0 | -1 | 1/3(1+M) | 2/3(1-2M) | 0 | M | 0 | 4/3(2M-1) | -20(M+4) |
| | x5 | 1 | 0 | -1/3 | 4/3 | 0 | -1 | 1 | -5/3 | 20 |
| | x3 | 2 | 0 | 2/3 | 1/3 | 1 | 0 | 0 | 1/3 | 20 |
| 2 | z | 0 | -1 | 1/2 | 0 | 0 | 1/2 | M-1/2 | M-1/2 | -90 |
| | x2 | 1 | 0 | -1/4 | 1 | 0 | -3/4 | 3/4 | -5/4 | 15 |
| | x3 | 2 | 0 | 3/4 | 0 | 1 | 1/4 | -1/4 | 3/4 | 15 |

Figure 5: Big M method to Question 4.6-9

| Iteration | Basic Variable | Eq. | Coefficient of: z | x1 | x2 | x3 | x4 | x5 | x6 | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | z | 0 | -1 | -5 | -4 | -8 | 1 | 0 | 0 | -180 |
| | x5 | 1 | 0 | 3 | 3 | 5 | -1 | 1 | 0 | 120 |
| | x6 | 2 | 0 | 2 | 1 | 3 | 0 | 0 | 1 | 60 |
| 1 | z | 0 | -1 | 1/3 | -4/3 | 0 | 1 | 0 | 8/3 | -20 |
| | x5 | 1 | 0 | -1/3 | 4/3 | 0 | -1 | 1 | -5/3 | 20 |
| | x3 | 2 | 0 | 2/3 | 1/3 | 1 | 0 | 0 | 1/3 | 20 |
| 2 | z | 0 | -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| | x2 | 1 | 0 | -1/4 | 1 | 0 | -3/4 | 3/4 | -5/4 | 15 |
| | x3 | 2 | 0 | 3/4 | 0 | 1 | 1/4 | -1/4 | 3/4 | 15 |

Figure 6: Phase 1 of the two-phase method to Question 4.6-9

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$.
Then Phase 1 of the two-phase method is shown in Figure 6.
The phase 2 problem is:

$$Minimize \quad z = 3x_1 + 2x_2 + 4x_3 \tag{14}$$

subject to

$$3x_1 + 3x_2 + 5x_3 - x_4 = 120 \tag{15}$$

$$2x_1 + x_2 + 3x_3 = 60 \tag{16}$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$.
Then Phase 2 of the two-phase method is shown in Figure 7. The final tableau shows that the final BF solution in Phase 1 is already optimal because all the coefficients in row 0 are positive, thus increasing either nonbasic variable will decrease z.

(c) We can see that the sequence of BF solutions obtained in parts (a) and (b) are the same. And except for the last solution in the final tableau, the solutions obtained through the big M method and the two-phase method are not feasible for the real problem.

| Iteration | Basic Variable | Eq. | Coefficient of: z | x1 | x2 | x3 | x4 | x5 | x6 | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|
| Final Phase 1 tableau | z | 0 | -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| | x2 | 1 | 0 | -1/4 | 1 | 0 | -3/4 | 3/4 | -5/4 | 15 |
| | x3 | 2 | 0 | 3/4 | 0 | 1 | 1/4 | -1/4 | 3/4 | 15 |
| Drop x5 and x6 | z | 0 | -1 | 0 | 0 | 0 | 0 | | | 0 |
| | x2 | 1 | 0 | -1/4 | 1 | 0 | -3/4 | | | 15 |
| | x3 | 2 | 0 | 3/4 | 0 | 1 | 1/4 | | | 15 |
| Substitute phase 2 objective function | z | 0 | -1 | 3 | 2 | 4 | 0 | | | 0 |
| | x2 | 1 | 0 | -1/4 | 1 | 0 | -3/4 | | | 15 |
| | x3 | 2 | 0 | 3/4 | 0 | 1 | 1/4 | | | 15 |
| Restore proper form from Gaossian elimination | z | 0 | -1 | 3 | 0 | 0 | 1/2 | | | -90 |
| | x2 | 1 | 0 | -1/4 | 1 | 0 | -3/4 | | | 15 |
| | x3 | 2 | 0 | 3/4 | 0 | 1 | 1/4 | | | 15 |

Figure 7: Phase 2 of the two-phase method to Question 4.6-9

# A Relevant Files

Relevant files can be found in my GitHub repository:
`https://github.com/OpenGHz/TBSI-MyHomework.git`.