LECTURE # 12

Problem # 1

(a) Find the optimal value using Lagrange or substitution method.
(b) Maximization or minimization problem'
(c) Quasi and Quasi concave?

Optimize $U(x,y) = x + \ln y + e^1$
Subject to : $4x + 6y = 24$

Solution: Doing by substitution method
(a) From constant $4x + 6y = 24$

$$4x = 24 - 6y$$

$$x = \frac{24}{4} - \frac{6y}{4}$$

$$x = 6 - \frac{3}{2}y$$

Putting the value of 'x' in $x + \ln y + e^1$

$$U(y) = x + \ln y + e^1$$

$$U(y) = 6 - \frac{3}{2}y + \ln y + e^1$$

$$y' = \frac{dx}{dy}\ 6 - \frac{3}{2}y + \ln y + e^1$$

$$y' = 0 - \frac{3}{2}(1) + \frac{1}{y} + 0 = -\frac{3}{2} + \frac{1}{y}$$

$y' = 0$ for optimization

$$\frac{-3}{2} + \frac{1}{y} = 0$$

$$\frac{1}{y} = \frac{-3}{2} + \frac{1}{y} = 0$$

$$\frac{-3y + 2}{2y} = 0$$

$$-3y + 2 = 0$$

$$\cancel{-}3y = \cancel{-}2$$

$$\boxed{y = \frac{2}{3}}$$

Putting the value of y in $x = 6 - \frac{3}{2}y$

$$x = 6 - \frac{\cancel{3}}{\cancel{2}}\left(\frac{\cancel{2}}{\cancel{3}}\right)$$

$$x = 6 - 1$$

$$\boxed{x = 5}$$

So

$(5, \frac{2}{3})$ Optimal values.

b) Maximization or minimization

$$\Delta = U_{xx}U_{yy} - (U_{xy})^2$$

So,

$$U_{xx} = 0$$

$$U_{yy} = \frac{-1}{y^2}$$

Put in formula

$$U_{xy} = 0$$

$\Delta = (0)\left(-\frac{1}{y^2}\right) - (0)^2$

$\Delta = 0$

Test fails here.

(c) Quasi concave or convex

$\frac{dy}{dx}\left(4 - \frac{2x}{3}\right)$

$= 0 - \frac{2}{3}(1)$

$= -\frac{2}{3}$

$\frac{d^2y}{dx^2}\left(-\frac{2}{3}\right) = 0$

Quasi concave

$x \leq 0$

(2) Optimize $U(x,y) = x^{1/7}y^{1/3}$
Subject to : $4x + 6y = 24$

Solution:

We will solve this problem using the Lagrange method

$L(x, y, \lambda) = U(x,y) - \lambda.(g(x,y))$

Where $U(x,y) = x^{1/7}y^{1/3}$

Constraint is $x + 2y = 24$

$g(x,y) = x + 2y - 24$

$h(x,y,\lambda) = x^{1/7} y^{1/3} - \lambda \cdot (x+2y-24)$

Partial derivative of $h(x,y,\lambda)$

Partial w.r.t x

$$\frac{\partial h}{\partial x} = \frac{1}{7} x^{-\frac{6}{7}} y^{\frac{1}{3}} - \lambda = 0$$

$$\frac{1}{7} x^{-\frac{6}{7}} y^{\frac{1}{3}} = \lambda \quad \text{(Equation 1)}$$

Partial derivative w.r.t to y

$$\frac{\partial h}{\partial y} = \frac{1}{3} x^{1/7} y^{-2/3} - 2\lambda = 0$$

$$\frac{1}{3} x^{1/7} y^{-2/3} = 2\lambda \quad \text{(Equation 2)}$$

Partial derivative w.r.t to λ

$$\frac{\partial h}{\partial \lambda} = -(x+2y-24) = 0$$

$$x + 2y = 24 \quad \text{(Equation 3)}$$

From eq(1) and (2) we have

$$\frac{1}{7} x^{-\frac{6}{7}} y^{1/3} = \lambda \text{ and } \frac{1}{3} x^{1/7} y^{-2/3} = 2\lambda$$

$$\frac{2\lambda}{\lambda} = \frac{1/3\, x^{1/7} y^{-2/3}}{1/7\, x^{-6/7} y^{1/3}}$$

$$\frac{7}{3} \cdot \frac{x^{1/7} y^{-2/3}}{x^{-\frac{6}{7}} y^{1/3}} = 2$$

$$\frac{7}{3} \cdot x^{\frac{1}{7}+\frac{6}{7}} y^{-\frac{2}{3}-\frac{1}{3}} = 2$$

$\frac{7}{3} \cdot x^{1} y^{-1/3} = 2$

$7xy^{-1/3} = 6$

$x = \frac{6}{7} y^{1/3}$

Substitute $x = \frac{6}{7} y^{1/3}$

into the constraint

$\frac{6}{7} y^{1/3} + 2y = 24$

$6y^{1/3} + 14y = 168$

b) Maximization or Minimization?

We need to compute

$\Delta = U_{xx} U_{yy} - U^2_{xy}$

• $U_{x} = \frac{1}{7} x^{-6/7} y^{1/3}$ • $U_y = \frac{1}{3} x^{1/7} y^{-2/3}$

Second derivative

• $U_{xx} = -\frac{6}{49} x^{-13/7} y^{1/3}$

• $U_{yy} = -\frac{2}{9} x^{1/7} y^{-5/3}$

• $U_{xy} = \frac{1}{21} x^{-6/7} y^{-2/3}$

$\Delta = \left(-\frac{6}{49} x^{-\frac{13}{7}} y^{1/3}\right) \cdot \left(-\frac{2}{9} x^{1/7} y^{-\frac{5}{3}}\right) - \left(\frac{1}{21} x^{-\frac{6}{7}} y^{-\frac{2}{3}}\right)^2$

$U_{xx}U_{yy} = \frac{12}{441} x^{-12/7} y^{-4/3}$

$\Delta = \frac{12}{441} x^{-12/7} y^{-4/3} - \frac{1}{441} x^{-12/7} y^{-4/3}$

$\Delta = \frac{11}{441} x^{-12/7} y^{-4/3}$

$\Delta > 0 \sim$ minimization

(c) Quasi concave or convex

$U(x, y) = x^{1/7} y^{1/3}$

Constraint $= x + 2y = 24$

$\frac{dy}{dx} = -\frac{1}{2}$

$\frac{d^2x}{dx^2} = \frac{d^2y}{dx^2} = 0$

Neither quasi-convex nor quasi-concave.

(6) Optimize $U(x, y) = 2x + 3y$

Subject to : $3x + 6y = 45$

$f(x, y, \lambda) = 2x + 3y + \lambda(45 - 3x - 6y)$

First order condition

$\frac{\partial f}{\partial x} = 2 - 3\lambda = 0 \quad \lambda = \frac{2}{3}$

$\frac{\partial f}{\partial y} = 3 - 6\lambda = 0 \quad \lambda = \frac{1}{2}$

$\frac{\partial L}{\partial \lambda} = 45 - 3x - 6y = 0 \qquad 3x + 6y = 45$

$\lambda = \frac{2}{3} \qquad \lambda = \frac{1}{2}$ No local maximum or minimum

(c) Quasi convex or quasi concave

Solve for $\frac{dy}{dx}$ from the constraint

$3x + 6y = 45 \qquad y = \frac{45 - 3x}{6}$

$\frac{dy}{dx} = -\frac{1}{2}$ Neither concave nor convex

$\frac{d^2y}{dx^2} = 0$

2 Problem

a) $f(x,y) = xy + x - y$

Partial derivative w.r.t. x

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy + x - y) = y + 1$

Partial derivative w.r.t. y

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy + x - y) = x - 1$

Solve for $\frac{dy}{dx} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{y+1}{x-1}$

$$\frac{dy}{dx} = -\frac{y+1}{x-1}$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{(x-1)\left(\frac{dy}{dx}\right)(y+1)(1)}{(x-1)^2}$$

Substitute $\frac{dy}{dx} = -\frac{y+1}{x-1}$ into the expression

$$\frac{d^2y}{dx^2} = \frac{(x-1)\left(\frac{y+1}{x-1}\right)+y+1}{(x-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y+1+y+1}{(x-1)^2} = \frac{2(y+1)}{(x-1)^2}$$

Since

$\frac{d^2y}{dx^2} = \frac{2(y+1)}{(x-1)^2}$ is always positive for $y > 1$ (and $x \neq 1$)

the function is quasi-convex for $y > -1$

(b) Monotonic transformation and Convexity / concavity e will transform the function $f(x,y) = xy + x - y$ by taking the natural log

$$f(z) = \ln(xy + x - y)$$

First derivative

$$\frac{d}{dx}(\ln(xy+x-y)) = \frac{1}{xy+x-y} \cdot \frac{d}{dx}(xy+x-y)$$

$$\frac{d}{dx}(xy+x-y) = y+1$$

So

$$\frac{d}{dx}(\ln(xy+x-y)) = \frac{y+1}{xy+x-y}$$

Second derivative

$$\frac{d^2}{dx^2}(\ln(xy+x-y)) = -\frac{(y+1)^2}{(xy+x-y)^2}$$

Since, the second derivative is negative, we conclude that the function $\ln(xy+x-y)$ is concave.

$$f(x,y) = x^4 + y^3 + x^2y + 7$$

(a) Quasi-convex or Quasi concave?

we will take partial derivative

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^4+y^3+x^2y+7)$$

$$f_x = 4x^3 + 2xy$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^4+y^3+x^2y+7)$$

$f_y = 3y^2 + x^2$

Second order Partial derivative

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 12x^2 + 2y$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6y$$

$$f_{xy} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 2x$$

The hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x^2 + 2y & 2x \\ 2x & 6y \end{bmatrix}$$

The determinent of the Hessian is

$$D = (12x^2 + 2y)(6y) - (2x)(2x)$$

$$D = 72x^2y + 12y^2 - 4x^2$$

$$D = 12y^2 + 72x^2y - 4x^2$$

$$D = 12(-1)^2 + 72(1)^2(-1) - 4(1)^2 = 12 - 72 - 4$$

$$= -64$$

$D < 0$ function is neither convex nor concave at this point.

b) monotonic transformation

$$\ln(f(x,y)) = \ln(x^4 + y^3 + x^2y + 7)$$

First order partial derivatives

$$g_x = \frac{1}{f(x,y)} \cdot \frac{\partial f}{\partial x}$$

$$g_y = \frac{1}{f(x,y)} \cdot \frac{\partial f}{\partial y}$$

$f_x = 4x^3 + 2xy$ $\qquad$ $f_y = 3y^2 + x^2$

Thus

$$g_x = \frac{4x^3 + 2xy}{x^4 + y^3 + x^2y + 7}$$

$$g_y = \frac{3y^2 + x^2}{x^4 + y^3 + x^2y + 7}$$

Second order partial derivative w.r.t to x

$$g_{xx} = \frac{(x^4+y^3+x^2y+7)(12x^2+2y) - (4x^3+2xy)(4x^3+2xy)}{(x^4+y^3+x^2y+7)^2}$$

Second partial derivative w.r.t. y

$$g_{yy} = \frac{(x^4+y^3+x^2y+7)(6y) - (3y^2+x^2)(3y^2+x^2)}{(x^4+y^3+x^2y+7)^2}$$

$$g_{xy} = \frac{(x^4+y^3+x^2y+7)(2x) - (4x^3+2xy)(3y^2+x^2)}{(x^4+y^3+x^2y+7)^2}$$

$$D = g_{xx}\, g_{yy} - g_{xy}^2$$

$D \leq 0$ then $g(x,y)$ is concave.

Since the log transformation of a quasi-convex function usually results in a concave function.

Problem3

For each of the following, show the steps to determine:

(a) Determine whether the input quantities x and y are complemetary or competitive with each other.

(b) Find the quantities and prices that maximize profit and the maximum profit for each of the following sets

13. $x = 1 - p + 2q$

$y = 11 + p - 3q$

$C = 4x + y$

$\frac{dx}{\partial q} = 2$ $\frac{\partial y}{\partial q} = -3$

Profit maximization

$p = 1 + 2q - x$

$q = \frac{y - 11 - p}{-3}$

Revenue $= R = p \cdot x + q \cdot y$

Cost function $= C = 4x + y$

Maximize profit $= \pi = R - C$

Put $p = 1 + 2q - x$ in $q = \frac{y - 11 - p}{-3}$

$q = \frac{y - 11 - (1 + 2q - x)}{-3}$

$$q = \frac{y-11-(1+2q-x)}{-3}$$

$$q = \frac{y-11-1-2q+x}{+3}$$

$$3q = y-11-1-2q+x$$

$$3q+2q = y-11-1+x$$

$$5q = y-12+x$$

$$q = \frac{y-12+x}{5}$$

Substitute 'q' and 'p'

$$R = (1+2q-x)-x+\left(\frac{y+x-12}{5}\right)\cdot y$$

$$R = x+2qx-x^2+\frac{y^2+xy-12x}{5}$$

Profit function

$$\pi = R-C$$

$$\pi = x+2qx-x^2+\frac{y^2+xy-12y}{5}-(4x+y)$$

$$\pi = 21x-3x^2-3xy+\cancel{42}y-y^2$$

$$\frac{\partial \pi}{\partial x} = 21-6x-3y=0$$

$$6x+3y=21 \Rightarrow 2x+y=7 \rightarrow (1)$$

$$\frac{\partial \pi}{\partial y} = -3x+11-2y=0 \qquad 3x+2y=11 \rightarrow (2)$$

From eq (1)

$$y = 7-2x$$

Put this in equation

$$3x+2(7-2x)=11$$

$3x + 14 - 4x = 11$

$14 - x = 11$

$x = 14 - 11$

$x = 13$

$y = 7 - 2(3)$

$y = 7 - 6$

$y = 1$

Finding p and q

From $x = 1 - p + 2q$

$p = 1 - x + 2q$

From $y = 11 + p - 3q$

$p = y - 11 + 3q$

Equating both equation

$1 - x + 2q = y - 11 + 3q$

$1 + 11 - x + 2q - 3q - y = 0$

$12 - x - q - y = 0$

$q = 12 - x - y$

Substitute values of x and y

in $q = 12 - x - y$

$q = 12 - 3 - 1$

$q = 8$

For p

$p = 1 - 11 + 3(8)$

$p = 1 - 11 + 24$

$p = 14$

Profit maximization

$\pi = 21(3) - 3(3)^2 - 3(3)(1) + 11(1) - 1^2$

$= 63 - 27 - 9 + 11 - 1 = 37$

14. $x = 11 - 2p - 2q$

$y = 16 - 2p - 3q$

$C = 3x + y$

$\frac{dx}{dq} = -2 \quad \frac{dy}{dq} = -3$

Since an increase in q decreases both x and y so they are complementary

Profit $\pi = Px + Py - C$

Substitute C

$$\pi = Px + qy - 3x - y$$

From the demand functions

$$P = \frac{11 - x - 2q}{2}$$

$$q = \frac{16 - y - 2p}{3}$$

From $x = 11 - 2p - 2q$

$$p + q = \frac{11 - x}{2} \rightarrow (1)$$

From $y = 16 - 2p - 3q$

$$2p + 3q = 16 - y \rightarrow (2)$$

Multiplying eq (1) by (2)

$$2p + 2q = 2\left(\frac{11 - 2x}{x}\right)$$

$$2p + 3q = 2p - 2q = 16 - y - 11 + x$$

$$q = 5 + x - y$$

Put $q = 5 + x - y$ in eq (1)

$$P + 5 - y + x = \frac{11 - 2x}{2} \Rightarrow P = \frac{11 - x - 5 + y - x}{2}$$

$$P = 11 - x - 10 + 2y - 2x = \frac{1 - 3x + 2y}{2}$$

Substituting P & q in profit function

$$\pi = \left(\frac{1-3x+2y}{2}\right)x + (5 - y + x)y - 3x - y$$

$$\pi = \frac{x - 3x^2 + 2xy}{2} + 5y - y^2 + xy - 3x - y$$

$$\pi = \frac{x}{2} - \frac{3x^2}{2} + 2xy + 5y - y^2 + xy - 5x - y$$

$$\pi = \frac{-3x^2}{2} + 2xy - y^2 - \frac{5y}{2} + 4y$$

$$\frac{\partial \pi}{\partial x} = -3x + 2y - \frac{5}{2} = 0 \Rightarrow -3x + 2y = \frac{5}{2}$$

$$\frac{\partial \pi}{\partial y} = 2x - 2y + 4 = 0 \Rightarrow 2x - 2y = -4$$

$$2x - 2y = -4$$

$$4 - y = -2 \Rightarrow y = x + 2$$

Substitute into $-3x + 2y = \frac{5}{2}$

$$-3x + 2(x+2) = \frac{5}{2}$$

$$-3x + 2x + y = \frac{5}{2} \Rightarrow -x = \frac{5}{2} - 4$$

$$-x = \frac{-3}{2} \Rightarrow x = \frac{3}{2}$$

$$y = \frac{3}{2} + 2 = \frac{7}{2}$$

Finding p and q

$$p = \frac{1 - 3(3/2) + 2(7/2)}{2} \Rightarrow 1.75$$

$$q = 5 - \frac{7}{2} + \frac{3}{2} = 5 - 2 = 3$$

Profit maximization

$$\pi = -3\left(\frac{3}{2}\right)^2 + 2\left(\frac{3}{2}\right)\left(\frac{7}{2}\right) - \left(\frac{7}{2}\right)^2 - \frac{5(7/2)}{2} + 4(7/2)$$

$$\boxed{\pi = 5.125}$$