Active Calculus
Carroll College

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Chapter 1

Review of Pre Calculus Materials

1.1 Lines, Slope, and Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a function and what do we mean by its domain and range?
- What is the slope of a line? What are linear functions and families of linear functions?

Introduction

We begin our study of calculus by reminding the reader of several pre-requisite topics. The study of calculus depends on a thorough understanding of these topics and it is imperative that the reader become as familiar as possible with these topics. In the present section we remind the reader about the concepts of functions, slope, and lines, but first, there are a few things that you should do to get your self ready to use this text.

Preview Activity 1.1. This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where can you find the full textbook?

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?

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Functions

Let's start with the fundamental mathematical idea of a function.

Definition 1.1 (Function).

A function is a mathematical rule that assigns exactly one output for every input.

It is easy to give many common examples of functions:

- The area of a circle A is a function of the radius of the circle: $A(r) = \pi r^2$.
- The amount M in your savings account is a function of the rate of interest the bank pays.
- The fuel efficiency in your car is a function of many things, e.g. the speed at which you drive, the number of cylinders in your engine, the type of driving conditions, etc.
- The pressure on a diver is a function of the depth of the diver under water.

Definition 1.2 (Domain of a Function).

The domain is the set of all possible inputs for a function.

Definition 1.3 (Range of a Function).

The range is the set of all possible outputs for a function.

Example 1.1. Find the domain and range of the functions $f(x) = \sin(x)$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$.

Solution. For $f(x) = \sin(x)$ we recall that the sine function is defined for every possible value of x but the output is strictly between $y = -1$ and $y = 1$. Therefore, the domain for $f(x) = \sin(x)$ is $-\infty < x < \infty$ and the range is $-1 \le y \le 1$. See the left plot in Figure 1.1.

For $g(x) = \sqrt{x}$ we recall that the square root of a negative number results in an imaginary number. In this text we are interested in real-valued output for functions so we must omit all of the negative

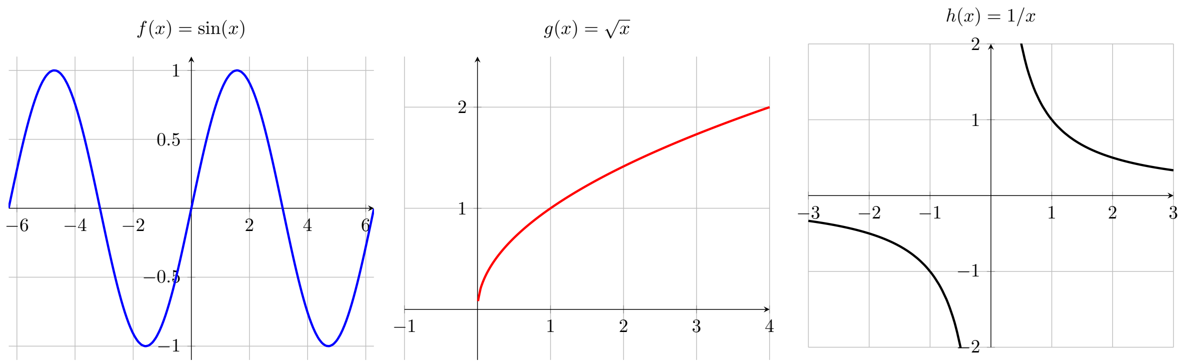


Figure 1.1: Graphs of the function $f(x) = \sin(x)$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$.

numbers from the domain and hence $0 \le x < \infty$. For the range we recall that the square root of a number will always be a non-negative number. As such, the range is $0 \le y < \infty$. See the middle plot in Figure 1.1.

For $h(x) = \frac{1}{x}$ we recall that division by zero is mathematically impossible. That is the only troublesome point in the domain so $-\infty < x < 0$ or $0 < x < \infty$. A moment's reflection also reveals that it is impossible to get zero out of the function $h(x)$ but it is possible to get any other number. Hence $-\infty < y < 0$ or $0 < y < \infty$. See the right plot in Figure 1.1.

It is also important to recall the notation for functions. When we write $f(x) = \sqrt{x}$ we are saying several things. First, the "f" is the name of the function that we're defining. The naming convention gives us a convenient way to refer to functions without having to explicitly state their algebraic form. Next, the "(x)" is an explicit statement to the reader that the variable "x" is the independent variable for the function f. Lastly, the right-hand side of the definition tells us exactly what to do with the independent variable algebraically.

When we write $f(25)$ we are referring to the already defined function f and explicitly saying to replace the independent variable with the number 25. In this instance, $f(25) = \sqrt{25} = 5$. Similarly, if we write $f(\sin(x))$ we mean to find the independent variable x in the function f and replace it with the function $\sin(x)$. In this case, $f(\sin(x)) = \sqrt{\sin(x)}$. This is simply a new function.

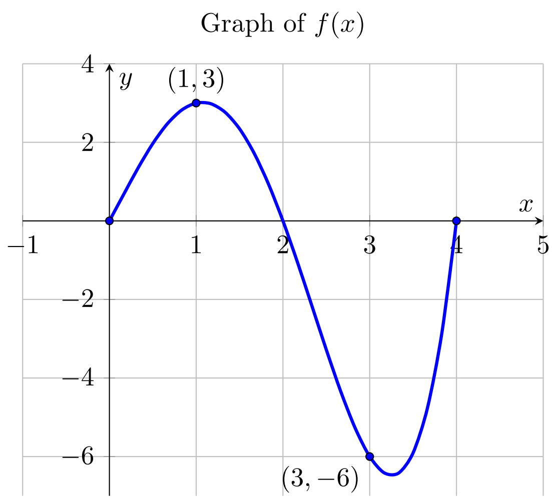
A function does not always need to be given algebraically. The three primary representations of a function are the algebraic form, the graphical form, and the tabular form. For example, for the function $f(x) = \sqrt{x}$ we are explicitly giving the algebraic form and the middle plot of Figure 1.1 shows the graphical form. Table 1.1 shows a portion of the table of values. The distinct disadvantage for a table of values on many functions is that there are infinitely many possible input values and a table can naturally only show finitely many of them.

Activity 1.1.

The graph of a function $f(x)$ is shown in the plot below.

x	0	1	2	3	4	5
$f(x)$	0	1	1.414	1.732	2	2.236

Table 1.1: Tabular form of the function $f(x) = \sqrt{x}$.



(a) What is the domain of $f(x)$?

(b) Approximate the range of $f(x)$.

(c) What are $f(0)$, $f(1)$, $f(3)$, $f(4)$, and $f(5)$?

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Slope and Linear Functions

One of the basic graphical ideas of calculus is that if we zoom in close enough to a curved function it will look approximately linear. The words "zoom" and "close enough" will be made explicit later. We now review the features of linear functions so that the idea of "zoomed in linearity" can flow naturally later in the course.

Every linear function is characterized by a constant rate of change; the slope. The slope of a linear function is a measure of the "steepness" of the line. We use the symbols Δx and Δy which mean respectively the "change in x" and the "change in y".

Definition 1.4.

The **slope**, m of a (non-vertical) linear function f which passes through any two points (x_1, y_1), (x_2, y_2) can be found using the formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}}$$

As shown in Figure 1.2, the slope of a linear function has the following characteristics:

- if the line rises from left to right then the slope is positive,

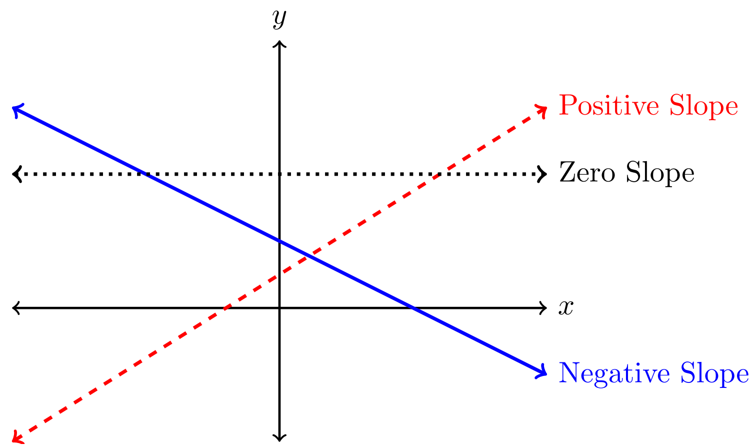

Figure 1.2: Characteristics of slope.

- if the line falls from left to right then the slope is negative,
- if the line is horizontal then the slope is zero, and
- if the line is vertical then the slope is undefined.

Depending on the information given there are several convenient forms of the equation of a line. Given the definition of the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and letting $(x, y) = (x_2, y_2)$ be any arbitrary point we get the point-slope form of a linear function by observing that $m = \frac{y-y_1}{x-x_1}$ which implies that $y - y_1 = m(x - x_1)$.

Definition 1.5.

If the linear function f has slope m and passes through the point (x_1, y_1), then the **point-slope form of the equation of a line** is given by:

$$y - y_1 = m(x - x_1).$$

An alternate form of a linear function which is probably very familiar to most readers is the slope-intercept form of a line.

Definition 1.6.

If the linear function f has slope m and y-intercept b, then the **slope-intercept form of the equation of a line** is given by:

$$y = mx + b.$$

In a calculus class the point-slope form is often the most useful. If you have a linear function written in the point-slope form you can always rearrange to get it into the slope-intercept form

$$y - y_1 = m(x - x_1) \quad \Longrightarrow \quad y = mx - mx_1 + y_1.$$

Hence we see that the y intercept of a line can be given as $b = -mx_1 + y_1$. The symbols and geometry used in each of the above definitions are shown in Figure 1.3.

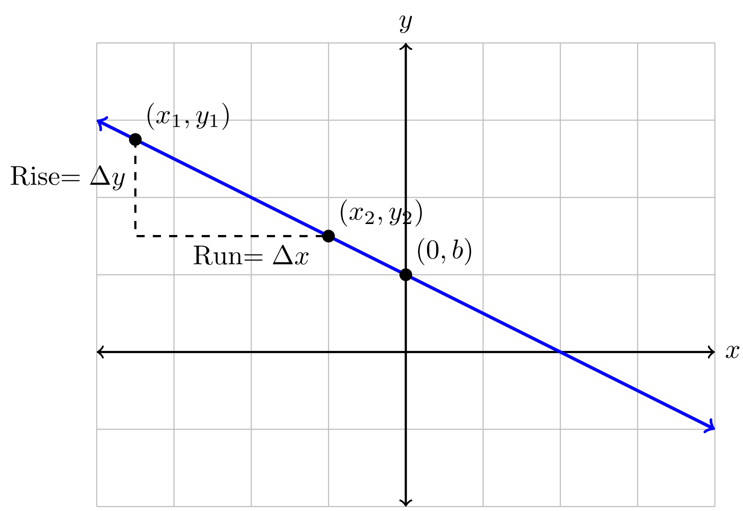

Figure 1.3: Anatomy of a linear function.

Activity 1.2.

Find the equation of the line with the given information.

(a) The line goes through the points $(-2, 5)$ and $(10, -1)$.

(b) The slope of the line is $3/5$ and it goes through the point $(2, 3)$.

(c) The y-intercept of the line is $(0, -1)$ and the slope is $-2/3$.

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Example 1.2. Write the equation of the line going through the points $(5, 7)$ and $(-3, 2)$.

Solution. First we calculate the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{7-2}{5-(-3)} = \frac{5}{8}.$$

Since we have two points and neither is the y intercept of the linear function we choose to use the point-slope form of the line. Letting $(x_1, y_1) = (5, 7)$ we see that

$$y - 7 = \frac{5}{8}(x - 5)$$

is one form of the linear function. It is often conventient to solve for y giving us

$$y = \frac{5}{8}(x-5) + 7.$$

Notice that we do not necessarily need to simplify all the way to the slope-intercept form of the line.

Linear Functions From Data

A feature of every linear function is that the slope is the same no matter where you are on the line. When given a table of data that you suspect might represent a linear function the slope manifests itself as a constant common difference between successive y-values.

Example 1.3. Consider the data in the table below.

x	5	6	7	8	9
y	12.2	17.5	22.8	28.1	33.4

Demonstrate that this data is linear and write an equation that fits the data.

Solution. The common differences can be found for each successive y-values

x	5	6	7	8	9
y	12.2	17.5	22.8	28.1	33.4
Common Difference	$\frac{17.5-12.2}{6-5} = 5.3$	$\frac{22.8-17.5}{7-6} = 5.3$	$\frac{28.1-22.8}{8-7} = 5.3$	$\frac{33.4-28.1}{9-8} = 5.3$	-

The successive differences are clearly the same throughout the data set and the slope for this data set is $m = 5.3$. Picking any convenient point, say $(5, 12.2)$, then allows us to write the equation of the line as

$$y - 12.2 = 5.3(x - 5).$$

This could be simplified to point-slope form, but there is typically no need for this algebraic simplification.

Activity 1.3.

An apartment manager keeps careful record of the rent that he charges as well as the number of occupied apartments in his complex. The data that he has is shown in the table below.

Monthly Rent	\$650	\$700	\$750	\$800	\$850	\$900
Occupied Apartments	203	196	189	182	175	168

(a) Just by doing simple arithmetic justify that the function relating the number of occupied apartments and the rent is linear.

(b) Find the linear function relating the number of occupied apartments to the rent.

(c) If the rent were to be increased to \$1000, how many occupied apartments would the apartment manager expect to have?

(d) At a \$1000 monthly rent what net revenue should the apartment manager expect?

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Example 1.4. The Old Farmer's Almanac tells us that you can tell the temperature by counting the chirps of a cricket. It is a linear function $T = f(C)$ given by T (in degrees Fahrenheit)=# of chirps in 15 seconds +40. We can approximate this with the formula

$$T = \frac{C}{4} + 40$$

where C is the number of chirps/minute and T is in °F.

(a) If the chirp rate is 120 chirps/minute, what is the temperature?

(b) Suppose that crickets will not chirp if the temperature is below 56°F. We can also suppose that crickets will not chirp above 136°F since that is the highest temperature ever recorded at a weather station. With these parameters, what is the domain of this function?

Solution.

(a) If $C = 120$ chirps/minute, substitute this into the function $T(C)$ to obtain

$$T(120) = \frac{120}{4} + 40 = 30 + 40 = 70^\circ F.$$

(b) To find the domain we need to find the appropriate values of C for the $T(C)$ function. Solve $56 = C/4 + 40$ and get $C = 64$. Solve $136 = C/4 + 40$ and get $C = 384$. So the domain of $T(C)$ is $64 \leq$ chirps/minute ≤ 384 or, in interval notation, $[64, 136]$.

Families of Linear Functions

We noted above that a linear function has the form $f(x) = mx + b$, where m is the slope of the line, and b is the y-intercept. Since m and b can take on various values, taken together, they represent a family of functions. For example, we could fix $b = 2$, and then draw the graphs of $f(x) = mx + 2$ for various values of m; for example, $m = -1, -2, 2, 1$. Doing so would give the functions in the family $f(x) = mx + 2$ shown in the left image of Figure 1.4.

Similarly, we could set m to be 2 and let b take on the values $b = -1, 1, 4, -6$ and we would get some examples from the family of functions for $y = f(x) = 2x + b$ shown in the right image of Figure 1.4.

From the right image in Figure 1.4 it should be clear to the reader that parallel lines have the same slope. What can you say about the slopes of perpendicular lines? Here is the result that we state without proof.

Theorem 1.1.

If line ℓ_1 has slope m_1 and line ℓ_2 has slope m_2, then

- lines ℓ_1 and ℓ_2 are parallel if the slopes are the same: $m_1 = m_2$, and
- lines ℓ_1 and ℓ_2 are perpendicular if the slopes are opposite reciprocals: $m_2 = -\frac{1}{m_1}$.

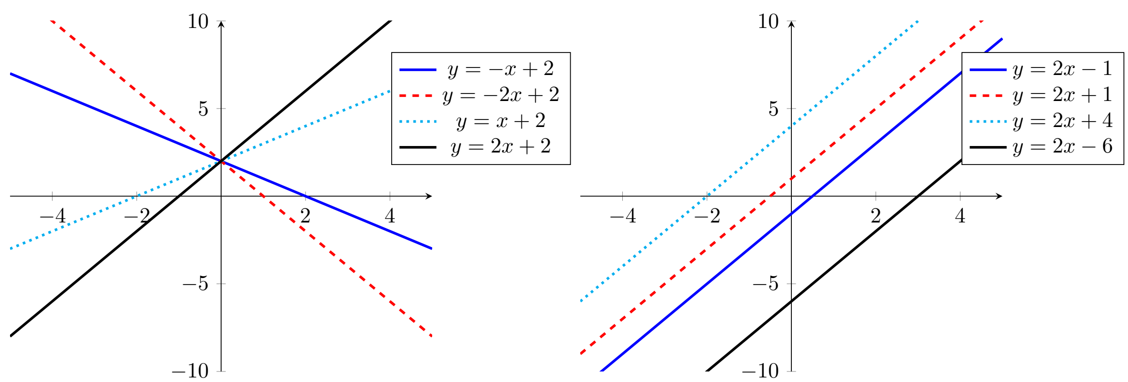

Figure 1.4: Several members of the family of linear functions $f(x) = mx + 2$ (left) and the family $f(x) = 2x + b$ (right).

Activity 1.4.

Write the equation of the line with the given information.

(a) Write the equation of a line parallel to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(b) Write the equation of a line perpendicular to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(c) Write the equation of a line with y-intercept $(0,-3)$ that is perpendicular to the line $y = -3x - 1$.

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Summary

In this section, we encountered the following important ideas:

- A function assigns one y value to each x value.
- The slope of a linear function can be written as

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_2}$$

- A linear function can be written in the forms

$$y = mx + b \quad \text{or} \quad y - y_1 = m(x - x_1)$$

- When examining linear data, the differences between successive y-values reveals the slope.

Exercises

1. (modified from NCTM Illuminations) The table below displays data that relate the number of oil changes per year and the cost of engine repairs. To predict the cost of repairs from the number of oil changes, use the number of oil changes as the x variable and the engine repair cost as the y variable.

Oil Changes Per Year	Cost of Repairs ($)
3	300
5	300
2	500
3	400
1	700
4	400
6	100
4	250
3	450
2	650
0	600
10	0
7	150

(a) Using graph paper make a plot of the data on appropriate axes.

(b) Do the data appear linear? Why or why not?

(c) Pick two representative points from the data and use them to write the equation of a line that *fits* the data. Plot your line on top of your data and discuss how well your line fits the data. (This may take a few attempts.)

(d) Despite how well your data fit a linear model, it is not entirely sensible to use a linear model for this data. Why?

2. The population of a city, P, in millions, is a function of t, the number of years since 1960, so $\mathrm{P} = f(t)$. Which of the following statements explains the meaning of $f(38) = 8$ in terms of the population of this city?

(a) The population of this city in the year 38 is 8 million people.

(b) The population of this city in the year 8 is 38 million people.

(c) The population of this city in the year 1968 is 38 million people.

(d) The population of this city in the year 1998 is 8 million people.

3. Determine the slope and y-intercept of the line whose equation is $-4y + 6x + 8 = 0$.

4. The value of a car in 1990 is $13,100 and the value is expected to go down by $80 per year for the next 7 years. Write a linear function for the value, V, of the 1990 car as a function of the number of years from 1990, x.

1.2 Exponential Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can exponential functions be used to model growth and decay of populations, investments, radioactive isotopes, and many other physical phenomena?
- How can we build exponential functions from data?

Introduction

The exponential function is a powerful tool in the mathematician's arsenal for modeling growth and decay phenomena. The common applications of the exponential funciton range from population modeling, to tracking drug levels in the blood stream, to using carbon dating to estimate the age of an artifact. The common mathematical fact about all of these situations is that the growth (or decay) rate is a constant multiple. For example, if we are measuring exponential population growth then the ratio of two successive populations must be constant. Linear functions have a similar behavior, except that in linear functions the difference (not the ratio) between two successive values is constant (the slope).

Preview Activity 1.2. Suppose that the populations of two towns are both growing over time. The town of Exponentia is growing at a rate of 2% per year, and the town of Lineola is growing at a rate of 100 people per year. In 2014, both of the towns have 2,000 people.

(a) Complete the table for the population of each of these towns over the next several years.

	2014	2015	2016	2017	2018	2019	2020	2021	2022
Exponentia	2000								
Lineola	2000								

(b) Write a linear function for the population of Lineola. Interpret the slope in the context of this problem.

(c) The ratio of successive populations for Exponentia should be equal. For example, dividing the population in 2015 by that of 2014 should give the same ratio as when the population from 2016 is divided by the population of 2015. Find this ratio. How is this ratio related to the 2% growth rate?

(d) Based on your data from part (a) and your ratio in part (c), write a function for the population of Exponentia.

(e) When will the population of Exponentia exceed that of Lineola?

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Exponential Functions

Consider the example where the population of a bacteria colony is doubling every week. If in the first week there are 100 bacteria, then there are 200 bacteria by the end of the second week, 400 by the end of the third and so on. In Table 1.2 and Equation (1.1) we can see a simple way to model this type of growth.

Week	Bacteria
0	100
1	$100 \cdot 2 = 200$
2	$200 \cdot 2 = 100 \cdot 2^2 = 400$
3	$400 \cdot 2 = 100 \cdot 2^3 = 800$
$\vdots$	$\vdots$

Table 1.2: Bacteria population doubling

$$\mathrm{P}(t) = 100 \cdot 2^t \quad (t = \text{number of weeks}) \tag{1.1}$$

The time, t in equation (1.1) is measured in weeks. It is easy to see that the ratio of the populations for each successive week is constant at $\mathrm{P}(t+1)/\mathrm{P}(t) = 2$. This is indicative of exponential growth. Of course, this population growth could have been modeled using time measured in days instead. The population still doubles every week so for this new model the value at $t = 7$ should be double the value at $t = 0$. Equation (1.2) shows this new model with only a slight modification adjusting for the new time measurement.

$$\mathrm{P}(t) = 100 \cdot 2^{t/7} \quad (t = \text{number of days}) \tag{1.2}$$

This type of modeling and thought process can be used to describe most exponential growth and decay situations. One general formula for an exponential function is

$$f(x) = \mathrm{A} \cdot r^{kx}. \tag{1.3}$$

where A is some given initial value, r is the common ratio, and k is a constant given by the frequency in which the common ratio is applied. In the previous population doubling example, $\mathrm{A} = 100$, $r = 2$, and $k = 1/7$.

A few simple guidelines should make it clear when an exponential function is modeling growth or decay.

- If $r > 1$ then the function exhibits exponential growth.
- If $0 < r < 1$ then the function exhibits exponential decay.
- If a population is growing by $p\%$ per unit time, then $r = 1 + p/100$.
- If a population is decreasing by $p\%$ per unit time, then $r = 1 - p/100$.

Activity 1.5.

Consider the exponential functions plotted in Figure 1.5

(a) Which of the functions have common ratio $r > 1$?

(b) Which of the functions have common ratio $0 < r < 1$?

(c) Rank each of the functions in order from largest to smallest r value.

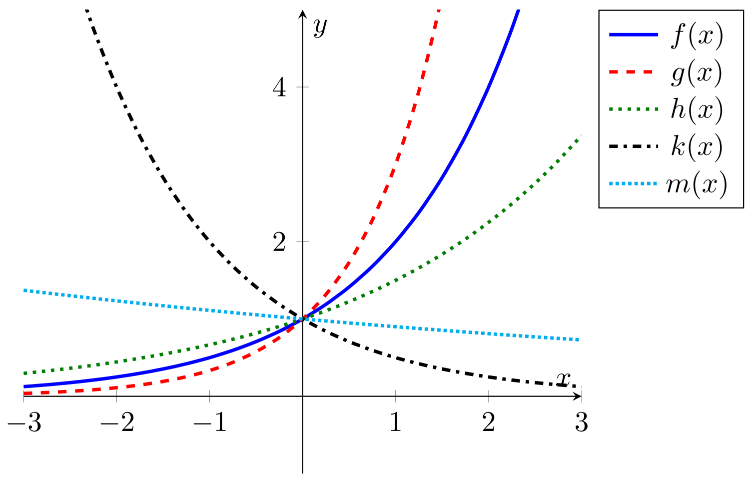

Figure 1.5: Exponential growth and decay functions

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Example 1.5. One application to exponential decay is to calculate the intensity of radiation from radioactive isotopes. Most isotopes emit particles and decay into stable forms. We measure the rate of decay from the particles by the isotope's **half-life**, which is how long it takes half of the isotope to decay. The half-life for Sodium-25 (Na^{25}) is almost exactly one minute. Write a function that models that amount of Na^{25} over time if you start with exactly 36 grams.

Solution. If you begin with 36 grams of Na^{25} then the number of grams remaining after t minutes, $S(t)$, can be represented by the function

$$S(t) = 36\left(\frac{1}{2}\right)^t,$$

where t is measured in minutes. Figure 1.6 shows this exponential decay function with an initial value of 36 and a value of 18 after 1 day.

Activity 1.6.

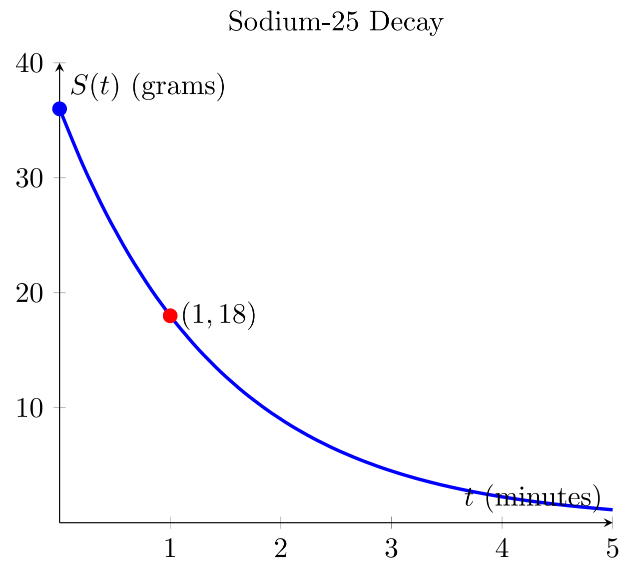

Figure 1.6: The grams of Sodium-25 remaining as a function of time. The blue point represents the initial value $(0, 36)$ and the red point represents the value after 1 minute $(1, 18)$.

A sample of $\mathrm{N}i^{56}$ has a half-life of 6.4 days. Assume that there are 30 grams present initially.

(a) Write a function describing the number of grams of $\mathrm{N}i^{56}$ present as a function of time. Check your function based on the fact that in 6.4 days there should be 50% remaining.

(b) What percent of the substance is present after 1 day?

(c) What percent of the substance is present after 10 days?

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Activity 1.7.

Uncontrolled geometric growth of the bacterium *Escherichia coli (E. Coli)* is the theme of the following quote taken from the best-selling author Michael Crichton's science fiction thriller, *The Andromeda Strain:*

> "The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."

(a) Write an equation for the number of E. coli cells present if a single cell of E. coli divides every 20 minutes.

(b) How many E. coli would there be at the end of 24 hours?

(c) The mass of an E. coli bacterium is 1.7×10^{-12} grams, while the mass of the Earth is 6.0×10^{27} grams. Is Michael Crichton's claim accurate? Approximate the number of hours we should have allowed for this statement to be correct?

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Investments

Interest bearing bank accounts and investments follow exponential growth and decay models. In the case of a savings account the interest is typically compounded several times per year. This means that the investor is getting interest on their interest every time the bank computes the interest.

If the money is gaining $p\%$ interest compounded n times per year then the common ratio for the exponential function is $1+p/n$. The exponent needs to reflect the fact that the interest occurs at monthly intervals. This means that the exponential function is

$$\mathrm{A}(t) = \mathrm{A}_0\left(1+\frac{p}{n}\right)^{nt} \quad (t = \text{number of years}). \tag{1.4}$$

In Equation (1.4), A_0 is the initial investment, $\mathrm{A}(t)$ is the value of the investment over time, p is the interest rate, and n is the number of times the interest is compounded per year.

Example 1.6. If \$100 are invested into a bank account earning 2% interest compounded 12 times per year, how much does the investor have at the end of 1 year? 5 years? at retirement age? How does this change is we compound quarterly or daily instead of monthly?

Solution. In the present situation the function modeling the value of the investment is

$$\mathrm{A}(t) = 100\left(1+\frac{0.02}{12}\right)^{12t}.$$

Table 1.3 shows the value of the investment over the first 5 years. It is clear that this is very slow growth, but it is exponential none the less. The common ratio in this case is $r = (1+0.02/12) \approx 1.0017$, and this means that you are really gaining 0.17% interest per month.

Year	0	1	2	3	4	5
Value	\$100	\$102.02	\$104.08	\$106.18	\$ 108.32	\$ 110.51

Table 1.3: Value of \$100 investment for the first 5 years

Assume that our investor was an 18 year old and extrapolate this to retirement age, let's say 65 years old. That is 47 years worth of interest, and the initial \$100 investment becomes

$$\mathrm{A}(47) = 100\left(1+\frac{0.02}{12}\right)^{12\cdot 47} \approx \$256.$$

If the number of times the bank compounds the interest changes the function will still have essentially the same form: $A(t) = 100(1+\frac{0.02}{n})^{nt}$. In Table 1.4 the same investment is considered for several values of n. While more compoundings per year generally gives a higher rate of return on the investment, the impact is small for larger values of n.

Year	0	1	2	3	4	5	$\cdots$	47
Value ($n=1$)	\$100	\$102.00	\$104.04	\$106.12	\$ 108.24	\$ 110.41	$\cdots$	\$253.63
Value ($n=4$)	\$100	\$102.02	\$104.07	\$106.17	\$ 108.31	\$ 110.49	$\cdots$	\$255.40
Value ($n=12$)	\$100	\$102.02	\$104.08	\$106.18	\$ 108.32	\$ 110.51	$\cdots$	\$255.80
Value ($n=365$)	\$100	\$102.02	\$104.08	\$106.18	\$ 108.33	\$ 110.52	$\cdots$	\$255.99

Table 1.4: Value of \$100 investment for various values of n.

Exponential Functions with Base e

Exponential functions are commonly written with a base of $e \approx 2.718281828459045\ldots$. This may seem like an arbitrary and bizarre choice at first glance, but we will see that this famous number (called Euler's Number [1]) plays a central role in Calculus.

Euler's number can be derived from Equation (1.4) if we assume that a fictitious bank gives 100% interest compounded infinitely many times per year on a one dollar investment. Mathematically this is written as

$$e = 1\cdot\left(1+\frac{1}{n}\right)^n \text{ as } n\to\infty. \tag{1.5}$$

n	1	10	100	1000	$\cdots$	10^{10}	$\cdots$
$(1+\frac{1}{n})^n$	2	2.5935	2.7048	2.7169	$\cdots$	2.71828	$\cdots$

Table 1.5: Approximations of Euler's number, e, using equation (1.5) with various values of n

Any exponential function can be rewritten in terms of Euler's number in the form

$$f(x) = Ae^{kx}. \tag{1.6}$$

In Equation (1.6), k is called the **continuous rate**.

- If $k>0$ then $f(x) = Ae^{kx}$ models exponential growth.
- If $k<0$ then $f(x) = Ae^{kx}$ models exponential decay.

[1] Euler's number is named after the famous 17^{th} century mathematician Leonhard Euler. Euler was the first mathematician to introduce the notion of a function, and he is responsible for a large amount of the development of Calculus.

Example 1.7. A population of a city is 5000 people and is doubling in size every 5 years. Use equations (1.3) and (1.6) to write two different functions modeling this population; one with base 2 and one with base e.

Solution. If the population is doubling every 5 years we can use equation (1.3) to write

$$\mathrm{P}(t) = 5000 \cdot 2^{t/5}.$$

In order to use equation (1.6) we need to find the value of "k". This is done by using the fact that at year 5 the population will be 10000 and solving the equation

$$10000 = 5000 \cdot e^{5k}.$$

Rearranging we see that $e^{5k} = 2$. In order to solve this algebraic equation we need to use logarithms. These important functions will be discussed in more detail in the logarithms section In this case we see that $k = \ln(2)/5 \approx 0.139$. Therefore,

$$\mathrm{P}(t) = 5000 \cdot e^{0.139t}.$$

Since these two equations model the same population they must be identical. Indeed,

$$5000 \cdot 2^{t/5} = 5000 \cdot \left(2^{1/5}\right)^t \approx 5000 \cdot (1.149)^t,$$

and

$$5000 \cdot e^{kt} = 5000 \cdot \left(e^k\right)^t \approx 5000 \cdot (1.149)^t.$$

Summary

In this section, we encountered the following important ideas:

- An exponential function can be written in the form $f(x) = \mathrm{A}r^{kx}$ or $g(x) = \mathrm{A}e^{kx}$.
    - In $f(x)$, if $k > 0$ and $r > 1$ then $f(x)$ models exponential growth.
    - In $f(x)$, if $k > 0$ and $0 < r < 1$ then $f(x)$ models exponential decay.
    - In $g(x)$, if $k > 0$ then $g(x)$ models exponential growth.
    - In $g(x)$, if $k < 0$ then $g(x)$ models exponential decay.
- Exponential functions have a constant common ratio for successive time values.

Exercises

1. Suppose that $h(t) = \mathrm{A} \cdot r^t$. If $h(3) = 4$ and $h(5) = 40$,

(a) find r.

(b) find A.

(c) Does this function model exponential growth or decay? How can you tell?

2. The half-life of Br^{77} is 57 hours.

(a) If the initial amount is 150 grams, find the amount remaining after 171 hours.

(b) Write an equation to predict the amount remaining after t hours.

(c) Estimate within one hour how long it will take the amount to decrease to 10 grams.

3. Consider the data in Table 1.6

(a) Which (if any) of the functions could be linear? Explain how you know that these functions are linear, and find formulas for these functions.

(b) Which (if any) of the functions could be exponential? Explain how you know that these functions are linear, and find formulas for these functions.

x	$f(x)$	$g(x)$	$h(x)$
-2	12	16	37
-1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

Table 1.6: Data tables for $f(x)$, $g(x)$, and $h(x)$

1.3 Transformations, Compositions, and Inverses

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can new functions be generated by shifts, stretches, and transformations of well-known functions?
- How can we mathematically describe symmetric functions?
- How can we build inverse functions, and when do those functions exist?

Introduction

There are infinitely many functions that can be generated using the basic mathematical operations (addition, subtraction, multiplication, division, and exponentiation) along with *simple* functions such as roots, exponentials, and trigonometric functions. In fact, we can build entire families of functions based only on these simple building blocks.

Preview Activity 1.3. The goal of this activity is to explore and experiment with the function

$$F(x) = Af(B(x-C)) + D.$$

The values of A, B, C, and D are constants and the function $f(x)$ will be henceforth called the *parent function*. To facilitate this exploration, use the applet located at
http://www.geogebratube.org/student/m93018.

(a) Let's start with a simple parent function: $f(x) = x^2$.

(1) Fix $B = 1$, $C = 0$, and $D = 0$. Write a sentence or two describing the action of A on the function $F(x)$.

(2) Fix $A = 1$, $B = 1$, and $D = 0$. Write a sentence of two describing the action of C on the function $F(x)$.

(3) Fix $A = 1$, $B = 1$, and $C = 0$. Write a sentence of two describing the action of D on the function $F(x)$.

(4) Fix $A = 1$, $C = 0$, and $D = 0$. Write a sentence of two describing the action of B on the function $F(x)$.

(b) In part (a) you have made conjectures about what A, B, C, and D do to a parent function graphically. Test your conjectures with the functions $f(x) = |x|$ (typed `abs(x)`), $f(x) = x^3$, $f(x) = \sin(x)$, $f(x) = e^x$ (typed `exp(x)`), and any other function you find interesting.

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Function Transformations

In Preview Activity 1.3 we experimented with the four main types of function transformations. You no doubt noticed that the values of C and D *shift* the parent function and the values of A and B *stretch* the parent function. More descriptively, if $f(x)$ is a parent function and

$$F(x) = Af(B(x - C)) + D$$

then the actions of each parameter are described in Table 1.7.

Parameter	Action
A	Stretch the parent function vertically
B	Stretch the parent function horizontally
C	Shift the parent function horizontally
D	Shift the parent function vertically

Table 1.7: Actions of stretch- and shift-type transformations

---

Example 1.8. Consider the function $f(x)$ in the left-hand plot of Figure 1.7. Plot the functions $g(x) = 2f(x)$, $j(x) = f(2x)$, $h(x) = f(x) + 1$, and $k(x) = f(x - 1)$.

Solution. You should notice the following features of these solutions:

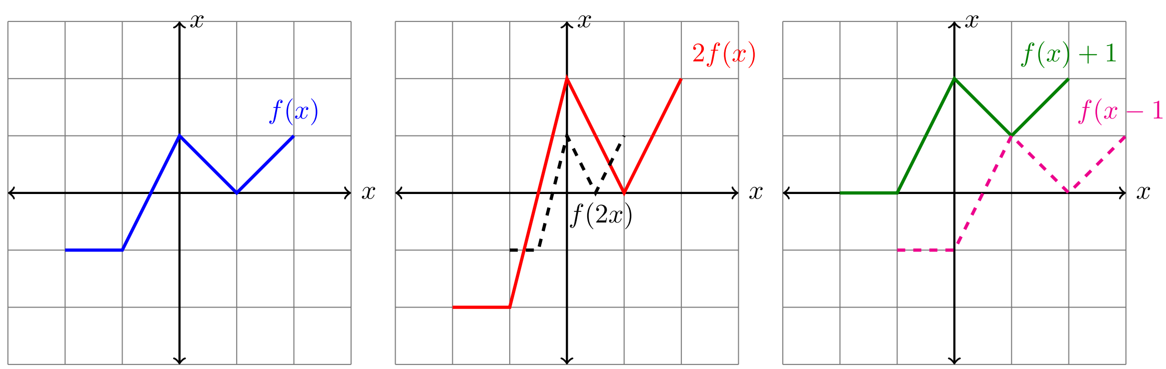

Figure 1.7: A function with transformations.

- In $g(x) = 2f(x)$, the "2" simply doubled all of the y-values from $f(x)$.
- In $j(x) = f(2x)$, the "2" actually cut all of the x-values in half from $f(x)$. This is potentially contrary to what you might expect. Verify this by substituting values in for x.

- In $h(x) = f(x) + 1$, the "+1" simply adds 1 unit to all of the y-values from $f(x)$.
- In $k(x) = f(x-1)$, the "−1" actually moves the graph of $f(x)$ to the right. This is potentially contrary to what you might expect. Verify this by substituting values in for x.

Activity 1.8.

Consider the function $f(x)$ displayed in Figure 1.8.

(a) Plot $g(x) = -f(x)$ and $h(x) = f(x) - 1$.

(b) Define the function $k(x) = -f(x) - 1$. Does it matter which order you complete the transformations from part (a) to result in $k(x)$? Plot the functions resulting from doing the two transformation in part (a) in opposite orders. Which of these functions is $k(x)$?

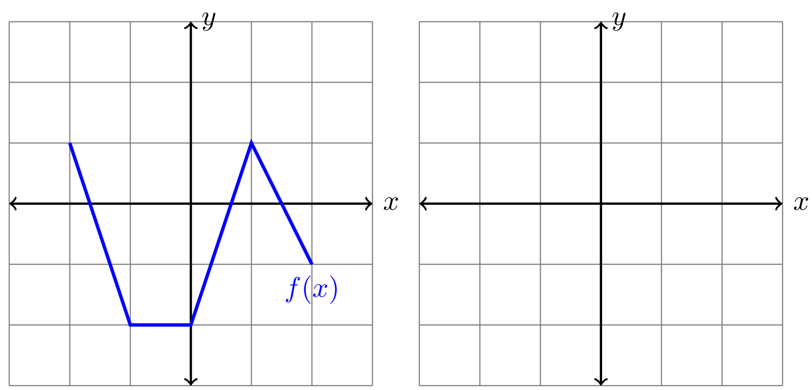

Figure 1.8: Function transformation for Activity 1.8

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Composition of Functions

When multiple transformations are applied in sequence, like in Activity 1.8, the resulting function is actually the **composition** of function transformations. The concept of a composition encompasses more than just transformations though. If $f(x)$ and $g(x)$ are functions where the range of $g(x)$ is a subset of the domain of $f(x)$ we can form a new function $h(x) = f(g(x))$. This literally means that you are substituting $g(x)$ in for every instance of the variable x in $f(x)$. For example, if $f(x) = x^2$ and $g(x) = e^x$ then $h(x) = f(g(x)) = (e^x)^2$ and $k(x) = g(f(x)) = e^{(x^2)}$.

Example 1.9. If $f(x) = x^2$ and $g(x) = x - 1$ then find $f(g(3))$, $g(f(3))$, $f(g(x))$, and $g(f(x))$.

Solution. To evaluate $f(g(3))$ we consider that $g(3) = 2$ and $g(2) = 4$. Therefore, $f(g(3)) = 4$. Similarly, $g(f(3)) = g(9) = 8$. The function compositions $f(g(x))$ and $g(f(x))$ are $f(g(x)) = (x-1)^2$ and $g(f(x)) = x^2 - 1$. Notice the difference between these resulting functions; the order that the composition takes places matters!

Activity 1.9.

(a) Let $f(x) = x^2$ and $g(x) = x + 8$. Find the following:

$f(g(3)) =$ ______________, $g(f(3)) =$ ______________, $f(g(x)) =$ ______________,

$g(f(x)) =$ ______________, $f(x)g(x) =$ ______________

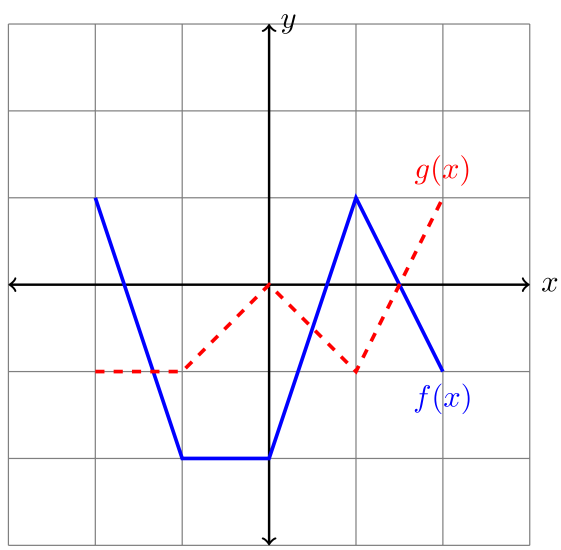
(b) Now let $f(x)$ and $g(x)$ be defined as in the table below. Use the data in the table to find the following compositions.

x	-3	-2	-1	0	1	2	3
$f(x)$	3	1	-1	-3	-1	1	3
$g(x)$	-2	-1	0	1	0	1	2

$f(-3) =$ ______________, $g(3) =$ ______________,

$f(g(-3)) =$ ______________, $f(g(f(-3))) =$ ______________

(c) Now let $f(x)$ and $g(x)$ be defined as in the plots below. Use the plots to find the following compositions.



$f(1) =$ ______________

$g(2) =$ ______________

$g(f(1)) =$ ______________

$f(g(1)) =$ ______________

$g(f(f(0))) =$ ______________

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Symmetry

There are many ways that a function can be symmetric, but two important symmetries are (1) reflective symmetry over the y-axis, and (2) 180° rotational symmetry about the origin. A function that has reflective symmetry over the y-axis is called an **even function** and a function with rotational symmetry about the origin is called an **odd function**. The reasoning for these names will be evident after completing Activity 1.10.

Activity 1.10.

(a) Based on symmetry alone, is $f(x) = x^2$ an even or an odd function?

(b) Based on symmetry alone, is $g(x) = x^3$ an even or an odd function?

(c) Find $f(-x)$ and $g(-x)$ and make conjectures to complete these sentences:

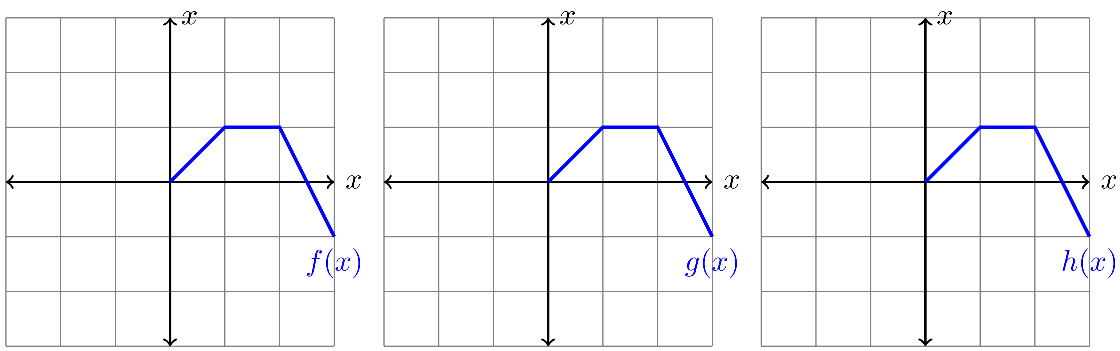
- If a function $f(x)$ is <u>even</u> then $f(-x) =$_______________.
- If a function $f(x)$ is <u>odd</u> then $f(-x) =$_______________.

Explain why the composition $f(-x)$ is a good test for symmetry of a function.

(d) Classify each of the following functions as even, odd, or neither.

$$h(x) = \frac{1}{x}, \quad j(x) = e^x, \quad k(x) = x^2 - x^4, \quad n(x) = x^3 + x^2.$$

(e) Each figure below shows only half of the function. Draw the left half so $f(x)$ is even. Draw the left half so $g(x)$ is odd. Draw the left half so $h(x)$ is neither even nor odd.



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Inverse Functions

We conclude this section by discussing an important question: If we know the action of a function is it possible to undo that action? This question can be rephrased by saying: If we know the output of a function can we tell exactly what the input was? The answer to these questions is that it depends on the type of function.

Consider, for example, the function $f(x) = x^2$. If we know that $f(a) = 4$ do we the value of a? Of course not! It is obvious that $f(2) = f(-2) = 4$, so just by knowing the output of the function $f(x) = x^2$ we cannot invert the function and find the input. What about the function $g(x) = x^3$? If we know that $g(b) = 8$ then there is only one unique value of b, $b = 2$, such that $g(b) = 8$. Therefore it seems like we can invert the cubic function.

The act of *reversing the action of a function* can be explored geometrically. Indeed, in Figure 1.9 we see that if we can simply switch the values of x and y we will get a plot that shows how to undo the action of a function. Geometrically, switching the role of the x and the y in the function is the same as reflecting over the line $y = x$.

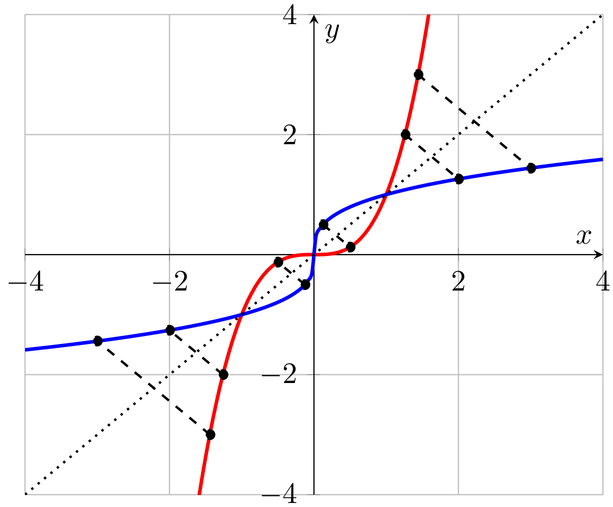

Figure 1.9: Inverse functions. If (x, y) is on one end of one of the dashed segments, then (y, x) is on the other side.

The question that remains is when an inverse function actually exists. This is the same as asking: "if I reflect over $y = x$ is the end result a function?" The answer to this question is certainly "no" if the function is $f(x) = x^2$ (as seen in the left-hand plot of Figure 1.10), but if we restrict the domain on $f(x) = x^2$ to $0 \le x < \infty$ then the result is a function (as seen in the right-hand plot of Figure 1.10). This leads us to the following results:

- If a horizontal line passes through a function only once, then it has a unique inverse found by interchanging the x and the y.
- The inverse of a function can be found geometrically by reflecting the graph of the function over the line $y = x$.

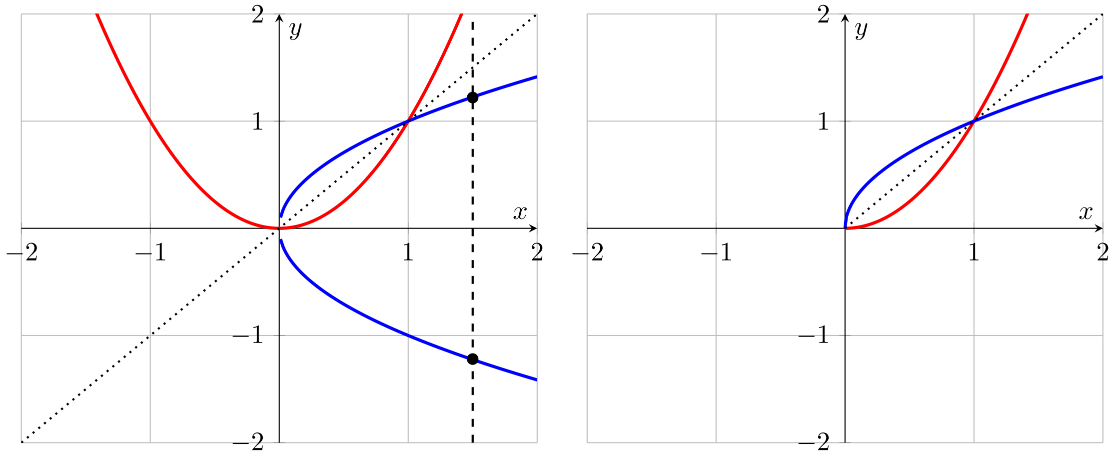

Figure 1.10: The left-hand plot shows that after reflecting $f(x) = x^2$ across $y = x$ the result is not a function. The right-hand plot shows that under a restriction of the domain the result can be a function.

Example 1.10. Find the inverse of the following functions. If necessary, restrict the domain on the function so that the inverse exists.

(a) $f(x) = x^2 + 1$

(b) $g(x) = ax + b$

(c) $h(x) = (2x + 8)^3$

Solution.

(a) To find the inverse of $f(x)$ we first interchange the x and y. Then we solve for y. That is:

$$\text{Solve for } y\text{: } x = y^2 + 1 \quad \Longrightarrow \quad y = \pm\sqrt{x-1}.$$

Obviously the resulting solution is two equations. By convention we choose the positive square root and note that the inverse only makes sense if $x \geq 1$. Hence, in order for the inverse to make sense we need a restriction on the domain of $f(x)$: If $0 \leq x < \infty$ then any horizontal line only crosses the graph of $f(x)$ once, and hence the inverse exists and is unique.

$$f^{-1}(x) = \sqrt{x-1}, \quad x \geq 1$$

(b) Interchanging the x and y in this equation gives

$$\text{Solve for } y\text{: } x = ay + b \quad \Longrightarrow \quad y = \frac{x-b}{a}.$$

There is no need to restrict the domain of $g(x)$ in this instance since the resulting equation is a function.

$$g^{-1}(x) = \frac{x-b}{a} = \frac{1}{a}x - \frac{b}{a}$$

(c) Interchanging the x and y in this equation gives

$$\text{Solve for } y\text{: } x = (2y+8)^3 \quad \implies \quad y = \frac{x^{1/3}-8}{2}.$$

In this instance there is no restriction on the domain of $h(x)$ since (as in part (b)) the resulting equation is a function.

$$h^{-1}(x) = \frac{x^{1/3}-8}{2}$$

Finally, to tie the ideas of composition and inverses together we observe that if the inverse of a function switches the roles of x and y then the composition $f^{-1}(f(x))$ should simply give x back. The logical argument is as follows:

$$f \text{ maps } x \text{ to } y \quad \text{then} \quad f^{-1} \text{ maps } y \text{ to } x$$

That is,

$$f^{-1}(f(x)) = x.$$

Similarly,

$$f^{-1} \text{ maps } x \text{ to } y \quad \text{then} \quad f \text{ maps } y \text{ to } x$$

which is written more compactly as

$$f(f^{-1}(x)) = x.$$

These two equations provide a nice algebraic check when finding inverses.

Activity 1.11.

(a) Find the inverse of each of the following functions by interchanging the x and y and solving for y. Be sure to state the domain for each of your answers.

$$y = \sqrt{x-1}, \quad y = -\frac{1}{3}x + 1, \quad y = \frac{x+4}{2x-5}$$

(b) Verify that the functions $f(x) = 3x-2$ and $g(x) = \frac{x}{3} + \frac{2}{3}$ are inverses of each other by computing $f(g(x))$ and $g(f(x))$.

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Summary

In this section, we encountered the following important ideas:

- A function can be transformed by $\mathrm{F}(x) = \mathrm{A}f(\mathrm{B}(x-\mathrm{C})) + \mathrm{D}$ where C and D shift the function and A and B stretch the function.
- If $f(-x) = f(x)$ then f is an even function.
- If $f(-x) = -f(x)$ then f is an odd function.

- To find the inverse of a function we switch the roles of the x and y variables. Geometrically this is the same as reflecting over the line $y = x$. Occasionally it is essential to restrict the domain of the original function in order for the inverse to exist.
- The composition of a function and its inverse is the original input:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Exercises

1. The functions $f(x)$ and $g(x)$ are defined in the table below. Use these function values to answer the following questions.

x	-3	-2	-1	0	1	2	3
$f(x)$	3	1	-1	-3	-1	1	3
$g(x)$	-2	-1	0	1	0	1	2

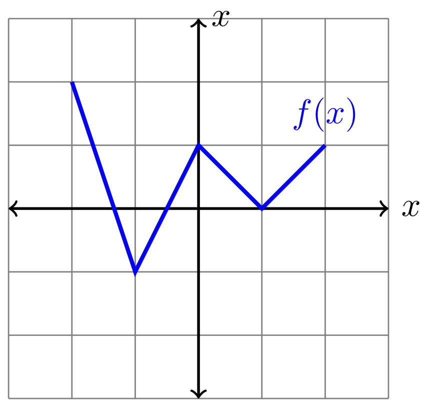
(a) $f(-3)$, (b) $g(3)$, (c) $f(g(-3))$, (d) $g(f(3))$, (e) $f(g(f(-3)))$
(f) Write a list of value of $f(-x)$ for $x = -3, -2, \ldots, 2, 3$. Based on this list is $f(x)$ an even function, an odd function, or neither?
(g) Repeat part (f) for $g(x)$.

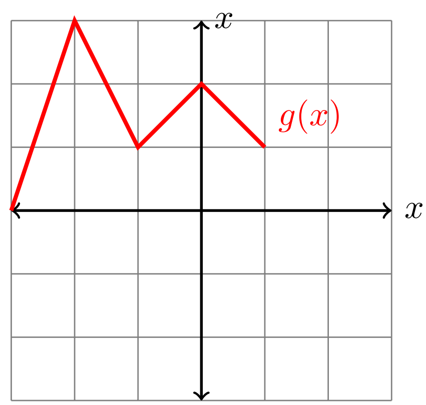
2. Find the inverse of each of the following functions. If necessary state a restriction on the domain of $f(x)$ so that the inverse actually exists.

(a) $f(x) = (2x-3)^2$

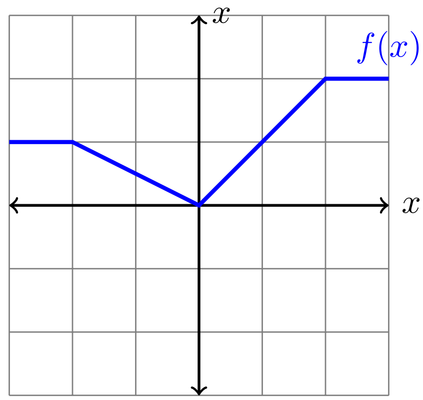
(b) $g(x) = x^2 - 2x + 1$

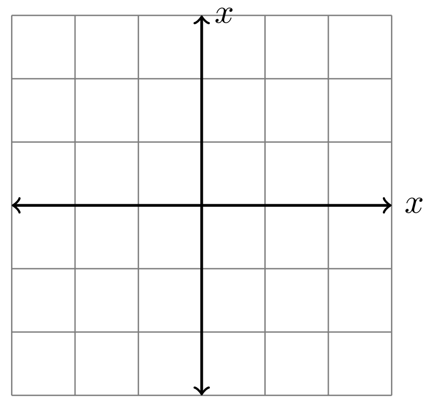
3. The plot on the left shows the function $f(x)$ and the plot on the right shows $g(x) = \mathrm{A}f(\mathrm{B}(x-\mathrm{C})) + \mathrm{D}$. Find the appropriate values of A, B, C, and D.





4. Use the function below to plot
(a) $f(x) - 3$, (b) $f(x+1)$, (c) $\frac{1}{2}f(x)$, (d) $-f(x)$, and (e) $\frac{1}{f(x)}$.


x
f(x)
x


x
x

1.4 Logarithmic Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can we "undo" the effects of exponentiation?
- How can we solve equations involving exponential and logarithmic expressions?
- What are the properties of logarithmic functions?

Introduction

In section 1.2 we studied exponential functions to model a variety of different settings. It is straightforward to verify that the graph of an exponential function passes the "horizontal line test" described in section 1.3, and so we should expect exponential functions to have corresponding inverse functions. In this section we will define the *logarithm* to be the inverse function for an exponential.

Preview Activity 1.4. Carbon-14 (^{14}C) is a radioactive isotope of carbon that occurs naturally in the Earth's atmosphere. During photosynthesis, plants take in ^{14}C along with other carbon isotopes, and the levels of ^{14}C in living plants are roughly the same as atmospheric levels. Once a plant dies, it no longer takes in any additional ^{14}C. Since ^{14}C in the dead plant decays at a predictable rate (the half-life of ^{14}C is approximately 5,730 years), we can measure ^{14}C levels in dead plant matter to get an estimate on how long ago the plant died. Suppose that a plant has 0.02 milligrams of ^{14}C when it dies.

(a) Write a function that represents the amount of ^{14}C remaining in the plant after t years.

(b) Complete the table for the amount of ^{14}C remaining t years after the death of the plant.

t	0	1	5	10	100	1000	2000	5730
^{14}C Level	0.02							

(c) Suppose our plant died sometime in the past. If we find that there are 0.014 milligrams of ^{14}C present in the plant now, estimate the age of the plant to within 50 years.

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Logarithms

Definition 1.7.

Let $b > 0$ with $b \neq 1$. The **logarithm of x with base b** is defined by

$$\log_b x = y \qquad \text{if and only if} \qquad x = b^y.$$

The expression $\log_b x$ represents the power to which b needs to be raised in order to get x. Two frequently used logarithmic functions are $\log_{10} x$ (frequently written $\log x$) and the natural logarithm $\log_e x$ (frequently written $\ln x$).

Note that we have specifically defined logarithms to be inverse functions for exponentials. For instance, $\log_{10} 1000 = 3$, since $10^3 = 1000$. Logarithmic functions give us a way to re-write exponential expressions and, more importantly, solve equations involving variables in an exponent.

Properties of Logarithms

Since logarithms and exponentials are inverse functions, many of the properties of logarithmic functions can be deduced directly from the properties of exponential functions. For example, the domain of all logarithmic functions is $(0, \infty)$ and the range of all logarithmic functions is $(-\infty, \infty)$ because those are the range and domain, respectively, of exponential functions. Similarly, logarithmic functions have a vertical asymptote at $x = 0$ because exponential functions have a horizontal asymptote at $y = 0$. These features can be seen in Figure 1.11.

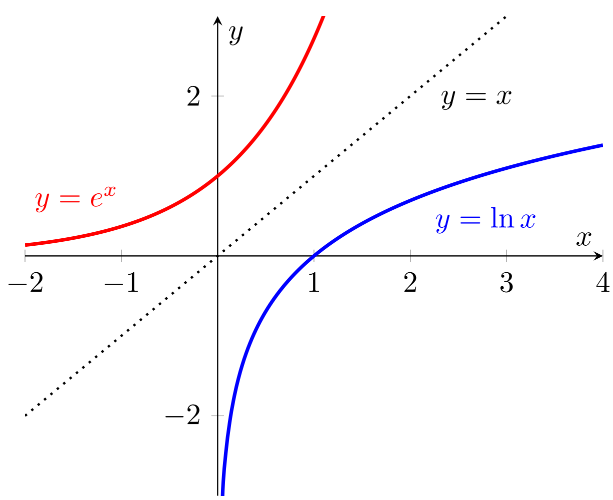


Figure 1.11: Graphs of the functions $y = e^x$ and $y = \ln x$.

The following properties of logarithms can be deduced from the properties of exponential functions and the definition of the logarithm. These properties are especially useful in simplifying or solving logarithmic and exponential equations.

Properties of logarithms: For $b > 0$, $b \neq 1$, and $x, y > 0$:

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b (xy) = \log_b x + \log_b y$
4. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
5. $\log_b x^r = r \log_b x$
6. $\log_b b^x = x$
7. $b^{\log_b x} = x$
8. $\log_b x = \log_b y$ if and only if $x = y$
9. $\log_a b = \frac{\log_c a}{\log_c b}$ (this is called the change of base formula)

Euler's number, e, shows up so often that we give a special name to the associated logarithm. The logarithm $\log_e(x)$ is called the natural logarithm and is written as $\ln(x)$. We will see later in this course that the exponential function $f(x) = e^x$ has very special calculus properties and as such the natural logarithm has very special properties as well. With this in mind, most mathematicians and scientists use e^x and $\ln(x)$ as their preferred exponential and logarithmic functions.

Activity 1.12.

Use the definition of a logarithm along with the properties of logarithms to answer the following.

(a) Write the exponential expression $8^{1/3} = 2$ as a logarithmic expression.

(b) Write the logarithmic expression $\log_2 \frac{1}{32} = -5$ as an exponential expression.

(c) What value of x solves the equation $\log_2 x = 3$?

(d) What value of x solves the equation $\log_2 4 = x$?

(e) Use the laws of logarithms to rewrite the expression $\log\left(x^3 y^5\right)$ in a form with no logarithms of products, quotients, or powers.

(f) Use the laws of logarithms to rewrite the expression $\log\left(\frac{x^{15} y^{20}}{z^4}\right)$ in a form with no logarithms of products, quotients, or powers.

(g) Rewrite the expression $\ln(8) + 5\ln(x) + 15\ln(x^2 + 8)$ as a single logarithm.

◁

Example 1.11. Find a value of x for which $3^x = 13$.

Solution. To isolate the variable x, we should take the logarithm of both sides. For convenience, let's choose to use the natural logarithm.

$$\ln(3^x) = \ln(13).$$

Applying logarithm property (5), we find

$$x \ln 3 = \ln 13.$$

Solving algebraically for x yields

$$x = \frac{\ln 13}{\ln 3} \approx 2.3347.$$

NOTE: Our choice to use the natural logarithm was arbitrary. We could have chosen any base for our logarithm to solve this equation.

Example 1.12. In 1970, the population of the United States was approximately 205.1 million people. Since that time, the population has grown at a continuous growth rate of approximately 1.05%. Assuming that this growth rate continues, when would we expect the population of the United States to reach 350 million?

Solution. Since the rate of growth of the population is proportional to the size of the population, we should use an exponential model for this problem. That is, we want

$$\mathrm{P}(t) = \mathrm{P}_0 e^{rt}$$

where t is the number of years after 1970, $\mathrm{P}(t)$ is the population (in millions) of the United States at time t, P_0 is the population (in millions) of the United States in 1970 (i.e. $t = 0$) and r is the continuous rate of growth of the population. To determine when the population will reach 350 million, we must solve the equation

$$350 = 205.1 e^{0.0105t}.$$

To solve for t, we need to first solve for the exponential expression by itself and then use logarithms. Dividing both sides of the equation by 205.1 gives

$$\frac{350}{205.1} = e^{0.0105t}.$$

Taking the natural logarithm of both sides gives

$$\ln\left(\frac{350}{205.1}\right) = \ln\left(e^{0.0105t}\right).$$

Applying logarithm property (6), we find

$$\ln\left(\frac{350}{205.1}\right) = 0.0105t.$$

Finally, solving algebraically for t gives

$$t = \frac{1}{0.0105}\ln\left(\frac{350}{205.1}\right) \approx 50.9\,\text{years}.$$

Thus, we expect the population of the United States to reach 350 million in 2021 (approximately 51 years after 1970).

Activity 1.13.

Solve each of the following equations for t, and verify your answers using a calculator.

(a) $\ln t = 4$

(b) $\ln(t+3) = 4$

(c) $\ln(t+3) = \ln 4$

(d) $\ln(t+3) + \ln(t) = \ln 4$

(e) $e^t = 4$

(f) $e^{t+3} = 4$

(g) $2e^{t+3} = 4$

(h) $2e^{3t+2} = 3e^{t-1}$

◁

Activity 1.14.

Consider the following equation:

$$7^x = 24$$

(a) How many solutions should we expect to find for this equation?

(b) Solve the equation using the log base 7.

(c) Solve the equation using the log base 10.

(d) Solve the equation using the natural log.

(e) Most calculators have buttons for $\log_{10}$ and ln, but none have a button for $\log_7$. Use your previous answers to write a formula for $\log_7 x$ in terms of $\log x$ or $\ln x$.

◁

Activity 1.15.

(a) In the presence of sufficient resources the population of a colony of bacteria exhibits exponential growth, doubling once every three hours. What is the corresponding continuous (percentage) growth rate?

(b) A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling so its temperature, T (measured in degrees Fahrenheit) after t minutes is given by

$$T(t) = 65 + 186e^{-0.06t}.$$

How long will it take from the time the food is served until the temperature is 120°F?

(c) The velocity (in ft/sec) of a sky diver t seconds after jumping is given by

$$v(t) = 80\left(1 - e^{-0.2t}\right).$$

After how many seconds is the velocity 75 ft/sec?

◁

Summary

In this section, we encountered the following important ideas:

- A logarithmic function can be written in the form $f(x) = \log_b x$ where $b > 0$, $b \neq 1$, and $x > 0$.
- Logarithmic functions are defined to be inverse functions for exponentials. That is

$$\log_b x = y \qquad \text{if and only if} \qquad x = b^y.$$

- Solving equations that contain exponential expressions frequently requires the use of logarithms; solving equations that contain logarithmic expressions frequently requires the use of exponentials.

Exercises

1. Use the laws of logarithms to rewrite the expression

$$\ln\left(x^{13}\sqrt{\frac{y^9}{z^2}}\right)$$

in a form with no logarithm of a power, product, or quotient.

2. Solve $\ln(5x^2 + 2) = 4$ for x. Give only an exact answer (no decimal approximations).

3. A wooden artifact from an ancient tomb consists of 20% of the carbon-14 that is present in living trees. How long ago was the artifact made assuming that the half-life of carbon-14 is 5,730 years?

1.5 Trigonometric Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can we model systems that vary in a smooth, wavelike cycle, rising and falling again and again?
- How can we model the shape of waves in water, sound waves, radio waves, the motion of the tides in and out over the course of a day, the shaking of an earthquake, or the varying time of sunrise over the course of a year?

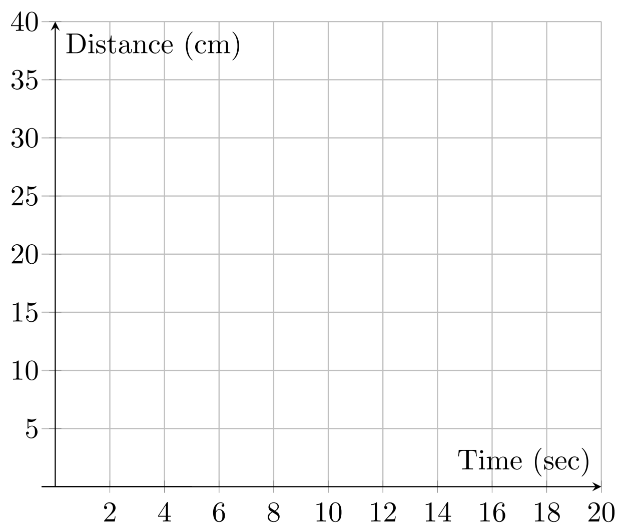
Introduction

You probably first learned about sines, cosines, and tangents when you were studying triangles. However, these functions are amazingly useful in an enormous variety of contexts. These functions are so handy that scientists and mathematicians always keep them in mind as part of our standard toolbox. We use sines and cosines whenever we see anything that varies in a smooth wave cycle, going up and down by the same amount, again and again on a regular basis.

Preview Activity 1.5. A tall water tower is swaying back and forth in the wind. Using an ultrasonic ranging device, we measure the distance from our device to the tower (in centimeters) every two seconds with these measurements recorded below.

Time (sec)	0	2	4	6	8	10	12	14	16	18	20
Distance (cm)	30.9	23.1	14.7	12.3	17.7	26.7	32.3	30.1	21.8	13.9	12.6

(a) Use the coordinate plane below to create a graph of these data points.



(b) What is the water tower's maximum distance away from the device?

(c) What is the smallest distance measured from the tower to the device?

(d) If the water tower was sitting still and no wind was blowing, what would be the distance from the tower to the device? We call this the tower's equilibrium position.

(e) What is the maximum distance that the tower moves away from its equilibrium position? We call this the amplitude of the oscillations.

(f) How much time does it take the water tower to sway back and forth in a complete cycle? We call this the period of oscillation.

$\bowtie$

Measuring Angles with Radians

Sines, cosines, and tangents are very useful when studying triangles. The input into each of these functions is an angle, and the output tells us the ratio of the lengths of the sides of the triangle. There are two commonly used units for measuring angles, degrees and radians, and so there are two commonly used versions of the trigonometric functions. There's $\sin x$ where x is in degrees, and there's $\sin x$ where x is in radians. Calculus is a lot easier if we measure angles in radians, so that's what we'll use throughout this course. If you ever have trouble getting the right numbers from your calculator, you may want to double check that your calculator is in radian mode.

So, what is a radian?

Definition 1.8.

A **radian** is a measure of angle which is defined so that if we have an angle with a size of one radian on a unit circle (with a radius $r = 1$), then the length of the arc along the circumference of the circle is also equal to one, as we see in Figure 1.12. Because the circumference of a circle is $2\pi r$, this means that for one complete circle,

$$360^\circ = 2\pi \text{ radians.}$$

Similarly half a circle is

$$180^\circ = \pi \text{ radians}$$

and a right angle is

$$90^\circ = \frac{\pi}{2} \text{ radians.}$$

So one radian is

$$57.3^\circ \approx \frac{180^\circ}{\pi} = 1 \text{ radian.}$$

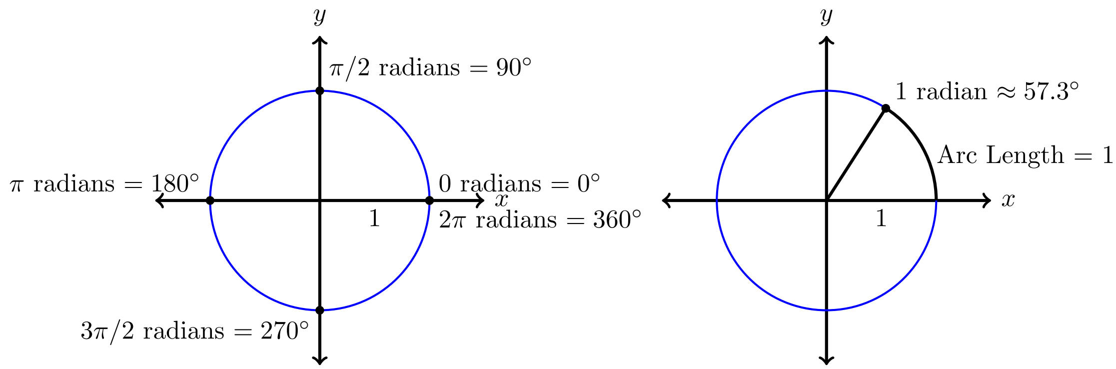


Figure 1.12: Common radian measures.

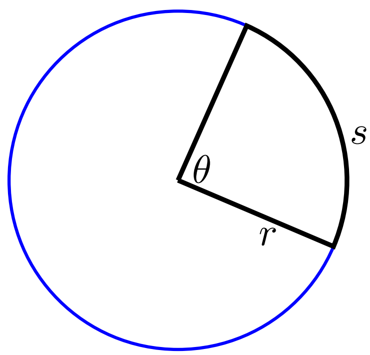


Figure 1.13: Arc length, angle, and radius on a circle.

Because we define the radian in this way, this means that the arc length s along the circumference of a circle with radius r over angle θ can be calculated as $s = r\theta$ as long as the angle θ is measured in radians.

Sine and Cosine on the Unit Circle

Let's draw a unit circle with its center at the origin and think about a point moving along the circumference of this circle. We start with the point on the x axis with coordinates $(1,0)$, as shown in the figure below, and define this location to be an angle of $\theta = 0$ radians. Then, we let this point move up, so that our point is at an angle θ above the x axis. The sine and cosine functions are defined so that they give us the coordinates of our point:

$$x = \cos\theta \quad \text{and} \quad y = \sin\theta$$

This means that an angle of $\theta = 2\pi$ carries us around one full circle and brings us back to our starting point on the x axis again, with coordinates $(1,0)$. That means that $\sin 2\pi = \sin 0 = 0$. Similarly $\theta = 4\pi$ carries us around two full circles, and $\theta = 6\pi$ carries us around three full circles.

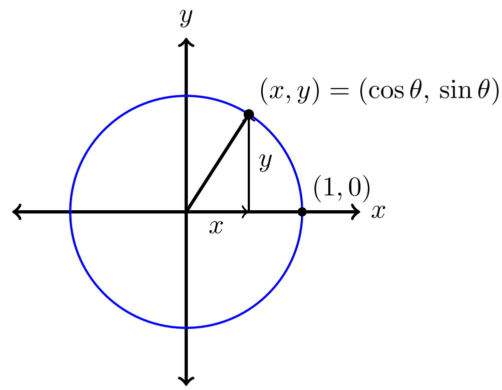


Figure 1.14: Sine and cosine on a unit circle.

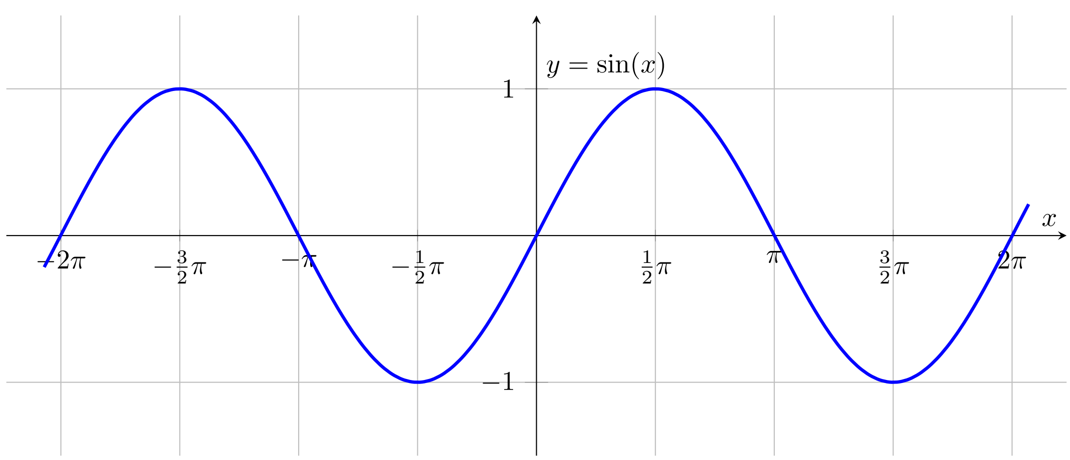
Next we can use the Pythagorean Theorem, and remember that our hypotenuse is equal to one to see that

$$\sin^2\theta + \cos^2\theta = 1$$

This is a very useful relationship!

Sine and Cosine as Functions

To get beyond trigonometry, rather than using the angle θ as the input to our sine and cosine functions, instead we will put our function input on the x axis. Then we can plot the output of on the y axis. This produces the following graph:

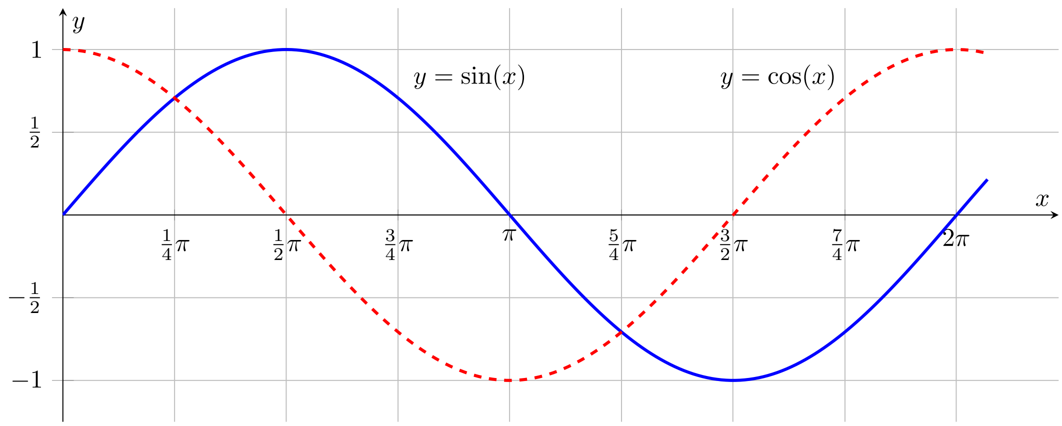


Here we can see that the range (output) of the sine function is the interval from -1 to $+1$. ($\sin x$ can never equal 2!) The domain (input) of the sine function is extends from $-\infty$ to $+\infty$, but the cycle repeats every 2π along the x axis. (Do you understand how the definition of sine on the unit circle makes both of these facts true?)

The following terms will be very important when we describe functions like this:

- The *period* of a function is how far along the x axis it takes to complete one full cycle.
- The *amplitude* of a function is how far it goes on the y axis above and below its average value.

The function $f(x) = \sin x$ has a period of 2π and an amplitude of 1. If we plot both the sine and the cosine functions together we see the following graph:



From this we see that the function $g(x) = \cos x$ also has a period of 2π and an amplitude of 1. The difference is that $\sin 0 = 0$, so the sine function starts at its average value, halfway between a peak and a trough. On the other hand $\cos 0 = 1$, so the cosine function starts at a peak. This means that we can turn a cosine function into a sine function and visa versa simply by shifting:

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right) \quad \text{and} \quad \cos(x) = \sin\left(x + \frac{\pi}{2}\right).$$

Sinusoidal Functions in the Real World

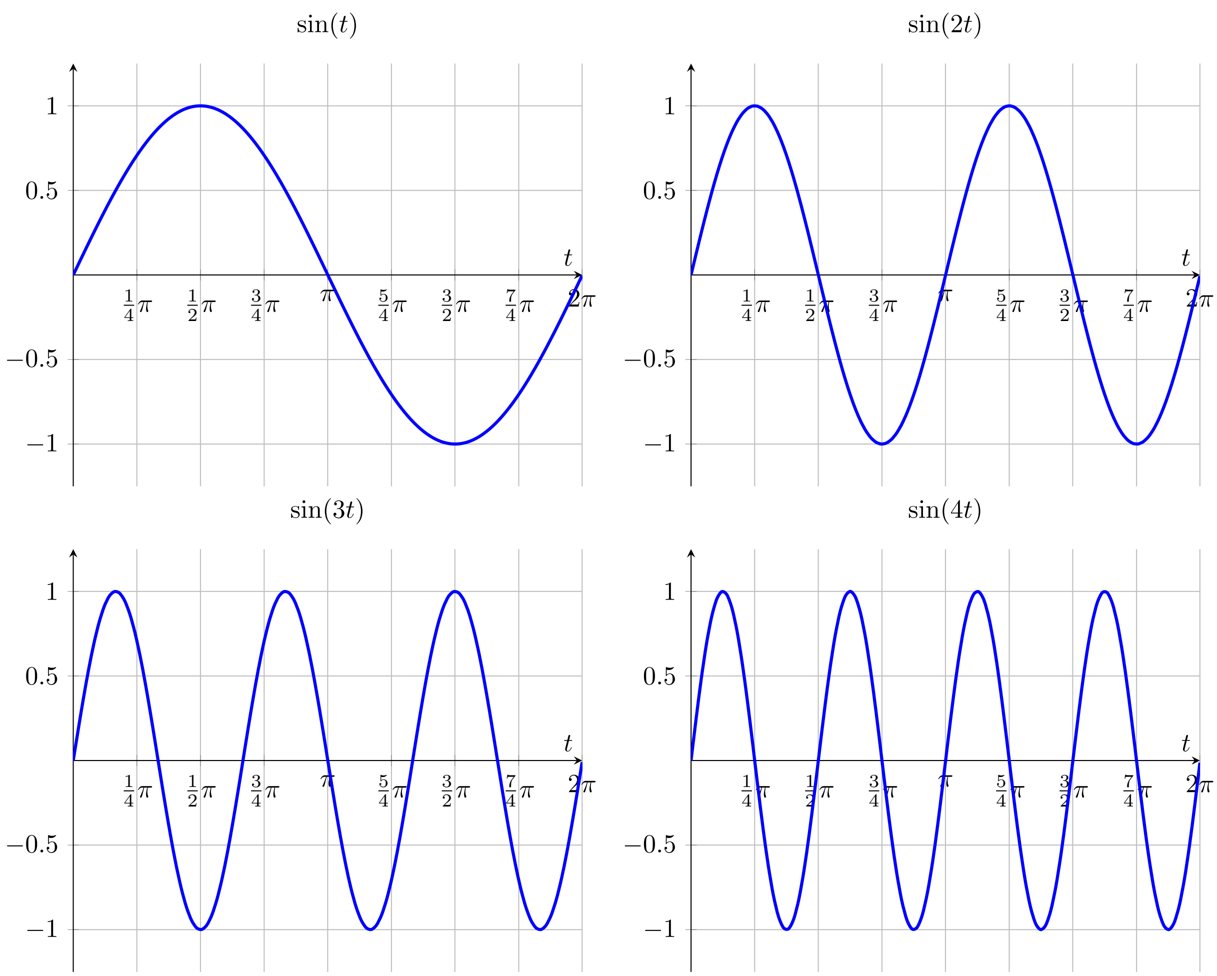
To model real data with the sine function, we must be able to change the amplitude, the period, and the average value of our wave, to get what we call a *sinusoidal* function. Every sinusoidal function can be written either of these two forms:

$$f(t) = A\sin(B(t - t_0)) + C \text{ or } f(t) = A\sin(Bt + \phi) + C$$

- A is the amplitude.
- B is the angular frequency, which determines the period, with $B = \frac{2\pi}{\text{Period}}$.
- C is the average value.
- t_0 is the shift along the t axis, a time when f is at an average value and increasing

- ϕ is the shift in radians, the angle at which the oscillations begin

The parameter B can be a little surprising. Because B is inversely related to the period, this means that larger values of B result in a shorter period, and smaller values of B result in a longer period, as we see in the graphs below:

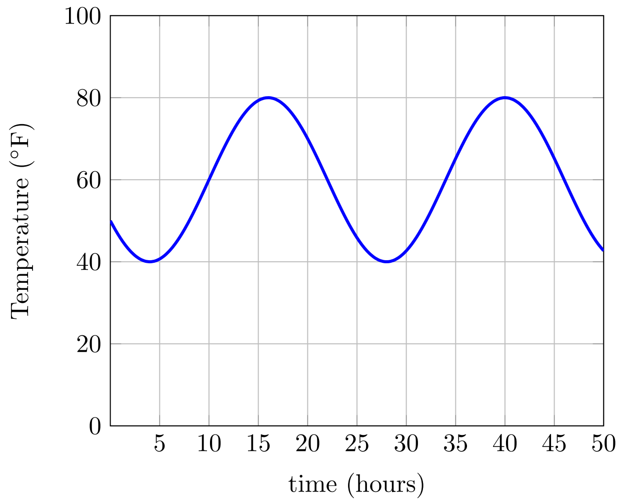


Example 1.13. Suppose we measure the temperature every hour throughout a day and find that T varies in a smooth sinusoidal pattern. We find that the average temperature is 60°, the amplitude is 20°, and the period is 24 hours. The minimum temperature is at 4am, the maximum temperature is at 4pm, and so it is at the average temperature and increasing at 10am. How would we model the temperature as a sinusoidal function?

Solution. Given the information above we could model these temperatures with the following formula:

$$T(t) = 20\sin\left(\frac{2\pi}{24}(t-10)\right) + 60.$$

A graph of the function looks like this:



Notice that the maximum temperature is 80° and the minimum temperature is 40°. We could tell this directly from the formula, because output of the sine function varies between −1 and +1, and this is multiplied by 20. As a result, the most we ever add to 60 is 20 to get a maximum temperature of 80, and the most we ever subtract from 60 is 20, to get a minimum temperature of 40.

Activity 1.16.

Figure 1.15 gives us the voltage produced by an electrical circuit as a function of time.

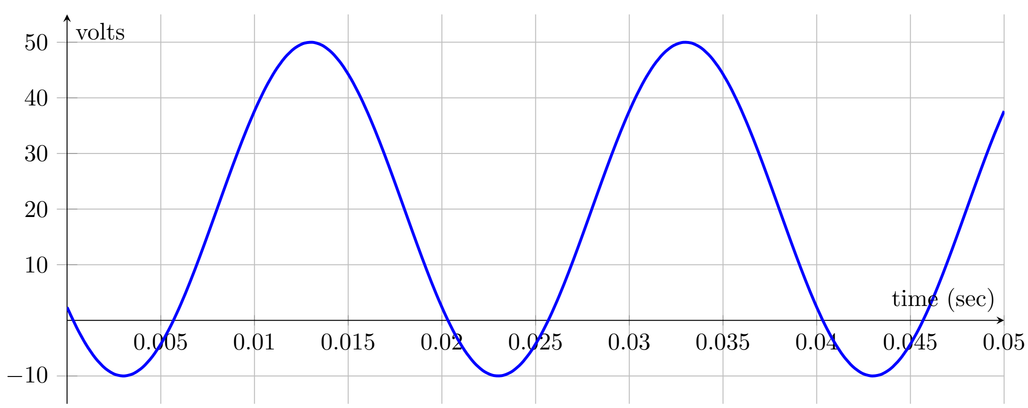


Figure 1.15: Voltage as a function of time.

(a) What is the amplitude of the oscillations?

(b) What is the period of the oscillations?

(c) What is the average value of the voltage?

(d) What is the shift along the t axis, t_0?

(e) What is a formula for this function?

◁

Units

Whenever we use mathematics in the real world, most numbers have units, like meters, minutes, dollars, pounds, or degrees. Units are a very useful tool that helps us understand the meaning of the numbers that we are using. For example, suppose we use the following sinusoidal function to model the water level on a pier in the ocean as it changes due to the tides during a certain day.

$$w(t) = 4.3\sin(0.51t + 0.82) + 10.6$$

This function isn't very useful to us unless we know what units the input t must have (Minutes? Seconds? Hours? Days?) and what units the output w will have (Centimeters? Feet? Meters? Yards?). In this case t is in hours since midnight, and w is in feet. Now, we can evaluate the function at noon to find that the water level is $w(12) = 13.3$ feet.

Here's how units work in equations:

- If two numbers are equal, or if we add or subtract two numbers, they must have the same units: 3 seconds plus 5 feet doesn't make any sense.
- If we multiply or divide two numbers, both units go into the result: 6 meters divided by 2 seconds equals a speed of 3 meters per second.

The parameters in our water level function ($A = 4.3$, $B = 0.51$, $\phi = 0.82$, $C = 10.6$) all have units, which help us interpret their meaning. The sine, cosine, and tangent functions all take angles in radians as their input, and return numbers with no units as output. $C = 10.6$ must be in feet, because we add it to something else, and then it equals the output, which is in feet. Similarly $A = 4.3$ must also be in feet, because the sine function does not have any units on its output. $\phi = 0.82$ must be in radians, because we add it to something else and use it as input to the sine function. Similarly $0.51t$ must be in radians. However, we know that t is in hours, so $B = 0.51$ must have units of radians per second.

Activity 1.17.

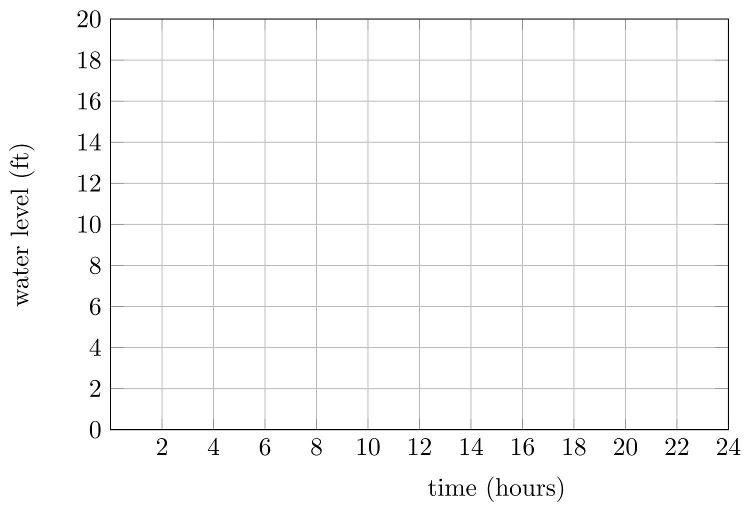
Suppose the following sinusoidal function models the water level on a pier in the ocean as it changes due to the tides during a certain day.

$$w(t) = 4.3\sin(0.51t + 0.82) + 10.6$$

(a) Using the formula above, make a table showing the water level every two hours for a 24 hour period starting at midnight.

time (hours)	0	2	4	6	8	10	12	14	16	18	20	22	24
water level (ft)													

(b) Sketch a graph of this function using the data from your table in part (a).



(c) What is the period of oscillation of this function?

(d) What time is high tide?

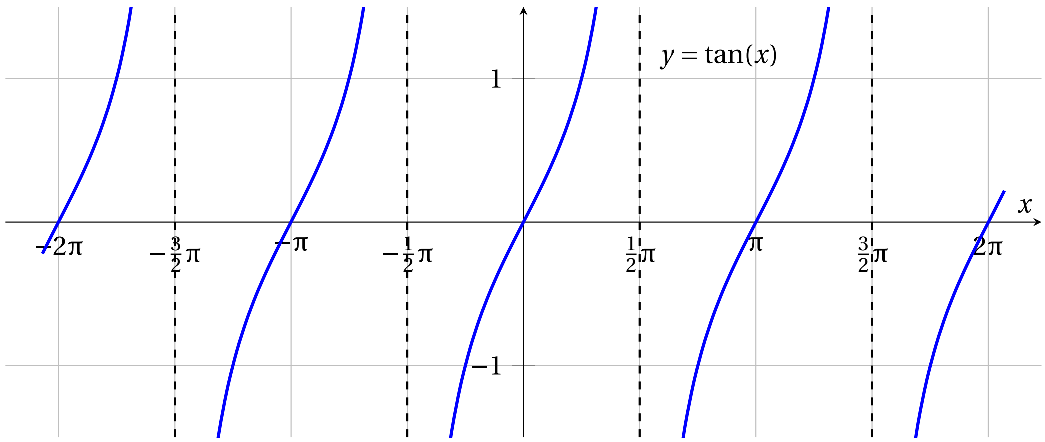
$\triangleleft$

The Tangent Function

The tangent function has a completely different shape than the sine and cosine functions because it is defined to be

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

as long as $\cos\theta \neq 0$, so that we don't have a divide-by-zero problem. This means that the tangent function is undefined at $\pm\pi/2, \pm 3\pi/2, \cdots$, and the function has vertical asymptotes at these points.

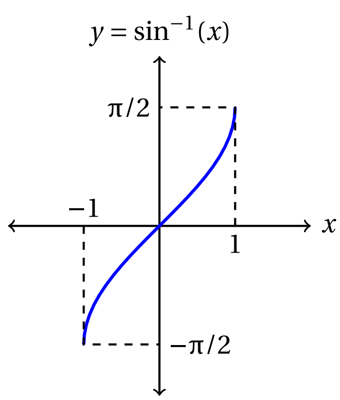


Because the tangent function blows up to infinity, we don't often fit tangent functions to real world data.

Inverse Sine Function

Suppose we want to find the value of x so that $\sin x = 0.75$. To find this, we use an inverse sine function, written as either arcsin or $\sin^{-1}$. In this case $\sin^{-1} 0.75 \approx 0.848$, because $\sin 0.848 \approx 0.75$. The range of the sine function only goes from -1 to $+1$, so that means the domain of arcsin only goes from -1 to $+1$. It is impossible to find a solution to $\sin x = 3$, because the sine function doesn't go that high!

It is important to remember that the sine function repeats itself in the same pattern again and again, every 2π, so $x = 0.848$ is not the only solution to $\sin x = 0.75$. Another solution is $x = 0.848 + 2\pi \approx 7.13$. Another solution is $x = 0.848 + 4\pi \approx 13.4$.. And of course we could subtract multiples of 2π as well to get $x = 0.848 - 2\pi \approx -5.44$. As a result, we have the arcsin function output the value between $-\pi/2$ and $+\pi/2$. This means that the arcsine function has a very limited domain and range, only existing for $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$.



Domain: $-1 \le x \le 1$, Range: $-\pi/2 \le y \le \pi/2$

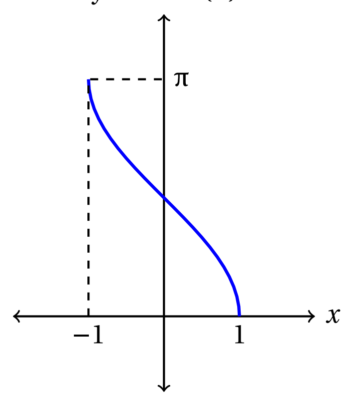
Warning: Even though we write the inverse sine function as $\sin^{-1} x$, it is a completely different thing than $1/\sin x$.

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

Inverse Cosine Function

We can define a similar inverse function for the cosine, which we call arccos or $\cos^{-1}$. The domain of this function is $-1 \leq x \leq 1$, and we choose the range to be $0 \leq y \leq \pi$.

$y = \cos^{-1}(x)$

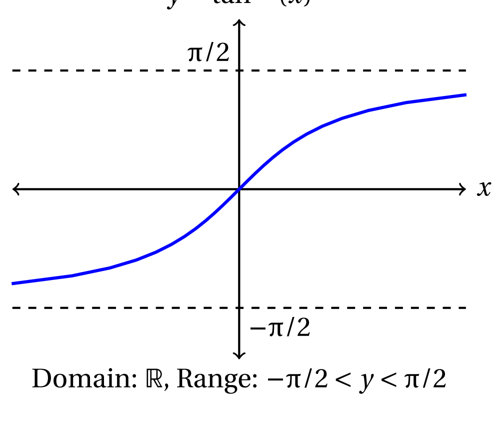


Domain: $-1 \leq x \leq 1$, Range: $0 \leq y \leq \pi$

Inverse Tangent Function

Recall that the tangent function has a range going all the way from negative infinity to positive infinity, as x goes from $-\pi/2$ to $+\pi/2$. As a result, the inverse tangent function has a domain of $-\infty < x < \infty$, and a range of $-\pi/2 \leq y \leq \pi/2$.

$y = \tan^{-1}(x)$



Domain: $\mathbb{R}$, Range: $-\pi/2 < y < \pi/2$

Summary

In this section, we encountered the following important ideas:

- Trigonometric functions are utilized to model periodic behavior such as tides, sound waves, or voltage through an electrical circuit.
- Converting between radian measure and degree measure can be achieved by remembering that

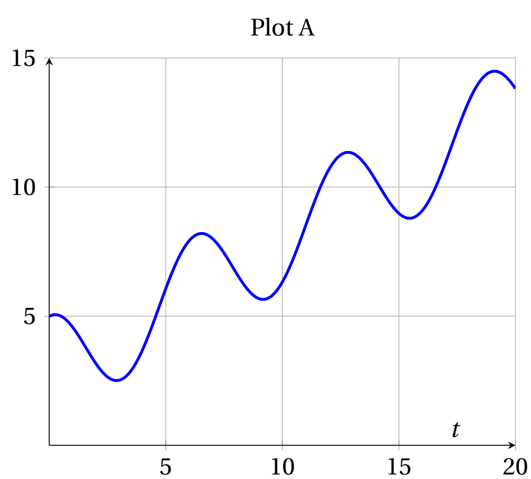
$$2\pi \text{ radians} = 360^\circ$$

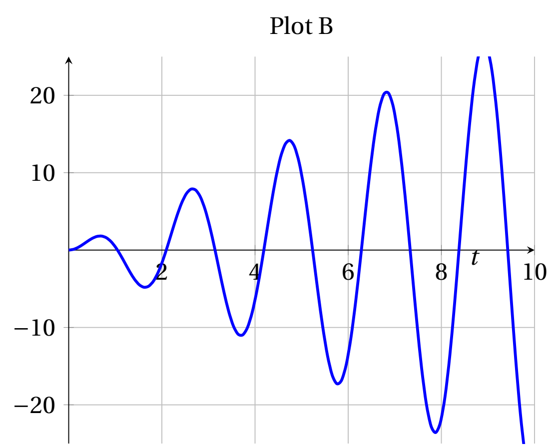
- for $f(t) = A\sin(B(t - t_0)) + C$ or $f(t) = A\sin(Bt + \phi) + C$
  - A is the amplitude.
  - B is the angular frequency, which determines the period, with $B = \frac{2\pi}{\text{Period}}$.
  - C is the average value.
  - t_0 is the shift along the t axis, a time when f is at an average value and increasing
  - ϕ is the shift in radians, the angle at which the oscillations begin

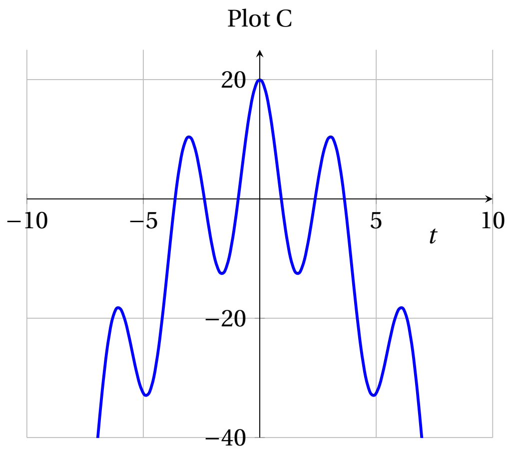
Exercises

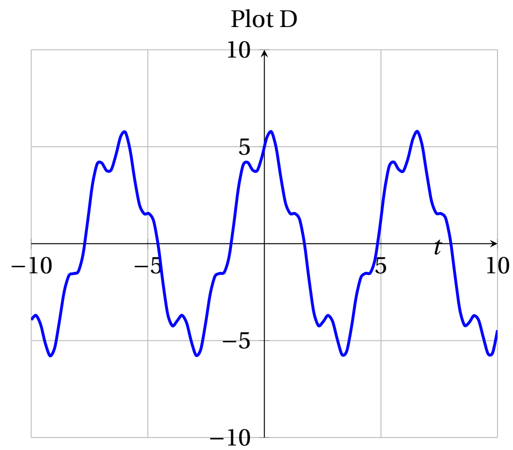
1. What is the formula for a sinusoidal function that has a minimum at coordinates $(3, 6)$ followed by a maximum at $(5, 9)$?

2. If we take a sinusoidal function $f(t) = A\sin(B(t - t_0)) + C$ and replace one of the parameters with a another function, say a linear function, we can create more complicated shapes. Find formulas for the function plotted in the following graphs.









3. The number of hours of daylight varies sinusoidally throughout the year. The maximum occurs on the summer solstice, June 21, when we have 15 hours and 50 minutes of daylight. The minimum occurs on the winter solstice, December 21, when we have only 8 hours and 33 minutes of daylight. Find the formula for a function to describe this. The input to your function should be d, the number of days since the beginning of the year, so that $d = 5$ on January 5. The output of your function should be the amount of daylight, in minutes. Assume that this is not a leap year. Hint: Because we know the date of maximum, it is easier to write this in terms of a cosine function.