Active Calculus & Mathematical Modeling
Carroll College

Active Calculus: Matt Boelkins, Lead Author and Editor
Department of Mathematics
Grand Valley State University
boelkinm@gvsu.edu
http://faculty.gvsu.edu/boelkinm/
David Austin, Contributing Author
http://merganser.math.gvsu.edu/david/
Steven Schlicker, Contributing Author
http://faculty.gvsu.edu/schlicks/

Carroll College: Eric Sullivan, Contributing Author and Editor
esullivan@carroll.edu
http://www.carroll.edu/esullivan

Last Updated: July 11, 2017

Contents

Chapter 1

Review of Pre Calculus Materials

1.1 Lines, Slope, and Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a function and what do we mean by its domain and range?
- What is the slope of a line? What are linear functions and families of linear functions?

Introduction

We begin our study of calculus by reminding the reader of several pre-requisite topics. The study of calculus depends on a thorough understanding of these topics and it is imperative that the reader become as familiar as possible with these topics. In the present section we remind the reader about the concepts of functions, slope, and lines, but first, there are a few things that you should do to get your self ready to use this text.

Preview Activity 1.1. This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where can you find the full textbook?

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?

⋈

Functions

Let's start with the fundamental mathematical idea of a function.

Definition 1.1 (Function)**.**

A function is a mathematical rule that assigns exactly one output for every input.

It is easy to give many common examples of functions:

- The area of a circle A is a function of the radius of the circle: $A(r) = \pi r^2$.
- The amount M in your savings account is a function of the rate of interest the bank pays.
- The fuel efficiency in your car is a function of many things, e.g. the speed at which you drive, the number of cylinders in your engine, the type of driving conditions, etc.
- The pressure on a diver is a function of the depth of the diver under water.

Definition 1.2 (Domain of a Function)**.**

The domain is the set of all possible inputs for a function.

Definition 1.3 (Range of a Function)**.**

The range is the set of all possible outputs for a function.

Example 1.1. Find the domain and range of the functions $f(x) = \sin(x)$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$.

Solution. For $f(x) = \sin(x)$ we recall that the sine function is defined for every possible value of x but the output is strictly between $y = -1$ and $y = 1$. Therefore, the domain for $f(x) = \sin(x)$ is $-\infty < x < \infty$ and the range is $-1 \le y \le 1$. See the left plot in Figure 1.1.

For $g(x) = \sqrt{x}$ we recall that the square root of a negative number results in an imaginary number. In this text we are interested in real-valued output for functions so we must omit all of the negative

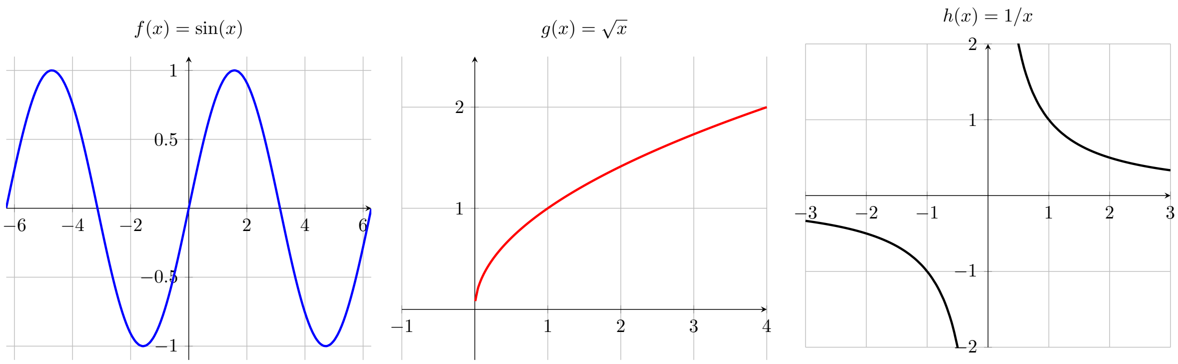


Figure 1.1: Graphs of the function $f(x) = \sin(x)$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$.

numbers from the domain and hence $0 \le x < \infty$. For the range we recall that the square root of a number will always be a non-negative number. As such, the range is $0 \le y < \infty$. See the middle plot in Figure 1.1.

For $h(x) = \frac{1}{x}$ we recall that division by zero is mathematically impossible. That is the only troublesome point in the domain so $-\infty < x < 0$ or $0 < x < \infty$. A moment's reflection also reveals that it is impossible to get zero out of the function $h(x)$ but it is possible to get any other number. Hence $-\infty < y < 0$ or $0 < y < \infty$. See the right plot in Figure 1.1.

It is also important to recall the notation for functions. When we write $f(x) = \sqrt{x}$ we are saying several things. First, the "f" is the name of the function that we're defining. The naming convention gives us a convenient way to refer to functions without having to explicitly state their algebraic form. Next, the "(x)" is an explicit statement to the reader that the variable "x" is the independent variable for the function f. Lastly, the right-hand side of the definition tells us exactly what to do with the independent variable algebraically.

When we write $f(25)$ we are referring to the already defined function f and explicitly saying to replace the independent variable with the number 25. In this instance, $f(25) = \sqrt{25} = 5$. Similarly, if we write $f(\sin(x))$ we mean to find the independent variable x in the function f and replace it with the function $\sin(x)$. In this case, $f(\sin(x)) = \sqrt{\sin(x)}$. This is simply a new function.

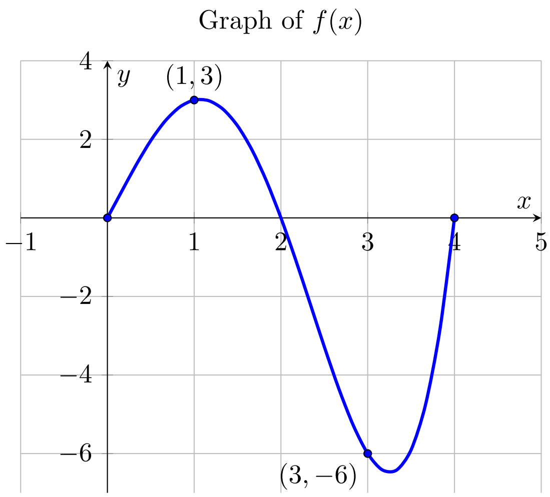
A function does not always need to be given algebraically. The three primary representations of a function are the algebraic form, the graphical form, and the tabular form. For example, for the function $f(x) = \sqrt{x}$ we are explicitly giving the algebraic form and the middle plot of Figure 1.1 shows the graphical form. Table 1.1 shows a portion of the table of values. The distinct disadvantage for a table of values on many functions is that there are infinitely many possible input values and a table can naturally only show finitely many of them.

Activity 1.1.

The graph of a function $f(x)$ is shown in the plot below.

x	0	1	2	3	4	5
$f(x)$	0	1	1.414	1.732	2	2.236

Table 1.1: Tabular form of the function $f(x) = \sqrt{x}$.



(a) What is the domain of $f(x)$?

(b) Approximate the range of $f(x)$.

(c) What are $f(0)$, $f(1)$, $f(3)$, $f(4)$, and $f(5)$?

◁

Slope and Linear Functions

One of the basic graphical ideas of calculus is that if we zoom in close enough to a curved function it will look approximately linear. The words "zoom" and "close enough" will be made explicit later. We now review the features of linear functions so that the idea of "zoomed in linearity" can flow naturally later in the course.

Every linear function is characterized by a constant rate of change; the slope. The slope of a linear function is a measure of the "steepness" of the line. We use the symbols Δx and Δy which mean respectively the "change in x" and the "change in y".

Definition 1.4.

The **slope**, m of a (non-vertical) linear function f which passes through any two points (x_1, y_1), (x_2, y_2) can be found using the formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}}$$

As shown in Figure 1.2, the slope of a linear function has the following characteristics:

- if the line rises from left to right then the slope is positive,

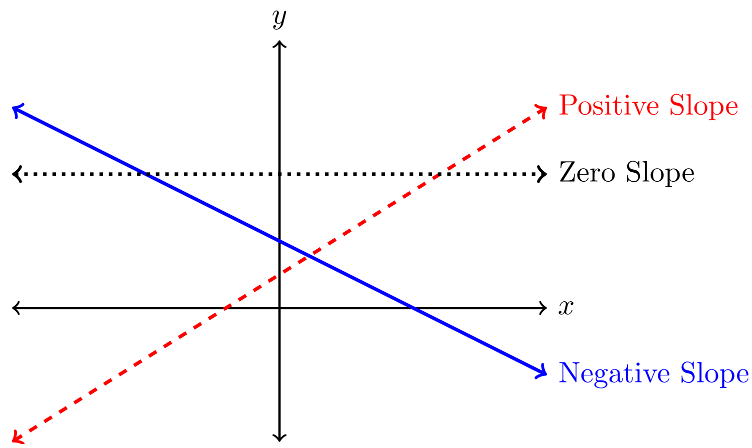


Figure 1.2: Characteristics of slope.

- if the line falls from left to right then the slope is negative,
- if the line is horizontal then the slope is zero, and
- if the line is vertical then the slope is undefined.

Depending on the information given there are several convenient forms of the equation of a line. Given the definition of the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and letting $(x, y) = (x_2, y_2)$ be any arbitrary point we get the point-slope form of a linear function by observing that $m = \frac{y-y_1}{x-x_1}$ which implies that $y - y_1 = m(x - x_1)$.

Definition 1.5.

If the linear function f has slope m and passes through the point (x_1, y_1), then the **point-slope form of the equation of a line** is given by:

$$y - y_1 = m(x - x_1).$$

An alternate form of a linear function which is probably very familiar to most readers is the slope-intercept form of a line.

Definition 1.6.

If the linear function f has slope m and y-intercept b, then the **slope-intercept form of the equation of a line** is given by:

$$y = mx + b.$$

In a calculus class the point-slope form is often the most useful. If you have a linear function written in the point-slope form you can always rearrange to get it into the slope-intercept form

$$y - y_1 = m(x - x_1) \quad \Longrightarrow \quad y = mx - mx_1 + y_1.$$

Hence we see that the y intercept of a line can be given as $b = -mx_1 + y_1$. The symbols and geometry used in each of the above definitions are shown in Figure 1.3.

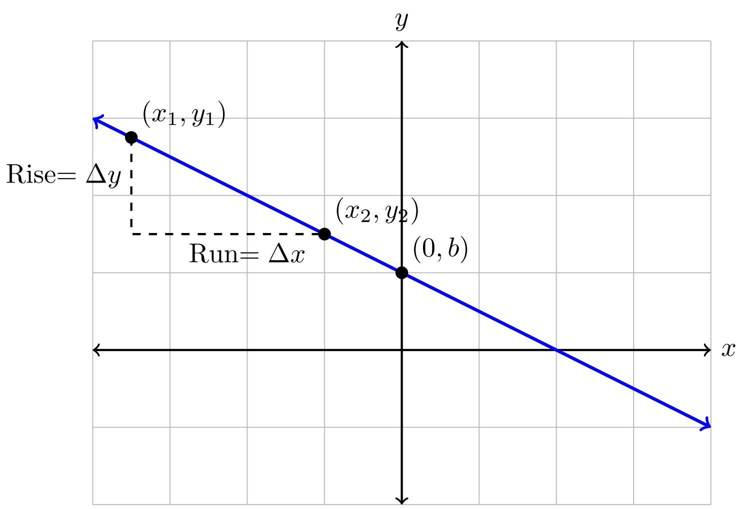


Figure 1.3: Anatomy of a linear function.

Activity 1.2.

Find the equation of the line with the given information.

(a) The line goes through the points $(-2,5)$ and $(10,-1)$.

(b) The slope of the line is $3/5$ and it goes through the point $(2,3)$.

(c) The y-intercept of the line is $(0,-1)$ and the slope is $-2/3$.

◁

Example 1.2. Write the equation of the line going through the points $(5,7)$ and $(-3,2)$.

Solution. First we calculate the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{7-2}{5-(-3)} = \frac{5}{8}.$$

Since we have two points and neither is the y intercept of the linear function we choose to use the point-slope form of the line. Letting $(x_1, y_1) = (5,7)$ we see that

$$y - 7 = \frac{5}{8}(x-5)$$

is one form of the linear function. It is often conventient to solve for y giving us

$$y = \frac{5}{8}(x-5) + 7.$$

Notice that we do not necessarily need to simplify all the way to the slope-intercept form of the line.

Linear Functions From Data

A feature of every linear function is that the slope is the same no matter where you are on the line. When given a table of data that you suspect might represent a linear function the slope manifests itself as a constant common difference between successive y-values.

Example 1.3. Consider the data in the table below.

x	5	6	7	8	9
y	12.2	17.5	22.8	28.1	33.4

Demonstrate that this data is linear and write an equation that fits the data.

Solution. The common differences can be found for each successive y-values

x	5	6	7	8	9
y	12.2	17.5	22.8	28.1	33.4
Common Difference	$\frac{17.5-12.2}{6-5} = 5.3$	$\frac{22.8-17.5}{7-6} = 5.3$	$\frac{28.1-22.8}{8-7} = 5.3$	$\frac{33.4-28.1}{9-8} = 5.3$	-

The successive differences are clearly the same throughout the data set and the slope for this data set is $m = 5.3$. Picking any convenient point, say $(5, 12.2)$, then allows us to write the equation of the line as

$$y - 12.2 = 5.3(x-5).$$

This could be simplified to point-slope form, but there is typically no need for this algebraic simplification.

Activity 1.3.

An apartment manager keeps careful record of the rent that he charges as well as the number of occupied apartments in his complex. The data that he has is shown in the table below.

Monthly Rent	\$650	\$700	\$750	\$800	\$850	\$900
Occupied Apartments	203	196	189	182	175	168

(a) Just by doing simple arithmetic justify that the function relating the number of occupied apartments and the rent is linear.

(b) Find the linear function relating the number of occupied apartments to the rent.

(c) If the rent were to be increased to \$1000, how many occupied apartments would the apartment manager expect to have?

(d) At a \$1000 monthly rent what net revenue should the apartment manager expect?

◁

Example 1.4. The Old Farmer's Almanac tells us that you can tell the temperature by counting the chirps of a cricket. It is a linear function $T = f(C)$ given by T (in degrees Fahrenheit)=# of chirps in 15 seconds +40. We can approximate this with the formula

$$T = \frac{C}{4} + 40$$

where C is the number of chirps/minute and T is in °F.

(a) If the chirp rate is 120 chirps/minute, what is the temperature?

(b) Suppose that crickets will not chirp if the temperature is below 56°F. We can also suppose that crickets will not chirp above 136°F since that is the highest temperature ever recorded at a weather station. With these parameters, what is the domain of this function?

Solution.

(a) If $C = 120$ chirps/minute, substitute this into the function $T(C)$ to obtain

$$T(120) = \frac{120}{4} + 40 = 30 + 40 = 70^{\circ}\text{F}.$$

(b) To find the domain we need to find the appropriate values of C for the $T(C)$ function. Solve $56 = C/4 + 40$ and get $C = 64$. Solve $136 = C/4 + 40$ and get $C = 384$. So the domain of $T(C)$ is $64 \le$ chirps/minute ≤ 384 or, in interval notation, $[64, 136]$.

Families of Linear Functions

We noted above that a linear function has the form $f(x) = mx + b$, where m is the slope of the line, and b is the y-intercept. Since m and b can take on various values, taken together, they represent a family of functions. For example, we could fix $b = 2$, and then draw the graphs of $f(x) = mx + 2$ for various values of m; for example, $m = -1, -2, 2, 1$. Doing so would give the functions in the family $f(x) = mx + 2$ shown in the left image of Figure 1.4.

Similarly, we could set m to be 2 and let b take on the values $b = -1, 1, 4, -6$ and we would get some examples from the family of functions for $y = f(x) = 2x + b$ shown in the right image of Figure 1.4.

From the right image in Figure 1.4 it should be clear to the reader that parallel lines have the same slope. What can you say about the slopes of perpendicular lines? Here is the result that we state without proof.

Theorem 1.1.

If line ℓ_1 has slope m_1 and line ℓ_2 has slope m_2, then

- lines ℓ_1 and ℓ_2 are parallel if the slopes are the same: $m_1 = m_2$, and
- lines ℓ_1 and ℓ_2 are perpendicular if the slopes are opposite reciprocals: $m_2 = -\frac{1}{m_1}$.

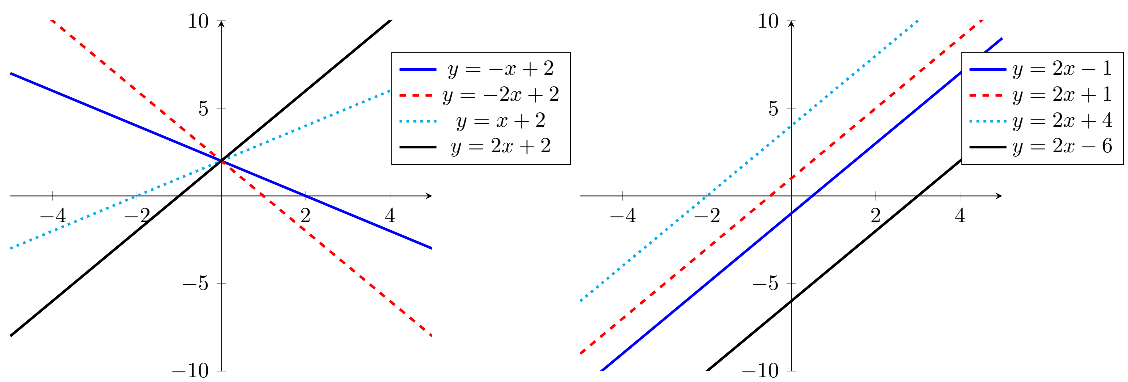


Figure 1.4: Several members of the family of linear functions $f(x) = mx + 2$ (left) and the family $f(x) = 2x + b$ (right).

Activity 1.4.

Write the equation of the line with the given information.

(a) Write the equation of a line parallel to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(b) Write the equation of a line perpendicular to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(c) Write the equation of a line with y-intercept $(0,-3)$ that is perpendicular to the line $y = -3x - 1$.

◁

Summary

In this section, we encountered the following important ideas:

- A function assigns one y value to each x value.
- The slope of a linear function can be written as

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_2}$$

- A linear function can be written in the forms

$$y = mx + b \quad \text{or} \quad y - y_1 = m(x - x_1)$$

- When examining linear data, the differences between successive y-values reveals the slope.

Exercises

1. (modified from NCTM Illuminations) The table below displays data that relate the number of oil changes per year and the cost of engine repairs. To predict the cost of repairs from the number of oil changes, use the number of oil changes as the x variable and the engine repair cost as the y variable.

Oil Changes Per Year	Cost of Repairs ($)
3	300
5	300
2	500
3	400
1	700
4	400
6	100
4	250
3	450
2	650
0	600
10	0
7	150

(a) Using graph paper make a plot of the data on appropriate axes.

(b) Do the data appear linear? Why or why not?

(c) Pick two representative points from the data and use them to write the equation of a line that *fits* the data. Plot your line on top of your data and discuss how well your line fits the data. (This may take a few attempts.)

(d) Despite how well your data fit a linear model, it is not entirely sensible to use a linear model for this data. Why?

2. The population of a city, P, in millions, is a function of t, the number of years since 1960, so $\mathrm{P} = f(t)$. Which of the following statements explains the meaning of $f(38) = 8$ in terms of the population of this city?

(a) The population of this city in the year 38 is 8 million people.

(b) The population of this city in the year 8 is 38 million people.

(c) The population of this city in the year 1968 is 38 million people.

(d) The population of this city in the year 1998 is 8 million people.

3. Determine the slope and y-intercept of the line whose equation is $-4y + 6x + 8 = 0$.

4. The value of a car in 1990 is \$13,100 and the value is expected to go down by \$80 per year for the next 7 years. Write a linear function for the value, V, of the 1990 car as a function of the number of years from 1990, x.

1.2 Exponential Functions

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can exponential functions be used to model growth and decay of populations, investments, radioactive isotopes, and many other physical phenomena?
- How can we build exponential functions from data?

Introduction

The exponential function is a powerful tool in the mathematician's arsenal for modeling growth and decay phenomena. The common applications of the exponential funciton range from population modeling, to tracking drug levels in the blood stream, to using carbon dating to estimate the age of an artifact. The common mathematical fact about all of these situations is that the growth (or decay) rate is a constant multiple. For example, if we are measuring exponential population growth then the ratio of two successive populations must be constant. Linear functions have a similar behavior, except that in linear functions the difference (not the ratio) between two successive values is constant (the slope).

Preview Activity 1.2. Suppose that the populations of two towns are both growing over time. The town of Exponentia is growing at a rate of 2% per year, and the town of Lineola is growing at a rate of 100 people per year. In 2014, both of the towns have 2,000 people.

(a) Complete the table for the population of each of these towns over the next several years.

	2014	2015	2016	2017	2018	2019	2020	2021	2022
Exponentia	2000								
Lineola	2000								

(b) Write a linear function for the population of Lineola. Interpret the slope in the context of this problem.

(c) The ratio of successive populations for Exponentia should be equal. For example, dividing the population in 2015 by that of 2014 should give the same ratio as when the population from 2016 is divided by the population of 2015. Find this ratio. How is this ratio related to the 2% growth rate?

(d) Based on your data from part (a) and your ratio in part (c), write a function for the population of Exponentia.

(e) When will the population of Exponentia exceed that of Lineola?

⋈

Exponential Functions

Consider the example where the population of a bacteria colony is doubling every week. If in the first week there are 100 bacteria, then there are 200 bacteria by the end of the second week, 400 by the end of the third and so on. In Table 1.2 and Equation (1.1) we can see a simple way to model this type of growth.

Week	Bacteria
0	100
1	$100 \cdot 2 = 200$
2	$200 \cdot 2 = 100 \cdot 2^2 = 400$
3	$400 \cdot 2 = 100 \cdot 2^3 = 800$
$\vdots$	$\vdots$

Table 1.2: Bacteria population doubling

$$\mathrm{P}(t) = 100 \cdot 2^t \quad (t = \text{number of weeks}) \tag{1.1}$$

The time, t in equation (1.1) is measured in weeks. It is easy to see that the ratio of the populations for each successive week is constant at $\mathrm{P}(t+1)/\mathrm{P}(t) = 2$. This is indicative of exponential growth. Of course, this population growth could have been modeled using time measured in days instead. The population still doubles every week so for this new model the value at $t = 7$ should be double the value at $t = 0$. Equation (1.2) shows this new model with only a slight modification adjusting for the new time measurement.

$$\mathrm{P}(t) = 100 \cdot 2^{t/7} \quad (t = \text{number of days}) \tag{1.2}$$

This type of modeling and thought process can be used to describe most exponential growth and decay situations. One general formula for an exponential function is

$$f(x) = \mathrm{A} \cdot r^{kx}. \tag{1.3}$$

where A is some given initial value, r is the common ratio, and k is a constant given by the frequency in which the common ratio is applied. In the previous population doubling example, $\mathrm{A} = 100$, $r = 2$, and $k = 1/7$.

A few simple guidelines should make it clear when an exponential function is modeling growth or decay.

- If $r > 1$ then the function exhibits exponential growth.
- If $0 < r < 1$ then the function exhibits exponential decay.
- If a population is growing by p% per unit time, then $r = 1 + p/100$.
- If a population is decreasing by p% per unit time, then $r = 1 - p/100$.

Activity 1.5.

Consider the exponential functions plotted in Figure 1.5

(a) Which of the functions have common ratio $r > 1$?

(b) Which of the functions have common ratio $0 < r < 1$?

(c) Rank each of the functions in order from largest to smallest r value.

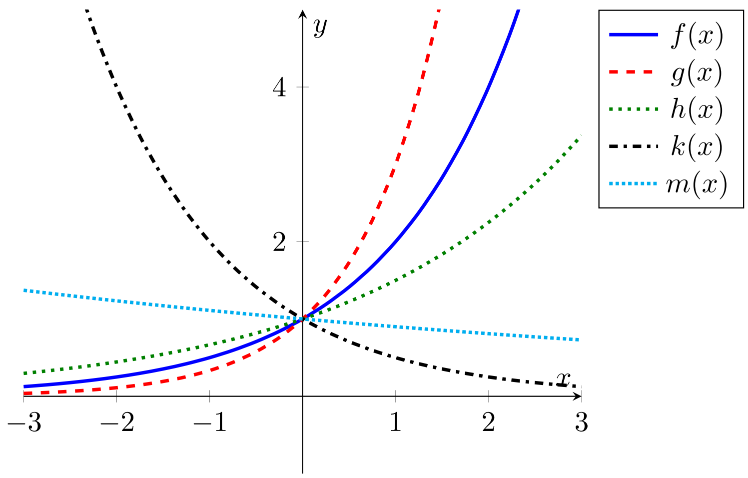


Figure 1.5: Exponential growth and decay functions

◁

Example 1.5. One application to exponential decay is to calculate the intensity of radiation from radioactive isotopes. Most isotopes emit particles and decay into stable forms. We measure the rate of decay from the particles by the isotope's **half-life**, which is how long it takes half of the isotope to decay. The half-life for Sodium-25 (Na^{25}) is almost exactly one minute. Write a function that models that amount of Na^{25} over time if you start with exactly 36 grams.

Solution. If you begin with 36 grams of Na^{25} then the number of grams remaining after t minutes, $S(t)$, can be represented by the function

$$S(t) = 36\left(\frac{1}{2}\right)^t,$$

where t is measured in minutes. Figure 1.6 shows this exponential decay function with an initial value of 36 and a value of 18 after 1 day.

Activity 1.6.

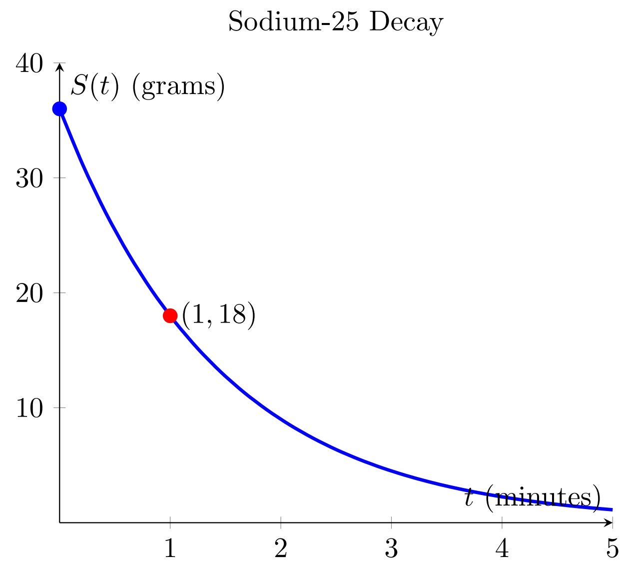


Figure 1.6: The grams of Sodium-25 remaining as a function of time. The blue point represents the initial value $(0,36)$ and the red point represents the value after 1 minute $(1,18)$.

A sample of $\mathrm{N}i^{56}$ has a half-life of 6.4 days. Assume that there are 30 grams present initially.

(a) Write a function describing the number of grams of $\mathrm{N}i^{56}$ present as a function of time. Check your function based on the fact that in 6.4 days there should be 50% remaining.

(b) What percent of the substance is present after 1 day?

(c) What percent of the substance is present after 10 days?

$\triangleleft$

Activity 1.7.

Uncontrolled geometric growth of the bacterium *Escherichia coli (E. Coli)* is the theme of the following quote taken from the best-selling author Michael Crichton's science fiction thriller, *The Andromeda Strain:*

> "The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."

(a) Write an equation for the number of E. coli cells present if a single cell of E. coli divides every 20 minutes.

(b) How many E. coli would there be at the end of 24 hours?

(c) The mass of an E. coli bacterium is 1.7×10^{-12} grams, while the mass of the Earth is 6.0×10^{27} grams. Is Michael Crichton's claim accurate? Approximate the number of hours we should have allowed for this statement to be correct?

◁

Investments

Interest bearing bank accounts and investments follow exponential growth and decay models. In the case of a savings account the interest is typically compounded several times per year. This means that the investor is getting interest on their interest every time the bank computes the interest.

If the money is gaining p% interest compounded n times per year then the common ratio for the exponential function is $1+p/n$. The exponent needs to reflect the fact that the interest occurs at monthly intervals. This means that the exponential function is

$$A(t) = A_0\left(1+\frac{p}{n}\right)^{nt} \quad (t = \text{number of years}). \tag{1.4}$$

In Equation (1.4), A_0 is the initial investment, $A(t)$ is the value of the investment over time, p is the interest rate, and n is the number of times the interest is compounded per year.

Example 1.6. If \$100 are invested into a bank account earning 2% interest compounded 12 times per year, how much does the investor have at the end of 1 year? 5 years? at retirement age? How does this change is we compound quarterly or daily instead of monthly?

Solution. In the present situation the function modeling the value of the investment is

$$A(t) = 100\left(1+\frac{0.02}{12}\right)^{12t}.$$

Table 1.3 shows the value of the investment over the first 5 years. It is clear that this is very slow growth, but it is exponential none the less. The common ratio in this case is $r = (1+0.02/12) \approx 1.0017$, and this means that you are really gaining 0.17% interest per month.

Year	0	1	2	3	4	5
Value	\$100	\$102.02	\$104.08	\$106.18	\$ 108.32	\$ 110.51

Table 1.3: Value of \$100 investment for the first 5 years

Assume that our investor was an 18 year old and extrapolate this to retirement age, let's say 65 years old. That is 47 years worth of interest, and the initial \$100 investment becomes

$$A(47) = 100\left(1+\frac{0.02}{12}\right)^{12\cdot 47} \approx \$256.$$

If the number of times the bank compounds the interest changes the function will still have essentially the same form: $A(t) = 100(1+\frac{0.02}{n})^{nt}$. In Table 1.4 the same investment is considered for several values of n. While more compoundings per year generally gives a higher rate of return on the investment, the impact is small for larger values of n.

Year	0	1	2	3	4	5	$\cdots$	47
Value ($n = 1$)	\$100	\$102.00	\$104.04	\$106.12	\$ 108.24	\$ 110.41	$\cdots$	\$253.63
Value ($n = 4$)	\$100	\$102.02	\$104.07	\$106.17	\$ 108.31	\$ 110.49	$\cdots$	\$255.40
Value ($n = 12$)	\$100	\$102.02	\$104.08	\$106.18	\$ 108.32	\$ 110.51	$\cdots$	\$255.80
Value ($n = 365$)	\$100	\$102.02	\$104.08	\$106.18	\$ 108.33	\$ 110.52	$\cdots$	\$255.99

Table 1.4: Value of \$100 investment for various values of n.

Exponential Functions with Base e

Exponential functions are commonly written with a base of $e \approx 2.718281828459045\ldots$. This may seem like an arbitrary and bizarre choice at first glance, but we will see that this famous number (called Euler's Number [1]) plays a central role in Calculus.

Euler's number can be derived from Equation (1.4) if we assume that a fictitious bank gives 100% interest compounded infinitely many times per year on a one dollar investment. Mathematically this is written as

$$e = 1 \cdot \left(1 + \frac{1}{n}\right)^n \text{ as } n \to \infty. \tag{1.5}$$

n	1	10	100	1000	$\cdots$	10^{10}	$\cdots$
$(1+\frac{1}{n})^n$	2	2.5935	2.7048	2.7169	$\cdots$	2.71828	$\cdots$

Table 1.5: Approximations of Euler's number, e, using equation (1.5) with various values of n

Any exponential function can be rewritten in terms of Euler's number in the form

$$f(x) = Ae^{kx}. \tag{1.6}$$

In Equation (1.6), k is called the **continuous rate**.

- If $k > 0$ then $f(x) = Ae^{kx}$ models exponential growth.
- If $k < 0$ then $f(x) = Ae^{kx}$ models exponential decay.

[1] Euler's number is named after the famous 17^{th} century mathematician Leonhard Euler. Euler was the first mathematician to introduce the notion of a function, and he is responsible for a large amount of the development of Calculus.

Example 1.7. A population of a city is 5000 people and is doubling in size every 5 years. Use equations (1.3) and (1.6) to write two different functions modeling this population; one with base 2 and one with base e.

Solution. If the population is doubling every 5 years we can use equation (1.3) to write

$$P(t) = 5000 \cdot 2^{t/5}.$$

In order to use equation (1.6) we need to find the value of "k". This is done by using the fact that at year 5 the population will be 10000 and solving the equation

$$10000 = 5000 \cdot e^{5k}.$$

Rearranging we see that $e^{5k} = 2$. In order to solve this algebraic equation we need to use logarithms. These important functions will be discussed in more detail in the logarithms section In this case we see that $k = \ln(2)/5 \approx 0.139$. Therefore,

$$P(t) = 5000 \cdot e^{0.139t}.$$

Since these two equations model the same population they must be identical. Indeed,

$$5000 \cdot 2^{t/5} = 5000 \cdot \left(2^{1/5}\right)^t \approx 5000 \cdot (1.149)^t,$$

and

$$5000 \cdot e^{kt} = 5000 \cdot \left(e^k\right)^t \approx 5000 \cdot (1.149)^t.$$

Summary

In this section, we encountered the following important ideas:

- An exponential function can be written in the form $f(x) = Ar^{kx}$ or $g(x) = Ae^{kx}$.
  - In $f(x)$, if $k > 0$ and $r > 1$ then $f(x)$ models exponential growth.
  - In $f(x)$, if $k > 0$ and $0 < r < 1$ then $f(x)$ models exponential decay.
  - In $g(x)$, if $k > 0$ then $g(x)$ models exponential growth.
  - In $g(x)$, if $k < 0$ then $g(x)$ models exponential decay.
- Exponential functions have a constant common ratio for successive time values.

Exercises

1. Suppose that $h(t) = A \cdot r^t$. If $h(3) = 4$ and $h(5) = 40$,

(a) find r.

(b) find A.

(c) Does this function model exponential growth or decay? How can you tell?

2. The half-life of $\mathrm{B}r^{77}$ is 57 hours.

(a) If the initial amount is 150 grams, find the amount remaining after 171 hours.

(b) Write an equation to predict the amount remaining after t hours.

(c) Estimate within one hour how long it will take the amount to decrease to 10 grams.

3. Consider the data in Table 1.6

(a) Which (if any) of the functions could be linear? Explain how you know that these functions are linear, and find formulas for these functions.

(b) Which (if any) of the functions could be exponential? Explain how you know that these functions are linear, and find formulas for these functions.

x	$f(x)$	$g(x)$	$h(x)$
-2	12	16	37
-1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

Table 1.6: Data tables for $f(x)$, $g(x)$, and $h(x)$