Active Calculus & Mathematical Modeling
Carroll College

Active Calculus: Matt Boelkins, Lead Author and Editor
Department of Mathematics
Grand Valley State University
boelkinm@gvsu.edu
http://faculty.gvsu.edu/boelkinm/
David Austin, Contributing Author
http://merganser.math.gvsu.edu/david/
Steven Schlicker, Contributing Author
http://faculty.gvsu.edu/schlicks/

Carroll College: Eric Sullivan, Contributing Author and Editor
esullivan@carroll.edu
http://www.carroll.edu/esullivan

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Chapter 1

Preliminaries: A Library of Functions

1.1 Functions, Slope, and Lines

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- What is a function and what do we mean by its domain and range?
- What is the slope of a line? What are linear functions and families of linear functions?
- What are difference equations and the delta notation and why are these useful?

Introduction

We begin the study of calculus by reminding the reader of several pre-requisite topics. The study of calculus depends on a thorough understanding of these topics and it is imperative that the reader become as familiar as possible with these topics. In the present section we remind the reader about the concepts of functions, slope, and lines, but first, there are a few things that you should do to get your self ready to use this text.

Preview Activity 1.1. This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where is the full textbook stored? Find it and save a copy to your computer.

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?

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Functions

Definition 1.1.

A **function** f defined on a set A, is a rule that assigns to each element x in A, exactly one element, denoted $f(x)$, from a set B.
The set A is called the **domain** of the function f. The **range** of f is the set of values of $f(x)$ as x takes on all the values of A. Another way to state it is that the range of f is the set of all y such that $y = f(x)$ for some x in A.

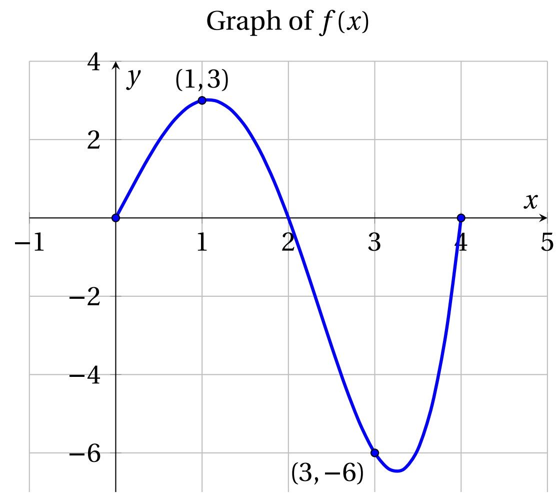
It is easy to give many common examples of functions:

- The area of a circle A is a function of the radius of the circle: $A(r) = \pi r^2$.
- The amount M in your savings account is a function of the rate of interest the bank pays.
- Your miles per gallon in your car depends on many things, e.g. the speed at which you drive.
- The pressure on a diver is a function of the depth of the diver under water.

Probably the most common method for representing a function is with a graph. If the domain of function f is set A, then the graph of f is the collection of all ordered pairs of the form $(x, f(x))$ where x comes from the domain A.

Activity 1.1.

The graph of a function $f(x)$ is shown below.



(a) What is the domain of $f(x)$?

(b) Approximate the range of $f(x)$.

(c) What are $f(0)$, $f(1)$, $f(3)$, $f(4)$, and $f(5)$?

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Example 1.1. Find the domain and range of the functions

$$f(x) = \sin(x), \quad g(x) = \sqrt{x}, \quad \text{and} \quad h(x) = \frac{1}{x}.$$

Solution. For $f(x) = \sin(x)$ we recall that the sine function is defined for every possible value of x but the output is strictly between $y = -1$ and $y = 1$. Therefore, the domain for $f(x) = \sin(x)$ is $-\infty < x < \infty$ and the range is $-1 \le y \le 1$. See the left plot in Figure 1.1.

For $g(x) = \sqrt{x}$ we recall that the square root of a negative number results in an imaginary number. In this text we are interested in real-valued output for functions so we must omit all of the negative numbers from the domain and hence $0 \le x < \infty$. For the range we recall that the square root of a number will always be a non-negative number. As such, the range is $0 \le y < \infty$. See the middle plot in Figure 1.1.

For $h(x) = \frac{1}{x}$ we recall that division by zero is mathematically impossible. That is the only troublesome point in the domain so $-\infty < x < 0$ or $0 < x < \infty$. A moment's reflection also reveals that it is impossible to get zero out of the function $h(x)$ but it is possible to get any other number. Hence $-\infty < y < 0$ or $0 < y < \infty$. See the right plot in Figure 1.1.

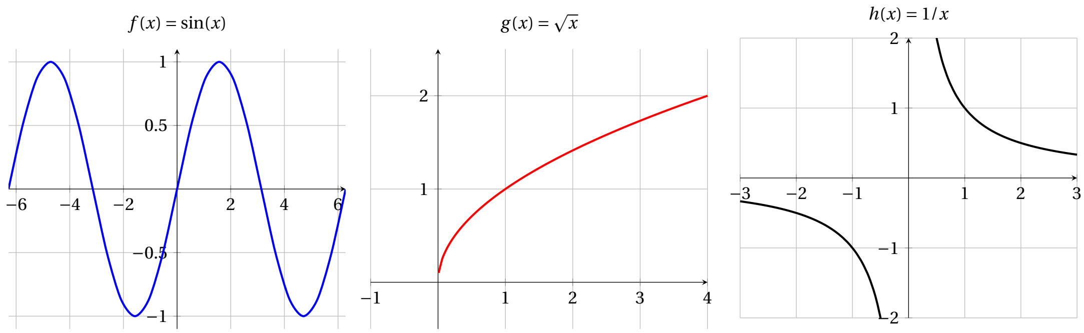


Figure 1.1: Graphs of the function $f(x) = \sin(x)$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$.

Slope and Linear Functions

In Calculus we are often interested in writing the equation for a linear function. As such, we should review the features of linear functions. Every linear function is characterized by a constant rate of change; the slope. The slope of a linear function is a measure of the "steepness" of the line. We use the symbols Δx, Δy which mean respectively the "change in x" and the "change in y".

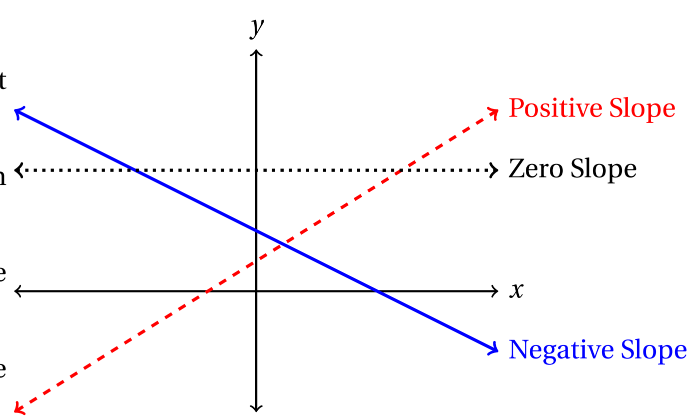
Definition 1.2.

The **slope**, m of a (non-vertical) linear function f which passes through any two points (x_1, y_1), (x_2, y_2) can be found using the formula

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}}$$

Recall that

- if the line rises from left to right then the slope is positive,
- if the line falls from left to right then the slope is negative,
- if the line is horizontal then the slope is zero, and
- if the line is vertical then the slope is undefined.



Depending on the information given there are several convenient forms of the equation of a line.

Given the definition of the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and letting $(x, y) = (x_2, y_2)$ be any arbitrary point we get the point-slope form of a linear function.

Definition 1.3.

If the linear function f has slope m and passes through the point (x_1, y_1), then the **point-slope form of the equation of a line** is given by:

$$y - y_1 = m(x - x_1).$$

An alternate form of a linear function which is probably very familiar to most readers is the slope-intercept form of a line.

Definition 1.4.

If the linear function f has slope m and y-intercept b, then the **slope-intercept form of the equation of a line** is given by:

$$y = mx + b.$$

In a calculus class the point-slope form is often the most useful. The symbols and geometry used in each of the above definitions are shown in Figure 1.2.

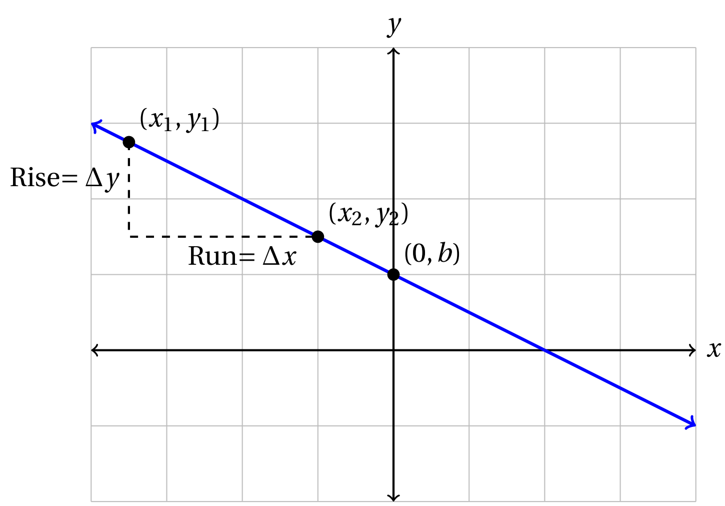


Figure 1.2: Anatomy of a linear function.

Activity 1.2.

Find the equation of the line with the given information.

(a) The line goes through the points $(-2,5)$ and $(10,-1)$.

(b) The slope of the line is $3/5$ and it goes through the point $(2,3)$.

(c) The y-intercept of the line is $(0,-1)$ and the slope is $-2/3$.

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Example 1.2. Write the equation of the line going through the points $(5,7)$ and $(-3,2)$.

Solution. First we calculate the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{7-2}{5-(-3)} = \frac{5}{8}.$$

Since we have two points and neither is the y intercept of the linear function we choose to use the point-slope form of the line. Letting $(x_1, y_1) = (5,7)$ we see that

$$y - 7 = \frac{5}{8}(x-5)$$

is one form of the linear function. It is often conventient to solve for y giving us

$$y = \frac{5}{8}(x-5) + 7.$$

Notice that we do not necessarily need to simplify all the way to the slope-intercept form of the line.

Linear Functions From Data

A feature of every linear function is that the slope is the same no matter where you are on the line. When given a table of data that you suspect might represent a linear function the slope manifests itself as a constant common difference between successive y-values.

Example 1.3. Consider the data in the table below.

x	5	6	7	8	9
y	12.2	17.5	22.8	28.1	33.4

Demonstrate that this data is linear and write an equation that fits the data.

Solution. The common differences can be found for each successive y-values

x	5	6	7	8	9
y	12.2	17.5	22.8	28.1	33.4
Common Difference	$\frac{17.5-12.2}{6-5}=5.3$	$\frac{22.8-17.5}{7-6}=5.3$	$\frac{28.1-22.8}{8-7}=5.3$	$\frac{33.4-28.1}{9-8}=5.3$	-

The successive differences are clearly the same throughout the data set and the slope for this data set is $m = 5.3$. Picking any convenient point, say $(5, 12.2)$, then allows us to write the equation of the line as

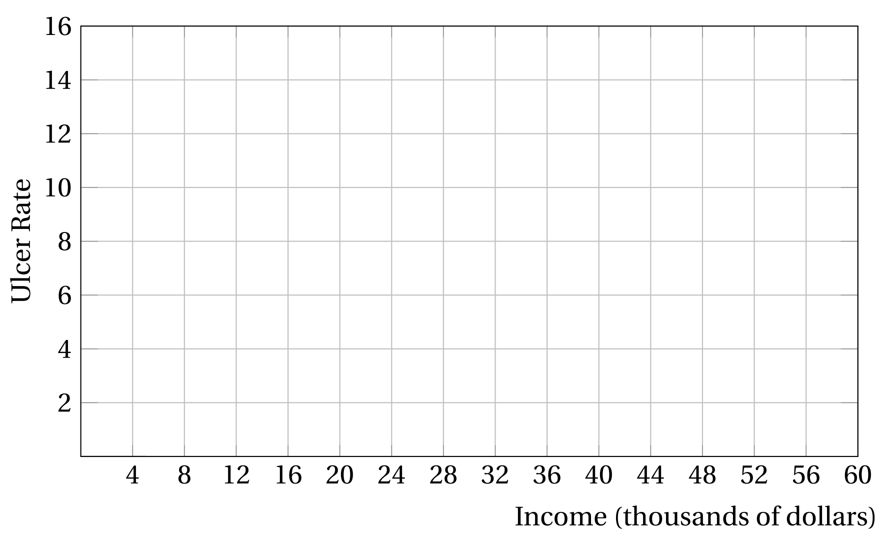
$$y - 12.2 = 5.3(x - 5).$$

This could be simplified to point-slope form, but there is typically no need for this algebraic simplification.

Activity 1.3.

A simulation shows lifetime peptic ulcer rates per 100 population for different family incomes as given in the following table.

Income	Ulcer Rate
\$4000	14.1
\$8000	13.4
\$12000	12.5
\$16000	12
\$20000	12.4
\$24000	11.6
\$28000	10.8
\$32000	10.3
\$36000	10.4
\$40000	9.6
\$44000	9.2
\$48000	8.8
\$52000	8.5
\$56000	8.4
\$60000	8.2



This data does not represent a straight line, but it is close.

(a) Just by doing simple arithmetic, how can you tell the function is not a straight line?

(b) Make a scatter plot of the data. Do you think a linear model can be a good approximation? Why or why not?

(c) Use just the first and the last data points, what is the equation of the straight line that these two points determine? Graph this equation.

(d) Using the model in part (c), estimate the ulcer rate for an income of \$26000.

(e) Using the model in part (c), how likely is someone with an income of \$100,000 will suffer from peptic ulcers? Note your answer will be a percent and remember that the ulcer rate is given per 100 people of population.

(f) Do you think it would be reasonable to apply this model to a person with an income of \$200,000? Why or why not?

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Example 1.4. The Old Farmer's Almanac tells us that you can tell the temperature by counting the chirps of a cricket. It is a linear function $T = f(C)$ given by T (in degrees Fahrenheit)=# of chirps in 15 seconds +40. We can approximate this with the formula

$$T = \frac{C}{4} + 40$$

where C is the number of chirps/minute and T is in °F.

(a) If the chirp rate is 120 chirps/minute, what is the temperature?

(b) Suppose that crickets will not chirp if the temperature is below 56°F. We can also suppose that crickets will not chirp above 136°F since that is the highest temperature ever recorded at a weather station. With these parameters, what is the domain of this function?

Solution.

(a) If $C = 120$ chirps/minute, substitute this into the function $T(C)$ to obtain

$$T(120) = \frac{120}{4} + 40 = 30 + 40 = 70^{\circ}F.$$

(b) To find the domain we need to find the appropriate values of C for the $T(C)$ function. Solve $56 = C/4 + 40$ and get $C = 64$. Solve $136 = C/4 + 40$ and get $C = 384$. So the domain of $T(C)$ is $64 \le \text{chirps/minute} \le 384$ or, in interval notation, $[64, 136]$.

Families of Linear Functions

We noted above that a linear function has the form $y = f(x) = mx+b$, where m is the slope of the line, and b is the y-intercept. Since m and b can take on various values, taken together, they represent a family of functions. For example, we could fix $b = 2$, and then draw the graphs of $f(x) = mx + 2$ for various values of m; for example, $m = -1, -2, 2, 1$. Doing so would give the functions in the family $f(x) = mx + 2$ shown in the left image of Figure 1.3.

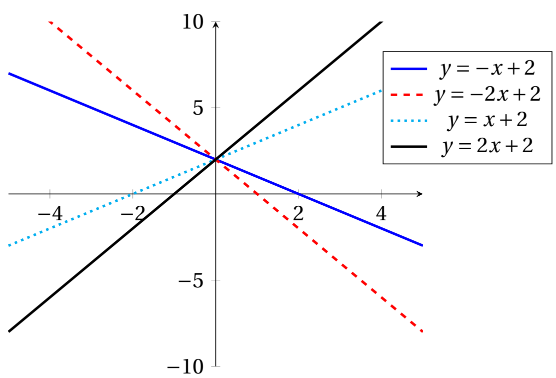
Similarly, we could set m to be 2 and let b take on the values $b = -1, 1, 4, -6$ and we would get some examples from the family of functions for $y = f(x) = 2x + b$ shown in the right image of Figure 1.3.

From the right image in Figure 1.3 it should be clear to the reader that parallel lines have the same slope. What can you say about the slopes of perpendicular lines? Here is the result that we state without proof.

Theorem 1.1.

If line ℓ_1 has slope m_1 and line ℓ_2 has slope m_2, then

- lines ℓ_1 and ℓ_2 are parallel if the slopes are the same: $m_1 = m_2$, and
- lines ℓ_1 and ℓ_2 are perpendicular if the slopes are opposite reciprocals: $m_2 = -\frac{1}{m_1}$.



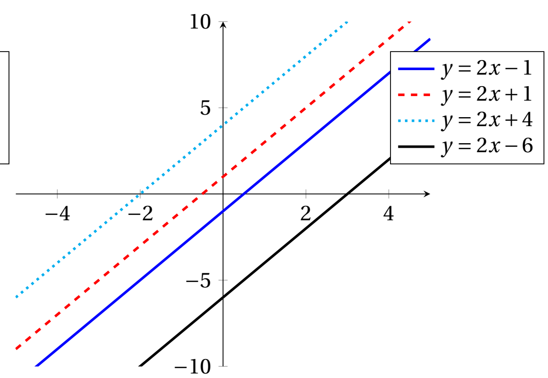


Figure 1.3: Several members of the family of linear functions $f(x) = mx + 2$ (left) and the family $f(x) = 2x + b$ (right).

Activity 1.4.

Write the equation of the line with the given information.

(a) Write the equation of a line parallel to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(b) Write the equation of a line perpendicular to the line $y = \frac{1}{2}x + 3$ passing through the point $(3, 4)$.

(c) Write the equation of a line with y-intercept $(0, -3)$ that is perpendicular to the line $y = -3x - 1$.

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Summary

In this section, we encountered the following important ideas:

- A function assigns one y value to each x value.
- The slope of a linear function can be written as

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_2}$$

- A linear function can be written in the forms

$$y = mx + b \quad \text{or} \quad y - y_1 = m(x - x_1)$$

- When examining linear data, the differences between successive y-values reveals the slope.

Exercises

1. (modified from NCTM Illuminations) The table below displays data that relate the number of oil changes per year and the cost of engine repairs. To predict the cost of repairs from the number of oil changes, use the number of oil changes as the x variable and the engine repair cost as the y variable.

Oil Changes Per Year	Cost of Repairs ($)
3	300
5	300
2	500
3	400
1	700
4	400
6	100
4	250
3	450
2	650
0	600
10	0
7	150

(a) Using graph paper make a plot of the data on appropriate axes.

(b) Do the data appear linear? Why or why not?

(c) Pick two representative points from the data and use them to write the equation of a line that *fits* the data. Plot your line on top of your data and discuss how well your line fits the data. (This may take a few attempts.)

(d) Despite how well your data fit a linear model, it is not entirely sensible to use a linear model for this data. Why?