Active Calculus
Activities Book

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Contents

Chapter 1

Preliminaries: A Library of Functions

1.1 Functions, Slope, and Lines

Preview Activity 1.1. This is the first Preview Activity in this text. Your job for this activity is to get to know the textbook.

(a) Where can you find the full textbook?

(b) What chapters of this text are you going to cover this semester. Have a look at your syllabus!

(c) What are the differences between Preview Activities, Activities, Examples, Exercises, Voting Questions, and WeBWork? Which ones should you do before class, which ones will you likely do during class, and which ones should you be doing after class?

(d) What materials in this text would you use to prepare for an exam and where do you find them?

(e) What should you bring to class every day?

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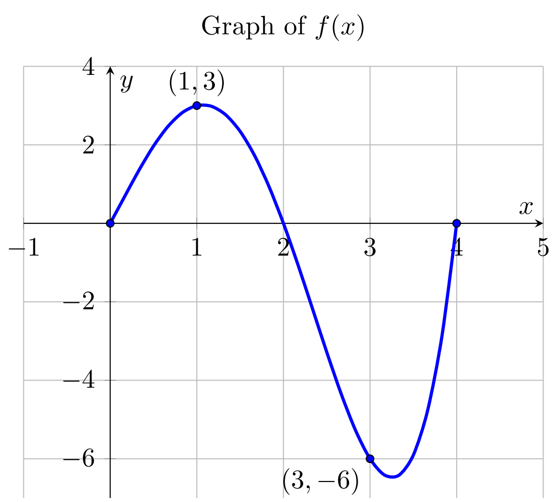
Activity 1.1.

The graph of a function $f(x)$ is shown in the plot below.



(a) What is the domain of $f(x)$?

(b) Approximate the range of $f(x)$.

(c) What are $f(0)$, $f(1)$, $f(3)$, $f(4)$, and $f(5)$?

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Activity 1.2.

Find the equation of the line with the given information.

(a) The line goes through the points $(-2,5)$ and $(10,-1)$.

(b) The slope of the line is $3/5$ and it goes through the point $(2,3)$.

(c) The y-intercept of the line is $(0,-1)$ and the slope is $-2/3$.

$\triangleleft$

Activity 1.3.

An apartment manager keeps careful record of the rent that he charges as well as the number of occupied apartments in his complex. The data that he has is shown in the table below.

Monthly Rent	\$650	\$700	\$750	\$800	\$850	\$900
Occupied Apartments	203	196	189	182	175	168

(a) Just by doing simple arithmetic justify that the function relating the number of occupied apartments and the rent is linear.

(b) Find the linear function relating the number of occupied apartments to the rent.

(c) If the rent were to be increased to \$1000, how many occupied apartments would the apartment manager expect to have?

(d) At a \$1000 monthly rent what net revenue should the apartment manager expect?

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Activity 1.4.

Write the equation of the line with the given information.

(a) Write the equation of a line parallel to the line $y = \frac{1}{2}x + 3$ passing through the point $(3,4)$.

(b) Write the equation of a line perpendicular to the line $y = \frac{1}{2}x + 3$ passing through the point $(3,4)$.

(c) Write the equation of a line with y-intercept $(0,-3)$ that is perpendicular to the line $y = -3x - 1$.

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Voting Questions

1.2 Exponential Functions

Preview Activity 1.2. Suppose that the populations of two towns are both growing over time. The town of Exponentia is growing at a rate of 2% per year, and the town of Lineola is growing at a rate of 100 people per year. In 2014, both of the towns have 2,000 people.

(a) Complete the table for the population of each of these towns over the next several years.

	2014	2015	2016	2017	2018	2019	2020	2021	2022
Exponentia	2000								
Lineola	2000								

(b) Write a linear function for the population of Lineola. Interpret the slope in the context of this problem.

(c) The ratio of successive populations for Exponentia should be equal. For example, dividing the population in 2015 by that of 2014 should give the same ratio as when the population from 2016 is divided by the population of 2015. Find this ratio. How is this ratio related to the 2% growth rate?

(d) Based on your data from part (a) and your ratio in part (c), write a function for the population of Exponentia.

(e) When will the population of Exponentia exceed that of Lineola?

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Activity 1.5.

Consider the exponential functions plotted in Figure 1.1

(a) Which of the functions have common ratio $r > 1$?

(b) Which of the functions have common ratio $0 < r < 1$?

(c) Rank each of the functions in order from largest to smallest r value.

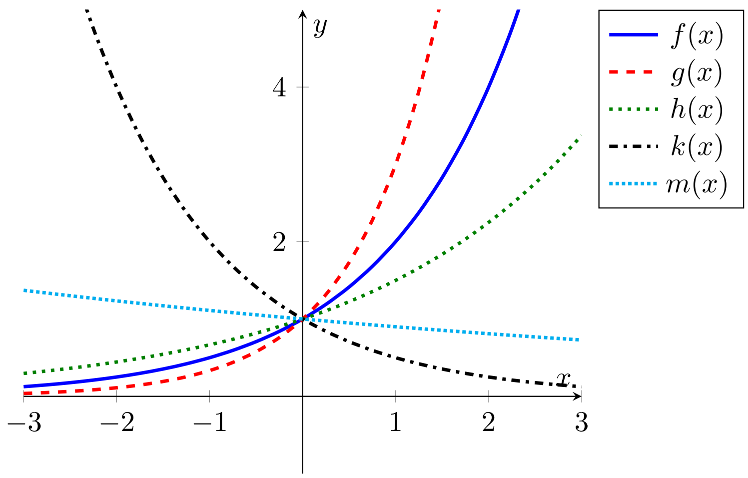

Figure 1.1: Exponential growth and decay functions

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Activity 1.6.

A sample of Ni^{56} has a half-life of 6.4 days. Assume that there are 30 grams present initially.

(a) Write a function describing the number of grams of Ni^{56} present as a function of time. Check your function based on the fact that in 6.4 days there should be 50% remaining.

(b) What percent of the substance is present after 1 day?

(c) What percent of the substance is present after 10 days?

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Activity 1.7.

Uncontrolled geometric growth of the bacterium *Escherichia coli (E. Coli)* is the theme of the following quote taken from the best-selling author Michael Crichton's science fiction thriller, *The Andromeda Strain:*

> "The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."

(a) Write an equation for the number of E. coli cells present if a single cell of E. coli divides every 20 minutes.

(b) How many E. coli would there be at the end of 24 hours?

(c) The mass of an E. coli bacterium is 1.7×10^{-12} grams, while the mass of the Earth is 6.0×10^{27} grams. Is Michael Crichton's claim accurate? Approximate the number of hours we should have allowed for this statement to be correct?

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Voting Questions

1.3 Transformations of Functions

Preview Activity 1.3. The goal of this activity is to explore and experiment with the function

$$F(x) = Af(B(x - C)) + D.$$

The values of A, B, C, and D are constants and the function $f(x)$ will be henceforth called the *parent function*. To facilitate this exploration, use the applet located at
http://www.geogebratube.org/student/m93018.

(a) Let's start with a simple parent function: $f(x) = x^2$.

(1) Fix $B = 1$, $C = 0$, and $D = 0$. Write a sentence or two describing the action of A on the function $F(x)$.

(2) Fix $A = 1$, $B = 1$, and $D = 0$. Write a sentence of two describing the action of C on the function $F(x)$.

(3) Fix $A = 1$, $B = 1$, and $C = 0$. Write a sentence of two describing the action of D on the function $F(x)$.

(4) Fix $A = 1$, $C = 0$, and $D = 0$. Write a sentence of two describing the action of B on the function $F(x)$.

(b) In part (a) you have made conjectures about what A, B, C, and D do to a parent function graphically. Test your conjectures with the functions $f(x) = |x|$ (typed `abs(x)`), $f(x) = x^3$, $f(x) = \sin(x)$, $f(x) = e^x$ (typed `exp(x)`), and any other function you find interesting.

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Activity 1.8.

Consider the function $f(x)$ displayed in Figure 1.2.

(a) Plot $g(x) = -f(x)$ and $h(x) = f(x) - 1$.

(b) Define the function $k(x) = -f(x) - 1$. Does it matter which order you complete the transformations from part (a) to result in $k(x)$? Plot the functions resulting from doing the two transformation in part (a) in opposite orders. Which of these functions is $k(x)$?

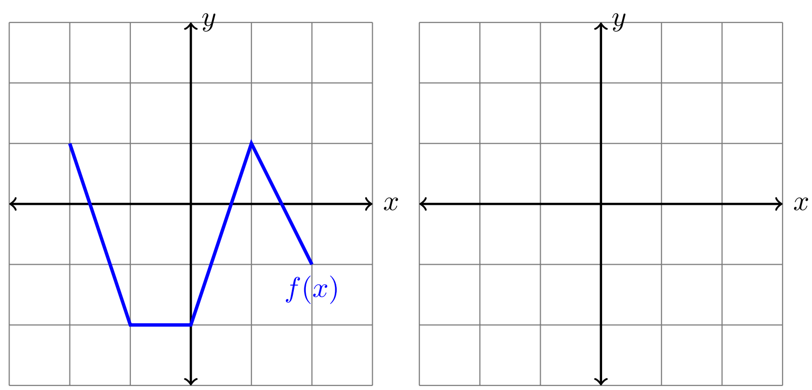

Figure 1.2: Function transformation for Activity 1.9

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Activity 1.9.

(a) Let $f(x) = x^2$ and $g(x) = x + 8$. Find the following:

$f(g(3)) =$ ____________, $g(f(3)) =$ ____________, $f(g(x)) =$ ____________,

$g(f(x)) =$ ____________, $f(x)g(x) =$ ____________

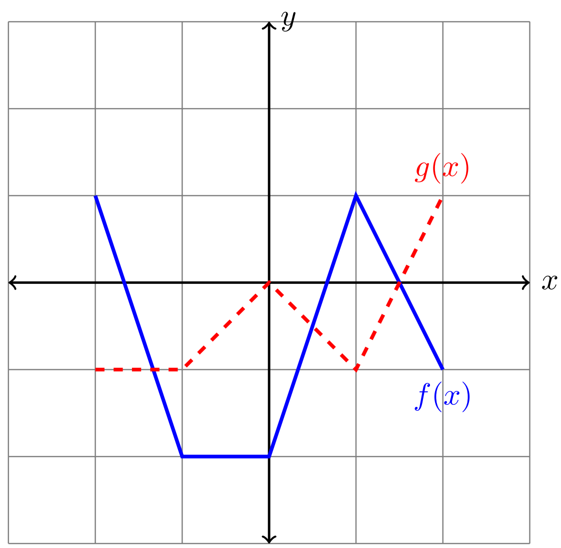
(b) Now let $f(x)$ and $g(x)$ be defined as in the table below. Use the data in the table to find the following compositions.

x	-3	-2	-1	0	1	2	3
$f(x)$	3	1	-1	-3	-1	1	3
$g(x)$	-2	-1	0	1	0	1	2

$f(-3) =$ ____________, $g(3) =$ ____________,

$f(g(-3)) =$ ____________, $f(g(f(-3))) =$ ____________

(c) Now let $f(x)$ and $g(x)$ be defined as in the plots below. Use the plots to find the following compositions.



$f(1) =$ ____________

$g(2) =$ ____________

$g(f(1)) =$ ____________

$f(g(1)) =$ ____________

$g(f(f(0))) =$ ____________

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Activity 1.10.

(a) Based on symmetry alone, is $f(x) = x^2$ an even or an odd function?

(b) Based on symmetry alone, is $g(x) = x^3$ an even or an odd function?

(c) Find $f(-x)$ and $g(-x)$ and make conjectures to complete these sentences:

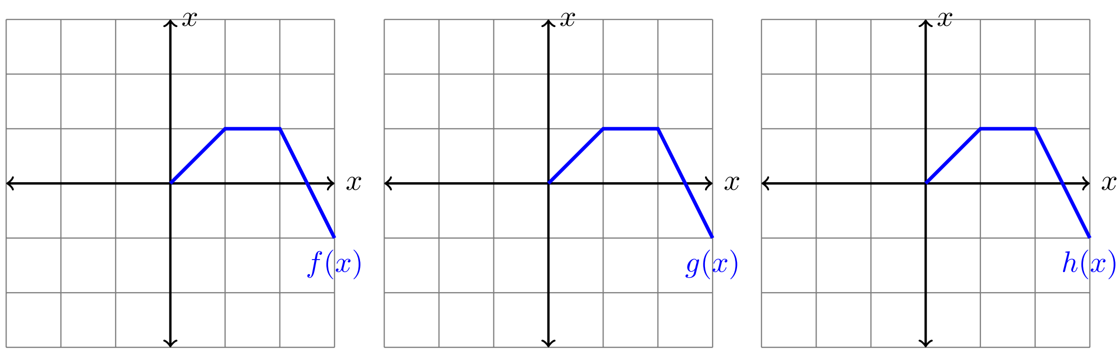
- If a function $f(x)$ is even then $f(-x) =$______________.
- If a function $f(x)$ is odd then $f(-x) =$______________.

Explain why the composition $f(-x)$ is a good test for symmetry of a function.

(d) Classify each of the following functions as even, odd, or neither.

$$h(x) = \frac{1}{x}, \quad j(x) = e^x, \quad k(x) = x^2 - x^4, \quad n(x) = x^3 + x^2.$$

(e) Each figure below shows only half of the function. Draw the left half so $f(x)$ is even. Draw the left half so $g(x)$ is odd. Draw the left half so $h(x)$ is neither even nor odd.



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Activity 1.11.

(a) Find the inverse of each of the following functions by interchanging the x and y and solving for y. Be sure to state the domain for each of your answers.

$$y = \sqrt{x-1}, \quad y = -\frac{1}{3}x + 1, \quad y = \frac{x+4}{2x-5}$$

(b) Verify that the functions $f(x) = 3x - 2$ and $g(x) = \frac{x}{3} + \frac{2}{3}$ are inverses of each other by computing $f(g(x))$ and $g(f(x))$.

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Voting Questions

1.4 Logarithmic Functions

Preview Activity 1.4. Carbon-14 (^{14}C) is a radioactive isotope of carbon that occurs naturally in the Earth's atmosphere. During photosynthesis, plants take in ^{14}C along with other carbon isotopes, and the levels of ^{14}C in living plants are roughly the same as atmospheric levels. Once a plant dies, it no longer takes in any additional ^{14}C. Since ^{14}C in the dead plant decays at a predictable rate (the half-life of ^{14}C is approximately 5,730 years), we can measure ^{14}C levels in dead plant matter to get an estimate on how long ago the plant died. Suppose that a plant has 0.02 milligrams of ^{14}C when it dies.

(a) Write a function that represents the amount of ^{14}C remaining in the plant after t years.

(b) Complete the table for the amount of ^{14}C remaining t years after the death of the plant.

t	0	1	5	10	100	1000	2000	5730
^{14}C Level	0.02							

(c) Suppose our plant died sometime in the past. If we find that there are 0.014 milligrams of ^{14}C present in the plant now, estimate the age of the plant to within 50 years.

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Activity 1.12.

Use the definition of a logarithm along with the properties of logarithms to answer the following.

(a) Write the exponential expression $8^{1/3} = 2$ as a logarithmic expression.

(b) Write the logarithmic expression $\log_2 \frac{1}{32} = -5$ as an exponential expression.

(c) What value of x solves the equation $\log_2 x = 3$?

(d) What value of x solves the equation $\log_2 4 = x$?

(e) Use the laws of logarithms to rewrite the expression $\log\left(x^3 y^5\right)$ in a form with no logarithms of products, quotients, or powers.

(f) Use the laws of logarithms to rewrite the expression $\log\left(\frac{x^{15} y^{20}}{z^4}\right)$ in a form with no logarithms of products, quotients, or powers.

(g) Rewrite the expression $\ln(8) + 5\ln(x) + 15\ln(x^2 + 8)$ as a single logarithm.

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Activity 1.13.

Solve each of the following equations for t, and verify your answers using a calculator.

(a) $\ln t = 4$

(b) $\ln(t+3) = 4$

(c) $\ln(t+3) = \ln 4$

(d) $\ln(t+3) + \ln(t) = \ln 4$

(e) $e^t = 4$

(f) $e^{t+3} = 4$

(g) $2e^{t+3} = 4$

(h) $2e^{3t+2} = 3e^{t-1}$

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Activity 1.14.

Consider the following equation:

$$7^x = 24$$

(a) How many solutions should we expect to find for this equation?

(b) Solve the equation using the log base 7.

(c) Solve the equation using the log base 10.

(d) Solve the equation using the natural log.

(e) Most calculators have buttons for $\log_{10}$ and ln, but none have a button for $\log_7$. Use your previous answers to write a formula for $\log_7 x$ in terms of $\log x$ or $\ln x$.

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Activity 1.15.

(a) In the presence of sufficient resources the population of a colony of bacteria exhibits exponential growth, doubling once every three hours. What is the corresponding continuous (percentage) growth rate?

(b) A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling so its temperature, T (measured in degrees Fahrenheit) after t minutes is given by

$$\mathrm{T}(t) = 65 + 186e^{-0.06t}.$$

How long will it take from the time the food is served until the temperature is 120°F?

(c) The velocity (in ft/sec) of a sky diver t seconds after jumping is given by

$$v(t) = 80\left(1 - e^{-0.2t}\right).$$

After how many seconds is the velocity 75 ft/sec?

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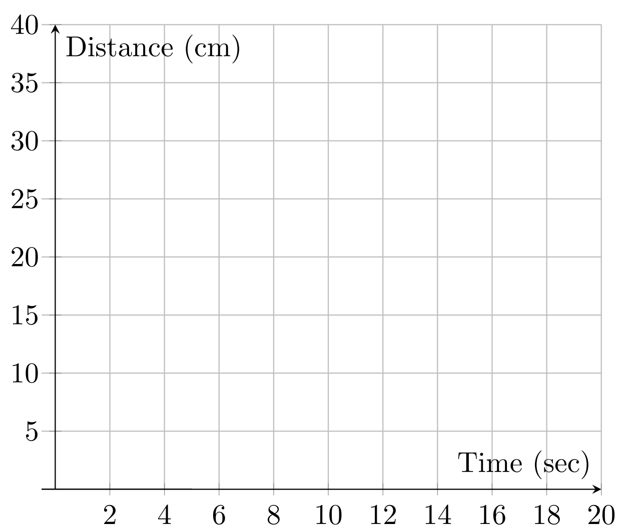
Voting Questions

1.5 Trigonometric Functions

Preview Activity 1.5. A tall water tower is swaying back and forth in the wind. Using an ultrasonic ranging device, we measure the distance from our device to the tower (in centimeters) every two seconds with these measurements recorded below.

Time (sec)	0	2	4	6	8	10	12	14	16	18	20
Distance (cm)	30.9	23.1	14.7	12.3	17.7	26.7	32.3	30.1	21.8	13.9	12.6

(a) Use the coordinate plane below to create a graph of these data points.



(b) What is the water tower's maximum distance away from the device?

(c) What is the smallest distance measured from the tower to the device?

(d) If the water tower was sitting still and no wind was blowing, what would be the distance from the tower to the device? We call this the tower's equilibrium position.

(e) What is the maximum distance that the tower moves away from its equilibrium position? We call this the amplitude of the oscillations.

(f) How much time does it take the water tower to sway back and forth in a complete cycle? We call this the period of oscillation.

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Activity 1.16.

Figure 1.3 gives us the voltage produced by an electrical circuit as a function of time.

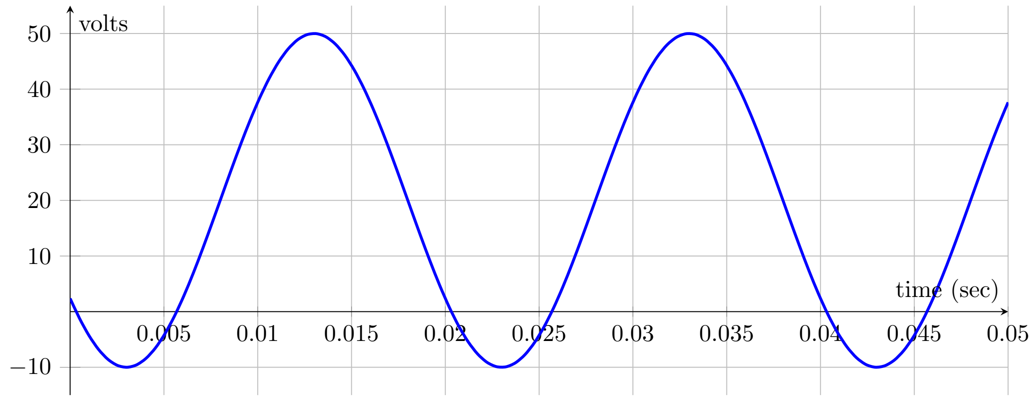

Figure 1.3: Voltage as a function of time.

(a) What is the amplitude of the oscillations?

(b) What is the period of the oscillations?

(c) What is the average value of the voltage?

(d) What is the shift along the t axis, t_0?

(e) What is a formula for this function?

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Activity 1.17.

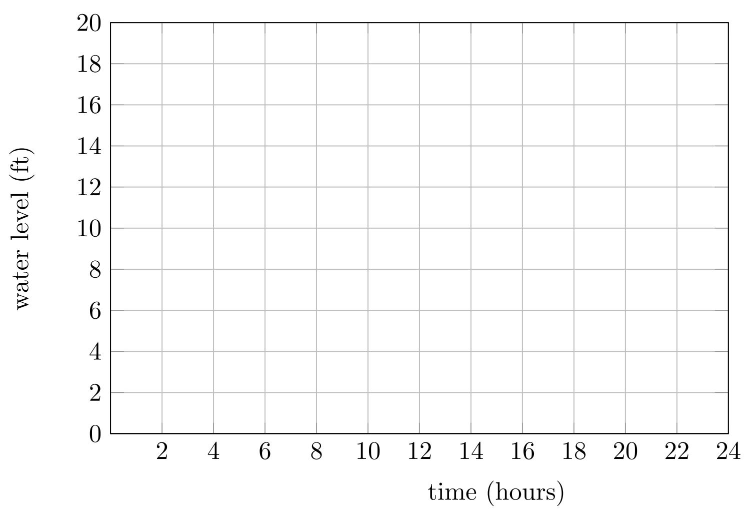
Suppose the following sinusoidal function models the water level on a pier in the ocean as it changes due to the tides during a certain day.

$$w(t) = 4.3\sin(0.51t + 0.82) + 10.6$$

(a) Using the formula above, make a table showing the water level every two hours for a 24 hour period starting at midnight.

time (hours)	0	2	4	6	8	10	12	14	16	18	20	22	24
water level (ft)													

(b) Sketch a graph of this function using the data from your table in part (a).



(c) What is the period of oscillation of this function?

(d) What time is high tide?

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Voting Questions

1.6 Powers, Polynomials, and Rational Functions

Preview Activity 1.6. Figure 1.4 shows the graphs of two different functions. Suppose that you were to graph a line anywhere along each of the two graphs.

1. Is it possible to draw a line that does not intersect the graph of f? g?
2. Is it possible to draw a line that intersects the graph of f an even number of times?
3. Is it possible to draw a line that intersects the graph of g an odd number of times?
4. What is the fewest number of intersections that your line could have with the graph of f? with g?
5. What is the largest number of intersections that your line could have with the graph of f? with g?
6. How many times does the graph of f change directions? How many times does the graph of g change directions?

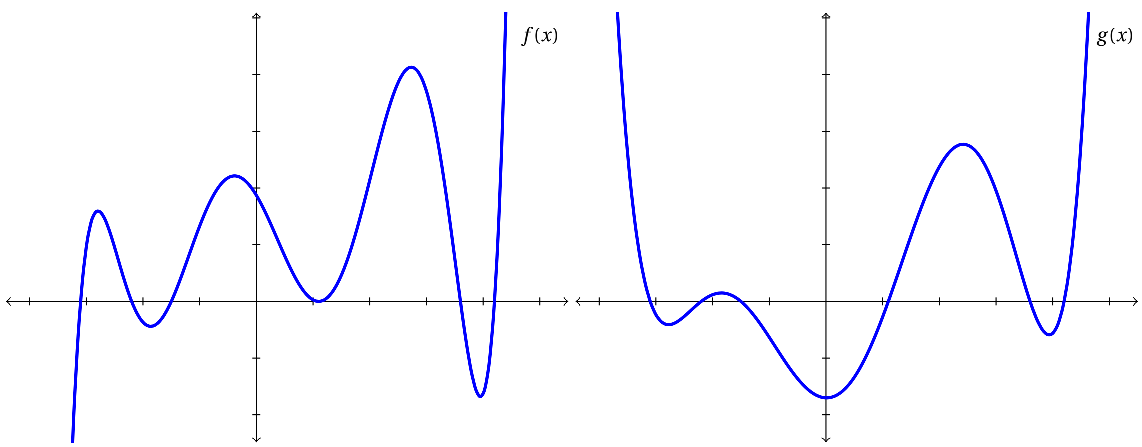

Figure 1.4: $f(x)$ and $g(x)$ for the preview activity.

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Activity 1.18.

Power functions and exponential functions appear somewhat similar in their formulas, but behave differently in many ways.

(a) Compare the functions $f(x) = x^2$ and $g(x) = 2^x$ by graphing both functions in several viewing windows. Find the points of intersection of the graphs. Which function grows more rapidly when x is large?

(b) Compare the functions $f(x) = x^{10}$ and $g(x) = 2^x$ by graphing both functions in several viewing windows. Find the points of intersection of the graphs. Which function grows more rapidly when x is large?

(c) Make a conjecture: As $x \to \infty$, which dominates, x^n or a^x?

(d) Suppose you are offered a job that lasts one month. You have the option of being paid in one of two ways: (1) One million dollars at the end of the month; or (2) One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the n^{th} day. Which option should you choose?

(e) How much different (shorter or longer) would the work period need to be for your answer to the previous question change?

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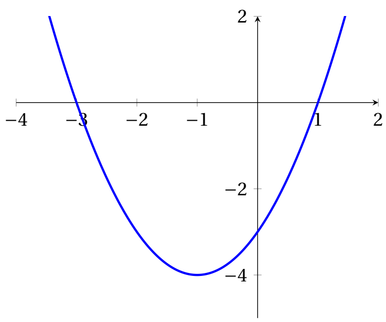
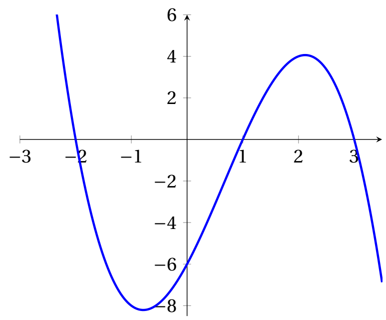
Activity 1.19.

For each of the following graphs, find a possible formula for the polynomial of lowest degree that fits the graph.

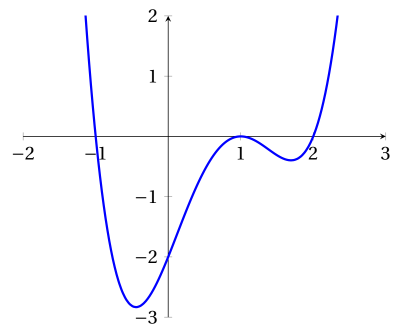
Plot (a)



Plot (b)



Plot (c)



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Activity 1.20.

(a) Suppose $f(x) = x^2 + 3x + 2$ and $g(x) = x - 3$.

1. What is the behavior of the function $h(x) = \dfrac{f(x)}{g(x)}$ near $x = -1$? (i.e. what happens to $h(x)$ as $x \to -1$?) near $x = -2$? near $x = 3$?
2. What is the behavior of the function $h(x) = \dfrac{g(x)}{f(x)}$ near $x = -1$? near $x = -2$? near $x = 3$?

(b) Suppose $f(x) = x^2 - 9$ and $g(x) = x - 3$.

1. What is the behavior of the function $h(x) = \dfrac{f(x)}{g(x)}$ near $x = -3$? (i.e. what happens to $h(x)$ as $x \to -3$?) near $x = 3$?
2. What is the behavior of the function $h(x) = \dfrac{g(x)}{f(x)}$ near $x = -3$? near $x = 3$?

(c) Suppose $f(x) = \sin x$ and $g(x) = x$.

1. What is $f(0)$? What is $g(0)$?
2. What is the behavior of the function $h(x) = \dfrac{f(x)}{g(x)}$ near $x = 0$? (i.e. what happens to $h(x)$ as $x \to 0$?)
3. What is the behavior of the function $h(x) = \dfrac{g(x)}{f(x)}$ near $x = 0$?

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Activity 1.21.

(a) Suppose $f(x) = x^3 + 2x^2 - x + 7$ and $g(x) = x^2 + 4x + 2$.

1. Which function dominates as $x \to \infty$?
2. What is the behavior of the function $h(x) = \dfrac{f(x)}{g(x)}$ as $x \to \infty$?
3. What is the behavior of the function $h(x) = \dfrac{g(x)}{f(x)}$ as $x \to \infty$?

(b) Suppose $f(x) = 2x^4 - 5x^3 + 8x^2 - 3x - 1$ and $g(x) = 3x^4 - 2x^2 + 1$

1. Which function dominates as $x \to \infty$?
2. What is the behavior of the function $h(x) = \dfrac{f(x)}{g(x)}$ as $x \to \infty$?
3. What is the behavior of the function $h(x) = \dfrac{g(x)}{f(x)}$ as $x \to \infty$?

(c) Suppose $f(x) = e^x$ and $g(x) = x^{10}$.

1. Which function dominates as $x \to \infty$ as $x \to \infty$?
2. What is the behavior of the function $h(x) = \dfrac{f(x)}{g(x)}$ as $x \to \infty$?
3. What is the behavior of the function $h(x) = \dfrac{g(x)}{f(x)}$ as $x \to \infty$?

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Activity 1.22.

For each of the following functions, determine (1) whether the function has a horizontal asymptote, and (2) whether the function crosses its horizontal asymptote.

(a) $f(x) = \dfrac{x+3}{x-2}$

(b) $g(x) = \dfrac{x^2+2x-1}{x-1}$

(c) $h(x) = \dfrac{x+1}{x^2+2x-1}$

(d) $k(x) = e^x \sin x$

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Voting Questions