MATH 322 **Homework 6 Solutions** Due: Wednesday, March 27

Instructions: Same rules as usual. Work together, write-up alone, no internet!

(6pts) 1. For any prime p, a *p-group* is a group of order p^n for some n.

(a) Explain why every element of a p group has order that is a power of p.

> **Solution:** This is by Lagrange's theorem. The order of the group is a power of the prime p. Since every element must have an order that divides the order of the group, every element of the group has order a power of p.

(b) Prove that for any group G, if every element has order some power of p, then G is a p-group. Hint: apply Cauchy's theorem.

> **Solution:** Suppose the order of G was not a power of p. Then there would be some other prime q which divides the order of G. By Cauchy's theorem, this means that there would be an element of order q, contradicting our assumption that every element has order some power of q (no prime is a power of any other prime).

(6pts) 2. Prove, using inner direct products, that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ if and only if $\gcd(m, n) = 1$. Note that the textbook has a proof of this using other methods, but you must use inner direct products for credit here.

> **Solution:** Consider $\mathbb{Z}_{mn} = \{0, 1, 2, \ldots, mn\}$. Consider the cyclic subgroups $\langle m \rangle = \{0, m, 2m, \ldots (n-1)m\} \cong \mathbb{Z}_n$ and $\langle n \rangle = \{0, n, 2n, \ldots, (m-1)n\} \cong \mathbb{Z}_m$. Is $\mathbb{Z}_{mn}$ the inner direct product of $\langle m \rangle$ and $\langle n \rangle$?
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> To make this so requires that everything in $\mathbb{Z}_{mn}$ can be written as the sum of an element from $\langle m \rangle$ and an element from $\langle n \rangle$, and that $\langle m \rangle \cap \langle n \rangle = \{0\}$. Notice that the first condition is equivalent to saying that for every $k \in \mathbb{Z}_{mn}$, there are integers r and s such that $rm + sn = k$. This is equivalent to saying that for some integers r and s, $rm + sn = 1$ (because then we could multiply both sides by k). By Bezout's lemma, this happens if and only if $\gcd(m, n) = 1$.
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> The second condition also requires that $\gcd(m, n) = 1$, since we need the least common multiple of m and n to be mn.

(6pts) 3. Consider the group $U_{35} = \{1, 2, 3, 4, 6, 8, 9, 11, 12, 13, 16, 17, 18, 19, 22, 23, 24, 26, 27, 29, 31, 32, 33, 34\}$ under the operation of multiplication modulo 35. The orders of the elements are:

g	1	2	3	4	6	8	9	11	12	13	16	17	18	19	22	23	24	26	27	29	31	32	33	34
ord(g)	1	12	12	6	2	4	6	3	12	4	3	12	12	6	4	12	6	6	4	2	6	12	12	2

(a) Find two p-groups H and K such that U_{35} is the internal direct product of H and K. Briefly explain why your groups work.

> **Solution:** The only primes which divide $|U_{35}|$ are 2 and 3, so we are looking for a 2-group and a 3-group. We take $G(2) = \{1, 6, 8, 13, 22, 27, 29, 34\}$ to be the group of elements whose orders are powers of 2 and $G(3) = \{1, 11, 16\}$ to be the group of

elements whose orders are powers of 3. Clearly $G(2) \cap G(3) = \{1\}$, and we can also show that $G(2) \cdot G(3) = U_{35}$ by simply multiplying out every pair of elements, one from $G(2)$ and the other from $G(3)$ (we will get exactly the elements of U_{35}).

(b) Let H be the larger of the two groups above. Show how to decompose it as the internal direct product of $\langle a \rangle$ and H' where a is of maximal order and H' is some other subgroup of H.

Solution: $H = G(2)$. We take any element of $G(2)$ of maximal order. That is, consider $\langle 8 \rangle = \{1, 8, 29, 22\}$. For H' we need a 2-element subgroup generated by an element not in $\langle 8 \rangle$. So let $H' = \{1, 6\}$. Again, the intersection is clearly just $\{1\}$ and multiplying the elements of $\langle 8 \rangle$ by 6 gives the other elements of H.

(c) Using the decompositions above (perhaps repeating the second step as needed), write U_{35} as the direct product of groups of the form $\mathbb{Z}_{p^k}$ (p prime).

Solution: $G(3) \cong \mathbb{Z}_3$ and $H' \cong \mathbb{Z}_2$. Since $\langle 8 \rangle$ contains an element of order 4 we have $\langle 8 \rangle \cong \mathbb{Z}_4$. Thus

$$U_{35} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4.$$

(6pts) 4. Describe all abelian groups of order 200 (up to isomorphism). Explain how you know you have them all.

Solution: We have $200 = 2^3 \cdot 5^2$, so each abelian group will be isomorphic to the direct product of a 2-group of order 8 and a 5-group of order 25. There are two 5-groups of order 25: $\mathbb{Z}_5 \times \mathbb{Z}_5$ and $\mathbb{Z}_{25}$ (the first has no element of order 25, the other does have such an element). There are three 2-groups of order 8, depending on whether there are elements of order 8, none of order 8 but some of order 4, and none of order 4: $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus the 6 abelian groups of order 200 are:

$$\mathbb{Z}_8 \times \mathbb{Z}_{25} \qquad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{25} \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$$

$$\mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \qquad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$$

(6pts) 5. Let G, H, and K be finite abelian groups. Suppose $G \times H \cong G \times K$. Prove that $H \cong K$.

Solution: The Fundamental Theorem of Finite Abelian Groups says that every finite abelian group can be written uniquely as the direct product of cyclic p-groups (each group in the product is $\mathbb{Z}_{p^k}$ for a prime p positive integer k). Suppose that $H \ncong K$. Then writing each as a direct product of cyclic p-groups would give different direct products. Now take the direct product of these decompositions with G, also written as the direct product of cyclic p-groups. This would say that the way we write $G \times H$ as the direct product of cyclic

p-groups is different than the way we write $G \times K$ is the direct product of cyclic p-groups, which says that $G \times H \ncong G \times K$.