MATH 322 **Exam 1** Spring 2021

Instructions: Write all answers in the space provided, showing work and providing explanations. Open notes, open book, but you must WORK ALONE and may NOT search the internet for answers.

By signing below, I certify that the work on this take-home exam is solely my own, that I did not receive assistance from anyone other than my instructor, and did not use resources other than my own notes and the course textbook.

Signature: ______________________________ **Date:** ____________________

(15pts) 1. Give an example of each of the following. For each, explain how you know your example is correct (that is, say why your example really is an example of what was requested).

(a) An algebraic number that is not constructible using a straight-edge and compass.

(b) An extension field E of $\mathbb{Q}$ such that every element of E is the root of a degree 4 polynomial.

(c) Two extension fields E and F of $\mathbb{Q}$ such that $[E : \mathbb{Q}] = [F : \mathbb{Q}]$ but E is not isomorphic to F. (Make sure to explain why your example satisfies both properties.)

(16pts) 2. Let's look at the field $E = \mathbb{Q}(\sqrt{5}, \sqrt[3]{5})$ and its subfields. Find the degree of the field extensions requested below and explain how you know you are correct.

(a) $[\mathbb{Q}(\sqrt{5}) : \mathbb{Q}]$.

(b) $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$.

(c) $[E : \mathbb{Q}]$.

(d) $[E : \mathbb{Q}(\sqrt{5})]$.

(6pts) 3. Use the previous problem to prove that there do not exists any rational numbers a, b, and c such that $\sqrt{5} = a + b\sqrt[3]{5} + c\sqrt[3]{5}^2$. Make sure to say what this has to do with fields.

(24pts) 4. The polynomial $p(x) = x^4 - 10x^2 + 25x - 5$ is irreducible. Say ϱ is one of the roots of $p(x)$. While ϱ is not an element of $\mathbb{Q}$, we can form a smallest possible extension field of $\mathbb{Q}$ that contains ϱ. There are two ways to represent E: one as a quotient ring, the other as $\mathbb{Q}(\varrho)$.

(a) Carefully describe these two representations: the general form of elements in each. Also, provide at least two specific examples of elements in both representations and how they are related.

(b) Write $\varrho^5 - 7\varrho^3 + 1$ in terms of the basis for $\mathbb{Q}(\varrho)$ (first find ϱ^4 and ϱ^5). Then use long division to find the remainder when $x^5 - 7x^3 + 1$ is divided by $p(x)$. What do you notice and why does it make sense?

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(c) The field E is really a field, so there is a multiplicative inverse for every non-zero element. Find the multiplicative inverse of $\varrho^3 - 4\varrho^2 + 6\varrho + 1$ using polynomials. Show all your work.

(d) Let K be the splitting field for $p(x)$ (the specific polynomial from the previous page). What are the three possible values for $[K : \mathbb{Q}]$? In particular, explain why $[K : \mathbb{Q}]$ cannot be 4. (It might be helpful to graph $p(x)$.)

(24pts) 5. As we saw in class, the polynomial $p(x) = x^3 - 2$. has three roots: $\sqrt[3]{2}$, $\sqrt[3]{2}\omega$, and $\sqrt[3]{2}\omega^2$ where $\omega = e^{i2\pi/3}$. The splitting field E has Galois group $G(E/\mathbb{Q})$ isomorphic to S_3. (Note, ω is a root of $x^2 + x + 1$ since $x^3 - 1 = (x - 1)(x^2 + x + 1)$)

(a) We can write $E = \mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{2}\omega)$ or $E = \mathbb{Q}(\sqrt[3]{2}, \omega)$. Explain how we know these are the same. Then write down two bases for E using these two representations.

(b) One element of the Galois group can be described as

$$\sigma = \begin{pmatrix} \sqrt[3]{2} & \omega \\ \sqrt[3]{2}\omega & \omega^2 \end{pmatrix}.$$

Write the five other elements of the Galois group in this form (this is different from the way we wrote them in class, but that is the point of this problem).

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(c) The polynomial $f(x) = x^6 - 3x^5 + 6x^4 - 11x^3 + 12x^2 + 3x + 1$ happens to have $\sqrt[3]{2} + \omega$ as a root. Use all the elements of the Galois group to find all the roots of the polynomial. Briefly explain why this works.

(d) What can we now conclude about the field $\mathbb{Q}(\sqrt[3]{2}+\omega)$ and how it relates to $\mathbb{Q}(\sqrt[3]{2}, \omega)$? Justify your answer.

(15pts) 6. Think back over the material we have discussed so far in this course and identify one topic, theorem, proof, part of a proof, or even example that you either still do not understand or really struggled with before understanding. Carefully explain the context and what your understanding of it is, and exactly where and why you are or were stuck. Then briefly reflect (write about) what you have learned about mathematics and yourself through your struggle with this topic.

You should use *at least* this full page for your response, using medium sized handwriting.

(15bn-pts) 7. Really fun BONUS! The polynomial $p(x) = x^7 - 1$ has a root $\alpha = e^{i2\pi/7}$. This is a primitive 7th root of unity; the other imaginary roots are $\alpha^2, \alpha^3, \ldots, \alpha^6$. On the other hand, recall that the group $U(7)$ is the group with set $\{1, 2, \ldots, 6\}$ under the operation multiplication mod 7.

Illustrate how $U(7)$ is isomorphic to the Galois group for $p(x)$. Further, pick a subgroup and find its fixed field, and pick a different subfield (of the splitting field) and find its fixer. Lots of partial credit available here.