Activity: Working with S_5

We are close to our goal of finding a degree 5 polynomial whose roots cannot be expressed in a nice way (using radicals and field operations). It turns out that a key step in this goal is understanding the group S_5. In this activity, we will get a better feel for this group by examining the "cycle structure" of its elements.

1. Every element in S_5 can be written as a single cycle or a product of disjoint cycles. Write down all the different possibilities for how these cycles or products of cycles might look (focusing on their "shape" rather than the specific numbers in the cycles).

2. Does S_5 contain a non-trivial subgroup that contains all the transpositions (2-cycles)? What is it, or why not?

3. Does S_5 contain a non-trivial subgroup that includes the elements (12), (13), (14), and (15)? What else would it contain? Hint: what is (12)(14)(12)?

4. Does S_5 contain a non-trivial subgroup that contains (24) and (12345)? Think about what else such a subgroup would contain.

5. What if you started with a different 2-cycle and a different 5-cycle?Would any pair of 5-cycle and 2-cycle work?

Now let's consider the alternating group A_5. Recall this is the group of all permutations in S_5 that can be written as the product of an even number of 2-cycles.

6. If you write elements of A_5 as the product of disjoint cycles, what sorts of cycle structures do you get?

7. Does A_5 contain a non-trivial subgroup that contains all the 3-cycles? Hint: show that every pair of transpositions can be written as a product of 3-cycles.

8. Now consider *normal* subgroups N of A_5. Remember, a normal subgroup is closed under conjugates (here the conjugate would be aba^{-1} where $a \in A_5$ and $b \in N$). Does A_5 contain a non-trivial normal subgroup that contains (123)?

9. Look at the different cycle structures of elements in A_n and start taking conjugates. Will you be able to get (123) starting from any non-identity element?