Extending Fields to Factor

Goal: Build the smallest field possible in which $p(x) = x^3 + 3x^2 - x + 2$ is NOT irreducible.

Note that $p(x)$ is irreducible over $\mathbb{Q}$ because it has no roots in $\mathbb{Q}$ (why is this and why is that enough?). So let's invent a new number, call it ξ, and insist that ξ is a root of $p(x)$. Then consider the smallest field E larger than $\mathbb{Q}$ that also contains ξ.

1. List five elements in E that are NOT already in $\mathbb{Q}$.

2. The element ξ^3 is in E, but this can also be written using smaller powers of ξ. How?

3. Describe E as a set using set builder notation. In other words, E is the set of all elements of the form ...

4. Wait: why are we doing this? Our goal is for $p(x)$ to factor. Does it? What would one of the factors be?

5. Wait again: we want E to be a field. Is it? What would we need to check?

6. List five elements in the quotient ring $\mathbb{Q}[x]/\langle p(x)\rangle$ (using the same $p(x)$ from the previous page). Remember, these will all be cosets.

7. The element $x^3 + \langle p(x)\rangle$ is an element of $\mathbb{Q}[x]/\langle p(x)\rangle$, but it can also be written as a "simpler" coset. How?

8. Describe $\mathbb{Q}[x]/\langle p(x)\rangle$ as a set using set builder notation. In other words, this quotient ring is the set of all cosets of the form ...

9. Wait: if we want to show that E is a field, and E is basically the same as $\mathbb{Q}[x]/\langle p(x)\rangle$, then we could just show $\mathbb{Q}[x]/\langle p(x)\rangle$ is a field. What would this mean? What do we need to verify?