Instructions: Write all answers in the space provided, showing work and providing explanations. Open notes, open book, but you must WORK ALONE and may NOT search the internet for answers.

By signing below, I certify that the work on this take-home exam is solely my own, that I did not receive assistance from anyone other than my instructor, and did not use resources other than my own notes and the course textbook.

Signature: ______________________________ **Date:** ______________

(8pts) 1. Consider the permutation (13)(14)(23). Rewrite this permutation as a product of three transpositions where 3 only occurs in the first transposition (reading from the left). Explain how you know what you found is equal to the original permutation.

(6pts) 2. Is it possible to rewrite (13)(14)(23) as the product of 6 transpositions? Of 9 transpositions? Explain.

(12pts) 3. Let G be a group containing elements $\{e, a, b, c, d, f, g, h\}$. Here is part of the table for G:

$$
\begin{array}{c|cccccccc}
 & e & a & b & c & d & f & g & h \\
\hline
\vdots & & & & \vdots & & & & \\
c & c & b & e & a & h & g & d & f \\
\vdots & & & & \vdots & & & &
\end{array}
$$

(a) Find the left regular representation of c as in Cayley's theorem. That is, find the permutation λ_c in S_8 corresponding to c.

(b) Suppose $\lambda_d = (1526)(37)(48)$ is the left regular representation of d. Write the row for d in the group table.

(c) If you compute $\lambda_c \lambda_d$ you get another permutation in S_8. Of which element in G is it the left regular representation? That is, find $x \in G$ such that $\lambda_x = \lambda_c \lambda_d$. Briefly explain how you know you are right.

(24pts) 4. Consider the group $\mathbb{Z}_{75}$.

(a) Give an example of a subnormal series which is not a composition series. Explain why your example works.

(b) Find a *refinement* of the series you gave above which is a composition series. That is, show how to extend your series into a composition series.

(c) Find *all* other composition series and briefly explain how you know you have them all.

(d) If $p(x)$ is a polynomial whose splitting field has Galois group over $\mathbb{Q}$ isomorphic to $\mathbb{Z}_{75}$, will the roots of $p(x)$ be expressible in terms of nth roots and field operations? Briefly explain (you can cite a result we discussed in class).

(12pts) 5. Suppose E is a splitting field whose Galois group over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_{75}$. (For this problem, you can refer to the work you did on the previous problem, part (c) in particular, as well as the Fundamental Theorem of Galois Theory.).

(a) How many different intermediate fields are there between $\mathbb{Q}$ and E? What degree extensions are these? Justify your answers. You may refer to the work you did on the previous problem (part (c) in particular)

(b) How many of the intermediate fields between $\mathbb{Q}$ and E are splitting fields for some polynomial? Briefly explain.

(12pts) 6. Consider the group $U(14297)$. In completing this question, you should NOT try to list out the elements of the group.

(a) What is the order of the group (i.e., how many elements are in the group)? Explain how you know. Hint: $14297 = 17 \cdot 29^2$

(b) Find 8640^{12992} modulo 14297, and explain how you know your answer is correct without using a calculator (and how this relates to the previous part).

(c) Let $E = 321$. Find a number D such that $8640^{ED} \equiv 8640 \pmod{14297}$. How do you know you are correct? If you use technology to find D, explain what the technology is giving you.

(8pts) 7. For how many abelian groups G of order less than 100, is $G = G(7)G(11)$? (Recall that $G(p)$ is the set of all elements in G of order p^k for some $k \geq 0$.) List all such groups and explain your answer.

(6pts) 8. Let G be a group of order $|G| = 16$, that acts on the set $X = \{1, 2, 3, 4, 5\}$. Prove that there is at least one element $x \in X$ such that $gx = x$ for all $g \in G$.

Hint 1: What do you need to show about $\mathcal{O}_x$?

Hint 2: What sizes could the stabilizer subgroups of G have?

(12pts) 9. You will create a cube by connecting 12 pipe cleaners of equal length together (to form the edges of the cube). The pipe cleaners come in 3 different colors (and you have at least 12 of each color you could use). How many different cubes could you make? Show your work.

(10bn-pts) 10. BONUS! Suppose you made a cube as in the previous problem but required that you used an equal number of each color (so four of each). Now how many different cubes could you make?