Name: ______________________________

Instructions: Answer each of the following questions, and make sure you SHOW ALL YOUR WORK! Answers without supporting work will be counted as incorrect. When asked to explain or prove your answers, use complete English sentences.

(12pts) 1. Suppose α is a real number such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. What does this tell you about...

(a) ...the number α in terms of polynomials? Be as specific as possible.

(b) ...about the field $\mathbb{Q}(\alpha)$? What does the field look like (i.e., what does a basis look like)?

(c) ...about the Galois group $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$, if you know also that $\mathbb{Q}(\alpha)$ is a splitting field? (You should say what sorts of things and how many of them are in the Galois group.)

(d) ...about whether it is possible to construct a line segment of length α using a compass and straight edge? Briefly explain.

(12pts) 2. For each item below, say whether the statement is TRUE or FALSE and justify your answer. If the statement is true, briefly explain why; if the statement is false, give a counterexample with brief explanation.

(a) Every algebraic number is constructible.

(b) If α is an algebraic number, then $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/\langle p(x)\rangle$ for some polynomial $p(x)$.

(c) If $\mathbb{Q}(\alpha)$ is a degree 2 extension of $\mathbb{Q}$, then then $\alpha = \sqrt{c}$ for some $c \in \mathbb{Q}$.

(d) If $p(\alpha) = 0$, then $\mathbb{Q}(\alpha)$ is the splitting field for $p(x)$.

(12pts) 3. Consider the field $E = \mathbb{Q}(\sqrt{5} + \sqrt[3]{7})$ and its subfields. In each part below, find the degree of the field extension and explain how you know you are correct.

(a) $[\mathbb{Q}(\sqrt{5}) : \mathbb{Q}]$.

(b) $[\mathbb{Q}(\sqrt[3]{7}) : \mathbb{Q}]$.

(c) $[E : \mathbb{Q}(\sqrt{5})]$. Hint: Use the fact that E contains both $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt[3]{7})$.

(d) $[E : \mathbb{Q}]$.

(4pts) 4. Use the previous question to prove that $x^6 - 15x^4 - 14x^3 + 75x^2 - 210x - 76$ is irreducible. Hint: the polynomial has $\sqrt{5} + \sqrt[3]{7}$ as a root.

5. Let $p(x) = (x^2 - 5)(x^3 - 7)$, and let E be the splitting field of $p(x)$.

(5pts) (a) Prove that there is no automorphism of E which sends $\sqrt{5}$ to $\sqrt[3]{7}$. Show specifically what goes wrong using the homomorphism property.

(5pts) (b) Give an example of a non-trivial automorphism of E and briefly explain how you know your example works.

(5bn-pts) (c) Bonus: Could $\mathrm{Gal}(E/\mathbb{Q})$ be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$? Explain.