Instructions: Same rules as usual. Work together, write-up alone, no internet!

(5pts) 1. We have seen that the group D_4 is isomorphic to a subgroup of S_4 (by numbering the vertices of the square). However, by Cayley's theorem, D_4 is also isomorphic to a subgroup of S_8. Find it. For at least one non-identity element, explain carefully how you found the corresponding element of S_8.

(10pts) 2. Practice working with cycles.

(a) For $\alpha = (123456)$, find α^2, α^3, α^4, etc.

(b) In S_5, find a cycle square root of each of the following cycles (that is, find a cycle α such that α^2 is the given element): (132), (12345), and (13)(24) (you will find different square roots for each, of course).

(c) Prove that if $\alpha = (a_1 a_2 \cdots a_s)$ for s odd (i.e., α has odd length), then α is the square of some cycle of length s.

(d) Prove that if α is of (even) length $s = 2t$, then α^2 is the product of two cycles of length t.

(e) Prove that if the length of α is prime, then every power of α is a cycle.

(6pts) 3. Recall that we say a permutation is *even* if it is possible to write it as the product of an even number of transpositions. On the other hand, the *length* of a cycle is the number of numbers appearing in the cycle.

(a) Prove that the product of two even permutations is even, the product of two odd permutations is even, and the product of an even and an odd permutation is odd.

(b) Prove that a cycle of length l is even if and only if l is odd. Note you must prove two directions here: if l is odd, then the cycle is even, and if l is even, then the cycle is odd.

(4pts) 4. Let $\alpha = (a_1 a_1 \cdots a_s)$ and $\beta = (b_1 b_2 \cdots b_r)$ be two disjoint cycles. Find a transposition γ such that $\alpha\beta\gamma$ is a cycle. Then show that $\alpha\gamma\beta$ and $\gamma\alpha\beta$ are also cycles.

(5pts) 5. The identity can be written as $\varepsilon = (13)(24)(35)(14)(12)(15)(34)(45)$. Mimic the proof that ε must be even and show how to eliminate $x = 5$ from the product of transpositions and write ε as the product of 2 fewer transpositions in the process. Show all intermediate steps.

(5bns-pts) 6. Bonus: Let $\alpha = (1372)(26374)(587)(1846)$. Write α^{2016} as a single cycle or the product of disjoint cycles. Explain how you know your answer is correct.