Instructions: Same rules as usual. Work together, write-up alone, no internet!

(5pts) 1. We have seen that the group D_4 is isomorphic to a subgroup of S_4 (by numbering the vertices of the square). However, by Cayley's theorem, D_4 is also isomorphic to a subgroup of S_8. Find it. For at least one non-identity element, explain carefully how you found the corresponding element of S_8.

Solution: The solutions depends on how you set up the group table (if you list elements in different orders, you will get different permutations). Here is one group table:

	R_0	R_1	R_2	R_3	H	V	D_1	D_2
R_0	R_0	R_1	R_2	R_3	H	V	D_1	D_2
R_1	R_1	R_2	R_3	R_0	D_1	D_2	V	H
R_2	R_2	R_3	R_0	R_1	V	H	D_2	D_1
R_3	R_3	R_0	R_1	R_2	D_2	D_1	H	V
H	H	D_2	V	D_1	R_0	R_2	R_3	R_1
V	V	D_1	H	D_2	R_2	R_0	R_1	R_3
D_1	D_1	H	D_2	V	R_1	R_3	R_0	R_2
D_2	D_2	V	D_1	H	R_3	R_1	R_2	R_0

Give this, the permutations are:

$\lambda_{R_0} = (1)$

$\lambda_{R_1} = (1234)(5768)$

$\lambda_{R_2} = (13)(24)(56)(78)$

$\lambda_{R_3} = (1432)(5867)$

$\lambda_H = (15)(28)(36)(47)$

$\lambda_V = (16)(27)(35)(48)$

$\lambda_{D_1} = (17)(25)(38)(46)$

$\lambda_{D_2} = (18)(26)(37)(45)$

(10pts) 2. Practice working with cycles.

(a) For $\alpha = (123456)$, find α^2, α^3, α^4, etc.

Solution: We have $\alpha^2 = (135)(246)$. $\alpha^3 = (14)(25)(36)$, $\alpha^4 = (153)(264)$, $\alpha^5 = (165432)$, $\alpha^6 = (1)$, and then the pattern repeats.

(b) In S_5, find a cycle square root of each of the following cycles (that is, find a cycle α such that α^2 is the given element): (132), (12345), and $(13)(24)$ (you will find different square roots for each, of course).

Solution: $(123)^2 = (132)$
$(14253)^2 = (12345)$.
$(1234)^2 = (13)(24)$.

(c) Prove that if $\alpha = (a_1 a_2 \cdots a_s)$ for s odd (i.e., α has odd length), then α is the square of some cycle of length s.

Solution: We need every other element of the cycle to be $a_1, a_2, a_3, \ldots$. Since s is odd, this will work and still give us just a single cycle. We get $(a_1 a_{\frac{s+1}{2}+1} a_2 a_{\frac{s+1}{2}+2} a_3 \cdots a_{\frac{s+1}{2}})$. Another way to see this: we know that $\alpha^{s+1} = \alpha$, since the order of α is s. But $s+1$ is even, so we can consider $\alpha^{\frac{s+1}{2}}$.

(d) Prove that if α is of (even) length $s = 2t$, then α^2 is the product of two cycles of length t.

Solution: The two cycles will be $(a_1 a_3 a_5 \cdots a_{s-1})$ and $(a_2 a_4 a_6 \cdots a_s)$. Each of these has length t.

(e) Prove that if the length of α is prime, then every power of α is a cycle.

Solution: If α^k can be written as a product of n cycles, then each cycle would have length l/n, where l is the length α. This means that l must be a multiple of n. But if the length is prime, that means n is either 1, or l. In the first case, we are done. In the second, we have that $\alpha^k = (1)$.

(6pts) 3. Recall that we say a permutation is *even* if it is possible to write it as the product of an even number of transpositions. On the other hand, the *length* of a cycle is the number of numbers appearing in the cycle.

(a) Prove that the product of two even permutations is even, the product of two odd permutations is even, and the product of an even and an odd permutation is odd.

Solution: Suppose we have two permutations α and β and that α can be written as k transpositions and β can be written as j transpositions. Then $\alpha\beta$ can be written as $k+j$ transpositions. If α and β are both even, then k and j must both be even, and so $k+j$ must also be even. Since the product can be written as an even number of transpositions, it *must* be written as an even number of transpositions, so $\alpha\beta$ is even. Similarly, if α and β are both odd, then $k+j$ is even (it is the sum of two odd numbers). Finally, if α is even and β is odd (or visa-versa) then $k+j$ is odd, so $\alpha\beta$ is an odd permutation.

(b) Prove that a cycle of length l is even if and only if l is odd. Note you must prove two directions here: if l is odd, then the cycle is even, and if l is even, then the cycle is odd.

Solution: We note that $(a_1 a_2 a_3 \cdots a_l)$ can be written as a product of transpositions as $(a_1 a_2)(a_2 a_3)(a_3 a_4) \cdots (a_{l-1} a_l)$. Thus a cycle of length l can be written as $l-1$

transpositions. If $l \geq 3$ is odd, then the permutation is even, since it can (and therefore must) be written as an even number $(l-1)$ of transpositions. If $l = 1$ then we are looking at the identity, which we proved is even since $(1) = (12)(12)$. Conversely, if l is even, then $l - 1$ is odd, so the permutation can (must) be written as an odd number of transpositions and is therefore odd.

(4pts) 4. Let $\alpha = (a_1 a_1 \cdots a_s)$ and $\beta = (b_1 b_2 \cdots b_r)$ be two disjoint cycles. Find a transposition γ such that $\alpha\beta\gamma$ is a cycle. Then show that $\alpha\gamma\beta$ and $\gamma\alpha\beta$ are also cycles.

Solution: We can take $\gamma = (a_s b_r)$. Then $\alpha\beta\gamma = (a_1 a_2 \cdots a_s b_1 b_2 \cdots b_r)$. Using this same γ, we have $\alpha\gamma\beta = (a_1 a_2 \cdots a_s b_r b_1 b_2 \cdots b_{r-1})$. Similarly, $\gamma\alpha\beta = (a_1 a_2 \cdots a_{s-1} b_r b_1 b_2 \cdots b_{r-1} a_s)$.

(5pts) 5. The identity can be written as $\varepsilon = (13)(24)(35)(14)(12)(15)(34)(45)$. Mimic the proof that ε must be even and show how to eliminate $x = 5$ from the product of transpositions and write ε as the product of 2 fewer transpositions in the process. Show all intermediate steps.

Solution: We look for the last occurrence of 5, which is in the final transposition. Now $(34)(45) = (35)(34)$, so we can write

$$\varepsilon = (13)(24)(35)(14)(12)(15)(35)(34)$$

Now $(15)(35) = (35)(13)$ so we gets

$$\varepsilon = (13)(24)(35)(14)(12)(35)(13)(34)$$

Then $(12)(35) = (35)(12)$ since the transpositions are disjoint so,

$$\varepsilon = (13)(24)(35)(14)(35)(12)(13)(34)$$

and similarly

$$\varepsilon = (13)(24)(35)(35)(14)(12)(13)(34)$$

But $(35)(35) = (1)$ so we end up with

$$\varepsilon = (13)(24)(14)(12)(13)(34)$$

(5bns-pts) 6. Bonus: Let $\alpha = (1372)(26374)(587)(1846)$. Write α^{2016} as a single cycle or the product of disjoint cycles. Explain how you know your answer is correct.

Solution: First write α as a product of disjoint cycles, because doing so makes taking powers much easier (or in this case *possible*). We get $\alpha = (14758)(263)$.

Now $\alpha^{2016} = (14758)^{2016}(263)^{2016}$. However, $2016 = 2015 + 1$ and 2015 is divisible by 5. Also 2016 is divisible by 3. So $\alpha^{2016} = (14758)^1(263)^0$ because raising a 5-cycle to a multiple of 5 will give the identity, and raising a 3-cycle to a power of 3 will give the identity. Thus

$$\alpha^{2016} = (14758)$$