Geometric Constructions

Last time we considered what geometric shapes we could or could not construct. We were left with three big questions: is it possible to *trisect an angle*, to *double a cube*, or to *square a circle*. To answer these questions, we must "algebratize" geometric constructions.

Start with two constructible points 0 and 1 {\em one unit} apart. We define **constructible** recursively from this base case:

(a) A **constructible line** is a line passing through two constructible points.

(b) A **constructible circle** is a circle whose radius is a constructible number and whose center is a constructible point.

(c) A **constructible point** is the intersection of two constructible lines, two constructible circles, or a constructible line and a constructible circle.

We say a number a is **constructible** provided $a = 0$ or there are two constructible points distance $|a|$ apart. So far we have constructible numbers 0, 1, and -1. What else is constructible?

For this activity, use GeoGebra. Use only the "new point" tool (to place points at the intersections of lines, circles, or both), the "line through two points" tool, and the "compass" tool (under the circle menu). You can also use the arrow to drag things around if you need to.

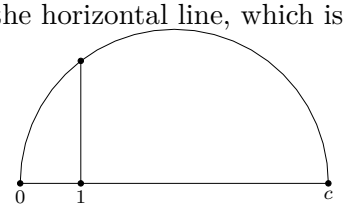
1. Show that the numbers 2 and 4 are constructible. Then show that the number $3 = 4 - 1$ is constructible.

2. If a and b are constructible numbers, are the numbers $a + b$ and $a - b$ also constructible?

3. Suppose a and b are constructible. Construct a triangle containing a base of unit length adjacent to a side of length a. Construct a similar triangle with where the side corresponding the unit length side now has length b. What is the length of the side corresponding to the a-length side?

4. Explain how you can modify the above construction to prove that if a and b are constructible, then a/b is constructible.

5. Given constructible number c, explain how you can construct the figure below. The vertical line should be perpendicular to the horizontal line, which is the diameter of the circle.



What is the length of the vertical line?

6. Let $\mathfrak{C}$ be the set of all constructible numbers. What sort of set is this? Is it a group? A ring? A field? Is it one of these we know about already?