Activity: Dimension and Degree

Recall that a *basis* for a vector space is a linearly independent spanning set, and that the *dimension* of a vector space is the size of a (any) basis for the space.

If K is an extension field of F, we can view K as a vector space over the field of scalars F. In this case, we say the **degree** of K over F, written $[K : F]$ is the dimension of this vector space.

1. Find a basis for $\mathbb{Q}(\sqrt{7})$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}]$?

2. Find a basis for $\mathbb{Q}(\sqrt[3]{5})$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$?

3. Suppose α is a root of $p(x) = x^5 - 6x^4 + 9x^2 + 3$. Find a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

4. What is the general rule here? Some things to think about: If you claim that you can always find a basis in some systematic way, how do you know it is really a basis? How do you know the basis is linearly independent? How do you know it spans?

5. The polynomial $q(x) = x^5 - 7x^3 - 5x^2 + 35$ has $\sqrt{7}$ and $\sqrt[3]{5}$ as roots. Does this mean $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = 5$? Why not?

We now have a fairly good idea how to work with $\mathbb{Q}(\alpha)$. What if we consider $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$, the smallest field containing $\mathbb{Q}$, $\sqrt{7}$, and also $\sqrt[3]{5}$?

6. We can think of this as an extension of an extension. Take $\mathbb{Q}(\sqrt[3]{5})$ as our base field. Adjoin to that $\sqrt{7}$ to get $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$. What is $[\mathbb{Q}(\sqrt[3]{5}, \sqrt{7}) : \mathbb{Q}(\sqrt[3]{5})]$? Use the general rule we discovered above and also find a basis

7. Using the basis above and the basis for $\mathbb{Q}(\sqrt[3]{5})$ over $\mathbb{Q}$, find a basis for $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$ over $\mathbb{Q}$.

8. What is $[\mathbb{Q}(\sqrt[3]{5}, \sqrt{7}) : \mathbb{Q}]$? What is the general rule for degrees of extensions of extensions?

9. What if we started with $\mathbb{Q}(\sqrt{7})$ and then adjoined $\sqrt[3]{5}$? Repeat the analysis you did above to make sure we get the same results about degree and basis.

10. What is $[\mathbb{Q}(\sqrt{2}, \sqrt[4]{2}) : \mathbb{Q}]$?