Activity: Iterated Extensions

We now have a fairly good idea how to work with $\mathbb{Q}(\alpha)$. What if we consider $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$, the smallest field containing $\mathbb{Q}$, $\sqrt[3]{5}$ and also $\sqrt{7}$?

1. We can think of this as an extension of an extension. Take $\mathbb{Q}(\sqrt[3]{5})$ as our base field. Adjoin to that $\sqrt{7}$ to get $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$. What is $[\mathbb{Q}(\sqrt[3]{5}, \sqrt{7}) : \mathbb{Q}(\sqrt[3]{5})]$?

2. What is a basis for $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$ over $\mathbb{Q}(\sqrt[3]{5})$? How do you know that this basis spans $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$?

3. Using the basis above and the basis for $\mathbb{Q}(\sqrt[3]{5})$ over $\mathbb{Q}$, find a basis for $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$ over $\mathbb{Q}$.

4. What is $[\mathbb{Q}(\sqrt[3]{5}, \sqrt{7}) : \mathbb{Q}]$? What is the general rule for degrees of extensions of extensions?

5. What is $[\mathbb{Q}(\sqrt{2}, \sqrt[4]{2}) : \mathbb{Q}]$?