Name: ____________________________

1. Find the degrees of field extensions, and briefly explain (using polynomials). Then give a basis for each larger field over the smaller field.

   (a) $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$

   Solution: The degree is 2. The minimum polynomial for $\sqrt{2}$ over $\mathbb{Q}$ is $x^2 - 2$, which is irreducible and has degree 2. A basis is $\{1, \sqrt{2}\}$.

   (b) $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{5+\sqrt{2}}) : \mathbb{Q}(\sqrt{2})]$

   Solution: The degree is 3. A minimum polynomial for $\sqrt[3]{5+\sqrt{2}}$ over $\mathbb{Q}(\sqrt{2})$ is $x^3 - 5 - \sqrt{2}$. To prove this is the minimum polynomial we need to show it is irreducible, which takes a little work.

   The basis is $\{1, \sqrt[3]{5+\sqrt{2}}, \sqrt[3]{5+\sqrt{2}}^2\}$.

   (c) $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{5+\sqrt{2}}) : \mathbb{Q}(\sqrt[3]{5+\sqrt{2}})]$

   Solution: The degree is 1. Since $\sqrt[3]{5+\sqrt{2}}^3 = 5 + \sqrt{2}$ and 5 is in $\mathbb{Q}$, we have that $\sqrt{2}$ is already in $\mathbb{Q}(\sqrt[3]{5+\sqrt{2}})$. The minimum polynomial is $x - \sqrt{2}$. A basis is $\{1\}$

   (d) $[\mathbb{Q}(\sqrt[3]{5+\sqrt{2}}) : \mathbb{Q}]$

   Solution: The degree is 6. Since $\mathbb{Q}(\sqrt[3]{5+\sqrt{2}}) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{5+\sqrt{2}})$ (as seen in part (c)), we can iterate the extensions using parts (a) and (b). In class we prove that $[E : F] = [E : L][L : F]$, so in this case we get the degree to be $2 \cdot 3 = 6$.

   The minimum polynomial is $x^6 - 10x^3 + 23$. A basis is $\{1, \sqrt[3]{5+\sqrt{2}}, \sqrt[3]{5+\sqrt{2}}^2, \sqrt[3]{5+\sqrt{2}}^3, \sqrt[3]{5+$