MATH 322 **Homework 5** Due: Wednesday, March 20

Instructions: Same rules as usual. Work together, write-up alone, no internet!

(6pts) 1. Consider the 362880 elements in S_9.

(a) What are the possible orders of elements in S_9. For each possible order, give an example of an element with that order. Explain how you know you have them all.

(b) Give an example of an element with order 3 that does not fix any element of $\{1, 2, \ldots, 9\}$.

(c) What do elements of order 8 look like? Bonus: how many elements of order 8 are there?

(8pts) 2. Prove the following basic facts about orders of elements. None of these are particularly difficult, so you should put most of your effort into writing a nice, clean proof of the fact. In each of the following, a is an element of a group G.

(a) If $\text{ord}(a) = n$ then for any $r < n$, $a^{n-r} = (a^r)^{-1}$.

(b) The order of a^{-1} is the same as the order of a.

(c) If $a^k = e$ where k is odd, then the order of a is odd.

(d) If $a \neq e$ and $a^p = e$ where p is prime, then $\text{ord}(a) = p$.

(12pts) 3. Let a and b be elements of a group G with $\text{ord}(a) = m$ and $\text{ord}(b) = n$.

(a) Assume a and b commute. Let $k = \text{ord}(ab)$ and $p = \text{lcm}(m, n)$. Prove k divides p.

(b) Assume m and n are relatively prime (i.e., $\gcd(m, n) = 1$). Prove that no power of a is equal to any power of b (other than e).

(c) Use the previous parts to prove that if a and b commute and m and n are relatively prime, then $\text{ord}(ab) = mn$.

(d) Give an example to show that part (a) is not true if a and b do not commute.

(4pts) 4. Suppose G is a group and H and K are distinct subgroups both with order the same prime number p. Prove that $H \cap K = \{e\}$. Hint: use Lagrange's theorem.