Activity: Geometric Constructions

Our first application of algebra (to other mathematics) will be to questions of classical geometry. We will look at geometry as it was done in ancient Greece, except that we will use GeoGebra for our constructions.

Our main question is, what can you *construct* using reasonable, fundamental tools. The tools are: an *unmarked straightedge* and a *compass*.

In GeoGebra, there are many more tools than these. Make sure you only use the "new point" tool (to place points at the intersections of lines and circles), the "line through two points" tool, and the "compass" tool (under the circle menu). You can also use the arrow to drag things around if you need to.

To get a feel for the sorts of things you can construct, and maybe things you cannot, here are a few challenges.

1. Can you construct a 60° angle? A 30° angle? If you have constructed any angle at all, can you construct an angle half its measure? That is, can you *bisect* an given angle?

2. Can you construct a square? Can you double the square? That is, if you can construct a square, can you construct a square of twice the area? Careful: this is not a square whose side length is twice the side length of the original.

3. Can you double the circle? That is, can you construct a circle and then construct a second circle of twice the area?

4. Here are three much harder, but related challenges. For each, play around enough to convince yourself these are really hard, if not impossible:

(a) Can you *trisect* and angle? That is, given a constructed angle, can you construct an angle 1/3 its measure?

(b) Can you double the *cube*? That is, if you can constructed a cube (or at least a line segment which is the length of the edge of a cube), can you construct cube with twice the volume of the original?

(c) Can you square the circle? That is, if you have constructed a circle, can you construct a square that has the same area as the circle?

Activity: Constructible Numbers

We have considered what geometric shapes we could or could not construct. We are left with three big questions: is it possible to *trisect an angle*, to *double a cube*, or to *square a circle*. To answer these questions, we must "algebratize" geometric constructions.

Start with two constructible points 0 and 1 *one unit* apart. We define **constructible** recursively from this base case:

(a) A **constructible line** is a line passing through two constructible points.

(b) A **constructible circle** is a circle whose radius is a constructible number and whose center is a constructible point.

(c) A **constructible point** is the intersection of two constructible lines, two constructible circles, or a constructible line and a constructible circle.

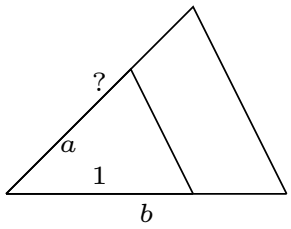
We say a number a is **constructible** provided $a = 0$ or there are two constructible points distance $|a|$ apart. So far we have constructible numbers 0, 1, and -1. What else is constructible?

For this activity, use GeoGebra. Use only the "new point" tool (to place points at the intersections of lines, circles, or both), the "line through two points" tool, and the "compass" tool (under the circle menu). You can also use the arrow to drag things around if you need to.

1. Show that the numbers 2 and 4 are constructible. Then show that the number $3 = 4 - 1$ is constructible.

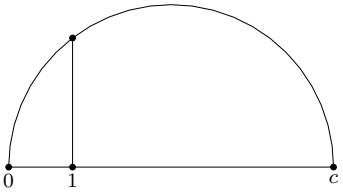
2. If a and b are constructible numbers, are the numbers $a + b$ and $a - b$ also constructible?

3. Suppose a and b are constructible. Construct a triangle containing a base of unit length adjacent to a side of length a. Construct a similar triangle with where the side corresponding the unit length side now has length b. What is the length of the side corresponding to the a-length side?



4. Explain how you can modify the above construction to prove that if a and b are constructible, then a/b is constructible.

5. Given constructible number c, explain how you can construct the figure below. The vertical line should be perpendicular to the horizontal line, which is the diameter of the circle.



What is the length of the vertical line?

6. Let $\mathfrak{C}$ be the set of all constructible numbers. What sort of set is this? Is it a group? A ring? A field? Is it one of these we know about already?

Activity: Review of Rings and Quotient Rings

The goal of this activity is to remind ourselves of basic but crucially important definitions we will need in our study of fields.

You are asked to provide definitions. Some definitions will include terms that also should be defined. Make sure that you know what every word in a definition means (if not, provide definitions for those words). For example, a **field** is a commutative division ring. If you have not yet defined commutative ring and division ring, you should say what these mean.

1. Give a definition of a **ring**.

2. What is a **commutative ring with unity**? How is this different from a ring? (Note, "unity" is also sometimes called "identity".)

3. What is a **commutative division ring**? What does "division" refer to here, and how is this different from a ring in general?

4. Give the definition of an **integral domain**. How does this relate to the other types of structures you defined above?

5. What is an **ideal**? What is the difference between an ideal and a **subring**?

6. Consider the integers $\mathbb{Z}$ (an integral doamin, right?). What does the notation $\langle 3 \rangle$ mean? What sort of thing is this? What is $\langle r \rangle$ in general?

7. What is $\mathbb{Q}[x]$? Then give an example of an ideal in $\mathbb{Q}[x]$, using proper notation and by listing out some of the elements in the ideal.

8. Give the definition of a **quotient ring** (i.e. a **factor ring**). What do elements of a quotient ring look like? How are the operations defined?

9. Illustrate you you wrote about quotient rings above using two examples: First, $\mathbb{Z}/\langle 3 \rangle$, and then $\mathbb{Q}[x]/\langle x^2 + 1 \rangle$. How many elements are in each of these quotient rings? What do the elements look like? Show how to add/multiply elements.

Activity: Review of the Euclidean Algorithm

The goal of this activity is to remember how to use the Eulcidean Algorithm to find the greatest common divisor of two elements in a ring (numbers or polynomials, for us) and write the gcd as a linear combination of the two elements (which Bezout's lemma tells us we can do).

Example 1 Let's find the gcd of 945 and 2415. Repeatedly use the division algorithm:

$$\begin{aligned} 2415 &= 945 \cdot 2 + 525 \\ 945 &= 525 \cdot 1 + 420 \\ 525 &= 420 \cdot 1 + 105 \\ 420 &= 105 \cdot 4 + 0. \end{aligned}$$

Check: 105 divides all the quotients and remainders, and any other divisor of 945 and 2415 would also divide 105. Therefore, $\gcd(945, 2415) = 105$.

Now work backwards to obtain numbers r and s such that $945r + 2415s = 105$.

$$\begin{aligned} 105 &= 525 + (-1) \cdot 420 \\ &= 525 + (-1) \cdot [945 + (-1) \cdot 525] \\ &= 2 \cdot 525 + (-1) \cdot 945 \\ &= 2 \cdot [2415 + (-2) \cdot 945] + (-1) \cdot 945 \\ &= 2 \cdot 2415 + (-5) \cdot 945. \end{aligned}$$

So $r = -5$ and $s = 2$. □

1. Find the greatest common divisor of 471 and 564 using the Euclidean Algorithm and then find integers r and s such that $\gcd(471, 564) = 471r + 564s$.

2. In the quotient ring $\mathbb{Z}/\langle 564\rangle$, find an element $a + \langle 564\rangle$ such that $(a + \langle 564\rangle)(471 + \langle 564\rangle) = 3 + \langle 564\rangle$. Explain why the previous question is helpful here.

3. Is $471 + \langle 564\rangle$ a unit in $\mathbb{Z}/\langle 564\rangle$? Explain.

4. In $\mathbb{Q}[x]$, find the gcd of the polynomials $a(x) = x^3 + 1$ and $b(x) = x^4 + x^3 + 2x^2 + x - 1$. Then express the gcd as a combination of the two polynomials (as in Bezout's lemma).

5. Find the greatest common divisor of $x^{24} - 1$ and $x^{15} - 1$ in $\mathbb{Q}[x]$, and then express the gcd as a combination of the two polynomials.

6. Find a coset $a(x) + \langle x^{24} - 1 \rangle$ of $\mathbb{Q}[x]/\langle x^{24} - 1 \rangle$ such that $(a(x) + \langle x^{24} - 1 \rangle)(x^{15} - 1 + \langle x^{24} - 1 \rangle) = x^3 - 1 + \langle x^{24} - 1 \rangle$.

Activity: Extending Fields to Factor

Goal: Build the smallest field possible in which $p(x) = x^3 + 3x^2 - x + 2$ is NOT irreducible.

Note that $p(x)$ is irreducible over $\mathbb{Q}$ because it has no roots in $\mathbb{Q}$ (why is this and why is that enough?). So let's invent a new number, call it ϱ, and insist that ϱ is a root of $p(x)$. Then consider the smallest field E larger than $\mathbb{Q}$ that also contains ϱ.

1. List five elements in E that are NOT already in $\mathbb{Q}$.

2. The element ϱ^3 is in E, but this can also be written using smaller powers of ϱ. How?

3. Describe E as a set using set builder notation. In other words, E is the set of all elements of the form …

4. Wait: why are we doing this? Our goal is for $p(x)$ to factor. Does it? What would one of the factors be?

5. Wait again: we want E to be a field. Is it? What would we need to check?

6. List five elements in the quotient ring $\mathbb{Q}[x]/\langle p(x)\rangle$ (using the same $p(x)$ from the previous page). Remember, these will all be cosets.

7. The element $x^3 + \langle p(x)\rangle$ is an element of $\mathbb{Q}[x]/\langle p(x)\rangle$, but it can also be written as a "simpler" coset. How?

8. Describe $\mathbb{Q}[x]/\langle p(x)\rangle$ as a set using set builder notation. In other words, this quotient ring is the set of all cosets of the form …

9. Wait: if we want to show that E is a field, and E is basically the same as $\mathbb{Q}[x]/\langle p(x)\rangle$, then we could just show $\mathbb{Q}[x]/\langle p(x)\rangle$ is a field. What would this mean? What do we need to verify?

Activity: Quotient Rings that are Extension Fields

Let's explore the connection between extension fields and quotient rings. We will see that $F[x]/\langle p(x)\rangle \cong F(\alpha)$, where α is a root of its minimal polynomial $p(x)$ (i.e., $p(x)$ is the unique monic irreducible polynomial that has α as a root). The goal of this activity is to see how working in quotient rings help us realize $E = F(\alpha)$ as a field.

We will start easy. For now, let $E = \mathbb{Q}(\sqrt{2})$.

1. What quotient ring is E isomorphic to?

2. One element in E is $1 + 3\sqrt{2}$. What element in the quotient ring does this correspond to?

3. What will $\gcd(3x + 1, x^2 - 2)$ be? How do you know? Then verify you are correct using the Euclidean algorithm.

4. Bezout's identity says that for any polynomials $a(x)$ and $b(x)$, there are polynomials $s(x)$ and $t(x)$ such that

$$\gcd(a(x), b(x)) = s(x)a(x) + t(x)b(x).$$

Find $s(x)$ and $t(x)$ in our case, by working backwards from the Euclidean algorithm above.

5. What does Bezout's identity have to do with the expression

$$1 + \langle x^2 - 2\rangle = (3x + 1 + \langle x^2 - 2\rangle)(t(x) + \langle x^2 - 2\rangle)$$

and what does this have to do with finding inverses? In particular, what is $(1 + 3\sqrt{2})^{-1}$ in E?

6. Now let's try this again with a more complicated polynomial. As in the earlier activity, take $p(x) = x^3 + 3x^2 - x + 2$ and let ϱ be a root. Use quotient rings to find the inverse of the element $2 + 3\varrho^2$ in $E = \mathbb{Q}(\varrho)$.

Activity: Dimension and Degree

Recall that a *basis* for a vector space is a linearly independent spanning set, and that the *dimension* of a vector space is the size of a (any) basis for the space.

If K is an extension field of F, we can view K as a vector space over the field of scalars F. In this case, we say the **degree** of K over F, written $[K : F]$ is the dimension of this vector space.

1. Find a basis for $\mathbb{Q}(\sqrt{7})$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}]$?

2. Find a basis for $\mathbb{Q}(\sqrt[3]{5})$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$?

3. Suppose α is a root of $p(x) = x^5 - 6x^4 + 9x^2 + 3$. Find a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

4. What is the general rule here? Some things to think about: If you claim that you can always find a basis in some systematic way, how do you know it is really a basis? How do you know the basis is linearly independent? How do you know it spans?

5. The polynomial $q(x) = x^5 - 7x^3 - 5x^2 + 35$ has $\sqrt{7}$ and $\sqrt[3]{5}$ as roots. Does this mean $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = 5$? Why not?

We now have a fairly good idea how to work with $\mathbb{Q}(\alpha)$. What if we consider $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$, the smallest field containing $\mathbb{Q}$, $\sqrt{7}$, and also $\sqrt[3]{5}$?

6. We can think of this as an extension of an extension. Take $\mathbb{Q}(\sqrt[3]{5})$ as our base field. Adjoin to that $\sqrt{7}$ to get $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$. What is $[\mathbb{Q}(\sqrt[3]{5}, \sqrt{7}) : \mathbb{Q}(\sqrt[3]{5})]$? Use the general rule we discovered above and also find a basis

7. Using the basis above and the basis for $\mathbb{Q}(\sqrt[3]{5})$ over $\mathbb{Q}$, find a basis for $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$ over $\mathbb{Q}$.

8. What is $[\mathbb{Q}(\sqrt[3]{5}, \sqrt{7}) : \mathbb{Q}]$? What is the general rule for degrees of extensions of extensions?

9. What if we started with $\mathbb{Q}(\sqrt{7})$ and then adjoined $\sqrt[3]{5}$? Repeat the analysis you did above to make sure we get the same results about degree and basis.

10. What is $[\mathbb{Q}(\sqrt{2}, \sqrt[4]{2}) : \mathbb{Q}]$?

Activity: Cayley Permutations

We have studied *permutation groups*, as well as groups that do not appear to be groups of permutations (such as $\mathbb{Z}_n$, $\mathbb{Z}_n \times \mathbb{Z}_m$, groups of functions or matrices, D_4, etc.). How distinct are these non-permutation groups from permutation groups?

1. Write the 4×4 group tables for $\mathbb{Z}_4$ and $U(8)$, the group of *units* of $\mathbb{Z}_8$ (numbers relatively prime to 8) under multiplication.

2. For each element in the groups above, we can see what adding or multiplying it by the other elements does to the other elements. For example, $5 \in U(8)$ corresponds to this function: $\lambda_5 = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 5 & 7 & 1 & 3 \end{pmatrix}$, since $5 \cdot 1 = 5$, $5 \cdot 3 = 7$, and so on.

For each element g in each group above, write down the corresponding function λ_g.

3. What happens when you compose two functions λ_g and λ_h for g and h in a group? Try this with a few examples you have above. What do you notice about the function you get?

4. Consider the set $\overline{G} = \{\lambda_g : g \in G\}$. Since each λ_g is a permutation of the elements of G, each of these will be a subset of S_n where $n = |G|$ (in our case, $n = 4$). Is $\overline{G}$ a subgroup of S_4 in both our cases? Will it be a subgroup in general?

Activity: Writing Cycles and Permutations

Goal: Understand how elements of S_n can be represented as cycles and products of cycles.

1. Can every permutation in S_n be represented using cycle notation? How could you represent the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 2 & 8 & 7 & 9 & 6 & 1 & 4 \end{pmatrix}$?

2. How should we define "cycle"? Write a definition. Some considerations: is (132)(45) a single cycle or two cycles? Is (3124) a cycle? What should we call the **length** of a cycle?

3. Here are a few permutations of S_7, written as products of cycles. Are there other ways to write each of these using cycle notation? What makes two products of cycles the *same*?

(142)(2534)(46)(135) (12546) (16)(14)(15)(12) (12)(37)(25)(37)(45)(46)

A *transposition* (or 2-cycle) is a cycle of *length* 2. That is, it swaps (transposes) a pair of elements.

4. Can the cycle (13254) be written as the product of transpositions? Can it be written as the product of transpositions in more than one way?

5. What about the product (124)(2354)?

6. Can you write (13)(24)(45)(15)(26)(13) as the product of more than 6 transpositions? Fewer than 6 transpositions? Can you write it as the product of an odd number of transpositions?

7. What numbers of transpositions can you write the identity (1) as?

Activity: (Sub)Normal Series

Given a group, we can look at subgroups. We say that a sequence of subgroups

$$G = H_n \supset H_{n-1} \supset \cdots \supset H_1 \supset H_0 = \{e\}$$

is a **subnormal series** provided each H_i is normal in H_{i+1}, and a **normal series** if each H_i is normal in G.

A non-trivial group G is called **simple** provided it has no non-trivial normal subgroups. We say that a subnormal series is a **composition series** and that a normal series is a **principle series** if every quotient group H_{i+1}/H_i is simple.

1. Find a subnormal series for D_4. Is it a normal series?

2. Find two different normal series for $\mathbb{Z}_{60}$ of length 3 (length is the number of proper inclusions).

3. Find the quotient groups H_{i+1}/H_i for both series above. How are these related? Are the series composition series?

4. Find a composition series for $\mathbb{Z}_{60}$. Can you take it to be a **refinement** of the normal series you found above?

Activity: Working with S_5

We are close to our goal of finding a degree 5 polynomial whose roots cannot be expressed in a nice way (using radicals and field operations). It turns out that a key step in this goal is understanding the group S_5. In this activity, we will get a better feel for this group by examining the "cycle structure" of its elements.

1. Every element in S_5 can be written as a single cycle or a product of disjoint cycles. Write down all the different possibilities for how these cycles or products of cycles might look (focusing on their "shape" rather than the specific numbers in the cycles).

2. Does S_5 contain a non-trivial subgroup that contains all the transpositions (2-cycles)? What is it, or why not?

3. Does S_5 contain a non-trivial subgroup that includes the elements (12), (13), (14), and (15)? What else would it contain? Hint: what is (12)(14)(12)?

4. Does S_5 contain a non-trivial subgroup that contains (24) and (12345)? Think about what else such a subgroup would contain.

5. What if you started with a different 2-cycle and a different 5-cycle?Would any pair of 5-cycle and 2-cycle work?

Now let's consider the alternating group A_5. Recall this is the group of all permutations in S_5 that can be written as the product of an even number of 2-cycles.

6. If you write elements of A_5 as the product of disjoint cycles, what sorts of cycle structures do you get?

7. Does A_5 contain a non-trivial subgroup that contains all the 3-cycles? Hint: show that every pair of transpositions can be written as a product of 3-cycles.

8. Now consider *normal* subgroups N of A_5. Remember, a normal subgroup is closed under conjugates (here the conjugate would be aba^{-1} where $a \in A_5$ and $b \in N$). Does A_5 contain a non-trivial normal subgroup that contains (123)?

9. Look at the different cycle structures of elements in A_n and start taking conjugates. Will you be able to get (123) starting from any non-identity element?

Powers mod Powers

Main Question: What is the remainder when you divide a^p by p?

1. Compute a^p and its remainder when divided by p, for various values of a and p. Everyone should do at least 5, and then share with the group. As a group, discuss any patterns you see and form a conjecture.

Find the remainders when you perform the following divisions. Try different values of a. You should first guess what the value is based on your conjecture and then verify (or refute) your guess.

2. a^6 divided by 10?

3. a^9 divided by 15?

4. a^{13} divided by 21?

5. a^{2321} divided by 2419?

Discuss in your groups: how might we think about the main question here in terms of group theory? What would we need to prove (about groups)?

Activity: Orders of Permutations

The *order* of an element g in a group G is the least natural number n such that $g^n = e$, if such a number exists (otherwise we say the order of g is infinite).

1. Find the orders of the elements of S_5 below:

$\alpha = (12)$ $\alpha = (123)$ $\alpha = (1234)$ $\alpha = (12345)$

2. Find an element of S_5 that has an order different from those found above.

3. Let α be an element of S_5. What is α^{120}?

4. Is there an element in S_5 that has order 120?

5. What is the largest order of any element in S_5?

Powers mod Powers

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Discuss in your groups: how might we think about the main question here in terms of group theory? What would we need to prove (about groups)?

Activity: Decomposing with Direct Products

Recall that last semester we saw that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. When does this sort of thing happen?

1. Given positive integers m and n, is it always true that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$? If this is not always true, for which m and n is it true? Try some (many) examples.

2. Consider $\mathbb{Z}_{12}$. Can we break this down as the direct product of two smaller $\mathbb{Z}_p$ groups? In other words is $\mathbb{Z}_{12} = \mathbb{Z}_m \times \mathbb{Z}_n$ for some values of m and n?

3. Suppose your absent minded professor claims the answer is "no" and you don't feel like arguing. Maybe we can do something similar. Find two subgroups of $\mathbb{Z}_{12}$, call them H and K, such that $H \cap K = \{0\}$ and $HK = \mathbb{Z}_{12}$. In general, $HK = \{h * k : h \in H, k \in K\}$; here it would be better to write $H + K$.

For any n, the group $U(n)$ is the set of all positive integers less than and relatively prime to n, under multiplication modulo n. For example we saw that $U(8) = \{1, 3, 5, 7\}$ is a group under multiplication modulo 8.

Consider the group $U(28)$. The table below gives the twelve elements with their orders:

g	1	3	5	9	11	13	15	17	19	23	25	27
$\text{ord}(g)$	1	6	6	3	6	2	2	6	6	6	3	2

4. Let $G(n)$ be the set of all elements of order n^k for some k (that is, elements with order some *power* of n). Find $G(2)$ and $G(3)$ for $U(28)$.

5. Are $G(2)$ and $G(3)$ subgroups of $U(28)$?

6. Do $G(2)$ and $G(3)$ have the property that $G(2) \cap G(3) = \{1\}$ and $U(28) = G(2)G(3)$?

7. Is $U(28) \cong G(2) \times G(3)$? Is $U(28) \cong \mathbb{Z}_m \times \mathbb{Z}_n$ for some values of m and n?

Activity: RSA Cryptography Lab

Let's implement the ideas behind RSA cryptography to create a public encryption key that you can use to have your friends send you secure messages.

We will use SAGE to compete this activity. You can use the interactive SAGE cells in the online version of this activity. Or use the online SAGE cell at https://sagecell.sagemath.org/. Of course if you have SAGE installed, you can use that.

1. First you will need to select two very large prime numbers. Let's shoot for 20-25 digits. Luckily, SAGE has the command `next_prime()`which will give you the next prime larger than your input. So pick a random input and get two primes. You will want to save these as a constant. For example (but you will need to find a much bigger numbers):

```
p = next_prime(20)
q = next_prime(30)
p, q
```

```
(23, 31)
```

2. Now you can compute $n = pq$ and $m = (p-1)(q-1)$.

```
n = p*q
m = (p-1)*(q-1)
n, m
```

```
(713, 660)
```

Notice that you *could* ask SAGE to find $m = \varphi(n)$ using `euler_phi(n)`, but you should think about why this is a really really really bad idea. If you want, you can try it below, but maybe first with only small values of p and q.

```
euler_phi(n)
```

```
660
```

3. Now we need E and D. Recall that we want $\gcd(E, m) = 1$ and $DE \equiv 1 \pmod{m}$. How are you going to find these?

SAGE has the command `gcd(E, m)`that will compute the gcd of the two inputs. You could try factoring m and looking for a reasonably large value of E, or just guess and check until you find a suitable E.

To find D, there is the command `inverse_mod(E, m)`which will run the Euclidean algorithm forwards and backwards on E and m.

```
# repeatedly pick E until gcd(E,m) = 1.
E = 35
gcd(E,m)
```

```
5
```

```
D = inverse_mod(E,m)
D
```

You can now publish the encryption pair (n, E) so your friends can send you messages. You will need to keep D private. But don't loose it! The whole point is that without p and q, you should not be able to find D again, even if you had n and E. Evaluate the cell below to refer to later:

```
print("Your public key, E =", E,"; n =", n)
print("Your private key, D =", D)
```

If someone has a message x to send you, they would simply need to compute $x^E \pmod{n}$. SAGE can do this with `mod(x^E,n)`, but if x and E are large, this would take a really long time. Luckily, there is a better way: `power_mod(x,E,n)`computes the same thing, but uses the method of repeated squaring to make this much more efficient.

For example:

```
a = mod(12345^35, 713)
b = power_mod(12345,35,713)
a, b
```

```
(470, 470)
```

You will need to use this function to decrypt the message y when you take $y^D \pmod{n}$.

4. When you get an encrypted message, you can assign it to the variable `encrypted`and then decrypt it using the `power_mod`function.

```
encrypted = #paste encrypted message here
decrypted = power_mod(encrypted, D, n)
decrypted
```

5. The last piece of the puzzle is what to do with the number you get out of the decryption. You will find some x, but need that to be translated into text you can read.

This depends on the agreed upon method for translating a string of symbols into numbers. For this lab, you can use the following code do decrypt the message:

```
digits = decrypted.digits(base=128)
letters = [chr(ascii) for ascii in digits]
''.join(letters)
```

Here is why the code above works. First, we need to agree upon how to translate the original message into a number.

Suppose we have a string called `message`. We can create a list of ASCII code values (0 through 127) using `digits = [ord(letter) for letter in message]`. Here `ord()`converts a single letter into its ASCII code, so we do that for each letter in the message.

We must then convert a collection of letters into a single number base 128. SAGE can do this using the `ZZ()`function. So `ZZ(digits,128)`. This will only work if $128^k < n$, where k is the number of digits. So in practice, we would break the longer message into chunks and encrypt each chunk separately.

To undo this coding, you can break down the received message `decrypted`using `digits = decrypted.digits(base=128)`, which makes a list of digits. Then you can create a list of letters using `letters = [chr(ascii) for ascii in digits]`. Finally, put these letters into a string using `''.join(letters)`.

To practice, you can try encoding and decoding messages below. The first set of cells allow you to encrypt and decrypt a very short message. The second pair show a way to break up the message word by word using for loops break up the message into word-long chunks so there is no limit to the length of the message.

```
message = #paste short message here.
digits = [ord(letter) for letter in message]
message_num = ZZ(digits,128)
```

```
encrypted = power_mod(message_num, E, n)
encrypted
```

```
decrypted = power_mod(encrypted, D, n)
digits = decrypted.digits(base=128)
letters = [chr(ascii) for ascii in digits]
''.join(letters)
```

For longer messages:

```
message = #paste message here, enclosed in quotes.
message_array = message.split()
encrypted = []
for word in message_array:
    digits = [ord(letter) for letter in word]
    word_num = ZZ(digits,128)
    encrypted.append(power_mod(word_num, E, n))
encrypted
```

```
dec_message_array = []
for num in encrypted:
    decrypted = power_mod(num, D, n)
    digits = decrypted.digits(base=128)
    letters = [chr(ascii) for ascii in digits]
    dec_message_array.append(''.join(letters))
'␣'.join(dec_message_array)
```

Here is an empty sage cell in case you want to experiment with other commands: