Review of the Euclidean Algorithm

The goal of this activity is to remember how to use the Eulcidea Algorithm to find the greatest common divisor of two elements in a ring (numbers or polynomials, for us) and write the gcd as a linear combination of the two elements (which Bezout's lemma tells us we can do).

Example 0.0.1 Let's find the gcd of 945 and 2415. Repeatedly use the division algorithm:

$$\begin{aligned} 2415 &= 945 \cdot 2 + 525 \\ 945 &= 525 \cdot 1 + 420 \\ 525 &= 420 \cdot 1 + 105 \\ 420 &= 105 \cdot 4 + 0. \end{aligned}$$

Check: 105 divides all the quotients and remainders, and any other divisor of 945 and 2415 would also divide 105. Therefore, $\gcd(945, 2415) = 105$.

Now work backwards to obtain numbers r and s such that $945r + 2415s = 105$.

$$\begin{aligned} 105 &= 525 + (-1) \cdot 420 \\ &= 525 + (-1) \cdot [945 + (-1) \cdot 525] \\ &= 2 \cdot 525 + (-1) \cdot 945 \\ &= 2 \cdot [2415 + (-2) \cdot 945] + (-1) \cdot 945 \\ &= 2 \cdot 2415 + (-5) \cdot 945. \end{aligned}$$

So $r = -5$ and $s = 2$. ▫

1. Find the greatest common divisor of 471 and 564 using the Euclidean Algorithm and then find integers r and s such that $\gcd(471, 564) = 471r + 564s$.

2. In the quotient ring $\mathbb{Z}/\langle 564\rangle$, find an element $a + \langle 564\rangle$ such that $(a + \langle 564\rangle)(471 + \langle 564\rangle) = 3 + \langle 564\rangle$. Explain why the previous question is helpful here.

3. Is $471 + \langle 564\rangle$ a unit in $\mathbb{Z}/\langle 564\rangle$? Explain.

4. In $\mathbb{Q}[x]$, find the gcd of the polynomials $a(x) = x^3 + 1$ and $b(x) = x^4 + x^3 + 2x^2 + x - 1$. Then express the gcd as a combination of the two polynomials (as in Bezout's lemma).

5. Find the greatest common divisor of $x^{24} - 1$ and $x^{15} - 1$ in $\mathbb{Q}[x]$, and then express the gcd as a combination of the two polynomials.

6. Find a coset $a(x) + \langle x^{24} - 1 \rangle$ of $\mathbb{Q}[x]/\langle x^{24} - 1 \rangle$ such that $(a(x) + \langle x^{24} - 1 \rangle)(x^{15} - 1 + \langle x^{24} - 1 \rangle) = x^3 - 1 + \langle x^{24} - 1 \rangle$.

If you have time.After completing the activity above, if you have time, you should look again at the activity distributed the first day of class, and try to answer the prompts on the second page.