Math 322 **Exam 2 – Take Home** Spring 2016

**Name:** ______________________________

**Instructions:** This is the take-home portion of the second exam. Here are my expectations:

- WORK ALONE! You may not collaborate or discuss problems with other students, either in or outside of this class. Also do not discuss with tutors, significant others, parents, kids, etc. Cats and dogs are okay, but only if they have not taken an abstract algebra class. If you need clarification on a problem, ask me.
- You may use your notes (including notes from last semester) or the textbook, but only notes you have taken in this class (or in Algebra I) and only the assigned textbook from this class. This is intended only for you to refresh your memory if you forget a definition, not for you to copy proofs (or even the style of proof) from your notes. Alternatively, if you do not remember a definition, send me an email.
- Other than your notes and textbook, do not use any outside sources. In particular, absolutely NO INTERNET.
- This is not a timed exam, and you may take as much time on it as you like. However, I do not intend for you to spend more than 3 hours total working on the exam.
- You should write up all solutions neatly on your own paper and staple this sheet to your solutions. Clearly number each problem, and put the problems in the usual numerical order (no non-identity permutations please).
- As always, you must show all your work to receive credit, and explanations and proofs should be written out in complete English sentences. A page of just equations and calculations will probably receive no credit.
- **Due Monday, April 15.**

Have Fun!

(12pts) 1. Both Lagrange's theorem and Cauchy's theorem deal with the relationship between the size of a group and the order of its elements.

(a) Explain the difference between the theorems in general terms and by using $S_7$ as an example. Your explanation should include what we can and cannot conclude from each theorem about $S_7$.

(b) Which theorem would allow you to prove that if a group contained only elements that had order some power of 2, then the order of the group must be a power of 2? You might want to consider contrapositives. Then **give the proof**.

(c) We have previously proved that if $G$ is a cyclic group of order $n$, then for every factor $m$ of $n$, $G$ contains elements of order $m$. Explain why neither Lagrange's theorem nor Cauchy's theorem can be used to prove this.

(16pts) 2. Consider groups generated by two elements $G = \langle a, b \rangle$. Note that these groups *could* still be cyclic, depending on the relationship between $a$ and $b$.

(a) Suppose $G = \langle a, b \rangle$ is abelian with $\text{ord}(a) = 4$ and $\text{ord}(b) = 5$. List all the elements of $G$. What familiar group is $G$ isomorphic to? Justify your answer using internal direct products. Hint: one of the elements you list should be either $a^2b^3$ or $b^3a^2$ (these are the same element).

(b) Find an example of an abelian group $G = \langle a, b \rangle$ where the order of $G$ is strictly less than $\text{ord}(a) \cdot \text{ord}(b)$. (Do not assume the orders are the same as the previous part.) Explain why your example works.

(c) Prove that if $G = \langle a, b \rangle$ is abelian, then $|G| \leq \text{ord}(a) \cdot \text{ord}(b)$.

(d) Give an example of a non-abelian group $G = \langle a, b \rangle$ that shows the previous part is not true for groups in general. Explain why your example works and what goes wrong when you try to use the proof you gave in the previous part.

(16pts) 3. Consider the group $\mathbb{Z}_{45}$.

(a) Give an example of a subnormal series which is not a composition series. Explain why your example works.

(b) Find a *refinement* of the series you gave above which is a composition series. That is, show how to extend your series into a composition series.

(c) Find *all* other composition series and briefly explain how you know you have them all.

(d) If $p(x)$ is a polynomial whose splitting field has Galois group over $\mathbb{Q}$ isomorphic to $\mathbb{Z}_{45}$, will the roots of $p(x)$ be expressible in terms of $n$th roots and field operations? Briefly explain (you can cite a result we discussed in class).

(6pts) 4. Suppose $E$ is a splitting field whose Galois group over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_{45}$. How many different intermediate fields are there between $\mathbb{Q}$ and $E$? What degree extensions are these? Justify your answers. You may refer to the work you did on the previous problem (part (c) in particular) as well as reference the Fundamental Theorem of Galois Theory.

(10bn-pts) 5. Bonus: Let $E$ be as in the previous problem, and $p(x)$ a polynomial for which $E$ is the splitting field. What can you say about $p(x)$? What could it's degree be? Could it be irreducible? Can you give an example of such a polynomial? Justify your answers. (The more you can say, the more bonus points you will get.)