Activity: Working with S_5

To produce a degree 5 polynomial that is not solvable by radicals, it is enough to produce a polynomial that has S_5 as its Galois group. This raises two questions: First, how do you know that S_5 really is the Galois group? Second, why is S_5 the group we are looking for?

Let's start with the first question. We claim that if the Galois group contains a 2-cycle and a 5-cycle, then the Galois group must contain all of S_5. This comes down to the question of what elements *generate* S_5.

1. Does S_5 contain a non-trivial subgroup that contains all the transpositions (2-cycles)? Explain.

2. Does S_5 contain a non-trivial subgroup that contains the set $\{(12), (13), (14), (15)\}$? Explain. Hint: what is (12)(14)(12)>

3. Does S_5 contain a non-trivial subgroup that contains $\{(24), (12345)\}$? Explain.

4. Conclude that S_5 can be generated by any 2-cycle and any 5-cycle.

For the second main question, we must show that A_5 is simple, because then the only normal series we can get for S_5 will be

$$S_5 \supset A_5 \supset \{(1)\}$$

but A_5 is not abelian, so S_5 is not solvable.

5. Show that A_5 is generated by the set of 3-cycles. You can do this by showing that every pair of transpositions can be written as a product of 3-cycles.

6. Show that if a normal subgroup N of A_5 contains even one 3-cycle, then it is all of A_5. Remember, normal subgroups are closed under conjugates (here the conjugate would be aba^{-1} where $a \in A_5$ and $b \in N$).

7. Finally, show that every non-trivial normal subgroup of A_n contains a 3-cycle.