The first exam (take-home due and in-class on Monday, February 18) will cover everything we have discussed so far this semester. In particular, this means polynomials, field extensions, and the relationship between the two. Here is a checklist of these topics.

- ☐ Field extensions and what they have to do with polynomials.
- ☐ Rings of polynomials and their ideals.
- ☐ Quotient rings of polynomial rings.
- ☐ Minimum polynomials.
- ☐ The degree of a field extension.
- ☐ Bases for algebraic field extensions.
- ☐ Constructible numbers.
- ☐ Splitting fields.
- ☐ Field automorphisms of splitting fields.
- ☐ Galois groups.
- ☐ Topics from last semester which might be helpful:
  - ○ The Division Algorithm for polynomials (along with long divison).
  - ○ The Euclidean Algorithm for polynomials.
  - ○ Greatest Common Divisors and Bezout's Lemma.
  - ○ Irreducible polynomials.
  - ○ Factoring in $\mathbb{Q}$ (including the rational roots theorem and Eisenstein's criterion).
  - ○ Homomorphisms (the homomorphism property, and the Fundamental Homomorphism Theorem).

The activities, quizzes, and homework should give you a good idea of the types of questions to expect. Copies of these assignments, with solutions, are available on Canvas.

Additionally, the questions below would all make fine exam questions.[1]

Sample Questions

1. Give an example of an ideal $J \subseteq \mathbb{Q}[x]$ and two non-zero polynomial $a(x), b(x) \in \mathbb{Q}[x]$ such that $(J + a(x))(J + b(x)) = J + 0$. Does such an example prove that $\mathbb{Q}[x]/J$ is NOT an integral domain? Explain.
2. Consider the following fact about polynomials:
$$(3x^2 - 14x + 24)(x^2 + 3x + 6) - (3x - 5)(x^3 - 4) = 124.$$
   (a) Does the polynomial $x^2 + 3x + 6$ have an inverse in $\mathbb{Q}[x]$? Find it or explain why not.
   (b) Does the coset $\langle x^3 - 4\rangle + x^2 + 3x + 6$ have an inverse in $\mathbb{Q}[x]/\langle x^3 - 4\rangle$? Find it or explain why not.
   (c) Does the number $6 + 3\sqrt[3]{4} + \sqrt[3]{4}^2$ have an inverse in the field $\mathbb{Q}(\sqrt[3]{4})$? Find it or explain why not.
   (d) Explain the relationship between parts (b) and (c) and what this has to do with the fact about polynomials above.

[1] Disclaimer: Question on the actual exam may be easier or harder than those given her. There might be types of questions on this study guide not covered on the exam and questions on the exam not covered in this study guide. Questions on the exam might be asked in a different way than here. If solving a question lasts longer than four hours, contact your professor immediately.

3. Let $\alpha = \frac{3+\sqrt{5}}{2}$.
   (a) Describe the field $\mathbb{Q}(\alpha)$. List two elements of the field (which are not in $\mathbb{Q}$).
   (b) Find a quotient ring of $\mathbb{Q}[x]$ which is isomorphic to $\mathbb{Q}(\alpha)$. Prove your answer is correct using a particular Fundamental theorem.
   (c) What elements in the quotient ring correspond to the two elements you listed in part (a)? Explain.
   (d) One of the elements of $\mathbb{Q}(\alpha)$ is $\alpha^4 + 2\alpha^3 + 7$. Which element (in standard form) is this? Use the correspondence to polynomials to help answer this question.
   (e) How does $\mathbb{Q}(\alpha)$ relate to $\mathbb{Q}(\sqrt{5})$? Explain.

4. Use polynomials to find the inverse of $\sqrt[3]{7} + 4$ in the field $\mathbb{Q}(\sqrt[3]{7})$. Write your answer in the form $a + b\sqrt[3]{7} + c\sqrt[3]{7}^2$

5. Give an example of a field E that is a degree 7 extension of $\mathbb{Q}$. Justify your answer.

6. Let a be a root of $x^6 + 1$ (in $\mathbb{C}$ say). What is the degree of $\mathbb{Q}(a)$ over $\mathbb{Q}$? Explain.

7. Consider the element $a = \sqrt{3 + \sqrt{2}}$.
   (a) Is a algebraic over $\mathbb{Q}$? Explain.
   (b) Is a algebraic over $\mathbb{Q}(\sqrt{2})$? Explain.
   (c) Find the degree of $\mathbb{Q}(a)$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{2})$.
   (d) Find a basis for $\mathbb{Q}(a)$ over $\mathbb{Q}$ and a basis for $\mathbb{Q}(a)$ over $\mathbb{Q}(\sqrt{2})$.

8. Let α be a root of the polynomial $x^3 + 4x^2 + 10x + 6$. Prove that $\{1, \alpha, \alpha^2, \alpha^3\}$ is linearly dependent in $\mathbb{Q}(\alpha)$. Does this mean that $\alpha^3 \in \mathbb{Q}$?

9. Give an example of a polynomial $p(x)$ whose Galois group over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Give the splitting field and describe the elements of the Galois group.

10. In the previous problem, the Galois group has subgroups isomorphic to $H_1 = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}$ and to $H_2 = \{(0,0,0), (1,0,0)\}$ (among many others). Which subfields of the splitting field do these correspond to. That is, what are the fixed fields for these subgroups.

11. Decide which of the statements below are true and which are false. For the true statements, give a short proof or explanation. For the false statements, provide a counterexample and explain why it is a counterexample.
    (a) It is possible to construct a line segment of length $\sqrt{3 + \sqrt{7}}$ using a straight edge and compass.
    (b) It is possible to construct a line segment of length $\sqrt[5]{7}$ using a straight edge and compass.
    (c) $\mathbb{Q}[x]/\langle x^4 + 3x - 6\rangle$ is a field
    (d) $\mathbb{R}[x]/\langle x^4 + 3x - 6\rangle$ is a field.
    (e) If E is a degree 4 field extension of $\mathbb{Q}$, then there is always a degree 2 field extension K of $\mathbb{Q}$ between $\mathbb{Q}$ and E (i.e., $\mathbb{Q} \subseteq K \subseteq E$).