MATH 322 **Homework 3** Due: Wednesday, February 13

Instructions: Carefully write up solutions to the questions below. A solution should consist of both the answer and a careful explanation for why that answer must be correct. Any solutions without an explanation written out in English prose will receive no credit. You are welcome to work together, but write up solutions in your own, individual rules. Also, NO INTERNET!

(4pts) 1. Suppose E is a degree 2 extension of F. Prove that E is the splitting field for some polynomial in $F[x]$.

(6pts) 2. Let $p(x)$ be a polynomial of degree n in $F[x]$ for some field F. Prove that there is a splitting field E for $p(x)$ such that $[E : F] \leq n!$. Your proof must use **mathematical induction**! Let $P(n)$ be the statement, "If $p(x)$ has degree n, then there is a splitting field with degree no more than $n!$" and prove $P(n)$ is true for all $n \geq 1$.

(8pts) 3. Consider the field $\mathbb{Q}(\sqrt[4]{3}, i)$.

   (a) Is this a splitting field for some polynomial in $\mathbb{Q}[x]$? If so, what is the degree of that polynomial?

   (b) What is the degree $[\mathbb{Q}(\sqrt[4]{3}, i) : \mathbb{Q}]$? Explain how you know.

   (c) Draw as much of a complete tower diagram as you can describing the fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[4]{3}, i)$.

   (d) Prove that the fields $\mathbb{Q}(\sqrt[4]{3})$ and $\mathbb{Q}(\sqrt[4]{3}i)$ are isomorphic, but not equal. This might help with the previous parts.

(12pts) 4. Consider the polynomial $a(x) = x^4 - 10x^2 + 21$ in $\mathbb{Q}[x]$.

   (a) Find the splitting field E over $\mathbb{Q}$. Draw a tower diagram including all intermediate fields, and their degrees. Hint: start by factoring $a(x)$.

   (b) Let $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$ be different intermediate fields between $\mathbb{Q}$ and E. Explain why there is NOT an isomorphism from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}(\beta)$. In particular, say why no isomorphism can send α to β.

   (c) Describe a non-trivial *automorphism* of E. Explain how you know your example works. Remember, an automorphism is an isomorphism from E to itself (that is, it moves some of the elements of E around, but is still a bijection satisfying the homomorphism property).

   (d) Describe the Galois group $\mathrm{Gal}(E : \mathbb{Q})$ for E over $\mathbb{Q}$. Be sure to explicitly say what each element of the group is, as well as say what "standard" group it is isomorphic to.