Instructions: Carefully write up solutions to the questions below. A solution should consist of both the answer and a careful explanation for why that answer must be correct. Any solutions without an explanation written out in English prose will receive no credit.

You are encouraged to work with classmates, but your write up should be in your own words. Also, other than our textbook, you may NOT use the internet to search for solutions.

(5pts) 1. Find a single generator for the smallest ideal in $\mathbb{Q}[x]$ which contains the polynomials x^3+3x^2+3x+2 and $2x^3-3x^2-11x+6$. Explain how you know that this generator is in the ideal.

(9pts) 2. Consider the polynomial $p(x) = x^3 - 5$ in $\mathbb{Q}[x]$.

(a) Explain how we know that the quotient ring $\mathbb{Q}[x]/\langle x^3 - 5\rangle$ is actually a field. That is, show that every non-zero element of the quotient ring has a multiplicative inverse. Hint: you will want to use Bezout's lemma.

(b) Let $E = \{a+b\sqrt[3]{5}+c\sqrt[3]{5}^2 \ : \ a, b, c \in \mathbb{Q}\}$. How does this set relate to the field $\mathbb{Q}[x]/\langle x^3-5\rangle$? Be explicit (for example, if you say they are isomorphic, give the isomorphism).

(c) Find the element of E (in $a + b\sqrt[3]{5} + c\sqrt[3]{5}^2$ form) equal to $1/(3 - 2\sqrt[3]{5} + \sqrt[3]{5}^2)$ using polynomials. That is, use the relationship you described in part (b) so you can work in $\mathbb{Q}[x]/\langle x^3 - 5\rangle$ instead of in E.

(6pts) 3. Let A be a commutative ring with unity. Let J be an ideal of A. We say that J is prime provided for any $a, b \in A$, if $ab \in J$ then $a \in J$ or $b \in J$.

(a) Prove that if J is prime, then A/J is an integral domain.

(b) Prove that if A/J is an integral domain, then J is prime.

(8pts) 4. Let A be a commutative ring with unity. An ideal J is *proper* if $A \neq J$. We say that a proper ideal J is *maximal* if no proper ideal of A strictly contains J (that is, if K is a proper ideal of A and $J \subseteq K$ then $J = K$).

(a) Prove that if J is maximal, then A/J is a field (you may assume that A/J is a commutative ring with unity). Here are some hints: first, explain why you want to show that for any $a \notin J$, that there is some element x such that $(J + a)(J + x) = J + 1$. Then let $K = \{xa + j \ : \ x \in A, j \in J\}$, and prove that K is an ideal strictly larger than J. In particular, $1 \in K$. Finally, explain why this is enough to finish the proof.

(b) Prove that if A/J is a field, then J is maximal.

(2pts) 5. Assuming the results from the previous two questions, prove that every maximal ideal is prime. This should be a 3-sentence proof.

(3 bns) 6. Bonus: it is not true that every prime ideal is maximal (although this does hold for $\mathbb{Z}$ and for $F[x]$). Find an example of a ring A with an ideal J that is prime but not maximal. Justify your answer. Hint: look at a polymoial ring that for which the coefficients do not belong to a field.