30 Cayley Digraphs of Groups

The important thing in science is not so much to obtain new facts as to discover new ways of thinking about them.

Sir William Lawrence Bragg, *Beyond Reductionism*

The changing of a vague difficulty into a specific, concrete form is a very essential element in thinking.

J. P. Morgan

Motivation

In this chapter, we introduce a graphical representation of a group given by a set of generators and relations. The idea was originated by Cayley in 1878. Although this topic is not usually covered in an abstract algebra book, we include it for five reasons: It provides a method of visualizing a group; it connects two important branches of modern mathematics—groups and graphs; it has many applications to computer science; it gives a review of some of our old friends—cyclic groups, dihedral groups, direct products, and generators and relations; and, most importantly, it is fun!

Intuitively, a directed graph (or digraph) is a finite set of points, called *vertices*, and a set of arrows, called *arcs*, connecting some of the vertices. Although there is a rich and important general theory of directed graphs with many applications, we are interested only in those that arise from groups.

The Cayley Digraph of a Group

Definition Cayley Digraph of a Group

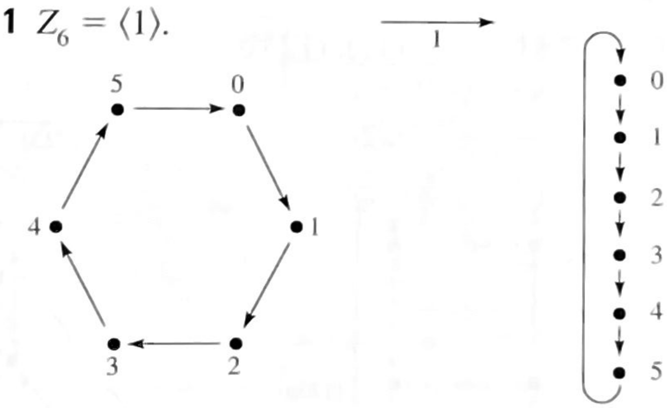
Let G be a finite group and let S be a set of generators for G. We define a digraph $\text{Cay}(S:G)$, called the *Cayley digraph of G with generating set S*, as follows.

1. Each element of G is a vertex of $\text{Cay}(S:G)$.
2. For x and y in G, there is an arc from x to y if and only if $xs = y$ for some $s \in S$.

To tell from the digraph which particular generator connects two vertices, Cayley proposed that each generator be assigned a color, and that the arrow joining x to xs be colored with the color assigned to s. He called the resulting figure the *color graph of the group*. This terminology is still used occasionally. Rather than use colors to distinguish the different generators, we will use solid arrows, dashed arrows, and dotted arrows. In general, if there is an arc from x to y, there need not be an arc from y to x. An arrow emanating from x and pointing to y indicates that there is an arc from x to y.

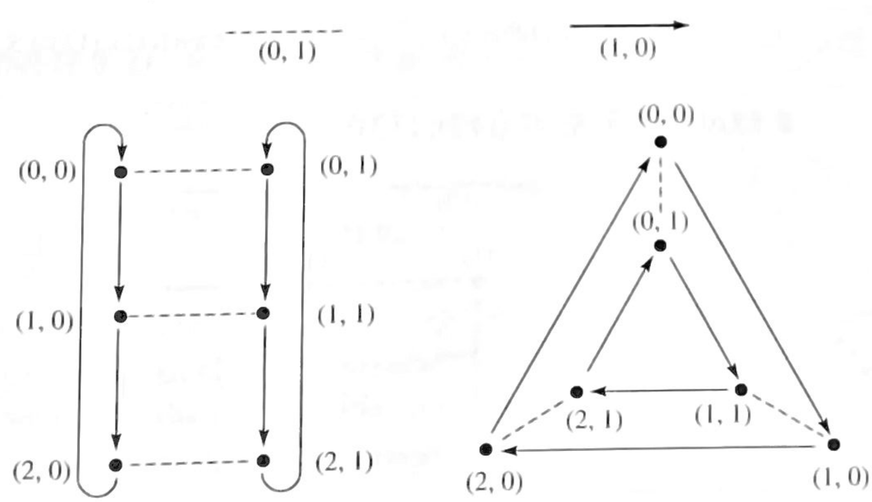
Following are numerous examples of Cayley digraphs. Note that there are several ways to draw the digraph of a group given by a particular generating set. However, it is not the appearance of the digraph that is relevant but the manner in which the vertices are connected. These connections are uniquely determined by the generating set. Thus, distances between vertices and angles formed by the arcs have no significance. (In the digraphs below, a headless arrow joining two vertices x and y indicates that there is an arc from x to y and an arc from y to x. This occurs when the generating set contains both an element and its inverse. For example, a generator of order 2 is its own inverse.)

EXAMPLE 1 $Z_6 = \langle 1 \rangle$.



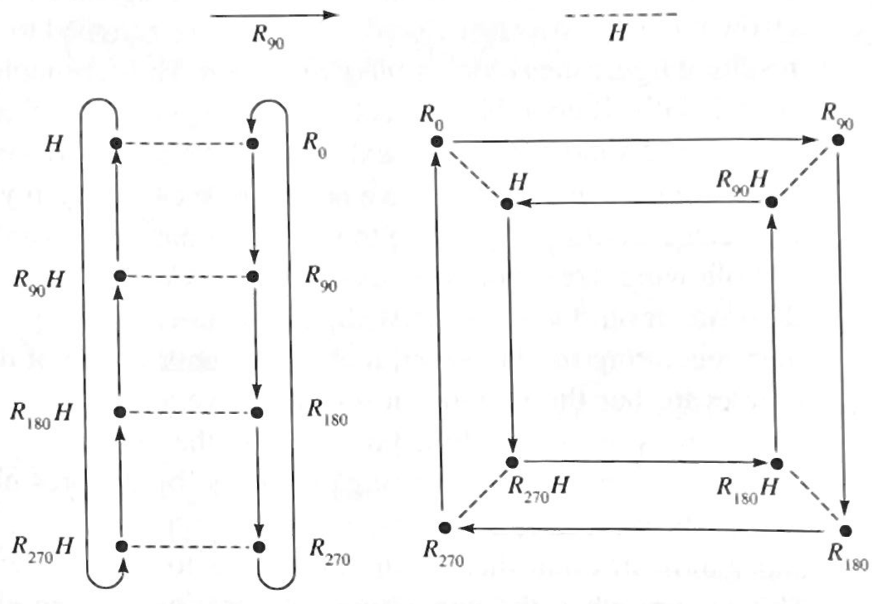
Cay({1}:Z_6) Cay({1}:Z_6) ∎

EXAMPLE 2 $Z_3 \oplus Z_2 = \langle (1, 0), (0, 1) \rangle$.



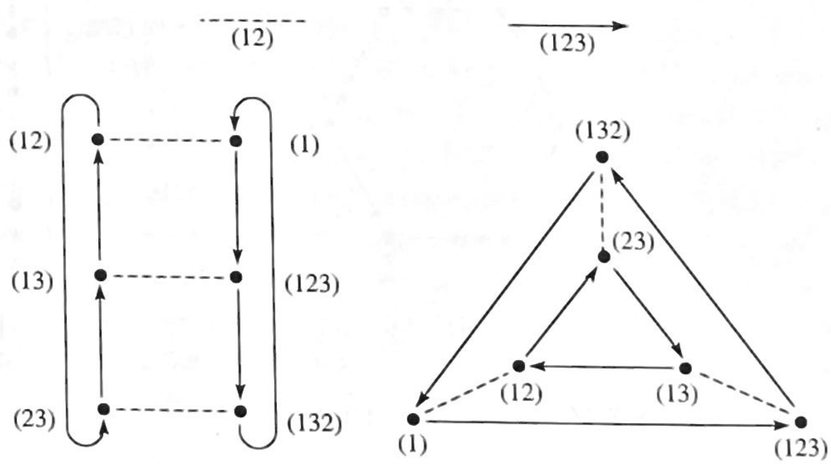
Cay({(1, 0), (0, 1)}:$Z_3 \oplus Z_2$) Cay({(1, 0), (0, 1)}:$Z_3 \oplus Z_2$) ∎

■ **EXAMPLE 3** $D_4 = \langle R_{90}, H \rangle$.



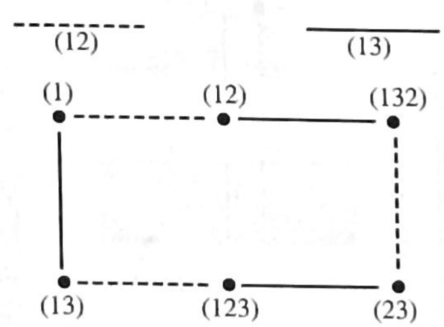
Cay($\{R_{90}, H\}$:D_4) Cay($\{R_{90}, H\}$:D_4) ■

■ **EXAMPLE 4** $S_3 = \langle (12), (123) \rangle$.



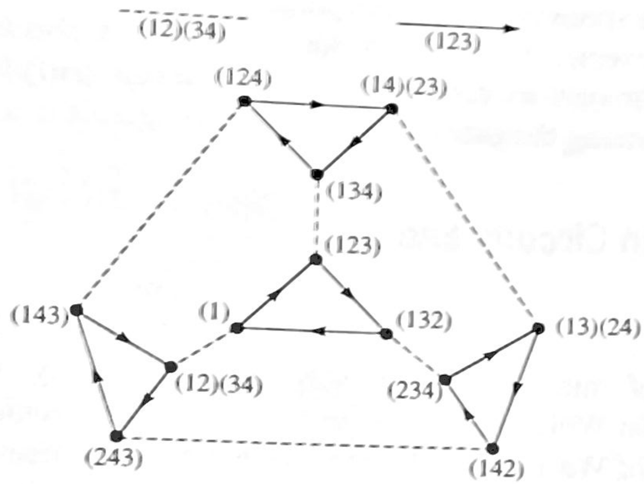
Cay({(12), (123)}:S_3) Cay({(12), (123)}:S_3) ■

■ **EXAMPLE 5** $S_3 = \langle (12), (13) \rangle$.



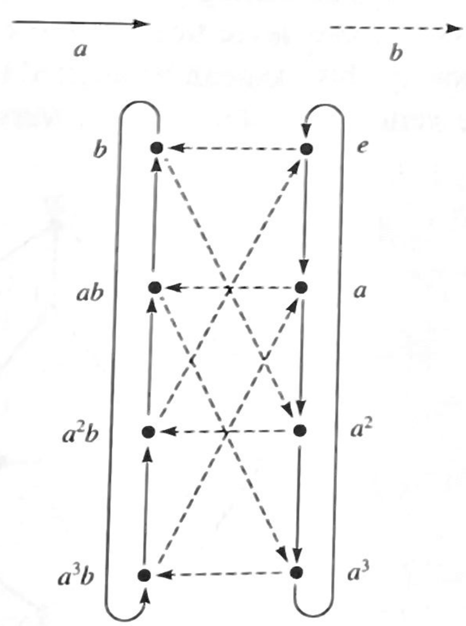
Cay({(12), (13)}:S_3) ■

EXAMPLE 6 $A_4 = \langle(12)(34), (123)\rangle$.



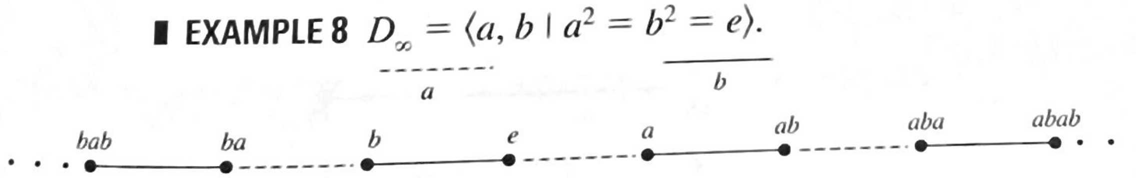
Cay({(12)(34), (123)}:A_4)

EXAMPLE 7 $Q_4 = \langle a, b \mid a^4 = e, a^2 = b^2, b^{-1}ab = a^3\rangle$.



Cay({a, b}:Q_4)

EXAMPLE 8 $D_\infty = \langle a, b \mid a^2 = b^2 = e\rangle$.



Cay({a, b}:D_∞)

The Cayley digraph provides a quick and easy way to determine the value of any product of the generators and their inverses. Consider, for example, the product ab^3ab^{-2} from the group given in Example 7. To reduce this to one of the eight elements used to label the vertices, we need only begin at the vertex e and follow the arcs from each vertex to the next as specified in the given product. Of course, b^{-1} means traverse the b arc in reverse. (Observations such as $b^{-3} = b$ also help.) Tracing the product through, we obtain b. Similarly, one can verify or discover other relations among the generators.

Hamiltonian Circuits and Paths

Now that we have these directed graphs, what is it that we care to know about them? One question about directed graphs that has been the object of much research was popularized by the Irish mathematician Sir William Hamilton in 1859, when he invented a puzzle called "Around the World." His idea was to label the 20 vertices of a regular dodecahedron with the names of famous cities. One solves this puzzle by starting at any particular city (vertex) and traveling "around the world," moving along the arcs in such a way that each other city is visited exactly once before returning to the original starting point. One solution to this puzzle is given in Figure 30.1, where the vertices are visited in the order indicated.

Obviously, this idea can be applied to any digraph; that is, one starts at some vertex and attempts to traverse the digraph by moving along

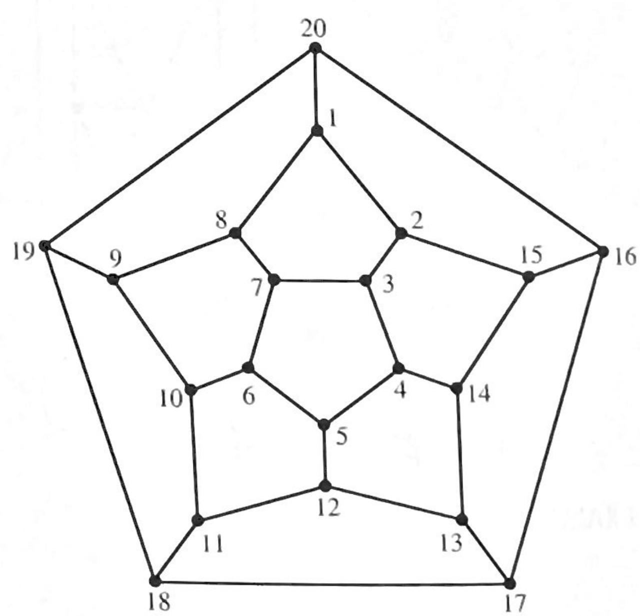


Figure 30.1 Around the World.

arcs in such a way that each vertex is visited exactly once before returning to the starting vertex. (To go from x to y, there must be an arc from x to y.) Such a sequence of arcs is called a *Hamiltonian circuit* in the digraph. A sequence of arcs that passes through each vertex exactly once without returning to the starting point is called a *Hamiltonian path*. In the rest of this chapter, we concern ourselves with the existence of Hamiltonian circuits and paths in Cayley digraphs.

Figures 30.2 and 30.3 show a Hamiltonian path for the digraph given in Example 2 and a Hamiltonian circuit for the digraph given in Example 7, respectively.

Is there a Hamiltonian circuit in

$$\text{Cay}(\{(1, 0), (0, 1)\}:Z_3 \oplus Z_2)?$$

More generally, let us investigate the existence of Hamiltonian circuits in

$$\text{Cay}(\{(1, 0), (0, 1)\}:Z_m \oplus Z_n),$$

where m and n are relatively prime and both are greater than 1. Visualize the Cayley digraph as a rectangular grid coordinatized with $Z_m \oplus Z_n$, as

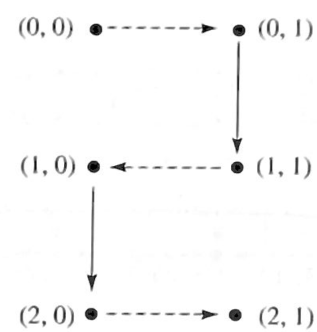


Figure 30.2 Hamiltonian path in $\text{Cay}(\{(1, 0), (0, 1)\}:Z_3 \oplus Z_2)$ from (0, 0) to (2, 1).

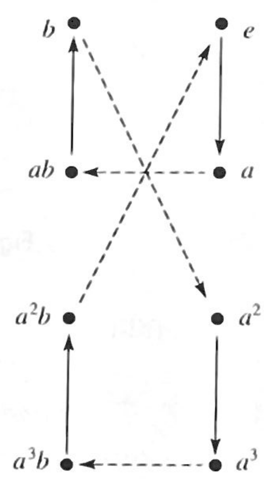


Figure 30.3 Hamiltonian circuit in $\text{Cay}(\{a, b\}:Q_4)$.

in Figure 30.4. Suppose there is a Hamiltonian circuit in the digraph and (a, b) is some vertex from which the circuit exits horizontally. (Clearly, such a vertex exists.) Then the circuit must exit $(a - 1, b + 1)$ horizontally also, for otherwise the circuit passes through $(a, b + 1)$ twice—see Figure 30.5. Repeating this argument again and again, we see that the circuit exits horizontally from each of the vertices (a, b), $(a - 1, b + 1)$, $(a - 2, b + 2), \ldots,$ which is just the coset $(a, b) + \langle(-1, 1)\rangle$. But when m and n are relatively prime, $\langle(-1, 1)\rangle$ is the entire group. Obviously, there cannot be a Hamiltonian circuit consisting entirely of horizontal moves. Let us record what we have just proved.

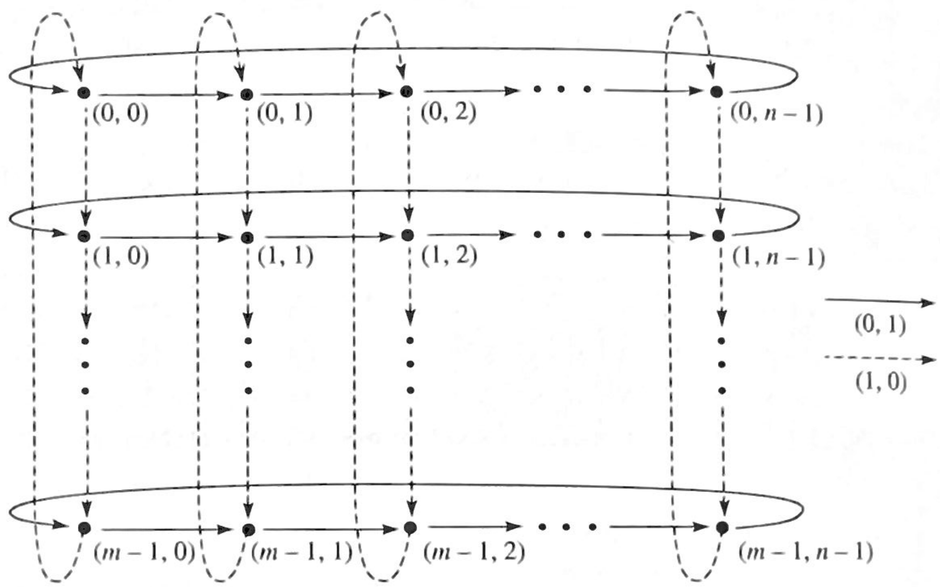


Figure 30.4 Cay({(1, 0), (0, 1)}:$Z_m \oplus Z_n$).

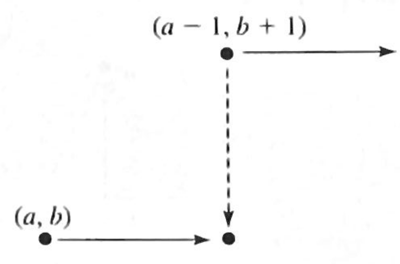


Figure 30.5

Theorem 30.1 A Necessary Condition

Cay({(1, 0), (0, 1)}:$Z_m \oplus Z_n$) does not have a Hamiltonian circuit when m and n are relatively prime and greater than 1.

What about when m and n are not relatively prime? In general, the answer is somewhat complicated, but the following special case is easy to prove.

Theorem 30.2 A Sufficient Condition

$Cay(\{(1, 0), (0, 1)\}:Z_m \oplus Z_n)$ has a Hamiltonian circuit when n divides m.

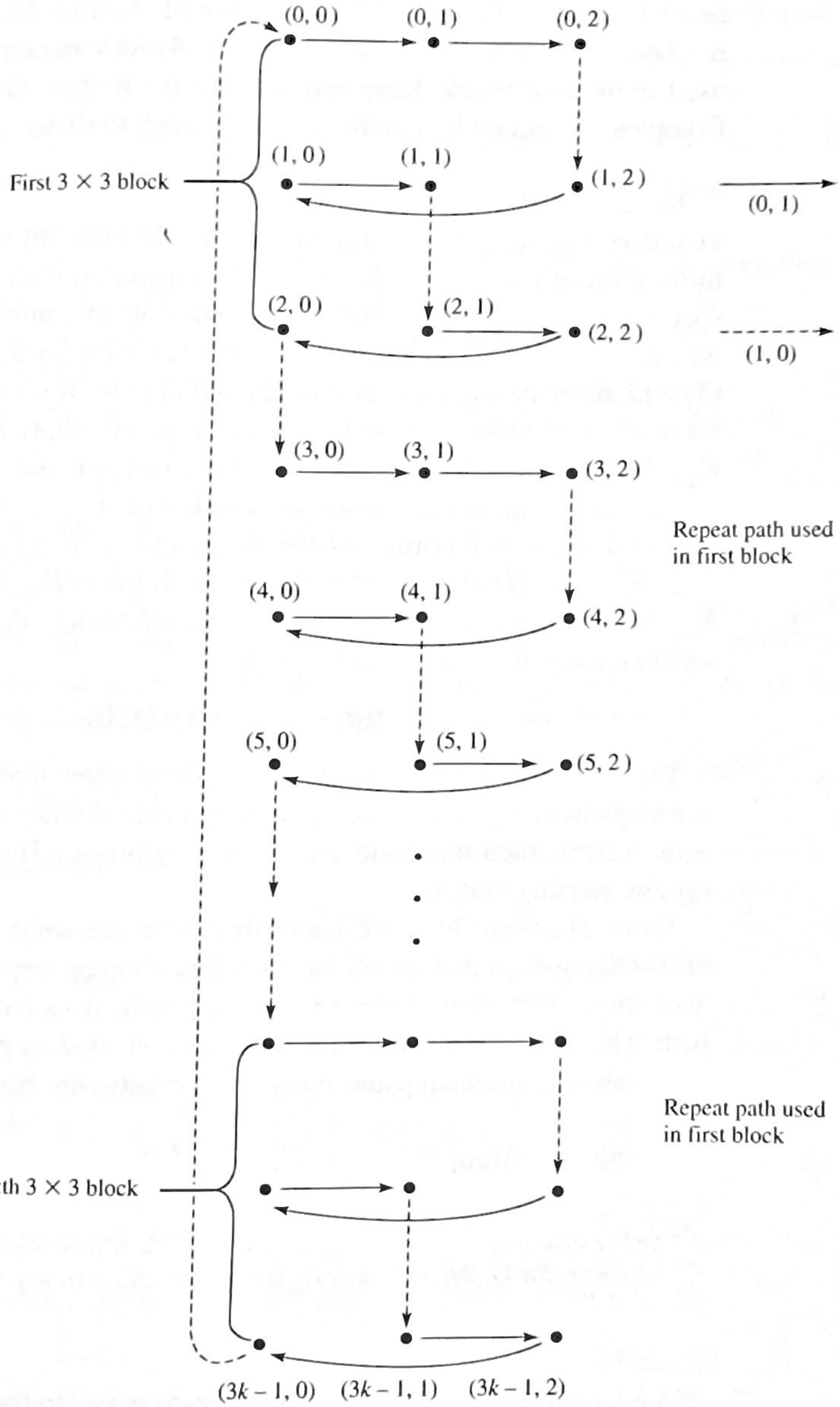


Figure 30.6 $Cay(\{(1, 0), (0, 1)\}:Z_{3k} \oplus Z_3)$.

PROOF Say $m = kn$. Then we may think of $Z_m \oplus Z_n$ as k blocks of size $n \times n$. (See Figure 30.6 for an example.) Start at $(0, 0)$ and cover the vertices of the top block as follows. Use the generator $(0, 1)$ to move horizontally across the first row to the end. Then use the generator $(1, 0)$ to move vertically to the point below, and cover the remaining points in the second row by moving horizontally. Keep this process up until the point $(n - 1, 0)$—the lower left-hand corner of the first block—has been reached. Next, move vertically to the second block and repeat the process used in the first block. Keep this up until the bottom block is covered. Complete the circuit by moving vertically back to $(0, 0)$. ∎

Notice that the circuit given in the proof of Theorem 30.2 is easy to visualize but somewhat cumbersome to describe in words. A much more convenient way to describe a Hamiltonian path or circuit is to specify the starting vertex and the sequence of generators in the order in which they are to be applied. In Example 5, for instance, we may start at (1) and alternate the generators (12) and (13) until we return to (1). In Example 3, we may start at R_0 and successively apply R_{90}, R_{90}, R_{90}, H, R_{90}, R_{90}, R_{90}, H. When k is a positive integer and $a, b, \ldots, c$ is a sequence of group elements, we use $k * (a, b, \ldots, c)$ to denote the concatenation of k copies of the sequence $(a, b, \ldots, c)$. Thus, $2 * (R_{90}, R_{90}, R_{90}, H)$ and $2 * (3 * R_{90}, H)$ both mean $R_{90}, R_{90}, R_{90}, H, R_{90}, R_{90}, R_{90}, H$. With this notation, we may conveniently denote the Hamiltonian circuit given in Theorem 30.2 as

$$m * ((n - 1) * (0, 1), (1, 0)).$$

We leave it as an exercise (Exercise 11) to show that if $x_1, x_2, \ldots, x_n$ is a sequence of generators determining a Hamiltonian circuit starting at some vertex, then the same sequence determines a Hamiltonian circuit for any starting vertex.

From Theorem 30.1, we know that there are some Cayley digraphs of Abelian groups that do not have any Hamiltonian circuits. But Theorem 30.3 shows that each of these Cayley digraphs does have a Hamiltonian path. There are some Cayley digraphs for *non-Abelian* groups that do not even have Hamiltonian paths, but we will not discuss them here.

Theorem 30.3 Abelian Groups Have Hamiltonian Paths

Let G be a finite Abelian group, and let S be any (nonempty†) generating set for G. Then $Cay(S{:}G)$ has a Hamiltonian path.

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†If S is the empty set, it is customary to define $\langle S \rangle$ as the identity group. We prefer to ignore this trivial case.

PROOF We use induction on $|S|$. If $|S| = 1$, say, $S = \{a\}$, then the digraph is just a circle labeled with $e, a, a^2, \ldots, a^{m-1}$, where $|a| = m$. Obviously, there is a Hamiltonian path for this case. Now assume that $|S| > 1$. Choose some $s \in S$. Let $T = S - \{s\}$—that is, T is S with s removed—and set $H = \langle s_1, s_2, \ldots, s_{n-1}\rangle$ where $S = \{s_1, s_2, \ldots, s_n\}$ and $s = s_n$. (Notice that H may be equal to G.)

Because $|T| < |S|$ and H is a finite Abelian group, the induction hypothesis guarantees that there is a Hamiltonian path $(a_1, a_2, \ldots, a_k)$ in Cay$(T{:}H)$. We will show that

$$(a_1, a_2, \ldots, a_k, s, a_1, a_2, \ldots, a_k, s, \ldots, a_1, a_2, \ldots, a_k, s, a_1, a_2, \ldots, a_k),$$

where $a_1, a_2, \ldots, a_k$ occurs $|G|/|H|$ times and s occurs $|G|/|H| - 1$ times, is a Hamiltonian path in Cay$(S{:}G)$.

Because $S = T \cup \{s\}$ and T generates H, the coset Hs generates the factor group G/H. (Since G is Abelian, this group exists.) Hence, the cosets of H are $H, Hs, Hs^2, \ldots, Hs^n$, where $n = |G|/|H| - 1$. Starting from the identity element of G, the path given by $(a_1, a_2, \ldots, a_k)$ visits each element of H exactly once [because $(a_1, a_2, \ldots, a_k)$ is a Hamiltonian path in Cay$(T{:}H)$]. The generator s then moves us to some element of the coset Hs. Starting from there, the path $(a_1, a_2, \ldots, a_k)$ visits each element of Hs exactly once. Then, s moves us to the coset Hs^2, and we visit each element of this coset exactly once. Continuing this process, we successively move to $Hs^3, Hs^4, \ldots, Hs^n$, visiting each vertex in each of these cosets exactly once. Because each vertex of Cay$(S{:}G)$ is in exactly one coset Hs^i, this implies that we visit each vertex of Cay$(S{:}G)$ exactly once. Thus, we have a Hamiltonian path. ∎

We next look at Cayley digraphs with three generators.

EXAMPLE 9 Let

$$D_3 = \langle r, f \mid r^3 = f^2 = e, rf = fr^2\rangle.$$

Then a Hamiltonian circuit in

$$\text{Cay}(\{(r, 0), (f, 0), (e, 1)\}{:}D_3 \oplus Z_6)$$

is given in Figure 30.7. ∎

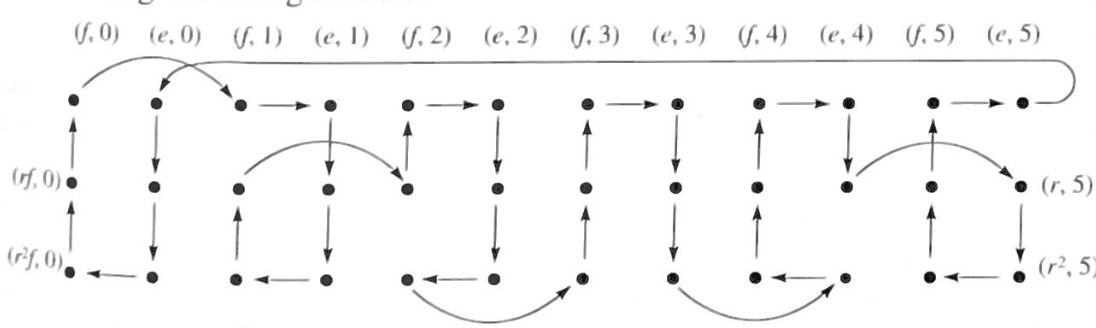


Figure 30.7

Although it is not easy to prove, it is true that

$$\mathrm{Cay}(\{(r, 0), (f, 0), (e, 1)\}:D_n \oplus Z_m)$$

has a Hamiltonian circuit for all n and m. (See [3].) Example 10 shows the circuit for this digraph when m is even.

EXAMPLE 10 Let

$$D_n = \langle r, f \mid r^n = f^2 = e, rf = fr^{-1}\rangle.$$

Then a Hamiltonian circuit in

$$\mathrm{Cay}(\{(r, 0), (f, 0), (e, 1)\}:D_n \oplus Z_m)$$

with m even is traced in Figure 30.8. The sequence of generators that traces the circuit is

$$m * [(n - 1) * (r, 0), (f, 0), (n - 1) * (r, 0), (e, 1)].$$

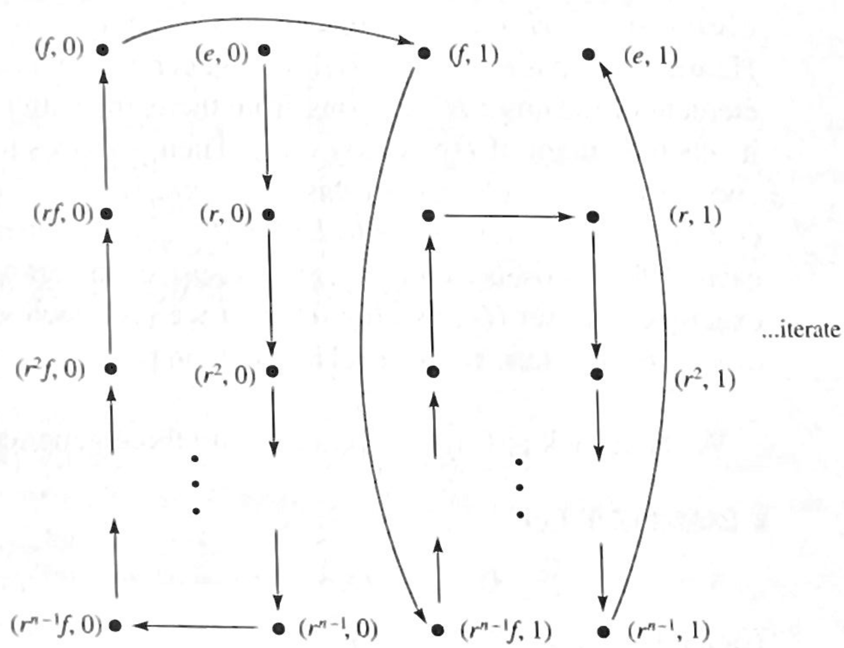


Figure 30.8

Some Applications

Cayley digraphs are natural models for interconnection networks in computer designs, and Hamiltonicity is an important property in relation to sorting algorithms on such networks. One particular Cayley digraph that is used to design and analyze interconnection networks of parallel machines is the symmetric group S_n with the set of all transpositions as the generating set. Hamiltonian paths and circuits in Cayley digraphs arise in a variety of group theory contexts. A Hamiltonian path in a Cayley digraph of a group is simply an ordered listing of the group elements without repetition. The vertices of the digraph are the group elements, and the arcs of the path are generators of the group. In 1948, R. A. Rankin used these ideas (although not the terminology) to prove that certain bell-ringing exercises could not be done by the traditional methods employed by bell ringers. (See [1, Chap. 22] for the group theoretic aspects of bell ringing.) In 1981, Hamiltonian paths in Cayley digraphs were used in an algorithm for creating computer graphics of Escher-type repeating patterns in the hyperbolic plane [2]. This program can produce repeating hyperbolic patterns in color from among various infinite classes of symmetry groups. The program has now been improved so that the user may choose from many kinds of color symmetry. The 2003 Mathematics Awareness Month poster featured one such image (see **http://www.mathaware.org/mam/03/index.html**). Two Escher drawings and their computer-drawn counterparts are given in Figures 30.9 through 30.12.

In this chapter, we have shown how one may construct a directed graph from a group. It is also possible to associate a group—called the *automorphism group*—with every directed graph. In fact, several of the 26 sporadic simple groups were first constructed in this way.

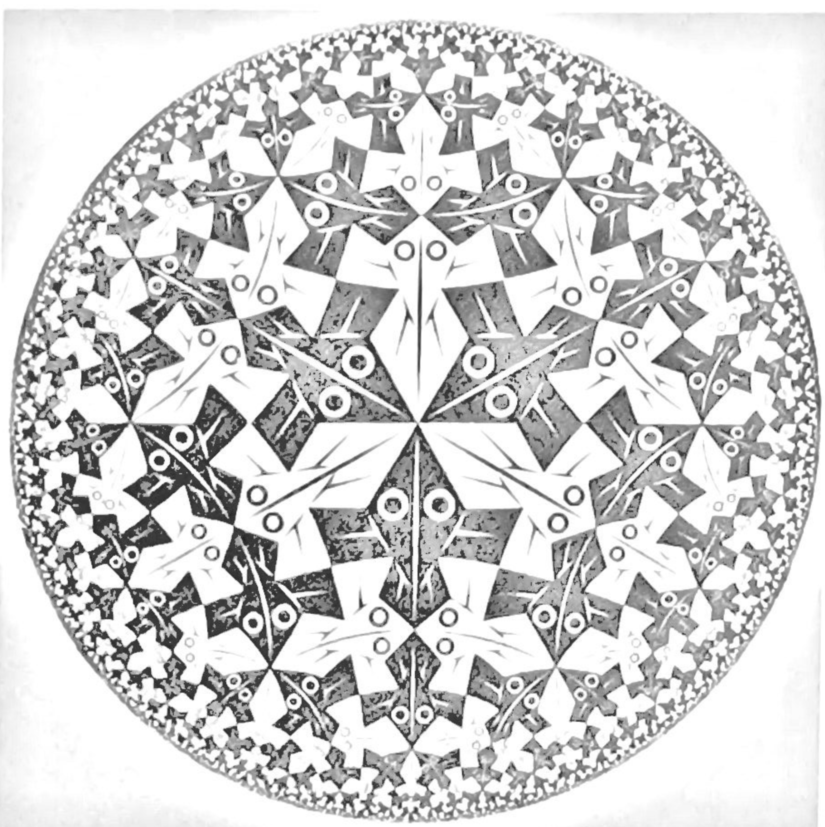


Figure 30.9 M. C. Escher's *Circle Limit I*.

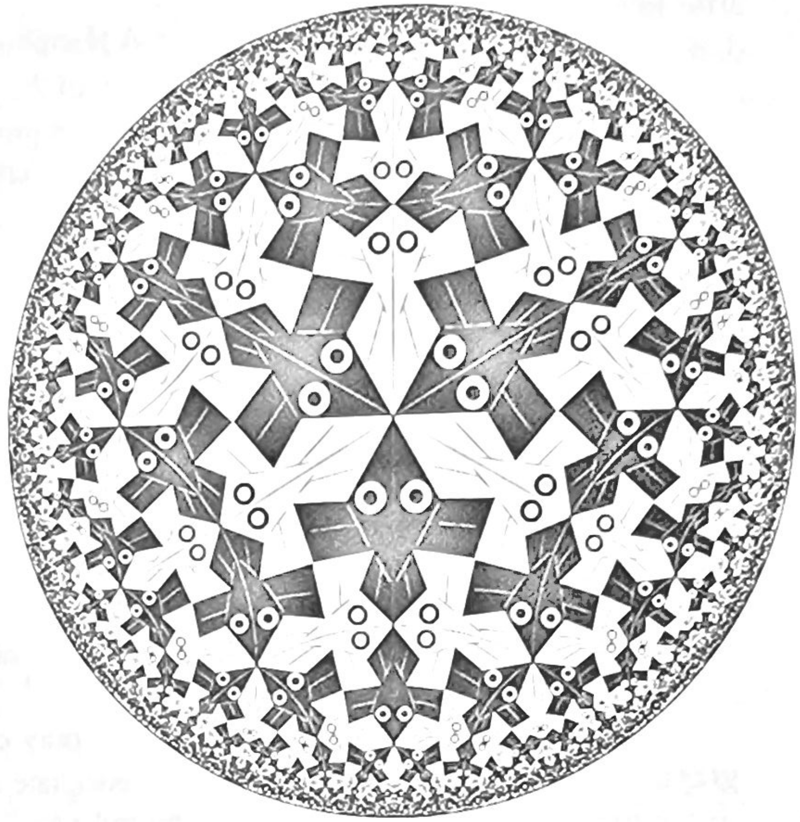


Figure 30.10 A computer duplication of the pattern of M. C. Escher's *Circle Limit I* [2]. The program used a Hamiltonian path in a Cayley digraph of the underlying symmetry group.

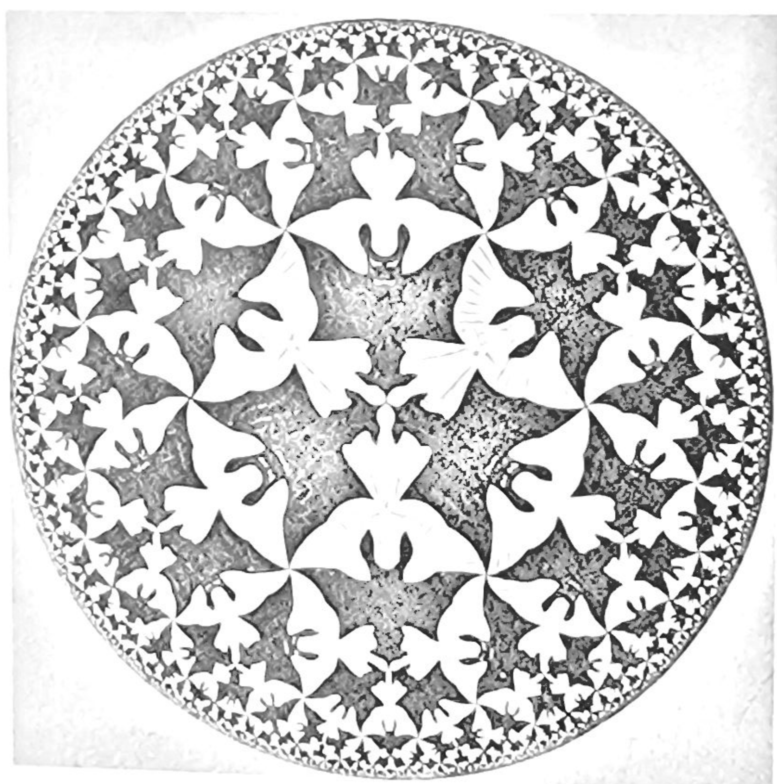


Figure 30.11 M. C. Escher's *Circle Limit IV*.

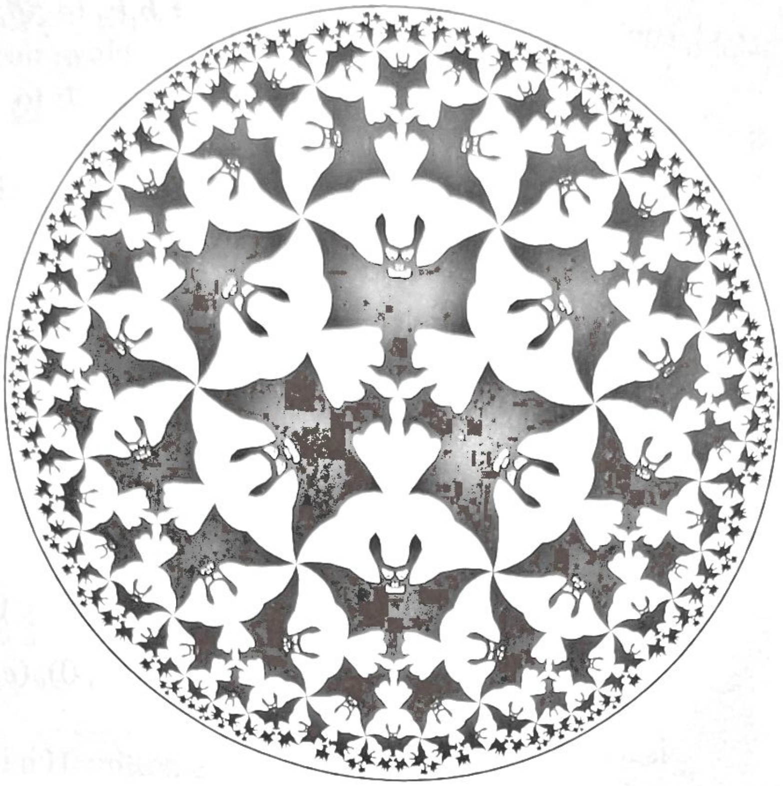


Figure 30.12 A computer drawing inspired by the pattern of M. C. Escher's *Circle Limit IV* [2]. The program used a Hamiltonian path in a Cayley digraph of the underlying symmetry group.

Exercises

A mathematician is a machine for turning coffee into theorems.

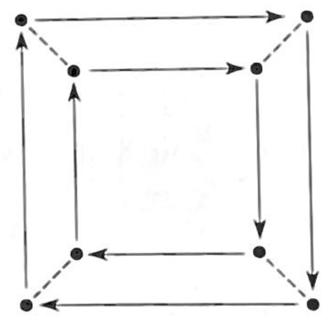
Paul Erdős

1. Find a Hamiltonian circuit in the digraph given in Example 7 different from the one in Figure 30.3.
2. Find a Hamiltonian circuit in
$$\mathrm{Cay}(\{(a, 0), (b, 0), (e, 1)\}:Q_4 \oplus Z_2).$$
3. Find a Hamiltonian circuit in
$$\mathrm{Cay}(\{(a, 0), (b, 0), (e, 1)\}:Q_4 \oplus Z_m)$$
where m is even.
4. Write the sequence of generators for each of the circuits found in Exercises 1, 2, and 3.
5. Use the Cayley digraph in Example 7 to evaluate the product $a^3ba^{-1}ba^3b^{-1}$.
6. Let x and y be two vertices of a Cayley digraph. Explain why two paths from x to y in the digraph yield a group relation—that is, an

equation of the form $a_1a_2 \cdots a_m = b_1b_2 \cdots b_n$, where the a_i's and b_j's are generators of the Cayley digraph.

7. Use the Cayley digraph in Example 7 to verify the relation $aba^{-1}b^{-1}a^{-1}b^{-1} = a^2ba^3$.

8. Identify the following Cayley digraph of a familiar group.



9. Let $D_4 = \langle r, f \mid r^4 = e = f^2, rf = fr^{-1} \rangle$. Verify that

$$6 * [3 * (r, 0), (f, 0), 3 * (r, 0), (e, 1)]$$

is a Hamiltonian circuit in

$$\text{Cay}(\{(r, 0), (f, 0), (e, 1)\}:D_4 \oplus Z_6).$$

10. Draw a picture of Cay($\{2, 5\}:Z_8$).

11. If $s_1, s_2, \ldots, s_n$ is a sequence of generators that determines a Hamiltonian circuit beginning at some vertex, explain why the same sequence determines a Hamiltonian circuit beginning at any point. (This exercise is referred to in this chapter.)

12. Show that the Cayley digraph given in Example 7 has a Hamiltonian path from e to a.

13. Show that there is no Hamiltonian path in

$$\text{Cay}(\{(1, 0), (0, 1)\}:Z_3 \oplus Z_2)$$

from (0, 0) to (2, 0).

14. Draw Cay($\{2, 3\}:Z_6$). Is there a Hamiltonian circuit in this digraph?

15. **a.** Let G be a group of order n generated by a set S. Show that a sequence $s_1, s_2, \ldots, s_{n-1}$ of elements of S is a Hamiltonian path in Cay(S:G) if and only if, for all i and j with $1 \leq i \leq j < n$, we have $s_is_{i+1} \cdots s_j \neq e$.

    b. Show that the sequence $s_1s_2 \cdots s_n$ is a Hamiltonian circuit if and only if $s_1s_2 \cdots s_n = e$, and that whenever $1 \leq i \leq j < n$, we have $s_is_{i+1} \cdots s_j \neq e$.

16. Let $D_4 = \langle a, b \mid a^2 = b^2 = (ab)^4 = e \rangle$. Draw Cay($\{a, b\}:D_4$). Why is it reasonable to say that this digraph is undirected?

17. Let D_n be as in Example 10. Show that $2 * [(n - 1) * r, f]$ is a Hamiltonian circuit in Cay($\{r, f\}:D_n$).

18. Let $Q_8 = \langle a, b \mid a^8 = e, a^4 = b^2, b^{-1}ab = a^{-1}\rangle$. Find a Hamiltonian circuit in $\text{Cay}(\{a, b\}:Q_8)$.

19. Let Q_8 be as in Exercise 18. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a, 0), (b, 0), (e, 1)\}:Q_8 \oplus Z_5).$$

20. Prove that the Cayley digraph given in Example 6 does not have a Hamiltonian circuit. Does it have a Hamiltonian path?

21. Find a Hamiltonian circuit in

$$\text{Cay}(\{(R_{90}, 0), (H, 0), (R_0, 1)\}:D_4 \oplus Z_3).$$

Does this circuit generalize to the case $D_{n+1} \oplus Z_n$ for all $n \geq 3$?

22. Let Q_8 be as in Exercise 18. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a, 0), (b, 0), (e, 1)\}:Q_8 \oplus Z_m) \text{ for all even } m.$$

23. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a, 0), (b, 0), (e, 1)\}:Q_4 \oplus Z_3).$$

24. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a, 0), (b, 0), (e, 1)\}:Q_4 \oplus Z_m) \text{ for all odd } m \geq 3.$$

25. Write the sequence of generators that describes the Hamiltonian circuit in Example 9.

26. Let D_n be as in Example 10. Find a Hamiltonian circuit in

$$\text{Cay}(\{(r, 0), (f, 0), (e, 1)\}:D_4 \oplus Z_5).$$

Does your circuit generalize to the case $D_n \oplus Z_{n+1}$ for all $n \geq 4$?

27. Prove that $\text{Cay}(\{(0, 1), (1, 1)\}:Z_m \oplus Z_n)$ has a Hamiltonian circuit for all m and n greater than 1.

28. Suppose that a Hamiltonian circuit exists for $\text{Cay}(\{(1, 0), (0, 1)\}: Z_m \oplus Z_n)$ and that this circuit exits from vertex (a, b) vertically. Show that the circuit exits from every member of the coset $(a, b) + \langle(1, -1)\rangle$ vertically.

29. Let $D_2 = \langle r, f \mid r^2 = f^2 = e, rf = fr^{-1}\rangle$. Find a Hamiltonian circuit in $\text{Cay}(\{(r, 0), (f, 0), (e, 1)\}:D_2 \oplus Z_3)$.

30. Let Q_8 be as in Exercise 18. Find a Hamiltonian circuit in $\text{Cay}(\{(a, 0), (b, 0), (e, 1)\}:Q_8 \oplus Z_3)$.

31. In $\text{Cay}(\{(1, 0), (0, 1)\}:Z_4 \oplus Z_5)$, find a sequence of generators that visits exactly one vertex twice and all others exactly once and returns to the starting vertex.

32. In $\text{Cay}(\{(1, 0), (0, 1)\}:Z_4 \oplus Z_5)$, find a sequence of generators that visits exactly two vertices twice and all others exactly once and returns to the starting vertex.

33. Find a Hamiltonian circuit in Cay($\{(1, 0), (0, 1)\}:Z_4 \oplus Z_6$).

34. Let G be the digraph obtained from Cay($\{(1, 0), (0, 1)\}:Z_3 \oplus Z_5$) by deleting the vertex (0, 0). [Also, delete each arc to or from (0, 0).] Prove that G has a Hamiltonian circuit.

35. Prove that the digraph obtained from Cay($\{(1, 0), (0, 1)\}:Z_4 \oplus Z_7$) by deleting the vertex (0, 0) has a Hamiltonian circuit.

36. Let G be a finite group generated by a and b. Let $s_1, s_2, \ldots, s_n$ be the arcs of a Hamiltonian circuit in the digraph Cay($\{a, b\}:G$). We say that the vertex $s_1 s_2 \cdots s_i$ *travels by* a if $s_{i+1} = a$. Show that if a vertex x travels by a, then every vertex in the coset $x\langle ab^{-1}\rangle$ travels by a.

37. A finite group is called *Hamiltonian* if all of its subgroups are normal. (One non-Abelian example is Q_4.) Show that Theorem 30.3 can be generalized to include all Hamiltonian groups.

38. (Factor Group Lemma) Let S be a generating set for a group G, let N be a cyclic normal subgroup of G, and let

$$\overline{S} = \{sN \mid s \in S\}.$$

If $(a_1N, \ldots, a_rN)$ is a Hamiltonian circuit in Cay($\overline{S}:G/N$) and the product $a_1 \cdots a_r$ generates N, prove that

$$|N| * (a_1, \ldots, a_r)$$

is a Hamiltonian circuit in Cay($S:G$).

References

1. F. J. Budden, *The Fascination of Groups,* Cambridge: Cambridge University Press, 1972.
2. Douglas Dunham, John Lindgren, and David Witte, "Creating Repeating Hyperbolic Patterns," *Computer Graphics* 15 (1981): 215–223.
3. David Witte, Gail Letzter, and Joseph A. Gallian, "On Hamiltonian Circuits in Cartesian Products of Cayley Digraphs," *Discrete Mathematics* 43 (1983): 297–307.

Suggested Readings

Frank Budden, "Cayley Graphs for Some Well-Known Groups," *The Mathematical Gazette* 69 (1985): 271–278.

This article contains the Cayley graphs of A_4, Q_4, and S_4 using a variety of generators and relations.

E. L. Burrows and M. J. Clark, "Pictures of Point Groups," *Journal of Chemical Education* 51 (1974): 87–90.

Chemistry students may be interested in reading this article. It gives a comprehensive collection of the Cayley digraphs of groups important to chemists.

Douglas Dunham, John Lindgren, and David Witte, "Creating Repeating Hyperbolic Patterns," *Computer Graphics* 15 (1981): 215–223.

In this beautifully illustrated paper, a process for creating repeating patterns of the hyperbolic plane is described. The paper is a blend of group theory, geometry, and art.

Joseph A. Gallian, "Circuits in Directed Grids," *The Mathematical Intelligencer* 13 (1991): 40–43.

This article surveys research done on variations of the themes discussed in this chapter.

Joseph A. Gallian and David Witte, "Hamiltonian Checkerboards," *Mathematics Magazine* 57 (1984): 291–294.

This paper gives some additional examples of Hamiltonian circuits in Cayley digraphs. It is available at **http://www.d.umn.edu/~jgallian/checker.pdf**

Henry Levinson, "Cayley Diagrams," in *Mathematical Vistas: Papers from the Mathematics Section,* New York Academy of Sciences, J. Malkevitch and D. McCarthy, eds., 1990: 62–68.

This richly illustrated article presents Cayley digraphs of many of the groups that appear in this text.

A. T. White, "Ringing the Cosets," *The American Mathematical Monthly* 94 (1987): 721–746.

This article analyzes the practice of bell ringing by way of Cayley digraphs.

Suggested Website

http://www.d.umn.edu/~ddunham/

This website has copies of several articles that describe the mathematics involved in creating Escher-like repeating patterns in the hyperbolic plane as shown in Figure 30.10.

Suggested DVD

N is a Number, Mathematical Association of America, 58 minutes.

In this documentary, Erdős discusses politics, death, and mathematics. Many of Erdős's collaborators and friends comment on his work and life. It is available for purchase at **http://www.amazon.com**