The Chromatic Function

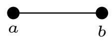
For any graph G, let $P(G,k)$ denote the *number* of proper colorings of G that use up to k colors (think of having k colors in your crayon box you can use to color the vertices; you don't have to use all of them). For a fixed graph G, $P(G,k)$ is a function of k.

1. For each graph G below, find $P(G,2)$, $P(G,3)$, and $P(G,4)$. Note, these graphs are *labeled*, so coloring vertex a red and b blue is a different coloring than coloring a blue and b red.

RB RY RG RP
BR BY BG BP
YR YB YG YP
GR GB GY GP

$P=4$

$P(G,k) = k^2 - k = k(k-1)$



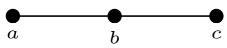
$P(G,2) = 2$ RB BR

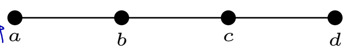
$P(G,3) = 6$

$P(G,4) = 12$

$P(G,5) = 20$

4, 6, 8; 2, 2

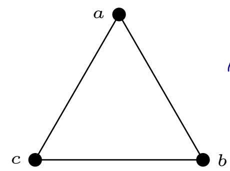




G

$P(G,4) = 4 \cdot 3 \cdot 3 \cdot 3$

$P(G,k) = k \cdot (k-1)^3$



$P(G,2) = 0$

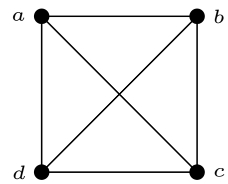
$P(G,3) = 6 = 3!$

$P(G,4) = 24 = 4!$

$P(G,5) = \binom{5}{3} \cdot 3!$

$= 5 \cdot 4 \cdot 3$

$= P(5,3)$



$P(K_n, k) = P(k, n)$

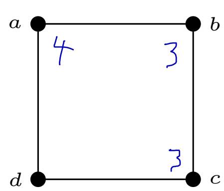
2. Generalize: Find $P(P_3,k)$ for any k (recall P_3 is the path with 4 vertices and 3 edges) and $P(C_3,k)$ for any k (where C_3 is a 3-edge cycle, the same as K_3). That is, find a closed formula for each of these.

$P(P_3, k) = k \cdot (k-1)^3$

$P(C_3, k) = k(k-1)(k-2)$

$= \binom{k}{3} \cdot 3!$

3. Use your answers to the previous problem to find $P(C_4,k)$. Hint: If there wasn't an edge between a and d, then we this would be coloring P_3. The colorings that we don't want to allow are those in which a and d are the same: how many of those are there?



$P(C_4, 4)$