Math 322 **Homework 7** Due: Wednesday, April 10

Instructions: Same rules as usual. Work together, write-up alone, no internet!

(9pts) 1. Consider the normal series below for $\mathbb{Z}_{24}$:

$$\mathbb{Z}_{24} \supset \langle 12 \rangle \supset \{0\}$$

   (a) Find the two quotient groups for the series. Find the "standard" abelian groups each is isomorphic to.

   (b) For the quotient group that is not simple found above, find a non-trivial normal subgroup, and realize it as a quotient group $G'/\langle 12 \rangle$ for some G'.

   (c) Demonstrate/explain how this shows us how to build a longer normal series for $\mathbb{Z}_{24}$.

(6pts) 2. Find two different composition series for $\mathbb{Z}_{28}$. Then use quotient groups to demonstrate that the two series are "isomorphic" (and explain what this means).

(4pts) 3. Suppose G is a group that contains a normal subgroup H which is itself a non-abelian simple group. Explain how you know that G is not solvable. Note, this is not difficult at all if you know the definitions of simple and solvable.

(6pts) 4. Consider the polynomial $p(x) = x^7 - 1 = (x-1)(x^6 + x^5 + \cdots + x + 1)$. Let $\omega = e^{2\pi i/7}$ be a root of $p(x)$. Then $\mathbb{Q}(\omega)$ is the splitting field for $p(x)$.

   (a) Explain how we know that the Galois group $\mathrm{Gal}(\mathbb{Q}(\omega) : \mathbb{Q})$ is isomorphic to $\mathbb{Z}_7^*$. Give two examples of elements in $\mathrm{Gal}(\mathbb{Q}(\omega) : \mathbb{Q})$ and say what elements in Z_7^* they correspond to.

   (b) $\mathbb{Z}_7^*$ has a composition series $\mathbb{Z}_7^* \supset \{1, 6\} \supset \{1\}$. Find the corresponding series of extension fields of $\mathbb{Q}$. In other words, find the intermediate field E such that $\mathrm{Gal}(\mathbb{Q}(\omega) : E) \cong \{1, 6\}$.

(5pts) 5. Find a degree 5 polynomial whose Galois group is isomorphic to S_5. Explain how you know your example works. Your example should be different from the one we discuss in class.

(5000bns-pts) 6. Bonus: express the roots of the polynomial you found in the previous question in terms of rational numbers, field operations and roots (e.g., square roots, cube roots, etc.)