Math 322 **Homework 2** Due: Wednesday, February 6

Instructions: Carefully write up solutions to the questions below. A solution should consist of both the answer and a careful explanation for why that answer must be correct. Any solutions without an explanation written out in English prose will receive no credit. You are welcome to work together, but write up solutions in your own, individual rules. Also, NO INTERNET!

(6pts) 1. Consider the field $\mathbb{Q}(\sqrt{3}+\sqrt{5})$, as well as the fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$.

(a) Is $\mathbb{Q}(\sqrt{3}+\sqrt{5})$ an extension field of $\mathbb{Q}(\sqrt{3})$? Is it an extension field of $\mathbb{Q}(\sqrt{5})$? Explain. Hint: does $\mathbb{Q}(\sqrt{3}+\sqrt{5})$ contain $\sqrt{15}$? If it does, then must is also contain $\sqrt{15}(\sqrt{3}+\sqrt{5})-3(\sqrt{3}+\sqrt{5})$?

(b) If we take $\mathbb{Q}(\sqrt{3})$ and adjoin $\sqrt{5}$, do we get a bigger field? We write $\mathbb{Q}(\sqrt{3})(\sqrt{5}) = \mathbb{Q}(\sqrt{3},\sqrt{5})$. Draw a tower diagram showing how all the fields in this problem are related and explain why your diagram is accurate.

(6pts) 2. Find the minimum polynomials for:

(a) $\sqrt{3}+i$ over $\mathbb{Q}(i)$

(b) $\sqrt{3}+i$ over $\mathbb{Q}$.

(12pts) 3. Recall that we say an element a is *algebraic* over a field F if it is the root of a polynomial in $F[x]$.

(a) Prove that $a=\sqrt{3+\sqrt[3]{5}}$ is algebraic over $\mathbb{Q}$.

(b) Find a basis for $\mathbb{Q}(a)$ over $\mathbb{Q}(\sqrt[3]{5})$. What is $[\mathbb{Q}(a):\mathbb{Q}(\sqrt[3]{5})]$?

(c) Find a basis for $\mathbb{Q}(a)$ over $\mathbb{Q}$. What is $[\mathbb{Q}(a):\mathbb{Q}]$?

(d) Find the *minimum* polynomial for a over $\mathbb{Q}$ and prove it is irreducible (use part (c)).

(6pts) 4. Which square roots are irrational? Let's find out.

(a) Prove that if c is a root of an irreducible polynomial of degree greater than 1 in $\mathbb{Q}[x]$, then c is irrational.

(b) Let $m,n\in\mathbb{Z}$ and consider $\sqrt{m/n}$. Suppose there is a prime p which divides m but not n, and that p^2 does not divide m. Prove that $\sqrt{m/n}$ is irrational.

(6pts) 5. Prove that a regular 9-gon is not constructible (with a compass and straight-edge), but a regular 20-gon is. For the impossibility of the of the regular 9-gon, your proof should mimic the proof we did in class that 20° is not constructible. That is, your proof should talk about field extensions and polynomials (do not just argue that it is impossible because if we could construct it, we would then be able to construct 20° which we know is impossible).