Activity: Dimension and Degree

Recall that a *basis* for a vector space is a linearly independent spanning set, and that the *dimension* of a vector space is the size of a (any) basis for the space.

If K is an extension field of F, we can view K as a vector space over the field of scalars F. In this case, we say the **degree** of K over F, written $[K : F]$ is the dimension of this vector space.

1. Find a basis for $\mathbb{Q}(\sqrt[3]{5})$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$?

2. Find a basis for $\mathbb{Q}(\sqrt{7})$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}]$?

3. Suppose α is a root of $p(x) = x^5 - 6x^4 + 9x^2 + 3$. Find a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

4. What is the connection between the degree of a field extension and polynomials? Investigate this. In particular, if you claim that you can always find a basis in some systematic way, how do you know it is really a basis? How do you know the basis is linearly independent? How do you know it spans?

5. The polynomial $q(x) = x^7 - 6x^6 - 7x^5 + 51x^4 - 60x^2 - 21$ has α (from question 3) and $\sqrt{7}$ as roots. Does this mean $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 7$? Why not?