Activity: Quotient Rings that are Extension Fields

Let's explore the connection between extension fields and quotient rings. We will see that $F[x]/\langle p(x)\rangle \cong F(\alpha)$, where α is a root of its minimal polynomial $p(x)$ (i.e., $p(x)$ is the unique monic irreducible polynomial that has α as a root). The goal of this activity is to see how working in quotient rings help us realize $E = F(\alpha)$ as a field.

We will start easy. For now, let $E = \mathbb{Q}(\sqrt{2})$.

1. What quotient ring is E isomorphic to?

2. One element in E is $1 + 3\sqrt{2}$. What element in the quotient ring does this correspond to?

3. What will $\gcd(3x + 1, x^2 - 2)$ be? How do you know? Then verify you are correct using the Euclidean algorithm.

4. Bezout's identity says that for any polynomials $a(x)$ and $b(x)$, there are polynomials $s(x)$ and $t(x)$ such that

$$\gcd(a(x), b(x)) = s(x)a(x) + t(x)b(x).$$

Find $s(x)$ and $t(x)$ in our case, by working backwards from the Euclidean algorithm above.

5. What does Bezout's identity have to do with the expression

$$1 + \langle x^2 - 2\rangle = (3x + 1 + \langle x^2 - 2\rangle)(t(x) + \langle x^2 - 2\rangle)$$

and what does this have to do with finding inverses? In particular, what is $(1 + 3\sqrt{2})^{-1}$ in E?

6. Now let's try this again with a more complicated polynomial. As in the earlier activity, take $p(x) = x^3 + 3x^2 - x + 2$ and let ϱ be a root. Use quotient rings to find the inverse of the element $2 + 3\varrho^2$ in $E = \mathbb{Q}(\varrho)$.