Part I: Geometric Constructions

Our first application of algebra (to other mathematics) will be to questions of classical geometry. We will look at geometry as it was done in ancient Greece, except that we will use GeoGebra for our constructions.

Our main question is, what can you *construct* using reasonable, fundamental tools. The tools are: an unmarked straightedge and a compass.

In GeoGebra, there are many more tools than these. Make sure you only use the "new point" tool (to place points at the intersections of lines, circles, or both), the "line through two points" tool, and the "compass" tool (under the circle menu). You can also use the arrow to drag things around if you need to.

To get a feel for the sorts of things you can construct, and maybe things you cannot, here are a few challenges.

1. Can you construct a 60° angle? A 30° angle? If you have constructed any angle at all, can you construct an angle half its measure? That is, can you *bisect* an given angle?

2. Can you construct a square? Can you double the square? That is, if you can construct a square, can you construct a square of twice the area?

3. Can you double the circle? That is, can you construct a circle and then construct a second circle of twice the area?

4. Here are three much harder, but related challenges. For each, play around enough to convince yourself these are really hard, if not impossible:

(a) Can you *trisect* and angle? That is, given a constructed angle, can you construct an angle $1/3$ its measure?

(b) Can you double the *cube*? That is, if you can constructed a cube (or at least a line segment which is the length of the edge of a cube), can you construct cube with twice the volume of the original?

(c) Can you square the circle? That is, if you have constructed a circle, can you construct a square that has the same area as the circle?

Part II: Constructible Numbers

Last time we considered what geometric shapes we could or could not construct. We were left with three big questions: is it possible to *trisect an angle*, to *double a cube*, or to *square a circle*. To answer these questions, we must "algebratize" geometric constructions.

Start with two constructible points 0 and 1 {\em one unit} apart. We define **constructible** recursively from this base case:

(a) A **constructible line** is a line passing through two constructible points.

(b) A **constructible circle** is a circle whose radius is a constructible number and whose center is a constructible point.

(c) A **constructible point** is the intersection of two constructible lines, two constructible circles, or a constructible line and a constructible circle.

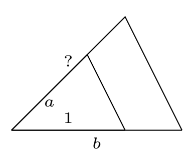
We say a number a is **constructible** provided $a = 0$ or there are two constructible points distance $|a|$ apart. So far we have constructible numbers 0, 1, and -1. What else is constructible?

For this activity, use GeoGebra. Use only the "new point" tool (to place points at the intersections of lines, circles, or both), the "line through two points" tool, and the "compass" tool (under the circle menu). You can also use the arrow to drag things around if you need to.

1. Show that the numbers 2 and 4 are constructible. Then show that the number $3 = 4 - 1$ is constructible.

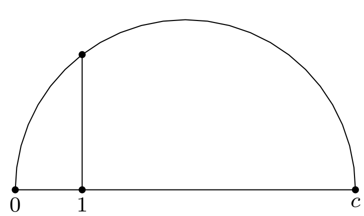
2. If a and b are constructible numbers, are the numbers $a + b$ and $a - b$ also constructible?

3. Suppose a and b are constructible. Construct a triangle containing a base of unit length adjacent to a side of length a. Construct a similar triangle with where the side corresponding the unit length side now has length b. What is the length of the side corresponding to the a-length side?



4. Explain how you can modify the above construction to prove that if a and b are constructible, then a/b is constructible.

5. Given constructible number c, explain how you can construct the figure below. The vertical line should be perpendicular to the horizontal line, which is the diameter of the circle.



What is the length of the vertical line?

6. Let $\mathfrak{C}$ be the set of all constructible numbers. What sort of set is this? Is it a group? A ring? A field? Is it one of these we know about already?