

Recall that a *basis* for a vector space is a linearly independent spanning set, and that the *dimension* of a vector space is the size of a (any) basis for the space.

If K is an extension field of F , we can view K as a vector space over the field of scalars F . In this case, we say the **degree** of K over F , written $[K : F]$ is the dimension of this vector space.

1. Find a basis for $\mathbb{Q}(\sqrt{7})$ over \mathbb{Q} . What is $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}]$?

2. Find a basis for $\mathbb{Q}(\sqrt[3]{5})$ over \mathbb{Q} . What is $[\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}]$?

3. Suppose α is a root of $p(x) = x^5 - 6x^4 + 9x^2 + 3$. Find a basis for $\mathbb{Q}(\alpha)$ over \mathbb{Q} . What is $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

4. What is the general rule here? Some things to think about: If you claim that you can always find a basis in some systematic way, how do you know it is really a basis? How do you know the basis is linearly independent? How do you know it spans?

5. The polynomial $q(x) = x^5 - 7x^3 - 5x^2 + 35$ has $\sqrt{7}$ and $\sqrt[3]{5}$ as roots. Does this mean $[\mathbb{Q}(\sqrt{7}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{5}) : \mathbb{Q}] = 5$? Why not?

6. We can think of this as an extension of an extension. Take $\mathbb{Q}(\sqrt[3]{5})$ as our base field. Adjoin to that $\sqrt{7}$ to get $\mathbb{Q}(\sqrt[3]{5}, \sqrt{7})$. What is $[\mathbb{Q}(\sqrt[3]{5}, \sqrt{7}) : \mathbb{Q}(\sqrt[3]{5})]$? Use the general rule we discovered above and also find a basis

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