For any n, the group $U(n)$ is the set of all positive integers less than and relatively prime to n, under multiplication modulo n. For example we saw that $U(8) = \{1, 3, 5, 7\}$ is a group under multiplication modulo 8.

Consider the group $U(28)$. The table below gives the twelve elements with their orders:

g	1	3	5	9	11	13	15	17	19	23	25	27
ord(g)	1	6	6	3	6	2	2	6	6	6	3	2

4. Let $G(n)$ be the set of all elements of order n^k for some k (that is, elements with order some *power* of n). Find $G(2)$ and $G(3)$ for $U(28)$.

$G(2) = \{1, 13, 15, 27\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad G(7) = \{1\}$

$G(3) = \{1, 9, 25\} \cong \mathbb{Z}_3$

p-groups: all elements have order p^k for some k.

$H = \{g \in G : g^p = e\}$ $\quad e = g_1^p \cdot g_2^p = (g_1 g_2)^p$

5. Are $G(2)$ and $G(3)$ subgroups of $U(28)$?

yes.

$13 \cdot 25$

6. Do $G(2)$ and $G(3)$ have the property that $G(2) \cap G(3) = \{1\}$ and $U(28) = G(2)G(3)$?

$U(28)$ is the internal direct product of $G(2)$ and $G(3)$

$$\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{G(2)} \times \underbrace{\mathbb{Z}_3}_{G(3)}$$

7. Is $U(28) \cong G(2) \times G(3)$? ✓ Is $U(28) \cong \mathbb{Z}_m \times \mathbb{Z}_n$ for some values of m and n? ✗ $\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

external direct product

$\varphi : G(2) \times G(3) \longrightarrow U(28)$

$\varphi((a, b)) = a \cdot b$

$\{(1,1), (13,1), (15,1), (27,1), (1,9), (13,9), (15,9), (27,9), (1,25), (13,25), (15,25), (27,25)\}$

$\varphi((a,b) \cdot (c,d)) = \varphi((ac, bd)) = ac \cdot bd = abcd = \varphi((a,b)) \cdot \varphi((c,d))$

$\mathbb{Z}_n$ is cyclic. it is $\langle a \rangle$ for a of order n.

If G is a group with $|G| = 8$. G abelian.
what is G?

- If $a \in G$ has $\text{ord}(a) = 8$, Then $G = \langle a \rangle \cong \mathbb{Z}_8$
- If there is no order 8 element, is there $a \in G$ with $\text{ord}(a) = 4$? If so, $\langle a \rangle \cong \mathbb{Z}_4$

  Let H be a 2-element subgroup so that $\langle a \rangle H = G$

  So $H = \{e, h\}$ $h \notin \langle a \rangle$ $\langle a \rangle \cap H = \{e\}$

  $\langle a \rangle = \{e, a, a^2, a^3\}$ $\langle a \rangle \cdot H = G$.

  G is the internal direct product of $\langle a \rangle$ and H.

  So $G \cong \langle a \rangle \times H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$
- If largest order in G is 2, let $a \in G$ have $\text{ord}(a) = 2$, look at $\langle a \rangle = \{e, a\} \cong \mathbb{Z}_2$

  Find H such that $\langle a \rangle \cap H = \{e\}$, $\langle a \rangle \cdot H = G$.

  $|H| = 4$, $G \cong \langle a \rangle \times H \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

  repeat.

$$G \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_n^{k_n}}$$

Fundamental Theorem of Finite abelian Groups.

Find all abelian groups of order $540 = 2^2 \cdot 3^3 \cdot 5$

$\times$ $\mathbb{Z}_{540}$

$G(2)$ $G(3)$ $G(5)$

max orders 4 2 | 27 9 3 | 5

$G \cong G(2) \times G(3) \times G(5)$

2^5

$\mathbb{Z}_2 \times \mathbb{Z}_2$

$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$= 2 \cdot 2 \cdot 2 \cdot 4$

$2 \cdot 2 \cdot 8$

$2 \cdot 4 \cdot 4$

$4 \cdot 8$

$2 \cdot 16$

32

$\mathbb{Z}_4 \times \mathbb{Z}_{27} \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27} \times \mathbb{Z}_5$

$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$