MATH 322 **Quiz 3 Solutions** Wednesday, February 13

Name: ____________________

1. Find the minimal polynomial for $\frac{1+\sqrt{3}}{5}$ in $\mathbb{Q}[x]$.

> **Solution:** We have $x = \frac{1+\sqrt{3}}{5}$ so $5x - 1 = \sqrt{3}$ which gives $25x^2 - 10x + 1 = 3$. Thus a polynomial that has the number as a root is
>
> $$25x^2 - 10x - 2$$
>
> This polynomial must be irreducible since $\frac{1+\sqrt{3}}{5} \notin \mathbb{Q}$ (or by Eisenstien), which makes it the minimal polynomial.

2. Is $\frac{1+\sqrt{3}}{5} \in \mathbb{Q}(\sqrt{3})$? Briefly explain.

> **Solution:** Yes. Because $\mathbb{Q}(\sqrt{3})$ is a field, it is closed under field operations.

3. Is $\mathbb{Q}(\sqrt{3})$ the splitting field for the polynomial you found in question 1? Briefly explain.

> **Solution:** Yes, it must be. Certainly $\mathbb{Q}(\frac{1+\sqrt{3}}{5})$ is the splitting field, since the polynomial will factor completely in it. But this is the same field as $\mathbb{Q}(\sqrt{3})$.

4. Is there an automorphism of $\mathbb{Q}(\sqrt{3})$ which sends $\sqrt{3}$ to $\frac{1+\sqrt{3}}{5}$? Explain how you know using polynomials.

> **Solution:** No there is not, since these are not the roots of the same irreducible polynomial.

5. Find a non-identity element of the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q})$, and say what it does to $\frac{1+\sqrt{3}}{5}$.

> **Solution:** We must send $\sqrt{3}$ to another root of *its* minimal polynomial $x^2 - 3$. So $\sigma(\sqrt{3}) = -\sqrt{3}$. Then we know that $\sigma(\frac{1+\sqrt{3}}{5}) = \frac{1-\sqrt{3}}{5}$.
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> Note that this tells us that $\frac{1-\sqrt{3}}{5}$ is the other root of the polynomial from question 1.