Activity: Cayley Permutations

We have studied *permutation groups*, as well as groups that do not appear to be groups of permutations (such as $\mathbb{Z}_n$, $\mathbb{Z}_n \times \mathbb{Z}_m$, groups of functions or matrices, D_4, etc.). How distinct are these non-permutation groups from permutation groups?

1. Write the 4×4 group tables for $\mathbb{Z}_4$ and $U(8)$, the group of *units* of $\mathbb{Z}_8$ (numbers relatively prime to 8) under multiplication.

2. For each element in the groups above, we can see what adding or multiplying it by the other elements does to the other elements. For example, $5 \in U(8)$ corresponds to this function: $\lambda_5 = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 5 & 7 & 1 & 3 \end{pmatrix}$, since $5 \cdot 1 = 5$, $5 \cdot 3 = 7$, and so on.

For each element g in each group above, write down the corresponding function λ_g.

3. What happens when you compose two functions λ_g and λ_h for g and h in a group? Try this with a few examples you have above. What do you notice about the function you get?

4. Consider the set $\overline{G} = \{\lambda_g : g \in G\}$. Since each λ_g is a permutation of the elements of G, each of these will be a subset of S_n where $n = |G|$ (in our case, $n = 4$). Is $\overline{G}$ a subgroup of S_4 in both our cases? Will it be a subgroup in general?