Chapter 1

Fake chapter

1.1 Graphs of Functions

1.1.1 Reading Function Values from a Graph

The graph in Figure 1.1.1 shows the Dow-Jones Industrial Average (the average value of the stock prices of 500 major companies) during the stock market correction of October 1987. The Dow-Jones Industrial Average (DJIA) is given as a function of time during the 8 days from October 15 to October 22; that is, $f(t)$ is the DJIA recorded at noon on day t.

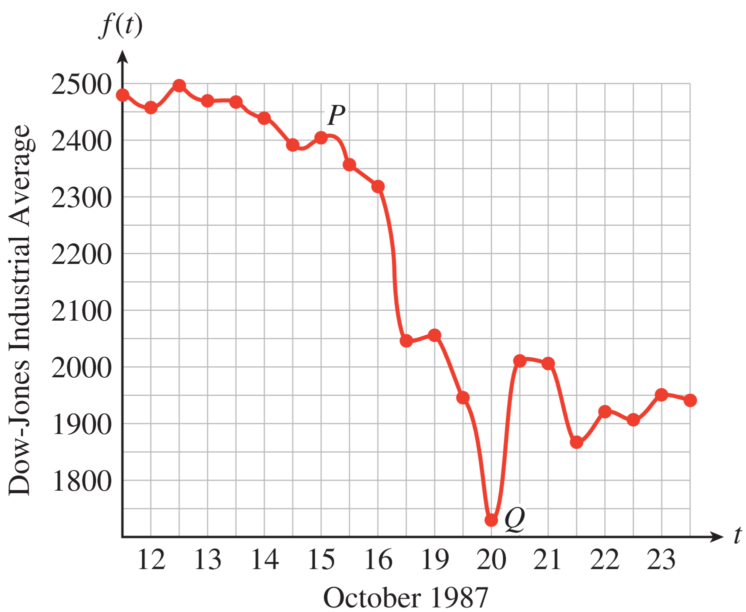

Figure 1.1.1

The values of the input variable, time, are displayed on the horizontal axis, and the values of the output variable, DJIA, are displayed on the vertical axis. There is no formula that gives the DJIA for a particular day; but it is still a function, defined by its graph. The value of $f(t)$ is specified by the vertical coordinate of the point with the given t-coordinate.

Example 1.1.2.

a The coordinates of point P in Figure 1.1.1 are $(15, 2412)$. What do the coordinates tell you about the function f?

b If the DJIA was 1726 at noon on October 20, what can you say about the graph of f?

Solution.

a The coordinates of point P tell us that $f(15) = 2412$, so the DJIA was 2412 at noon on October 15.

b We can say that $f(20) = 1726$, so the point $(20, 1726)$ lies on the graph of f. This point is labeled Q in Figure 1.1.1.

Thus, the coordinates of each point on the graph of the function represent a pair of corresponding values of the two variables. In general, we can make the following statement.

Graph of a Function The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

Example 1.1.3. Figure 1.1.4 shows the graph of a function g.

a Find $g(2)$ and $g(5)$.

b For what value(s) of t is $g(t) = 2$?

c What is the largest, or maximum, value of $g(t)$? For what value of t does the function take on its maximum value?

d On what intervals is g increasing?

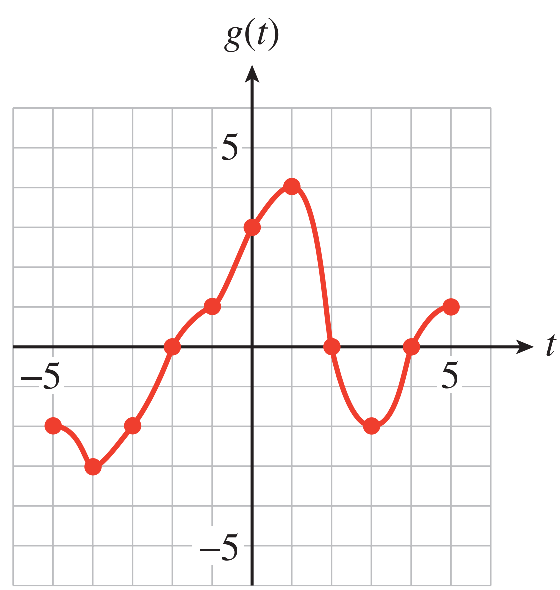

Figure 1.1.4

Solution.

a To find $g(2)$, we look for the point with t-coordinate 2. The point $(2, 0)$ lies on the graph of g, so $g(2) = 0$. Similarly, the point $(5, 1)$ lies on the graph, so $g(5) = 1$.

b We look for points on the graph with y-coordinate 2. Because the points $(5, 2)$, $(3, 2)$, and $(3, 2)$ lie on the graph, we know that $g(5) = 2$, $g(3) = 2$, and $g(3) = 2$. Thus, the t-values we want are 5, 3, and 3.

c The highest point on the graph is $(1, 4)$, so the largest y-value is 4. Thus, the maximum value of $g(t)$ is 4, and it occurs when $t = 1$.

d A graph is increasing if the y-values get larger as we read from left to right. The graph of g is increasing for t-values between 4 and 1, and between 3 and 5. Thus, g is increasing on the intervals $(4, 1)$ and $(3, 5)$.

Exercise 1.1.5. Refer to the graph of the function g shown in Figure 1.1.4 in Example 1.1.3.

a Find $g(0)$.

b For what value(s) of t is $g(t) = 0$?

c What is the smallest, or minimum, value of $g(t)$? For what value of t does the function take on its minimum value?

d On what intervals is g decreasing?

Remark 1.1.6 (**.** images/icon-GC.pdfFinding Coordinates with a Graphing Calculator] We can use the **TRACE** feature of the calculator to find the coordinates of points on a graph. For example, graph the equation $y = 2.6x5.4$ in the window

Xmin= -5	Xmax= 4.4
Ymin= -20	Ymax= 15

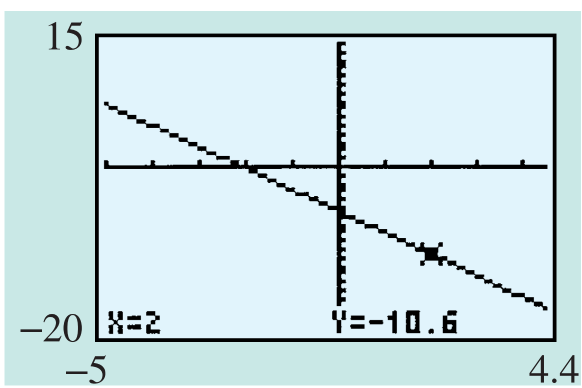

Figure 1.1.7

Press **TRACE**, and a bug begins flashing on the display. The coordinates of the bug appear at the bottom of the display, as shown in Figure 1.1.7. Use the left and right arrows to move the bug along the graph. You can check that the coordinates of the point $(2, 10.6)$ do satisfy the equation $y = 2.6x5.4$.

The points identified by the Trace bug depend on the window settings and on the type of calculator. If we want to find the y-coordinate for a particular x-value, we enter the x-coordinate of the desired point and press **ENTER**.

1.1.2 Constructing the Graph of a Function

Although some functions are defined by their graphs, we can also construct graphs for functions described by tables or equations. We make these graphs the same way we graph equations in two variables: by plotting points whose coordinates satisfy the equation.

Example 1.1.8. Graph the function $f(x) = \sqrt{x+4}$.

Solution. Choose several convenient values for x and evaluate the function to find the corresponding $f(x)$-values. For this function we cannot choose x-values less than 4, because the square root of a negative number is not a real number.

$$f(\mathbf{4}) = \sqrt{\mathbf{4}+4} = \sqrt{0} = 0$$

$$f(\mathbf{3}) = \sqrt{\mathbf{3}+4} = \sqrt{1} = 1$$

$$f(\mathbf{0}) = \sqrt{\mathbf{0}+4} = \sqrt{4} = 2$$

$$f(\mathbf{2}) = \sqrt{\mathbf{2}+4} = \sqrt{6} \approx 2.45$$

$$f(\mathbf{5}) = \sqrt{\mathbf{5}+4} = \sqrt{9} = 3$$

The results are shown in the table.

x	$f(x)$
-4	0
-3	1
0	2
2	$\sqrt{6}$
5	3

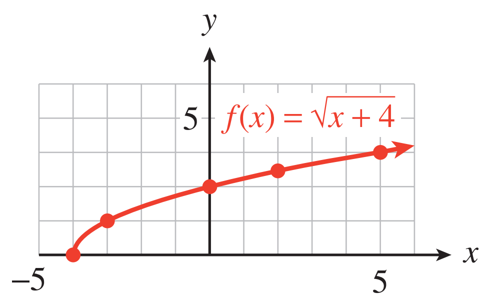

Figure 1.1.9

Remark 1.1.10 (. images/icon-GC.pdfUsing a Calculator to Graph a Function] We can also use a graphing calculator to obtain a table and graph for the function in Example 1.1.8. We graph a function just as we graphed an equation. For this function, we enter

$$Y_1 = \sqrt{\quad(X+4)}$$

and press **ZOOM** 6 for the standard window. (See ⟨⟨appendix-b⟩⟩ for details.) The calculators graph is shown in Figure 1.1.11.

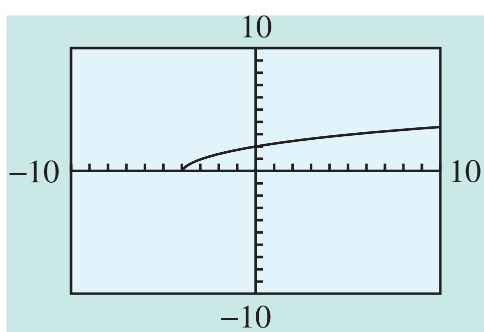

Figure 1.1.11

Exercise 1.1.12.

$$f(x) = x^3 2$$

a Complete the table of values and sketch a graph of the function.

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f(x)$							

b Use your calculator to make a table of values and graph the function.

1.1.3 The Vertical Line Test

In a function, two different outputs cannot be related to the same input. This restriction means that two different ordered pairs cannot have the same first coordinate. What does it mean for the graph of the function?

Consider the graph shown in Figure 1.1.13a. Every vertical line intersects the graph in at most one point, so there is only one point on the graph for each x-value. This graph represents a function. In Figure 1.1.13b, however, the line $x = 2$ intersects the graph at two points, $(2, 1)$ and $(2, 4)$. Two different y-values, 1 and 4, are related to the same x-value, 2. This graph cannot be the graph of a function.

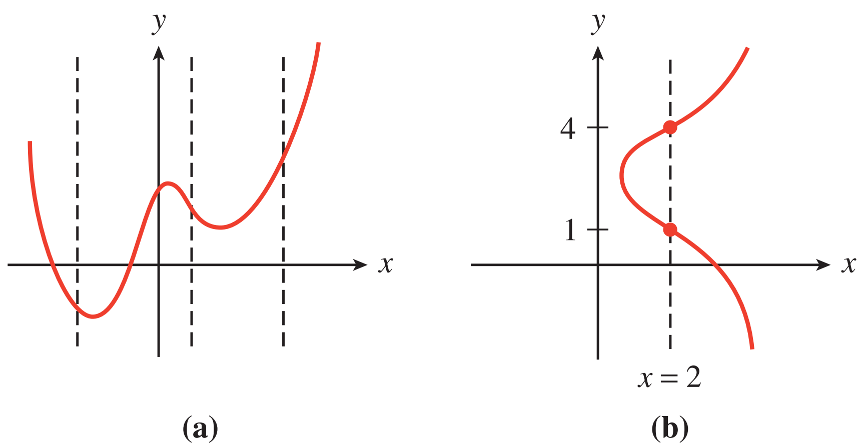

Figure 1.1.13

We summarize these observations as follows.

The Vertical Line Test A graph represents a function if and only if every vertical line intersects the graph in at most one point.

Example 1.1.14. Use the vertical line test to decide which of the graphs in Figure 1.1.15 represent functions.

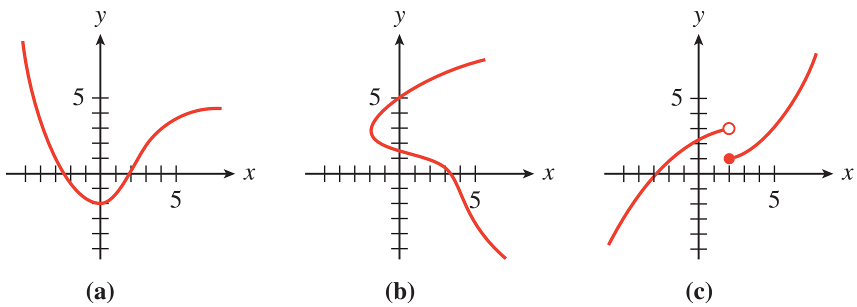

Figure 1.1.15

Solution. Graph (a) represents a function, because it passes the vertical line test. Graph (b) is not the graph of a function, because the vertical line at (for example) $x = 1$ intersects the graph at two points. For graph (c), notice the break in the curve at $x = 2$: The solid dot at $(2, 1)$ is the only point on the graph with $x = 2$; the open circle at $(2, 3)$ indicates that $(2, 3)$ is not a point on the graph. Thus, graph (c) is a function, with $f(2) = 1$.

Exercise 1.1.16. Use the vertical line test to determine which of the graphs in Figure 1.1.17 represent functions.

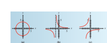

Figure 1.1.17

1.1.4 Graphical Solution of Equations and Inequalities

The graph of an equation in two variables is just a picture of its solutions. When we read the coordinates of a point on the graph, we are reading a pair of x- and y-values that make the equation true.

For example, the point $(2, 7)$ lies on the graph of $y = 2x+3$ shown in Figure 1.1.18 , so we know that the ordered pair $(2, 7)$ is a solution of the equation $y = 2x + 3$. You can verify algebraically that $x = \mathbf{2}$ and $y = \mathbf{7}$ satisfy the equation:

$$\text{Does } \mathbf{7} = 2(\mathbf{2}) + 3? \text{ Yes}$$

We can also say that $x = 2$ is a solution of the one-variable equation $2x + 3 = 7$. In fact, we can use the graph of $y = 2x + 3$ to solve the equation $2x + 3 = k$ for any value of k. Thus, we can use graphs to find solutions to equations in one variable.

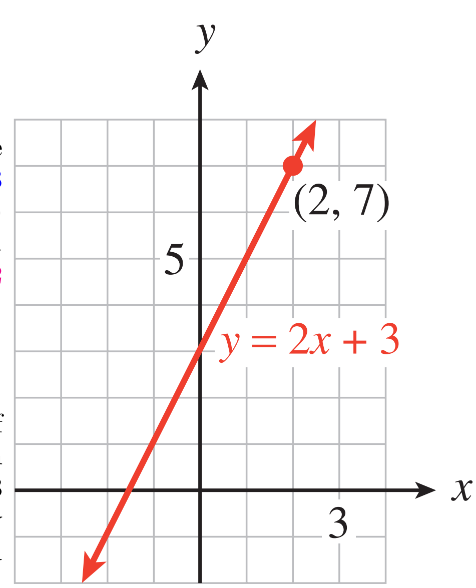

Figure 1.1.18

Example 1.1.19. Use the graph of $y = 28515x$ to solve the equation $150 = 28515x$.

Solution.

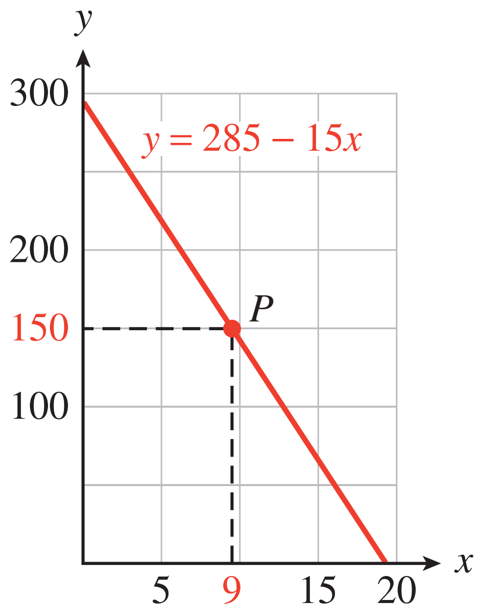

Figure 1.1.20

Begin by locating the point P on the graph for which $y = 150$, as shown in Figure 1.1.20. Now find the x-coordinate of point P by drawing an imaginary line from P straight down to the x-axis. The x-coordinate of P is $x = 9$. Thus, P is the point $(9, 150)$, and $x = 9$ when $y = 150$. The solution of the equation $150 = 28515x$ is $x = 9$. You can verify the solution algebraically by substituting $x = \mathbf{9}$ into the equation:

Does $150 = 28515(\mathbf{9})$?

$$28515(\mathbf{9}) = 285135 = 150. \quad \textbf{Yes}$$

The relationship between an equation and its graph is an important one. For the previous example, make sure you understand that the following three statements are equivalent:

1. The point $(9, 150)$ lies on the graph of $y = 28515x$.
2. The ordered pair $(9, 150)$ is a solution of the equation $y = 28515x$.
3. $x = 9$ is a solution of the equation $150 = 28515x$.

Exercise 1.1.21.

a Use the graph of $y = 308x$ shown in Figure 1.1.22 to solve the equation

$$308x = 50$$

b Verify your solution algebraically.

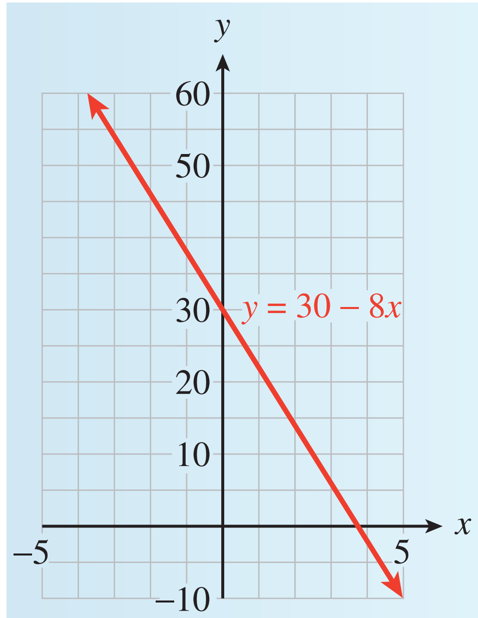

Figure 1.1.22

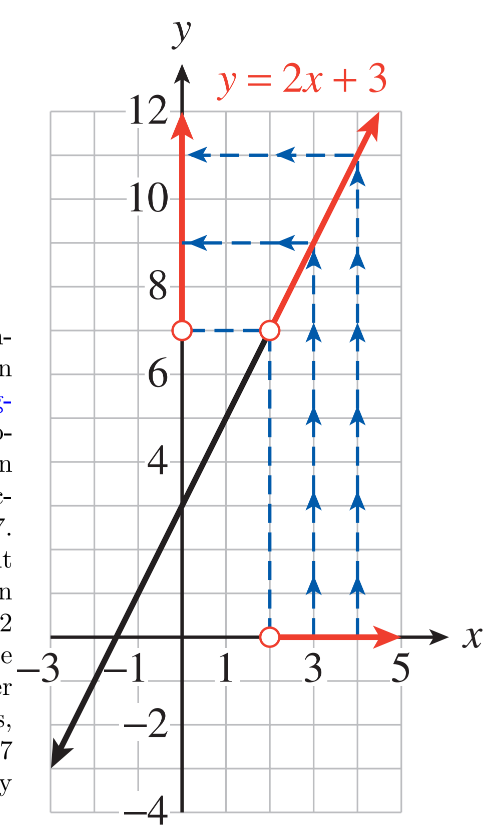

Figure 1.1.23

In a similar fashion, we can solve inequalities with a graph. Consider again the graph of $y = 2x + 3$, shown in Figure 1.1.23. We saw that $x = 2$ is the solution of the equation $2x + 3 = 7$. When we use $x = 2$ as the input for the function $f(x) = 2x + 3$, the output is $y = 7$. Which input values for x produce output values greater than 7? You can see in Figure 1.1.23 that x-values greater than 2 produce y-values greater than 7, because points on the graph with x-values greater than 2 have y-values greater than 7. Thus, the solutions of the inequality $2x + 3 > 7$ are $x > 2$. You can verify this result by solving the inequality algebraically.

Example 1.1.24. Use the graph of $y = 28515x$ to solve the inequality

$$28515x > 150$$

Solution. We begin by locating the point P on the graph for which $y = 150$ and $x = 9$ (its x-coordinate). Now, because $y = 28515x$ for points on the graph, the inequality $28515x > 150$ is equivalent to $y > 150$. So we are looking for points on the graph with y-coordinate greater than 150. These points are shown in Figure 1.1.25. The x-coordinates of these points are the x-values that satisfy the inequality. From the graph, we see that the solutions are $x < 9$.

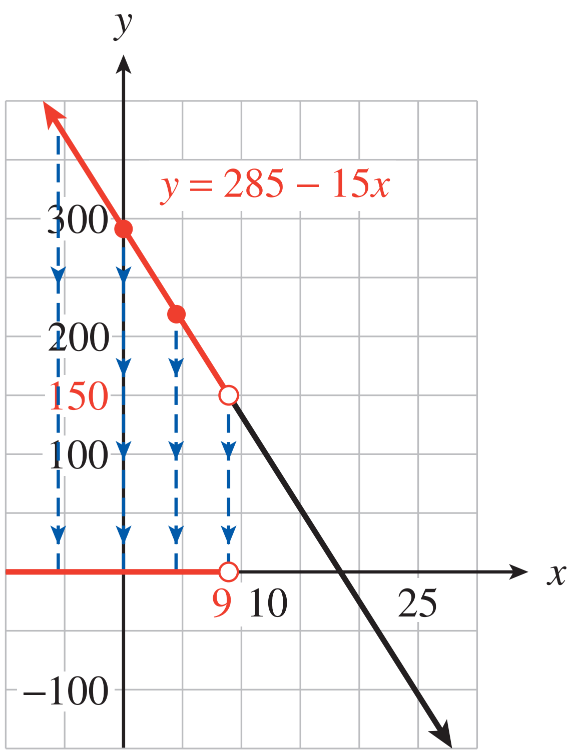

Figure 1.1.25

Exercise 1.1.26.

a Use the graph of $y = 308x$ in Figure 1.1.22 to solve the inequality

$$308x \leq 50$$

b Solve the inequality algebraically.

We can also use this graphical technique to solve nonlinear equations and inequalities.

Example 1.1.27. Use a graph of $f(x) = 2x^3 + x^2 + 16x$ to solve the equation

$$2x^3 + x^2 + 16x = 15$$

Solution. If we sketch in the horizontal line $y = 15$, we can see that there are three points on the graph of f that have y-coordinate 15, as shown in Figure 1.1.28. The x-coordinates of these points are the solutions of the equation

$$2x^3 + x^2 + 16x = 15$$

From the graph, we see that the solutions are $x = 3$, $x = 1$, and approximately $x = 2.5$. We can verify the solutions algebraically. For example, if $x = \mathbf{3}$, we have

$$f(\mathbf{3})\&2(\mathbf{3})3 + (\mathbf{3})^2 + 16(\mathbf{3})nn\& = 2(27) + 948nn\& = 54$$

so 3 is a solution.

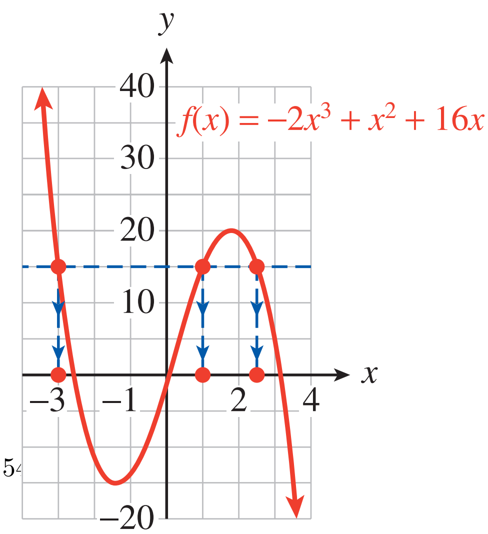

Figure 1.1.28

Exercise 1.1.29. Use the graph of $y = \frac{1}{2}n^2 + 2n10$ shown in Figure 1.1.30 to solve

$$\frac{1}{2}n^2 + 2n10 = 6$$

and verify your solutions algebraically.

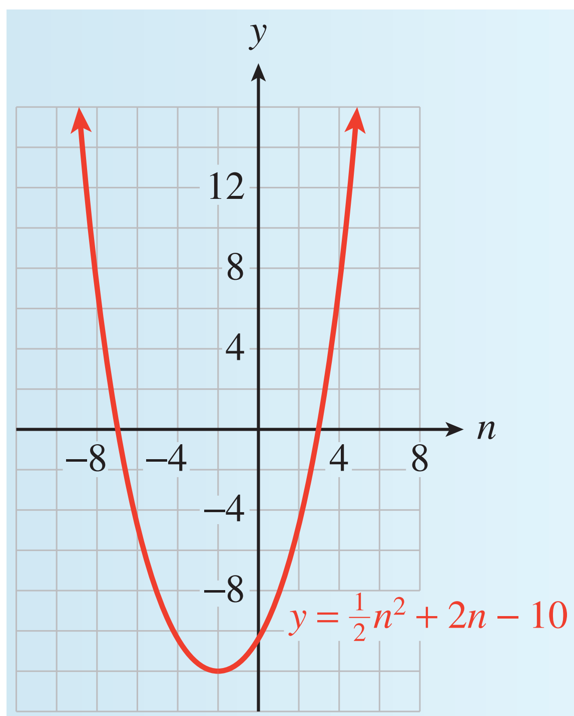

Figure 1.1.30

Remark 1.1.31 (. images/icon-GC.pdfUsing the Trace Feature] You can use the Trace feature on a graphing calculator to approximate solutions to equations. Graph the function $f(x)$ in Example 1.1.27 in the window

Xmin= -4	Xmax= 4
Ymin= -20	Ymax= 40

and trace along the curve to the point $(2.4680851, 15.512401)$. We are close to a solution, because the y-value is close to 15. Try entering x-values close to 2.4680851, for instance, $x = 2.4$ and $x = 2.5$, to find a better approximation for the solution.

We can use the intersect feature on a graphing calculator to obtain more accurate estimates for the solutions of equations. See ⟨⟨appendix-b⟩⟩ for details.

Example 1.1.32. Use the graph in Example 1.1.27 to solve the inequality

$$2x^3 + x^2 + 16x \geq 15$$

Solution. We first locate all points on the graph that have y-coordinates greater than or equal to 15. The x-coordinates of these points are the solutions of the inequality. Figure 1.1.33 shows the points, and their x-coordinates as intervals on the x-axis. The solutions are $x \leq 3$ and $1 \leq x \leq 2.5$, or in interval notation, $(-, 3][1, 2.5]$.

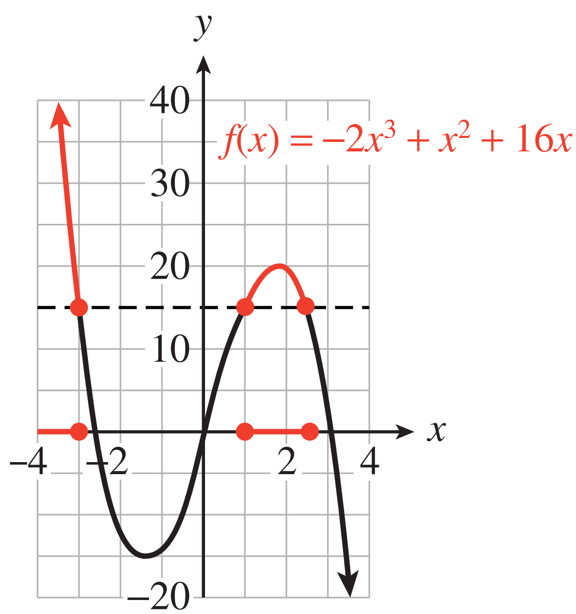

Figure 1.1.33

Exercise 1.1.34. Use Figure 1.1.30 in Exercise 1.1.29 to solve the inequality

$$\frac{1}{2}n^2 + 2n10 < 6$$

1.2 Domain and Range

1.2.1 Definitions of Domain and Range

In Example 1.1.8 of Section 1.1, we graphed the function $f(x) = \sqrt{x+4}$ and observed that $f(x)$ is undefined for x-values less than 4. For this function, we must choose x-values in the interval $[4, \infty)$.

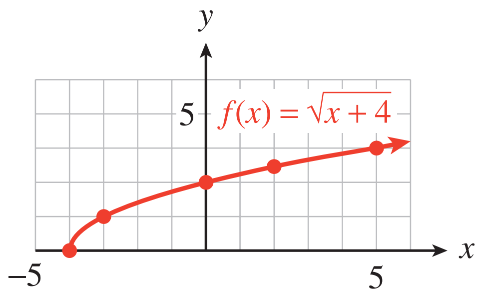

Figure 1.2.1

All the points on the graph have x-coordinates greater than or equal to 4, as shown in Figure 1.2.1. The set of all permissible values of the input variable is called the **domain** of the function f.

We also see that there are no points with negative $f(x)$-values on the graph of f: All the points have $f(x)$-values greater than or equal to zero. The set of all outputs or function values corresponding to the domain is called the **range** of the function. Thus, the domain of the function $f(x) = \sqrt{x+4}$ is the interval $[4, \infty)$, and its range is the interval $[0, \infty)$. In general, we make the following definitions.

Domain and Range The **domain** of a function is the set of permissible values for the input variable. The **range** is the set of function values (that is, values of the output variable) that correspond to the domain values.

Using the notions of domain and range, we restate the definition of a function as follows.

Definition of Function A relationship between two variables is a **function** if each element of the domain is paired with exactly one element of the range.

1.2.2 Finding Domain and Range from a Graph

We can identify the domain and range of a function from its graph. The domain is the set of x-values of all points on the graph, and the range is the set of y-values.