Chapter 1

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1.1 Nonlinear Models

In Chapter 1, we considered models described by linear functions. In this chapter, we begin our study of nonlinear models.

Solving Nonlinear Equations

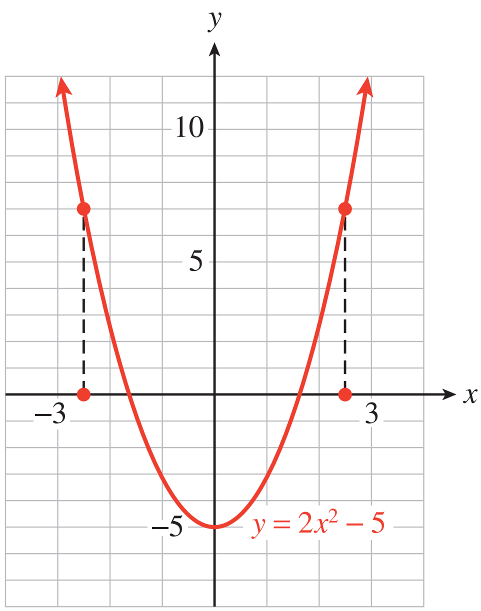

Figure 1.1.1

When studying nonlinear models, we will need to solve nonlinear equations. For example, in Investigation 3 we used a graph to solve the quadratic equation

$$18xx^2 = 80$$

Here is another example. Figure 1.1.1 shows a table and graph for the function $y = 2x^2 5$.

x	-3	-2	-1	0	1	2	3
y	13	3	-3	-5	-3	3	13

You can see that there are two points on the graph for each y-value greater than 5. For example, the two points with y-coordinate 7 are shown. To solve the equation

$$2x^2 5 = 7$$

we need only find the x-coordinates of these points. From the graph, the solutions appear to be about 2.5 and 2.5.

How can we solve this equation algebraically? The opposite operation for squaring a number is taking a square root. So we can undo the operation of squaring by extracting square roots. We first solve for x^2 to get

$$2x^2 = 12 \tag{1.1}$$
$$x^2 = 6 \tag{1.2}$$
$$\tag{1.3}$$

and then take square roots to find

$$x = \pm\sqrt{6}$$

Don't forget that every positive number has two square roots. The symbol $\pm$ (read plus or minus) is a shorthand notation used to indicate both square roots of 6. The exact solutions are thus $\sqrt{6}$ and $-\sqrt{6}$. We can also find decimal approximations for the solutions using a calculator. Rounded to two decimal places, the approximate solutions are 2.45 and 2.45.

In general, wecan solve equations of the form $ax^2 + c = 0$ by isolating x^2 on one side of the equation and then taking the square root of each side. This method for solving equations is called **extraction of roots**.

Extraction of Roots To solve the equation

$$ax^2 + c = 0$$

1. Isolate x^2.

2. Take square roots of both sides. There are two solutions.

Example 1.1.2. If a cat falls off a tree branch 20 feet above the ground, its height t seconds later is given by $h = 2016t^2$.

a What is the height of the cat 0.5 second later?

b How long does the cat have to get in position to land on its feet before it reaches the ground?

Solution.

a In this question, we are given the value of t and asked to find the corresponding value of h. To do this, we evaluate the formula for $t = 0.5$. We substitute **0.5** for t into the formula and simplify.

$$\begin{aligned} h &= 2016(\mathbf{0.5})^2 && \text{Compute the power.} \\ &= 2016(0.25) && \text{Multiply; then subtract.} \\ &= 204 = 16 \end{aligned}$$

The cat is 16 feet above the ground after 0.5 second.

b We would like to find the value of t when the height, h, is known. We substitute $h = \mathbf{0}$ into the equation to obtain

$$\mathbf{0} = 2016t^2$$

To solve this equation, we use extraction of roots. First isolate t^2 on one side of the equation.

$$16t^2 = 20 \qquad \text{Divide by 16.}$$
$$t^2 = \frac{20}{16} = 1.25$$

Now take the square root of both sides of the equation to find

$$t = \pm\sqrt{1.25} \approx \pm 1.118$$

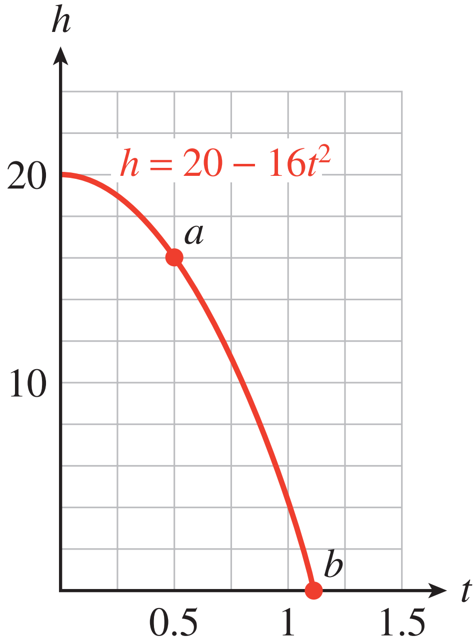

Figure 1.1.3

Only the positive solution makes sense here, so the cat has approximately 1.12 seconds to get into position for landing. A graph of the cats height after t seconds is shown in Figure 2.3. The points corresponding to parts (a) and (b) are labeled.

Note that in Example 1.1.2 we evaluated the expression $2016t^2$ to find a value for h, and in part (b) we solved the equation $0 = 2016t^2$ to find a value for t.

Exercise 1.1.4.

a

Solve by extracting roots $\frac{3x^2 8}{5} = 10$.

First, isolate x^2.Take the square root of both sides.

b Give exact answers; then give approximations rounded to two decimal places.

Solving Formulas

We can use extraction of roots to solve many formulas involving the square of the variable.

Example 1.1.5. The formula $V = \frac{1}{3}r^2h$ gives the volume of a cone in terms of its height and radius. Solve the formula for r in terms of V and h.

Solution. Because the variable we want is squared, we use extraction of roots. First, multiply both sides by 3 to clear the fraction.

$$\begin{aligned} 3V &= 3(\frac{1}{3}r^2h) \\ 3V &= r^2h && \text{Divide both sides by } h. \\ \frac{3V}{h} &= r^2 && \text{Take square roots.} \\ \pm\sqrt{\frac{3V}{h}} &= r \end{aligned}$$

Because the radius of a cone must be a positive number, we use only the positive square root: $r = \sqrt{\frac{3V}{h}}$.

Exercise 1.1.6. Find a formula for the radius of a circle in terms of its area.Start with the formula for the area of a circle: $A =$ ________

Solve for r in terms of A.

More Extraction of Roots

Equations of the form

$$a(px+q)^2 + r = 0$$

can also be solved by extraction of roots after isolating the squared expression, $(px+q)^2$.

Example 1.1.7. Solve the equation $3(x2)^2 = 48$.

Solution. First, isolate the perfect square, $(x2)^2$.

$$\begin{aligned} 3(x2)^2 &= 48 && \text{Divide both sides by 3.} \\ (x2)^2 &= 16 && \text{Take the square root of each side.} \\ x2 &= \pm\sqrt{16} = \pm 4 \end{aligned}$$

This gives us two equations for x,

$$\begin{aligned} x2 &= 4 \quad \text{or} \quad x2 = 4 \quad \text{Solve each equation.} \\ x &= 6 \quad \text{or} \quad x = 2 \end{aligned}$$

The solutions are 6 and 2.

Extraction of Roots To solve the equation

$$a(px+q)^2 + r = 0$$

1. Isolate the squared expression, $(px+q)^2$.
2. Take the square root of each side of the equation. Remember that a positive number has two square roots.
3. Solve each equation. There are two solutions.

Exercise 1.1.8. Solve $2(5x+3)^2 = 38$ by extracting roots.

a Give your answers as exact values.

b Find approximations for the solutions to two decimal places.

Compound Interest and Inflation

Many savings institutions offer accounts on which the interest is *compounded annually*. At the end of each year, the interest earned is added to the principal, and the interest for the next year is computed on this larger sum of money.

Compound Interest If interest is compounded annually for n years, the amount, A, of money in an account is given by

$$A = P(1 + r)^n$$

where P is the principal and r is the interest rate, expressed as a decimal fraction.

Example 1.1.9. Carmella invests \$3000 in an account that pays an interest rate, r, compounded annually.

a Write an expression for the amount of money in Carmellas account after two years.

b What interest rate would be necessary for Carmella's account to grow to \$3500 in two years?

Solution.

a Use the formula above with $P = 3000$ and $n = 2$. Carmellas account balance will be

$$A = 3000(1 + r)^2$$

b Substitute 3500 for A in the equation.

$$\mathbf{3500} = 3000(1 + r)^2$$

We can solve this equation in r by extraction of roots. First, isolate the perfect square.

$$3500 = 3000(1 + r)^2 \quad \text{Divide both sides by 3000.} \tag{1.4}$$
$$1.1\overline{6} = (1 + r)^2 \quad \text{Take the square root of both sides.} \tag{1.5}$$
$$\pm 1.0801 \approx 1 + r \quad \text{Subtract 1 from both sides.} r \approx 0.0801 \quad \text{or } r \approx 2.0801 \tag{1.6}$$

Because the interest rate must be a positive number, we discard the negative solution. Carmella needs an account with interest rate $r \approx 0.0801$, or just over 8%, to achieve an account balance of \$3500 in two years.

The formula for compound interest also applies to the effects of inflation. For instance, if there is a steady inflation rate of 4% per year, in two years an item that now costs \$100 will cost

$$A = P(1 + r)^2 \tag{1.7}$$
$$= 100(1 + 0.04)2 = \$108.16 \tag{1.8}$$

Exercise 1.1.10. Two years ago, the average cost of dinner and a movie was \$24. This year the average cost is \$25.44. What was the rate of inflation over the past two years?

Other Nonlinear Equations

Because squaring and taking square roots are opposite operations, we can solve the equation

$$\sqrt{x} = 8.2$$

by squaring both sides to get

$$(\sqrt{x})^2 = 8.2^2 \tag{1.9}$$
$$x = 67.24 \tag{1.10}$$

Similarly, we can solve

$$x^3 = 258$$

by taking the cube root of both sides, because cubing and taking cube roots are opposite operations. Rounding to three places, we find

$$\sqrt[3]{x^3} = 258 \tag{1.11}$$
$$x \approx 6.366 \tag{1.12}$$

The notion of undoing operations can help us solve a variety of simple nonlinear equations. The operation of taking a reciprocal is its own opposite, so we solve the equation

$$\frac{1}{x} = 50$$

by taking the reciprocal of both sides to get

$$x = \frac{1}{50} = 0.02$$

Example 1.1.11. Solve $\frac{3}{x2} = 4$.

Solution. We begin by taking the reciprocal of both sides of the equation to get

$$\frac{x2}{3} = \frac{1}{4}$$

We continue to undo the operations in reverse order. Multiply both sides by 3.

$$x2 = \frac{3}{4} \qquad \text{Add 2 to both sides.} \tag{1.13}$$
$$x = 2 + \frac{3}{4} = \frac{11}{4} \qquad \frac{2}{1} = \frac{8}{4}, \text{ so } \frac{2}{1} + \frac{3}{4} = \frac{8}{4} + \frac{3}{4} = \frac{11}{4} \tag{1.14}$$

The solution is $\frac{11}{4}$, or 2.75.

Exercise 1.1.12. Solve $2\sqrt{x+4} = 6$.

Remark 1.1.13 (**.** images/icon-GC.pdfUsing the Intersect Feature] We can use the *intersect* feature on a graphing calculator to solve equations.

Example 1.1.14. Use a graphing calculator to solve $\frac{3}{x2} = 4$.

Solution. We would like to find the points on the graph of $y = \frac{3}{x2}$ that have y-coordinate equal to 4. Graph the two functions

$$Y_1 = 3/(X2) \tag{1.15}$$
$$Y_2 = 4 \tag{1.16}$$

in the window

$$\text{Xmin} = 9.4 \qquad \text{Xmax} = 9.4 \tag{1.17}$$

$$\text{Ymin} = 10 \qquad \text{Ymax} = 10 \tag{1.18}$$

The point where the two graphs intersect locates the solution of the equation. If we trace along the graph of Y_1, the closest we can get to the intersection point is $(2.8, 3.75)$, as shown in Figure 1.1.15a. We get a better approximation using the *intersect* feature.

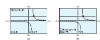

Figure 1.1.15

Use the arrow keys to position the Trace bug as close to the intersection point as you can. Then press **2nd** **TRACE** to see the Calculate menu. Press 5 for intersect; then respond to each of the calculator's questions, *First curve?*, *Second curve?*, and *Guess?* by pressing **ENTER**. The calculator will then display the intersection point, $x = 2.75$, $y = 4$, as shown in Figure 1.1.15b. The solution of the original equation is $x = 2.75$.

Exercise 1.1.16. Use the intersect feature to solve the equation $2x^2 5 = 7$. Round your answers to three decimal places.

1.2 Solving Quadratic Equations

Not every quadratic equation can be solved by factoring or by extraction of roots. For example, the expression $x^2 + x1$ cannot be factored, so the equation $x^2 + x1 = 0$ cannot be solved by factoring. For other equations, factoring may be difficult. In this section we learn two methods that can be used to solve any quadratic equation.

Squares of Binomials

In Section 1.1 we used extraction of roots to solve equations of the form

$$a(px + q)^2 + r = 0$$

where the left side of the equation includes the square of a binomial, or a **perfect square**. We can write any quadratic equation in this form by completing the square.

Consider the following squares of binomials.

Square of binomial $(x + p)^2$	p	$2p$	p^2
1. $(x + \mathbf{5})^2 = x^2 + 10x + 25$	$\mathbf{5}$	$2(\mathbf{5}) = 10$	$\mathbf{5}^2 = 25$
2. $(x - \mathbf{3})^2 = x^2 - 6x + 9$	$-\mathbf{3}$	$2(-\mathbf{3}) = -6$	$-\mathbf{3}^2 = 9$
3. $(x - \mathbf{12})^2 = x^2 - 24x + 144$	$-\mathbf{12}$	$2(-\mathbf{12}) = -24$	$-\mathbf{12}^2 = 144$

In each case, the square of the binomial is a quadratic trinomial,

$$(x+p)^2 = x^2 + 2px + p^2$$

The coefficient of the linear term, $2p$, is twice the constant in the binomial, and the constant term of the trinomial, p^2, is its square.

We would like to reverse the process and write a quadratic expression as the square of a binomial. For example, what constant term can we add to

$$x^2 16x$$

to produce a perfect square trinomial? Compare the expression to the formula above:

$$\begin{aligned} x^2 + 2px + p^2 &= (x+p)^2 \\ x^2 - 16x + \ \ ? &= (x+?)^2 \end{aligned}$$

We see that $2p = 16$, so $p = \frac{1}{2}(16) = 8$, and $p^2 = (8)^2 = 64$. Substitute these values for p^2 and p into the equation to find

$$x^2 16x + 64 = (x8)^2$$

Notice that in the resulting trinomial, the constant term is equal to *the square of one-half the coefficient of x*. In other words, we can find the constant term by taking one-half the coefficient of x and then squaring the result. Adding a constant term obtained in this way is called **completing the square**.

Example 1.2.1. Complete the square by adding an appropriate constant; write the result as the square of a binomial.

a $x^2 12x +$ ______

b $x^2 + 5x +$ ______

Solution.

a One-half of 12 is 6, so the constant term is $(6)^2$, or 36. Add 36 to obtain

$$x^2 12x + \mathbf{36} = (x6)^2 \qquad \underset{p^2=(6)^2=36}{p = \frac{1}{2}(12) = 6}$$

b One-half of 5 is $\frac{5}{2}$, so the constant term is $\left(\frac{5}{2}\right)^2$, or $\frac{25}{4}$. Add $\frac{25}{4}$ to obtain

$$x^2 + 5x + \mathbf{\frac{25}{4}} = \left(x + \frac{5}{2}\right)^2 \qquad \underset{p^2=(\frac{5}{2})^2=\frac{25}{4}}{p = \frac{1}{2}(5) = \frac{5}{2}}$$

You may find it helpful to visualize completing the square geometrically. We can think of the expression $x^2 + 2px$ as the area of a rectangle with dimensions x and $x + 2p$. For example, the rectangle with length $x + 10$ and width x has area $x(x+10) = x^2 + 10x$, as shown in Figure 1.2.2a. We would like to cut the rectangle into pieces and rearrange them so that we can make a square. In Figure 1.2.2b, we move half of the x-term so that each side of the square has length $x + 5$ (note that $p = \frac{1}{2}(10) = 5$), and in Figure 1.2.2c we see that the missing corner piece has area $p^2 = 5^2 = 25$.

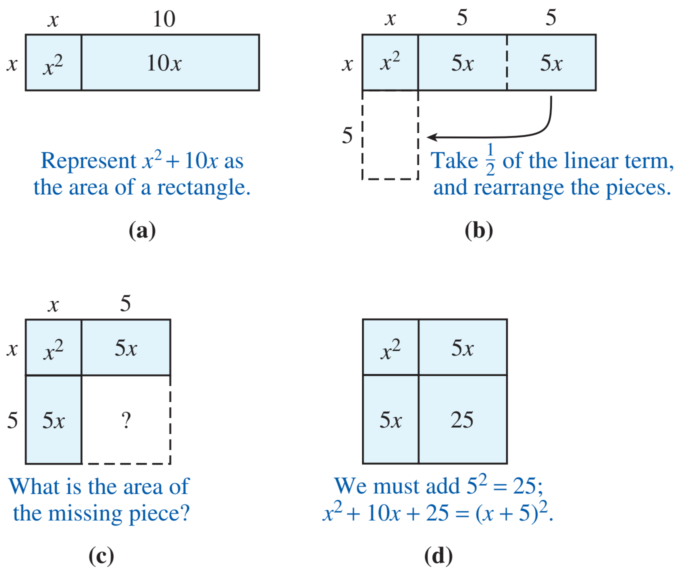

Figure 1.2.2

Exercise 1.2.3. Complete the square by adding an appropriate constant; write the result as the square of a binomial.

a $x^2 18x +$ ______ $= (x$______$)^2 p = \frac{1}{2}(-18) =$ ______, $p^2 =$ ______

b $x^2 + 9x +$ ______ $= (x$______$)^2 p = \frac{1}{2}(9) =$ ______, $p^2 =$ ______

Solving Quadratic Equations by Completing the Square

Now we will use completing the square to solve quadratic equations. First, we will solve equations in which the coefficient of the squared term is 1. Consider the equation

$$x^2 6x 7 = 0$$

1. Begin by moving the constant term to the other side of the equation, to get

$$x^2 6x \_\_\_\_\_\_ = 7$$

2. Now complete the square on the left. Because

$$p = \frac{1}{2}(6) = 3 \quad \text{and} \quad p^2 = (3)^2 = 9$$

we add 9 to *both* sides of our equation to get

$$x^2 6x \mathbf{+ 9} = 7 \mathbf{+ 9}$$

3. The left side of the equation is now the square of a binomial, namely $(x3)^2$. We write the left side in its square form and simplify the right side, which gives us

$$(x3)^2 = 16$$

(You can check that this equation is equivalent to the original one; if you expand the left side and collect like terms, you will return to the original equation.)

4. We can now use extraction of roots to find the solutions. Taking square roots of both sides, we get

$$\begin{aligned} x3 &= 4 &\quad \text{or} \quad& x3 = 4 &\quad& \text{Solve each equation.} \\ x &= 7 &\quad \text{or} \quad& x = 1 \end{aligned}$$

   The solutions are 7 and 1.

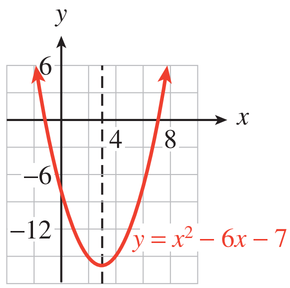

Figure 1.2.4

The graph of $y = x^2 6x7$ is shown in Figure 1.2.4. Note that the x-intercepts of the graph are $x = 7$ and $x = 1$, and the parabola is symmetric about the vertical line halfway between the intercepts, at $x = 3$.

We can also solve $x^2 6x7 = 0$ by factoring instead of completing the square. Of course, we get the same solutions by either method. In Example 2, we will solve an equation that cannot be solved by factoring.

Example 1.2.5. Solve $x^2 4x3 = 0$ by completing the square.

Solution.

Step 1: Write the equation with the constant term on the right side.

$$x^2 4x$$

$_{=3}$

Step 2: Now complete the square on the left side. The coefficient of x is 4, so

$$p = \frac{1}{2}(4) = 2 \quad \text{and} \quad p^2 = (2)^2 = 4$$

We add 4 to both sides of our equation:

$$x^2 4x + \mathbf{4} = 3 + \mathbf{4}$$

Step 3: Write the left side as the square of a binomial, and combine terms on the right side:

$$(x2)^2 = 7$$

Step 4: Finally, use extraction of roots to obtain

$$\begin{aligned} x2 &= \sqrt{7} &\quad \text{or} \quad& x2 = \sqrt{7} &\quad& \text{Solve each equation.} \\ x &= 2 + \sqrt{7} &\quad \text{or} \quad& x = 2\sqrt{7} \end{aligned}$$

The solutions are $2 + \sqrt{7} \approx 4.646$ and $2\sqrt{7} \approx 0.646$. The graph of $y = x^2 4x3$ is shown in Figure **??**.

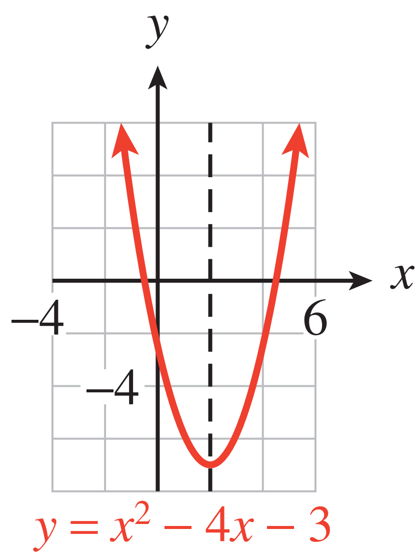

Figure 1.2.6

Exercise 1.2.7.

a Follow the steps to solve by completing the square: $x^2 1 = 3x$.

Step 1: Write the equation with the constant on the right.

Step 2: Complete the square on the left:

$$p = \frac{1}{2}(3) = \qquad p^2 =$$

Add p^2 to both sides.

Step 3: Write the left side as a perfect square; simplify the right side.

Step 4: Solve by extracting roots.

b Find approximations to two decimal places for the solutions.

c Graph the parabola $y = x^2 3x 1$ in the window

$$\text{Xmin} = 4.7 \qquad \text{Xmax} = 4.7 \tag{1.19}$$
$$\text{Ymin} = 5 \qquad \text{Ymax} = 5 \tag{1.20}$$

The General Case

Our method for completing the square works only if the coefficient of x^2 is 1. If we want to solve a quadratic equation whose lead coefficient is not 1, we first divide each term of the equation by the lead coefficient.

Example 1.2.8. Solve $2x^2 6x 5 = 0$.

Solution.

Step 1: Because the coefficient of x^2 is 2, we must divide each term of the equation by 2.

$$x^2 3x \frac{5}{2} = 0$$

Now we proceed as before. Rewrite the equation with the constant on the right side.

$$x^2 3x$$

$$=\frac{5}{2}$$

Step 2: Complete the square:

$$p=\frac{1}{2}(3)=\frac{-3}{2} \quad \text{and} \quad p^2=\left(\frac{-3}{2}\right)^2=\frac{9}{4}$$

Add $\frac{9}{4}$ to both sides of our equation:

$$x^2 3x+\mathbf{\frac{9}{4}}=\frac{5}{2}+\mathbf{\frac{9}{4}}$$

Step 3: Rewrite the left side as the square of a binomial and simplify the right side to get

$$\left(x\frac{3}{2}\right)^2=\frac{19}{4}$$

Step 4: Finally, extract roots and solve each equation for x.

$$x\frac{3}{2}=\sqrt{\frac{19}{4}} \quad \text{or} \quad x\frac{3}{2}=\sqrt{\frac{19}{4}}$$

The solutions are $\frac{3}{2}+\sqrt{\frac{19}{4}}$ and $\frac{3}{2}\sqrt{\frac{19}{4}}$. Using a calculator, we can find decimal approximations for the solutions: 3.679 and 0.679.

1. Because the coefficient of x^2 is 2, we must divide each term of the equation by 2.

$$x^2 3x\frac{5}{2}=0$$

Now we proceed as before. Rewrite the equation with the constant on the right side.

$$x^2 3x$$

$$=\frac{5}{2}$$

2. Complete the square:

$$p=\frac{1}{2}(3)=\frac{-3}{2} \quad \text{and} \quad p^2=\left(\frac{-3}{2}\right)^2=\frac{9}{4}$$

Add $\frac{9}{4}$ to both sides of our equation:

$$x^2 3x+\mathbf{\frac{9}{4}}=\frac{5}{2}+\mathbf{\frac{9}{4}}$$

3. Rewrite the left side as the square of a binomial and simplify the right side to get

$$\left(x\frac{3}{2}\right)^2=\frac{19}{4}$$

4. Finally, extract roots and solve each equation for x.

$$x\frac{3}{2}=\sqrt{\frac{19}{4}} \quad \text{or} \quad x\frac{3}{2}=\sqrt{\frac{19}{4}}$$

The solutions are $\frac{3}{2}+\sqrt{\frac{19}{4}}$ and $\frac{3}{2}\sqrt{\frac{19}{4}}$. Using a calculator, we can find decimal approximations for the solutions: 3.679 and 0.679.

Caution 1.2.9. In Example 1.2.5, it is essential that we first divide each term of the equation by 2, the coefficient of x^2. The following attempt at a solution is *incorrect.*

$$\begin{aligned} 2x^2 6x &= 5 \\ 2x^2 6x + 9 &= 5 + 9 \\ (2x3)^2 &= 14 \quad \rightarrow \quad \text{Incorrect!} \end{aligned}$$

You can check that $(2x3)^2$ is not equal to $2x^2 6x + 9$. We have not written the left side of the equation as a perfect square, so the solutions we obtain by extracting roots will not be correct.