Chapter 1

Fake chapter

1.1 Logarithms

In this section, we introduce a new mathematical tool called a *logarithm*, which will help us solve exponential equations.

Suppose that a colony of bacteria doubles in size every day. If the colony starts with 50 bacteria, how long will it be before there are 800 bacteria? We answered questions of this type in ⟨⟨Unresolved xref, reference "Exponential-Functions"; check spelling or use "provisional" attribute⟩⟩ by writing and solving an exponential equation. The function

$$P(t) = 50 \cdot 2^t$$

gives the number of bacteria present on day t, so we must solve the equation

$$800 = 50 \cdot 2^t$$

Dividing both sides by 50 yields

$$16 = 2^t$$

The solution of this equation is the answer to the following question: To what power must we raise 2 in order to get 16?

The value of t that solves the equation is called the base 2 **logarithm** of 16. Since $2^4 = 16$, the base 2 logarithm of 16 is 4. We write this as

$$\log_2 16 = 4$$

In other words, we solve an exponential equation by computing a logarithm. You can check that $t = \mathbf{4}$ solves the problem stated above:

$$P(\mathbf{4}) = 50 \cdot 2^{\mathbf{4}} = 800$$

Thus, the unknown exponent is called a logarithm. In general, for positive values of b and x, we make the following definition.

Definition 1.1.1 (Logarithm)**.** The **base** b **logarithm of** x, written $\log_b x$, is the exponent to which b must be raised in order to yield x.

It will help to keep in mind that a logarithm is just an exponent. Some logarithms, like some square roots, are easy to evaluate, while others require a calculator. We will start with the easy ones.

Example 1.1.2. Compute the logarithms.

a $\log_3 9$

b $\log_5 125$

c $\log_4 \frac{1}{16}$

d $\log_5 \sqrt{5}$

Solution.

a To evaluate $\log_3 9$, we ask what exponent on base 3 will produce 9. In symbols, we want to fill in the blank in the equation $3^{\underline{?}} = 9$. The exponent we need is 2, so

$$\log_3 9 = 2 \text{ because } 3^2 = 9$$

We use similar reasoning to compute the other logarithms.

b $\log_5 125 = 3$ because $5^3 = 125$

c $\log_4 \frac{1}{16} = 2$ because $4^2 = \frac{1}{16}$

d $\log_5 \sqrt{5} = \frac{1}{2}$ because $5^{1/2} = \sqrt{5}$

Exercise 1.1.3. Find each logarithm.

a $\log_3 81$

b $\log_{10} \frac{1}{1000}$

From the definition of a logarithm and the examples above, we see that the following two statements are equivalent.

Logarithms and Exponents: Conversion Equations If $b > 0$ and $x > 0$,

$$y = \log_b x \text{ if and only if } x = b^y$$

In other words, the logarithm y, is the same as the *exponent* in $x = b^y$. We see again that *a logarithm is an exponent*; it is the exponent to which b must be raised to yield x.

These equations allow us to convert from logarithmic to exponential form, or vice versa. You should memorize the conversion equations, because we will use them frequently.

As special cases of the equivalence in (1), we can compute the following useful logarithms. For any base $b > 0$,

Some Useful Logarithms

$$\log_b b = 1 \text{ because } b^1 = b$$
$$\log_b 1 = 0 \text{ because } b^0 = 1$$
$$\log_b bx = x \text{ because } b^x = b^x$$

Example 1.1.4.

a $\log_2 2 = 1$

b $\log_5 1 = 0$

c $\log_3 3^4 = 4$

Exercise 1.1.5. Find each logarithm.

a $\log_n 1$

b $\log_n n^3$

1.1.1 Using the Conversion Equations

We use logarithms to solve exponential equations, just as we use square roots to solve quadratic equations. Consider the two equations

$$x^2 = 25 \quad \text{and} \quad 2^x = 8$$

We solve the first equation by taking a square root, and we solve the second equation by computing a logarithm:

$$x = \pm\sqrt{25} = \pm 5 \quad \text{and} \quad x = \log_2 8 = 3$$

The operation of taking a base b logarithm is the inverse operation for raising the base b to a power, just as extracting square roots is the inverse of squaring a number.

Every exponential equation can be rewritten in logarithmic form by using the conversion equations. Thus,

$$3 = \log_2 8 \text{ and } 8 = 2^3$$

are equivalent statements, just as

$$5 = \sqrt{25} \text{ and } 25 = 5^2$$

are equivalent statements. Rewriting an equation in logarithmic form is a basic strategy for finding its solution.

Example 1.1.6. Rewrite each equation in logarithmic form.

a $2^1 = \frac{1}{2}$

b $a^{1/5} = 2.8$

c $6^{1.5} = T$

d $M^v = 3K$

Solution. First identify the base b, and then the exponent or logarithm y. Use the conversion equations to rewrite $b^y = x$ in the form $\log_b x = y$.

a The base is 2 and the exponent is 1. Thus, $\log_2 \frac{1}{2} = 1$.

b The base is a and the exponent is $\frac{1}{5}$. Thus, $\log_a 2.8 = \frac{1}{5}$.

c The base is 6 and the exponent is 1.5. Thus, $\log_6 T = 1.5$.

d The base is M and the exponent is v. Thus, $\log_M 3K = v$.

Exercise 1.1.7. Rewrite each equation in logarithmic form.

a $8^{1/3} = \frac{1}{2}$

b $5^x = 46$

1.1.2 Approximating Logarithms

Suppose we would like to solve the equation

$$2^x = 26$$

The solution of this equation is $x = \log_2 26$, but can we find a decimal approximation for this value? There is no integer power of 2 that equals 26, because

$$2^4 = 16$$
$$\text{and } 2^5 = 32$$

Thus, $\log_2 26$ must be between 4 and 5. We can use trial and error to find the value of $\log_2 26$ to the nearest tenth. Use your calculator to make a table of values for $y = 2^x$, starting with $x = 4$ and using increments of 0.1.

x	2^x
4	$2^4 = 16$
4.1	$2^{4.1} = 17.148$
4.2	$2^{4.2} = 18.379$
4.3	$2^{4.3} = 19.698$
4.4	$2^{4.4} = 21.112$

x	2^x
4.5	$2^{4.5} = 22.627$
4.6	$2^{4.6} = 24.251$
4.7	$2^{4.7} = 25.992$
4.8	$2^{4.8} = 27.858$
4.9	$2^{4.9} = 29.857$

Table 1.1.8

From Table **??**, we see that 26 is between 24.7 and 24.8, and is closer to 24.7. To the nearest tenth, $\log_2 26 \approx 4.7$.

Trial and error can be a time-consuming process. In Example 4, we illustrate a graphical method for estimating the value of a logarithm.

Example 1.1.9. Approximate $\log_3 7$ to the nearest hundredth.

Solution. If $\log_3 7 = x$, then $3^x = 7$. We will use the graph of $y = 3^x$ to approximate a solution to $3^x = 7$. Graph $Y_1 = 3$^ X and $Y_2 = 7$ in the standard window (`ZOOM` 6) to obtain the graph shown in Figure **??**. Activate the intersect feature to find that the two graphs intersect at the point $(1.7712437, 7)$. Because this point lies on the graph of $y = 3^x$, we know that

$$31.7712437 \approx 7, \text{ or } \log_3 7 \approx 1.7712437$$

To the nearest hundredth, $\log_3 7 \approx 1.77$.

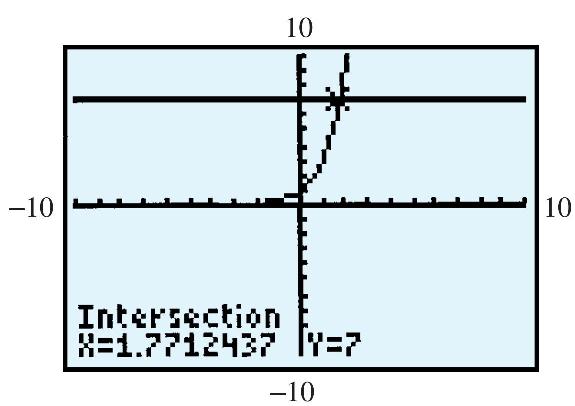

Figure 1.1.10

Exercise 1.1.11.

a Rewrite the equation $3^x = 90$ in logarithmic form.

b Use a graph to approximate the solution to the equation in part (a). Round your answer to three decimal places.

1.1.3 Base 10 Logarithms

Some logarithms are used so frequently in applications that their values are programmed into scientific and graphing calculators. These are the base 10 logarithms, such as

$$\log_{10} 1000 = 3 \quad \text{and} \quad \log_{10} 0.01 = 2$$

Base 10 logarithms are called **common logarithms**, and the subscript 10 is often omitted, so that $\log x$ is understood to mean $\log_{10} x$.

To evaluate a base 10 logarithm, we use the `LOG` key on a calculator. Many logarithms are irrational numbers, and the calculator gives as many digits as its display allows. We can then round off to the desired accuracy. This book was authored in MathBook XML.