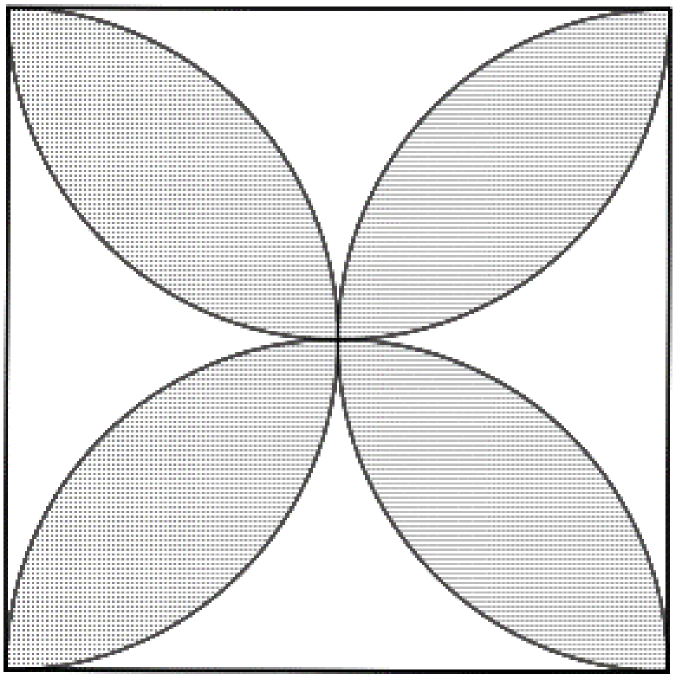
Problem Solving

Exploring Paths Less Traveled



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PREFACE

Goals and Objectives of This Workbook

The inclusion of nonroutine problems in the elementary and secondary school curriculum provides an opportunity for growth in mathematical maturing. Our approach here allows a student to enter into a situation where a certain level of difficulty is encountered, where a considerable number of tools are provided to attack that difficulty, and where a certain level of confidence is developed that persistence will be rewarded upon achieving a satisfactory resolution.

The NCTM PROCESS STANDARDS 2000 give clear evidence of the importance of modifying school curriculum to allow for a more inquiry-based, student-centered approach to understanding mathematics:

STANDARD 6: Problem Solving
STANDARD 7: Reasoning and Proof
STANDARD 8: Communication
STANDARD 9: Connections
STANDARD 10: Representation

It is not sufficient to simply do problems that illustrate the interfacing of these five standards. What is most important, but also most difficult to achieve, is the final step in the Polya process, the step we call TIE-TOGETHER, the inquiry about inquiry step. Inquiry-oriented teaching requires that a student engaged in exercising process skills reflect upon what has happened in a problem-solving episode and asks, "how did it happen, why did it happen, and will my technique apply in other situations?" The worksheets in the workbook address this questioning process.

Organization of the Workbook

Fifteen problem-solving heuristics are illustrated with completely written out solutions, showing typical thought processes engaged in, followed by worksheets with brief hints given for another fifteen problems. These are followed by thirty problems, two per strategy. Finally, a large collection of a variety of problems is included separated into elementary and advanced classifications.

Who Is the Book Aimed At?

A wide audience can benefit. The workbook can be used for:
A pre-service teacher course in problem solving.
An in-service teacher's workshop.
A supplement to many mathematics courses in grades 6 – 12.
A motivated student needing an unguided challenge.
Your Math Counts participants – just give it to them, they will run with it.

The Problems

The problems in this workbook range in difficulty from those that would present a mild challenge to good fifth graders, to those that would seriously challenge a talented mathematics major.

Acknowledgement

Many of the problems presented are modifications of oft used problems, or versions of problems from Math Counts, The Mathematics Magazine, The Mathematics Teacher, The College Journal of Mathematics, or various national and statewide mathematics contests. Some are new; some are special cases of problems from the Putnam Intercollegiate Mathematics Competition. The value of this book derives from how you use it in your classroom setting; or how it is used in workshops for in-service teachers, both elementary and secondary.

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TABLE OF CONTENTS

INTRODUCTION

THE ANATOMY AND DISSECTION OF A FEW GOOD PROBLEMS

Many posed problems are centered on a core mathematical structure or theme that we will call the essence of the problem. A problem's essence provides the key to solving the problem and is the essential idea that the problem was designed to teach or to reinforce during the problem-solving episode. Pólya said, "an empty mind cannot solve problems," meaning basic skills and content knowledge within the field that you are solving the problem is necessary to successfully solve a problem. Creativity rarely emerges from an unprepared mind. Thus, an awareness of the various ideas that serve as the essence of a problem can guide your efforts. The following is a partial listing of basic mathematical themes. As you progress through the problems in this text, you may identify other themes you will want to add to the list.

Number themes:
Arithmetic growth
Differencing
Gauss forward and backward sum
Geometric growth
Geometric ratios
Shift and subtract technique
Greatest common divisors
Least common multiple
Special sequences of numbers
Odds and Evens
Squares
Triangular numbers
Prime and composite numbers
Parity

Algebraic themes:
Factoring
Factor theorem
Remainder theorem
Rational root theorem
Add-in and subtract-out
Telescoping or collapsing sums/products
Averages

Geometric themes:
Symmetry
Properties of diagonals in polygons
Pythagorean theorem
Congruent triangles

Counting themes:
Binomial coefficients
Permutations
Compositions
Principle of inclusion-exclusion
Pigeonhole principle
Mutually exclusive and exhaustive partitions of sets

Problem solving needn't be synonymous with real-life application problems. Indeed the majority of real-life problems are far too intractable for the average high school student to tackle. However, problems designed around important mathematical themes that also have applications in real-life problems can provide students with adequate background and practice with essential mathematical skills within a "safe" environment. Merely containing a mathematical theme, however, does not imply a problem is a good problem. According to a sample of secondary students and teachers good problems have the following characteristics:

- The problem is interesting and challenging
- The problem lends itself to a variety of solutions
- The problem can be solved using multiple strategies
- The problem is open ended in that it affords an opportunity for extension
- A surprise occurs somewhere in the solution
- A discovery is made during its solution which leads to an "aha" or "aaah"
- The solution of the problem leads to a generalization
- The solution of the problem involves the understanding of distinct mathematical concepts or the use of mathematical skills
- The solution allows for various representations, i.e. algebraic, geometric, graphical, data chart, spreadsheet, etc.
- The solution provides connections within mathematics or to an application outside of mathematics

This list is not intended to be exhaustive by any means. As you solve more and more problems, you may find other characteristics that make a problem "good" for you and/or your students. It is also not reasonable to expect that a problem would possess all of the above characteristics. Problems are as individual as the people who solve them. A problem that is "good" for one solver may be the bane of another's existence.

Many students lament that they have never been good at story problems. Problem solving is not the same as the story problems you ran into in high school algebra. Story problems are exercises that practice a particular skill within the context of a real-life situation, such as using the formula $D = rt$ to find the average speed if you travel 152 miles in 3 hours. In our case, problem solving involves non-routine problems for which there is no apparent and direct solution. This means that problems that may be non-routine to you today, after some practice, will no longer be true problem solving episodes for you. With practice, you will become more adept at solving problems and the difficulty of the problems presented will need to escalate to continue to be challenging at a high level.

Fortunately, regardless of your expertise at problem solving there are heuristics (strategies) that can assist you in organizing your thoughts. George Pólya identified four basic steps needed to approach any problem – understand the problem, devise a plan, carry out the plan, and look back. The one presented and used in this text is an adaptation of the Pólya 4-step process. This plan uses the acronym US v. IT to help you remember the four steps. The acronym stands for:

Understand →Strategize→ Implement→ Tie-Together

The Understand phase of the problem-solving episode is essential to the success of the process. During this stage you need to ask yourself a variety of questions.

- What is the problem asking?
- What kind of answer should I get?
- If I phrased the answer in a sentence what would it say?
- What information is given to me? Is there any information missing? Is there any extra information contained in the problem that I don't need?
- Can I restate the problem in my own words?
- Is there a particular skill that I might need to solve this problem?
- Should I refresh in my mind definitions of vocabulary words I need before continuing?
- What assumptions might I make about this problem that are different from those that the author of the problem might have made?

The thoroughness of your questioning your understanding of the problem can determine your success at solving the problem. It is possible that during subsequent stages in the problem-solving episode, you may find that your understanding was insufficient or inaccurate. You may need to return to this stage numerous times before you successfully solve the problem.

The Strategize phase is the most frequently undervalued phase for novice problem solvers and is the phase in which expert problem solvers spend the most time. During this stage you should list out the different types of heuristics you think could be used to solve the problem. Once you have created a list of potential heuristics, you can return to this list and decide which of the methods, if any, would be the best to use. “Best” will be relative. For one person, heuristics involving data collection may always be ranked higher than heuristics involving drawing a picture. What is important is that a list is developed and then refined with some commentary as to benefits or detriments associated with each of the potential heuristics. This list may prove helpful if a selected heuristic proves ineffective. During this phase, you are creating a plan of attack. You are not actually carrying out the plan. You may create an outline for your plan, or a list of tasks to do, or a decision tree. This is the stage where you organize your problem solving efforts and allocate your resources.

The Implement phase is the phase of the most obvious action to the onlooker. Much of the initial groundwork has been laid and now you are beginning to collect data, draw diagrams, act out a situation, look for a pattern in data, etc. Keeping a record of your efforts becomes crucial. In part, this record is used to document your success. It is also used to document failures. If a particular path is not fruitful, you will want to remember how you ended up on that train of thought and how to avoid it in your future attempts at solving this problem. Your records should be organized so you know exactly how and why you collected your data. During this phase you will also want to regularly refer to the plan you developed in the Strategize phase and to the base of knowledge you accumulated in the Understand phase. This way you know you are not off your intended course and that critical misunderstandings have not crept into your problem-solving episode.

During the Tie-together phase, you will present your solution to the problem. At this point you should state your solution to the problem in a complete sentence and verify that you have indeed answered the question stated in the problem. You will want to look back over your calculations and check for any logical errors. Many students will stop once they have answered the questions; however, the answer is not the solution or, more properly, the resolution of the problem. In the final stage of the process, you need to search for the essence of the problem and determine if you developed or utilized any tools that were particularly helpful. When searching for a problem’s essence, you will need to remain open to many possibilities. A single problem may contain multiple themes or perhaps contain no apparent core. You should look for ways that the problem could be extended as a means for creating new problems to be solved. These extensions can either use the basic statement of the original problem and make it somewhat more difficult or could create different problems that use the same tools. Be diligent during the tie-together phase. A problem hasn’t been completely solved until you can extract its essence (if one exists) and create a new problem that is distinctly different from the original problem but incorporates the same theme.

In the next few pages we will present several strategies that can also assist you in your problem solving efforts. Each strategy will be demonstrated with one problem. A second problem using the same strategy will be stated and a worksheet will guide you through the resolution of the problem. For those problems that are completely worked out, comments related to the heuristic or to the use of the Pólya 4-step appear in the left column. A list of the heuristics that will be demonstrated appear below. Good luck and happy hunting. ☺

- Act it out
- Guess and check
- Brute force
- Draw a picture, make a model
- Exploit symmetry
- Try small values for n
- Collect data, search for a pattern
- Establish a subgoal
- Use technology
- Resolve a similar but simpler problem
- Pursue parity
- Convert words into mathematical notation
- What happens if…?
- Place in a more general setting
- Argue by contradiction or contrapositive

MATHEMATICAL THEMES

George Pólya: "An empty mind cannot solve problems."

There are a number of mathematical themes that appear in the problems in this book. Some of these themes you may be familiar with and some you may not.

1. Arithmetic Growth

   An arithmetic progression is a sequence of the form *a*, *a + d*, *a + 2d*, *a + 3d*… where *a* is a beginning term, and d is the common difference.

   To sum terms in an arithmetic sequence like 1, 3, 5, 7, … reverse and add:

$$\begin{aligned} S &= 1 + 3 + 5 + \cdots + (2n - 1) \\ &\underline{S = (2n - 1) + \cdots + 3 + 1} \\ 2S &= 2n + 2n + \cdots + 2n \end{aligned}$$

Write sum down
Write in reverse order
Add and solve for S

$$S = \frac{(2n)n}{2} = n^2$$

2. Geometric Growth

   To sum a geometric series, multiply by the ratio, shift and subtract.

$$S = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n}$$

Write sum down

$$\frac{1}{3}S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$$

Multiply by common ratio, shifting all the terms

$$\frac{2}{3}S = 1 - \frac{1}{3^{n+1}}$$

Subtract and solve for S

$$S = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right)$$

3. Greatest Common Divisor

The GCD of two integers is the largest integer that divides each.
GCD (12, 8) = 4   GCD (7, 13) = 1   GCD (6^7, 8^5) = 2^7

4. Least Common Multiple

The LCM of two integers is the smallest integer that each will divide into.
LCM (4, 6) = 12   LCM (5, 7) = 35   LCM (12^4, 18^5) = $2^8 \cdot 3^{10}$

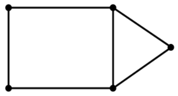
If m and n are positive integers mn = LCM (m, n) · GCD (m, n).

5. Special sequences of numbers

ODD NUMBERS: 1, 3, 5, 7, …
EVENS: 0, 2, 4, 6, 8, …
SQUARES: 0, 1, 4, 9, 16, …
TRIANGULAR: 1, 3, 6, 10, 15, …
PRIMES: 2, 3, 5, 7, 11, 13, …
COMPOSITES: 4, 6, 8, 9, 10, 12, …
CUBES: 1, 8, 27, 64, 125, 216, …
FIBONACCI: 1, 1, 2, 3, 5, 8, …
POWERS OF TWO: 1, 2, 4, 8, 16, …
FACTORIALS: 1, 1, 2, 6, 24, 120, …
POWERS OF THREE: 1, 3, 9, 27, 81, …
PERFECT NUMBERS: 6, 28, 496, 8128,…

6. Parity

This concept refers to evenness or oddness of an enumeration. The binary strings 01101 and 11001 both have the same odd weight. The graph has 3 vertices of even degree, and 2 of odd degree.



7. Factoring

$x^n - y^n = (x - y)\,(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$

If *a*, *b*, *c* are the three roots of $x^3 + px^2 + qx + r = 0$ then
$p = -(a + b + c)$ $q = ab + ac + bc$ $r = -abc$

8. Factor theorem

If r is a root of p(x) = 0 then x - r is a factor of p(x); i.e., $p(x) = (x - r)q(x)$ for some (quotient) polynomial $q(x)$.

9. Remainder Theorem

The remainder upon dividing $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ by $x - r$ is $p(r)$.

10. Rational Root Theorem

If the rational root $\frac{p}{q}$ is a root of $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ where $a_0 \neq 0$ and $a_k \in \mathbb{Z}$, then $\mathrm{p} \mid \mathrm{a_n}$ and $\mathrm{q} \mid \mathrm{a_0}$. This theorem can be used to show that $\sqrt{2}$ is irrational. Since $\sqrt{2}$ is a root of $x^2 - 2 = 0$ and the only possible rational roots are ± 1, ± 2, then $\sqrt{2}$ must be irrational.

11. Add-in and subtract-out

To factor $x^4 + 4$ add $4x^2$ in and subtract $4x^2$ back out:
$x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2 = (x^2 + 2x + 2)(x^2 - 2x + 2)$

To rewrite a fraction: $\frac{2a}{a-4} = \frac{2a-8+8}{a-4} = 2 + \frac{8}{a-4}$

To evaluate $\frac{x}{x+1}dx$, write it as $\frac{x+1-1}{x+1}dx$.

To find the center of the circle $x^2 - 4x + y^2 + 6y - 3 = 0$, complete the square by adding-in and subtracting-out.

12. Telescoping or collapsing sums/products

The equation $(n+1)! - n! = n \cdot n!$ converts products into differences. With this, the sum $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n!$ can be rewritten as the sum of differences that collapses $(2! - 1!) + (3! - 2!) + (4! - 3!) + \cdots + (n+1)! - n! = (n+1)! - 1$ after telescoping.

Similarly, $\ln\frac{x}{y} = \ln x - \ln y$, the Fibonacci numbers $F_{n+1} = F_n - F_{n-1}$ and (using partial fraction decomposition) $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ will introduce a subtraction allowing for a collapsing sum.

13. Averages

The average of the n positive integers $a_1, a_2, a_3, \ldots, a_n$ is $\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$.

The Arithmetic-Geometric mean inequality states that

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n}$$

Using this inequality, an easy non-calculus proof that $x + \frac{1}{x} \geq 2$ if $x \geq 0$ follows: just let $a_1 = x$, $a_2 = \frac{1}{x}$ and n=2.

14. Symmetry

Many formulas in algebra and ideas in geometry hold within themselves a notion of symmetry. Symbols can be permuted in $a^2 + b^2 + c^2 = ab + bc + ca$, $x^2 + y^2 = 1$, and Area $=\frac{1}{2}ab$. Geometric symmetry allows different views of the same concept. Three consecutive terms in an arithmetic sequence could be written as $a - d$, a, $a + d$ rather than a, $a + d$, $a + 2d$.

15. Properties of diagonals in polygons

Connectivity, triangulation, preservation of area. Either diagonal separates a parallelogram into congruent triangles.

16. Pythagorean Theorem

In a right triangle the square of the hypotenuse equals the sum of the squares of the two legs: $c^2 = a^2 + b^2$. The converse is also true.

17. Congruent triangles

Two triangles are congruent if they can satisfy either of the following three correspondence conditions: SSS, SAS or ASA. Consequences: The areas are the same, corresponding parts are congruent.

18. Binomial coefficients

The binomial coefficient $\binom{n}{k}$ gives the number of ways of selecting *k* items out of a set of *n* items. They form the rows of the Pascal triangle, appearing as the coefficients in the expansion of $(x+1)^n$. A recursion for generating $\binom{n}{k}$ is $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. A closed formula is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. The value $\binom{n}{k}$ counts selections.

19. Permutations

A permutation of the distinct objects *a, b, c, d* is any arrangement of these objects. There are $n!$ permutations of $a_1, a_2, \dots, a_n$.

20. Compositions

A composition of a positive integer *n* is an ordered *k*-tuple $(a_1, a_2, \dots, a_k)$ such that $a_1, +a_2 + \cdots + a_k = n$.

21. Principle of Inclusion – Exclusion

This counting procedure says that $\#(A \cup B) \ = \ \#A + \#B - \#(A \cap B)$ where *A* and *B* are sets of objects. Include everything in *A* and *B* but subtract your duplications in $A \cap B$. For three sets we have:

$$\#(A \cup B \cup C) = \#A + \#B + \#C - \#(A \cap B) - \#(A \cap C) - \#(B \cap C) + \#(A \cap B \cap C)$$

22. Pigeonhole principle

One version is: Let *m* objects be distributed into *n* containers. If $m > n$, then some container will end up with at least two objects in it. Example: In any set of 22 people there must be 4 or more people who were born on the same day of the week.

23. Set Partitions

A partition of a set *S* is a set of sets *A, B, C,* … such that they are pairwise disjoint and their union is all of *S*. The sets of all even integers and all odd integers is a partition of all integers.

SAMPLE SOLUTIONS FOR VARIOUS GRADE LEVELS

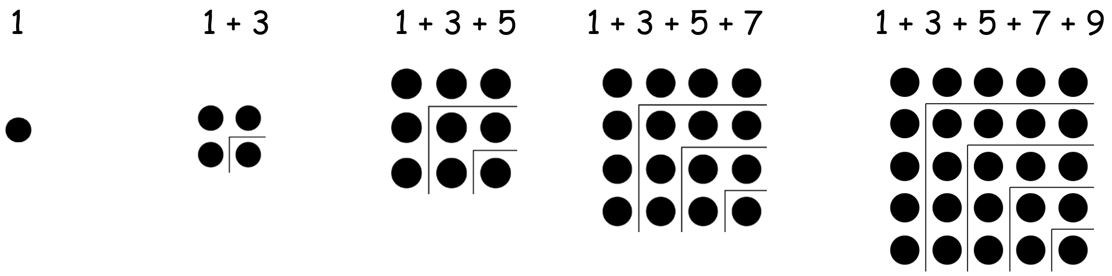
OBJECTIVE: Try to analyze how you would present each problem to students at various levels, say 4th, 9th, and 12th grade.

SAMPLE PROBLEM: Conjecture a formula for $1 + 3 + 5 + \dots + (2n - 1)$

<u>4th Grade Presentation</u>: What do you observe about the following pattern of number sums?

$$\begin{array}{r}
1 = 1 \\
1 + 3 = 4 \\
1 + 3 + 5 = 9 \\
1 + 3 + 5 + 7 = 16 \\
1 + 3 + 5 + 7 + 9 = 25
\end{array}$$

POSSIBLE SOLUTION: Noticing that 1 = 1 times 1, 4 = 2 times 2, 9 = 3 times 3, etc., we could express the left hand sums as arrangements of dots:



At each stage a square is formed.

9^{th} Grade Presentation: Is there a simple way of expressing sums of consecutive odd integers? In other words, compute 1 + 3, 1 + 3 + 5, 1 + 3 + 5 + 7, etc., and guess at a pattern. Can you verify your pattern?

SOLUTION: Use the Gauss technique of writing each sum forward and backward and adding.

$$\begin{array}{ll|l} \text{If} & S = 1 + 3 + 5 + 7 & \\ \text{and} & \underline{S = 7 + 5 + 3 + 1} & S = \frac{4 \cdot 8}{2} = 42 = 16 \\ \text{then} & 2S = 8 + 8 + 8 + 8 & \end{array}$$

Try this with S = 1 + 3 + 5 + 7 + 9. Why does this work? Will this technique work on the sum 3 + 7 + 11 + 15? Generalize.

12^{th} Grade Presentation: Conjecture and prove a closed formula for the sum 1 + 3 + 5 + … + (2n – 1).

SOLUTION 1:

$$\begin{array}{llcccccccccc} \text{Let} & S = & 1 & + & 3 & + & \dots & + & (2n - 3) & + & (2n - 1) \\ \text{and} & \underline{S =} & \underline{(2n - 1)} & \underline{+} & \underline{(2n - 3)} & \underline{+} & \underline{\dots} & \underline{+} & \underline{3} & \underline{+} & \underline{1} \\ & 2S = & 2n & + & 2n & + & \dots & + & 2n & + & 2n \end{array}$$

So, $S = \frac{1}{2}n(2n) = n^2$.

SOLUTION 2: Tabulating data, you can guess that the sum is n^2. This can be proven by mathematical induction.

SOLUTION 3: Use the result that $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.

$$\begin{aligned} 1 + 3 + 5 + \dots + 2n - 1 &= 1 + 2 + 3 + \dots + 2n - (2 + 4 + \dots + 2n) \\ &= \frac{2n(2n+1)}{2} - \frac{2(n)(n+1)}{2} \\ &= n\,(2n + 1) - n\,(n + 1) \\ &= n\,(2n + 1 - n - 1) \\ &= n^2. \end{aligned}$$

The reverse and add Gauss technique in Solution 1 works on any arithmetic sum, i.e., on sums of terms coming from an arithmetic sequence. The ADD IN – SUBTRACT OUT technique in Solution 3 is a useful algebraic tool.

SAMPLE PROBLEM SET

FACTORIALS

Definition: $n! = 1 \cdot 2 \cdot 3 \cdots n, 0! = 1.$ Example: $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

1. Determine the following:
   (a) $7! =$ (b) $(3!)^2 =$ (c) $(3^2)! =$

   (d) $(2!)\,(3!) =$ (e) $(2 \cdot 3)! =$

2. Is $(mn)! = m!\,n!$?

3. Express as single factorials
   (a) $3! \cdot 4 \cdot 5 =$ (b) $n!\,(n + 1) =$ (c) $a!\,(a^2 + 3a + 2) =$

4. Express $\frac{(n+3)!}{n!}$ as a polynomial in *n*.

5. Show that $7! - 6! = 6! \cdot 6$.

6. Show that $(n + 1)! - n! = n! \cdot n$.

7. Use #6 to derive a compact expression for $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \ldots + n! \cdot n$.

8. Come up with a formula for $(n + 2)! - n!$ similar to the one in #6.

9. Use the formula in #8 to derive a compact expression for
$$0! + 11\,(2!) + 29\,(4!) + \ldots + (4m^2 + 6m + 1)\,[(2m)!].$$

<u>QUESTIONS:</u>

A. What is the purpose of Problem #1?

B. How are 3a and 3c related?

C. How are problems 5 and 6 related?

D. What is the purpose of Problem #9?

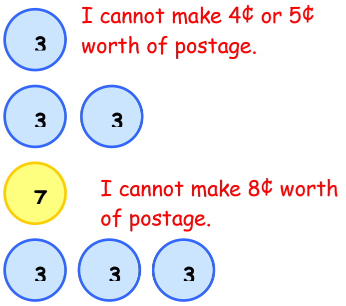
Strategy 1 – ACT IT OUT

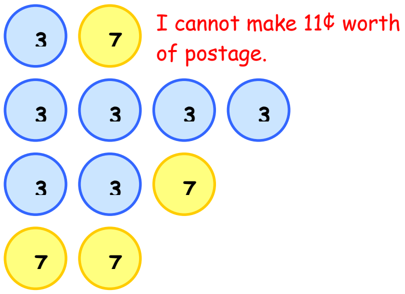
Problem 1.1 – A small post office has in its lobby a machine that dispenses only 3¢ and 7¢ stamps. If the window at the post office is closed, what is the largest possible postage amount that could not be made using stamps purchased from the machine in the lobby?

Questions to ask myself: Are there values of postage that I cannot make? Is there any significance to the fact that 3 and 7 are relatively prime (no common divisors)? Are there values of postage that cannot be made that are larger than values of postage that can be made?	**Understand the Problem** – The only source of stamps I have is from the stamp machine. I cannot "split" up a stamp's value. Are there totals that cannot be made? Yes, I cannot make 1¢ or 2¢. I can make any multiple of 3 and any multiple of 7, but I will need to look at combinations of those two values. The two numbers do not have any common factors. The largest value of postage that cannot be made will occur after values of postage that can be made. For example, I cannot make 4¢ worth of postage even though I can make 3¢ worth of postage. I have an unlimited number of stamps of both denominations available to me.
*The **ACT IT OUT** involves the problem solver kinesthetically by having him/her work with manipulatives or with other people physically on the problem. You might form a human chain, or build a triangle with toothpicks. In some manner this heuristic involves physical action by the problem solver. You might ask yourself, are there any objects I could use as props to model this situation? Could I build a model?*	**Select a Strategy** – I know there are values of postage I can and cannot make. I would like to keep a record of the postage values I can create and how many stamps of each denomination I have used to create it. I am going to use the ACT IT OUT strategy and use blue tiddlywinks to represent 3¢ stamps and yellow tiddlywinks to represent the 7¢ stamps. Other potential strategies: Brute force, guess and check, convert words into mathematical notation Related strategies: Brute force, Collect data

I am going to try to keep my work organized by not using the chips from previous postage values to form new postage values. I am also going to start with the lowest value I can make and work up from there.

Implement – I will use blue and yellow tiddlywinks to represent the 3 and 7¢ stamps, respectively.





Now that I have three consecutive postage values I can form any value of postage greater than 12¢. To create the next three postage values all I need to do is add a 3¢ stamp onto the three values of 12¢, 13¢, and 14¢.

This strategy was effective for this problem; however, if the number of combinations needed before identifying the maximum value was much greater this heuristic would start to get clumsy and frustrating.

This problem needs to be explored further by experimenting with a variety of different stamp value combinations.

Tie-Together – The largest value of postage that cannot be made is 11¢. Once I had three consecutive values of postage, I knew I could make any subsequent value of postage by adding to one of those three postage values a pile of 3¢ stamps. For example, if I wanted to make 43¢ worth of postage I need to look at 43-12=31, 43-13=30, and 43-14=29. The value 43-13=30 is a multiple of 3 and so to the pile of stamps that has a value of 13¢ I can add on ten 3¢ stamps. This is not the fewest number of stamps I would need to make 43¢ worth of postage but it would work. The essence of this problem may be the fact that the two stamp values are relatively prime. The essence could also have to do with creating "linear combinations" of two values 3a + 7b.

An extension for this problem would be to devise an algorithm that would always give you the fewest number of stamps of each denomination to form a given postage value. Is there a connection between the values of 3 and 7 and the maximum unattainable postage? What would happen if you had stamps that had common factors but were not multiples of each other, like 4¢ and 6¢ stamps?

Problem 1.2 – Given 6 points in a plane (no three collinear), how many straight lines can you draw if each line must pass through exactly 2 of the six points?

Can you restate this problem in your own words? What information do you need to know to solve this problem? What does it mean to be collinear? Other than not being collinear, are there any restrictions that need to be placed on where the points appear in the plane? This problem involves counting, what might be the essence of this problem or are there any counting tools you could bring to bear on this problem?	**Understand the Problem** –
*Use the **ACT IT OUT** heuristic on this problem. How could you model the situation posed in this problem? Would you be able to physically act it out by having classmates serve as points in the plane? What would lines look like if people were the points in your model? Other than using classmates, are there materials at my disposal with which I can model this problem? What could I draw using pencil and paper? What other strategies might be appropriate to solve this problem?*	**Select a Strategy** –

During the implement phase you need to keep your work organized so you know where to find it when you need it. Begin by acting out the problem in the manner you devised.

How many different lines will a single point lie on? Does it matter which point you consider?

Implement –

Tie-Together –

The tie together phase allows you to look back at work and summarize your results.

What seems to be the essence of this problem? Is it related to any other problems you have solved?

Can you create a new problem that is similar to this one? Can you create a new problem that is quite different from this one but contains the same essence?

Strategy 2 – GUESS AND CHECK

Problem 2.1 – I am thinking of three numbers. Their product is 72, their sum is a multiple of 7. If the three numbers are all different what are they?

Can I assume the three numbers are all integers?	**Understand the Problem** – I am going to assume that the three numbers are positive integers. The problem does not state this, but I think it would be far too difficult to solve if I allowed any real numbers.
Can I name the numbers using variables?	I will represent the three numbers as a, b, and c. This means I have two equations a•b•c=72 and a+b+c=7m. This is rough having two equations and four unknowns.
How can I find a unique answer if I have two equations and four unknowns? Are there any other restrictions I can place on this problem that would get me two more equations?	I know that the multiples of 7 are {0, 7, 14, 21, 28, 35, 42, 49, 56, …}. I guess I can eliminate 0 from the set.
The strategy of ***GUESS AND CHECK*** *does not require that you know any sophisticated methods. You can start anywhere, with any guess and learn something about the problem or possibly you'll get lucky. Polya emphasized the importance of both skill and luck when problem solving.*	***Select a Strategy*** – I cannot remember any techniques for solving this system of equations and I cannot think of any other equations I could include as restrictions. I think the best approach for this problem is to make a GUESS AND CHECK to see if I am correct (or really lucky).
Making sure you have a solid mathematical background and have honed your skills can sometimes create the luck you need.	So what am I going to guess? There are an infinite number of multiples of 7, not to mention numerous ways to write any single multiple as a sum of three positive integers. I don't think I want to guess which multiple it is and then on top of that guess which composition I want. I know that there are a finite number of triples (a, b, c) whose product is 72. I will guess the three factors that give me 72, then check their sum. I will write out the prime factorization of 72 and use it as a reference when I am trying to create different triples.
	Other potential strategies: Establish a subgoal, collect data, convert words into mathematical notation Related strategies: Brute force, act it out, what happens if

The prime factors will need to be grouped to create non-prime divisors. This does not mean that a, b, and c cannot be a prime. I need to also remember that 1 can always be used as a factor and might be a nice “filler.” *Do you feel “lucky?” Nope!*	**Implement** – The prime factorization of 72 is $2^{3}\bullet 3^{2}$. I need to remember as I chose my triples that the factor of 1 is sort of a “gimme” factor. It doesn’t affect the product but it can help me in the sum side of the problem. There are only two groupings of three factors that include 6, so I will start with those triples.
The choice of 6 was capricious. It is my favorite factor of 72.	Guess #1 – (3, 4, 6) Check #1 – 3•4•6=72, but 3+4+6=13 ☹ Guess #2 – (1, 6, 12) Check #2 – 1•6•12=72, but 1+6+12=19 ☹
Intuition is a powerful ally. Sometimes you need to follow your hunches even if you don’t have any “good reasoning” behind them. *This seems like I’m returning to an approach I eliminated in my Strategize phase, but I am using the multiples of 7 only as an intuition helper not as the main approach.*	I don’t know why, but I don’t think the multiple of 7 I am going to get will be one of the lower ones like 7, 14, or 21. Maybe I should try some triples that have larger factors and include a 1 as one of the factors. If 1 were one of the factors, I could subtract 1 away from the sequence of multiples of 7 and look at the new sequence. Maybe one of those numbers will jump out as the sum of two numbers whose product is 72. (This reminds me a lot of factoring binomials.)
The technique of subtracting or adding one to a sequence is useful when you have a set of data for which you are trying to find a formula.	I will subtract 1 from each term in the sequence {7, 14, 21, 28, 35, 42, 49, 56, …} to get {6, 13, 20, 27, 34, 41, 48, 55, …}. I am looking for pairs (the 1 is already part of the triple) of numbers whose product is 72 and I want one of the numbers big. Guess #3 – (1, 2, 36) Check #3 – 1•2•36=72, but 2+36=38 ☹ is not in the 7m-1 listing
Does this triple satisfy all of the conditions listed in the problem?	Guess #4 – (1, 3, 24) Check #4 – 1•3•24=72 and 3+24=27 is in the 7m-1 listing ☺

Are there any other triples? *Is the problem solved if you don't know if the answer is unique?*	**<u>Tie-Together</u>** – I don't know if this answer is unique. I suppose I could have listed out all of the possible trios of factors and then found their sums. This would have been a more brute force approach to the problem and would put an end to the problem entirely.
What role (if any) does commutativity play in selection of the factors?	For the sake of being complete, I will make such a listing and put the sums in parentheses behind the factor triple. 1•2•36 (39) 2•3•12 (17) 3•4•6 (13) 1•3•24 (28) 2•4•9 (15) 1•4•18 (23) 1•6•12 (19) 1•8•9 (18)
This confirms my solution and demonstrates that it is unique. It also gives me a way to extend the problem.	The triple I found is the only one that satisfies the given conditions.
It appears in this problem that we are being asked to solve a system with two equations and four unknowns. Actually there are two other conditions you can place on the system. What are they?	I could extend this problem by asking for a triple whose sum is a multiple of 3. Or I could have the sum be a factor of the number 45. The first extension would not have a unique answer but the second one would.
	The essence of this problem seems to be factorization.

Problem 2.2 – An army of ants is organizing a peace march across a room. If they form columns of 8 ants there are 4 left over. If they form columns of either 3 or 5 there are 2 left over. What is the smallest number of ants that could in this army?

Are there any classes of numbers that you could eliminate as potential guesses? *If you focused only on the "8 ants in a column," then what kinds of numbers could be the solution to the number of ants in the army?* *Would the smallest of the number of ants for the 8 ants in a column satisfy the 3 and 5 ants restriction?* *How are division and remainders being used in this problem?*	**Understand the Problem** –
*The **GUESS AND CHECK** strategy is appropriate here since the numbers we are working with are fairly small.* *Perhaps you could establish a subgoal: What numbers have a remainder of 4 when divided by 8? Or, what numbers have a remainder of 2 when divided by 3 or 5?*	**Select a Strategy** –
What numbers have a remainder of 2 when divided by 3? What numbers have a remainder of 2 when divided by 5? Which of these two restrictions will help you make an "educated" guess about the number of ants in the army?	**Implement** –

What is your initial guess? Why did you choose this number? *How will you determine if this number satisfies all three of the criteria?* *How will you determine if this number is the smallest?* *What are the themes in this problem?* *What other characteristics of numbers could you isolate as a way of extending this problem?* *How could you extend the problem so that only the theme was contained in the new problem?*	**Implement (cont'd)** - **Tie-Together** –

Strategy 3 – BRUTE FORCE

Problem 3.1 – If the 120 permutations of 1, 2, 3, 4, 5 are listed in increasing numerical order from the smallest 12345 to the largest 54321, what is the 49th number on that list?

What is a permutation and how can I be certain that there are only 120 permutations of the digits 1, 2, 3, 4, 5? When creating a permutation do I have to use all of the numbers 1-5?

Understand the Problem – I remember that permutations involve ordering and combinations involve selection. So I want to look at the possible orderings of the digits {1, 2, 3, 4, 5}. I have to use each of the digits {1, 2, 3, 4, 5} exactly once and so I will have $5\bullet4\bullet3\bullet2\bullet1 = 5!$ permutations. These numbers will all be of the form ab,cde.

If I used the digits {1, 2, 3}, I would have $3\bullet2\bullet1 = 6$ permutations the smallest being 123 and the largest 321. The complete listing is 123, 132, 213, 231, 312, and 321. The number 213 is the third number in this list of six permutations.

*Sometimes you may be at a loss as to which strategy is best to use or to see how any of the previous strategies may work. In these cases the strategy of **BRUTE FORCE** gives the problem solver a place to start tackling a problem. This technique essentially means "do what the problem says to do until you find a short cut or receive insight into a different approach."*

Select a Strategy – I am sure that I could using BRUTE FORCE make a complete listing of these permutations provided I proceed methodically and carefully.

I hope that as I list the numbers out I may figure out a short cut to the process. Otherwise, it would be very tedious to list all 120 permutations. I will need to double check my listing to ensure I do not miss any permutations or list any of them twice.

Other potential strategies: Create a table, look at a similar problem, look at cases, collect data, establish a subgoal
Related strategies: Collect data, look at cases, use technology

I started with the number 12,345 and then to create the next largest number I had to reverse the last two digits. That lists all of the numbers

Implement – Since 49 is not half way to 120, I will start at the beginning of the list, working my way up from the smallest number 12,345. Here goes:

that look like 12,3_ _. The next smallest group of numbers will look like 12,4_ _. There are also two of this type. That means I should have two of the type 12,5_ _. I get six different permutations that look like 12,_ _ _. Will this happen every time?

I partitioned the permutations into four sets:
12,_ _ _
13,_ _ _
14,_ _ _
15,_ _ _

Are these sets mutually exclusive? Are these sets exhaustive for the permutations that start with the digit 1?

I ended up with 24 permutations that start with the digit 1. Can you explain why? Will I have 24 permutations that start with the digit 2?

<u>Implement (cont'd)</u> -

12,345 12,354 12,435
12,453 12,534 12,543

This has to be a complete listing of the first six permutations because all I am doing is permuting the last three digits. In my Understand phase I wrote down the six permutations of {1, 2, 3}. If I add two to each of the digits in that list I would produce the listing above. I should have six numbers that look like 13,_ _ _. These are:

13,245 13,254 13,425
13,452 13,524 13,542

I've accounted for 12 of the 120 permutations so far. If I kept writing out this list (which I am not), I would have six permutations starting with 14 and six starting with 15. That means I have accounted for 24 permutations so far and these are the permutations that start with 12, 13, 14, or 15.

I guess I should say these are the 24 permutations that have a value between 10,000 and 19,999 (i.e. they start with 1). Does this number make sense? Sure if the permutations all start with 1, then I must permute the last four digits {2, 3, 4, 5}. There will be 4•3•2•1 = 24 "endings" for the numbers in the 10,000 grouping. Using this reasoning I will also have 24 permutations in the 20,000 grouping. The first one is 21,345 and the last is 25,431.

The 10,000 and 20,000 permutations account for 48 of the 120 permutations. I need to identify the 49th permutation, which will be the first permutation in the 30,000 group. The smallest number that starts with a 3 is 31,245 and this is the 49th permutation in my listing.

Does the sequence of factorials 1, 1, 2, 6, 24, 120, … play a role in the solution to this problem? Did knowing or not knowing the factorials make a difference in your ability to solve or follow this problem?	**Tie-Together** – I got lucky in this problem in a couple of different ways. One way is that the 49th permutation was the first permutation in a subgrouping. The second way is that I developed the subgroupings of 10 thousands, 20 thousands, 30 thousands, etc. By looking at these subgroupings the listing and the counting was much easier than just listing the numbers out by hand.
How else could this problem be extended? *Can you create an extension that uses factorials but not permutations?*	An easy extension of this problem would be to ask for the 54th number or the 73rd permutation in the listing. You could complicate the listing by ordering the permutations in descending order within the ascending subgroupings. That is, the 10 thousands would still come first but that subgroup would be ordered from 15,432 to 12,345 instead.
This is an example of a problem that has multiple essences.	This problem seems to be developed around permutations and factorials. I also needed to focus on numerical ordering and counting techniques. I used the concept of partitioning sets as well.

Problem 3.2 – Which number(s) from 1 to 100 has(have) the largest number of positive factors.

What is a factor? What process do you use to ensure you have accounted for all the factors of a number? How do you determine how many factors a number has? *Will larger numbers always have more factors than smaller numbers?*	**Understand the Problem** –
*Use the **BRUTE FORCE** heuristic on this problem along with trying cases.*	**Select a Strategy** –
Choose three or four numbers between 1 and 100 that you think have a large number of factors. Write down all of the factors of those numbers. *Why did you choose these numbers?* *Are there numbers you can eliminate from consideration because you already know how many factors they have?*	**Implement** –

The number $8=2^3$ has four factors: 1, 2, 2^2, and 2^3. How many factors does 12 have? *Is there a connection between the number of prime factors a number has and the number of factors a number has? What is the difference between a prime factor and a factor?* *Perhaps you could establish a subgoal: How many factors does the number $p^a \bullet q^b$ have given that p and q are primes? What role do the primes play?* *Do you need to write out all of the factors of a number to establish a reliable count?* *How will you determine if this number has the largest number of factors without checking all of the numbers between 1 and 100?* *Is it possible that two different numbers could have the same number of factors?*	**Implement (cont'd)** -
Is there a unique answer to this question? *What are the themes in this problem?* *What other characteristics of numbers could you isolate as a way of extending this problem? For example: Which number from 1 to 100 has the greatest number of prime factors?* *How could you extend the problem so that only the theme was contained in the new problem?*	**Tie-Together** –

Strategy 4 – DRAW A PICTURE, MAKE A MODEL

Problem 4.1 – A 2 by 6 rectangle is subdivided into 12 1 by 1 squares. If a diagonal is drawn it will pass through 6 of these 12 squares. Through how many squares will a diagonal pass in a subdivided 4 by 6?

What happens if the diagonal passes exactly through the corner of a square? Do I count all four of the squares or do I count none of the squares?

Using the given information to increase your understanding is always an excellent idea.

Passing through a vertex does not count; the diagonal must pass through the interior of the square.

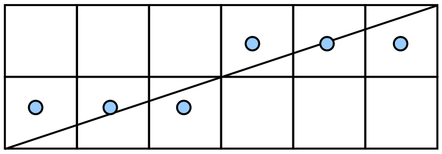
Visualization is an important skill to develop. Often drawing an accurate diagram is crucial aid to honing your visualization ability.

*Since trying to do this mentally, or with formulas, does not seem easy or reasonable, the strategy **DRAW A PICTURE, MAKE A MODEL** will be used on this problem. You may need to draw various smaller models before completely solving a problem. Using a manipulative such as a geoboard or graph paper will help increase the accuracy of your model.*

Understand the Problem – I understand why a subdivided 2 by 6 rectangle would have 12 small 1 by 1 squares. Similarly, the subdivided 4 by 6 would have 24 1 by 1 squares.

I am not sure what happens if the diagonal passes through the corner where four squares meet. Should I count all four squares or none? I am going to draw a picture of the 2 by 6 case and double check the count I am given.

From the picture I can see that if the diagonal passes exactly through a corner, I do not count the squares that converge at that point.



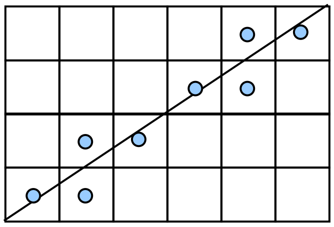
Can I solve this problem by stacking two of my 2 by 6 grids from above on top of each other, making the number of squares hit by the diagonal 6+6=12 squares instead of 6? No, that picture would have two different sloping lines neither of which is a diagonal.

Select a Strategy – Drawing a picture of the 2 by 6 rectangle was very helpful in counting the squares "touched" by the diagonal. I will DRAW A PICTURE on graph paper of the 4 by 6 case and then as I count I will place a dot in the squares that are "touched" by the diagonal.

Other potential strategies: Solve a similar, simpler problem; establish a subgoal, use technology, collect data
Related strategies:

What can I learn about from how the diagonal passes through the squares? How does a 2 by 3 differ from a 2 by 4?

Implement – My picture shows that the diagonal passes through one vertex exactly. There seems to be two 2 by 3 subrectangles through which the diagonal passes and two 2 by 3 subrectangles through which the diagonal does not pass.



The answer to the question is: there are 2•4 = 8 1 by 1 squares through which the diagonal passes.

Another strategy that played a role in solving this problem was establish a subgoal. The count in the smaller 2 by 3 rectangle was used to find the overall count.

How else could you extend this problem?

The theme or essence of this problem centers around fractions, slopes of lines, reducible fractions, and divisibility of integers. You should watch for the extensions of this problem in the problems in the back of the text. You might want to review this solution and see where the themes listed above appear in the solution presented.

Are there other themes contained in this problem?

Tie-Together – Symmetry of rectangles must be playing a role in the solution of this problem. I do not believe this is the end of this problem; however, because there seems to be an unasked question. Given any a by b subdivided rectangle, through how many squares will the diagonal of the rectangle pass? This would be an excellent extension.

Drawing a picture disposed of this problem really quickly. If I created a drawing of an 8 by 12, I would expect there would be 4 of the little 2 by 3 subrectangles appearing along the diagonal. That means I should get a total of 4•4=16 1 by 1 squares along the diagonal. I can solve two classes of problems with the two pictures I have solved so far – those whose subrectangles are 1 by 3 and those whose are 2 by 3s. I would need to collect more data before I could come up with a general formula.

Problem 4.2 – Five friends were sitting on one side of a table. Gary sat next to Bill. Mike sat next to Tom. Howard sat in the third seat from Bill. Gary sat in the third seat from Mike. Who sat on the other side of Tom?

Which friends must sit next to one another? *Which friends cannot sit next to one another?* *How many seats must be between Howard and Bill?*	**Understand the Problem** –
Try the strategy ***DRAW A PICTURE, MAKE A MODEL****. What kind of model/picture can you use to solve this problem?*	**Select a Strategy** –
Choose letters to represent each of the five friends. Use those letters to represent pairs of friends that must sit together. *Does the order of the friends (e.g. Tom then Mike or Mike then Tom) matter?*	**Implement** –

How can you represent the empty seats between friends?	**Implement (cont'd)** -
Is there a unique ordering for the friends? *How many different orderings (without restrictions on who sit next to whom) of the friends were there?* *What seems to be the essence of this problem?* *What extensions of this problem could you create that would involve the essence of the problem?*	**Tie-Together** –

Strategy 5 – EXPLOIT SYMMETRY

Problem 5.1 – A ship must travel from point A = (0, 20) to a point C = (80, 50) touching the x-axis first. At what point B = (k, 0) should the ship touch so that the total distance from A to B to C is minimized?

How do I find the distance between two points? Why would a ship need to "touch" the shore? What happens to the distance between the points as I move the axis point?

Understand the Problem – As I move the point on the x-axis to the left it is closer to point A but moves farther away from point B. The formula for distance is $d(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$.

*The **EXPLOIT SYMMETRY** strategy allows you to simplify a problem by using the symmetric properties of figures or equations. You might ask yourself the question what kinds of symmetry does this figure display? If I used symmetry, what portion of the figure would I need to attend to and what information about it would I need?*

Select a Strategy – I could develop a distance function using the variable k as the input and then attempt to find the minimum value of the function using calculus techniques. This approach doesn't appeal to me so I will search out an alternative.

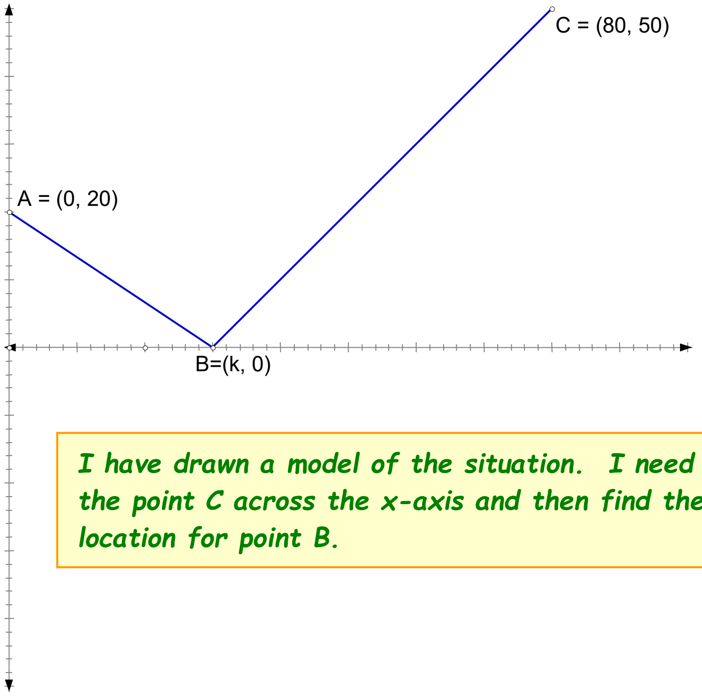
I know that the shortest distance between two points is a straight line. That means that if the point B did not have to be on the x-axis, I could place B on the line between A and C and the distance would be minimized.

Using the idea of symmetry I could imagine that the x-axis is a line of reflection. The image of point C (label it C') would appear in the fourth quadrant. The distance between point B and C' would be equal to the distance between points B and C. I will graph the points A and C and then use a Mira to find the reflection of point C across the x-axis. When I connect A with C' using a straight line, the point of intersection of this line with the x-axis should give me the placement of point B.

Other potential strategies: Establish a subgoal, act it out, use technology, introduce a variable
Related strategies: none

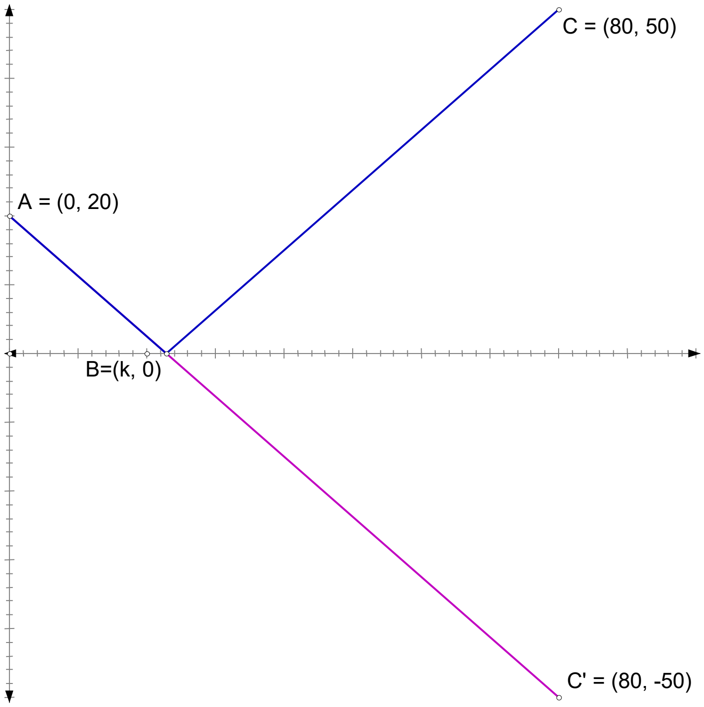
Implement –

During the implement phase you need to keep your work organized so you know where to find it when you need it.



I have drawn a model of the situation. I need to reflect the point C across the x-axis and then find the correct location for point B.

Now that I have found the location for point B, I need to check to see if I have answered the question.



Implement (cont'd) –

I need to determine the coordinates of point B. I know it lies on the line between A and C'. The slope for the line is (–50 - 20)/(80 - 0) = -70/80 = -7/8. Thus the equation for the line is y = (-7/8)x + 20. To find the x-intercept for this line I need to set y = 0. Thus I have:

(-7/8)x + 20 = 0
(-7/8)x = -20
x = (-8/7)(-20) = 160/7

Point B should be placed at (160/7, 0) to minimize the distance of the path traveled from A to B to C.

The tie together phase allows you to look back at your work and summarize all of your accomplishments. This is also your opportunity to extract the essence of the problem and take note of any tools you may have developed during the implementation phase. As a final statement of your understanding of the problem and its solution, you should extend the problem by stating other problems that contain the same essence.

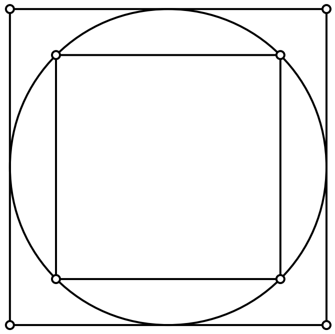
Tie-Together – I not only needed to use the concept of symmetry but I also needed to make use of the formula for slope and for determining the x-intercept of a line. This was a good review of a variety of geometric concepts.

I never had to use the distance formula to solve the problem. I could find the distance traveled just as a wrap up of the problem.

$d(A, C') = \text{sqrt}[(80 - 0)^2 + (-50 – 20)^2] = 10[\text{sqrt}(113)]$

This is definitely a situation you might run into in real life. Ships have to dock on the shore and you might want to locate the pier in the optimal location between two islands. If this were a billiards table, the point A would represent the cue ball and point C would represent the ball I am trying to hit. The location of point B tells me where to bank my shot off the bumper.

Problem 5.2 – A square is inscribed in a circle that is inscribed in a square. What is the ratio of the areas of the two squares?

What does it mean to inscribe? Do the figures that are inscribed inherit any particular properties? How do you find the area of a square and of a circle? Do I need to find the area of the circle?	**<u>Understand the Problem</u>** – 
*The **<u>EXPLOIT SYMMETRY</u>** strategy is the strategy you are to use on this problem. What are the different kinds of symmetry that are present in the figure? How does these symmetries assist you in finding the ratio? What other strategies might also be appropriate to use on this problem?*	**<u>Select a Strategy</u>** –

Verbalize clearly how you are using symmetry to assist you in finding the ratio of the areas. *Draw a sketch with the lines of symmetry that you are using included in the sketch.* *What is the ratio of the areas?*	**Implement (cont'd)** –
Did you need to find the area of the circle? *What seems to be the essence of this problem? Is it related to any other problems you have solved?* *Is there more than one way to approach this problem?* *Can you create a new problem that is similar to this one? Can you create a new problem that is quite different from this one but contains the same essence?*	**Tie-Together** –

Strategy 6 – ESTABLISH A SUBGOAL

Problem 6.1 – Two non-intersecting lines are drawn across the face of a clock. The sums of the numbers appearing in each of the three regions formed are equal. How are the lines drawn and what is the regional sum?

During the understand phase of the problem solving process, you need to identify important facts contained in the statement of the problem. You will also need to clarify your understanding of terms. You may be able to identify the essence of the problem at this stage.

Understand the Problem – A regular clock contains the numbers 1 through 12. My lines need to be non-intersecting on the face of the clock, but they don't have to be parallel from a Euclidean geometry point-of-view. Any number will appear in only one of the three regions. I will need to "separate" the larger numbers. Regions won't necessarily contain the same number of terms. One of the three regions will contain non-consecutive numbers.

Novice problem solvers typically spend little time exploring different possible strategies for solving problems. Rather they select the first strategy that occurs to them (or the one they feel most comfortable with) and plunge ahead. This may cause them to suffer set backs early on in the problem solving episode or expend considerable energy on a path that leads to a dead end.

*The **ESTABLISH A SUBGOAL** strategy allows you to attack a problem that is somewhat easier. You might ask yourself the questions such as what is a reasonable next to the last step? What would a helpful preliminary or intermediate step look like? It would be helpful if I only knew . . .*

Select a Strategy – Since I know each of the three regions will add up to the same amount, it would be helpful to know what the sum will be. I will ESTABLISH A SUBGOAL of determining the regional sum. I will then draw a diagram and experiment with a few pairs of lines.

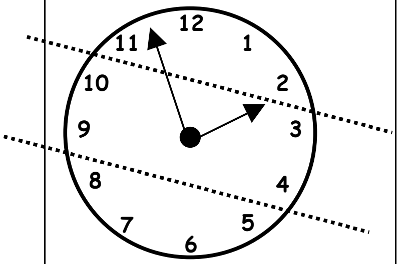
Other potential strategies: Guess and check
Related strategies: Collect data, look at a simpler case

During the implement phase you need to keep your work organized so you know where to find it when you need it.

Implement – I need to find the sum 1+2+3+…+12. This is an arithmetic sum and I can use the Gauss forward and backward sum technique on it.

$S = 1+2+3+4+5+6+7+8+9+10+11+12$
$\underline{S= 12+11+10+9+8+7+6+5+4+3+2+1}$
$2S = 13+13+\ldots+13$
$S=(\frac{1}{2})(13)(12)=78$
This means each regional sum is (1/3)(78)=26.

With this in mind and also knowing I must keep the big numbers apart if I can, I will test a line that passes between 12 and 11. If I do that then 12 must be "paired" with numbers that add up to 14=26-12. The sum 1+2+3+4=10 is too small and the sum 1+2+3+4+5=15 is too big. That means the line cannot separate 12 and 11. If the line is drawn between 11 and 10, keeping 12 and 11 together in one region, I need to "pair" 11 and 12 with numbers that add up to 3=26-(11+12). This can be done using 1 and 2. That means I draw my first line between the 11 and the 10 and exiting between the 2 and the 3.



Now I know that 10 and 3 will be in the same region. That means I need to "pair" 10 and 3 with numbers that add up to 13=26-(10+3). Placing 9 and 4 in the same region can do this. That means I should draw my second line between the 9 and the 8 and exiting between the 4 and the 5.

The tie together phase allows you to look back at work and summarize your accomplishments. This is also your opportunity to extract the essence of the problem and take note of any tools you may have developed during the implementation phase. As a final statement of your understanding of the problem and its solution, you should extend the problem by stating other problems that contain the same essence.

Tie-Together – I can check my three sums to make sure my answer is correct.

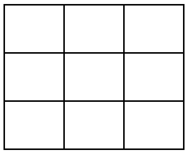
11+12+1+2=26 and 10+9+3+4=26 and 8+7+6+5=26

Have I completely answered the question? Yes I have drawn the non-intersecting lines and I have identified the regional sum.

The essence of this problem seems to be the Gauss forward and backward sum technique. I also used the idea of "consecutive" integers from a clock's perspective.

I could extend this problem by looking at more general clocks, like the military clock. I might also try to draw more lines. Could I divide it up into 4 or 5 regions? What would happen if I allowed a number to appear in multiple regions or if I allowed the lines to intersect? What if I abandoned the clock and just considered drawing lines on a circle to see how many regions I could create with n lines?

Problem 6.2 – In the 3 by 3 figure, place one of the integers 1, 2, 3, . . ., 9 in each box so that each row, column, and diagonal sum is the same.



Can you restate this problem in your own words? What information do you need to know to solve this problem? Are there any definitions or terms you need to clarify before you can progress? What might the essence of this problem be?	**Understand the Problem** –
The ***ESTABLISH A SUBGOAL*** *strategy is the strategy you are to use on this problem. List two subgoals whose resolution may be important in the solving of this problem. What other strategies might also be appropriate to use on this problem?*	***Select a Strategy*** –
During the implement phase you need to keep your work organized so you know where to find it when you need it. Begin by accomplishing your stated subgoals and keeping track of your work. *How many different sums will each integer appear in? Does it matter which integer is in the middle box?*	**Implement** –

Implement (cont'd) –

Is that your final answer?

Tie-Together –

The tie together phase allows you to look back at work and summarize your results. Check the row, column and diagonal sums. Are they all equal?

Did each integer appear in one and only one box?

Is there more than one possible solution?

What seems to be the essence of this problem? Is it related to any other problems you have solved?

Can you create a new problem that is similar to this one? Can you create a new problem that is quite different from this one but contains the same essence?

Strategy 7 – TRY SMALL VALUES FOR n, LOOK AT A SIMILAR SIMPLER CASE

Problem 7.1 – Determine a closed formula for the sum below.

$$\frac{1}{1\bullet 2} + \frac{1}{2\bullet 3} + \frac{1}{3\bullet 4} + \cdots + \frac{1}{n\bullet(n+1)}$$

What is your understanding of the word "closed"? Sometimes the phrase "nice" or compact expression is used instead of closed. A closed formula would not use the summation symbol Σ or the product symbol Π.

Understand the Problem – Each summand in the expression is a fraction whose numerator is 1. Each of the denominators is the product of two consecutive integers. I will have to add fractions and reduce (if necessary) appropriately. I can see that it will be even more important that I have my work organized and neat in the implementation phase of this problem.

A closed formula would be a formula for the sum that is a function of n and that does not contain any symbols such as Σ, Π, or $\ldots$ During the Tie-Together phase, I should be able to check my formula by choosing a value for n and plugging it into both the expression above and the formula I derive.

*The strategy **TRY SMALL n, SOLVE A SIMPLER PROBLEM** allows the problem solver to develop his or her intuition by essentially diving into the problem quickly in the hopes of predicting a solution based on the findings. This strategy is helpful in this problem since the sum in question is a function of n, and the fractional summands have a compact, nice form.*

Select a Strategy – I'll try small values of n and collect data to help build up my understanding of the problem. Once I have some data collected, I will try to guess at what is happening before proceeding. First it may help me to give a name to the sum; perhaps S(n) would suffice.

With $S(n) = \frac{1}{1\bullet 2} + \frac{1}{2\bullet 3} + \frac{1}{3\bullet 4} + \cdots + \frac{1}{n\bullet(n+1)}$

I can try to evaluate this sum for n=1, 2, 3, 4, and so on.

Other potential strategies: Create a table, place in a more general setting, look for a pattern

Related strategies: Collect data, create a table, look for a pattern

What does S(4) mean? Does it mean 1/(4• 5)? Does it mean the expression whose final summand is 1/(4• 5)?

I can use my understanding and value for S(3) to find S(4). Could I develop a recursion to generate values for the sum? Would this help me solve the problem?

Implement – The sums and data for S(1), S(2), and S(3) are

$$S(1) = \frac{1}{1\bullet 2}$$

$$S(2) = \frac{1}{1\bullet 2} + \frac{1}{2\bullet 3}$$

$$S(3) = \frac{1}{1\bullet 2} + \frac{1}{2\bullet 3} + \frac{1}{3\bullet 4}$$

Continuing in this fashion we can make the table below.

Neatly organized data facilitates pattern recognition.

n	1	2	3	4	5	6	7	8
S(n)	1/2	2/3	3/4	4/5	5/6	6/7	7/8	??

Be sure to test the conjecture with a new value of n; try n=8. To test a conjecture you must calculate separately each piece.

It is reasonable to conjecture that S(n) = n/(n+1). As a small test, I will try n=8. The sum gives 8/9 and my formula S(8) gives me 8/9.

Is this a proof? A conjecture may settle the problem for younger students; however, you would always want to justify your beliefs through proper reasoning. There are mathematical statements that are true for hundreds of cases and then become false in general.

Tie-Together – The data checks with the new formula; however, this conjecture needs a proof. One approach I could use is to try mathematical induction. Alternatively, the form 1/n(n+1) might reminds me of partial fractions from calculus. Since 1/n(n+1) = 1/n – 1/(n+1) I can convert each product into a difference. Then the sum will collapse.

What is striking about this new form? How does the sum collapse? What is the difference with whose second term will cancel out 1/n in the final term of the sum? Why are 1 and 1/(n+1) left after the collapsing has taken place?

$$S(n) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \ \dots \ + \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$

The second term in each difference will cancel with the first term in the following difference. This proves the closed formula is correct.

Can this technique work on other problems? What type?

The essence of this problem seems to be collapsing sums.

Problem 7.2 – Determine a closed formula for (1-1/4)(1-1/9)(1-1/16)…(1-1/n^2).

How does this problem differ from the example in 8.1? *What are the numbers in the sequence 4, 9, 16, …?*	**Understand the Problem** –
Use the ***TRY SMALL n, SOLVE A SIMILAR SIMPLER PROBLEM*** *heuristic on this problem.* *Would it be beneficial to call this expression P(n)? What does the P stand for?* *What other strategies could you use on this problem?*	**Select a Strategy** –
Compute P(n) for n=1, 2, 3, 4, 5, 6. Can you conjecture a formula for P(n)?	**Implement** –

Did you check your formula with n=7 or 8? How did you perform your check? Did you calculate the product and then also calculate the value using the formula? *Is what you have accomplished thus far a proof of your conjecture?*	**Implement (cont'd)** -
How could you prove your conjecture? What proof techniques have you used in the past? *Does the proof you have written seem clear and convincing? Would a friend understand your reasoning if you read it to him or her?* *What seems to be the essence of this problem?* *What extensions of this problem could you create that would use this heuristic or the essence of the problem?*	**Tie-Together** –

Strategy 8 – COLLECT DATA, SEARCH FOR A PATTERN

Problem 8.1 – Determine the maximum number of regions into which n lines can divide the plane.

What does maximum mean in this context?

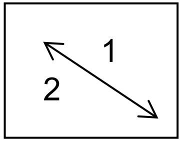
Does a plane end? Does a line have an end?

What happens when three lines are collinear?

Does the angle between lines make a difference?

Will there be more than one way to draw the lines that will give you the maximum number of regions?

Understand the Problem –Even though I may draw line segments on a piece of paper, I need to keep in mind that planes and lines are infinite objects. If I draw one line, there are two regions. I can draw the line in any position I want (horizontal, vertical, slanted) and I will still only have two regions.



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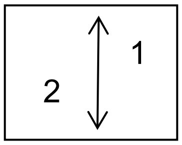

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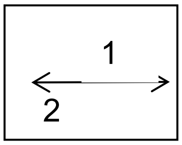
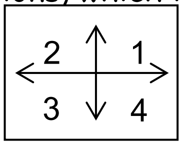


For two lines I could draw them just like the coordinate plane I use to graph points. Two intersecting lines create four regions. Once again I don't think the angle between the two lines matters. I could have them perpendicular or I could change the angles; but these two intersecting lines would still create four regions. I could have drawn the two parallel lines, but this would have only created three regions, which is not maximal.



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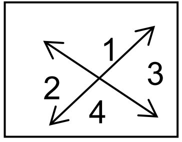

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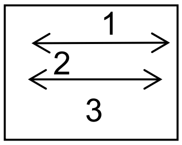
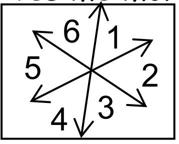


There will be even more ways for three lines. I don't think I will want any of the lines to be parallel, because then that pair of lines wouldn't be creating the maximum number of regions they could. I don't think I will want three lines to intersect at a single point (collinear). If I compare the cases of three collinear and no three collinear, the second case gives me more regions.



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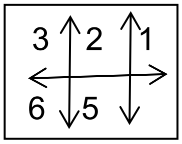

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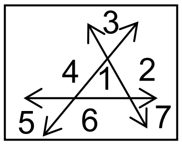


As I proceed I need to remember that the angle between the lines isn't going to make a difference in the number of regions created. There may be more that one way to draw the lines as n gets larger; however I won't want my picture to contain any parallel lines nor do I want my picture to contain a set of three collinear lines.

*Mathematics is the science of patterns. The strategy of **COLLECT DATA, SEARCH FOR A PATTERN** involves collecting data, and then organizing this data in a chart or table to allow you to detect a pattern if one exists. Once you find a pattern, you will need to confirm your conjecture using a form of proof.*

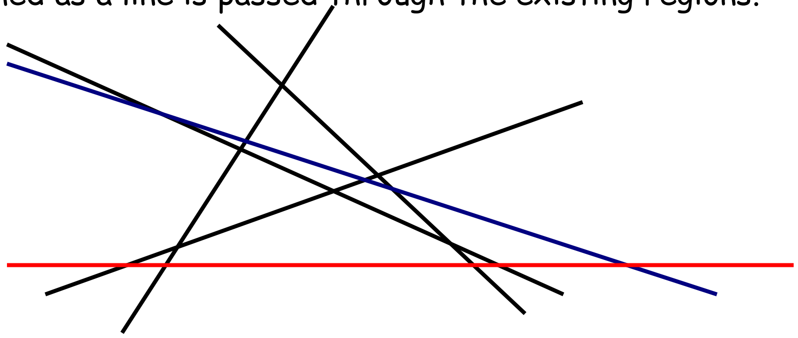
Select a Strategy – In order to understand the problem I drew several pictures and collected some preliminary data. This method seems to be very effective. The strategy of COLLECT DATA, SEARCH FOR A PATTERN will be useful in attacking this problem. I can continue to draw pictures for the cases of n=4, 5, and 6 and then examine the data to see if I can determine a recursion or if I could develop a closed formula which relates the value of n to the number of regions.

Other potential strategies: Place in a more general setting
Related strategies: Brute force, use technology

I can use the data I collected during the understand phase and continue collecting data to determine any patterns.

First guess at the next entry without actually drawing the lines. Then confirm (or refute) your guess by counting the regions. Be careful not to let your guess influence your count.

Implement – I already have the data for the first three cases when n=1, 2, and 3. I will draw a picture for n=4 and then add a blue line for the fifth line and a red line for the sixth. Maybe after determining a pattern from the data I could explain the formula by seeing how new regions are formed as a line is passed through the existing regions.



In collecting data, and organizing this data in chart form, you will begin to see a pattern. Pay attention during your collecting to see if you can find any clues as to why the pattern holds.

I can create the following data chart from the pictures I created. I have also chosen to denote the maximum number of regions as P(n) and the number of lines as n.

I can use the "free data" when n=0 to increase the number of terms in my data chart. This may help me determine a pattern more quickly and also check any formula using a case that is not too difficult to check by hand.

n	0	1	2	3	4	5	6	7
P(n)	1	2	4	7	11	16	22	??

There are a couple of different approaches that can be followed to find a pattern. We can guess noticing that each entry is 1 more than the elements in the sequence of triangular numbers. We could fit a polynomial to the data by using differencing.	Now I need to search for a pattern that tells how to make the general term of the sequence 1, 2, 4, 7, 11, 16, 22, … Each term seems to be increasing by 1, then 2, then by 3, and so on. My guess is that the next term is 29. I also notice that if I subtract 1 from each term the new sequence is 0, 1, 3, 6, 10, 15, 21, … These are the triangular numbers, shifted by one term. The formula for P(n) is P(n)= $\frac{1}{2}n\bullet(n+1)+1$.
I first need to check my formula against the data I have collected.	**Tie-Together** – I need to check to see if this formula works with the data I collected. Letting n=5 I get P(5)= $\frac{1}{2}(5)\bullet(5+1)+1 = \frac{1}{2}(30)+1 = 15+1=16$. This checks with the data I collected empirically so this gives me a little more faith in my formula although this is not sufficient proof.
Exploring alternate methods for solving the problem may give you added faith in your solution or provide you with insight into why your solution is true.	I could have differenced the data to determine if a polynomial would fit the sequence. The original sequence of 1, 2, 4, 7, 11, 16, … has a first difference of 1, 2, 3, 4, 5, … and a second difference of 1, 1, 1, 1, 1, … These differences tell us that a quadratic polynomial, $an^2+bn+c=P(n)$, will fit the data. If I expanded the formula I came up with for P(n), it is quadratic so I have more support for my answer.
Looking at the differences helped me develop a recursion for the data as well. This recursion helped me explain how the regions were formed when a new line was passed through the picture. *Is this explanation a proof?*	Based on the first differences, I now notice that I could have described the sequence of data using the recursion $P(n)=P(n-1) + n$. Essentially this recursion tells me how the new regions are formed. I must keep all of the regions created by the other lines and that passing the new line through creates n new regions. Looking back at my pictures I notice that when I pass the new line through, it intersects each of the previous lines. For each intersection I split a region into two pieces. The last intersection would give me n-1 new regions and then when the new line emerges it splits that final region into two pieces, giving me n-1+1 = n new regions.
The triangular numbers were the essence of the problem. What other problems could	The essence of the problem centers on the triangular numbers 1, 3, 6, 10, 15, 21,… which have the general form

have the triangular numbers as the essence? A second essence could be the use of the differencing technique for determining formulas. *What type of extensions are these? Are they ones that keep the setting of the problem essentially unchanged but increases the difficulty of the problem by changing the dimension or numbers involved? Are they extensions that use the same basic setting but tweaks the initial given conditions to create a different problem? Are they changing the setting of the problem so that the new problem does not resemble the old but incorporates the same essence? This last type of extension provides surprises to people who solve your problems. They aren't expecting the two problems to be connected and yet they are. To an avid problem solver, that's cool!*	$\frac{1}{2}$n(n+1). In this problem we are off by 1 and the sequence is also shifted. If I hadn't recognize the triangular numbers I would not have been able to come up with a formula for the number of regions so quickly. I wonder if there is a combinatorial proof for my formula using binomial coefficients as counting tools. This region problem could be stated in various dimensional settings: 1 dimensional, 2 dimensional (which was done), and 3 dimensional. Anything beyond 3 dimensional would be very hard to visualize. Also we could hold k of the n lines parallel and ask for the maximum number of regions possible.

Problem 8.2 – Let each of m points on the positive y-axis be joined (by line segments) to each of n points on the positive x-axis. Assuming that no three segments are concurrent (except at the axes) determine a formula for the number of interior intersection points.

What does it mean for three line segments to be concurrent? *Does it matter where the points are placed on the axes? Can m and n be equal? Could you have exterior intersection points?* *Will my formula depend on both m and n? Is it possible to have distinct ordered pairs of numbers (m_1, n_1) and (m_2, n_2) and have the same number of interior intersection points?*	**Understand the Problem** –
*Use the **COLLECT DATA, SEARCH FOR A PATTERN** heuristic on this problem. You might also try using a manipulative like a geoboard to help you with the data collection. Make sure you decide on a way to organize your collection and display of the data. Such organization will make spotting a pattern, if one exists, easier.*	**Select a Strategy** –
Start by collecting some "cheap" data. How many intersection points would you get when either m or n equaled 1? *Use some other small numbers to help build your intuition. Try m=2 and n=2.*	**Implement** –

Could you exploit symmetry in this problem?

Try fixing m=1 and let n range. Can you develop a formula for this data? Repeat this process when m=2. Can you develop a similar formula for this data? Can you connect the two formulas together?

It is important to create useful symbolism!! For the pair (m, n) let f(m, n) denote the number of interior intersection points. Your data should reflect the values f(2,3) = 3 and f(2,4)=6.

Implement (cont'd) -

Tie-Together –

Did you test your formula? Can you explain to a classmate how you came up with your formula and why you think it is correct?

How could you prove your formula is correct?

What seems to be the essence of this problem? Spotting the essence of this problem may help you come up with a combinatorial proof of your general formula.

How would you extend or place this idea in a different setting?

Strategy 9 – USE TECHNOLOGY

Problem 9.1 – For which n is 1! + 2! + 3! + … + n! a perfect square?

What is the definition of n!? *What do perfect squares look like? Can I determine if the continued sum is a perfect square without finding its value?* *A sum, such as 1! + 2! + 3! +…+ n!, is called a partial sum.*	**Understand the Problem** – This problem involves a sum of factorials. Recall that 5! = 5•4•3•2•1. The … in the middle of the sum indicates that there are terms missing; however, the missing terms follow the established pattern. This means if I let n=5 the sum would be 1! + 2! + 3! + 4! + 5! I cannot tell by looking whether or not this is a perfect square. Whatever strategy I chose, I will need to collect data. The sequence of perfect squares is 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, … I am also going to need to set up good notation. I will let S(n) denote the sum 1! + 2! + 3! + … + n!
*The strategy try small n would be helpful, however this problem requires a lot of computations. The strategy of **USE TECHNOLOGY** allows you to collect lots of data quickly. Don't expect the use of technology to organize your data for you. In fact, you will need to think about what kind of data you will be collecting and how you want the data organized before you make a choice of technology.*	**Select a Strategy** – There doesn't seem to be an apparent pattern to the sequence of squares so I may need considerable data to solve this problem. Rather than doing the data collection by hand, I will USE TECHNOLOGY, specifically the spreadsheet capabilities of the TI-92. I will use the data/matrix editor and have columns representing the counting numbers, the factorials and the partial sum of the factorials. Once I have collected the data, I will see if there is any pattern. Other potential strategies: Create a table, place in a more general setting, try small n, collect data Related strategies: Try small n, collect data

Setting up appropriate notation early in the problem solving process is essential to successfully organizing your data, especially when using technology. Otherwise you may be overloaded with data that has no meaningful organization or connection to the problem at hand.

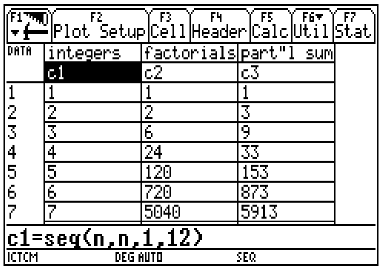
<u>Implement</u> – In column c1 in the data/matrix editor of the TI-92, I will generate the positive integers from 1 to 12. I will generate the factorials in column c2 and the partial sum in column c3. Column c3 is where the values S(n) appear and the value of n will appear in c1.

The commands are:

c1 is seq(n, n, 1, 12)

c2 is c1!

c3 is cumsum(c2)



DATA	integers	factorials	part"l sum
	c1	c2	c3
1	1	1	1
2	2	2	3
3	3	6	9
4	4	24	33
5	5	120	153
6	6	720	873
7	7	5040	5913

```
c1=seq(n,n,1,12)
```

How are the partial sums being created? Could you write a recursion for the partial sums? What do the terms you are adding onto the partial sum look like after the fourth sum?

The first seven partial sums are 1, 3, 9, 33, 153, 873, 5913. I notice that after the third term all the partial sums I have collected end in a 3. This pattern does continue in the remaining entries, but will it continue from this point onward? And, is it significant with respect to the problem?

Why do factorials from 5! on end in 0's?

This pattern will continue. The fourth partial sum is 33. You could write a recursion for the partial sum, namely S(n)=S(n-1) + n! Looking at the sequence of factorials {1, 2, 6, 24, 120, 720, 5040, 40320, …} I notice that all of the factorials after 5! end with a zero. This means that the unit's digit will remain a 3 because I will always be adding to the previous partial sum a term that has a zero as its unit's digit.

Why do the units digits of the perfect squares cycle in this manner?

So, what is the significance in this observation? What do perfect squares look like? The sequence of perfect squares was {1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, …} none of which end in a 3. Will this always be true? Yes, it will be true because the unit's digit cycles as 1, 4, 9, 6, 5, 6, 9, 4, 1, 0 (repeat).

Since no perfect squares end in a 3, the only possible perfect squares are when n=1 and n=3. These partial sums are S(1)=1 and S(3)=9.

Technology can only provide you with sufficient data from which you may draw conclusions. The real work in this problem was not the data collection. This problem required several deep observations. Critics of using calculators in secondary classrooms argue that technology will do "all of the work" leaving the student with little meaningful mathematics to think about. This is a good example to illustrate the appropriate use of technology and to demonstrate that the use of technology in a mathematics classroom will not degrade students' mathematical abilities. On the contrary, the presence of technology in this problem allows younger students to tackle a more difficult problem successfully.

What other types of extension are there?

What sequences could you use to create your partial sums?

Tie-Together – The partial sums S(2)=3 and S(4)=33 also end in a 3 but for a reason different from those for $n \geq 5$. This was a great problem. The technology was a handy way to collect a good chunk of data, but the problem didn't end with the data collection. I had to observe the pattern of units' digits, provide reasoning that this pattern would continue, and then justify that no partial sum ending with a unit's digit of 3 could be a perfect square.

Extensions of this problem could be to investigate a variety of partial sums using as the summands terms of familiar sequences. For example:

When is $1^2+2^2+3^2+\ldots+n^2$ a perfect square?
When is $1^2+1^2+2^2+\ldots+F_n^2$ a perfect square, where F_n is the nth Fibonacci number?

I could also extend the problem to look at sequences of numbers and determine if any term in a sequence is a perfect square. For example:

Are any triangular numbers perfect squares?
Which of the perfect cubes are also perfect squares?
Which of the Fibonacci numbers are perfect squares?

The essence of this problem is two-fold: perfect squares and factorials. I also learned a lot about taking partial sums of numbers. I was surprised that the command on the TI-92 was cumsum and not parsum.

Problem 9.2 – Which of the following polynomials can be factored over the integers? $x^2 + 1$, $x^4 + x^2 + 1$, $x^6 + x^4 + x^2 + 1$, $x^8 + x^6 + x^4 + x^2 + 1$, … ?

What is the pattern on the exponents in the polynomials? The first four polynomials are listed. What are the fifth and sixth polynomials in this listing? *What does it mean to say a polynomial is prime?* *What techniques for factoring polynomials do you know?* *Try factoring by grouping on the polynomial $x^6+ x^4+ x^2+1$. What do the first two terms $x^6+ x^4$ have in common?*	**Understand the Problem –**
*The **USE TECHNOLOGY** heuristic may be most effective on this problem as we suspect that these polynomials will grow quite large before we solve the problem. You might want to use a CAS like Derive or the built in functions on the TI-92.*	**Select a Strategy –**
Are there any groupings of polynomials you can immediately eliminate from the listing? For example, could you say the 2nd, 4th, 6th, etc. are all factorable? *Keep your data organized so that you can spot a pattern if one exists.*	**Implement –**

When working with the TI-92 you may need to experiment with the MODE and FACTOR functions. *Have you identified any patterns in your data? How can you prove (demonstrate) this pattern will hold?*	**Implement (cont'd)** -
What factoring techniques did you know or did you develop that helped you solve this problem? Does this problem remind you of any other problems you have solved? *What seems to be the essence of this problem?* *What extensions of this problem could you create that would use the essence of the problem?*	**Tie-Together** –

Strategy 10 – RESOLVE A SIMILAR BUT SIMPLER PROBLEM

Problem 10.1 – How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^{30} and 20^{20}?

What does it mean to be a divisor of another number?	**Understand the Problem** – A divisor in this problem must be an integer that divides into a larger (or equal to) number without remainder. If d is a divisor of at least one of the numbers 10^{30} and 20^{20}, then d must divide either 10^{30}, 20^{20} or both of the numbers.
What does "at least" mean? Is there a connection between the set operation $A \cup B$ and the phrase "at least?" *How many divisors does 6 have? How many divisors does 2^3 have?*	The conjunction "or" signals I am looking for the union of the two sets of divisors. But I don't want to double count any of them. If I were working with 10^{20} and 20^{20}, then all of the divisors of 20^{20} would also be divisors of 10^{20}. Some of the larger divisors of 20^{20} would not be divisors of 10^{20}. In that case the divisors of 10^{20} would be a subset of the divisors of 20^{20}. I will need to watch for overlap between the two sets of divisors for the numbers in the problem I am trying to solve. It may be helpful to write out the prime factorization of the two numbers I am using in this problem.
*The strategy of **RESOLVE A SIMILAR BUT SIMPLER PROBLEM** allows the problem solver to indirectly attack a difficult problem. Making up a similar but somewhat simpler problem improves your understanding of the problem and may help you develop tools or techniques you can apply to the original problem.*	**Select a Strategy** – The numbers in this problem are very large. If I consider the bases only, I notice that the prime factors of 10 and 20 are the same. The difference being the exponent on the 2 in the prime factorization of 20. Rather than working with such large exponents on the bases 10 and 20, I will RESOLVE A SIMILAR BUT SIMPLER PROBLEM by changing the exponents to 3 and 2 respectively. Seems like a minor change such as this will mimic and preserve relationships between 10^{30} and 20^{20} that I haven't even noticed yet. Other potential strategies: Establish a subgoal, brute force Related strategies: Try small n, look at cases

The number of divisors can be determined directly from the prime factorization of a number. Let p and q be primes. The number $p^a \bullet q^b$ has $(a+1)(b+1)$ factors.

Implement – My new, "easier" question is: How many positive integers n are there such that n is an exact divisor of at least one of the numbers 10^3 and 20^2? Writing out the prime factorizations of 10^3 and 20^2 will help me count how many divisors they each have. The number of divisors of at least one of these two will be less than or equal to the sum of the number of divisors of 10^3 and the number of divisors of 20^2. The prime factorizations of 10^3 and 20^2 are $2^3 \bullet 5^3$ and $2^4 \bullet 5^2$.

Since I am finding the cardinality of the union of sets, I may find I need to use PIE. PIE tells me that $\#(A \cup B) = \#A + \#B - \#(A \cap B)$. So I need to find $\#(A \cap B)$.

If I let A be the set of divisors of 10^3 and B be the set of divisors of 20^2 then $\#A=16$ and $\#B=15$. This means $\#(A \cup B) \leq 31$. Now I need to find out where A and B overlap so I can use PIE to find $\#(A \cup B) = \#A + \#B - \#(A \cap B)$. The prime factorizations of 10^3 and 20^2 have $2^3 \bullet 5^2$ in common. This is the greatest common factor of 10^3 and 20^2 and $2^3 \bullet 5^2$ has $4 \bullet 3=12$ factors. These twelve factors will be contained in the set $A \cap B$. Thus $\#(A \cup B) = 16 + 15 - 12 = 19$.

Double checking your work by solving a problem a different way is a good way of monitoring your accuracy and progress in a problem. Monitoring your progress is a form of metacognition or thinking about your thinking.

I'm going to count these a second way to double check my work before continuing. Which factors of 10^3 are not also factors of $2^3 \bullet 5^2$? Since 10^3 contained 5^3 and not 5^2, I need to count all of the factors of 10^3 that include the factor 5^3. There are 4 such factors $\{5^3, 2 \bullet 5^3, 2^2 \bullet 5^3, 2^3 \bullet 5^3\}$. None of these are divisors of 20^2 because the highest power of 5 dividing 20^2 is 5^2. Similarly, 20^2 contained 2^4 and not 2^3. Thus the 3 factors $\{2^3, 5 \bullet 2^3, 5^2 \bullet 2^3\}$ are divisors of 20^2 but not 10^3. This gives us a total of 12+4+3=19 divisors of either 10^3 or 20^2.

The simpler problem mimicked all of the important aspects of the original problem nicely. This match up does not always occur, but as you practice with this strategy you will get more adept at spotting the key factors you want your simpler problem to contain.

How has all this extra work helped me? I can determine the number of divisors of and the greatest common factor of 10^{30} and 20^{20}. Then using PIE, I can quickly calculate the number of integers that divides either 10^{30} or 20^{20}. I will now work on the original problem.

Now back to the problem at hand.

The prime factorizations of 10^{30} and 20^{20} are $2^{30} \bullet 5^{30}$ and $2^{40} \bullet 5^{20}$ respectively. If I let A be the set of divisors of

Can you double check my answer using the method I illustrated in the simpler problem?	10^{30} and B be the set of divisors of 20^{20} then $\#A=31^2$ and $\#B=41\bullet 21$. This means $\#(A\cup B) \le 31^2+41\bullet 21$. Now I need to find out where A and B overlap so I can use PIE to find $\#(A\cup B) = \#A + \#B - \#(A\cap B)$. The greatest common factor of 10^{30} and 20^{20} is $2^{30}\bullet 5^{20}$ which has $31\bullet 21$ factors. These factors will be contained in the set $A\cap B$. Thus $\#(A\cup B) = 31^2 + 41\bullet 21 - 31\bullet 21 = 31^2 + 10\bullet 21$ factors of either 10^{30} or 20^{20}.
Celebrate your victories with gusto! ☺ *Early metacognitive monitoring pays off with a higher level of confidence in the work you did.*	<u>**Tie-Together**</u> – This was a neat problem to solve. I had to use several techniques such as my technique for counting the divisors of a number based on its prime factorization and PIE. I was glad I developed a method for checking my reasoning in the simpler problem I made up. This gives me more confidence in my "final answer" even though I didn't double check it.
How do you find the GCD of two numbers using their prime factorization?	I was surprised that it was the divisors of the greatest common factor that were contained in the intersection of sets A and B. I also had to recall how to find the GCD of two numbers based on their prime factorizations.
How can you use the prime factorizations of numbers to find their LCM?	I could extend this problem by investigating the LCM as well. If I look at the set of divisors of the LCM, what is its relationship to the sets A and B. Just trying to create a problem involving this concept is a good extension of this problem. I also wonder what would have happened if I had numbers whose prime factors were different or that had more prime factors. I could extend the problem by having three different numbers instead of two.
What was special about the pairs of exponents chosen in this problem? *Try on your own 30^4 and 60^{20}. More PIE?*	I notice that the exponents on the prime factorization and the exponents on the bases of 10 and 20 were picked carefully to make sure the set $A\cap B$ was not trivial so the problem will be interesting. What other pairs of numbers could I pick that would have this same property?

Problem 10.2 – Demonstrate that the square root of $10^4 + 5\bullet10^{15} + 6$ is not an integer.

What might happen if you use a calculator to find the square root of a very large number? If you cannot use your calculator, how can you determine that the square root of a large, strangely configured number not an integer? *Is this number an integer? What do numbers that are perfect squares look like?*	**Understand the Problem** –
*Use the **MAKE UP A SIMILAR, SIMPLER PROBLEM** heuristic on this problem. What characteristics of this problem will you want to mimic, albeit in a simpler manner?* *State the simpler problem in the same manner as the original problem. Why do you think the problem you created will give you insight into this problem?* *Are there other strategies that might prove equally effective at solving this problem?*	**Select a Strategy** –

What are the characteristics of this problem you chose to mimic? How do they extract the essential elements? *Has creating a simpler problem given you insights into the original problem? What was the solution to your simpler problem?* *Could you work backwards and create a strangely configured, large number that is a perfect square?* *Would creating a second problem that is slightly more difficult but still simpler than the original problem help you explore the problem?*	**Implement** –
What seems to be the essence of this problem? *What did you learn as you solved this problem?* *Did you need to find the square root of a number numerically?* *Can you create a new problem that is similar to this one?*	**Tie-Together** –

Strategy 11 – PURSUE PARITY

Problem 11.1 – Given any set S of 5 lattice points in the plane, explain why there must exist a pair of lattice points in S so that the midpoint of the line segment connecting those two points is also a lattice point.

In the statement of the problem the term lattice point is used but not defined. Do you know the definition of this term? This may be a case where you would need to look up a definition to be able to completely understand the problem. *How do you find the midpoint of a segment given its endpoints?* *Why does the set need to contain 5 points? If there were only 4 would the statement be false?* *Can I resolve this by just considering examples?*	**Understand the Problem** – A lattice point in the plane is a point (m, n) whose coordinates are integers. A few examples of lattice points are (0, 2), (-4, 3) and (8, 8). I need to recall the formula for the midpoint of a line segment. For the points (a, b) and (x, y) the midpoint of the segment would be ($\frac{1}{2}(a+x)$, $\frac{1}{2}(b+y)$). I notice that the statement is not true if I have just three lattice points. None of the three midpoints between the points in the set S = {(1, 1), (2, 2), (3, 4)} are lattice points. What if I include the point (6, 3) in the set S? Would there be a midpoint that is a lattice point in this new set? Now I have six pairs of points to check, but none of these pairings have a lattice midpoint. In the examples I generated it seems like for one of the coordinates I am always getting an odd sum that I must divide by two, resulting in a fraction.
*The strategy of **PURSUE PARITY** allows the problem solver to simplify the problem by concentrating on the evenness or oddness of mathematical objects. Use the idea of evenness and oddness as a way to decide when repetitions in lattice point types are forced.*	**Select a Strategy** – Since I am using the PURSUE PARITY, I need to examine the different types of lattice points from a parity point of view. Once I have these types identified I can look at the nature of their midpoints and when the midpoint would be a lattice point. Other potential strategies: Convert words into mathematical notation, place in a more general setting Related strategies: Convert words into mathematical notation, place in a more general setting

	Implement – There are four types of lattice points from a parity point-of-view. Using E to represent even and O to represent odd these four classes are:
When using the PURSUE PARITY strategy, you need to remember to explain why the problem can be reduced to the basic parity classes you form in your argument.	(E, E) (E, O) (O, E) (O, O)
What happens when you find the midpoint of segments with points from these classes? Let me consider what happens when I try to find a midpoint of a segment whose endpoints are in the classes of (E, E) and (O, O) such as (2, 4) and (7, 5).	I need to look at what happens when I add odd and even numbers together, because I will need to find the sum of the coordinates when calculating the midpoints. I am going to look at a few specific examples before going on with a general argument. The points (2, 4) and (7, 5) come from the classes (E, E) and (O, O) respectively. The midpoint between these two points is (9/2, 9/2) which is not a lattice point. The points (2, 7) and (5, 4) come from the classes (E, O) and (O, E) respectively. The midpoint between these two points is (7/2, 11/2) which is also not a lattice point. I think if I had two points coming from the same class, then their midpoint would be a lattice point. For example, (2, 7) and (4, 5) both come from the same class, namely (E, O). Their midpoint is (3, 6) which is a lattice point! ☺
I am going to make some general statements about the parity of the sums of integers. This means I will need to prove my statements are true before I use them in the solution of my problem.	I am going to check all of the different parity sums. Recall that all even numbers have the form 2m and odd numbers have the form 2m+1. Note that 2m + 2n = 2(m + n). Since m and n are integers then m + n is also an integer. Thus the sum of two evens is an even, or notationally, E + E = E. Note that (2m +1) + (2n + 1) = 2m + 2n + 2 = 2(m + n + 1). Since m and n are integers then m + n + 1 is also an integer. So, the sum of two odds is an even, O + O = E.
	Looking at the mixed parity case we note that 2m + (2n + 1) = 2(m + n) + 1. Since m and n are integers then m + n is an integer and the number 2(m + n) + 1 is odd. Hence, the sum of an even and an odd integer is odd, or E + O = O. Since addition is commutative I do not need to examine the O + E case.

Implement (cont'd) - Apparently, when parities match the sum is even and when parities are mismatched the sum is odd. This means that to create a midpoint that is a lattice point, I need to have two points whose coordinates "match up" from a parity point-of-view. Meaning, the two points will need to come from the same class.

Now, given any set S of 5 lattice points, it is possible that there will be several ways to create this lattice point midpoint. However, by the pigeonhole principle I am sure that there is at least one such midpoint. This is because I have only four different classes of points forcing a minimum of two points coming from one of these classes.

Can this technique be applied to other, similar problems?

What was special about 5? You could experiment with a few arbitrary sets of 5 lattice points to see if you do get a midpoint that is also a lattice point.

Tie-Together – There were two themes for this problem. One was the use of parity and the other was the pigeonhole principle. I could extend this problem to look at $\Re^3$ instead of just the plane. Would a set of 5 points be sufficient in $\Re^3$ or would I need even more points? Just how many points would I need if I were working in $\Re^3$? Points in $\Re^3$ look like (x, y, z). The ordered triple (1, 2, 3) would be in the class (O, E, O). How many different classes would there be?

Problem 11.2 – Each point in the plane is colored red or green. Show that some rectangle has its vertices all the same color.

Can you really do such a coloring? This is a theoretical coloring that indicates the density of real points in the plane.	**Understand the Problem –**
*Use the **PURSUE PARITY** heuristic on this problem. What are the different classes of points this time? Is it also even versus odd?*	**Select a Strategy –**
Draw three parallel lines and then nine lines perpendicular to these three. Color the intersection points red or green. You may need to use the pigeonhole principle in your explanation.	**Implement –**

. *Can you generalize your argument? Make sure you draw sketches to support and clarify your explanation.*	**Implement (cont'd)** –
What, if any, underlying theme does this problem contain? *How is this problem related to other problems you have solved?* *Can you extend the problem? Can you create a similar problem that does not involve coloring but does use a parity or pigeonhole argument?*	**Tie-Together** –

Strategy 12 – CONVERT WORDS INTO MATHEMATICAL NOTATION

Problem 12.1 – Demonstrate that all six-digit numbers of the form abcabc are divisible by 13.

What do numbers like abcabc look like? Are they really divisible by 13? What does it mean to be divisible? *Can I resolve this by just considering examples? Since I am going to make some general statements about the divisibility of a class of numbers I will need to prove my statements are true in general.* *Does the dividend give me any hint as to the solution to the problem?* *The strategy of **CONVERT WORDS INTO MATHEMATICAL NOTATION** allows the problem solver to concentrate on the general principles illustrated by the problem rather than the detail in the statement of the problem.*	**Understand the Problem** – I am trying to look at numbers like 240240. The digit a must be restricted to being nonzero. The fact that numbers that look like abcabc are divisible by 13 is not obvious. I will test out a few using a calculator to build my intuition. $240240 \div 13 = 18480$ $129129 \div 13 = 9933$ $905905 \div 13 = 69685$ $732732 \div 13 = 56364$ It seems to be true that 13 does divide evenly into numbers that look like abcabc. There does not seem to be a particular pattern in the resulting dividend. **Select a Strategy** – Since I am using the CONVERT WORDS INTO MATHEMATICAL NOTATION, I need to see what in the problem may lend itself to mathematical notation. I will also want to see if the conversion into mathematical notation shines any light on the solution of the problem. Is there a general form for six-digit numbers that might be helpful in approaching this problem? I could convert the number abcabc into base ten notation and then see if any further simplification would help. I could also use voicing to see if the way I say a number would help me convert it into appropriate mathematical notation. The number 294,294 said out loud as 294 thousand and 294. This could be written mathematically as 294•(1000) + 294. I could then factor out the common term of 294 to get 294•(1001) and see where that leads me. Other potential strategies: Place in a more general setting Related strategies: Place in a more general setting

Base 10 converts the number by multiplying the digit by its place value.

Implement – I will start by converting the number abcabc into base 10. I get the following

I can use my knowledge of factoring by grouping. If I group the first three terms together they have a common factor of 10^3. The last three terms are the same as what is left after I factor 10^3 out of the first three terms. I can factor a second time.

$$\begin{aligned} abcabc &= a\bullet 10^5 + b\bullet 10^4 + c\bullet 10^3 + a\bullet 10^2 + b\bullet 10 + c \\ &= 10^3(\mathbf{a\bullet 10^2 + b\bullet 10 + c}) + \mathbf{a\bullet 10^2 + b\bullet 10 + c} \\ &= (10^3 + 1)(a\bullet 10^2 + b\bullet 10 + c) \end{aligned}$$

The second term in my factorization is just a three-digit number. Not all three-digit numbers are divisible by 13 for example 137 is not. This means the other term, 1001 is the significant factor.

I don't notice anything special about numbers of the form abc with regard to divisibility by 13. Some three-digit numbers are divisible by 13 and some are not. I also recognize the first term in the factorization above. In my strategize section I used voicing to break up the number 294,294 into the product 294•(1001). The first term from the above factorization $10^3 + 1 = 1001$ appeared in my voicing strategy! This may be the key to the problem's resolution.

Since $1001 \div 13 = 77$, if I can factor out 1001 from all six-digit numbers of the form abc,abc then the number is divisible by 13.

Tie-Together – Clearly the voicing technique I referred to during the strategize phase was the best approach to take. I should have pursued that strategy further before deciding to convert the number abc,abc into base 10. The essence of this problem seems to be factoring by grouping.

Numbers that look like abccba are called palindromes. Palindromes are words or numbers or sentences that read the same both forward and backward.

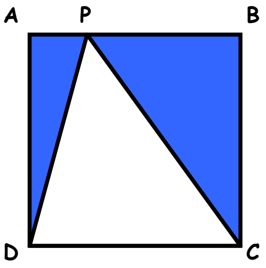
There seems to be a variety of potential extensions for this problem. Are the numbers abc,abc divisible by other integers? Is there a divisibility rule for 13 that could have helped me in the resolution of the problem? Could I create one? Are there other classes of numbers such as abc,cba that are also divisible by 13? Do the numbers have to be six-digit numbers or would numbers such as abab or ab,cda,bcd also be divisible by 13?

Problem 12.2 – The sum of three consecutive integers in an arithmetic progression is 24. The sum of the squares is 642. What are the three integers?

Is there sufficient information to find the numbers? Will the answer be unique? *What is an arithmetic progression?* *What does it mean to be consecutive integers in the arithmetic progression? Would they have to be like 1, 2, 3?* *How would you describe the following two expressions in words – $(a+b+c)^2$ and $a^2+b^2+c^2$? Are they equal?*	**Understand the Problem –**
*Use the **CONVERT INTO MATHEMATICAL NOTATION** heuristic on this problem.* *What will you need to use mathematical notation to represent?* *What property do all arithmetic progressions possess? Would you name the three integers a, b, and c or can you make use of the concept of arithmetic progression to reduce the number of variables you would have to introduce?*	**Select a Strategy –**
How did you represent your three integers? *How can you write the sum of their squares?*	**Implement –**

What techniques do you know that will help you solve the equation you have developed? *Are the three integers consecutive terms in an arithmetic sequence? What is the common difference?* *Does the sum of their squares equal 642?*	**Implement (cont'd)** –
Can you generalize your argument? Could you develop a method for constructing other problems that are identical to this one only with different numbers? *Was the particular value for the common difference significant in this problem? Could you do something similar for the terms in a geometric sequence?* *What seems to be the essence of this problem?*	**Tie-Together** –

Strategy 13 – WHAT HAPPENS IF …



Problem 13.1 – ABCD is a square with side length 8. A point P is placed on side AB. If segments DP and CP are drawn as shown, what is the area of the shaded region?

How can I find the area of the triangular shaded regions?

Why does the statement of the problem not specify the location of the point P? The location of P seems to be essential to being able to proceed.

WHAT HAPPENS IF… *is a peculiar title for a strategy and yet such open exploration is exactly what is needed in some problems. If you spot something in a problem that appears to be unrestricted or missing, you can explore the impact of its value or interpretation. Do the decisions you make change the general thrust of the problem or does the problem seem unaffected? If you find the problem is basically unaffected, perhaps the information is not needed.*

Try moving the point P and determine the area of the shaded region. What affect does the position of P have on the area?

Are there any particularly "good" locations for P's initial placement?

Understand the Problem – The location of point P is not specified. This problem would be very easy if I knew where P was located. I would need to find the areas of the two triangular regions using the formula $A=\frac{1}{2}b\bullet h$. I know the area of the shaded region will be less than the total area of the square, which is 64 units^2.

Select a Strategy – There seems to be missing data in this problem. I want the location of the point P. Perhaps I can use a WHAT HAPPENS IF approach, choose a couple of locations for the point P and calculate the area of the shaded region for each choice. I will then see what impact the location of P has on the shaded area. Maybe the shaded area ranges between two values depending on P's placement.

I might also use Geometer's Sketchpad to model the situation. If I set the sketch up correctly, all of the calculations will be made and tabulated for me by the program.

Other potential strategies: Create a table, place in a more general setting, establish a subgoal, exploit symmetry
Related strategies:

Implement – A "good" placement to choose for P is the midpoint of the side AB. The two triangles formed, ΔDAP and ΔCBP, are congruent by SAS and will have the same area. The height is 8 and the base is 4, making the area of a single triangle is $A=\frac{1}{2}\bullet 4\bullet 8=16$ units^2. This means the total area of the shaded region is $2A=2\bullet 16=32$ units^2.

The area is the same even though I changed the location of the point P. Will I always get half the area of the square? Could I prove this in general?

Drawing a line of symmetry really emphasized the fact that the shaded area is half the area of the square. I don't know why I didn't see it this clearly at the start of the problem.

Is there another way I could prove this? Was creating a table of values using GSP sufficient as a proof?

Creating multiple proofs of the same conjecture may seem redundant, but sometimes different proof techniques will illuminate different facets or themes contained in a problem.

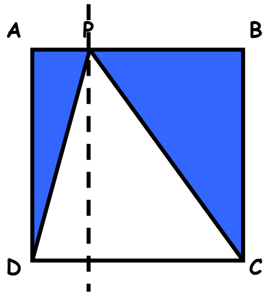
Observation and conjecture are important components but they are never the end to solving a problem. Unless you can prove your conjecture, you have not completely solved the problem.

The Principle of Inclusion-Exclusion could be used to solve this problem. If you subtract from the total area of the square the area of the unshaded region, you will get the area of the shaded regions as desired. To apply PIE, however, you must be willing to look at the piece you are "throwing away" instead of the piece you are interested in.

The themes in this problem are symmetry, congruent triangles, using technology and hidden assumptions.

<u>Implement (cont'd)</u> – A second place I could locate P is at point A. That would create only two triangles, one shaded and the other not. It is easy to see that the shaded region would be exactly half of the area of the square, which is the value I got for the other placement of point P.

I'm going to draw a line of symmetry through point P that is perpendicular to side AB. I notice that this line of symmetry will split the square into two rectangles. The segments DP and CP are diagonals of the two rectangles. One half of each of the two rectangles is shaded. This means that the area of the shaded region is exactly half of the square.



<u>Tie-Together</u> – Noticing that the area of the shaded region did not depend on where I placed P and that the area was exactly half the area of the square prompted me to prove it. I used a line of symmetry as part of my proof, but I could have used a similar triangle argument as well.

I notice now that I could have found the area of the non-shaded region instead. That triangle will always have the same height regardless of where I place its apex P. The base also remains unchanged because it is the length of side CD.

This is an important strategy I should remember: finding the part that is leftover instead of finding the part shaded.

I wonder if this would work with rectangles or if I could create a similar problem using circles that overlap.

Problem 13.2 – Given a point P somewhere inside the equilateral triangle ABC, determine the sum of the distances from P to the three sides.

How do you find the distances between the two points?

If you want to find the distance between the point P and a side of the triangle, where do you draw the segment that connects P to the side?

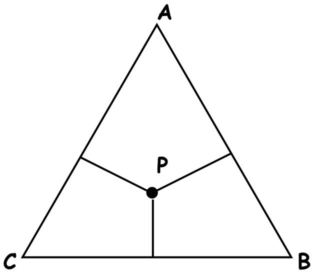
Won't the distance vary depending on where point P is placed? If the individual distances between the sides and the point P vary, won't the sum of the distances vary?

Does the side length of the triangle affect the sum of the distances?

*The wording of the problem provides you with a strong clue as to which strategy would be best to use. The **WHAT HAPPENS IF …** heuristic is the best approach for this problem. What in the statement of the problem is vague? (There is more than one piece of information that is vague.)*

What would happen if you moved the point P to a more desirable location? What is a "more desirable location" for the point P? Why is this "more desirable?" How does it alleviate some of your calculation burden?

Understand the Problem –



Select a Strategy –

Where did you choose to locate the point P? *Could you use Geometer's Sketchpad to help you out?* *Does the side length of the triangle matter?* *Would placing the point P so that it triangulates the figure be helpful?* *Does the symmetry of an equilateral triangle play a role in the solution of this problem?*	**Implement** –
What affect did the placement of the point P have on the sum of the distances? What affect did the side length of the triangle have on the sum of the distances? *Could you have done this problem if the triangle had been an isosceles triangle or scalene?* *What seems to be the essence of this problem?* *What extensions of this problem could you create? Can you think of an extension that involves higher dimensions?*	**Tie-Together** –

Strategy 14 – PLACE IN A MORE GENERAL SETTING

Problem 14.1 – Show that $9^{13} - 3^{13}$ is divisible by 6.

Questions to ask myself: What does it mean to be divisible? When would a number not be divisible by 6? Is there a test for divisibility by 6? How is the value 6 related to the difference in the problem?	**Understand the Problem** – An integer x is divisible by another integer y, if there is no remainder or if I can find a third integer z so that x = zy. The value 9^{13} means I have thirteen 9s multiplied together. Using the order of operations means I will need to first find both 9^{13} and 3^{13} before taking their difference. An integer is divisible by 6 if it is divisible by both 2 and 3.
The ***PLACE IN A MORE GENERAL SETTING*** *strategy allows the problem solver a chance to use his/her abstract mathematical training. You may need to introduce a variable. You could look at slope as the derivative or area as an integral. You could look at how polynomials can be factored or general formulas for geometric sequences. A calculator could resolve this problem quickly, although there would be a potential error from the rounding or algorithmic procedure the calculator would utilize. By placing the problem in a more general setting, I may be able to utilize the power of algebra or still appeal to technology only via a computer algebra system (CAS) like Derive.*	**Select a Strategy** – Since I am using the strategy PLACE IN A MORE GENERAL SETTING, does the form $9^{13} - 3^{13}$ remind me of any algebraic form? If I were to replace 13 with n then I would be looking at $9^n - 3^n$. If each of the powers of 9 and 3 contained a 6, then I could factor it out and be done. The lack of factor of 2 precludes this. However, $9^n - 3^n$ looks a lot like $x^n - y^n$. Now I can look at factoring this polynomial. Other potential strategies: Brute force, guess and check, convert words into mathematical notation, use technology Related strategies: Convert words into mathematical notation **Implement** – Replace 13 by n, 9 by x and 3 by y. The question now becomes show that x-y divides $x^n - y^n$. I can use either long division or a calculator with symbolic manipulation capabilities like the TI-92. I will use the long division route but practice on a few simple cases before tackling the nth case. $x^2 - y^2 = (x - y)(x + y)$

Implement (cont'd) -

$$\begin{array}{r|l} & x^2 + xy + y^2 \\ \hline x - y & x^3 - 0x^2y - 0xy^2 - y^3 \\ & x^3 - x^2y \\ \hline & x^2y \qquad - y^3 \\ & x^2y - xy^2 \\ \hline & xy^2 \quad - y^3 \\ & xy^2 \quad - y^3 \\ \hline & 0 \end{array}$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3)$$

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1})$$

When using the PLACE IN A MORE GENERAL SETTING strategy, you need to remember to return to the special case from the statement of the original problem.

The above factorization shows that $x^n - y^n$ is divisible by the difference $x - y$. Now replacing these variables with the appropriate values of 13, 9 and 3 we can see that our original problem is solved.

Did I accomplish my goal? Is the problem solved?

$$\begin{aligned} 9^{13} - 3^{13} &= (9 - 3)(9^{12} + 9^{11}\bullet 3 + 9^{10}\bullet 3^2 + \ldots + 9\bullet 3^{11} + 3^{12}) \\ &= 6 \bullet \text{(a really big integer)} \end{aligned}$$

Can this technique be applied to other, similar problems?

This strategy was effective for this problem; however, I needed to solve a simpler problem to ensure I was doing long division with polynomials correctly and to establish a pattern for me to use in the general case of long division.

Tie-Together – I now can solve an entire class of problems relatively quickly (i.e. they become exercises practicing a particular mathematical skill). For example, I know that 5 divides the difference $9^{17} - 4^{17}$. I could extend this by determining if 13 divides $9^{17} + 4^{17}$. This would require my investigating the factorization of the sum $x^n + y^n$ instead. I could also investigate combinations that involve both sums and differences such as $1^n + 8^n - 3^n - 6^n$.

The essence of this problem seems to be geometric sums. The factor $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1}$ is a geometric sum with a ratio of y/x. I could have used mathematical induction to prove my factorization as well. This problem was neat because it tied together the concepts of geometric sums, long division of polynomials, factorization of polynomials and divisibility of integers.

Problem 14.2 – How many of the numbers in the following infinite sequence are prime: 101, 1010101, 10101010101, 101010101010101, ... ?

What does it mean to be prime versus composite? How do factors play a role? *How many 1's are there in each number in the sequence?*	**Understand the Problem** –
Use the ***PLACE IN A MORE GENERAL SETTING*** *heuristic on this problem. What other strategies might be appropriate to solve this problem?* *What does factorable mean in algebra? Note that* $101=10^2+1$*. What could you replace 10 with? How could you rewrite 1010101? Is 101 prime? Will a calculator help?*	**Select a Strategy** –
State clearly your process indicating how you plan on following your selected strategy. State your goals, so that you can check them later.	**Implement** –

During the implement phase you need to keep your work organized so you know where to find it when you need it.

State your answer clearly. Exactly which numbers in the sequence are prime?

Test your process on 101010101. What features of a number indicate that it is not prime?

State the essence or mathematical core of the problem.

Can you create a new problem that is similar to this one that uses this same solution technique?

Can you generalize and extend the algebra?

Implement (con'td) –

Tie-Together –

Strategy 15 – ARGUE BY CONTRADICTION OR CONTRAPOSITIVE

Problem 15.1 – For the game below, identify the location of all of the ten mines. If a square cannot be clearly identified as containing a mine, then explain why its contents are unclear.

What are the rules of the game?	**Understand the Problem** – The object in the game Minesweeper © is to mark the locations of all of the mines in the minefield without setting any of the mines off.
What does the number inside a square mean? *What does it mean if a square doesn't contain a number?*	The number inside a given square tells you how many of the eight adjacent squares contain a mine. If a square is empty, then there were no mines in the adjacent squares.

☐	1					1	☐	☐
☐	1					2	☐	☐
☐	1	1	1		1	3	☐	☐
☐	☐	☐	1	1	2	☐	3	2
☐	1	1	1	1	☐	2	1	
☐	1			1	1	1		
☐	2		1	1	1			
☐	2		1	☐	1			
☐	1		1	☐	1			

How many mines do I have to mark?	For the beginner level, there are only 10 mines that need to be marked. There are 21 squares that need to be marked.
Sometimes it is very difficult to prove or show something directly. The strategy of ***ARGUE BY CONTRADICTION OR CONTRAPOSITIVE*** *allows the problem solver to indirectly attack a problem.*	**Select a Strategy** – Each square either contains a mine or it does not. Since the game pictured in incomplete, it is possible that I cannot determine whether or not a square contains a mine. The best approach to this problem would be to ARGUE BY CONTRADICTION.

Arguing by contrapositive means you must negate the conclusion and use it to prove the negation of the hypothesis. For example: if x^2 is odd then x is odd could be argued by trying to prove the statement – If x is even then x^2 is even.	First, I will identify those squares that contain mines based solely on the number of mines adjacent to the square. Then, I will use indirect reasoning to determine whether mines are hidden in the rest of the squares.
I need a way of identifying squares that remain unmarked.	**Implement** – I will use another old game, Battleship, to describe the unmarked squares. I will label the columns with the letters A, B, C, …, I and I will label the rows with the numbers 1, 2, 3, …, 9. That means the square in the uppermost left corner is A1 and the square in the lowermost right corner is I9. Right now, A1 is unknown and I9 is empty.
I will try to identify those unmarked squares that are adjacent to squares containing a 1 first.	Squares D5, E6, D7 and H5 each contain a 1 and have only a single unmarked adjacent square. Those four adjacent squares (C4, F5, E8 and G4) must each contain a mine. I will mark those squares in black.
If I know mines appear in those squares, then I can determine which squares cannot contain mines.	
It is essential that I follow the rules of logic and use only true statements in my reasoning. That way when I reach a contradiction I know that the only place where I could have "gone wrong" is when I assumed to the contrary.	Either E9 contains a mine or it does not. I will prove by contradiction E9 does not contain a mine. Proof: Suppose, to the contrary that E9 contains a mine. Then D9 is

adjacent to two mines. However, D9 is marked with a 1 and thus, it can only be adjacent to one mine (which is in E8). Therefore, E9 does not contain a mine.

Hint: Use the B3 and B5 as the squares that are marked with a 1 and already adjacent to a marked mine.

Using the same kind of argument, I can establish that A2, A3, A4, A5, A6 and B4 do not contain mines. I will mark these mine-free spots with white.

I can once again look for squares that are isolated and contain a 1. There are two such squares, namely B1 and B6. The adjacent squares, A1 and A7, must contain mines. I will once again mark these squares black. Marking these mines does not create any isolated squares, so I will look for squares that are marked with a 2 or 3.

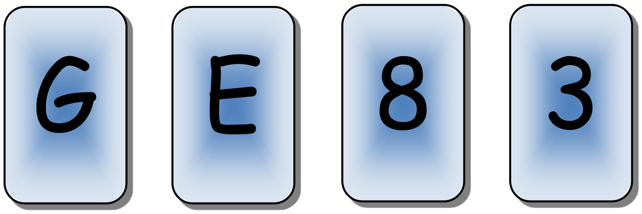
The square B7 contains a 2. The adjacent square, A6 is marked as containing a mine. Thus, the only unmarked square that could contain a mine that is also adjacent to B7 is A8. So, A8 must contain a mine. A8 is also adjacent to B9, which contains a 1. Thus A9 does not contain a mine. The squares G3 and H4 both contain a 3. They are currently adjacent to only 1 marked mine and have exactly two other unmarked adjacent squares, one of which is common. These three squares H2, H3 and I3 must each contain a mine.

There are now ten mines marked in the drawing. Thus the remaining three unmarked squares (H1, I1, and I2) must be mine free.

Tie-Together –

The official results of the game!

Problem 15.2 – Each of four cards has a single letter (A-Z) on one side and a single digit (0-9) on the other. Shuffle the cards and turn them up on a table. Suppose you see:



Now, consider the following theorem about the four cards: If a card has a vowel on one side, then it has an odd digit on the other. Which cards MUST be turned over to verify whether or not this theorem is true? For each of the cards explain why you decided whether to turn the card over or not.

If the theorem is a true statement, then what kinds of numbers can be on the back of the card that has a G on its front? *If the theorem is a true statement, then what kinds of letters can be on the front of the card with a 3 on its back?*	**Understand the Problem** –
*Use the **ARGUE BY CONTRADITION OR CONTRAPOSITIVE** heuristic on this problem.* *What is the "opposite" of a letter being a vowel?* *What is the hypothesis of the theorem? What is the negation of the hypothesis?*	**Select a Strategy** –

What is the conclusion of this statement? What is the negation of the conclusion? *What is the contrapositive of the original statement?*	
What facts can you assume in the beginning of your proof? *Could the theorem still be a true statement if you found that there was a consonant on the other side of the card with an odd number?* *What two possibilities could happen if you turned over the card with the E on its face? What would you know in each case?* *What two possibilities could happen if you turned over the card with the G on its face? What would you know in each case?*	**Implement** –
What seems to be the essence of this problem? *Do you find your proof convincing?* *Could you read it to another person and would your reasoning also convince them?* *What extensions of this problem could you create that would use either reasoning by contrapositive or the essence of the problem?*	**Tie-Together** –

Some Final Words

Using the US v IT format when writing up your problems may seem tedious. Consistently following this format however will make following this general heuristic for solving problems more natural and provide you with a ready made start when you face a particularly challenging problem. As you begin working on the problems in the next three sections, you may photocopy the next two pages as a template or create your own to suit your needs. The left column was included in the template so you can jot down the questions, emotional reactions (both irritations and elations), hunches, etc. that you have in your mind as you work on the problem. This reveals your thought processes to others and to yourself and serves as a meaningful record of how you attacked and eventually solved the problem.

Use the rest of this page to jot down characteristics of a good problem as you think of more and other potential heuristics you could use to solve problems.

Statement of Problem:

Metacognitive Banter (self-talk and questions to ask)	**Understand the Problem** – **Select a Strategy** –

Metacognitive Banter (self-talk and questions to ask)	**Implement** – **Tie-Together** –

PROBLEM SET 1 – SPECIFIED HEURISTICS

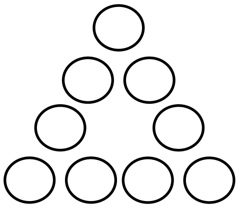
The following collection of 30 problems provides additional examples and practice with each of the 15 strategies. The numbering of the problems is consistent with those in the strategies section. Solve each problem. Make sure you identify the mathematical theme or essence of each problem and indicate how you might extend or generalize.

1.3. ACT IT OUT – Rene has 9 pockets and 44 silver dollars. She wants to put her dollars into her pockets so that each pocket contains a different (but positive) number of dollars. Can she accomplish this?

1.4. ACT IT OUT – How many different tips are possible using exactly three coins with exactly one penny, one nickel, one dime, one quarter, and one half-dollar available?

2.3. GUESS AND CHECK – Place the integers 1, 2, 3, … , 9 in the triangular array so that the three side sums are the same and so that

a.) the side sum is as large as possible.
b.) the side sum is as small as possible.



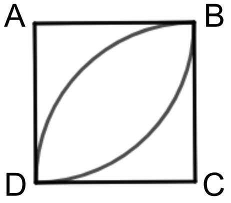
2.4. GUESS AND CHECK –Ekim Kasnay collects cats, beetles and worms. He has more worms than cats and beetles together. There are 12 heads altogether in the collection and 26 legs. How many cats does Ekim have?

3.3. BRUTE FORCE – How many positive 3-digit numbers abc are there such that $a + b = c$? For example, 202 and 178 have this property but 245 and 317 do not.

3.4. BRUTE FORCE – How many different isosceles triangles can you make if each side is a whole number length and the perimeter is 104 inches?

4.3. DRAW A PICTURE, MAKE A MODEL – Two cubes, A and B, have edges with integral lengths. The numerical value of the combined volumes of the cubes is equal to the combined lengths of all of their edges. What are the dimensions of A and B?

4.4. DRAW A PICTURE, MAKE A MODEL – An open box is to be formed from a 5 in. by 8 in. piece of tin by cutting off square corners of equal sizes and folding up the edges. What size square should be cut off of each corner in order to obtain an open box of maximum volume?



5.3. EXPLOIT SYMMETRY – A square has an area of 16 in^2. Two circles are drawn that overlap the square as shown. The vertices B and D of the square lie on the circumference of the circle and the vertices A and C are the centers of the circles. What is the area of the lemon shape region?

5.4. EXPLOIT SYMMETRY – How many four-digit palindromic numbers $abba$ are there?

6.3. ESTABLISH SUBGOAL – How many zeroes are there at the end of 126!? (Possible subgoal: How do you make a zero?)

6.4. ESTABLISH SUBGOAL – Subdivide an m by m square into one by one squares. How many squares of all sizes are there in this m by m square?

7.3. TRY SMALL n – Sally is having a party. The first time the doorbell rings, one guest enters. If on each successive ring a group enters that has 2 more guests than the group that entered on the previous ring, how many total guests will have arrived after the 20th ring?

7.4. TRY SMALL n – Show that every odd number $2n + 1$ can be written as the difference of two consecutive squares.

8.3. COLLECT DATA, SEARCH FOR PATTERN – Express as an integer the sum $1 + 3 + 5 + \cdots + n$, where n is the fiftieth odd positive integer.

8.4. COLLECT DATA, SEARCH FOR PATTERN – Determine a closed formula for the rising and falling sum of odd numbers where $2n - 1$ is the "peak":

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) + \ldots + 5 + 3 + 1$$

9.3. USE TECHNOLOGY – The five digits *a*, *b*, *c*, *d*, and *e* of 55225 are such that $a = b = e$ and $c = d$; in addition, $55225 = 2352 = (235)(235)$. Find another positive integer m such that m^2 is also a five-digit number $abcde$ that satisfies $a = b = e$ and $c = d$.

9.4. USE TECHNOLOGY – Which whole numbers are equal to the sum of the factorials of their digits? Here is one: 40585 = 4! + 0! + 5! + 8! + 5!. Find the other three answers. (Hint: one is a 3 digit number.)

10.3. RESOLVE A SIMILAR BUT SIMPLER PROBLEM – Let $R_n = 1^n + 2^n + 3^n + 4^n$. For how many n between 1 and 100 inclusive is R_n a multiple of 5?

10.4. RESOLVE A SIMILAR BUT SIMPLER PROBLEM – A group of students is sitting equally spaced forming a circle. The 43rd student is directly opposite the 89th student. Determine the number of students.

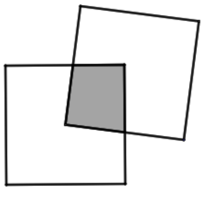
11.3. PURSUE PARITY – If n is a positive integer, then $n^3 – n$ is even.

11.4. PURSUE PARITY – How many lattice points would be needed in the plane in order to guarantee that a lattice point would exist on the segment joining some two of these points?

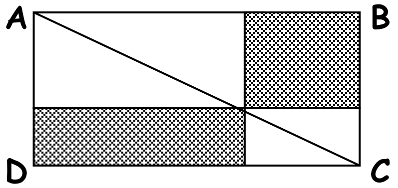
12.3. CONVERT WORDS INTO MATHEMATICAL NOTATION – A water main for a street is being laid using a particular kind of pipe that comes in either 18-foot sections or 20-foot sections. The designer has determined that the water main would require 14 fewer sections of 20-foot pipe than if 18-foot sections were used. What is the total length of the water main?

12.4. CONVERT WORDS INTO MATHEMATICAL NOTATION - The sum of two numbers is 7 and their product is 25. Determine the sum of their reciprocals. What is the "hard" way to do this problem?

13.3. WHAT HAPPENS IF… - Place two squares, each having side length s, so that a corner of one square lies on the center of the other. Determine the range of the possible area representing the intersection of the two squares.



13.4. WHAT HAPPENS IF… - In the rectangle ABCD what is the ratio of the one shaded area to the other shaded area?



14.3. PLACE IN MORE GENERAL SETTING – Give a convincing argument that $5^{17} - 1$ is a multiple of 4.

14.4. PLACE IN MORE GENERAL SETTING – Determine a closed expression for the infinite series $1 + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right)^3 + \cdots$.

15.3. ARGUE BY CONTRADICTION OR CONTRAPOSITIVE – If the product *mn* is odd, then both *m* and *n* are odd.

15.4. ARGUE BY CONTRADICTION OR CONTRAPOSITIVE – Let *n* be a positive integer. If n^2 is divisible by 3, then *n* is divisible by 3.

PROBLEM SET 2 – MIXED HUERISTICS USING CONTENT FROM THE LATE ELEMENTARY AND MIDDLE SCHOOL LEVEL

As you try this mixed collection of problems you should identify a strategy, apply it and after you have completed the problem try to identify other strategies that could have been used, and finally give the underlying mathematical theme. Remember: Sometimes one strategy will work better than another, and sometimes it is just a matter of taste and which strategy you are most comfortable with. As the master problem solver George Pólya said, you can solve these problems if you are willing to work hard, and if you have a lot of luck.

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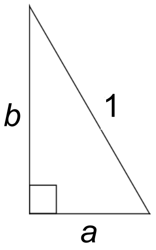
1. Consider the positive integers that have all of their digits restricted to the set {2, 3, 4, 5, 6, 7, 8}. In their natural order these integers form the unending sequence: 2, 3, 4, 5, 6, 7, 8, 22, 23, 24, 25, 26, 27, 28, 32, 33, …
    a.) What number follows 888?
    b.) What is the 2817th number in the sequence?

2. A collection of 25 consecutive positive integers adds to 1000. What are the smallest and largest integers in this collection?

3. Starting with 8, arrange the fifteen integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 in a row so that the sum of any two adjacent integers is a perfect square.

4. The right triangle has hypotenuse of length 13 and legs *a* and *b*. If the area of the triangle is 14, what is the sum $a + b$?

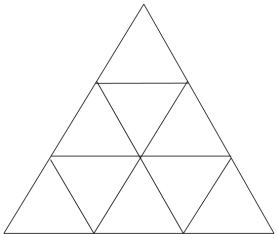


5. An unending arithmetic progression is generated by adding a fixed number to one term to obtain the next term. Find the smallest integer that is common to each of the following two arithmetic progressions of positive integers:

    4, 11, 18, 25, 32, …
    5, 13, 21, 29, 37, …

6. Determine which even numbers can be expressed as the difference of two squares. Prove your result.

7. The equilateral triangle of side length 3 is subdivided into 9 smaller triangles. Place the integers 1, 2, 3, 4, 5, 6, 7, 8, 9 in them so that the sum of all four numbers in any of the equilateral triangles with side length 2 is the same. What is the <u>smallest</u> value of this sum?

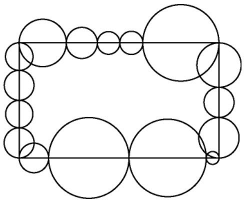


8. What number should be removed from the list 1, 2, 3, 4,…, 21 so that the average of the remaining numbers is $10\frac{3}{4}$?

9. Fill in the four missing numbers so that after the first two terms each number is the sum of the two before it:

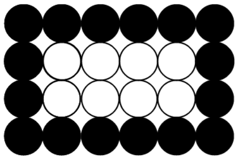
4, _____, _____, _____, _____, 47

10. The rectangle in the picture has dimensions 4 by 7. Determine the sum of all the circumferences of all the circles as drawn. The center of each circle lies on the rectangle.

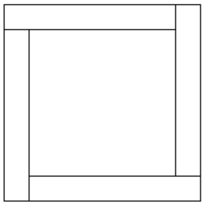


11. Fifty tickets numbered consecutively from 1 to 50 are placed in a jar. Two are drawn at random (without replacement). What is the probability that the difference of the two numbers is 10 or less?

12. The picture shown is made by making a rectangle of white disks and then creating a border of black disks around the rectangle. Give the dimensions of the outside rectangle for <u>all</u> such configurations where the number of black disks equals the number of white disks. (The picture shows a configuration with a 4 by 6 outside rectangle that does <u>not</u> work since it uses 8 white disks and 16 black disks.)



13. A square is surrounded by four congruent rectangles as is depicted in the figure to the right. If each of the rectangles has a perimeter of 18 units, what is the total sum of the areas of the four rectangles and interior square? If the dimensions of the rectangles are restricted to integer values, what are the four possible areas for just the interior square?



14. Determine those integers *n* for which $n^2 + n + 1$ is a perfect square.

15. The sum of the lengths of the three sides of a right triangle is 90 in. The sum of the squares of the lengths of the three sides is 3362 sq. in. Determine the area of the triangle.

16. Choose a number at random from {1, 2, 3, …, 100}. What is the probability that it has an odd number of divisors?

17. Express $\left(1-\frac{2}{3}\right)\cdot\left(1-\frac{2}{5}\right)\cdot\left(1-\frac{2}{7}\right)\cdot\left(1-\frac{2}{9}\right)\cdot\ldots\cdot\left(1-\frac{2}{211}\right)$ as a fraction $\frac{a}{b}$.

18. The integer *n* is 124 less than one perfect square and 56 less than another perfect square. Determine *n*.

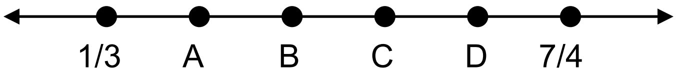
19. To swim a mile in a certain rectangular swimming pool, one must either swim the long length 80 times or negotiate the perimeter of the pool 22 times. What is the total number of square yards in the area of the pool?

20. Determine the digit in the 623rd place after the decimal point in the repeating decimal for the sum $\frac{1}{9}+\frac{2}{99}+\frac{3}{999}$.

21. Suppose you have three uncalibrated cooking pots, one holding exactly 5 quarts, one 3 quarts and one 8 quarts. If the 8 quart pot is full (of water) show how you can measure exactly 4 quarts using the three pots.

22. Previously you made a magic square using addition. Can you make a 3 by 3 multiplicative magic square? All nine entries must be different but they need not be 1, 2, …, 9.

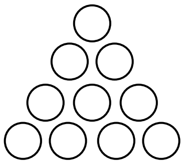
23. The six points shown are evenly spaced on the number line. What reduced fraction corresponds to the point *C*? Express your answer as a fraction $\frac{a}{b}$.



24. There are 720 permutations of the digits 1, 2, 3, 4, 5, 6. If they are arranged in order of size when interpreted as 6 digit numbers, from the smallest 123456 to the largest 654321 what is the 358th 6 digit number in that list?

25. A post office vending machine dispenses 4¢ and 9¢ stamps. What is the greatest amount of postage you are unable to make using just these two values? (e.g. you cannot make 7¢).

26. Ten pennies are arranged in a triangle. By moving only three pennies, make the triangle point in the opposite direction.



27. You are given two pieces of rope and told that each will burn for exactly one hour. The rope does not burn at a constant rate. Explain how you can measure 30 minutes. Can you measure 15 minutes? Are any other durations possible to measure?

28. Waldo ate 100 peanuts over a period of five days. On each of the last four days he ate six more than the day before. How many peanuts did Waldo eat on the first day?

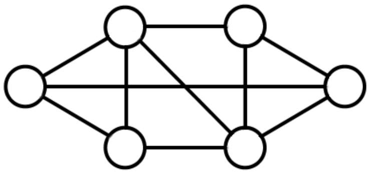
29. The letters *a*, *b*, *c*, *d*, and *e* represent natural numbers.

   a.) Given that $\{a, b, c\} = \{1, 2, 3\}$ what is the largest possible value of $ab + bc + ca$?

   b.) Repeat with the set $\{a, b, c, d\} = \{1, 2, 3, 4\}$ and find the maximum of $ab + bc + cd + da$.

   c.) Repeat with the set $\{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}$ and find the maximum of $ab + bc + cd + de + ea$.

30. Place one of the numbers $\frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, 1\frac{2}{3}, 2$ in each of the circles so that the two numbers on each line differ by more than $\frac{1}{3}$.



31. Find a 5 digit number such that the leftmost digit tells how many zeros are in the number, the second digit tells how many ones are in the number, the third how many twos, the fourth how many threes and the rightmost digit tells how many fours. Repeat the problem for a 10 digit number.

32. Sally is having a party. The first time the doorbell rings, one guest enters. If on each successive ring a group enters that has 2 more guests than the group that entered on the previous ring, how many total guests will have arrived after the 20th ring?

33. Study the triangular arrangement to the right. Suppose the pattern is continued.

- a.) In which row will the integer 45 appear?
- b.) Where will 999 appear?
- c.) Determine the sum of the integers on the 50th row.

1
2 3
4 5 6
7 8 9 10
:
:

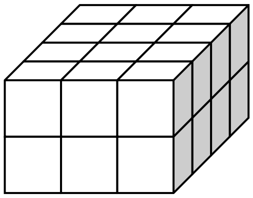
34. Consider the triangle made up of the odd numbers.

- a.) In which row will 169 appear and where?
- b.) What are the row sums? Explain your answer.

1
3 5
7 9 11
13 15 17 19
:
:

35. Let *S* = {1, 2, 3, 4, 5}. Determine the probability of selecting a subset of *S* that consists of consecutive integers. Clarifications: You should count {2, 3} but not {2, 4, 5}; any singleton subset satisfies the condition.

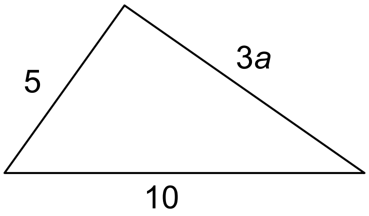
36. Suppose a 3 by 2 by 4 rectangular prism, constructed using twenty-four 1″ cubes, is dipped in paint and then disassembled.



- a.) How many cubes would have three faces covered with paint?
- b.) How many cubes would have two faces covered with paint?
- c.) How many cubes would have one face covered with paint?
- d.) How many cubes would not be painted?

37. Ten volumes of a set of encyclopedias are sitting on a shelf. They are numbered #1 to #10 left to right. The thickness of each cover is 0.5 cm; each page is 0.03 cm thick. The pages in each volume are numbered 1-1000. If a bookworm is on page 1 of volume #1 and eats his way through page 1000 of volume #10, what distance does it travel?

38. The lengths of three segments are 5, 10, and 3*a* units long. Determine three integer values of *a* so that 5, 10, and 3*a* are the measures of the sides of a triangle.

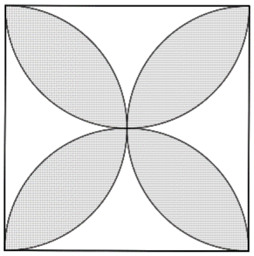


39. Explain how you can make a 3 by 3 magic square using any 9 consecutive integers.

40. Color each point in the plane red or green. Prove that there must be two points exactly one mile apart that are colored the same.

41. Given a chain with 7 links what is the fewest number of links that need to be cut so that you could hand a person any number of links from 1 to 7? A cut link is counted as a link (weld it back together after you remove it). Repeat this problem with 23 links.

42. A square has an area of 16 in^2. Four circles are drawn each having as its diameter one of the four sides of the square. The intersections of the four circles in the interior of the square forms four lemon shaped regions. What is the total area of the four lemon shaped regions?



43. What is the smallest multiple of 30 that has exactly 36 divisors?

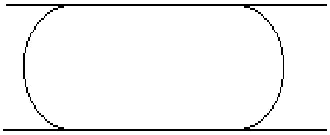
44. Which of the natural numbers less than 200 have exactly 3 factors (including 1 and the number itself)?

45. Let $S_n = \{1, 2, 3, \ldots, n\}$. For some *n*, the set S_n can be split into two disjoint sets so that the sums of the elements in the two sets are equal. For example, when $n = 3$ the set $S_3 = \{1, 2, 3\}$ can be split into the two sets {1, 2} and {3} and the sum of the elements in each of the sets is 3.

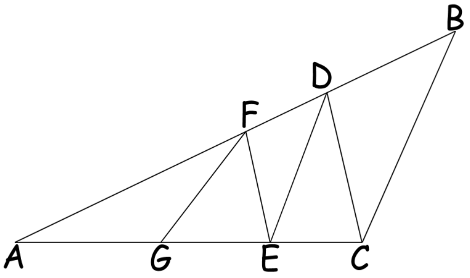
- a.) Create a splitting for the set $S_4 = \{1, 2, 3, 4\}$.
- b.) What is the next value for n that allows S_n to be split in such a manner?
- c.) For which *n* will you be able to split S_n?

46. Through how many squares will a diagonal pass in an m by n rectangle that is subdivided into 1 by 1 squares?

47. When a circle of radius 2 cm is "squeezed" between two parallel lines 2 cm apart, as shown, its area changes but its perimeter remains constant. What fraction of the old area is the new area if the "ends" are semicircles?

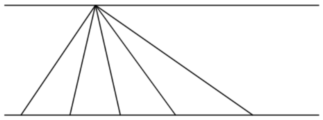


48. Describe how the broken line CDEFG can be drawn in ΔABC so that the five (small) triangles obtained have the same area.



49. How many ordered pairs (a, b) are there with $a > b > 0$ and each of a and b are integers such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{8}$. Hint: Use the Add In – Subtract Out technique.

50. How many different triangles are there in the picture? Generalize.



51. A 104 in. long piece of wire is cut into two pieces, and two squares are formed. The sum of the areas of the two squares is 388 sq. in. If the sides have integral lengths, what are the lengths of the sides of each square?

52. How can a chain with 63 links be cut in three places so that you could hand a person any number of links from 1 to 63 (a cut link is counted as a link). What formula relates n, the number of cut links, to the maximum length of the chain?

53. In the pattern to the right we see that cubes can be written as sums of consecutive odd integers.

$$1^3 = 1$$
$$2^3 = 3 + 5$$
$$3^3 = 7 + 9 + 11$$
$$4^3 = 13 + 15 + 17 + 19$$
$$\vdots$$

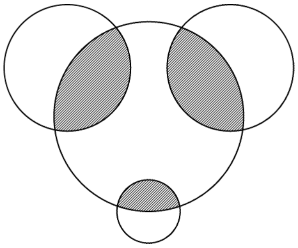
a.) Express 5^3 in similar manner as demonstrated by the pattern in the figure.

b.) State and prove a formula for n^3.

c.) Illustrate your formula geometrically with cubic blocks.

54. Determine the total number of 5's preceding the 98th 2 in the decimal 0.52552555255552…?

55. The circles to the right have radii 6, 4, 4, and 2. How does the unshaded area inside the largest circle compare with the unshaded area outside the largest circle, assuming that the small circles are disjoint?

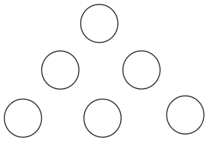


56. Each student in a classroom shakes hands with everyone else exactly one time. How many students are there if there were a total of 91 handshakes?

57. Suppose that *n* rooms are located along a long corridor, numbered consecutively from 1 to *n*. The *n* guests proceed in the following action: Guest 1 opens all the doors, guest 2 closes every second door beginning with door 2, guest 3 changes the position of every third door starting with the third door, guest 4 changes positions of doors 4, 8, 12, … Continue this until each of the *n* guests has walked the length of the corridor. Guest *n*, of course, merely strolls the corridor and changes the position of door *n*. Which of the doors were left open at the end of the process?

58. Using the integers 1, 2, 3, 4, 5, 6 two players alternate in selecting numbers, one at a time, until the sum reaches 50. The winner is the person who first reaches exactly 50. Describe a winning strategy.

59. Place 1, 2, 3, 4, 5, and 6 in the circles so that each side of the triangle adds up to 10.



60. One of nine gold coins is counterfeit and weighs less than the other eight. Using a balance scale, how can you determine which coin is the light one? Can you do it in one weighing? Two weighings?

61. The difference between the squares of the two consecutive odd counting numbers is 800. What are the numbers?

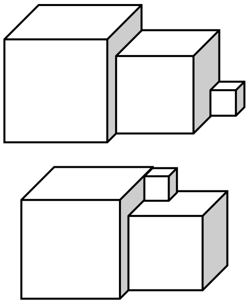
62. One triangle has side lengths 5, 5, and 6. Another has side lengths 5, 5, 8. How are their areas related? Find another pair of noncongruent isosceles triangles with integer sides where this happens.

63. Which integers have the property that they can be expressed as the sum of a string of consecutive positive integers? For example, 22 = 4 + 5 + 6 + 7. Sometimes there are several ways: 18 = 5 + 6 + 7 and 18= 3 + 4 + 5 + 6.

64. Consider the triangle to the right.
- a.) What are the next three rows?
- b.) How is the triangle made?
- c.) What are the row sums?
- d.) Where do the perfect squares appear in the triangle?

1
2 2
3 4 3
4 6 6 4
5 8 9 8 5
:
:

65. Three cubes whose edges are 2, 6, and 8 centimeters long are glued together at their faces as shown.
- a.) What is the surface area for the figure?
- b.) What is the surface area if the smallest cube is placed on top of the medium cube and glued to both the medium and the large cubes, as shown?



66. Determine all positive integers *a* and *b* whose product is equal to their sum. Repeat this problem on find those integers *a* and *b* whose product is equal to twice their sum.

67. An ant crawls around the outside of a square, whose sides measure 1 inch, to mark its territory. At all times, the ant remains exactly one inch from the boundary of the square.
- a.) What figure does the ant's path trace?
- b.) Determine the area of the ant's territory.

68. Consider the arrangement (to the right) of the integers greater than 1.
- a.) In which column will 100 fall?
- b.) In which column will 1000 fall?
- c.) In which column will 2011 fall?

A	B	C	D	E
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
	⋮	⋮	⋮	

69. Find unequal positive integers *m* and *n* so that $\frac{1}{m}+\frac{1}{n}=\frac{1}{5}$.

70. Consider the three integers 2718, 3875, and 4486. Find the largest integer *n* which yields the same remainder when each of these three integers is divided by *n*. As an example, the largest integer *n* for the three integers 12, 17, and 37 is $n = 5$.

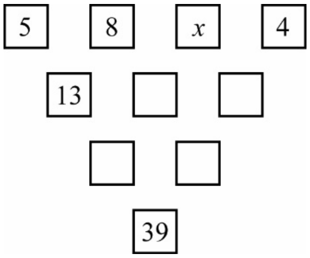
71. The arithmetic mean of a set of 50 numbers is 32. The arithmetic mean of a second set of 70 numbers is 53. What is the arithmetic mean of the sets combined?

72. Alyson purchased 100 farm animals for a total of $4500. Calves cost $120 each, goats $50 each, and lambs $25 each. If Alyson obtained at least one animal of each type, how many of each type did she purchase?

73. A jar contains a bunch of pennies, nickels, dimes and quarters. The average value of all these coins is 11¢. If you toss in one more nickel and three more pennies, the average value drops to 7¢. How many dimes were in the jar to begin with?

74. What is the 2011th number in the following sequence:
1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, …?

75. The number in an empty square box is found by adding the two numbers in the row directly above. What is the value of *x*? As shown, 5 + 8 = 13.



76. Find three different odd positive integers *a*, *b* and *c* so that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{3}$.

77. A unit fraction is a fraction of the form $\frac{1}{a}$, where *a* is a positive integer. Express $\frac{5}{11}$ as a sum of three different unit fractions with odd denominators. That is, find different odd integers *a*, *b*, and *c* so that $\frac{5}{11} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

78. The integer 12 has six positive divisors. They are 1, 2, 3, 4, 6 and 12. Determine the number of positive divisors of 1008; including 1 and 1008.

79. Express the following sum as a whole number:

$$1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + 10 + 11 - 12 + \cdots + 2005 + 2006 - 2007.$$

80. What is the 2011th number in the following sequence:

1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, …?

81. Six friends are sitting around a campfire. Each person in turn announces the total of the ages of the other five people. If 104, 105, 108, 114, 115, and 119 gives the six sums of each group of five people, what is the age of the oldest person?

82. In the number spiral, give the next <u>three</u> numbers that appear in the third row to the right of the numbers 11, 2, 1, 6, 19.

13	14	15	16	17
12	3	4	5	18
11	2	1	6	19
10	9	8	7	20
		…	22	21

PROBLEM SET 3 – MIXED HUERISTICS USING CONTENT FROM THE LATE MIDDLE SCHOOL AND HIGH SCHOOL LEVEL CONTENT

Identify a reasonable strategy (or two), implement it, identify the theme and indicate potential extensions, generalizations and connections to other problems. Those problems with an asterisk * are particularly challenging.

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1. Determine all positive integers n such that $1 + 3^n + 3^{2n} = 757$.

2. A set of consecutive integers beginning with 1 is given. One of the numbers is erased and the arithmetic mean of the remaining numbers is $35\frac{7}{17}$. Which number was erased?

3. How many rectangles are there in an m by m square? How many rectangles are there in an m by n rectangle?

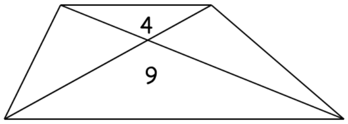
4. Determine all positive integers a, b such that $\left(1+\frac{1}{a}\right)\bullet\left(1+\frac{1}{b}\right)=\frac{3}{2}$

5. If m + n = 5 and mn = 2 determine the value of:
   - a.) $m^2 + n^2$
   - b.) $m^3 + n^3$
   - c.) $m^4 + n^4$

6. An integer is called decreasing if each digit is less than the one to its left. For example 6320 is a 4-digit decreasing number.
   - a.) How many 3-digit decreasing integers are there? That is, how many decreasing integers occur between 100 and 999?
   - b.) How many 4-digit decreasing integers are there?

7. Ten stools are arranged in a row. Three friends go in each day and always sit so that at least one stool is between two people. In how many ways can the stools be occupied, assuming the friends are indistinguishable (assume they were identical triplets)? For example, if there are just five stools, the following diagram shows that there is just one way of sitting: ⊗**O** ⊗ **O** ⊗. Generalize to n stools.

8. A trapezoid is divided into four triangles by its diagonals. Two of the areas are given. Determine the total area of the trapezoid.

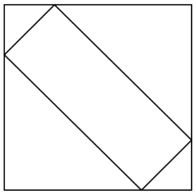


9. Factor each as a product of two quadratics:

a.) $n^4 + 4$

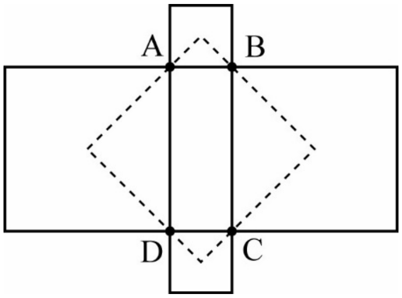
b.) $n^4 + 6n^3 + 11n^2 + 6n + 1$

10. Inscribe a rectangle in a square so that each of the four triangles formed is isosceles. Determine the length of the diagonal of the rectangle if the sum of the areas of these four triangles is 200 square inches.



11. For which n is $x^{2n} + x^n + 1$ divisible by $x^2 + x + 1$?

12. Place a square on each side of rectangle ABCD. Connect the centers of each of these four squares forming the quadrilateral shown with the dotted segments. Find the area of this quadrilateral, if the dimensions of rectangle ABCD are 3 and 8.



*13. A five-digit number is a "peak number" if the first three digits are in strict <u>ascending</u> order and the last three are in strict <u>descending</u> order. The middle digit is the peak. For example, 24840 is a "peak number" but 14722 and 32752 are not.

a.) Determine the number of five-digit "peak numbers ".

b.) Determine the number of five-digit "peak numbers " that do <u>not</u> contain zeros. (Try to express your answer in terms of binomial coefficients).

c.) Extend this problem to seven-digit numbers.

14. Express tan(nx) in terms of tan x for n = 1, 2, 3, 4, 5. Generalize and try to prove your generalization.

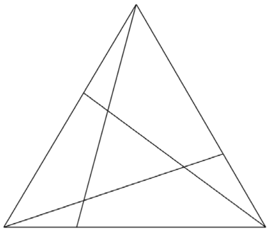
15. Given that n is a natural number:

a.) For which n is $1^2 + 2^2 + \ldots + n^2$ a perfect square?

b.) For which n is $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \ldots + \binom{n}{2}$ a perfect square?

16. Demonstrate that the set of all integers that are sums of two perfect squares is closed under multiplication. For example, if $m = 2^2 + 3^2 = 13$ and $n = 1^2 + 5^2 = 26$, $mn = 13 \cdot 26 = 7^2 + 17^2$.

*17. A cevian of a triangle is a line drawn from a vertex to the opposite side. Let R(n) denote the maximum number of regions formed within a triangle by drawing n cevians from each vertex. The picture for the case n = 1 is shown. Determine, with proof, a formula for R(n). Generalize.



18. The product $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$ is called n factorial. For which values of n is $n! + 1$ a perfect square?

19. Determine the number of positive integers less than 1 million whose digits are in strictly increasing order. Single digit numbers automatically satisfy this condition.

20. Discover properties of the triangle whose first six rows are given. Try to prove your conjectures.

1
1 2
1 3 2
1 4 5 2
1 5 9 7 2
1 6 14 16 9
2
:

*21. Suppose you have 15 envelopes and 15 letters addressed to match. Closing your eyes, you randomly stuff one letter into each envelope. In how many ways can you insert precisely 2 letters into the wrong envelopes (and the remaining 13 into their correct envelopes)? What if you have 30 envelopes, and 30 letters and precisely 3 are inserted incorrectly?

22. Consider the "integer triangle." Give, with proof, a formula for:

1
1 3 1
1 3 5 3 1
1 3 5 7 5 3 1
1 3 5 7 9 7 5 3 1
:

a.) The sum of the elements in the nth row.
b.) The sum of all the elements through the nth row.
c.) Determine the other properties of this triangle.

23. An isosceles trapezoid has sides 15, 28, 15, and 52.
   - a.) Determine its area.
   - b.) Determine the length of its diagonal.

24. Determine the next two rows of the triangle. What is the sum of the integers in the n^{th} row? List other properties.

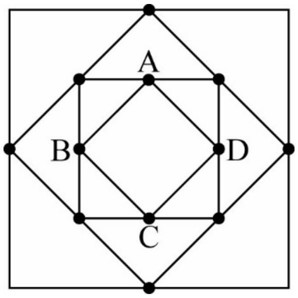
1
1 2
1 4 4
1 6 12 8
1 8 16 16 16
:
:

25. Determine a compact expression for each of the infinite sums below.

a.) $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\dots+\frac{1}{T_n}+\dots$ (where T_n is the n^{th} triangular number)

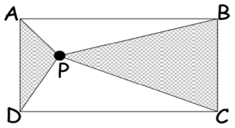
b.) $\frac{1}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\dots+\frac{n^2}{2^n}+\dots$ (start with $1/(1-x)=1+x+x^2+x^3+\dots$)

26. Consider a square having side length 40 with three successive inscribed squares formed by connecting midpoints of sides of previous squares.
   - a.) What is the perimeter of the smaller square ABCD?
   - b.) What is the area of square ABCD?



27. The set Z_7 consists of the integers 0, 1, 2, 3, 4, 5, 6. Show that if you add and multiply in Z_7 using ordinary addition and multiplication *but* reduced modulo 7 (that is, take the remainder upon division by 7) then $x^7 - x = x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$. (Hint: Avoid brute force.)

28. Where should the point P be placed in the rectangle BCD to maximize the shaded area?



29. Determine a closed formula for the sum $\ln\frac{2}{3}+\ln\frac{3}{4}+\ln\frac{4}{5}+\cdots+\ln\frac{n}{n+1}$. Here, ln stands for the natural logarithm.

30. Determine a closed formula for each of the following sums:
   - a.) $1\cdot 2+2\cdot 3+3\cdot 4+\dots+n(n+1)$
   - b.) $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\dots+n(n+1)(n+2)$.

31. Determine a closed formula for

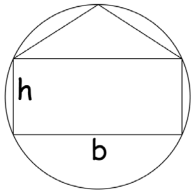
$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\ldots+\frac{1}{n(n+1)(n+2)}.$$

32. The odometer of a car shows 15951 miles, a palindromic number. The driver said, "It will be a long time before that happens again." But, two hours later the odometer showed a new palindromic number. What was the average speed during those two hours?

*33. (a.) How many of 1001, 1001001, 1001001001, … are prime?
(b.) Prove that there are no prime numbers in the infinite sequence of integers: 10001, 100010001, 1000100010001, …

34. A line segment joining the points (0, 120) and (200, 0) contains how many points having both coordinates as integers? In your count, include the two endpoints (0, 120) and (200, 0).

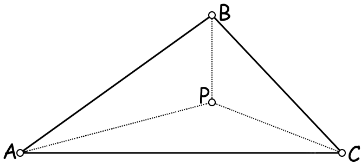
35. Inscribe a rectangle having base b and height h and an isosceles triangle of base b in a circle of radius 1 as shown in the figure. For what value of h do the rectangle and triangle have the same area?



36. How many different fractions $\frac{m}{n}$ can you make if m and n are positive integers, m < n, m + n = 665, and also, each fraction is reduced to lowest terms? Generalize.

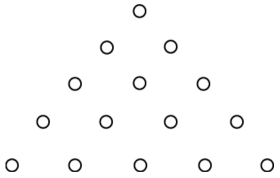
37. Six points, no 3 collinear, are drawn in a plane. The fifteen line segments joining them in pairs are drawn and then painted, some red, some blue. Prove that some triangle (whose vertices are among the six points) has all its sides the same color.

38. Given any triangle ABC, determine the location of point P so that the areas of the three triangles formed by connecting P to the vertices are equal.



39. There are a total of *n* students in a middle school. Four elevenths of *n* are in the 7th grade, a few sevenths of *n* are in the 8th grade and the remaining 324 students are in grade 9. Determine the total enrollment *n*.

40. The 15 points are painted red or blue. Prove that some equilateral triangle within this arrangement must have all its vertices the same color.



41. Show that $2^m - 1$ is a factor of $2^{mn} - 1$ for m and n positive integers.

42. Determine all polynomials p(x) such that (x-1)p(x+1)=(x+2)p(x).

43. Let $P(x) = ax^3 + bx^2 + cx + d$ and $Q(x) = dx^3 + cx^2 + bx + a$. What is the relationship between the roots of P(x) = 0 and Q(x) = 0? Generalize.

44. Let a, b, and c be the roots of $x^3 + 3x + 3 = 0$. Determine the value of the product (a + 1)(b + 1)(c + 1).

45. If n is a positive integer such that 2n + 1 is a perfect square, show that n + 1 is the sum of two consecutive perfect squares. Hint: If $2n + 1 = s^2$, s must be odd.

46. Prove that the reciprocal of an irrational number is also irrational.

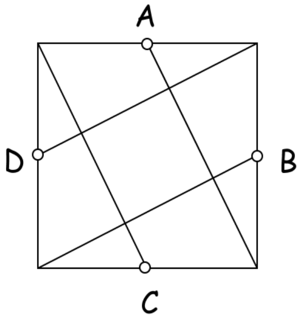
47. Take a square and draw a straight line across it. Draw several more straight lines in any arrangement, thus dividing the squares into regions. Color the regions so that adjacent regions are never colored the same. What is the minimum number of colors needed to color any such arrangement?

48. Let $a_1, a_2, a_3, \ldots, a_n$ denote the roots of $x^n - a_1^n = 0$. Show that $(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n) = na_1^{n-1}$. Hint: What does na_1^{n-1} look like?

49. Three congruent rectangles form a larger rectangle, as shown, with an area of 2400 cm^2. Determine the area of a square that has the same perimeter as the larger rectangle.

50. For each positive integer n, let S_n be the sum of all the positive integer factors of 2010^n (including 2010 itself). Determine $\lim_{n\to\infty}\frac{S_n}{2010^n}$. Perhaps first try $\lim_{n\to\infty}\frac{T_n}{12^n}$ where T_n is the sum of such factors of 12^n.

51. A, B, C, and D are midpoints of the sides of a square of side length 8. If these midpoints are connected by line segments to the vertices of the square as shown, a square is formed. Determine the area of the square. What happens if you have a parallelogram? What if A, B, C, D trisect the side?

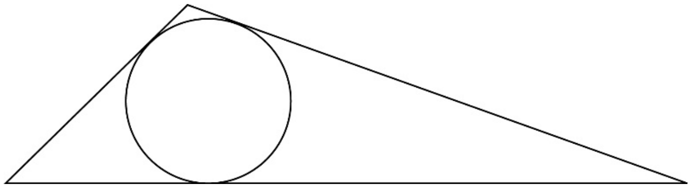


*52. Let $V_1, V_2, \ldots, V_n$ be n equally spaced points on a unit circle and let P be any point on the circle. Conjecture and prove a formula for the sum $\overline{PV_1^2}+\overline{PV_2^2}+\overline{PV_3^2}+\ldots+\overline{PV_n^2}$.

53. The length of each leg of an isosceles triangle is $2x-1$. If the base is $5x-7$ determine all possible values so that x is an integer.

54. Let a, b, c, d be distinct integers such that $(x - a)(x - b)(x - c)(x - d) - 4 = 0$ has an integral root r. Show that $4r = a + b + c + d$. (Hint: If a, b, c, d are all different so are r – a, r – b, r – c, r – d.)

55. The perimeter of the triangle is 24 in, and its area is 8 in^2. What is the exact area of the inscribed circle? [That is, express the area as a fractional multiple of π].



56. The roots of $2x^3 - 6x^2 + 8x - 3 = 0$ are a, b, and c. Determine the value of $a+b+c+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ac}$.

57. Determine all n for which x^n+y^n factors. [As a reminder, x^3+y^3 factors as $(x+y)(x^2-xy+y^2)$, and x^4-y^4 factors as $(x^2-y^2)(x^2+y^2)$].

58. Does there exist a convex (nondegenerate) pentagon, all of whose vertices are lattice points in the plane, with no lattice points in the interior?

59. Determine the maximum number of points of intersection formed inside a circle by connecting n points on the circumference in all possible ways. What is the maximum number of regions formed?

60. Let n be any odd integer. Show that n^2 leaves a remainder of 1 when divided by 8.

61. Find all the roots of $x^3 + x^2 + x + 1 = 0$. Place in a graphical setting. Discuss complex roots of unity. (Hint: Notice that $(x-1)(x^3 + x^2 + x + 1) = x^4 - 1$.)

62. Prove that for every integer n, $n \geq 2$, the equation $x^2 - y^2 = n^3$ has a solution (x, y) with x and y positive integers. For example, if we let $n = 2$, $x^2 - y^2 = 8$ has $x = 3$, $y = 1$ as a solution.

63. An International Conference on Global Warming has 5 diplomats from the US, 3 diplomats from Russia and 4 diplomats from China. These 12 diplomats are to be seated at the head table in a single row. Determine the number of possible seating arrangements if the diplomats from each country must be seated together as a group. Express your answer using the n! notation.

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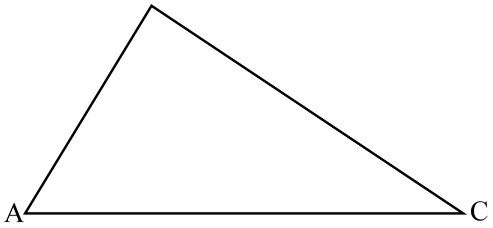
64. For which value(s) of *n*, are *n*, 3*n*+1, 3*n*-1 the lengths of the three sides of a right triangle?

65. A polynomial p(x) has a remainder of 3 when divided by $x - 1$ and a remainder of 5 when divided by $x - 3$. What is the remainder when p(x) is divided by the product$(x - 1)(x - 3)$. (Hint: Express p(x) as $p(x) = (x - 1)(x - 3)q(x) + r(x)$ and analyze the nature of r(x).)

*66. Triangle ABC has integer side lengths. One side has length 13 and a second side is twice the length of the third side. What is the greatest possible perimeter?



67. For which n is $1^3 - 2^3 + 3^3 - 4^3 + \ldots + (-1)^{n+1}n^3$ a perfect square?

68. Determine the number of positive integer solutions to $\frac{1}{m} + \frac{1}{n} = \frac{1}{2^{30} \bullet 3^{20}}$.

69. Equilateral triangles whose sides are 1, 3, 5, 7, 9, … are placed so that their bases lie corner to corner in a straight line. Show that the vertices lie on a parabola and are all at integer distances from its focus.

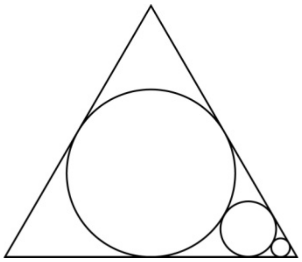
70. Find all positive integers n such that $n^3 - 12n^2 + 40n - 29$ is a prime number.

71. What is the minimum distance between points on the graph of $y = \frac{9}{x}$ and points on the graph of $y = \sqrt{2 - x^2}$?

72. Determine a closed formula for $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n!$.

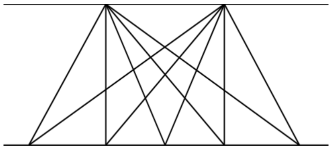
73. Let n be a natural number. For which n is $n \bullet (n + 1) \bullet (n + 2) \bullet (n + 3)$ the square of an integer.

74. A circle is inscribed in an equilateral triangle whose side length is 2. Then another circle is inscribed externally tangent to the first circle but inside the triangle as shown. And then another, and another. If this process continues forever what is the total area of all the circles? Express your answer as an exact multiple of π (and not as a decimal approximation).



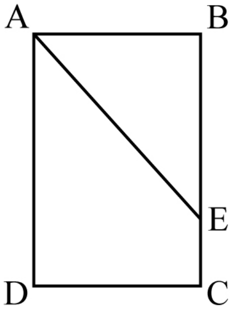
75. Given any 9 lattice points in 3-dimensional Euclidean space, show that there is a lattice point on the interior of a line segment joining a pair of these points.

76. Connect the 2 points on one line to each of the other 5 points with line segments. Determine the maximum number of interior points of intersection. Redo the problem with 3 points on top and 4 points below.



77. Demonstrate that the cube root of $10^{21} + 5 \bullet 10^{14} + 9 \bullet 10^{7} + 1$ is not an integer.

78. Give a convincing argument that if a/b is a reduced fraction, so is 1 – a/b.



79. ABCD is a rectangle with AB=5, EC=2, AE+AD. Find AD.

80. Determine the sum of the first 50 terms of

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}.$$

81. Determine all integer-sided right triangles whose perimeter equals their area.

82. Verify that each of the numbers 1225, 112225, 11122225, …, consisting of n 1's, n+1 2's followed by a 5, is a perfect square.

83. Let *A* be a 7 by 7 symmetric matrix such that each row and column of *A* consists of some permutation of the integers 1, 2, 3, 4, 5, 6, 7. Show that each one of the integers 1, 2, …, 7 must appear on the main diagonal of *A*. What can happen if *A* is not symmetric?

84. Four people – Andrew, Carolyn, Darcey, and Rene – play a game that requires them to split up into two teams of two players. In how many ways can they split up? Now suppose that Euler and Fibonacci join these four people. In how many ways can these six people split up into three teams of two?

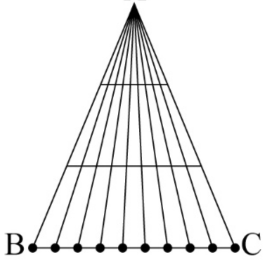
85. Determine the number of 3 by 3 square arrays whose row and column sums are equal to 2, using 0, 1, 2 as entries. Entries may be repeated, and not all of 0, 1, 2 need be used as the two examples show.

$$\begin{matrix}1 & 1 & 0\\ 0 & 0 & 2\\ 1 & 1 & 0\end{matrix}\qquad\begin{matrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{matrix}$$

86. In the figure there are 8 line segments drawn from vertex A to the base BC (not counting the segments AB or AC).

a.) Determine the total number of triangles of all sizes.

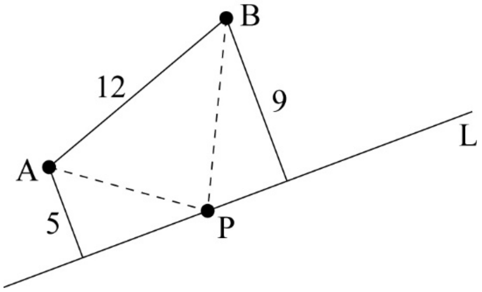
b.) How many triangles are there if there are *n* lines drawn from A to *n* interior points on BC?



87. Let $S = \{a,b,c,d\}$ be a set of four positive integers. If pairs of distinct elements of S are added, the following six sums are obtained: 5, 10, 11, 13, 14, 19. Determine the values of *a, b, c,* and *d.* (Hint: there are two possibilities.)

88. The sum of 400, 3, 500, 800, and 308 is 2011 and the product of these five numbers is 147,840,000,000.

- a.) Determine the largest number which is the product of positive integers whose sum is 10.
- b.) Determine the largest number which is the product of positive integers whose sum is 2011.
- c.) Generalize.



89. Triangle ABC has integer side lengths. One side is twice the length of a second side.

- a.) If the third side has length 40 what is the greatest possible perimeter?
- b.) If the third side has length n what is the greatest possible perimeter?
- c.) Now suppose one side is three times the length of a second side and the third side has length of 40. What is the maximum perimeter?
- d.) Generalize.

90. Let $f(n, 2)$ be the number of ways of splitting $2n$ people into n groups, each of size 2. As an example, the 4 people A, B, C, D can be split into 3 groups: $\boxed{AB}$ $\boxed{CD}$; $\boxed{AC}$ $\boxed{BD}$; and $\boxed{AD}$ $\boxed{BC}$. Hence $f(2, 2) = 3$.

- a.) Compute $f(3, 2)$ and $f(4, 2)$.
- b.) Conjecture a formula for $f(n, 2)$.
- c.) Let $f(n, 3)$ be the number of ways of splitting $\{1, 2, 3, \ldots, 3n\}$ into n subsets of size 3. Compute $f(2,3)$, $f(3,3)$ and conjecture a formula for $f(n, 3)$.

HINTS AND ANSWERS – PROBLEM SET 1

1.3 No.

1.4 $\binom{5}{3} = 10.$

2.3 a.) The largest side sum is 23.
  b.) For a minimum sum, 17.

2.4 $c = 2.$

3.3 Start listing.

3.4 Start listing, the answer is 25.

4.3 2 and 4

4.4 Draw a picture. Use technology.

5.3 $8\pi - 16.$

5.4 90.

6.3 31.

6.4 $\binom{m+1}{3} + \binom{m+2}{3}.$

7.3 400.

7.4 Tabulate lots of data.

8.3 2500.

8.4 $n^2 + (n-1)^2.$

9.3 Try trial and error with a calculator.

9.4 Cases and technology help here.

10.3 75.

10.4 92.

11.3 There are two cases.

11.4 5.

12.3 2520.

12.4 $\frac{7}{25}$.

13.3 $\frac{1}{4}$.

13.4 1:1.

14.3 Replace *x* by 5 and use long division.

14.4 $\frac{9}{4}$.

15.3 State the contrapositive.

15.4 State the contrapositive.

HINTS AND ANSWERS – PROBLEM SET 2

1. a.) 2222.
   b.) 22244.

2. The largest is 52.

3. The possible sums are 1, 4, 9, 16, and 25. Write out all the squares that can be made with 8 as one of the summands.

4. $a + b = 15$. Hint: don't find *a* and *b* individually.

5. Write out more terms.

6. Collect data – it appears that multiples of 4 can be expressed as the different of two squares.

7. 17 is the smallest. What is the largest?

8. 16 should be removed. Why?

9. Try *x* in the second spot.

10. 22π

11. $\frac{445}{1225}$.

12. 6 by 8 or 5 by 12.

13. Total sum is 81. There are four possibilities for the interior square.

14. For <u>no</u> positive *n* is $n^2 + n + 1$ a perfect square; $n = 0$ works, as does $n = -1$.

15. 180.

16. $\frac{1}{10}$.

17. $\frac{1}{211}$.

18. $n = 200$; look at a listing of squares and see where they are 68 apart.

19. 18 by 22.

20. 1.

21. Start with the triple (0, 0, 8) denoting an empty 3 quart pot, an empty 5 quart pot and a full 8 quart pot and start pouring.

22. One clever way is to use the additive table entries for exponents. There are many other ways.

23. $\frac{71}{60}$.

24. 365,241.

25. 23.

26. There are several ways to accomplish this.

27. Start lighting.

28. 8.

29. a.) 11.
    b.) 25.
    c.) 48 (try to generalize this one).

30. Perhaps first multiply all numbers by 3.

31. a.) 21200.
    b.) You try this part.

32. 400.

33. a.) 45 appears in row 9.
    b.) 999 appears in row 45.
    c.) You need to sum an arithmetic sequence.

34. a.) Look for squares.
    b.) Collect a lot of data; $3 + 5 = 8 = 2^3$.

35. $\frac{15}{31}$. Generalize for the set $S = \{1, 2, 3, \ldots, n\}$.

36. a.) 8.
    b.) 12.
    c.) You can count these.
    d.) 0.

37. 129.03.

38. Try a few small *a*. Can you actually find 4 values of *a*?

39. Modify our earlier additive square.

40. Think about constructing a familiar geometric figure.

41. Make a model and try it.

42. $8\pi - 16$.

43. 1440 is close but not the smallest.

44. A few are: 4, 9, 25, … You should find all of them.

45. a.) {1, 4}, {2, 3}.
    b.) Collect data.
    c.) Collect data.

46. Try many rectangles using graph paper to count carefully.

47. The old area is 4π; the new area is 3π.

48. First make D so that $DB = \frac{1}{5}AB$; then make E so that $EC = \frac{1}{4}AC$, …

49. (72, 9); (40, 10); (24, 12).

50. 10.

51. 8, 18.

52. First cut is at 5th link. What are the others?

53. a.) Write it out.
 b.) Use $n^2 = 1 + 3 + 5 + \cdots + (2n - 1)$ and $n^3 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2$.
 c.) Start stacking.

54. 4851.

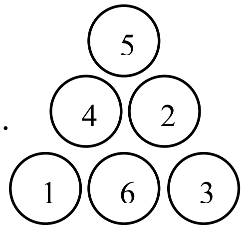
55. Try "what happens if" heuristic.

56. There are 14 students.

57. The doors labeled with perfect squares are open.

58. A winning strategy can be determined by thinking backwards. The player to reach 43 is guaranteed a win because the other player can at most achieve a sum of 49, and so on.

59. One possible solution is pictured to the right.



60. Try dividing the nine coins into three groups of three coins.

61. The two numbers are 199 and 201.

62. Their areas are equal. Look at other familiar Pythagorean Triples.

63. Try 1, 2, 3, …, 35.

64. a.) The next four rows are:

6 10 12 12 10 6
7 12 15 16 15 12 7
8 14 18 20 20 18 14 8
9 16 21 24 25 24 21 16 9

b.) The kth diagonal consists of the multiples of k.
c.) Try to find the row sums from this new triangle in Pascal's Triangle.

65. a.) The arrangement has a surface area of 544 cm^2.
b.) The arrangement has a surface area of 536 cm^2.

66. The pair (2, 2) has a product equal to its sum. The pairs (3, 6), (4, 4) and (6, 3) have a product equal to twice its sum.

67. a.) The ant's territory is a square-like figure with side length 3 cm, but whose corners are quarters of a circle with a 1 cm radius.
b.) The ant's territory has area $5 + \pi$ cm^2.

68. a.) 100 will fall in column D. Explain why.
b.) 1000 will fall in column B. Explain why.
c.) 2011 will fall in column C. Explain why.

69. What is the smallest m that will work?

70. The integer 13 will yield the same remainder. Why?

71. 44.25.

72. The triple (L, C, G) represents the number of lambs, calves, and goats. There are several potential solutions: (34, 5, 61), (48, 10, 42), (62, 15, 23), and (76, 20, 4).

73. There were 2 dimes.

74. The 2011th number is 58.

75. The value of x is 2.

76. $\frac{1}{3} = \frac{1}{5} + \frac{1}{9} + \frac{1}{45}$.

77. $\frac{5}{11} = \frac{1}{3} + \frac{1}{11} + \frac{1}{33}$.

78. The number 1008 has 30 divisors.

79. The sum is 670,338.

80. 31.

81. The oldest is 29. Explain.

82. The next three numbers are 40, 69, and 106.

HINTS AND ANSWERS – PROBLEM SET 3

1. $n = 3$.

2. $n = 7$.

3. $\binom{m+1}{2}\binom{m+1}{2}$.

4. 3 and 8; 4 and 5.

5.
   a.) 21.
   b.) 95.
   c.) 433.

6.
   a.) 120.
   b.) 210.

7. 56.

8. 25.

9.
   a.) Try the add in-subtract out technique.
   b.) Try $n = 1, 2, 3$. The polynomial looks like it is generating perfect squares.

10. $d = 20$.

11. $n = 3k + 1, 3k + 2$.

12. 60.5.

13. a.) $\binom{2}{2}\binom{3}{2}+\binom{3}{2}\binom{4}{2}+\binom{4}{2}\binom{5}{2}+\cdots+\binom{8}{2}\binom{9}{2}$.

b.) $\binom{2}{2}^2+\binom{3}{2}^2+\binom{4}{2}^2+\cdots+\binom{8}{2}^2$.

c.) Allowing 0s as in part a.) the summation is $\binom{3}{3}\binom{4}{3}+\binom{4}{3}\binom{5}{3}+\binom{5}{3}\binom{6}{3}+\cdots+\binom{8}{3}\binom{9}{3}$.

Without 0s as in part b.) the summation is $\binom{3}{3}^2+\binom{4}{3}^2+\binom{5}{3}^2+\cdots+\binom{8}{3}^2$.

14. $\tan 2A=\frac{2t}{1-t^2}$, $\tan 3A=\frac{3t-t^3}{1-3t^2}$, $\tan 4A=\frac{4t-4t^3}{1-6t^2+t^4}$ and $\tan 5A=\frac{5t-10t^3+t^5}{1-10t^2+5t^4}$.

15. a.) $n=1,24$ only.
b.) $n=1,2,48$ only; use technology.

16. $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$.

17. $R(n)=3n^2+3n+1$.

18. $n=4,5,7;\ 4!+1=25, 5!+1=121, 7!+1=5041$ using technology.

19. $\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\cdots+\binom{9}{6}$.

20. Compare with properties of Pascal's Triangle.

21. $\binom{15}{2}; 2\binom{30}{2}$.

22. a.) A quadratic fits the data 1, 5, 13, 25, 41, …
b.) A cubic will fit 1, 6, 19, 44, 85, …

23. a.) 360.
b.) $\sqrt{4177}$.

24. What binomial expansion will give these row elements?

25. a.) 2.
    b.) 6.

26. a.) $40\sqrt{2}$; the successive side lengths of the four squares are 40, $20\sqrt{2}$, 20, $10\sqrt{2}$.
    b.) 200.

27. Use $1 \equiv -6(mod\ 7), 2 \equiv -5(mod\ 7)$.

28. The shaded area is constant no matter where P is placed.

29. $\ln\frac{2}{n+1}$.

30. a.) $2\binom{n+2}{3}$.

    b.) $6\binom{n+3}{4}$.

31. $\frac{1}{4} - \frac{1}{2(n+1)(n+2)}$.

32. 55 mph

33. a.) Only $1001 = 7 \cdot 11 \cdot 13$ is composite.
    b.) Place in a more general setting.

34. 41.

35. $h = \frac{2}{5}$.

36. 216.

37. Choose one vertex and the 5 lines emanating from it. At least three of these must be the same color. Now examine the triangles formed by connecting the three vertices at the ends of these three line segments.

38. Place P at the intersection of the medians.

39. $n = 924$.

40. As a start, look at the inner small triangle. At least two of these points must be the same color. Now consider possible colors of points of the surrounding points.

41. $2^{mn} - 1 = (2^m - 1)(2^{m(n-1)} + 2^{m(n-2)} + \cdots + 2^m + 1)$.

42. This one is difficult. Try $x = 1$ and $x = -2$ and watch what happens.

43. If r is a root of $P(x) = 0$ then $\frac{1}{r}$ is a root of $Q(x) = 0$.

44. 1.

45. If $2n + 1 = s^2$, then s must be odd, say $s = 2k + 1$. Then $2n + 1 = 4k^2 + 4k + 1$ and $n = 2k(k + 1)$ and $n + 1 = 2k^2 + 2k + 1 = k^2 + (k + 1)^2$.

46. Reason by contradiction.

47. 2.

48. Start with $x^n - a_1^n = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$. Divide by $(x - a_1)$ and notice that $\lim\limits_{x \to a_1} \frac{x^n - a_1^n}{x - a_1} = nx^{n-1}|_{a_1} = na_1^{n-1}$, the derivative. Alternatively, use long division.

49. 2500.

50. $\lim\limits_{n \to \infty} \frac{T_n}{12^n} = 3; \frac{335}{88}$.

51. The area of the square is $\frac{64}{5}$.

52. $2n$. This is a hard problem. First investigate $n = 2, 3, 4$ and draw pictures.

53. Only $x = 2, 3, 4$ will make a triangle.

54. If r is a root, then $(r - a)(r - b)(r - c)(r - d) = 4$. These four factors must be 1, -1, 2 and -2 in some order; so, $(r - a) + (r - b) + (r - c) + (r - d) = 0$ and thus, $4r = a + b + c + d$.

55. $\frac{4}{9}\pi$.

56. 5.

57. When n is not a power of 2, $x^n + y^n$ factors. A few examples illustrate what happens. $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^5 + y^5 = (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

$x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$ which resembles the factorization of $x^3 + y^3$.

58. No.

59. a.) $\binom{n}{4}$.

b.) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4}$.

60. Let $n = 2k + 1$; Then $n^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1$. One of k or $k+1$ is a multiple of 2, so 8 divides $4k(k+1)$ leaving a remainder of 1.

61. The roots of $x^3 + x^2 + x + 1 = 0$ are the other three roots of $x^4 - 1 = 0$, ignoring $x = 1$. They are $-1, i, -i$.

62. Collect some data inspired by the identity: $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

63. $3! \cdot 5! \cdot 3! \cdot 4!$.

64. $n = 12$.

65. $r(x) = x + 2$.

66. 49.

67. 1, 5 and perhaps more.

68. $61 \cdot 41$.

69. Try finding the equation of the parabola by fitting the data.

70. $n = 2$ or $n = 4$.

71. $2\sqrt{2}$.

72. $(n+1)! - 1$.

73. Never.

74. $\frac{3\pi}{8}$.

75. There are only eight possible patterns for lattice points (p, q, r).

76. $\binom{2}{2} \cdot \binom{5}{2} = 10$; $\binom{3}{2} \cdot \binom{4}{2} = 18$.

77. The given number is between $(10^7 + 1)^3 = 10^{21} + 3 \cdot 10^{14} + 3 \cdot 10^7 + 1$ and $(10^7 + 2)^3 = 10^{21} + 6 \cdot 10^{14} + 12 \cdot 10^7 + 8$, two consecutive cubes.

78. Suppose $1 - \frac{a}{b}$ is not reduced.

79. $\frac{29}{4}$.

80. $\frac{n}{2n+1}$.

81. There are two: 5, 12, 13 and 6, 8, 10.

82. These numbers are squares 35, 335, 3335, … The proof in general is difficult.

83. Each integer appears 7 times, an odd number.

84. With four people there are 3 ways; with 6 there are 15 ways.

85. 21.

86. a.) $3 \cdot \binom{10}{2}$.

b.) $3 \cdot \binom{n+2}{2}$.

87. {1, 4, 9, 10} or {2, 3, 8, 11}.

88. a.) 36; $2 + 2 + 3 + 3 = 10$.
b.) $2 \cdot 3^{670}$; $3 + 3 + 3 + \cdots + 3 + 2 = 2011$.
c.) None of 4, 5, 6, 7, 8, 9 need be used as summands.

89. Reflect B across L.

90. a.) $f(3,2) = 15$; $f(4,2) = 7 \cdot 5 \cdot 3 = 105$.

b.) $f(n,2) = 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1) = \frac{(2n)!}{2^n \cdot n!}$.

c.) $f(2,3) = \binom{5}{2}$; $f(n,3) = \binom{5}{2}\binom{8}{2}\binom{11}{2} \ldots \binom{3n-1}{2}$.