

# GAME-DA Documentation

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# Chapter 1

## Introduction

GAME-DA provides *data assimilation (DA)* functionality for the GAME model. The theoretical derivations are presented in [1]. This documentation contains the technical details of the implementations of the data assimilation algorithms. Three different methods of DA are implemented in GAME-DA:

- optimum interpolation (OI)
- three-dimensional variational data assimilation (3D-Var)
- four-dimensional variational data assimilation (4D-Var)

Depending on the resolution and the type and amount of observations, certain methods are more suitable than others.

In any of the three cases, the basic workflow is as follows:

1. downloading observations
2. bringing the observations into a standardized format
3. executing the data assimilation itself, resulting in an input file for the model run

## Chapter 2

# Optimum interpolation

Let  $M \geq 1$  be the number of degrees of freedom of the model,  $N \geq 1$  the number of observations,  $\mathbf{h} \in \mathbb{R}^N$  the observations as reconstructed from the background state,  $\mathbf{y} \in \mathbb{R}^N$  the actual observations,  $\overleftarrow{\mathbf{H}} \in \mathbb{R}^{N \times M}$  the Jacobian of the observations operator,  $\overleftarrow{\mathbf{B}} \in \mathbb{R}^{M \times M}$  the background error covariance matrix and  $\overleftarrow{\mathbf{R}} \in \mathbb{R}^{N \times N}$  the observations error covariance matrix.  $\overleftarrow{\mathbf{R}}$  is usually assumed to be diagonal, which is an excellent approximation if the observation errors are of statistical nature. This does not lead to useful simplifications, however.

It is

$$\left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}}\right)_{i,j} = \sum_{k=1}^M \overleftarrow{\mathbf{H}}_{i,k} \overleftarrow{\mathbf{B}}_{k,j}. \quad (2.1)$$

For  $\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T$ , one obtains

$$\left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T\right)_{i,j} = \left[\left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}}\right) \overleftarrow{\mathbf{H}}^T\right]_{i,j} = \sum_{k=1}^M \left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}}\right)_{i,k} \overleftarrow{\mathbf{H}}_{k,j}^T = \sum_{k=1}^M \left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}}\right)_{i,k} \overleftarrow{\mathbf{H}}_{j,k} = \sum_{k=1}^M \left(\sum_{l=1}^M \overleftarrow{\mathbf{H}}_{i,l} \overleftarrow{\mathbf{B}}_{l,k}\right) \overleftarrow{\mathbf{H}}_{j,k} = \sum_{k,l=1}^M \overleftarrow{\mathbf{H}}_{i,l} \overleftarrow{\mathbf{B}}_{l,k} \overleftarrow{\mathbf{H}}_{j,k}. \quad (2.2)$$

For the final result, we obtain

$$\begin{aligned} x_i &= x_{B,i} + \sum_{j=1}^N \left[ \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T \left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T + \overleftarrow{\mathbf{R}}\right)^{-1} \right]_{i,j} (y_j - h_j) = x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left(\overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T\right)_{i,k} \left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T + \overleftarrow{\mathbf{R}}\right)^{-1}_{k,j} \right] (y_j - h_j) \\ &= x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T + \overleftarrow{\mathbf{R}}\right)^{-1}_{k,j} \left(\overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T\right)_{i,k} \right] (y_j - h_j) = x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T + \overleftarrow{\mathbf{R}}\right)^{-1}_{k,j} \left(\sum_{l=1}^M \overleftarrow{\mathbf{B}}_{i,l} \overleftarrow{\mathbf{H}}_{l,k}^T\right) \right] (y_j - h_j) \end{aligned}$$

$$\Rightarrow x_i = x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left(\overleftarrow{\mathbf{H}} \overleftarrow{\mathbf{B}} \overleftarrow{\mathbf{H}}^T + \overleftarrow{\mathbf{R}}\right)^{-1}_{k,j} \left(\sum_{l=1}^M \overleftarrow{\mathbf{B}}_{i,l} \overleftarrow{\mathbf{H}}_{k,l}\right) \right] (y_j - h_j). \quad (2.3)$$

### 2.1 Inclusion of moisture

So far, moisture is taken into account in a separate assimilation process for efficiency.

## Chapter 3

# 3D-Var

### 3.1 Technicalities

The observations used come from a time window  $[-\frac{T}{2}, \frac{T}{2}]$  around the analysis time and are all taken to be valid at the analysis time.  $T = 3$  h is a typical value..

# Chapter 4

## 4D-Var

### 4.1 Technicalities

The observations used come from a time window  $[-\frac{T}{2}, \frac{T}{2}]$  around the analysis time and are taken to be valid at individual time steps  $n\Delta t$ .  $T = 6$  h and  $\Delta t = 15$  min are typical values.

# Bibliography

- [1] M. H. Balsmeier. *Kompendium Theoretische Meteorologie*. 2021. URL: <https://raw.githubusercontent.com/MHBalsmeier/kompendium/master/kompendium.pdf>.