

# GAME-DA Documentation

GAME-DA Development Team

# Chapter 1

## Optimum interpolation

Let  $M \geq 1$  be the number of degrees freedom of the model,  $N \geq 1$  the number of observations,  $\mathbf{h} \in \mathbb{R}^N$  the the observations as reconstructed from the background state,  $\mathbf{y} \in \mathbb{R}^N$  the actual observations,  $\overrightarrow{H} \in \mathbb{R}^{N \times M}$  the Jacobian of the observations operator,  $\overleftarrow{H} \in \mathbb{B}^{M \times M}$  the background error covariance matrix and  $\overleftarrow{R} \in \mathbb{R}^{N \times N}$  the model error cavariance matrix. It is

$$\left(\overleftarrow{H} \overleftarrow{B}\right)_{i,j} = \sum_{k=1}^M \overleftarrow{H}_{i,k} \overleftarrow{B}_{k,j}. \quad (1.1)$$

If  $\overleftarrow{B}$  is diagonal, this reduces to

$$\left(\overleftarrow{H} \overleftarrow{B}\right)_{i,j} = \sum_{k=1}^M \overleftarrow{H}_{i,k} \overleftarrow{B}_{k,j} \delta_{k,j} = \sum_{k=1}^M \overleftarrow{H}_{i,k} \overleftarrow{B}_{j,j} \delta_{k,j} = \overleftarrow{H}_{i,j} \overleftarrow{B}_{j,j}. \quad (1.2)$$

For  $\overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T$ , one obtains

$$\left(\overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T\right)_{i,j} = \left[\left(\overleftarrow{H} \overleftarrow{B}\right) \overleftarrow{H}^T\right]_{i,j} = \sum_{k=1}^M \left(\overleftarrow{H} \overleftarrow{B}\right)_{i,k} \overleftarrow{H}_{k,j}^T = \sum_{k=1}^M \left(\overleftarrow{H} \overleftarrow{B}\right)_{i,k} \overleftarrow{H}_{j,k} = \sum_{k=1}^M \left(\sum_{l=1}^M \overleftarrow{H}_{i,l} \overleftarrow{B}_{l,k}\right) \overleftarrow{H}_{j,k} = \sum_{k,l=1}^M \overleftarrow{H}_{i,l} \overleftarrow{B}_{l,k} \overleftarrow{H}_{j,k}. \quad (1.3)$$

If  $\overleftarrow{B}$  is diagonal, this reduces to

$$\left(\overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T\right)_{i,j} = \sum_{k=1}^M \overleftarrow{H}_{i,k} \overleftarrow{B}_{k,k} \overleftarrow{H}_{j,k} = \sum_{k=1}^M \overleftarrow{B}_{k,k} \left(\overleftarrow{H}_{i,k} \overleftarrow{H}_{j,k}\right). \quad (1.4)$$

For the final result, we obtain

$$\begin{aligned} x_i &= x_{B,i} + \sum_{j=1}^N \left[ \overleftarrow{B} \overleftarrow{H}^T \left( \overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T + \overleftarrow{R} \right)^{-1} \right]_{i,j} (y_j - h_j) = x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left( \overleftarrow{B} \overleftarrow{H}^T \right)_{i,k} \left( \overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T + \overleftarrow{R} \right)^{-1}_{k,j} \right] (y_j - h_j) \\ &= x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left( \overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T + \overleftarrow{R} \right)^{-1}_{k,j} \left( \overleftarrow{B} \overleftarrow{H}^T \right)_{i,k} \right] (y_j - h_j) = x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left( \overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T + \overleftarrow{R} \right)^{-1}_{k,j} \left( \sum_{l=1}^M \overleftarrow{B}_{i,l} \overleftarrow{H}_{l,k}^T \right) \right] (y_j - h_j) \\ &= x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left( \overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T + \overleftarrow{R} \right)^{-1}_{k,j} \left( \sum_{l=1}^M \overleftarrow{B}_{i,l} \overleftarrow{H}_{k,l} \right) \right] (y_j - h_j). \end{aligned} \quad (1.5)$$

If  $\overleftarrow{B}$  is diagonal, this reduces to

$$x_i = x_{B,i} + \sum_{j=1}^N \left[ \sum_{k=1}^N \left( \overleftarrow{H} \overleftarrow{B} \overleftarrow{H}^T + \overleftarrow{R} \right)^{-1}_{k,j} \overleftarrow{B}_{i,i} \overleftarrow{H}_{k,i} \right] (y_j - h_j). \quad (1.6)$$

### 1.1 Inclusion of moisture

So far, moisture is taken into account in a separate assimilation process for efficiency.

## Chapter 2

# 3D-Var

### 2.1 Technicalities

The observations used come from a time window  $[-\frac{T}{2}, \frac{T}{2}]$  around the analysis time and are all taken to be valid at the analysis time.  $T = 3$  h is a typical value..

## Chapter 3

# 4D-Var

### 3.1 Technicalities

The observations used come from a time window  $[-\frac{T}{2}, \frac{T}{2}]$  around the analysis time and are taken to be valid at individual time steps  $n\Delta t$ .  $T = 6$  h and  $\Delta t = 15$  min are typical values.