GAME-DA Documentation

GAME-DA Development Team

Chapter 1

Optimum interpolation

Let $M \geq 1$ be the number of degrees freedom of the model, $N \geq 1$ the number of observations, $\mathbf{h} \in \mathbb{R}^N$ the the observations as reconstructed from the background state, $\mathbf{y} \in \mathbb{R}^N$ the actual observations, $\overleftrightarrow{H} \in \mathbb{R}^{N\times M}$ the Jacobian of the observations operator, $\overleftrightarrow{H} \in \mathbb{B}^{M\times M}$ the background error covariance matrix and $\overleftrightarrow{R} \in \mathbb{R}^{N\times N}$ the model error covariance matrix. It is

$$\left(\overleftrightarrow{H}\overleftrightarrow{B}\right)_{i,j} = \sum_{k=1}^{M} \overleftrightarrow{H}_{i,k}\overleftrightarrow{B}_{k,j}. \tag{1.1}$$

If $\overleftrightarrow{B}$ is diagonal, this reduces to

$$\left(\overleftrightarrow{H}\overleftrightarrow{B}\right)_{i,j} = \sum_{k=1}^{M} \overleftrightarrow{H}_{i,k}\overleftrightarrow{B}_{k,j}\delta_{k,j} = \sum_{k=1}^{M} \overleftrightarrow{H}_{i,k}\overleftrightarrow{B}_{j,j}\delta_{k,j} = \overleftrightarrow{H}_{i,j}\overleftrightarrow{B}_{j,j}. \tag{1.2}$$

For $\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T$, one obtains

$$\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T\right)_{i,j} = \left[\left(\overleftrightarrow{H}\overleftrightarrow{B}\right)\overleftrightarrow{H}^T\right]_{i,j} = \sum_{k=1}^{M}\left(\overleftrightarrow{H}\overleftrightarrow{B}\right)_{i,k}\overleftrightarrow{H}^T_{k,j} = \sum_{k=1}^{M}\left(\overleftrightarrow{H}\overleftrightarrow{B}\right)_{i,k}\overleftrightarrow{H}_{j,k} = \sum_{k=1}^{M}\left(\sum_{l=1}^{M}\overleftrightarrow{H}_{i,l}\overleftrightarrow{B}_{l,k}\right)\overleftrightarrow{H}_{j,k} = \sum_{k,l=1}^{M}\overleftrightarrow{H}_{i,l}\overleftrightarrow{B}_{l,k}\overleftrightarrow{H}_{j,k}. \tag{1.3}$$

If $\overleftrightarrow{B}$ is diagonal, this reduces to

$$\boxed{\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T\right)_{i,j} = \sum_{k=1}^{M}\overleftrightarrow{H}_{i,k}\overleftrightarrow{B}_{k,k}\overleftrightarrow{H}_{j,k} = \sum_{k=1}^{M}\overleftrightarrow{B}_{k,k}\left(\overleftrightarrow{H}_{i,k}\overleftrightarrow{H}_{j,k}\right).} \tag{1.4}$$

For the final result, we obtain

$$\begin{aligned}
x_i &= x_{B,i} + \sum_{j=1}^{N}\left[\overleftrightarrow{B}\overleftrightarrow{H}^T\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T + \overleftrightarrow{R}\right)^{-1}\right]_{i,j}\left(y_j - h_j\right) = x_{B,i} + \sum_{j=1}^{N}\left[\sum_{k=1}^{N}\left(\overleftrightarrow{B}\overleftrightarrow{H}^T\right)_{i,k}\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T + \overleftrightarrow{R}\right)^{-1}_{k,j}\right]\left(y_j - h_j\right) \\
&= x_{B,i} + \sum_{j=1}^{N}\left[\sum_{k=1}^{N}\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T + \overleftrightarrow{R}\right)^{-1}_{k,j}\left(\overleftrightarrow{B}\overleftrightarrow{H}^T\right)_{i,k}\right]\left(y_j - h_j\right) = x_{B,i} + \sum_{j=1}^{N}\left[\sum_{k=1}^{N}\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T + \overleftrightarrow{R}\right)^{-1}_{k,j}\left(\sum_{l=1}^{M}\overleftrightarrow{B}_{i,l}\overleftrightarrow{H}^T_{l,k}\right)\right]\left(y_j - h_j\right) \\
&= x_{B,i} + \sum_{j=1}^{N}\left[\sum_{k=1}^{N}\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T + \overleftrightarrow{R}\right)^{-1}_{k,j}\left(\sum_{l=1}^{M}\overleftrightarrow{B}_{i,l}\overleftrightarrow{H}_{k,l}\right)\right]\left(y_j - h_j\right).
\end{aligned} \tag{1.5}$$

If $\overleftrightarrow{B}$ is diagonal, this reduces to

$$\boxed{x_i = x_{B,i} + \sum_{j=1}^{N}\left[\sum_{k=1}^{N}\left(\overleftrightarrow{H}\overleftrightarrow{B}\overleftrightarrow{H}^T + \overleftrightarrow{R}\right)^{-1}_{k,j}\overleftrightarrow{B}_{i,i}\overleftrightarrow{H}_{k,i}\right]\left(y_j - h_j\right).} \tag{1.6}$$

1.1 Inclusion of moisture

So far, moisture is taken into account in a separate assimilation process for efficiency.

Chapter 2

3D-Var

2.1 Technicalities

The observations used come from a time window $\left[-\frac{T}{2},\frac{T}{2}\right]$ around the analysis time and are all taken to be valid at the analysis time. $T = 3$ h is a typical value..

Chapter 3

4D-Var

3.1 Technicalities

The observations used come from a time window $\left[-\frac{T}{2},\frac{T}{2}\right]$ around the analysis time and are taken to be valid at individual time steps $n\Delta t$. $T = 6$ h and $\Delta t = 15$ min are typical values.