Iterative Diagonalization Algorithm (IDA)

Consider a conventional trapped ion system, its potential has the following form,

$$V = V_{\text{trap}} + V_{\text{Coulomb}}$$

The normal modes of the system are obtained by diagonalizing,

$$A_{ij} = \frac{1}{\sqrt{m_i m_j}} \{\text{Hess}[V](x^*)\}_{ij}$$

Note: Hessian of the conventional trap potential is diagonal.

Therefore, only the Coulomb potential contributes to the off diagonal elements of A.

Iterative Diagonalization Algorithm (IDA)

Consider adding optical tweezers on each ion,

$$\tilde{V} = V_{\text{trap}} + V_{\text{Coulomb}} + V_{\text{optical}}$$

$$\tilde{A}_{ij} = \frac{1}{\sqrt{m_i m_j}} \{\text{Hess}[\tilde{V}](\tilde{x}^*)\}_{ij}$$

Note: Hessian of the optical tweezer potentials is diagonal.

Therefore, only the Coulomb potential contributes to the off diagonal elements of $\tilde{A}$.

Under the assumption that the equilibrium position does not change with the addition of the optical potentials, i.e. $x^* \approx \tilde{x}^*$,

$$A_{ij} \approx \tilde{A}_{ij}, \forall\, i \neq j$$

Iterative Diagonalization Algorithm (IDA)

Problem Statement:
Given a conventional trapped ion system and a target normal mode spectrum, ω^{target}.

By introducing optical tweezers on each ion, are we able to find the powers, such that, the new trapped ion system has the target normal mode spectrum.

Note: There may not exist a solution and solutions may not be unique.

Iterative Diagonalization Algorithm (IDA)

Initialize: $\omega^{\text{target}}, A_{ij}, b^{(0)}, \epsilon$

$$A^{(n)} = b^{(n)} \cdot \omega^{\text{target}} \cdot b^{T(n)}$$

$$\tilde{A}_{ij}^{(n)} = \begin{cases} A_{ii}^{(n)}, & \text{if } i = j \\ A_{ij}, & \text{if } i \neq j \end{cases}$$

Diagonalize $\tilde{A}^{(n)}$ to obtain $\omega^{(n+1)}, b^{(n+1)}$

If $\exists\, m$, s.t. $\omega_m^{(n+1)} - \omega_m^{\text{target}} > \epsilon$

If $\forall\, m$, $\omega_m^{(n+1)} - \omega_m^{\text{target}} \leq \epsilon$

Terminate: $w_i^{\text{opt}} = \sqrt{\left| \tilde{A}_{ii}^{(n)} - A_{ii} \right|}, \quad s_i^{\text{opt}} = \text{sign}\left(\tilde{A}_{ii}^{(n)} - A_{ii} \right)$

Iterative Diagonalization Algorithm (IDA)

Problems with IDA:
1) There exist other points of convergence.

2) Within IDA, we assumed that the equilibrium positions of the ions do not change. In actuality, the equilibrium position would change due to the optical potentials.

Iterative Diagonalization Algorithm (IDA)

The following condition define points of convergence for IDA,
A and $\tilde{A}$ both have the same eigenvectors, b.
The above condition implies that,

$$\omega_m^{\text{convergence}} = \omega_m^{\text{target}} + \delta_m$$

where $b_{im}\delta_m b_{mj}^T$ is hollow symmetric, i.e diagonal elements are 0.
Which further implies b_{im}^2 is not full rank and δ_m orthogonal to rows of b_{im}^2.

Iterative Diagonalization Algorithm (IDA)

Example:

$$\omega_m^{\text{target}} = \begin{pmatrix} 10 & 8 & 4 \end{pmatrix} \quad A_{i<j} = \begin{pmatrix} 3 & 1 & 3 \end{pmatrix}$$

$$b_{im} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \end{pmatrix} \quad A_{ij} = \begin{pmatrix} 8 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 8 \end{pmatrix}$$

$$\tilde{A}_{ij} = \begin{pmatrix} 8 & 3 & 1 \\ 3 & 6 & 3 \\ 1 & 3 & 8 \end{pmatrix} \quad \omega_m^{\text{convergence}} = \begin{pmatrix} 12 & 7 & 3 \end{pmatrix}$$

$$\delta_m = \begin{pmatrix} 2 & -1 & -1 \end{pmatrix} \quad b_{im}\delta_m b_{mj}^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \tilde{A}_{ij} - A_{ij}$$

$$b_{im}^2 = \frac{1}{6} \begin{pmatrix} 2 & 3 & 1 \\ 2 & 0 & 4 \\ 2 & 3 & 1 \end{pmatrix} \quad b_{im}^2\delta_m = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

Iterative Diagonalization Algorithm (IDA)

For problem 1, using the following property of the other convergence points, $\det(b_{im}^2) = 0$,we can check if our algorithm is converging to one of these points and avoid these points.

Iterative Diagonalization Algorithm (IDA)

Initialize: $\omega^{\text{target}}, A_{ij}, b^{(0)}, \epsilon, \delta$

$A^{(n)} = b^{(n)} \cdot \omega^{\text{target}} \cdot b^{T(n)}$

$$\tilde{A}^{(n)}_{ij} = \begin{cases} A^{(n)}_{ii}, & \text{if } i = j \\ A_{ij}, & \text{if } i \neq j \end{cases}$$

$$\tilde{A}^{(n)}_{ij} = \begin{cases} \text{Randomize}, & \text{if } i = j \\ A_{ij}, & \text{if } i \neq j \end{cases}$$

Diagonalize $\tilde{A}^{(n)}$ to obtain $\omega^{(n+1)}, b^{(n+1)}$

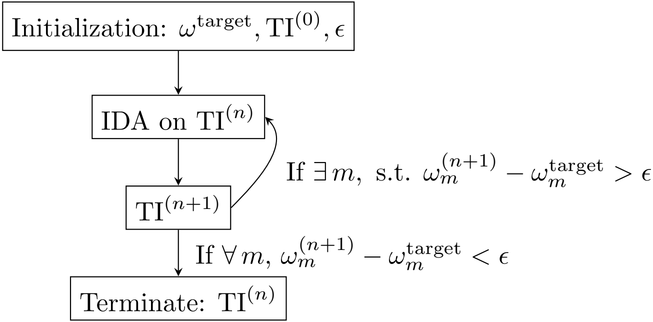
If $|\det(b^{(n+1)\,2}_{im})| < \delta$

If $\forall\, m,\ \omega^{(n+1)}_m - \omega^{\text{target}}_m \leq \epsilon$

Terminate: $w^{\text{opt}}_i = \sqrt{\left|\tilde{A}^{(n)}_{ii} - A_{ii}\right|}, \quad s^{\text{opt}}_i = \text{sign}\left(\tilde{A}^{(n)}_{ii} - A_{ii}\right)$

Iterative Diagonalization Algorithm (IDA)

For problem 2, supposing a small change in the equilibrium position due to the optical potentials, a method to compensate for the change in equilibrium position would be to perform IDA again using the new system. This could also be done iteratively.



It is yet to be determined if the above method for compensating for change in equilibrium position would work for larger changes in the equilibrium position.

Control of Normal Mode Frequencies

Iterative Diagonalization Algorithm (IDA)

Initialize: $\omega^{\text{target}}, A_{ij}, b^{(0)}, \epsilon$

$$A^{(n)} = b^{(n)} \cdot \omega^{\text{target}} \cdot b^{T(n)}$$

$$\tilde{A}_{ij}^{(n)} = \begin{cases} A_{ii}^{(n)}, & \text{if } i = j \\ A_{ij}, & \text{if } i \neq j \end{cases}$$

Diagonalize $\tilde{A}^{(n)}$ to obtain $\omega^{(n+1)}, b^{(n+1)}$

If $\exists\, m,$ s.t. $\omega_m^{(n+1)} - \omega_m^{\text{target}} > \epsilon$

If $\forall\, m,\ \omega_m^{(n+1)} - \omega_m^{\text{target}} \leq \epsilon$

Terminate: $w_i^{\text{opt}} = \sqrt{\left|\tilde{A}_{ii}^{(n)} - A_{ii}\right|}, \quad s_i^{\text{opt}} = \text{sign}\left(\tilde{A}_{ii}^{(n)} - A_{ii}\right)$

Control of Normal Mode Frequencies

Differential Evolution (DE)

Initialize: $\omega^{\text{target}}, A_{ij}, \tilde{A}^{(n)}_{(p)}, \lambda, \kappa, \eta$

$$x^{(n)}_{(p)} = \operatorname{diag}\left(\tilde{A}^{(n)}_{(p)}\right)$$

$$v^{(n)}_{(p)} = (1-\lambda)x^{(n)}_{(p)} + \lambda x^{(n)}_{(\text{best})} + \kappa(x^{(n)}_{(r_1)} - x^{(n)}_{(r_2)}), \quad r_1 \neq r_2 \neq p$$

$$u^{(n)}_{(p)} = \begin{cases} v^{(n)}_{(p)}, & \text{if uniform}(0,1) <= \eta \\ x^{(n)}_{(p)}, & \text{otherwise} \end{cases}$$

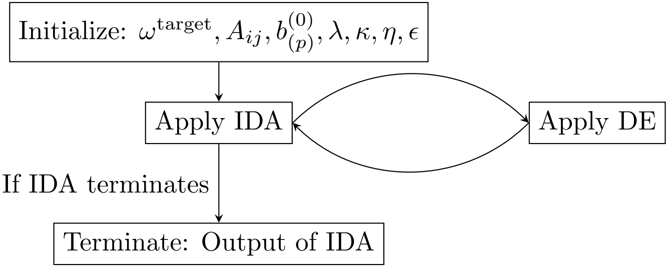
$$A'^{(n)}_{(p)ij} = \begin{cases} u^{(n)}_{(p)i}, & \text{if } i = j \\ A_{ij}, & \text{if } i \neq j \end{cases}$$

Diagonalize $\tilde{A}^{(n)}_{(p)}, A'^{(n)}_{(p)}$ to obtain $\tilde{\omega}^{(n)}_{(p)}, \tilde{b}^{(n)}_{(p)}, \omega'^{(n)}_{(p)}, b'^{(n)}_{(p)}$

$$b^{(n+1)}_{(p)} = \begin{cases} \tilde{b}^{(n)}_{(p)}, & \text{if } \tilde{\omega}^{(n)}_{(p)} \text{ closer to } \omega^{\text{target}} \\ b'^{(n)}_{(p)}, & \text{if } \omega'^{(n)}_{(p)} \text{ closer to } \omega^{\text{target}} \end{cases}$$

Control of Normal Mode Frequencies

Iterative Diagonalization Algorithm (IDA) with Differential Evolution (DE)



Control of Normal Mode Vectors

Iterative Diagonalization Algorithm (IDA)

Initialize: $b^{\text{target}}, A_{ij}, \omega^{(0)}, \epsilon$

$$A^{(n)} = b^{\text{target}} \cdot \omega^{(n)} \cdot b^{T\text{target}}$$

$$\tilde{A}_{ij}^{(n)} = \begin{cases} A_{ii}^{(n)}, & \text{if } i = j \\ A_{ij}, & \text{if } i \neq j \end{cases}$$

Diagonalize $\tilde{A}^{(n)}$ to obtain $\omega^{(n+1)}, b^{(n+1)}$

If $b^{(n+1)}$ is close to b^{target} by a measure ϵ

Terminate: $w_i^{\text{opt}} = \sqrt{\left|\tilde{A}_{ii}^{(n)} - A_{ii}\right|}, \quad s_i^{\text{opt}} = \text{sign}\left(\tilde{A}_{ii}^{(n)} - A_{ii}\right)$

Control of Normal Mode Vectors

Differential Evolution (DE)

Initialize: $b^{\text{target}}, A_{ij}, \tilde{A}^{(n)}_{(p)}, \lambda, \kappa, \eta$

$$x^{(n)}_{(p)} = \text{diag}\left(\tilde{A}^{(n)}_{(p)}\right)$$

$$v^{(n)}_{(p)} = (1-\lambda)x^{(n)}_{(p)} + \lambda x^{(n)}_{(\text{best})} + \kappa(x^{(n)}_{(r_1)} - x^{(n)}_{(r_2)}), \quad r_1 \neq r_2 \neq p$$

$$u^{(n)}_{(p)} = \begin{cases} v^{(n)}_{(p)}, & \text{if uniform}(0,1) <= \eta \\ x^{(n)}_{(p)}, & \text{otherwise} \end{cases}$$

$$A'^{(n)}_{(p)ij} = \begin{cases} u^{(n)}_{(p)i}, & \text{if } i = j \\ A_{ij}, & \text{if } i \neq j \end{cases}$$

Diagonalize $\tilde{A}^{(n)}_{(p)}, A'^{(n)}_{(p)}$ to obtain $\tilde{\omega}^{(n)}_{(p)}, \tilde{b}^{(n)}_{(p)}, \omega'^{(n)}_{(p)}, b'^{(n)}_{(p)}$

$$\omega^{(n+1)}_{(p)} = \begin{cases} \tilde{\omega}^{(n)}_{(p)}, & \text{if } \tilde{b}^{(n)}_{(p)} \text{ closer to } b^{\text{target}} \\ \omega'^{(n)}_{(p)}, & \text{if } b'^{(n)}_{(p)} \text{ closer to } b^{\text{target}} \end{cases}$$

Control of Normal Mode Vectors

Iterative Diagonalization Algorithm (IDA) with Differential Evolution (DE)