Chapter 8

S-period-lived Agent Problem with Exogenous Labor Supply, Demographic Dynamics, and Productivity Growth

In this chapter, we extend the S-period-lived agent model with exogenous labor supply from Chapter 3 to include a realistic demographic transition process as well as productivity growth. Both of these sources of growth will render the model nonstationary. In order to solve the model, we will have to be careful to correctly stationarize all of the characterizing equations.

The addition of mortality rates to the model adds some uncertainty to the household's expectation as to whether it will survive to the next period. This creates a situation in which some fraction of age-s households will die every period and leave an unintended bequest. For this reason, we will incorporate some of the modeling structures from Chapter 9 into this chapter. However, for simplicity, we will not incorporate the warm-glow bequest motive of Chapter 9.

Nishiyama (2015) and DeBacker et al. (2019) are examples of overlapping generations papers that carefully model the demographics of the households in the respective models.

8.1 Population Dynamics

We define $\omega_{s,t}$ as the number of households of age s alive at time t. A measure $\omega_{1,t}$ of households is born in each period t and live for up to $E+S$ periods, with $S \geq 4$.[1] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$. We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t} + i_1\omega_{1,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} + i_{s+1}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{8.1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific net immigration rate, ρ_s is an age-specific mortality hazard rate, and ρ_0 is an infant mortality rate.[2] The total population in the economy N_t at any period is simply the sum of households in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{8.2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{8.3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{8.4}$$

[1]Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), it is convenient for S to be at least 4.

[2]The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{8.5}$$

We discuss the approach to estimating fertility rates f_s, mortality rates ρ_s, and immigration rates i_s in Section 8.7.

8.2 Households

A measure $\omega_{1,t}$ of identical individuals are born each period, become economically relevant at age $s = E+1$ if they survive to that age, and live for up to $E+S$ periods (S economically active periods), with the population of age-s individuals in period t being $\omega_{s,t}$. Let the age of an individual be indexed by $s = \{1, 2, ...E+S\}$. An age-s individual faces the following budget constraint,

$$c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_s + \frac{BQ_t}{\tilde{N}_t} \quad \forall t \quad \text{and} \quad s \geq E+1 \tag{8.6}$$

where BQ_t represents total accidental bequests available in period t from households who died in period $t-1$. Dividing by the total economically relevant population $\tilde{N}_t$ implies that total bequests are equally distributed across the population. We will relax this assumption in Chapter 9. Total bequests are characterized by the following equation.

$$BQ_t = (1+r_t)\sum_{s=E+2}^{E+S}\rho_{s-1}\omega_{s-1,t-1}b_{s,t} \quad \forall t \tag{8.7}$$

As in Chapter 3, we assume the individuals supply a unit of labor inelastically in the first two thirds of working life ($s \leq E + \text{round}(2S/3)$) and are retired during the last third of life ($s > E + \text{round}(2S/3)$).

$$n_{s,t} = \begin{cases} 1 & \text{if} \quad E+1 \leq s \leq E + \text{round}\left(\frac{2S}{3}\right) \\ 0.2 & \text{if} \quad s > E + \text{round}\left(\frac{2S}{3}\right) \end{cases} \quad \forall t \tag{8.8}$$

Because exogenous labor in (8.8) is not dependent on the time period, we drop the t subscript from labor n_s for the rest of this section. We also assume that households are born with no wealth from savings $b_{E+1,t} = 0$ and have no incentive to save anything in the last period of life $b_{E+S+1,t} = 0$ for all periods t. Assume that $c_{s,t} \geq 0$ because negative consumption neither has an intuitive interpretation nor is it household utility defined for it. It is the latter condition that will make $c_{s,t} > 0$ in equilibrium.

Because a fraction ρ_s of all age-s households dies every period t, we must account for where these households' savings $b_{s+1,t+1}$ go. We have already shown the equation (8.7) for total bequests and in the household budget constraint (8.6) that these savings get bequeathed to the next generation. Because we add no warm-glow bequest motive as in Chapter 9, households simply save the amount they would otherwise save, and the fraction of that savings belonging to households that deceased becomes part of total bequests as in (8.7).

Let the utility of consumption in each period be defined by the constant relative risk aversion function (2.6) $u(c_{s,t})$, such that $u' > 0$, $u'' < 0$, and $\lim_{c\to 0} u(c) = -\infty$. Individuals choose lifetime consumption $\{c_{s,t+s-1}\}_{s=E+1}^{E+S}$ and savings $\{b_{s+1,t+s}\}_{s=E+1}^{E+S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$\begin{aligned} &\max_{\{c_{s,t+v}\}_{s=E+1}^{E+S},\{b_{s+1,t+v+1}\}_{s=E+1}^{E+S-1}} \sum_{s=E+1}^{E+S} \beta^{s-E-1}\left[\Pi_{u=E}^{s-1}(1-\rho_u)\right] u(c_{s,t+s-E-1}) \quad \forall s,t \\ &\text{s.t.} \quad c_{s,t} = (1+r_t)b_{s,t} + w_t n_s + \frac{BQ_t}{\tilde{N}_t} - b_{s+1,t+1} \quad \forall s,t \\ &\text{and} \quad b_{E+1,t}, b_{E+S+1,t} = 0 \quad \forall t \quad \text{and} \quad c_{s,t} \geq 0 \quad \forall s,t \end{aligned} \tag{8.9}$$

The product term in brackets $\Pi_{u=E}^{s-1}(1-\rho_u)$ multiplies each future period utility function by the cumulative probability of surviving to that period. For example, the third term in that sum $\beta^2(1-\rho_{E+1})(1-\rho_{E+2})u(c_{E+3,t+2})$ represents the discounted and mortality risk adjusted value of consumption utility in period $t+2$.

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (8.9). The optimal choice of how much to save in the each of the first $S-1$ periods of life $b_{s+1,t+1}$ is found by taking the derivative of the lifetime utility function with respect to each of the lifetime savings amounts $\{b_{s+1,t+s+1}\}_{s=E+1}^{E+S-1}$ and setting the derivatives equal to zero.

Similar to Chapter 3, the $S-1$ lifetime savings decisions $b_{s+1,t+1}$ are characterized by the following $S-1$ nonstationary dynamic Euler equations,

$$\begin{aligned} u'(c_{s,t}) = \beta(1+r_{t+1})(1-\rho_s)u'(c_{s+1,t+1}) \quad \forall t, \quad \text{and} \quad E+1\leq s\leq S-1 \\ \text{and} \quad b_{E+1,t}, b_{E+S+1,t} = 0 \quad \forall t \end{aligned} \tag{8.10}$$

and the S consumption decisions are directly implied by the S budget constraints over the household's lifetime (8.6). The difference in the Euler equations (8.10) is the presence of the extra discounting due to the mortality risk ρ_s—the risk that someone alive at age-s will die at the end of that period and not be alive for age-$s+1$. The policy functions for each of the savings decisions is a function of the individual's wealth at the beginning of the period $b_{s,t}$ and the time path of wages and interest rates over the remaining periods of the individual's life.

$$b_{s+1,t+1} = \psi_s\Bigl(b_{s,t}, \{r_v\}_{u=t}^{t+E+S-s}, \{w_u\}_{u=t}^{t+E+S-s}\Bigr) \quad \forall t \quad \text{and} \quad E+1\leq s\leq E+S-1 \tag{8.11}$$

To summarize the individual's problem, if one knows his initial savings or wealth $b_{s,t}$ and the time path of factor prices over his remaining lifetime, he can solve for all of his optimal savings levels $\{b_{s+1,t+s-E}\}_{s=E+1}^{E+S-1}$.

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates and wages $\{r_u, w_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Section 8.5, the equilibrium interest rate r_t and wage w_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \bigl\{b_{s,t}\bigr\}_{s=E+2}^{E+S} \quad \forall t \tag{8.12}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{8.13}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[3]

8.3 Firms

The production side of this economy is similar to the one in Chapter 2.2 with a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. A difference here is that we assume that the productivity of labor is growing at a constant rate g_y (labor augmenting technological change). Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology,

$$Y_t = F(K_t, L_t) \equiv AK_t^\alpha\big(e^{g_y t}L_t\big)^{1-\alpha} \quad \text{where} \quad \forall t \quad \alpha\in(0,1) \quad \text{and} \quad A>0 \tag{8.14}$$

The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t,L_t}\; AK_t^\alpha\big(e^{g_y t}L_t\big)^{1-\alpha} - (r_t+\delta)K_t - w_tL_t \tag{8.15}$$

where $\delta\in[0,1]$ is the rate of capital depreciation, and the two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha\left(\frac{Y_t}{K_t}\right) - \delta \tag{8.16}$$

$$w_t = (1-\alpha)\left(\frac{Y_t}{L_t}\right) \tag{8.17}$$

8.4 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. We will repeat the equation for total bequests (8.7) here, because it is similar to a market clearing condition in that it equates total bequests BQ_t with the accidental

[3]In Section 8.5 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 8.2.

individual bequests that comprise it. Each of these equations amounts to a statement of supply equals demand. But in their adding up, the market clearing conditions must now account for the population of all the labor supplied, savings invested, and goods consumed. Furthermore, we must account for the fact that immigrants are bringing capital with them in to the country every period.

$$L_t = \sum_{s=E+1}^{E+S} \omega_{s,t} n_s \tag{8.18}$$

$$K_t = \sum_{s=E+2}^{E+S} \Big(\omega_{s-1,t-1} b_{s,t} + i_s \omega_{s,t-1} b_{s,t}\Big) \tag{8.19}$$

$$\begin{aligned} Y_t &= C_t + I_t - \sum_{s=E+2}^{E+S} i_s \omega_{s,t} b_{s,t+1} \\ &\quad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \\ &\quad \text{and} \quad C_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} c_{s,t} \end{aligned} \tag{8.20}$$

$$BQ_t = (1+r_t) \sum_{s=E+2}^{E+S} \rho_{s-1} \omega_{s-1,t-1} b_{s,t} \forall t \tag{8.7}$$

The capital market clearing equation (8.19) includes the capital saved by the previous period's agents adjusted by the inflow of capital from immigrants ($i_s > 0$) or the outflow of capital from immigrants ($i_s < 0$). This specification has two main implications. First, we are assuming that immigrants of age-s have the same wealth $b_{s,t}$ as their domestic counterparts. This assumption greatly simplifies the state vector of the model. This specification also implies that capital investment or savings goes into the production process of the country of destination for agents either moving in or moving out. That is, capital taken out of the country by emigrants becomes productive in the foreign country's gross domestic product, and capital brought into the country by immigrants becomes productive in the domestic country's gross domestic product Y_t.

Note that the last term in the goods market clearing condition (8.20) represents the capital account portion of net exports. The only international transaction included in this model is capital imports or exports through immigration. Another way to think about the last term in (8.20) is that we must subtract out the part of K_{t+1} that is coming from immigrants. The goods market clearing equation (8.20) is redundant by Walras' Law.

8.5 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points

a. Households optimize according to equations (8.10).

b. Firms optimize according to (8.16) and (8.17).

c. Markets clear according to (8.18) and (8.19).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

Table 8.1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{s,t} \equiv \frac{c_{s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	n_s
$\hat{b}_{s,t} \equiv \frac{b_{s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{C}_t \equiv \frac{C_t}{e^{g_y t}\tilde{N}_t}$	
		$\hat{BQ}_t \equiv \frac{BQ_t}{e^{g_y t}\tilde{N}_t}$	

[a] The interest rate r_t in (8.16) is already stationary because Y_t and K_t grow at the same rate. Household labor supply n_s are exogenous, constant, and therefore stationary.

Because the variables in the equations that will characterize the equilibrium are all growing due to the labor augmenting technological change g_y in (8.14) and because of the population growth $\tilde{g}_{n,t}$ specified in (8.5), we have to stationarize the model. Table 8.1 characterizes the stationary versions of the variables of the model in terms of the variables that grow because of labor augmenting technological change, population growth, both, or none. With the definitions in Table 8.1, we must stationarize all the characterizing equations of equilibrium. The equations (8.6), (8.10), (8.16), (8.17), (8.18), (8.19), and (8.7) characterize the equilibrium, but we must implement a change of variables such that all the growth components are removed.

We start with the market clearing conditions. Aggregate labor (8.18) is a function of population weights $\omega_{s,t}$, which are growing at the population growth rate, and stationary exogenous labor supply n_s. We can stationarize the aggregate labor market clearing condition by dividing both sides of (8.18) by the total economically relevant population in period t, $\tilde{N}_t$.

$$\hat{L}_t = \sum_{s=E+1}^{E+S} \hat{\omega}_{s,t} n_s \tag{8.21}$$

The capital market clearing equation (8.19) and total bequests equations can be stationarized by dividing both sides of the respective equations by $e^{g_y t}\tilde{N}_t$.

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}} \sum_{s=E+2}^{E+S} \Big(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t} \Big) \quad \forall t \tag{8.22}$$

$$\hat{BQ}_t = \left(\frac{1+r_t}{1+\tilde{g}_{n,t}}\right) \sum_{s=E+2}^{E+S} \rho_{s-1}\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} \quad \forall t \tag{8.23}$$

Because the goods market clearing equation ends up being relatively complicated to stationarize and it is an important check whether the model is solving correctly, we show its stationary version here despite its being redundant by Walras' Law.

$$\begin{aligned} \hat{Y}_t &= \hat{C}_t + \hat{I}_t - e^{g_y} \sum_{s=E+2}^{E+S} i_s\hat{\omega}_{s,t}\hat{b}_{s,t+1} \\ &\text{where} \quad \hat{I}_t \equiv e^{g_y}(1+\tilde{g}_{n,t+1})\hat{K}_{t+1} - (1-\delta)\hat{K}_t \\ &\text{and} \quad \hat{C}_t \equiv \sum_{s=E+1}^{E+S} \hat{\omega}_{s,t}\hat{c}_{s,t} \end{aligned} \tag{8.24}$$

The price levels in the model are functions of aggregate labor L_t and the aggregate capital stock K_t through the production function Y_t. We get a stationary version of the production function by dividing both sides of (8.14) by $e^{g_y t}\tilde{N}_t$.

$$\hat{Y}_t = A\hat{K}_t^\alpha \hat{L}_t^{1-\alpha} \quad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad A > 0 \tag{8.25}$$

Note that there is no growth rate term in (8.25). The interest rate (8.16) is already stationary because aggregate output Y_t and the aggregate capital stock K_t grow at the same rate and appear in the function as a ratio.

$$r_t = \alpha\left(\frac{Y_t}{K_t}\right) - \delta = \alpha\left(\frac{\hat{Y}_t}{\hat{K}_t}\right) - \delta = \alpha A\left(\frac{\hat{L}_t}{\hat{K}_t}\right)^{1-\alpha} - \delta \tag{8.26}$$

The equilibrium wage (8.17), on the other hand, is a function of the ratio of aggregate output Y_t that grows with the population and with productivity and of aggregate labor L_t that grows only with the population. For this reason, we can see that the equilibrium wage grows only at the productivity growth rate because the population growth rates cancel out in the ratio of Y_t and L_t.

$$\hat{w}_t = (1-\alpha)\left(\frac{\hat{Y}_t}{\hat{L}_t}\right) = (1-\alpha)A\left(\frac{\hat{K}_t}{\hat{L}_t}\right)^\alpha \tag{8.27}$$

With these definitions, it can be shown that the Euler equations characterizing the equilibrium (8.10) can be written in stationary form by dividing both sides of the equation by $e^{-\sigma g_y t}$ and $e^{-\sigma g_y (t+1)}$,

$$\begin{aligned} &u'(\hat{c}_{s,t}) = e^{-\sigma g_y}\beta(1+r_{t+1})(1-\rho_s)u'(\hat{c}_{s+1,t+1}) \quad \forall t, \quad \text{and} \quad E+1 \leq s \leq S-1 \\ &\qquad \text{and} \quad \hat{b}_{E+1,t}, \hat{b}_{E+S+1,t} = 0 \quad \forall t \end{aligned} \tag{8.28}$$

and the stationarized budget constraint is,

$$\hat{c}_{s,t} = (1+r_t)\hat{b}_{s,t} + \hat{w}_t n_s + \hat{BQ}_t - e^{g_y}\hat{b}_{s+1,t+1} \quad \forall s,t \tag{8.29}$$

The easiest way to understand the equilibrium solution is to substitute the stationary market clearing conditions (8.21) and (8.22) into the firm's stationary first order conditions (8.26) and (8.27) which characterize the equilibrium wage and interest rate as functions of the distribution of capital $\hat{\mathbf{\Gamma}}_t$.

$$\hat{w}\big(\hat{\mathbf{\Gamma}}_t\big): \quad \hat{w}_t = (1-\alpha)A\left[\frac{\frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S}\big(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t}\big)}{\sum_{s=E+1}^{E+S}\hat{\omega}_{s,t}n_s}\right]^{\alpha} \quad \forall t \tag{8.30}$$

$$r\big(\hat{\mathbf{\Gamma}}_t\big): \quad r_t = \alpha A\left[\frac{\sum_{s=E+1}^{E+S}\hat{\omega}_{s,t}n_s}{\frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S}\big(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t}\big)}\right]^{1-\alpha} - \delta \quad \forall t \tag{8.31}$$

Now (8.30) and (8.31) can be substituted into household stationary Euler equations (8.28) to get the following $(S-1)$-equation system that completely characterizes the equilibrium.

$$\begin{aligned}&\left(\Big[1+r\big(\hat{\mathbf{\Gamma}}_t\big)\Big]\hat{b}_{s,t} + \hat{w}\big(\hat{\mathbf{\Gamma}}_t\big)n_s + \hat{BQ}_t\big(\hat{\mathbf{\Gamma}}_t\big) - e^{g_y}\hat{b}_{s+1,t+1}\right)^{-\sigma} = \\ &e^{-\sigma g_y}\beta\Big[1+r\big(\hat{\mathbf{\Gamma}}_{t+1}\big)\Big](1-\rho_s)\left(\Big[1+r\big(\hat{\mathbf{\Gamma}}_{t+1}\big)\Big]\hat{b}_{s+1,t+1} + \hat{w}\big(\hat{\mathbf{\Gamma}}_{t+1}\big)n_{s+1} + \hat{BQ}_{t+1}\big(\hat{\mathbf{\Gamma}}_{t+1}\big) - e^{g_y}\hat{b}_{s+2,t+2}\right)^{-\sigma} \\ &\qquad \forall t, \quad \text{and} \quad E+1 \leq s \leq E+S-1\end{aligned} \tag{8.32}$$

The system of $S-1$ nonlinear dynamic equations (8.32) characterizing the stationary lifetime savings decisions for each household $\{\hat{b}_{s+1,t+s}\}_{s=E+1}^{E+S-1}$ is not identified. Each individual knows the current distribution of stationary capital $\hat{\mathbf{\Gamma}}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\hat{\mathbf{\Gamma}}_{t+1} = \{\hat{b}_{s+1,t+1}\}_{s=E+1}^{E+S-1}$ for all agents alive next period, and for a policy function for the individual $\hat{b}_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of stationary endogenous variable $\hat{x}_t$ be characterized by $\hat{x}_{t+1} = \hat{x}_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the stationary steady-state equilibrium as follows.

Definition 8.1 (Stationary steady-state equilibrium). A stationary non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents, exogenous labor supply, productivity growth, and population dynamics is defined as constant allocations of stationary consumption $\{\bar{c}_s\}_{s=E+1}^{E+S}$, savings $\{\bar{b}_s\}_{s=E+2}^{E+S}$, and prices $\bar{w}$

and $\bar{r}$ such that:

i. households optimize according to (8.28),

ii. firms optimize according to (8.26) and (8.27),

iii. markets clear according to (8.21), (8.22), and (8.23),

iv. The population has reached its stationary steady-state distribution $\{\bar{\omega}_s\}_{s=E+1}^{E+S}$ and steady-state population growth rate $\bar{g}_n$ as characterized in Section 8.7.5.

The characterizing equations in Definition 8.1 reduce to the system (8.32) in which all the variables are their steady-state versions. These $S-1$ equations are exactly identified in the steady state. That is, they are $S-1$ equations and $S-1$ unknowns $\{\bar{b}_s\}_{s=E+2}^{E+S}$.

$$\begin{aligned}&\left(\Big[1+r\big(\bar{\mathbf{\Gamma}}\big)\Big]\bar{b}_s+\hat{w}\big(\bar{\mathbf{\Gamma}}\big)n_s+\bar{BQ}\big(\bar{\mathbf{\Gamma}}\big)-e^{g_y}\bar{b}_{s+1}\right)^{-\sigma}= \\ &e^{-\sigma g_y}\beta\Big[1+r\big(\bar{\mathbf{\Gamma}}\big)\Big]\big(1-\rho_s\big)\left(\Big[1+r\big(\bar{\mathbf{\Gamma}}\big)\Big]\bar{b}_{s+1}+\hat{w}\big(\bar{\mathbf{\Gamma}}\big)n_{s+1}+\bar{BQ}\big(\bar{\mathbf{\Gamma}}\big)-e^{g_y}\bar{b}_{s+2}\right)^{-\sigma} \\ &\qquad\text{for}\quad E+1\leq s\leq E+S-1\end{aligned} \tag{8.33}$$

We can solve for steady-state $\{\bar{b}_s\}_{s=E+2}^{E+S}$ by using an unconstrained root finder as we did in Chapter 3. Then we solve for $\bar{w}$, $\bar{r}$, and $\{\bar{c}_s\}_{s=E+1}^{E+S}$ by substituting $\{\bar{b}_s\}_{s=E+2}^{E+S}$ into the equilibrium firm first order conditions (8.30) and (8.31) and the substituting $\{\bar{b}_s\}_{s=E+2}^{E+S}$, $\bar{w}$, and $\bar{r}$ into the stationary household budget constraints (8.29).

Now we can describe the stationary non-steady-state functional equilibrium of the model in which each endogenous variable chosen in period t is a function of the state vector $\hat{\mathbf{\Gamma}}_t$, which is the distribution of stationary capital at time t.

Definition 8.2 (Stationary non-steady-state functional equilibrium). A stationary non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents, exogenous labor supply, productivity growth, and population dynamics is defined as stationary allocation functions of the state $\big\{\hat{b}_{s+1,t+1}=\psi_s\big(\hat{\mathbf{\Gamma}}_t\big)\big\}_{s=E+1}^{E+S-1}$ and stationary price functions $\hat{w}(\hat{\mathbf{\Gamma}}_t)$ and $r(\hat{\mathbf{\Gamma}}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (8.13), and those beliefs about the future distribution of stationary savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u}=\hat{\mathbf{\Gamma}}^e_{t+u}=\Omega^u\left(\hat{\mathbf{\Gamma}}_t\right)\quad\forall t,\quad u\geq 1$$

ii. households optimize according to (8.28),

iii. firms optimize according to (8.26) and (8.27),

iv. markets clear according to (8.21), (8.22), and (8.23).

We have already shown how to reduce the characterizing equations in Definition 8.2 to $S-1$ equations (8.32) and $S-1$ unknowns. But we have also seen that those $S-1$ equations are not identified. So how do we solve for these equilibrium functions? The solution to the non-steady-state equilibrium in Definition 8.2 is a fixed point in function space. Choose $S-1$ functions $\{\psi_s\}_{s=E+1}^{E+S-1}$ and verify that they satisfy the Euler equations for all points in the state space (all possible values of the state).

Figure 8.1: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$

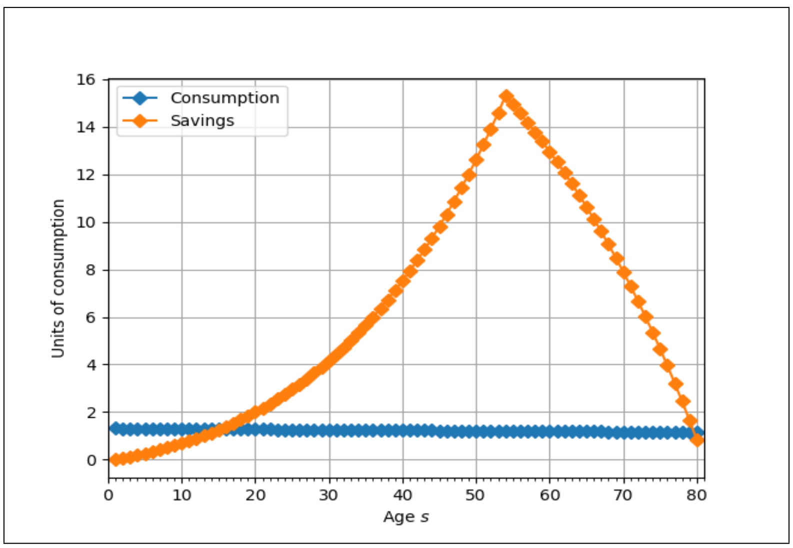


Table 8.2: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.037	Max. absolute savings Euler error	6.42e-09
$\bar{w}$	1.373	Resource constraint error	2.84e-14
$\bar{K}$	494.1	Serial computation time	0.0237 sec.

Figure 8.1 shows the steady-state distributions of consumption $\bar{c}_s$ and savings $\bar{b}_{s+1}$. The steady-state capital stock for this calibration is $\bar{K} = 494.1$, and the steady-state interest

rate and wage are $\bar{r} = 0.037$ and $\bar{w} = 1.373$, respectively. Table 8.2 lists the steady-state values of the aggregate variables as well as the characterizing equation errors of our solution as evidence that we have found the steady-state equilibrium.

8.6 Solution method: time path iteration (TPI)

The solution method is time path iteration (TPI). The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 8.1 in a finite number of periods $T < \infty$ regardless of the initial state $\hat{\boldsymbol{\Gamma}}_1$.

In Chapter 3, we only had to guess a time path for the aggregate capital stock $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, which then determined the respective time paths of prices $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. However, in this model, these time paths are not enough to be able to solve all the households' respective problems. We must also guess the time path for total bequests $\hat{\boldsymbol{BQ}} = \left\{\hat{BQ}_1^i, \hat{BQ}_2^i, ...\hat{BQ}_T^i\right\}$.

The first step is to assume a transition path for stationary aggregate capital $\hat{\boldsymbol{K}}^i$ and total bequests $\hat{\boldsymbol{BQ}}^i$ such that T is sufficiently large to ensure that $\hat{\boldsymbol{\Gamma}}_T = \bar{\boldsymbol{\Gamma}}$. The superscript i is an index for the iteration number. The exogenously supplied population demographics $\hat{\omega}_{s,t}$ reach a steady-state in a known number of periods (see Section 8.7.5). T must be greater than that number of periods.

The transition path for stationary aggregate capital determines the transition path for both the stationary wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the interest rate $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $\hat{\boldsymbol{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies $\hat{K}_1^i = \frac{1}{1+\tilde{g}_{n,1}}\sum_{s=E+2}^{E+S}\hat{\omega}_{s-1,0}\hat{b}_{s,1}$ according to market clearing condition (8.22) and satisfies $BQ_1^i = \left(\frac{1+r_t^i}{1+\tilde{g}_{n,1}}\right)\sum_{s=E+2}^{E+S}\rho_{s-1}\hat{\omega}_{s-1,0}\hat{b}_{s,1}$ according to market clearing condition (8.29). Note that both of these conditions require knowing the population growth rate from the period before the initial period to the initial period $\tilde{g}_{n,1}$ as well as the stationary population distribution from the period before the initial period $\{\hat{\omega}_{s,0}\}_{s=E+1}^{E+S-1}$. One could also first choose the initial stationary distribution of capital $\hat{\boldsymbol{\Gamma}}_1$ and then choose an initial stationary aggregate capital stock $\hat{K}_1^i$ and total bequests $\hat{BQ}_1^i$ that corresponds to that distribution. The only other restrictions on the initial transition paths for aggregate capital and total bequests is that they both equal the steady-state level $\hat{K}^i_T = \bar{K} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S}\big(\bar{\omega}_{s-1}\bar{b}_s + i_s\bar{\omega}_s\bar{b}_s\big)$ by period T and $\hat{BQ}^i_T = \left(\frac{1+\bar{r}}{1+\bar{g}_n}\right)\sum_{s=E+2}^{E+S}\rho_{s-1}\bar{\omega}_{s-1}\bar{b}_s$. But the initial guesses for the aggregate capital stocks $\hat{K}^i_t$ and total bequests $\hat{BQ}^i_t$ for periods $1 < t < T$ can be any level.

Given the initial capital distribution $\hat{\mathbf{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}^i_1, \hat{K}^i_2, ...\hat{K}^i_T\right\}$, total bequests $\hat{\boldsymbol{BQ}}^i = \left\{\hat{BQ}^i_1, \hat{BQ}^i_2, ...\hat{BQ}^i_T\right\}$, the wage $\hat{\boldsymbol{w}}^i = \{\hat{w}^i_1, \hat{w}^i_2, ...\hat{w}^i_T\}$, and the interest rate $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_T\}$, one can solve for the optimal savings decision for the initial age $s = E+S-1$ individual for the last period of his life $\hat{b}_{E+S,2}$ using his last stationary intertemporal Euler equation.

$$\begin{split}\Big([1+r^i_1]\hat{b}_{E+S-1,1} + \hat{w}^i_1 n_{E+S-1} + \hat{BQ}^i_1 - e^{g_y}\hat{b}_{E+S,2}\Big)^{-\sigma} = \\ e^{-\sigma g_y}\beta\big(1+r^i_2\big)\big(1-\rho_{E+S-1}\big)\Big([1+r^i_2]\hat{b}_{E+S,2} + \hat{w}^i_2 n_{E+S} + \hat{BQ}^i_2\Big)^{-\sigma}\end{split} \tag{8.34}$$

Notice that everything in equation (8.34) is known except for the savings decision $\hat{b}_{E+S,2}$. This is one equation and one unknown.

The next step is to solve for the remaining lifetime savings decisions for the next oldest individual alive in period $t = 1$. This individual is age $s = E+S-2$ and has two remaining savings decisions $b_{E+S-1,2}$ and $b_{E+S,3}$. From (8.32), we know that the two equations that characterize these two decisions are the following.

$$\begin{split}\Big([1+r^i_1]\hat{b}_{E+S-2,1} + \hat{w}^i_1 n_{E+S-2} + \hat{BQ}^i_1 - e^{g_y}\hat{b}_{E+S-1,2}\Big)^{-\sigma} = \\ e^{-\sigma g_y}\beta(1+r^i_2)(1-\rho_{E+S-2})\Big([1+r^i_2]\hat{b}_{E+S-1,2} + \hat{w}^i_2 n_{E+S-1} + \hat{BQ}^i_2 - e^{g_y}\hat{b}_{E+S,3}\Big)^{-\sigma}\end{split} \tag{8.35}$$

$$\begin{split}\Big([1+r^i_2]\hat{b}_{E+S-1,2} + \hat{w}^i_2 n_{E+S-1} + \hat{BQ}^i_2 - e^{g_y}\hat{b}_{E+S,3}\Big)^{-\sigma} = \\ e^{-\sigma g_y}\beta(1+r^i_3)(1-\rho_{E+S-1})\Big([1+r^i_3]\hat{b}_{E+S,3} + \hat{w}^i_3 n_{E+S} + \hat{BQ}^i_3\Big)^{-\sigma}\end{split} \tag{8.36}$$

Euler equations (8.35) and (8.36) represent two equations and two unknowns $\hat{b}_{E+S-1,2}$ and $\hat{b}_{E+S,3}$. Everything else is known.

We continue solving the remaining lifetime decisions of each individual alive between periods 1 and T. This includes all the individuals who were already alive in period 1 and therefore have fewer than $S-1$ savings decisions for which to solve. It also includes all the individuals born between periods 1 and T for whom we have the full set of $S-1$ lifetime decisions. Once we have solved for all the individual savings decisions for individuals alive between periods 1 and T, then we have the complete distribution of savings $\{\hat{\mathbf{\Gamma}}_t\}_{t=1}^T$ for each period between 1 and T. We can use this to compute new time paths of the aggregate capital stock and total bequests consistent with the individual savings decisions $\hat{K}_t^{i'} = \frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S}\left(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t}\right)$ and $\hat{BQ}_t^{i'} = \left(\frac{1+r_t}{1+\tilde{g}_{n,t}}\right)\sum_{s=E+2}^{E+S}\rho_{s-1}\hat{\omega}_{s-1,t-1}\hat{b}_{s,t}$ for all $1\leq t\leq T$. We place a "$'$" on the iteration index of these aggregate time series because, in general, $\hat{K}_t^{i'}\neq\hat{K}_t^i$ and $\hat{BQ}_t^{i'}\neq\hat{BQ}_t^i$. That is, the initial conjectured paths of the aggregate capital stock and total bequests from which the savings decisions were made is not necessarily equal to the paths of the aggregate capital stock and total bequests consistent with those savings decisions.[4]

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and total bequests. Common norms to use are the L^2 and the L^∞ norms. Then the fixed point necessary for the equilibrium transition path from Definition 8.2 has been found when the distance between $\left(\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{BQ}}^{i'}\right)$ and $\left(\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{BQ}}^{i}\right)$ is arbitrarily close to zero.

$$\left\|\left(\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{BQ}}^{i'}\right)-\left(\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{BQ}}^{i}\right)\right\|<\varepsilon\quad\text{for}\quad\varepsilon>0 \tag{8.37}$$

If the fixed point has not been found $\left\|\left(\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{BQ}}^{i'}\right)-\left(\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{BQ}}^{i}\right)\right\|>\varepsilon$, then new transition paths for the aggregate capital stock and total bequests are generated as a convex combination of $\left(\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{BQ}}^{i'}\right)$ and $\left(\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{BQ}}^{i}\right)$.

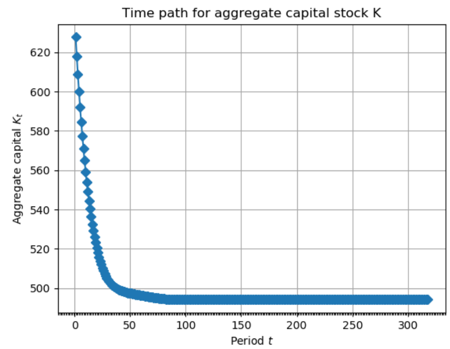
$$\left(\hat{\boldsymbol{K}}^{i+1},\hat{\boldsymbol{BQ}}^{i+1}\right)=\xi\left(\hat{\boldsymbol{K}}^{i'},\hat{\boldsymbol{BQ}}^{i'}\right)+(1-\xi)\left(\hat{\boldsymbol{K}}^{i},\hat{\boldsymbol{BQ}}^{i}\right)\quad\text{for}\quad\xi\in(0,1) \tag{8.38}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization. TPI solves for the equilibrium transition path from Definition 8.2 by finding a fixed point in the time path of the economy.

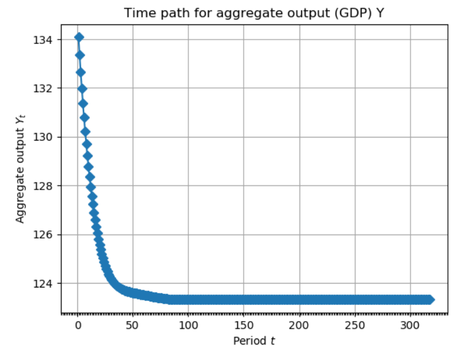
The four panels of Figure 8.2 show the equilibrium time paths of the stationary aggregate

[4] A check here for whether T is large enough is if $\hat{K}_T^{i'}=\bar{K}$ and $\hat{BQ}_T^{i'}=\bar{BQ}$ as well as those aggregate variables for the following periods. If not, then T needs to be larger.

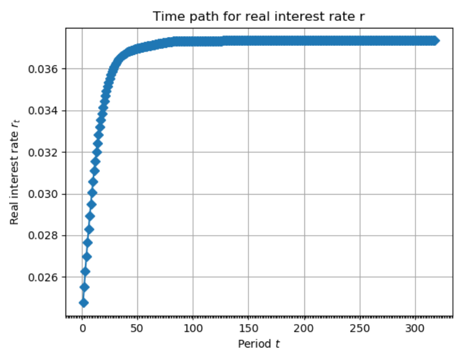
Figure 8.2: Equilibrium transition paths of prices and aggregate variables



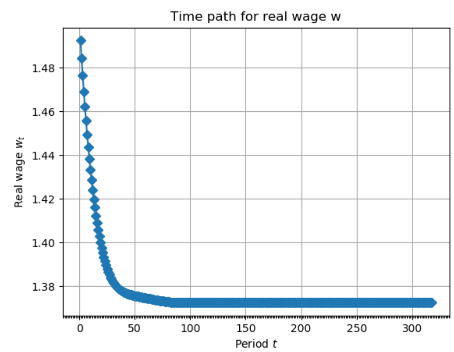
(a) Aggregate capital



(b) Aggregate output

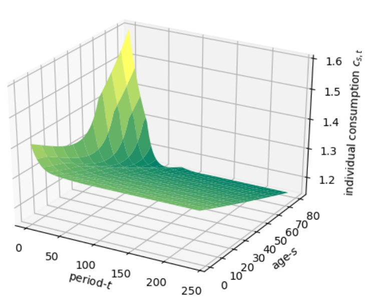


(c) Interest rate

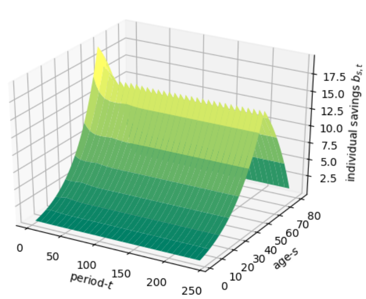


(d) Wage

Figure 8.3: Equilibrium transition paths of distributions of consumption and savings



(a) Household consumption



(b) Household savings

Table 8.3: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute savings Euler error	3.34e-14
Maximum absolute resource constraint error	?
Serial computation time	2 min. 0.2 sec.

variables r_t, $\hat{w}_t$, $\hat{K}_t$, and $\hat{Y}_t$. Figure 8.3 shows the equilibrium time path of stationary individual consumption $\hat{c}_{s,t}$ and savings $\hat{b}_{s+1,t+1}$. Table 8.3 lists the maximum absolute values across the time path of the characterizing equation errors of our solution as evidence that we have found the steady-state equilibrium.

8.7 Calibration

In this section, we discuss the calibration of the exogenous parameters of the model, with a special emphasis on estimating the parameters associated with the demographic dynamics. Assume that agents live for $E+S$ total periods, are not economically relevant for the first E periods, are born into economic relevance at age $s=E+1$ and live until a maximum age os $s=E+S$. This implies S economically relevant years. Let $E=20$ and $S=80$ such that each economically relevant model period represents a year from ages $s=E+1=21$ to $s=E+S=100$.

The first subsection 8.7.1 discusses the calibration of the non-population related model parameters. Subsections 8.7.2, 8.7.3, and 8.7.4 deal with a general approach for calibrating fertility rates f_s, mortality rates ρ_s, and immigration rate i_s, which are the inputs to solving for the time-path and steady state of of the population distribution $\{\hat{\omega}_{s,t}\}_{s=1}^{E+S}$ and growth rate $\tilde{g}_{n,t}$, which is the topic of Section 8.7.5.

8.7.1 Non-population parameters

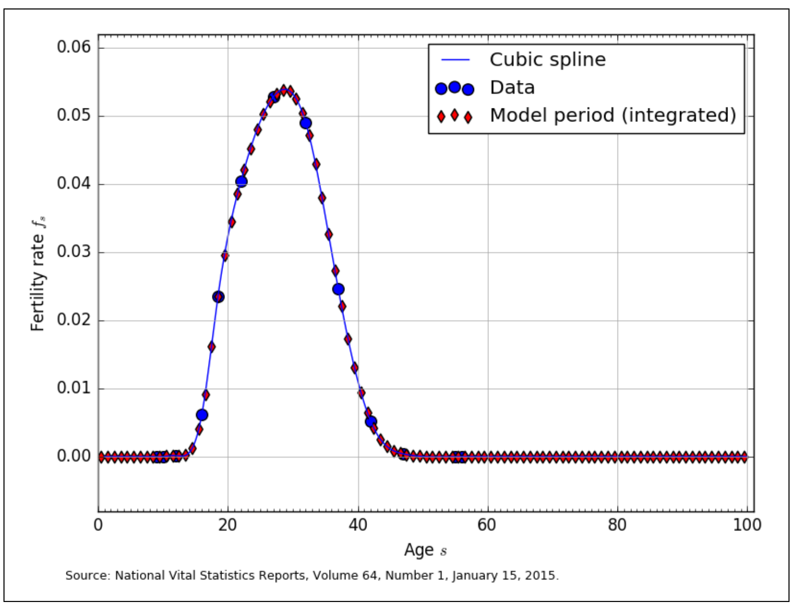
In general, the time-dependent parameters can be written as functions of total economically relevant lifetime model periods S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is

$\beta = 0.96^{80/S}$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1 - 0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 2.2$, let the productivity scale parameter of firms be $A = 1$, let the capital share of income be $\alpha = 0.35$, and let the growth rate of labor augmenting technological change be $g_y = 0.03$.

8.7.2 Fertility rates

In this model, we assume that the fertility rates for each age cohort f_s are constant across time. However, this assumption is conceptually straightforward to relax. Our data for U.S. fertility rates by age come from Martin et al. (2015, Table 3, p. 18) National Vital Statistics Report, which is final fertility rate data for 2013. Figure 8.4 shows the fertility-rate data and the estimated average fertility rates for $E + S = 100$.

Figure 8.4: Fertility rates by age (f_s) for $E + S =$ 100



The large blue circles are the 2013 U.S. fertility rate data from Martin et al. (2015). These are 9 fertility rates $[0.3, 12.3, 47.1, 80.7, 105.5, 98.0, 49.3, 10.4, 0.8]$ that correspond to the midpoint ages of the following age (in years) bins $[10-14, 15-17, 18-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49]$. In order to get our cubic spline interpolating function to fit better at the endpoints we added to fertility rates of zero to ages 9 and 10, and we added two fertility rates of zero to ages 55 and 56. The blue line in Figure 8.4 shows the cubic spline interpolated function of the data.

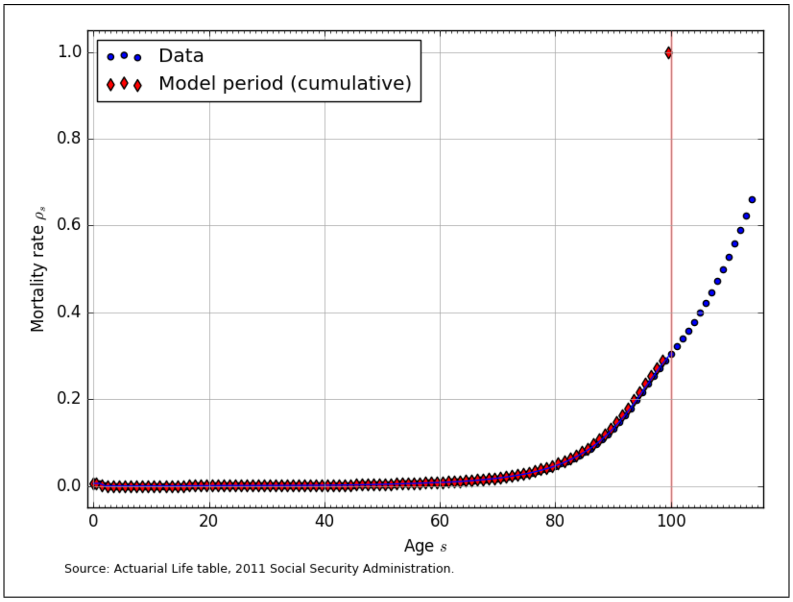
The red diamonds in Figure 8.4 are the average fertility rate in age bins spanning households born at the beginning of period 1 (time = 0) and dying at the end of their 100th year. Let the total number of model years that a household lives be $E + S \leq 100$. Then the span from the beginning of period 1 (the beginning of year 0) to the end of period 100 (the end of year 99) is divided up into $E + S$ bins of equal length. We calculate the average fertility rate in each of the $E+S$ model-period bins as the average population-weighted fertility rate in that span. The red diamonds in Figure 8.4 are the average fertility rates displayed at the midpoint in each of the $E + S$ model-period bins.

8.7.3 Mortality rates

The mortality rates in our model ρ_s are a one-period hazard rate and represent the probability of dying within one year, given that an household is alive at the beginning of period s. We assume that the mortality rates for each age cohort ρ_s are constant across time. The infant mortality rate of $\rho_0 = 0.00587$ comes from the 2015 U.S. CIA World Factbook. Our data for U.S. mortality rates by age come from the Actuarial Life Tables of the U.S. Social Security Administration (see Bell and Miller, 2015), from which the most recent mortality rate data is for 2011. Figure 8.5 shows the mortality rate data and the corresponding model-period mortality rates for $E + S = 100$.

The mortality rates in Figure 8.5 are a population-weighted average of the male and female mortality rates reported in Bell and Miller (2015). Figure 8.5 also shows that the data provide mortality rates for ages up to 111-years-old. We truncate the maximum age in years in our model to 100-years old. In addition, we constrain the mortality rate to be 1.0 or 100 percent at the maximum age of 100.

Figure 8.5: Mortality rates by age (ρ_s) for $E+S = 100$



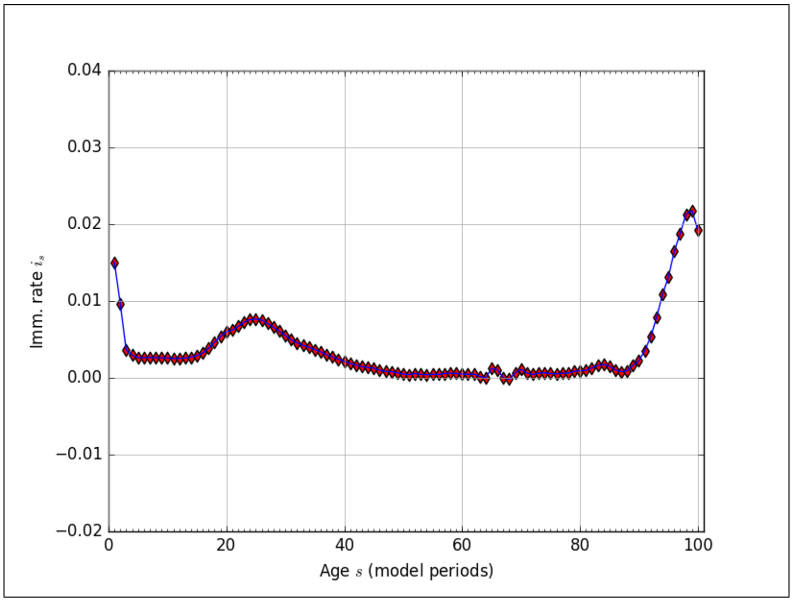
8.7.4 Immigration rates

Because of the difficulty in getting accurate immigration rate data by age, we estimate the immigration rates by age in our model i_s as the average residual that reconciles the current-period population distribution with next period's population distribution given fertility rates f_s and mortality rates ρ_s. Solving equations (8.1) for the immigration rate i_s gives the following characterization of the immigration rates in given population levels in any two consecutive periods $\omega_{s,t}$ and $\omega_{s,t+1}$ and the fertility rates f_s and mortality rates ρ_s.

$$\begin{aligned} i_1 &= \frac{\omega_{1,t+1} - (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t}}{\omega_{1,t}} \quad \forall t \\ i_{s+1} &= \frac{\omega_{s+1,t+1} - (1-\rho_s)\omega_{s,t}}{\omega_{s+1,t}} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{8.39}$$

We calculate our immigration rates for three different consecutive-year-periods of population distribution data (2010 through 2013). Our four years of population distribution by age data come from Bureau (2015). The immigration rates i_s that we use in our model are the residuals described in (8.39) averaged across the three periods. Figure 8.6 shows the

Figure 8.6: Immigration rates by age (i_s), residual, $E+S=100$



estimated immigration rates for $E+S=100$ and given the fertility rates from Section 8.7.2 and the mortality rates from Section 8.7.3.

At the end of Section 8.7.5, we describe a small adjustment that we make to the immigration rates after a certain number of periods in order to make computation of the transition path equilibrium of the model from Definition 8.2 compute more robustly.

8.7.5 Population steady-state and transition path

This model requires information about mortality rates ρ_s in order to solve for the household's problem each period. It also requires the steady-state stationary population distribution $\bar{\omega}_s$ and population growth rate $\bar{g}_n$ as well as the full transition path of the stationary population distribution $\hat{\omega}_{s,t}$ and population grow rate $\tilde{g}_{n,t}$ from the current state to the steady-state. To solve for the steady-state and the transition path of the stationary population distribution, we write the stationary population dynamic equations (8.40) and their matrix representation

(8.41).

$$\begin{aligned}\hat{\omega}_{1,t+1} &= \frac{(1-\rho_0)\sum_{s=1}^{E+S} f_s\hat{\omega}_{s,t} + i_1\hat{\omega}_{1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \\ \hat{\omega}_{s+1,t+1} &= \frac{(1-\rho_s)\hat{\omega}_{s,t} + i_{s+1}\hat{\omega}_{s+1,t}}{1+\tilde{g}_{n,t+1}} \qquad \forall t \quad \text{and} \quad 1\leq s\leq E+S-1\end{aligned} \tag{8.40}$$

$$\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...$$

$$\begin{bmatrix} (1-\rho_0)f_1+i_1 & (1-\rho_0)f_2 & (1-\rho_0)f_3 & \dots & (1-\rho_0)f_{E+S-1} & (1-\rho_0)f_{E+S} \\ 1-\rho_1 & i_2 & 0 & \dots & 0 & 0 \\ 0 & 1-\rho_2 & i_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & i_{E+S-1} & 0 \\ 0 & 0 & 0 & \dots & 1-\rho_{E+S-1} & i_{E+S} \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{8.41}$$

We can write system (8.41) more simply in the following way.

$$\boldsymbol{\hat{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}}\boldsymbol{\Omega}\boldsymbol{\hat{\omega}}_t \quad \forall t \tag{8.42}$$

The stationary steady-state population distribution $\boldsymbol{\bar{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (8.42).

$$(1+\bar{g}_n)\boldsymbol{\bar{\omega}} = \boldsymbol{\Omega}\boldsymbol{\bar{\omega}} \tag{8.43}$$

Proposition 8.1. If the age $s=1$ immigration rate is $i_1 > -(1-\rho_0)f_1$ and the other immigration rates are strictly positive $i_s > 0$ for all $s\geq 2$ such that all elements of $\boldsymbol{\Omega}$ are nonnegative, then there exists a unique positive real eigenvector $\boldsymbol{\bar{\omega}}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible. This can be easily shown. The matrix is of the form

$$\boldsymbol{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\boldsymbol{\Omega}^2 = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}; \quad \boldsymbol{\Omega}^{S+E-1} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

$$\boldsymbol{\Omega}^{S+E} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

Existence of an $m \in \mathbb{N}$ such that $(\boldsymbol{\Omega}^{\mathbf{m}})_{ij} \neq 0 \;\; (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

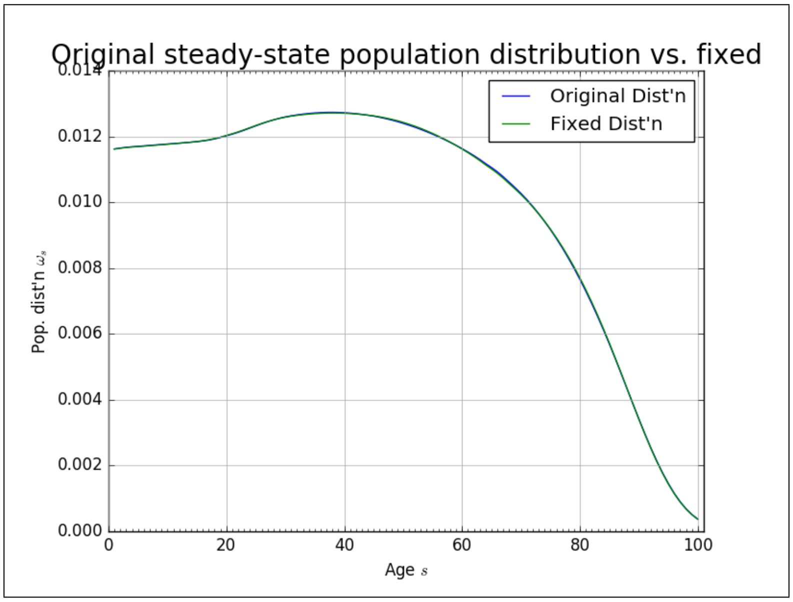
We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\boldsymbol{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. □

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$.

Figure 8.7 shows the steady-state population distribution $\bar{\boldsymbol{\omega}}$ and the population distribution after 120 periods $\hat{\boldsymbol{\omega}}_{120}$. Although the two distributions look very close to each other, they are not exactly the same.

Further, we find that the maximum absolute difference between the population levels $\hat{\omega}_{s,t}$ and $\hat{\omega}_{s,t+1}$ was 1.3852×10^{-5} after 160 periods. That is to say, that after 160 periods, given the estimated mortality, fertility, and immigration rates, the population has not achieved its steady state. For convergence in our solution method over a reasonable time horizon, we want the population to reach a stationary distribution after T periods. To do this, we artificially impose that the population distribution in period $t = 120$ is the steady-state. As

Figure 8.7: Theoretical steady-state population distribution vs. population distribution at period $t = 120$



can be seen from Figure 8.7, this assumption is not very restrictive. Figure 8.8 shows the change in immigration rates that would make the period $t = 120$ population distribution equal be the steady-state. The maximum absolute difference between any two corresponding immigration rates in Figure 8.8 is 0.0028.

The most recent year of population data come from Bureau (2015) population estimates for both sexes for 2013. We those data and use the population transition matrix (8.42) to age it to the current model year of 2015. We then use (8.42) to generate the transition path of the population distribution over the time period of the model. Figure 8.9 shows the progression from the 2013 population data to the fixed steady-state at period $t = 120$. The time path of the growth rate of the economically active population $\tilde{g}_{n,t}$ is shown in Figure 8.10.

Figure 8.8: Original immigration rates vs. adjusted immigration rates to make fixed steady-state population distribution

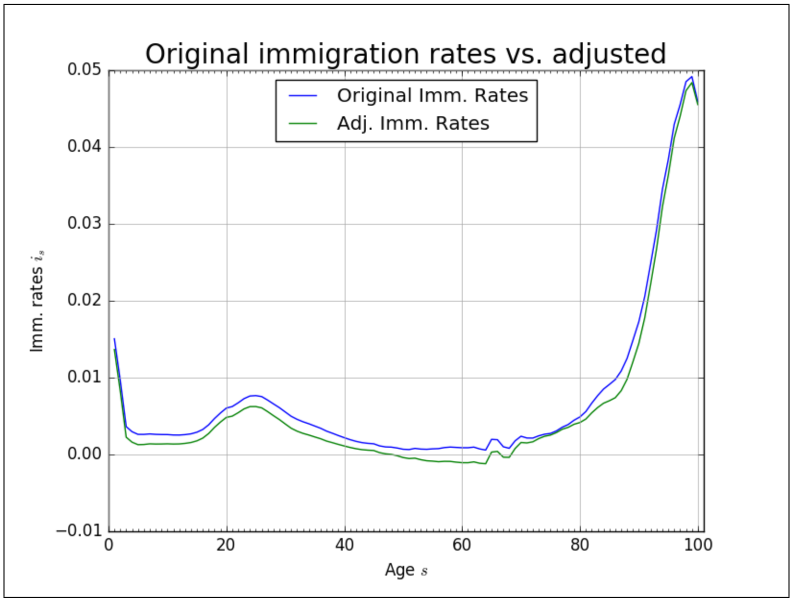


Figure 8.9: Stationary population distribution at periods along transition path

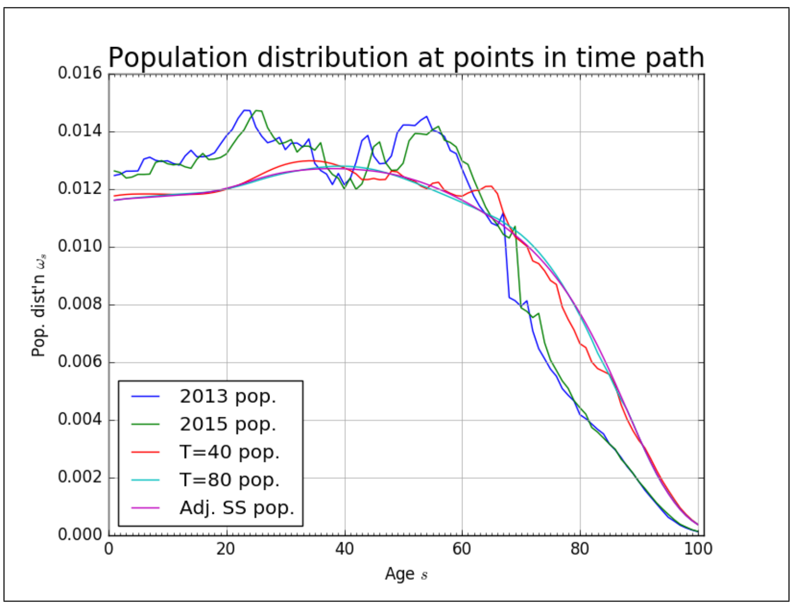
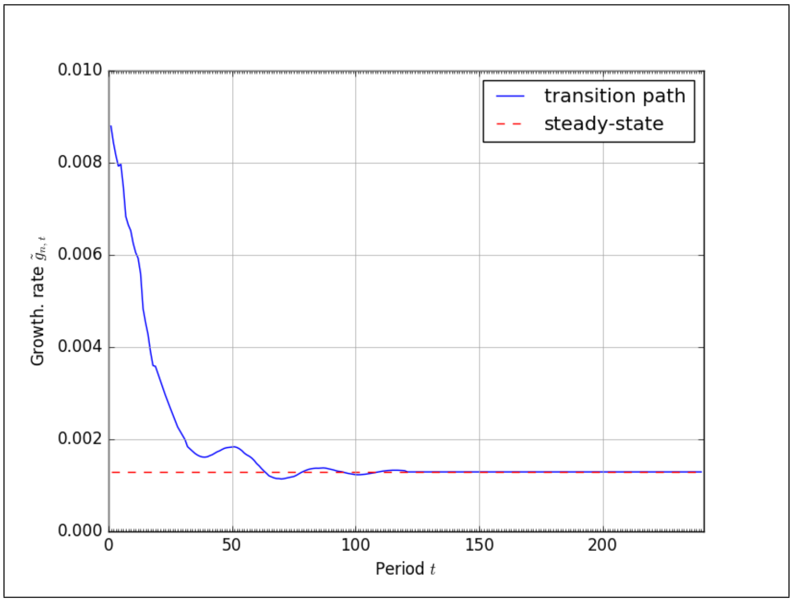


Figure 8.10: Time path of the population growth rate $\tilde{g}_{n,t}$



8.8 Exercises

Exercise 8.1. Fertility rates function. Get current U.S. fertility data (births per 1,000 women) from the National Vital Statistics Reports, volume 64, number 1. Use Table 3 from the appendix for fertility rates for the age categories 10-14, 15-17, 18-19, 20-24, 25-29, 30-34, 35-39, 40-44, and 45-49. Assume that the fertility value in each of those age bins is associated with the midpoint age in those bins. Convert the fertility rates to the form of births per person (both men and women) of a given age. For example, if the fertility value for women age 20-24 is 80.7 births per 1,000 women, then the fertility rate associated with the midpoint of age 22 is 0.04035 (80.7/2,000). Write a function `get_fert()` that has the following form,

```
fert_rates = get_fert(totpers, graph)
```

where `totpers` is the number of periods an agent lives ($E + S$), `fert_rates` is a vector of fertility rates of length `totpers` for each model period, and `graph` is a Boolean that equals `True` if you want to output a plot of the fertility rates. Inside the `get_fert()` function, use the function `scipy.interpolate.interp1d()` to fit a smoothly interpolated fertility rate function of age for 100 evenly spaced periods from 1 to 100. Then any number of periods greater than 100 or less than 100 evenly spans that period of years. Regardless of how many periods an individual's life is in the model (`totpers`), the number of years to which those total model periods correspond is 100. An agent is born at the beginning of year-1 and dies at the end of year-100. The total model periods (`totpers`) span that period of 100 years.

a. Plot the interpolated fertility rate function for `totpers`=100.

b. Plot the interpolated fertility rate function for `totpers`=80 as scatter points on top of the line you produced in part (a).

c. Plot the interpolated fertility rate function for `totpers`=20 as scatter points on top of the line you produced in part (a).

Exercise 8.2. Mortality rates function. Look up the infant mortality value (number of births per 1,000 births that die at birth or before they turn one year old). Transform that number into a mortality rate (percent of total births that die at birth or before they turn one year old). Get current U.S. mortality data by age from the Actuarial Life Table, 2011 from the U.S. Social Security Administration, which is available as `mort_rates2011.csv`. These data give mortality rates by age (percent of age-s individuals that died in that year) by gender. You need to calculate the total fertility rate (both men and women) by age using the population of men and women, respectively, in each of those age groups. Again, assume that agents in your model live for 100 years. Those years start at the beginning of year-1 and finish at the end of year-100. In the data, let the mortality rate from age=0 correspond to the mortality rate of year-1, and let the mortality rate in the data from age-98 correspond to the mortality rate of year-99. Set the mortality rate of year-100 to 1.0, such that everyone dies after 100 years. Write a function `get_mort()` that has the following form,

```
mort_rates, infmort_rate = get_mort(totpers, graph)
```

where `totpers` is the number of periods an agent lives ($E + S$), `mort_rates` is a vector of mortality rates of length `totpers` for each model period (percent of individuals age-s that die each period), and `graph` is a Boolean that equals `True` if you want to output a plot of the mortality rates. Note that when `totpers`$<$ 100, each model period has a duration that is longer than a year. So mortality rates will not just be a simple interpolation along the mortality rate curve when `totpers`=100, as in part (a). You will have to calculate the cumulative mortality rate for the model age bin given the annual mortality rates and total population levels for the given ages from the data.

a. Plot the mortality rate function with the infant mortality rate at age ($s = 0$) for `totpers`=100.

b. Plot the mortality rate function with the infant mortality rate at age ($s = 0$) for `totpers`=80 as scatter points in the same plot of the line you produced in part (a).

c. Plot the mortality rate function with the infant mortality rate at age ($s = 0$) for `totpers`=20 as scatter points in the same plot of the line you produced in part (a).

Exercise 8.3. Immigration rates function. We will treat immigration rates as a residual based on two-consecutive periods of population data by age, mortality rates by age, and fertility rates by age. Get population level data by age from the comma-delimited file pop_data.csv taken from the U.S. Census Bureau Population Estimates. This file has U.S. population totals by age (ages 0 to 99) for years 2012 and 2013. For the model with $E+S=100$, treat age-0 in the data as model age 1 and age-99 in the data as model age 99. Let the immigration rates i_s be implicitly defined by the population dynamics equations (8.1) and explicitly defined in Section 8.7.4 as equations (8.39). Write a function get_imm_resid() that has the following form,

```
imm_rates = get_imm_resid(totpers, graph)
```

where totpers is the number of periods an agent lives ($E+S$), imm_rates is a vector of immigration rates of length totpers for each model period (percent of individuals age-s immigrate into the country), and graph is a Boolean that equals True if you want to output a plot of the mortality rates. Note that negative immigration rates imply the people of age-s are moving out of the country. You will want to call your get_fert() and get_mort() functions inside of this get_imm_resid() function.

a. Plot the immigration rates for totpers=100.

b. Plot the immigration rates for totpers=80.

c. Plot the immigration rates for totpers=20.

Exercise 8.4. Time path of the population distribution. Let $E=20$ and $S=80$. For the following problems, use the fertility rates f_s, mortality rates ρ_s, and immigration rates i_s from Exercises 8.1, 8.2, and 8.3. You must use the stationary version of the population dynamics equations (8.40).

a. Use the instructions in Section 8.7.5 to solve for and plot the steady-state stationary population distribution $\bar{\omega}_s$.

b. Create an $80 \times T+S-2$ matrix that contains the time path of the stationarized population distribution $\hat{\omega}_{s,t}$ for the economically relevant population $E+1 \leq s \leq E+S$ over the time path $1 \leq t \leq T+S-2$. Let the first column be $\hat{\omega}_{s,1}$. Calculate $\hat{\omega}_{s,1}$ from the 2013 population by age data from the file pop_data.csv. And every column for $t > T$ should be the steady-state stationary population distribution $\bar{\omega}_s$.

c. Make a single plot that shows the stationary population distribution at time $t = 1, 10, 30$, and T.

d. Plot the population growth rate of the economically relevant population $\tilde{g}_{n,t}$ for $1 \leq t \leq T + S - 2$.

Exercise 8.5. Solve for the steady-state equilibrium. Use the calibration from Section 8.7 and the steady-state equilibrium Definition 8.1. Write a function named `get_SS()` that has the following form,

```
ss_output = get_SS(params, bvec_guess, SS_graphs)
```

where the inputs are a tuple of the parameters and objects for the model,

```
params = (beta, sigma, nvec, L, A, alpha, delta, g_y, mort_rates,
    imm_rates, omega_SS, g_n_SS, SS_tol)
```

an initial guess of the steady-state savings `bvec_guess`, and a Boolean `SS_graphs` that generates a figure of the steady-state distribution of consumption and savings if it is set to `True`.

The output object `ss_output` is a Python dictionary with the steady-state solution values for the following endogenous objects.

```
ss_output = {
  'b_ss': b_ss, 'c_ss': c_ss, 'w_ss': w_ss, 'r_ss': r_ss,
  'K_ss': K_ss, 'Y_ss': Y_ss, 'C_ss': C_ss,
  'EulErr_ss': EulErr_ss, 'RCerr_ss': RCerr_ss,
  'ss_time': ss_time}
```

Let `ss_time` be the number of seconds it takes to run your steady-state program. You can time your program by importing the time library. And let the object `EulErr_ss` be a length-$(S-1)$ vector of the Euler errors from the resulting steady-state solution given in ratio form $\frac{e^{-\sigma g_y}\beta(1+\bar{r})(1-\rho_s)u'(\bar{c}_{s+1})}{u'(\bar{c}_s)} - 1$ or difference form $e^{-\sigma g_y}\beta(1+\bar{r})(1-\rho_s)u'(\bar{c}_{s+1}) - u'(\bar{c}_s)$. The object `RCerr_ss` is a resource constraint error which should be close to zero. It is given by,

$$\bar{Y} - \bar{C} - \Big[(1+\bar{g}_n)e^{g_y} - 1 + \delta\Big]\bar{K} + e^{g_y}\sum_{s=E+2}^{E+S} i_s\bar{\omega}_s\hat{b}_{s+1}$$

a. Solve numerically for the steady-state equilibrium values of $\{\bar{c}_s\}_{s=E+1}^{E+S}$, $\{\bar{b}_s\}_{s=E+2}^{E+S}$, $\bar{w}$, $\bar{r}$, $\bar{K}$, $\bar{Y}$, $\bar{C}$, the $S-1$ Euler errors, and the resource constraint error. List those values. Time your function. How long did it take to compute the steady-state?

b. Generate a figure that shows the steady-state distribution of consumption and savings by age $\{\bar{c}_s\}_{s=E+1}^{E+S}$ and $\{\bar{b}_s\}_{s=E+2}^{E+S}$.

Exercise 8.6. Solve for the non-steady-state equilibrium time path. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy. Let the initial state of the economy be given by the following distribution of savings,

$$\{b_{s,1}\}_{s=E+2}^{E+S} = \{x(s)\bar{b}_s\}_{s=E+2}^{E+S} \quad \text{where} \quad x(s) = \frac{(1.5 - 0.87)}{S + E - 2}(s-2) + 0.87$$

where the function of age $x(s)$ is simply a linear function of age s that equals 0.87 for $s = E + 2$ and equals 1.5 for $s = E + S$. This gives an initial distribution where there is more inequality than in the steady state. The young have less than their steady-state values and the old have more than their steady-state values. You'll have to choose a guess for T and a time path updating parameter $\xi \in (0, 1)$, but you can be assured that $T < 320$. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear or parabolic initial guess for the time path of the aggregate capital stock from the initial state $\hat{K}_1^1$ to the steady state $\hat{K}_T^1$ at time T.

a. Plot the equilibrium time paths of the stationary aggregate capital stock $\{\hat{K}_t\}_{t=1}^{T+5}$, wage $\{\hat{w}_t\}_{t=1}^{T+5}$, and interest rate $\{r_t\}_{t=1}^{T+5}$.

b. How many periods did it take for the economy to get within 0.00001 of the steady-state aggregate capital stock $\bar{K}$? What is the period after which the aggregate capital stock never is again farther than 0.00001 away from the steady-state?