Chapter 17

Multiple Static Industries, Endogenous Labor

In this chapter, we take the S-period-lived agent model from Chapter 4 with endogenous labor and add to it M distinct perfectly competitive industries characterized by Cobb-Douglas production functions and static profit maximization incentives.

17.1 Household Industry-specific Consumption

A unit measure of identical individuals are born each period and live for S periods. Let the age of an individual be indexed by $s = \{1, 2, ...S\}$. Individuals in this economy choose how much to work each period $n_{s,t}$ and how much to consume among M different consumption goods $c_{m,s,t}$. However, we assume that this consumption can be aggregated into a composite numeraire consumption good $c_{s,t}$ every period in every individual's preferences according to the following Stone-Geary version of a Cobb-Douglas consumption aggregator,

$$c_{s,t} \equiv \prod_{m=1}^{M} \left(c_{m,s,t} - \underline{c}_m\right)^{\alpha_m} \quad \forall s,t \quad \text{with} \quad \sum_{m=1}^{M} \alpha_m = 1 \tag{17.1}$$

where $\underline{c}_m$ is the minimum consumption of good m allowed.[1]

[1] This functional form was originally proposed as a utility function by in a short comment by Geary (1950-51) that aggregates differentiated goods into a scalar utility value. It is differentiated from Cobb-Douglas utility by the subsistence consumption amount in each term of the product. This function was further

Assume that the non-normalized price of each individual consumption good is $\tilde{p}_{m,t}$. We can solve for the optimal good-m consumption demands $c_{m,s,t}$ as a function of composite consumption $c_{s,t}$ by minimizing the total expenditure on consumption given that individual consumption adds up to composite consumption according to (17.1). The Lagrangian for this expenditure minimization problem is the following.

$$\mathcal{L} = \sum_{m=1}^{M}\tilde{p}_{m,t}c_{m,s,t} + \lambda_{s,t}\Bigl[c_{s,t} - \prod_{m=1}^{M}\left(c_{m,s,t} - \underline{c}_m\right)^{\alpha_m}\Bigr] \quad \forall s,t \tag{17.2}$$

Because the Lagrangian multiplier on the constraint $\lambda_{s,t}$ represents the shadow price of an extra unit of composite consumption, we can re-label it as the price of composite consumption $\tilde{p}_{s,t}$.

$$\mathcal{L} = \sum_{m=1}^{M}\tilde{p}_{m,t}c_{m,s,t} + \tilde{p}_{s,t}\Bigl[c_{s,t} - \prod_{m=1}^{M}\left(c_{m,s,t} - \underline{c}_m\right)^{\alpha_m}\Bigr] \quad \forall s,t \tag{17.3}$$

Note that the price of composite consumption can be different for each age-s individual at this point.

The $M+1$ first order conditions of this constrained minimization problem are the following.[2]

$$\tilde{p}_{m,t} = \alpha_m\tilde{p}_{s,t}\left(\frac{c_{s,t}}{c_{m,s,t} - \underline{c}_m}\right) \quad \forall m,s,t \tag{17.4}$$

$$c_{s,t} = \prod_{m=1}^{M}\left(c_{m,s,t} - \underline{c}_m\right)^{\alpha_m} \quad \forall s,t \tag{17.1}$$

Solving (17.4) for $c_{m,s,t}$ gives the optimal demand function for consumption of good m by

developed and operationalized by Stone (1954).

[2]Note that the derivation for first order condition (17.4) is the following:

$$\tilde{p}_{m,t} = \tilde{p}_{s,t}\alpha_m\left(c_{m,s,t} - \underline{c}_m\right)^{\alpha_m - 1}\prod_{u\neq m}^{M}\left(c_{u,s,t} - \underline{c}_u\right)^{\alpha_u}$$

$$\tilde{p}_{m,t}\left(c_{m,s,t} - \underline{c}_m\right) = \tilde{p}_{s,t}\alpha_m\left(c_{m,s,t} - \underline{c}_m\right)^{\alpha_m}\prod_{u\neq m}^{M}\left(c_{u,s,t} - \underline{c}_u\right)^{\alpha_u}$$

$$\tilde{p}_{m,t}\left(c_{m,s,t} - \underline{c}_m\right) = \tilde{p}_{s,t}\alpha_m\prod_{m=1}^{M}\left(c_{m,s,t} - \underline{c}_m\right)^{\alpha_m} = \alpha_m\tilde{p}_{s,t}c_{s,t}$$

age-s individual in period t.

$$c_{m,s,t} = \alpha_m \left(\frac{\tilde{p}_{m,t}}{\tilde{p}_{s,t}}\right)^{-1} c_{s,t} + \underline{c}_m \quad \forall m,s,t \tag{17.5}$$

This household demand function for good-m shows that $c_{m,s,t}$ is a fraction of total consumption $c_{s,t}$, and that fraction is negatively correlated with the relative price of good-m.

Substituting the demand equations (17.5) back into the composite consumption definition (17.1) gives us the expression for the composite price $\tilde{p}_{s,t}$ as a function of each non-normalized industry-m good price $\tilde{p}_{m,t}$.

$$\tilde{p}_{s,t} = \prod_{m=1}^{M}\left(\frac{\tilde{p}_{m,t}}{\alpha_m}\right)^{\alpha_m} \quad \forall s,t \tag{17.6}$$

However, in this case, because nothing on the right-hand-side of (17.6) is a function of s, then $\tilde{p}_{s,t} = \tilde{p}_t$ for all s.

$$\tilde{p}_t = \prod_{m=1}^{M}\left(\frac{\tilde{p}_{m,t}}{\alpha_m}\right)^{\alpha_m} \quad \forall t \tag{17.7}$$

Finally, if we assume that the composite consumption good $c_{s,t}$ is the numeraire good, we can normalize the composite price $\tilde{p}_t = 1$ in every period t by dividing all the equations with prices by the composite numeraire price $\tilde{p}_t$. Then we can rewrite the optimal consumption demand (17.5) and composite price index (17.7) equations as the following,

$$c_{m,s,t} = \alpha_m \left(\frac{c_{s,t}}{p_{m,t}}\right) + \underline{c}_m \quad \forall m,s,t \tag{17.8}$$

$$1 = \prod_{m=1}^{M}\left(\frac{p_{m,t}}{\alpha_m}\right)^{\alpha_m} \quad \forall t \tag{17.9}$$

$$\text{where} \quad p_{m,t} \equiv \frac{\tilde{p}_{m,t}}{\tilde{p}_t} \quad \forall m,t \tag{17.10}$$

where $p_{m,t}$ defined in (17.10) are normalized industry prices with the composite good being the numeraire.

17.2 Household Composite Consumption and Labor

Because the individual-good problem from Section 17.1 as characterized by equations (17.1), (17.8), and (17.9) is determined by functions of composite consumption $c_{s,t}$ and normalized industry prices $p_{m,t}$, we can write the individual's utility maximization in terms of composite consumption $c_{s,t}$. An age-s individual faces the following per-period budget constraint.

$$c_{s,t} + \sum_{m=1}^{M} p_{m,t}\underline{c}_m + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \quad \forall s,t \tag{17.11}$$

Note that the normalized price of composite consumption $p_t = 1$ is implicit in this budget constraint and that we must add in the cost of minimum consumption $\underline{c}_m$ for all m because that amount is subtracted out of composite consumption in (17.1). In addition, the interest rate r_t and wage w_t are normalized and are in terms of composite good price $p_t = 1$.

We assume that each household is endowed with a measure of time $\tilde{l}$ each period that it can choose to spend as either labor $n_{s,t} \in [0,\tilde{l}]$ or leisure $l_{s,t} \in [0,\tilde{l}]$.

$$n_{s,t} + l_{s,t} = \tilde{l} \quad \forall s,t \tag{17.12}$$

We assume that individual utility in each period is the following additively separable function of period composite consumption $c_{s,t}$ and labor supply $n_{s,t}$,

$$u\big(c_{s,t},n_{s,t}\big) = \frac{(c_{s,t})^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \quad \forall s,t \tag{17.13}$$

where the first term on the right-hand-side is the standard CRRA utility of period consumption and the second term is the elliptical disutility of labor function described in Chapter 4.

We assume that households are born with no savings $b_{1,t} = 0$ and that individuals save no income in the last period of their lives $b_{S+1,t} = 0$ for all periods t. Assume that net industry-specific consumption is nonnegative $c_{m,s,t} - \underline{c}_m \geq 0$, which implies that composite consumption is nonnegative $c_{s,t} \geq 0$ because the composite consumption aggregator (17.1) is not defined for industry-specific consumption less-than-or-equal-to zero. However, it will also be the case that period household utility will not be defined for $c_{s,t} <= 0$.

Households choose lifetime composite consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$ and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and nonnegativity constraints.

$$\begin{aligned} &\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^S \beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \quad \forall s,t \\ &\text{s.t.} \quad c_{s,t} = (1+r_t)b_{s,t} + w_t n_{s,t} - b_{s+1,t+1} - \sum_{m=1}^M p_{m,t}\underline{c}_m \quad \forall s,t \\ &\text{and} \quad b_{1,t}, b_{S+1,t} = 0 \quad \forall t \quad \text{and} \quad c_{s,t} \geq 0 \quad \forall s,t \end{aligned} \tag{17.14}$$

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (17.14). The optimal choice of how much to save in the each of the first $S-1$ periods of life $b_{s+1,t+1}$ is found by taking the derivative of the lifetime utility function with respect to each of the lifetime savings amounts $\{b_{s+1,t+s+1}\}_{s=1}^{S-1}$ and setting the derivatives equal to zero.

Solving the household's lifetime optimal decisions by backward induction gives the following S policy functions for labor supply, each of which is characterized the the following Euler equations.

$$n_{s,t} = \phi_s\Bigl(b_{s,t}, \{r_u, w_u\}_{u=t}^{t+S-s}, \{p_{m,u}\}_{m=1,u=t}^{M,t+S-s}\Bigr) \quad \text{for} \quad 1 \leq s \leq S, \quad \forall t \tag{17.15}$$

$$w_t\bigl(c_{s,t}\bigr)^{-\sigma} = \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad 1 \leq s \leq S, \quad \forall t \tag{17.16}$$

and the following $S-1$ policy functions for savings, each of which is characterized by the following Euler equation,

$$b_{s+1,t+1} = \psi_s\Bigl(b_{s,t}, \{r_u, w_u\}_{u=t}^{t+S-s}\{p_{m,u}\}_{m=1,u=t}^{M,t+S-s}\Bigr) \quad \text{for} \quad 1 \leq s \leq S-1, \quad \forall t \tag{17.17}$$

$$\bigl(c_{s,t}\bigr)^{-\sigma} = \beta(1+r_{t+1})\bigl(c_{s+1,t+1}\bigr)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \forall t \tag{17.18}$$

with final age-S consumption being determined by the budget constraint in which the individual consumes all the resources.

$$c_{S,t} = (1+r_t)b_{S,t} + w_t n_{S,t} - \sum_{m=1}^{M} p_{m,t}\underline{c}_m \quad \forall t \tag{17.19}$$

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of prices $\big\{r_u, w_u, \{p_{m,u}\}_{m=1}^M\big\}_{u=t}^{t+S-s}$ over its remaining lifetime. As we will show in Section 17.5, the equilibrium prices in period t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \big\{b_{s,t}\big\}_{s=2}^{S} \quad \forall t \tag{17.20}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{17.21}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[3]

17.3 Industries and Firms

The production side of this economy is comprised of M industries indexed by $m \in \{1, 2, ...M\}$, each of which has a unit measure of identical, perfectly competitive firms. These firms rent investment capital from individuals for real return r_t and hire labor for real wage w_t. The interest rate r_t and wage w_t are equal across industries because labor and capital are perfectly mobile and households are indifferent among the industries. Firms in each industry m use their total capital $K_{m,t}$ and labor $L_{m,t}$ to produce output $Y_{m,t}$ every period according

[3] In Section 17.5 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 17.2.

to a constant elasticity of substitution (CES) production technology,

$$Y_{m,t} = F(K_{m,t},L_{m,t}) \equiv Z_{m,t}\biggl[(\gamma_m)^{\frac{1}{\varepsilon_m}}(K_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}(L_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}}\biggr]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \forall m,t \tag{17.22}$$

where $\gamma_m \in (0,1)$ is the capital share of income, $\varepsilon_m \geq 1$ is the elasticity of substitution between capital and labor, and total factor productivity $Z_{m,t} > 0$ for all t. Two useful transformations of the CES production function are the output-capital ratio and the output-labor ratio, which are both functions of the capital-labor ratio.

$$\frac{Y_{m,t}}{K_{m,t}} = Z_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{L_{m,t}}{K_{m,t}}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \forall m,t \tag{17.23}$$

$$\frac{Y_{m,t}}{L_{m,t}} = Z_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{K_{m,t}}{L_{m,t}}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \forall m,t \tag{17.24}$$

The representative firm in each industry m chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_{m,t},L_{m,t}} \; p_{m,t}F\bigl(K_{m,t},L_{m,t}\bigr) - (r_t + \delta_{M,t})K_{m,t} - w_t L_{m,t} \quad \forall m,t \tag{17.25}$$

where $p_{m,t}$ is the normalized price of output from industry m defined in (17.10), $\delta_{M,t} \in [0,1]$ is the rate of capital depreciation on the investment good from industry M, and r_t and w_t are the normalized interest rate and wage, respectively.[4] Note that all terms in the profit function (17.25) is in terms of composite good prices, because each of the prices $p_{m,t}$, r_t, w_t, and $\delta_{M,t}$ is normalized by the composite goods price.

It is important to note a strong assumption here. We are assuming that only good-M can be used as investment and capital. For this reason, there is only one depreciation rate $\delta_{M,t}$ that applies to any of the capital used in the other $m < M$ industries. As such, only the goods market clearing condition in industry M in (17.33) in Section 17.4 will have

[4]The interest rate on capital r_t and the wage w_t are equivalent across industries m because we assume that households are indifferent working for and investing in the different industries.

investment.[5]

The two first order conditions that characterize firm optimization in each industry m are the following.

$$r_t = p_{m,t}(Z_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}}\left(\gamma_m\frac{Y_{m,t}}{K_{m,t}}\right)^{\frac{1}{\varepsilon_m}} - \delta_{M,t} \quad \forall m,t \tag{17.26}$$

$$w_t = p_{m,t}(Z_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}}\left([1-\gamma_m]\frac{Y_{m,t}}{L_{m,t}}\right)^{\frac{1}{\varepsilon_m}} \quad \forall m,t \tag{17.27}$$

Each firm in industry m can make its capital and labor demand decision if it knows the normalized price of its good $p_{m,t}$, the normalized interest rate r_t, and the normalized wage w_t. For the entire supply side of the economy, all firms can optimize if r_t, w_t, and $\{p_{m,t}\}_{m=1}^M$ are known.

As a convenience for the solution method in Section 17.6, we show that the wage-interest rate ratio and the two firm first order conditions can be specified in terms of the industry-specific capital-labor ratio $K_{m,t}/L_{m,t}$ rather than just the capital-output and labor-output ratios. The wage-interest rate ratio is found by dividing (17.27) by (17.26). The two alternative specifications for the interest rate and wage are found by substituting the expressions (17.23) and (17.24) into (17.26) and (17.27).

$$\frac{w_t}{r_t+\delta_{M,t}} = \left[\left(\frac{1-\gamma_m}{\gamma_m}\right)\left(\frac{K_{m,t}}{L_{m,t}}\right)\right]^{\frac{1}{\varepsilon_m}} \quad \forall m,t \tag{17.28}$$

[5]Think of an economy that has two industries—delivery services and trucks. The delivery services industry uses trucks and labor to produce its output. The trucks industry uses trucks and labor to produce its output. Both industries face depreciation of their capital (trucks). But only in the trucks industry can the output be used for both consumption and investment.

$$r_t = \begin{cases} p_{m,t}(\gamma_m)^{\frac{1}{\varepsilon_m}} Z_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{L_{m,t}}{K_{m,t}}\right)^{\frac{\varepsilon_m - 1}{\varepsilon_m}}\right]^{\frac{1}{\varepsilon_m - 1}} - \delta_{M,t} \quad \text{for} \quad \varepsilon_m \neq 1 \\ p_{m,t}\gamma_m Z_{m,t}\left(\frac{L_{m,t}}{K_{m,t}}\right)^{1-\gamma_m} - \delta_{M,t} \quad \text{for} \quad \varepsilon_m = 1 \end{cases} \quad \forall m,t \tag{17.29}$$

$$w_t = \begin{cases} p_{m,t}(1-\gamma_m)^{\frac{1}{\varepsilon_m}} Z_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{K_{m,t}}{L_{m,t}}\right)^{\frac{\varepsilon_m - 1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\right]^{\frac{1}{\varepsilon_m - 1}} \quad \text{for} \quad \varepsilon \neq 1 \\ p_{m,t}(1-\gamma_m) Z_{m,t}\left(\frac{K_{m,t}}{L_{m,t}}\right)^{\gamma_m} \quad \text{for} \quad \varepsilon = 1 \end{cases} \quad \forall m,t \tag{17.30}$$

Note that (17.28) is true for both normalized interest rates $r_t + \delta_{M,t}$ and wages w_t as well as non-normalized interest rates $\tilde{r}_t + \tilde{\delta}_{M,t}$ and wages $\tilde{w}_t$. For this reason, if (17.28) was derived from non-normalized prices, then (17.29) and (17.30) must have the corresponding industry prices—non-normalized industry prices $\tilde{p}_{m,t}$ for non-normalized wages $\tilde{w}_t$ and interest rates $\tilde{r}_t + \tilde{\delta}_{M,t}$, and normalized industry prices $p_{m,t}$ for normalized wages w_t and interest rates $r_t + \delta_{M,t}$.

17.4 Market Clearing

The markets that must clear in this economy are the labor market, capital market, and M goods markets. In the labor market, total labor demand across M industries must equal the total labor supplied by households.

$$\sum_{m=1}^{M} L_{m,t} = \sum_{s=1}^{S} n_{s,t} \quad \forall t \tag{17.31}$$

This implies that households are indifferent regarding which industry they provide labor, and firms see household labor as perfectly substitutable across age cohorts.

In the capital market, total capital demand across M industries must equal the total capital supplied by household savings.

$$\sum_{m=1}^{M} K_{m,t} = \sum_{s=2}^{S} b_{s,t} \quad \forall t \tag{17.32}$$

This also implies that households are indifferent regarding to which industry they provide capital, and firms see household capital as perfectly substitutable across age cohorts.

Goods market clearing requires the amount of goods production $Y_{m,t}$ in industries $m = 1, ...M-1$ to equal the total household consumption of that good $C_{m,t}$. And in industry M, goods production $Y_{M,t}$ equals total household consumption $C_{M,t}$ plus the amount of investment $I_{M,t}$.

$$\begin{aligned} &Y_{m,t} = C_{m,t} \quad \forall t \quad \text{and} \quad \text{for} \quad 1 \leq m \leq M-1 \\ &Y_{M,t} = C_{M,t} + I_{M,t} \quad \forall t \\ &\quad \text{where} \quad C_{m,t} \equiv \sum_{s=1}^{S} c_{m,s,t} \quad \text{and} \quad I_{M,t} \equiv \frac{\sum_{m=1}^{M} K_{m,t+1} - (1-\delta_{M,t})\sum_{m=1}^{M} K_{m,t}}{p_{M,t}} \end{aligned} \tag{17.33}$$

Investment in industry M is divided by the price $p_{M,t}$ because that term needs to be in units of industry-M output. But the capital, which is supplied by the households (17.32) from their budget constraints (17.11) is in terms of the composite good.

One of these market clearing conditions is redundant by Walras' Law. We usually leave out the goods market clearing condition in industry M from our solution algorithms, then check to make sure it is satisfied after the equilibrium is solved.

17.5 Equilibrium

A rough description of the equilibrium solution to the model above is the following three points

i. Households optimize labor supply and savings decisions according to S labor supply Euler equations (17.16), $S-1$ savings Euler equations (17.18), $S\times M$ good-m consumption good demand equations (17.8), and one individual composite good price equation (17.9). Also, equilibrium assumes that each household can correctly forecast the time path of prices over their respective lifetimes.

ii. Firms in each of the M industries choose capital and labor demand to maximize profits characterized by first order conditions (17.29) and (17.30).

iii. Markets clear according to (17.31), (17.32), and M industry goods market clearing equations (17.33). And the price of the numeraire composite good $p_t = 1$ is a function of the industry-specific prices $p_{m,t}$ (17.9).

These equations characterize the equilibrium and constitute a system of nonlinear dynamic difference equations.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows. Section 17.6.1 describes the algorithm for solving for the steady-state equilibrium.

Definition 17.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents, endogenous labor supply, and M perfectly competitive static industries is defined as constant allocations of consumption $\{\bar{c}_s, \bar{n}_s\}_{s=1}^S$ and $\{\bar{c}_{m,s}\}_{m,s=1}^{M,S}$ and savings $\{\bar{b}_s\}_{s=2}^S$, and normalized prices $\bar{r}$, $\bar{w}$, and $\{\bar{p}_m\}_{m=1}^M$ such that:

i. households optimize according to (17.8), (17.16), and (17.18),

ii. numeraire composite goods consumption price is normalized to 1 and is a function of all M industry-specific goods prices (17.9),

iii. firms optimize in all M industries according to (17.29) and (17.30),

iv. and the labor market, capital market, and all M goods markets clear according to (17.31), (17.32), and (17.33) (with one of the M goods market clearing conditions being redundant.

The relevant examples of stationary functions in this model are the policy functions for labor supply, savings and for prices. Let the equilibrium policy functions for labor supply be represented by savings be represented by $n_{s,t} = \phi_s(\mathbf{\Gamma}_t)$ and for savings be represented by $b_{s+1,t+1} = \psi_s(\mathbf{\Gamma}_t)$. These are similar to the policy functions in (17.15) and (17.17), except that all the arguments in those functions are functions of the state $\mathbf{\Gamma}_t$. The arguments of the functions (the state) may change overtime causing the savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 17.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents, endogenous labor supply, and M perfectly competitive static industries is defined as stationary allocation functions of the state $\left\{n_{s,t} = \phi_s\big(\boldsymbol{\Gamma}_t\big)\right\}_{s=1}^{S}$ and $\left\{b_{s+1,t+1} = \psi_s\big(\boldsymbol{\Gamma}_t\big)\right\}_{s=1}^{S-1}$ and stationary normalized price functions $w(\boldsymbol{\Gamma}_t)$, $r(\boldsymbol{\Gamma}_t)$, and $p_m(\boldsymbol{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (17.21), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\boldsymbol{\Gamma}_{t+u} = \boldsymbol{\Gamma}^e_{t+u} = \Omega^u\left(\boldsymbol{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (17.8), (17.16), and (17.18) each period,

iii. numeraire composite goods consumption prices are normalized to 1 each period and are functions of all M industry-specific goods prices (17.9) in each period,

iv. firms optimize in all M industries according to (17.29) and (17.30) each period,

v. and the labor market, capital market, and all M goods markets clear according to (17.31), (17.32), and (17.33) each period (with one of the M goods market clearing conditions being redundant).

17.6 Solution Method

In this section we characterize computational approaches to solving for the steady-state equilibrium from Definition 17.1 and the non-steady-state transition path equilibrium from Definition 17.2.

17.6.1 Steady-state solution method

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 17.1. The parameters needed for the steady-state solution of this model are $\left\{S, \beta, \sigma, \{\chi^n_s\}_{s=1}^S, \{\underline{c}_m, \alpha_m\}_{m=1}^M, \tilde{l}, b, \upsilon\right\}$ for household parameters, and $\left\{M, \{\gamma_m, \varepsilon_m, \delta_M, Z_m\}_{m=1}^M\right\}$, and solution method parameters `SS_tol` and ξ_{ss}. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

There are several approaches to solving this system of equations. We follow the approach of previous models in this book by choosing outer loop variables that allow us to solve for all the household and firm decisions then using those decisions to get updated values of the outer loop variables. The following steps describe the fixed point algorithm for solving for the steady-state solution.

We refer to steps (1a) through (1d) as the "inner loop" of the steady-state solution technique. These steps solve for all the household and firm decisions, as well as some prices, given guesses for the interest rate and wage. We refer to steps (1), (2), and (3) as the "outer loop" of the solution method because they deal with updating the guesses for the non-normalized interest rate and wage given the inner-loop (household and firm optimization) outcomes.

1. Make a guess for the normalized steady-state interest rate $\bar{r}^i$ and wage $\bar{w}^i$. Solve for all the steady-state prices, household decisions, and industry decisions implied by that guess.

   (a) Given values for $\bar{r}^i$ and $\bar{w}^i$, solve for $m = 1, ...M-1$ industry-specific capital-labor ratios $\{\bar{K}_m/\bar{L}_m\}_{m=1}^{M-1}$, output-capital ratios $\{\bar{Y}_m/\bar{K}_m\}_{m=1}^{M-1}$, output-labor ratios $\{\bar{Y}_m/\bar{L}_m\}_{m=1}^{M-1}$, and normalized industry prices $\{\bar{p}_m\}_{m=1}^{M-1}$.

      i. Use $m = 1, ...M-1$ pairs of steady-state firm first order conditions combined to get the wage-interest rate ratio (17.28) to solve for the first $M-1$ capital-labor ratios $\{\bar{K}_m/\bar{L}_m\}_{m=1}^{M-1}$.

      $$\frac{\bar{K}_m}{\bar{L}_m} = \left(\frac{\gamma_m}{1-\gamma_m}\right)\left(\frac{\bar{w}^i}{\bar{r}^i + \delta_M}\right)^{\varepsilon_m} \quad \text{for} \quad m = 1, ...M-1 \tag{17.34}$$

      ii. Given the $m = 1, ...M-1$ capital-labor ratios from step (1.a.i) $\{\bar{K}_m/\bar{L}_m\}_{m=1}^{M-1}$, solve for the corresponding output-capital ratios using (17.23).

$$\frac{\bar{Y}_m}{\bar{K}_m} = Z_m\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{\bar{K}_m}{\bar{L}_m}\right)^{\frac{1-\varepsilon_m}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \text{for} \quad m = 1,...M-1 \tag{17.35}$$

iii. Given the $m = 1,...M-1$ capital-labor ratios from step (1.a.i) $\{\bar{K}_m/\bar{L}_m\}_{m=1}^{M-1}$, solve for the corresponding output-labor ratios using (17.24).

$$\frac{\bar{Y}_m}{\bar{L}_m} = Z_m\left[(\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{\bar{K}_m}{\bar{L}_m}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \text{for} \quad m = 1,...M-1 \tag{17.36}$$

iv. Given $\bar{w}^i$ and the $M-1$ output-labor ratios from step (1.a.iii) $\{\bar{Y}_m/\bar{L}_m\}_{m=1}^{M-1}$, solve for $m = 1,...M-1$ industry prices $\{\bar{p}_m\}_{m=1}^{M-1}$ by solving industry first order conditions for labor demand (17.27) for the price p_m.

$$\bar{p}_m = \bar{w}^i(Z_m)^{\frac{1-\varepsilon_m}{\varepsilon_m}}\left([1-\gamma_m]\frac{\bar{Y}_m}{\bar{L}_m}\right)^{-\frac{1}{\varepsilon_m}} \quad \text{for} \quad m = 1,...M-1 \tag{17.37}$$

v. Use the normalized composite price equation (17.9) to solve for the steady-state price $\bar{p}_M$ in industry M as a function of the prices in the other industries $m = 1,...M-1$ from step (1.a.iv).

$$\bar{p}_M = \alpha_M\prod_{m=1}^{M-1}\left(\frac{\alpha_m}{\bar{p}_m}\right)^{\frac{\alpha_m}{\alpha_M}} \tag{17.38}$$

(b) Given values for normalized $\bar{r}^i$ and $\bar{w}^i$, and resulting normalized industry-specific prices $\{\bar{p}_m\}_{m=1}^{M}$, solve for the households' steady-state lifetime labor supply $\{\bar{n}_s\}_{s=1}^{S}$, savings $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$, composite consumption $\{\bar{c}_s\}_{s=1}^{S}$, and industry-specific consumption $\{\bar{c}_{m,s}\}_{m,s=1}^{M,S}$.

i. Solve for steady-state labor supply $\{\bar{n}_s\}_{s=1}^{S}$ and savings $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ using Euler equations (17.16) and (17.18).

ii. Solve for steady-state composite consumption $\{\bar{c}_s\}_{s=1}^{S}$ by plugging $\{\bar{n}_s\}_{s=1}^{S}$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ into the budget constraints (17.11) for each age s.

iii. Given $\{\bar{p}_m\}_{m=1}^M$ and $\{\bar{c}_s\}_{s=1}^S$, solve for the steady-state industry specific consumptions for each age and industry $\{\bar{c}_{m,s}\}_{m,s=1}^{M,S}$ using (17.8).

(c) Given household decisions from the previous step (1.b), the capital-labor, output-capital, and output-labor ratios and the prices from step (1.a), solve for the implied steady-state total capital, labor, investment, and consumption demands, and output in each industry $\{\bar{Y}_m, \bar{K}_m, \bar{L}_m, \bar{I}_m, \bar{C}_m\}_{m=1}^M$.

i. Calculate steady-state total consumption in each industry $\{\bar{C}_m\}_{m=1}^M$ from the household industry-specific consumption decisions $\{\bar{c}_{m,s}\}_{m,s=1}^{M,S}$.

$$\bar{C}_m = \sum_{s=1}^S \bar{c}_{m,s} \quad \forall m \tag{17.39}$$

ii. Solve for output $\bar{Y}_m$ in industries $m = 1, ...M-1$ using the goods market clearing condition (17.33) in those industries.

$$\bar{Y}_m = \bar{C}_m \quad \text{for} \quad m = 1, ...M-1 \tag{17.40}$$

iii. Solve for capital $\bar{K}_m$ and labor $\bar{L}_m$ in industries $m = 1, ...M-1$ using the outputs $\bar{Y}_m$ from step (1.d.ii) and the output-capital ratios and capital-labor ratios $\{\bar{Y}_m/\bar{K}_m, \bar{K}_m/\bar{L}_m\}_{m=1}^{M-1}$ in those industries from steps (1.a.ii) and (1.a.i), respectively.

$$\bar{K}_m = \bar{Y}_m\left(\frac{\bar{K}_m}{\bar{Y}_m}\right) \quad \text{and} \quad \bar{L}_m = \bar{K}_m\left(\frac{\bar{L}_m}{\bar{K}_m}\right) \quad \text{for} \quad m = 1, ...M-1 \tag{17.41}$$

iv. Solve for capital $\bar{K}_M$ and labor $\bar{L}_M$ in industry M as a residual from the capital market clearing equation (17.32) and labor market clearing (17.31) equation, respectively.

$$\bar{K}_M = \sum_{s=1}^S \bar{b}_s - \sum_{m-1}^{M-1} \bar{K}_m \quad \text{and} \quad \bar{L}_M = \sum_{s=1}^S \bar{n}_s - \sum_{m-1}^{M-1} \bar{L}_m \tag{17.42}$$

v. Solve for output $\bar{Y}_M$ in industry M using the production function (17.22) and

$\bar{K}_M$ and $\bar{L}_M$ from step (1.d.iv).

$$\bar{Y}_M = Z_M\left[(\gamma_M)^{\frac{1}{\varepsilon_M}}(\bar{K}_M)^{\frac{\varepsilon_M-1}{\varepsilon_M}} + (1-\gamma_M)^{\frac{1}{\varepsilon_M}}(\bar{L}_M)^{\frac{\varepsilon_M-1}{\varepsilon_M}}\right]^{\frac{\varepsilon_M}{\varepsilon_M-1}} \tag{17.43}$$

vi. Solve for investment $\bar{I}_M$ in industry M as a function of the capital demands in all industries $\{\bar{K}_m\}_{m=1}^M$ and the depreciation rate δ_M and price $\bar{p}_M$ in industry M.

$$\bar{I}_M = \frac{\delta_M\left(\sum_{m=1}^M \bar{K}_m\right)}{\bar{p}_M} \tag{17.44}$$

2. Use the values for $\bar{Y}_M$, $\bar{K}_M$, and $\bar{L}_M$ from steps (1.d.iv) and (1.d.v) and p_M from step (1.a.v) to plug into the yet unused two first order conditions (17.26) and (17.27) from industry M to solve for updated implied values of the interest rate $\bar{r}^{i'}$ and wage $\bar{w}^{i'}$.

$$\bar{r}^{i'} = \bar{p}_M(Z_M)^{\frac{\varepsilon_M-1}{\varepsilon_M}}\left(\gamma_M\frac{\bar{Y}_M}{\bar{K}_M}\right)^{\frac{1}{\varepsilon_M}} - \delta_M \tag{17.45}$$

$$\bar{w}^{i'} = \bar{p}_M(Z_M)^{\frac{\varepsilon_M-1}{\varepsilon_M}}\left([1-\gamma_m]\frac{\bar{Y}_M}{\bar{L}_M}\right)^{\frac{1}{\varepsilon_M}} \tag{17.46}$$

3. Use root finder on guesses for normalized $(\bar{r}^i, \bar{w}^i)$ that goes through steps (1) and (2) to find the root of updating equations (17.45) and (17.46), such that the initial guess $(\bar{r}^i, \bar{w}^i)$ implies the same updated values $(\bar{r}^{i'}, \bar{w}^{i'})$.

   (a) As a check once the steady-state is found, make sure the goods market clearing condition in inustry M holds.

$$\bar{Y}_M = \bar{C}_M + \bar{I}_M \tag{17.47}$$

Table 17.1: Steady-state aggregate prices and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.057	Max. absolute savings Euler error	9.592e-14
$\bar{w}$	0.284	Max. absolute labor supply Euler error	2.398e-14
		Max. absolute ind.-M goods market clearing error	-4.974e-13
		Serial computation time	3.427 sec.

Figure 17.1: Steady-state distribution of composite consumption $\bar{c}_s$ and savings $\bar{b}_s$

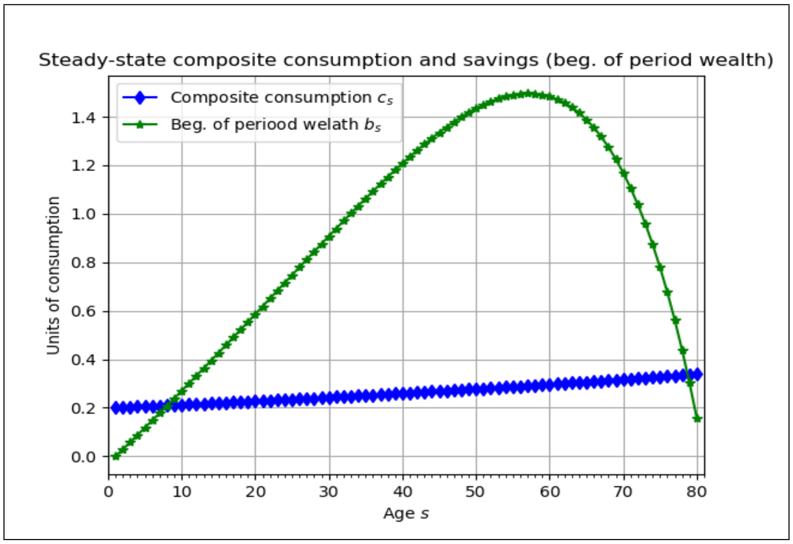


Table 17.2: Steady-state industry-specific prices and aggregate variables

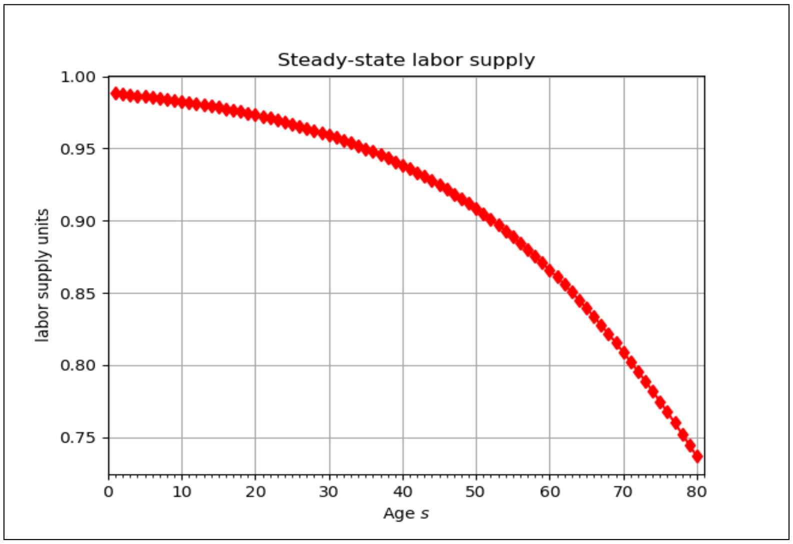
Variable	$m=1$	$m=2$	$m=3$
$\bar{p}_m$	0.342	0.351	0.342
$\bar{K}_m$	18.309	19.723	34.791
$\bar{I}_m$	0.000	0.000	12.770
$\bar{L}_m$	30.699	20.056	22.177
$\bar{Y}_m$	27.258	19.946	27.036
$\bar{C}_m$	27.258	19.946	14.266

Figures 17.1, 17.2, and 17.3 and Tables 17.1 and 17.2 show the steady-state equilibrium household outcomes by age and the steady-state equilibrium prices and aggregate variables.

17.6.2 Non-steady-state transition path solution method

This section outlines the steps for computing the solution to the non-steady-state transition path equilibrium in Definition 17.2. The parameters needed for the steady-state solution of this model are $\Big\{S, T, \beta, \sigma, \{\chi^n_s\}_{s=1}^S, \{\underline{c}_m, \alpha_m\}_{m=1}^M, \tilde{l}, b, \upsilon, \{b_{s,0}\}_{s=2}^S\Big\}$ for household parameters, and $\Big\{M, \{\gamma_m, \varepsilon_m, \tilde{\delta}_{m,t}\}_{m=1}^M, \{Z_{m,t}\}_{m,t=1}^{M,T}\Big\}$ for firm parameters, and solution method parameters `TP_tol` and ξ_{tp}. These parameters are chosen, calibrated, or estimated outside of the

Figure 17.2: Steady-state distribution of labor supply $\bar{n}_s$



model and are inputs to the solution method.[6]

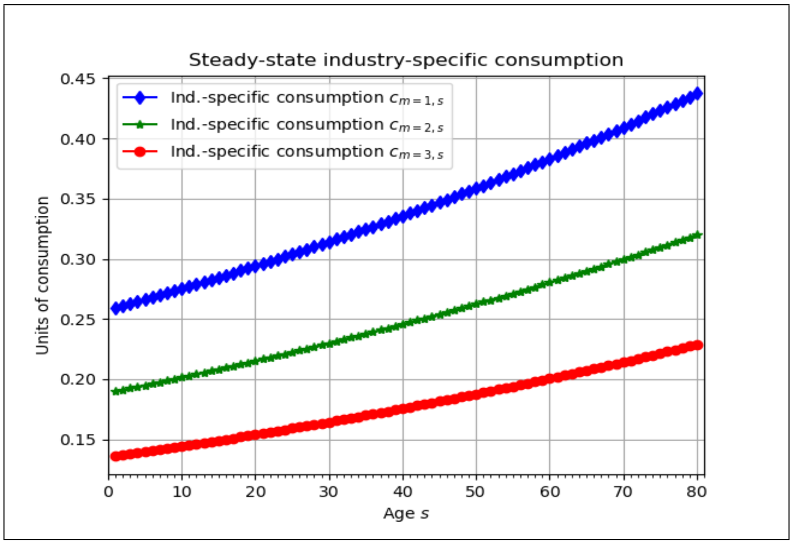
The solution method follows similar steps to those outlined in the steady-state solution method in Section 17.6.1. We refer to steps (2a) through (2c) as the "inner loop" of the non-steady-state transition path solution technique. These steps solve for all the household and firm decisions, as well as some prices, given guesses for the interest rate and wage. We refer to steps (2), (3), and (4) as the "outer loop" of the solution method because they deal with updating the guesses for interest rate and wage given the inner-loop (household and firm optimization) outcomes.

1. Make a guess $T > S$ for the number of periods it takes for the model to reach the steady-state defined in Definition 17.1. Period T is defined as the period for which $x_t = \bar{x}$ for all $t \geq T$

2. Make a guess for the transition path of the normalized interest rate $\boldsymbol{r}^i \equiv \{r_0^i, r_1^i, ...r_T^i\}$ and the normalized wage $\boldsymbol{w}^i \equiv \{w_0^i, w_1^i, ...w_T^i\}$, where T is the period on which and after which the solution is assumed to be in the steady-state. The only restrictions on the initial guesses for $\boldsymbol{r}^i$ and $\boldsymbol{w}^i$ are that $r_t^i + \delta_{M,t}, w_t^i > 0$ for all t and all iterations i

[6]Note that the initial distribution of capital across industries $\{K_{m,t=0}\}_{m=1}^M$ is not a state variable. It is endogenous. The initial distribution of capital across industries depends on the interest rate r_0 and wage w_0 in the first period.

Figure 17.3: Steady-state distribution of industry-specific consumption $\bar{c}_{m,s}$



and that the final values equal the steady-state values $r_T^i = \bar{r}$ and $w_T^i = \bar{w}$ solved for in the previous section. A smooth path between the initial values (r_0^i, w_0^i) and the steady-state $(r_T^i, w_T^i) = (\bar{r}, \bar{w})$ usually solves the most quickly. We must also take as exogenous the initial distribution of wealth across households in the first period $\{b_{s,t=0}\}_{s=1}^S$. The initial distribution of capital across firms $\{K_{m,t=0}\}_{m=1}^M$ depends endogenously on the initial distribution of household wealth $\{b_{s,t=0}\}_{s=1}^S$ and the initial interest r_0 rate and wage w_0.

(a) Given time paths $\boldsymbol{r}^i$ and $\boldsymbol{w}^i$, solve for the time paths of $m = 1, ...M-1$ industry-specific capital-labor ratios $\{K_{m,t}/L_{m,t}\}_{m=1,t=0}^{M-1,T}$, output-capital ratios $\{Y_{m,t}/K_{m,t}\}_{m=1,t=0}^{M-1,T}$, output-labor ratios $\{Y_{m,t}/L_{m,t}\}_{m=1,t=0}^{M-1,T}$, and normalized industry prices $\{p_{m,t}\}_{m=1,t=0}^{M-1,T}$.

i. Use $m = 1, ...M-1$ pairs of firm first order conditions in each period t combined to get the wage-interest rate ratio (17.28) to solve for the time paths of the first $M-1$ capital-labor ratios $\{K_{m,t}/L_{m,t}\}_{m=1,t=0}^{M-1,T}$.

$$\frac{K_{m,t}}{L_{m,t}} = \left(\frac{\gamma_m}{1-\gamma_m}\right)\left(\frac{w_t^i}{r_t^i + \delta_{M,t}}\right)^{\varepsilon_m} \quad \forall t \quad \text{and} \quad m = 1, ...M-1 \qquad (17.48)$$

ii. Given the $m = 1, ...M-1$ time paths of capital-labor ratios from step (2.a.i) $\{K_{m,t}/L_{m,t}\}_{m=1,t=0}^{M-1,T}$, solve for the corresponding time paths of output-capital ratios using (17.23).

$$\frac{Y_{m,t}}{K_{m,t}} = Z_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}}+(1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{K_{m,t}}{L_{m,t}}\right)^{\frac{1-\varepsilon_m}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \forall t \quad \text{and} \quad m=1,...M-1 \tag{17.23}$$

iii. Given the $m=1,...M-1$ capital-labor ratios from step (1.a.i) $\{\bar{K}_m/\bar{L}_m\}_{m=1}^{M-1}$, solve for the corresponding output-labor ratios using (17.24).

$$\frac{Y_{m,t}}{L_{m,t}} = Z_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{K_{m,t}}{L_{m,t}}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}}+(1-\gamma_m)^{\frac{1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \forall t \quad \text{and} \quad m=1,...M-1 \tag{17.24}$$

iv. Given $\boldsymbol{w}^i$ and the $M-1$ time paths of the output-labor ratio from step (2.a.iii) $\{Y_{m,t}/L_{m,t}\}_{m=1,t=0}^{M-1,T}$, solve for $m=1,...M-1$ time paths of normalized industry prices $\{p_{m,t}\}_{m=1,t=0}^{M-1,T}$ by solving the first order codition for labor demand (17.27) for the price $p_{m,t}$.

$$p_{m,t} = w_t^i(Z_{m,t})^{\frac{1-\varepsilon_m}{\varepsilon_m}}\left([1-\gamma_m]\frac{Y_{m,t}}{L_{m,t}}\right)^{-\frac{1}{\varepsilon_m}} \quad \forall t \quad \text{and} \quad m=1,...M-1 \tag{17.49}$$

v. Use the normalized composite price equation (17.9) to solve for the time path of prices $p_{M,t}$ in industry M as a function of the prices in the other industries $m=1,...M-1$ from step (1.a.iv).

$$p_{M,t} = \alpha_M\prod_{m=1}^{M-1}\left(\frac{\alpha_m}{p_{m,t}}\right)^{\frac{\alpha_m}{\alpha_M}} \quad \forall t \quad \text{and} \quad m=1,...M-1 \tag{17.50}$$

(b) Given transition paths for normalized interest rate $\{r_t^i\}_{t=0}^T$, wage $\{w_t^i\}_{t=0}^T$, and industry-specific prices $\{p_{m,t}\}_{m=1,t=0}^{M,T}$, solve for the transition paths of households' lifetime labor supply $\{n_{s,t}\}_{s=1,t=0}^{S,T}$, savings $\{b_{s+1,t+1}\}_{s=1,t=0}^{S,T}$, composite consumption $\{c_{s,t}\}_{s=1,t=0}^{S,T}$, and industry-specific consumption $\{c_{m,s,t}\}_{m=1,s=1,t=0}^{M,S,T}$.

i. Solve for the transition path of each household's labor supply $\{n_{s,t}\}_{s=1,t=0}^{S,T}$ and savings $\{b_{s+1,t+1}\}_{s=1,t=0}^{S,T}$ using Euler equations (17.16) and (17.18).

ii. Solve for the transition path of each household's composite consumption

$\{c_{s,t}\}_{s=1,t=0}^{S,T}$ by plugging $\{n_{s,t}\}_{s=1,t=0}^{S,T}$ and $\{b_{s+1,t+1}\}_{s=1,t=0}^{S,T}$ into the budget constraints (17.11) for each age s.

iii. Given $\{p_{m,t}\}_{m=1,t=0}^{M,T}$ and $\{c_{s,t}\}_{s=1,t=0}^{S,T}$, solve for the transition paths of industry specific consumptions for each age and industry $\{c_{m,s,t}\}_{m=1,s=1,t=0}^{M,S,T}$ using (17.8).

(c) Given the time paths of household decisions from step (2.b), the capital-labor, output-capital, and output-labor ratios and the prices from step (2.a), solve for the implied time paths of total capital, labor, investment, and consumption demands and output in each industry $\{Y_{m,t}, K_{m,t}, L_{m,t}, I_{m,t}, C_{m,t}\}_{m=1,t=0}^{M,T}$.

i. Calculate the time paths of total consumption in each industry $\{C_{m,t}\}_{m=1,t=0}^{M,T}$ from the household industry-specific consumption decisions $\{c_{m,s,t}\}_{m=0,s=1,t=0}^{M,S,T}$.

$$C_{m,t} = \sum_{s=1}^{S} c_{m,s,t} \quad \forall m, t \tag{17.51}$$

ii. Solve for time paths of output $Y_{m,t}$ in industries $m = 1, ...M - 1$ using the goods market clearing condition (17.33) in those industries.

$$Y_{m,t} = C_{m,t} \quad \forall t \quad \text{and} \quad m = 1, ...M - 1 \tag{17.33}$$

iii. Solve for the time paths of capital $K_{m,t}$ and labor $L_{m,t}$ in industries $m = 1, ...M - 1$ using the time paths of output $Y_{m,t}$ from step (2.d.ii) and the time paths of the output-capital ratios and capital-labor ratios $\{Y_{m,t}/K_{m,t}$ and $K_{m,t}/L_{m,t}\}_{m=1,t=0}^{M-1,T}$ in those industries from steps (2.a.ii) and (2.a.i), respectively.

$$K_{m,t} = Y_{m,t}\left(\frac{K_{m,t}}{Y_{m,t}}\right) \quad \text{and} \quad L_{m,t} = K_{m,t}\left(\frac{L_{m,t}}{K_{m,t}}\right) \forall t \quad \text{and} \quad m = 1, ...M - 1 \tag{17.52}$$

iv. Solve for the time paths of capital $K_{M,t}$ and labor $L_{M,t}$ in industry M as a residual from the capital market clearing equation (17.32) and labor market clearing (17.31) equation, respectively.

$$K_{M,t} = \sum_{s=1}^{S} b_{s,t} - \sum_{m-1}^{M-1} K_{m,t} \quad \text{and} \quad L_{M,t} = \sum_{s=1}^{S} n_{s,t} - \sum_{m-1}^{M-1} L_{m,t} \quad \forall t \tag{17.53}$$

v. Solve for the time path of output $Y_{M,t}$ in industry M using the production function (17.22) and time paths $K_{M,t}$ and $L_{M,t}$ from step (2.d.iv).

$$Y_{M,t} = Z_{M,t}\left[(\gamma_M)^{\frac{1}{\varepsilon_M}}(K_{M,t})^{\frac{\varepsilon_M-1}{\varepsilon_M}} + (1-\gamma_M)^{\frac{1}{\varepsilon_M}}(L_{M,t})^{\frac{\varepsilon_M-1}{\varepsilon_M}}\right]^{\frac{\varepsilon_M}{\varepsilon_M-1}} \quad \forall t \tag{17.54}$$

vi. Solve for the time path of investment $I_{M,t}$ in industry M as a function of the capital demands in all industries $\{K_{m,t}\}_{m=1}^{M}$, the depreciation rate δ_M, and time path of price $p_{M,t}$ in industry M.

$$I_{M,t} = \frac{\left(\sum_{m=1}^{M} K_{m,t+1}\right) - (1-\delta_{M,t})\left(\sum_{m=1}^{M} K_{m,t}\right)}{p_{M,t}} \tag{17.55}$$

3. Use the time paths for $Y_{M,t}$, $K_{M,t}$, and $L_{M,t}$ from steps (2.d.iv) and (2.d.v) and $p_{M,t}$ from step (2.a.v) to plug into the yet unused two first order conditions (17.26) and (17.27) from industry M to solve for updated implied time paths of the interest rate $\boldsymbol{r}^{i'}$ and wage $\boldsymbol{w}^{i'}$.

$$r_t^{i'} = p_{M,t}(Z_{M,t})^{\frac{\varepsilon_M-1}{\varepsilon_M}}\left(\gamma_M\frac{Y_{M,t}}{K_{M,t}}\right)^{\frac{1}{\varepsilon_M}} - \delta_{M,t} \quad \forall t \tag{17.56}$$

$$w_t^{i'} = p_{M,t}(Z_{M,t})^{\frac{\varepsilon_M-1}{\varepsilon_M}}\left([1-\gamma_m]\frac{Y_{M,t}}{L_{M,t}}\right)^{\frac{1}{\varepsilon_M}} \quad \forall t \tag{17.57}$$

4. Define a distance metric on the difference between the updated values for the time paths of interest rates and wages $(\boldsymbol{r}^{i'}, \boldsymbol{w}^{i'})$ from step (3) and the original guesses $(\boldsymbol{r}^{i}, \boldsymbol{w}^{i})$ from step (2). We choose the maximum absolute error across the two time series differences.

$$\begin{split} \texttt{TP\_dist} \equiv \max\left|\left\{\epsilon_{r,t}, \epsilon_{w,t}\right\}_{t=0}^{T}\right| \\ \text{where} \quad \epsilon_{r,t} \equiv r_t^{i'} - r_t^{i} \quad \text{and} \quad \epsilon_{w,t} \equiv w_t^{i'} - w_t^{i} \quad \forall t \end{split} \tag{17.58}$$

5. Repeat steps (2), (3), and (4), updating the guesses for the transition path of the normalized interest rate $\boldsymbol{r}^{i+1}$ and wage $\boldsymbol{w}^{i+1}$ until the maximum absolute error across both error time paths defined in (17.58)is less than some tolerance threshold `TP_toler` > 0.

   (a) If the TPI distance measure is greater than some tolerance `TP_dist` $>$ `TP_toler`, then update the guesses for the normalized interest rate $\boldsymbol{r}^{i+1}$ and wage $\boldsymbol{w}^{i+1}$ and repeat steps (2) through (4). The parameter $\xi \in (0,1]$ is a dampening parameter that reduces the stepsize of the updated guesses in each iteration.

$$r_t^{i+1} = \xi r_t^{i'} + (1-\xi) r_t^i \quad \forall t$$
$$w_t^{i+1} = \xi w_t^{i'} + (1-\xi) w_t^i \quad \forall t$$

   (b) If the TPI distance measure is less-than-or-equal to the tolerance level `TP_dist` $\leq$ `TP_toler`, then the problem has converged and the equilibrium has been found.

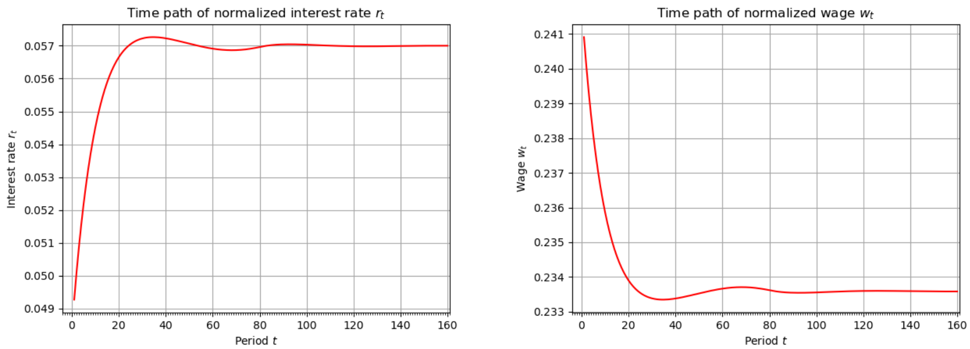
Table 17.3: Transition path maximum absolute error criteria and computation time

Equilibrium error	Value
Max. absolute $\epsilon_{r,t} = r_t^{i'} - r_t^i$	9.722e-08
Max. absolute $\epsilon_{r,t} = r_t^{i'} - r_t^i$	3.131e-08
Max. absolute labor and savings Euler errors	1.421e-13
Max. absolute ind.-M goods market clearing error	3.795e-04
Serial computation time	7 min. 57.6 sec.

Table 17.3 shows the equilibrium errors, which are minimized, and the computation time for the transition path equilibrium. Figure 17.4 shows the equilibrium transition path of interest rates r_t and wages w_t. Figure 17.5 shows the transition paths of household composite good consumption $c_{s,t}$, labor supply $n_{s,t}$, and wealth/savings $b_{s,t}$ by age.

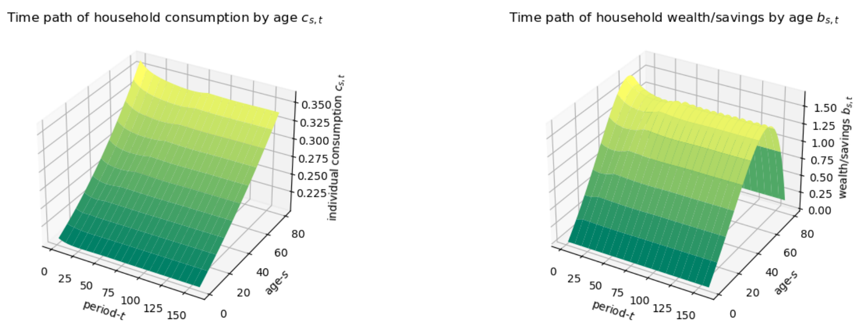
Figure 17.6 shows the initial distribution of household wealth in period $t = 0$ compared to the steady state distribution of household wealth. Figure 17.7 shows the time paths of output $Y_{m,t}$, capital $K_{m,t}$, labor $L_{m,t}$, and investment $I_{m,t}$ in each industry. And Figure 17.8 shows the time paths of prices $p_{m,t}$ and total consumption $C_{m,t}$ in each industry.

Figure 17.4: Equilibrium transition paths of interest rate and wage



(a) Interest rate r_t **(b)** Wage w_t

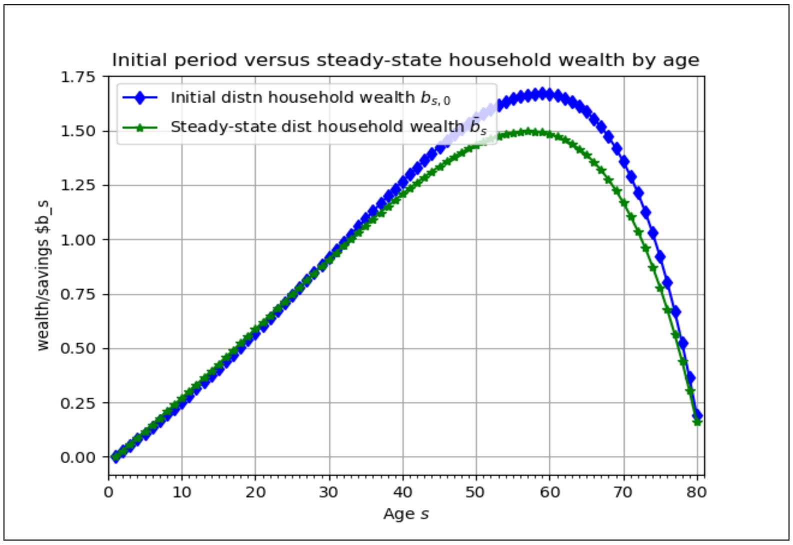
Figure 17.5: Equilibrium transition paths household consumption, savings, and labor supply by age



(a) Composite consumption by age $c_{s,t}$ **(b)** Savings by age $b_{s,t}$

(c) Labor supply by age $n_{s,t}$

Figure 17.6: Initial distribution of household wealth by age $b_{s,0}$ and the steady-state distribution of household wealth $\bar{b}_s$



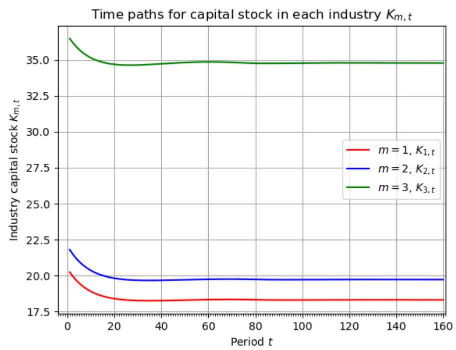
17.7 Calibration

[TODO]

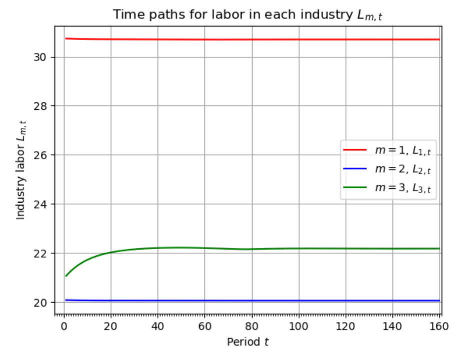
17.8 Exercises

[TODO]

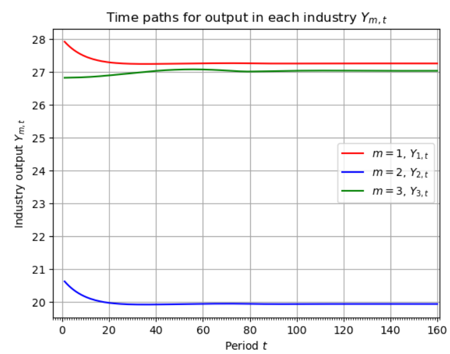
Figure 17.7: Equilibrium transition paths of aggregate variables by industry



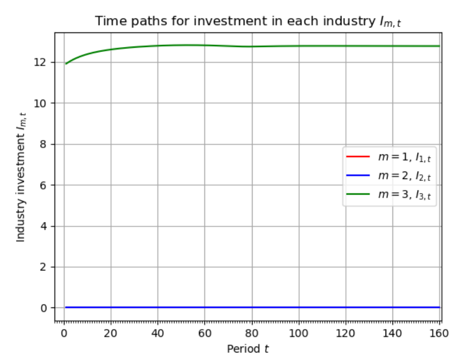
(a) Total capital in each industry $K_{m,t}$



(b) Total labor in each industry $L_{m,t}$

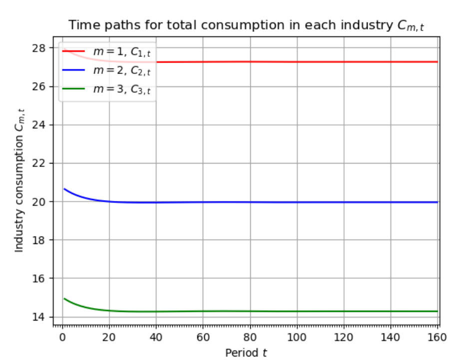


(c) Total output in each industry $Y_{m,t}$

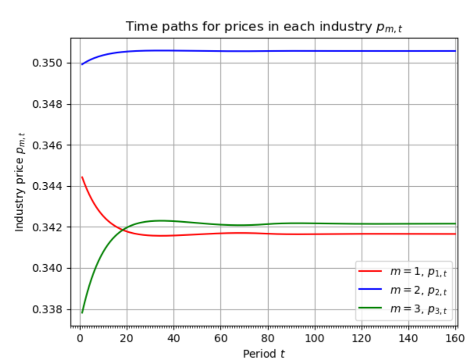


(d) Total investment in each industry $I_{m,t}$

Figure 17.8: Equilibrium transition paths of total consumption and prices by industry



(a) Total consumption in each industry $C_{m,t}$



(b) Prices in each industry $p_{m,t}$