Overlapping Generations Models for Policy Analysis: Theory and Computation

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2017

(version 2017.11.b)

Preface

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Part I

Introduction and Motivation

Chapter 1

Introduction

The overlapping generations (OG) model was first proposed by Samuelson (1958).[1] OG models represent a rich class of macroeconomic general equilibrium model that is extremely useful for answering questions in which inequality, demographics, and individual heterogeneity are important. The OG model is attractive for its clear portrayal of the realistic characteristics that each individual in an economy lives for a finite number of periods and that each individual coexists with individuals of different ages at every point in time.

Modeling individuals over different ages with an expectation of life ending at some point is important for matching microeconomic decision making. The savings decisions of 70-year-olds, who know they are close to the end of their lives, are very different from those of 20-year-olds. However, inclusion of finitely lived individuals in a model in which the economy can last forever increases the difficulty in solving the model. In classical macroeconomic theory, the Ramsey model with infinitely lived agents has dominated the field. The first welfare theorem of Pareto optimality of the competitive equilibrium holds in the Ramsey model. In contrast, the first welfare theorem notoriously does not hold in the OG framework. In other words, overlapping generations models have built in a dynamic inefficiency stemming from the inability of individuals to fully overcome initial conditions in their finite lives. Weil (2008) sketches a description of the proof of this inefficiency in the OG model, and you will work through an exercise on this at the end of this chapter.

[1] The most correct acronym for overlapping generations model is OG. However, these models have often also been referred to as OLG models.

The standard dynamic programming solution methods of value function iteration and policy function iterationIn addition, overlapping generations models assume that the lifetime of an individual is finite and must eventually end. This is a reality for all of us, and seems very intuitive. However, infinitely lived agent models have become important in economics because they are actually more analytically tractable, and the decisions an agent makes if they live forever might be a good approximation of what an agent would choose if he expected to live for 40 more years. But OG models are essential for answering questions about policies that affect age cohorts differently, the leading example of which is pension programs.

1.1 A Simple 2-period-lived Example

Assume that a unit measure of individual agents are born each period t and live for two periods, indexed by $s = \{1, 2\}$.[2] Table 1.1 shows the structure of this economy, with the economy beginning in period $t = 1$ and continuing in perpetuity. This asymmetric structure of time is important for many of the welfare properties that are unique to the OG framework.

Table 1.1: Two-period-lived OG structure

	Period					
Birthday	$\dots$	t	$t+1$	$t+2$	$t+3$	$\dots$
born $t-1$	$\dots$	$c_{2,t}$				
born t		$c_{1,t}$	$c_{2,t+1}$			
born $t+1$			$c_{1,t+1}$	$c_{2,t+2}$		
born $t+2$				$c_{1,t+2}$	$c_{2,t+3}$	
					$\vdots$	$\dots$

The rows represent the lifetime of a given cohort, and the columns represent a cross section of the population alive at time t.

Agents choose consumption in each period to maximize lifetime utility.

$$\max_{c_{1,t},c_{2,t+1}} (1-\beta)\ln\big(c_{1,t}\big) + \beta\ln\big(c_{2,t+1}\big) \quad \forall t \tag{1.1}$$

[2]Individuals face a zero probability of death between ages $s = 1$ and $s = 2$, and have a 100-percent probability of death at the end of age $s = 2$.

Agents receive a strictly positive endowment of the consumption good in both periods of life $e_{1,t}, e_{2,t+1} > 0$ for all t.

- Key papers: Samuelson (1958), Shell (1971), Diamond (1965, public debt), Solow (2006), Ball and Mankiw (2007)
- Compare overlapping generations model to standard infinite horizon model.
- First fundamental welfare theorem fails in OG (not a bad thing)
  - Define dynamic inefficiency as the inefficient oversaving that comes from the incomplete markets from young individuals not being able to perfectly risk share.
  - With certainty: dynamic inefficiency comes from ability to make perpetual transfers from young to old.
  - With uncertainty: extra inefficiency comes from young not being able to perfectly risk share.
  - All this inefficiency without assuming standard market frictions or incomplete markets.
  - Main takeaway, Weil (2008) In OG, "left to their own devices, markets will not properly allocate risk across generations."
- Give example why
  - Do simple example from beginning of Weil (2008).
- Give solutions: pay-go soc. sec., debt, currency

1.2 Equilibrium Solution

Stokey et al. (1989, Ch. 17) has a chapter treating overlapping generations models. They use a simple two-period-lived agent OG model and carefully prove existence and uniqueness of equilibria in this limited setting. These proofs can be extended to show the conditions under which the steady-state equilibrium exists and is unique.[3] But one difficulty with OG models is that no proof of uniqueness has been produced yet for non-steady-state equilibrium (transition path). The OG framework is difficult because these proofs do not exist for the more general S

1.3 Outline

This textbook will take you through successive iterations of OG models from the most simple 3-period-lived deterministic model with inelastic labor supply to an S-period-lived agent model with endogenous labor, a bequest motive, population dynamics, productivity growth, and taxes. All the models here will be deterministic in aggregate, but individuals will face mortality risk when we incorporate population dynamics. Eventually, although not in this document, we will include idiosyncratic stochastic income processes on individual ability. All along the way, we will build up these models with theory and computational exercises that you will complete.

[3] See Wendner (2004).

Part II

Simple OG Model with Exogenous Labor Supply

Chapter 2

3-period-lived Agents with Exogenous Labor Supply

We start the modeling portion of this book with nearly the simplest version of the OG model. It is "nearly" the simplest because we start with overlapping generations of agents who live for three periods. The simplest overlapping generations model is like the example in Chapter 1 in which agents live for two periods. Although the two-period-lived agent model seems like the natural starting place, that model is fundamentally different from any OG model in which agents live for three periods or more.[1] We start with the three-period-lived agent model because its solution method and results are similar for all life spans $S \geq 3$.

The model presented here is a perfect foresight, three-period-lived agent OG model. The model in this chapter has trivial demographics in that a unit measure of agents are born each period, and each generation of agents lives for three periods. There is no population growth because the population in each period equals 3, and the mortality rate is zero in every period except the last period in which it is one. And the agents inelastically supply labor. We also characterize "nearly" the simplest production sector with a unit measure of infinitely lived, perfectly competitive firms that rent capital and hire labor from households.[2]

[1] The main reason for this difference is that, in the two-period-lived agent OG model, young agents completely determine the supply of capital in the next period due to the fact that old people will not be around. This greatly simplifies the two-period-lived agent model and makes its solution method much easier than OG models with agents that live for three or more periods. However, two-period-lived OG models have been productively used in the literature to show many qualitatively interesting results.

[2] This production sector is "nearly" the simplest because many simple 2-period-lived OG models assume

There is no government sector in that agents and firms in the model pay no taxes nor receive any transfers.

2.1 Households

A unit measure of identical individuals are born each period and live for three periods. Let the age of an individual be indexed by $s = \{1, 2, 3\}$. In general, an age-s individual faces a budget constraint each period that looks like the following,

$$c_{s,t} + b_{s+1,t+1} = w_t n_{s,t} + (1 + r_t) b_{s,t} \quad \forall s, t \tag{2.1}$$

where $c_{s,t}$ is consumption and $n_{s,t}$ is labor supply by age-s individuals in period t. The variable $b_{s+1,t+1}$ is savings by age-s individuals in period t to be returned to them with interest in the next period, and $b_{s,t}$ is the savings with which the age-s agent entered period t that was chosen in the previous period. The current period wage w_t and interest rate r_t are the same for all agents (no age s subscript).

We assume the individuals supply a unit of labor inelastically in the first two periods of life and are retired in the last period of life.

$$n_{s,t} = \begin{cases} 1 & \text{if} \quad s = 1, 2 \\ 0.2 & \text{if} \quad s = 3 \end{cases} \qquad \forall s, t \tag{2.2}$$

We also assume that individuals are born with no savings $b_{1,t} = 0$ and that they save no income in the last period of their lives $b_{4,t} = 0$ for all periods t.

These assumptions give rise to the three age-specific budget constraints derived from the

a type of yeoman farmer that works and produces himself—a type of home production.

general version (2.1).[3]

$$c_{1,t} + b_{2,t+1} = w_t \tag{2.3}$$
$$c_{2,t+1} + b_{3,t+2} = w_{t+1} \quad + (1 + r_{t+1})b_{2,t+1} \tag{2.4}$$
$$c_{3,t+2} \quad = 0.2w_{t+2} + (1 + r_{t+2})b_{3,t+2} \tag{2.5}$$

We assume that $c_{s,t} \geq 0$ for all s and t because the utility function we will use will not be defined for negative consumption values. And we assume that $b_{2,t} + b_{3,t} > 0$ because the aggregate capital stock must be strictly positive.

Let the utility of consumption in each period be defined by a function $u(c_{s,t})$, such that $u' > 0$, $u'' < 0$, and $\lim_{c\to 0} u(c) = -\infty$. We will use the constant relative risk aversion (CRRA) utility function that takes the following form,

$$u(c_{s,t}) = \frac{(c_{s,t})^{1-\sigma} - 1}{1 - \sigma} \tag{2.6}$$

where the parameter $\sigma \geq 1$ represents the coefficient of relative risk aversion.

Individuals choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^3$, savings $\{b_{s+1,t+s}\}_{s=1}^2$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$\begin{aligned} &\max_{\{c_{s,t+s-1}\}_{s=1}^3,\{b_{s+1,t+s}\}_{s=1}^2} \quad u(c_{1,t}) + \beta u(c_{2,t+1}) + \beta^2 u(c_{3,t+2}) \\ &c_{1,t} = w_t - b_{2,t+1} \\ &c_{2,t+1} = w_{t+1} + (1 + r_{t+1})b_{2,t+1} - b_{3,t+2} \\ &c_{3,t+2} = 0.2w_{t+2} + (1 + r_{t+2})b_{3,t+2} \end{aligned} \tag{2.7}$$

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (2.7) and assuming

[3] Note that the 3-period-lived agent OLG model generalizes to the S-period-lived agent model. The more periods an agent lives, the more period budget constraints there are that look like (2.4).

that the nonnegativity constraints on the two capital stocks do not bind.[4]

$$\begin{split}\max_{b_{2,t+1},b_{3,t+2}}\:\mathcal{L} =&u\Bigl(w_t - b_{2,t+1}\Bigr) + \beta u\Bigl(w_{t+1} + [1 + r_{t+1}]b_{2,t+1} - b_{3,t+2}\Bigr)...\\ &+ \beta^2 u\Bigl([1+r_{t+2}]b_{3,t+2} + 0.2w_{t+2}\Bigr)\end{split} \tag{2.8}$$

The optimal choice of how much to save in the second period of life $b_{3,t+2}$ is given by taking the derivative of the Lagrangian (2.8) with respect to $b_{3,t+2}$ and setting it equal to zero.

$$\begin{split}\frac{\partial\mathcal{L}}{\partial b_{3,t+2}} = 0 \quad &\Rightarrow \quad u'\bigl(c_{2,t+1}\bigr) = \beta(1 + r_{t+2})u'\bigl(c_{3,t+2}\bigr) \\ &\Rightarrow \quad u'\Bigl(w_{t+1} + [1 + r_{t+1}]b_{2,t+1} - b_{3,t+2}\Bigr) = ... \\ &\qquad\qquad \beta(1 + r_{t+2})u'\Bigl([1+r_{t+2}]b_{3,t+2} + 0.2w_{t+2}\Bigr)\end{split} \tag{2.9}$$

Equation (2.9) implies that the optimal savings for age-2 individuals is a function $\psi_{2,t+1}$ of the wage and interest rate in that period, the interest rate in the next period, and how much capital the individual saved in the previous period.

$$b_{3,t+2} = \psi_{2,t+1}\Bigl(b_{2,t+1},w_{t+1},r_{t+1},r_{t+2},w_{t+2}\Bigr) \tag{2.10}$$

The optimal choice of how much to save in the first period of life $b_{2,t+1}$ is a little more involved. The first order condition of the Lagrangian includes derivatives of $b_{3,t+2}$ with respect to $b_{2,t+1}$ because (2.9) and (2.10) show that optimal middle-aged savings $b_{3,t+2}$ is a

[4]Notice that the individual's problem can be reduced from 5 choice variables to 2 choice variables because the choice in the first two periods between consumption and savings is really just one choice. And the choice of how much to consume in the last period is trivial, because an individual just consumes all their income in the last period.

function of savings when young $b_{2,t+1}$.

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial b_{2,t+1}} = 0 \quad \Rightarrow \quad & -u'\bigl(c_{1,t}\bigr) + \beta(1+r_{t+1})u'\bigl(c_{2,t+1}\bigr)... \\
& \quad - \beta u'\bigl(c_{2,t+1}\bigr)\frac{\partial\psi_{2,t+1}}{\partial b_{2,t+1}} + \beta^2(1+r_{t+2})u'\bigl(c_{3,t+2}\bigr)\frac{\partial\psi_{2,t+1}}{\partial b_{2,t+1}} = 0 \\
\Rightarrow \quad & u'\Bigl(w_t - b_{2,t+1}\Bigr) = \\
& \quad \beta(1+r_{t+1})u'\Bigl([1+r_{t+1}]b_{2,t+1} + w_{t+1} - b_{3,t+2}\Bigr)... \\
& \quad - \beta\frac{\partial\psi_{2,t+1}}{\partial b_{2,t+1}}\Bigl[u'(c_{2,t+1}) - \beta(1+r_{t+2})u'(c_{3,t+2})\Bigr]
\end{aligned}
\tag{2.11}
$$

Notice that the term in the brackets on the third line of (2.11) equals zero because of the optimality condition (2.9) for $b_{3,t+1}$. This is the envelope condition or the principle of optimality. The intuition is that I don't need to worry about the effect of my choice today on my choice tomorrow because I will optimize tomorrow given today. So the first order condition for optimal savings when young $b_{2,t+1}$ simplifies to the following expression.

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial b_{2,t+1}} = 0 \quad \Rightarrow \quad & u'\bigl(c_{1,t}\bigr) = \beta(1+r_{t+1})u'\bigl(c_{2,t+1}\bigr) \\
\Rightarrow \quad & u'\Bigl(w_t - b_{2,t+1}\Bigr) = ... \\
& \quad \beta(1+r_{t+1})u'\Bigl(w_{t+1} + [1+r_{t+1}]b_{2,t+1} - \psi_{2,t+1}\Bigr)
\end{aligned}
\tag{2.12}
$$

Equation (2.12) implies that the optimal savings for age-1 individuals is a function of the wages in that period and the next period and the interest rate in the next period and in the period after that.[5]

$$
b_{2,t+1} = \psi_{1,t}\Bigl(w_t, w_{t+1}, r_{t+1}, r_{t+2}, w_{t+2}\Bigr) \tag{2.13}
$$

Instead of looking at the age-1 and age-2 savings decisions of a particular individual, which happen in consecutive periods, we could look at the age-1 savings decisions of the young in period t as characterized in (2.12) and the age-2 savings decisions of the middle-aged in period t. This savings $b_{3,t+1}$ is characterized by the following first order condition,

[5] The presence of r_{t+2} and w_{t+2} in (2.13) comes from the fact that optimal $b_{2,t+1}$ depends on the optimal $b_{3,t+2}$ from (2.10).

which is simply Equation (2.9) iterated backward in time one period,

$$\begin{aligned} u'\big(c_{2,t}\big) &= \beta(1+r_{t+1})u'\big(c_{3,t+1}\big) \\ u'\Big(w_t + [1+r_t]b_{2,t} - b_{3,t+1}\Big) &= \beta(1+r_{t+1})u'\Big(0.2w_{t+1} + [1+r_{t+1}]b_{3,t+1}\Big) \end{aligned} \tag{2.14}$$

which implies that the period-t savings decision of the middle aged is a function of the wage and interest rate in period-t, the interest rate in the period $t+1$, and how much capital the individual saved in the previous period.

$$b_{3,t+1} = \psi_{2,t}\Big(b_{2,t}, w_t, r_t, r_{t+1}, w_{t+1}\Big) \tag{2.15}$$

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \big\{b_{2,t}, b_{3,t}\big\} \quad \forall t \tag{2.16}$$

As will be shown in Section 2.4, the state as defined in Definition 2.2 in every period t for the entire equilibrium system described in the non-steady-state equilibrium characterized in Definition 2.4 is the current distribution of individual savings $\mathbf{\Gamma}_t$ from (2.16). Because individuals must forecast wages and interest rates in every period in order to solve their optimal lifetime decisions and because each of those future variables depends on the entire distribution of savings in the future, we must assume some individual beliefs about how the entire distribution will evolve over time. Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[6]

[6] In Section 2.4 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 2.4.

2.2 Firms

The economy also includes a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology.

$$Y_t = F(K_t, L_t) \equiv AK_t^\alpha L_t^{1-\alpha} \quad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad A > 0 \tag{2.18}$$

We assume that the price of the output in every period $P_t = 1$.[7] The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t, L_t} AK_t^\alpha L_t^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{2.19}$$

where $\delta \in [0,1]$ is the rate of capital depreciation.[8] The two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha A \left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$

$$w_t = (1-\alpha) A \left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

[7]This is just a cheap way to assume no monetary policy. Relaxing this assumption is important in many applications for which price fluctuation is important.

[8]Note that it is equivalent whether we put depreciation on the firms' side as in equation (2.19) or on the household side making the return on capital savings $1 + r_t - \delta$. Depreciation must be in one place or the other, not both. We choose to put depreciation on the firm's side here because the tax model we are building up to includes taxes and subsidies to firms for depreciation expenses.

2.3 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{3} n_{s,t} = 2.2 \quad \forall t \tag{2.22}$$

$$K_t = \sum_{i=2}^{3} b_{s,t} = b_{2,t} + b_{3,t} \quad \forall t \tag{2.23}$$

$$\begin{aligned} &Y_t = C_t + I_t \quad \forall t \\ &\qquad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{aligned} \tag{2.24}$$

The goods market clearing equation (2.24) is redundant by Walras' Law.

2.4 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, I want to give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points

i. Households optimize according to (2.12) and (2.14).

ii. Firms optimize according to (2.20) and (2.21).

iii. Markets clear according to (2.22) and (2.23).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (2.22) and (2.23) into the firm's optimal conditions (2.20) and (2.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t\big(b_{2,t},b_{3,t}\big): \quad w_t = (1-\alpha)A\left(\frac{b_{2,t}+b_{3,t}}{2.2}\right)^{\alpha} \tag{2.25}$$

$$r_t\big(b_{2,t},b_{3,t}\big): \quad r_t = \alpha A\left(\frac{2.2}{b_{2,t}+b_{3,t}}\right)^{1-\alpha} - \delta \tag{2.26}$$

Now (2.25) and (2.26) can be substituted into household Euler equations (2.12) and (2.14) to get the following two-equation system that completely characterizes the equilibrium.

$$\begin{aligned} &u'\Big(w_t(b_{2,t},b_{3,t}) - b_{2,t+1}\Big) = \beta\Big(1 + r_{t+1}(b_{2,t+1},b_{3,t+1})\Big) \times ... \\ &\quad u'\Big(w_{t+1}(b_{2,t+1},b_{3,t+1}) + [1 + r_{t+1}(b_{2,t+1},b_{3,t+1})]b_{2,t+1} - b_{3,t+2}\Big) \end{aligned} \tag{2.27}$$

$$\begin{aligned} &u'\Big(w_t(b_{2,t},b_{3,t}) + [1 + r_t(b_{2,t},b_{3,t})]b_{2,t} - b_{3,t+1}\Big) = ... \\ &\quad \beta\Big(1 + r_{t+1}(b_{2,t+1},b_{3,t+1})\Big)u'\Big([1 + r_{t+1}(b_{2,t+1},b_{3,t+1})]b_{3,t+1} + 0.2w_{t+1}(b_{2,t+1},b_{3,t+1})\Big) \end{aligned} \tag{2.28}$$

The system of two dynamic equations (2.27) and (2.28) characterizing the decisions for $b_{2,t+1}$ and $b_{3,t+1}$ is not identified. These households know the current distribution of capital $b_{2,t}$ and $b_{3,t}$. However, we need to solve for policy functions for $b_{2,t+1}$, $b_{3,t+1}$, and $b_{3,t+2}$ from these two equations. It looks like this system is unidentified. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 2.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with 3-period lived agents is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^3$, capital $\{\bar{b}_s\}_{s=2}^3$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (2.12) and (2.14),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (2.22) and (2.23).

As we saw earlier in this section, the characterizing equations in Definition 2.1 reduce to (2.27) and (2.28). These two equations are exactly identified in the steady state. That is, they are two equations and two unknowns $(\bar{b}_2, \bar{b}_3)$.

$$u'\Bigl(w(\bar{b}_2,\bar{b}_3) - \bar{b}_2\Bigr) = \beta\Bigl(1 + r(\bar{b}_2,\bar{b}_3)\Bigr)u'\Bigl(w(\bar{b}_2,\bar{b}_3) + [1 + r(\bar{b}_2,\bar{b}_3)]\bar{b}_2 - \bar{b}_3\Bigr) \tag{2.29}$$

$$\begin{aligned} u'\Bigl(w(\bar{b}_2,\bar{b}_3) + [1 + r(\bar{b}_2,\bar{b}_3)]\bar{b}_2 - \bar{b}_3\Bigr) &= ... \\ \beta\Bigl(1 + r(\bar{b}_2,\bar{b}_3)\Bigr)u'\Bigl([1 + r(\bar{b}_2,\bar{b}_3)]\bar{b}_3 + 0.2w(\bar{b}_2,\bar{b}_3)\Bigr) \end{aligned} \tag{2.30}$$

We can solve for steady-state $\bar{b}_2$ and $\bar{b}_3$ by using a unconstrained optimization solver. Then we solve for $\bar{w}$, $\bar{r}$, $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_3$ by substituting $\bar{b}_2$ and $\bar{b}_3$ into the equilibrium firm first order conditions and into the household budget constraints.

Now we can get ready to define the non-steady-state equilibrium. To do this, we need to define two other important concepts.

Definition 2.2 (State of a dynamical system). The state of a dynamical system—sometimes called the state vector—is the smallest set of variables that completely summarizes all the information necessary for determining the future of the system at a given point in time.

In the 3-period-lived agent, perfect foresight, OLG model described in this section, the state vector can be seen in equations (2.27) and (2.28). What is the smallest set of variables that completely summarize all the information necessary for the three generations of all three generations living at time t to make their consumption and saving decisions? What information do they have at time t that will allow them to make their savings decisions? The state vector of this model in each period is the distribution of capital $(b_{2,t}, b_{3,t})$.

Definition 2.3 (Stationary function). We define a stationary function to be a function that only depends upon its arguments and does not depend upon time.

The relevant examples of stationary functions in this model are the policy functions for saving and investment. We defined the functions $\psi_{1,t}$ and $\psi_{2,t}$ generally in equations (2.13) and (2.15). But they were indexed by time as evidenced by the t in $\psi_{1,t}$ and $\psi_{2,t}$. The stationary versions of those functions would be ψ_1 and ψ_2, which do not depend upon time. The arguments of the functions (the state) may change overtime causing the savings levels to change over time, but the function of the arguments is constant across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state equilibrium of the model.

Definition 2.4 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with 3-period lived agents is defined as stationary allocation functions of the state $\psi_1\big(b_{2,t}, b_{3,t}\big)$ and $\psi_2\big(b_{2,t}, b_{3,t}\big)$ and stationary price functions $w(b_{2,t}, b_{3,t})$ and $r(b_{2,t}, b_{3,t})$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (2.12) and (2.14),

iii. firms optimize according to (2.20) and (2.21),

iv. markets clear according to (2.22) and (2.23).

We have already shown how to boil down the characterizing equations in Definition 2.4 to two equations (2.27) and (2.28). But we have also seen that those two equations are not identified. So how do we solve for these equilibrium functions? The solution to the non-steady-state equilibrium in Definition 2.4 is a fixed point in function space. Choose two functions ψ_1 and ψ_2 and verify that they satisfy the Euler equations for all points in the state space (all possible values of the state).

2.5 Solution method: time path iteration (TPI)

The benchmark conventional solution method for the non-steady-state rational expectations equilibrium transition path in OG models was originally outlined in a series of papers between 1981 and 1985[9] and in the seminal book Auerbach and Kotlikoff (1987, ch. 4) for the perfect foresight case and in Nishiyama and Smetters (2007, Appendix II) and Evans and Phillips (2014, Sec. 3.1) for the stochastic case. We call this method time path iteration (TPI). The idea is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating

[9]See Auerbach et al. (1981, 1983), Auerbach and Kotlikoff (1983c,b,a), and Auerbach and Kotlikoff (1985).

on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey et al. (1989, ch. 17)). Evans and Phillips (2014) give a good description of how to implement this method.

The key assumption is that the economy will reach the steady-state equilibrium $(\bar{b}_2, \bar{b}_3)$ described in Definition 2.1 in a finite number of periods $T < \infty$ regardless of the initial state $(b_{2,1}, b_{3,1})$. The first step is to assume a transition path for aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$ such that T is sufficiently large to ensure that $(b_{2,T}, b_{3,T}) = (\bar{b}_2, \bar{b}_3)$. The superscript i is an index for the iteration number. The transition path for aggregate capital determines the transition path for both the real wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $(b_{2,1}, b_{3,1})$ can be arbitrarily chosen as long as it satisfies $K_1^i = b_{2,1} + b_{3,1}$ according to market clearing condition (2.23). One could also first choose the initial distribution of capital $(b_{2,1}, b_{3,1})$ and then choose an initial aggregate capital stock K_1^i that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $K_T^i = \bar{K} = \bar{b}_2 + \bar{b}_3$ by period T. But the aggregate capital stocks K_t^j for periods $1 < t < T$ can be any level.

Given the initial capital distribution $(b_{2,1}, b_{3,1})$ and the transition paths of aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$, the real wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$, and the real return to investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal savings decision for the initial middle-aged $s = 2$ individual for the last period of his life $b_{3,2}$ using his intertemporal Euler equation (2.28).

$$u'\Bigl(w_1^i + [1 + r_1^i]b_{2,1} - b_{3,2}\Bigr) = \beta\Bigl(1 + r_2^i\Bigr)u'\Bigl([1 + r_2^i]b_{3,2} + 0.2w_2^i\Bigr) \tag{2.31}$$

Notice that everything in equation (2.31) is known except for the savings decision $b_{3,2}$. This is one equation and one unknown.

The next step is to solve for $b_{2,2}$ and $b_{3,3}$ for the initial young $s = 1$ agent at period 1 using the appropriately timed versions of (2.12) and (2.9) with the conjectured interest rates and real wages.

$$u'\Big(w_1^i - b_{2,2}\Big) = \beta(1+r_2^i)u'\Big(w_2^i + [1+r_2^i]b_{2,2} - b_{3,3}\Big) \tag{2.32}$$

$$u'\Big(w_2^i + [1+r_2^i]b_{2,2} - b_{3,3}\Big) = \beta(1+r_3^i)u'\Big([1+r_3^i]b_{3,3} + 0.2w_3^i\Big) \tag{2.33}$$

Everything is known in these two equations except for $b_{2,2}$ and $b_{3,3}$. So we can solve for those with a standard unconstrained solver. We next solve for $b_{2,t}$ and $b_{3,t+1}$ for the remaining $t \in \{3, 4, ...T+m\}$, where T represents the period in the future at which the economy should have converged to the steady-state and m represents some number of periods past that.[10]

At this point, we have solved for the distribution of capital $(b_{2,t}, b_{3,t})$ over the entire time period $t \in \{1, 2, ...T\}$. In each period t, the distribution of capital implies an aggregate capital stock $K_t^{i'} = b_{2,t} + b_{3,t}$. I put a "$'$" on this aggregate capital stock because, in general, $K_t^{i'} \neq K_t^i$. That is, the conjectured path of the aggregate capital stock is not equal to the optimally chosen path of the aggregate capital stock given $\boldsymbol{K}^i$.[11]

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock. Common norms to use are the L^2 and the L^∞ norms. Then the fixed point necessary for the equilibrium transition path from Definition 2.4 has been found when the distance between $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$ is arbitrarily close to zero.

$$\left\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\right\| < \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{2.34}$$

If the fixed point has not been found $\left\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\right\| > \varepsilon$, then a new transition path for the aggregate capital stock is generated as a convex combination of $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$.

$$\boldsymbol{K}^{i+1} = \xi\boldsymbol{K}^{i'} + (1-\xi)\boldsymbol{K}^i \quad \text{for} \quad \xi \in (0,1) \tag{2.35}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization. TPI solves for the equilibrium transition path from Definition 2.4 by finding a

[10] For models in which agents live for S periods, $m \geq S$ so that the full distribution of capital at time T can be solved for. In the 3-period-lived agent model described here, $m \geq 3$.

[11] A check here for whether T is large enough is if $K_T^{i'} = \bar{K}$ as well as $K_{T+1}^{i'}$ and $K_{T+2}^{i'}$. If not, then T needs to be larger.

fixed point in the time path of the economy.

2.6 Calibration

Use the following parameterization of the model for the problems below. Because agents live for only three periods, assume that each period of life is 20 years. If the annual discount factor is estimated to be 0.96, then the 20-year discount factor is $\beta = 0.96^{20} = 0.442$. Let the annual depreciation rate of capital be 0.05. Then the 20-year depreciation rate is $\delta = 1 - (1 - 0.05)^{20} = 0.6415$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

2.7 Exercises

Exercise 2.1. Using the calibration from Section 2.6, write a Python function named `feasible()` that has the following form,

```
b_cnstr, c_cnstr, K_cnstr = feasible(f_params, bvec_guess)
```

where the inputs are a tuple `f_params = (nvec, A, alpha, delta)`, and a guess for the steady-state savings vector `bvec_guess = np.array([scalar, scalar])`. The outputs should be Boolean (`True` or `False`, `1` or `0`) vectors of lengths 2, 3, and 1, respectively. `K_cnstr` should be a singleton Boolean that equals `True` if $K \leq 0$ for the given `f_params` and `bvec_guess`. The object `c_cnstr` should be a length-3 Boolean vector in which the sth element equals `True` if $c_s \leq 0$ given `f_params` and `bvec_guess`. And `b_cnstr` is a length-2 Boolean vector that denotes which element of `bvec_guess` is likely responsible for any of the consumption nonnegativity constraint violations identified in `c_cnstr`. If the first element of `c_cnstr` is `True`, then the first element of `b_cnstr` is `True`. If the second element of `c_cnstr` is `True`, then both elements of `b_cnstr` are `True`. And if the last element of `c_cnstr` is `True`, then the last element of `b_cnstr` is `True`.

a. Which, if any, of the constraints is violated if you choose an initial guess for steady-state savings of `bvec_guess = np.array([1.0, 1.2])`?

b. Which, if any, of the constraints is violated if you choose an initial guess for steady-state savings of `bvec_guess = np.array([0.06, -0.001])`?

c. Which, if any, of the constraints is violated if you choose an initial guess for steady-state savings of `bvec_guess = np.array([0.1, 0.1])`?

Exercise 2.2. Use the calibration from Section 2.6 and the steady-state equilibrium Definition 2.1. Write a function named `get_SS()` that has the following form,

```
ss_output = get_SS(params, bvec_guess, SS_graphs)
```

where the inputs are a tuple of the parameters for the model `params = ((beta, sigma, nvec, L, A, alpha, delta, SS_tol))`, an initial guess of the steady-state savings `bvec_guess`, and a Boolean `SS_graphs` that generates a figure of the steady-state distribution of consumption and savings if it is set to `True`. The output object `ss_output` is a Python dictionary with the steady-state solution values for the following endogenous objects.

```
ss_output = {
  'b_ss': b_ss, 'c_ss': c_ss, 'w_ss': w_ss, 'r_ss': r_ss,
  'K_ss': K_ss, 'Y_ss': Y_ss, 'C_ss': C_ss,
  'EulErr_ss': EulErr_ss, 'RCerr_ss': RCerr_ss,
  'ss_time': ss_time}
```

Let `ss_time` be the number of seconds it takes to run your steady-state program. You can time your program by importing the time library.

```
import time
...
start_time = time.clock() # Place at beginning of get_SS()
...
ss_time = time.clock() - start_time # Place at end of get_SS()
```

And let the object `EulErr_ss` be a length-2 vector of the two Euler errors from the resulting steady-state solution given in difference form $\beta(1+\bar{r})u'(\bar{c}_{s+1}) - u'(\bar{c}_s)$. The object `RCerr_ss` is a resource constraint error which should be close to zero. It is given by $\bar{Y} - \bar{C} - \delta\bar{K}$.

a. Solve numerically for the steady-state equilibrium values of $\{\bar{c}_s\}_{s=1}^3$, $\{\bar{b}_s\}_{s=2}^3$, $\bar{w}$, $\bar{r}$, $\bar{K}$, $\bar{Y}$, $\bar{C}$, the two Euler errors and the resource constraint error. List those values. Time your function. How long did it take to compute the steady-state?

b. Generate a figure that shows the steady-state distribution of consumption and savings by age $\{\bar{c}_s\}_{s=1}^3$ and $\{\bar{b}_s\}_{s=2}^3$.

c. What happens to each of these steady-state values if all households become more patient $\beta\uparrow$ (an example would be $\beta = 0.55$)? That is, in what direction does $\beta\uparrow$ move each steady-state value $\{\bar{c}_s\}_{s=1}^3$, $\{\bar{b}_s\}_{s=2}^3$, $\bar{w}$, and $\bar{r}$? What is the intuition?

Exercise 2.3. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from $(b_{2,1}, b_{3,1}) = (0.8\bar{b}_2, 1.1\bar{b}_3)$ to the steady-state $(\bar{b}_2, \bar{b}_3)$. You'll have to choose a guess for T and a time path updating parameter $\xi \in (0,1)$, but I can assure you that $T < 50$. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear initial guess for the time path of the aggregate capital stock from the initial state K_1^1 to the steady state K_T^1 at time T.

a. Report the maximum of the absolute values of all the Euler errors across the entire time path. Also report the maximum of the absolute value of all the aggregate resource constraint errors $Y_t - C_t - K_{t+1} + (1-\delta)K_t$ across the entire time path.

b. Plot the equilibrium time paths of the aggregate capital stock $\{K_t\}_{t=1}^{T+5}$, wage $\{w_t\}_{t=1}^{T+5}$, and interest rate $\{r_t\}_{t=1}^{T+5}$.

c. How many periods did it take for the economy to get within 0.00001 of the steady-state aggregate capital stock $\bar{K}$? What is the period after which the aggregate capital stock never is again farther than 0.00001 away from the steady-state?

Chapter 3

S-period-lived Agent Problem with Exogenous labor Supply

In this chapter, we extend the simple 3-period-lived agent problem from Chapter 2 and generalize it to an $S \in [3, 80]$-period-lived agent problem.

3.1 Households

A unit measure of identical individuals are born each period and live for S periods. Let the age of an individual be indexed by $s = \{1, 2, ...S\}$. In general, an age-s individual faces the same per-period budget constraint (2.1) as in the previous section.

$$c_{s,t} + b_{s+1,t+1} = (1 + r_t)b_{s,t} + w_t n_{s,t} \quad \forall s, t \tag{2.1}$$

We assume the individuals supply a unit of labor inelastically in the first two thirds of life ($s \leq \text{round}(2S/3)$) and are retired during the last third of life ($s > \text{round}(2S/3)$).

$$n_{s,t} = \begin{cases} 1 & \text{if} \quad s \leq \text{round}\left(\frac{2S}{3}\right) \\ 0.2 & \text{if} \quad s > \text{round}\left(\frac{2S}{3}\right) \end{cases} \quad \forall s, t \tag{3.1}$$

Because exogenous labor in (3.1) is not dependent on the time period, we drop the t subscript from labor n_s for the rest of this section. We also assume that households are born with no savings $b_{1,t} = 0$ and that individuals save no income in the last period of their lives $b_{S+1,t} = 0$ for all periods t. Assume that $c_{s,t} \geq 0$ because negative consumption neither has an intuitive interpretation nor is it household utility defined for it. It is the latter condition that will make $c_{s,t} > 0$ in equilibrium.

Let the utility of consumption in each period be defined by the constant relative risk aversion function (2.6) $u(c_{s,t})$ from the previous section, such that $u' > 0$, $u'' < 0$, and $\lim_{c\rightarrow 0} u(c) = -\infty$. Individuals choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^{S}$, savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$
\begin{aligned}
&\max_{\{c_{s,t+s-1}\}_{s=1}^{S},\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{u=0}^{S-s} \beta^u u(c_{s+u,t+u}) \quad \forall s,t \\
&\text{s.t.} \quad c_{s,t} = (1+r_t)b_{s,t} + w_t n_s - b_{s+1,t+1} \quad \forall s,t \\
&\text{and} \quad b_{1,t}, b_{S+1,t} = 0 \quad \forall t \quad \text{and} \quad c_{s,t} \geq 0 \quad \forall s,t
\end{aligned} \tag{3.2}
$$

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (3.2). The optimal choice of how much to save in the each of the first $S-1$ periods of life $b_{s+1,t+1}$ is found by taking the derivative of the lifetime utility function with respect to each of the lifetime savings amounts $\{b_{s+1,t+s+1}\}_{s=1}^{S-1}$ and setting the derivatives equal to zero.

In the last period of life, the household optimally chooses no savings $b_{S+1,t+1} = 0$ for all t because any positive savings only imposes a cost of reduced consumption, and negative savings (borrowing) would impose an automatic default on anyone lending to him. The final period S decision is simple. The individual enters the period with wealth $b_{S,t}$, he knows the interest rate r_t and the wage w_t, and inelatically supplies zero labor $n_{S,t} = 0$. Everything in the budget constraint (2.1) is determined except for $c_{S,t}$. In the final period, the individual simply consumes all his resources.

$$
c_{S,t} = (1+r_t)b_{S,t} + w_t n_S \quad \forall t \tag{3.3}
$$

In the second-to-last period of life $s = S-1$, the household has a savings decision to make. He enters the period with wealth $b_{S-1,t}$, he knows the current interest rate r_t and the current wage w_t, and he must know or be able to forecast next period's interest rate r_{t+1} and wage w_{t+1}. In this case, the household's lifetime utility function is one equation and one unknown.

$$\max_{b_{S,t+1}} u\Bigl([1+r_t]b_{S-1,t} + w_t n_{S-1} - b_{S,t+1}\Bigr) + \beta u\Bigl([1+r_{t+1}]b_{S,t+1} + w_{t+1}n_S\Bigr) \tag{3.4}$$

The first order condition, or dynamic Euler equation, for this second-to-last period of life savings decision is the following.

$$u'\Bigl([1+r_t]b_{S-1,t} + w_t n_{S-1} - b_{S,t+1}\Bigr) = \beta(1+r_{t+1})u'\Bigl([1+r_{t+1}]b_{S,t+1} + w_{t+1}n_S\Bigr) \tag{3.5}$$

The solution for savings $b_{S,t+1}$ in the second-to-last period of life to be returned with interest in the last period of life is characterized by the nonlinear dynamic Euler equation (3.5) and is a function of individual wealth $b_{S-1,t}$, the interest rate r_t, and the wage w_t at the beginning of the second-to-last period of life, as well as the interest rate r_{t+1} and wage w_{t+1} in the last period of life.

$$b_{S,t+1} = \psi_{S-1}\left(b_{S-1,t}, r_t, w_t, r_{t+1}, w_{t+1}\right) \quad \forall t \tag{3.6}$$

Call $\psi_{S-1}(\cdot)$ the policy function for savings $b_{S,t+1}$ in the second-to-last period of life.

In the third-to-last period of life $s = S-2$, the individual has two remaining lifetime decisions to make. He must choose how much to save in the third-to-last period of life $b_{S-1,t}$ and how much to save in the second-to-last period of life $b_{S,t+1}$. The latter of these two decisions will be characterized by the same function (3.6) that equates (3.5). However, the maximization problem is trickier for the third-to-last period savings $b_{S-1,t}$ because the individual must maximize utility over three periods.

$$\begin{aligned} &\max_{b_{S-1,t}} u\Bigl([1+r_{t-1}]b_{S-2,t-1} + w_{t-1}n_{S-2} - b_{S-1,t}\Bigr) + ... \\ &\quad \beta u\Bigl([1+r_t]b_{S-1,t} + w_t n_{S-1} - b_{S,t+1}\Bigr) + \beta^2 u\Bigl([1+r_{t+1}]b_{S,t+1} + w_{t+1}n_S\Bigr) \end{aligned} \tag{3.7}$$

It initially looks like the savings $b_{S-1,t}$ only shows up in two places, which should make this derivative very easy. However, we must remember that it is also in the optimal function for the second to last period savings $b_{S,t+1}$ from (3.6). The derivative of (3.7) with respect to $b_{S-1,t}$ and set equal to zero is, therefore,

$$-u'(c_{S-2,t-1}) + \beta\left(1 + r_t - \frac{\partial\psi_{S-1}}{\partial b_{S-1,t}}\right)u'(c_{S-1,t}) + \beta^2(1 + r_{t+1})\frac{\partial\psi_{S-1}}{\partial b_{S-1,t}}u'(c_{S,t+1}) = 0 \quad (3.8)$$

This looks very different from the equation characterizing optimal savings in the second-to-last period (3.4). However, factoring out the partial derivative terms gives the following version of the equation.

$$-u'(c_{S-2,t-1}) + \beta(1 + r_t)u'(c_{S-1,t}) = \beta\frac{\partial\psi_{S-1}}{\partial b_{S-1,t}}\Big[u'(c_{S-1,t}) - \beta(1 + r_{t+1})u'(c_{S,t+1})\Big] \quad (3.9)$$

Notice that the term on the right in brackets is zero from (3.4). This is the envelope theorem or the principle of optimality. It means that the savings decisions in all future periods will be made optimally, so the derivative of that function will be zero with respect to today's savings. The third-to-last period Euler equation in (3.9) reduces to the following due to the envelope theorem.

$$u'(c_{S-2,t-1}) = \beta(1 + r_t)u'(c_{S-1,t}) \quad (3.10)$$

Using the expressions for $c_{S-2,t-1}$ and $c_{S-1,t}$ from the budget constraint (2.1) and the function for second-to-last period savings $b_{S,t+1}$ from (3.6), it is simple to show that the policy function for third-to-last period savings $b_{S-1,t}$ characterized by nonlinear dynamic Euler equation (3.10) is the following.

$$b_{S-1,t} = \psi_{S-2}\left(b_{S-2,t-1}, r_{t-1}, w_{t-1}, r_t, w_t, r_{t+1}, w_{t+1}\right) \quad \forall t \quad (3.11)$$

By backward induction, it is straightforward to show that the $S-1$ savings decisions over an individual's lifetime are characterized by $S-1$ nonlinear dynamic Euler equations of the form,

$$
\begin{aligned}
u'(c_{s,t}) = \beta(1+r_{t+1})u'(c_{s+1,t+1}) \quad \forall t, \quad \text{and} \quad 1 \leq s \leq S-1 \\
\text{and} \quad c_{s,t} = w_t n_s + (1+r_t)b_{s,t} - b_{s+1,t+1} \quad \forall s,t \\
\text{and} \quad b_{1,t}, b_{S-1,t} = 0 \quad \forall t
\end{aligned}
\tag{3.12}
$$

Following the pattern of (3.6) and (3.11), the policy functions for each of the savings decisions is a function of the individual's wealth at the beginning of the period $b_{s,t}$ and the time path of wages and interest rates over the remaining periods of the individual's life.

$$
b_{s+1,t+1} = \psi_s\Big(b_{s,t}, \{r_v\}_{u=t}^{t+S-s}, \{w_u\}_{u=t}^{t+S-s}\Big) \quad \forall t \quad \text{and} \quad 1 \leq s \leq S-1 \tag{3.13}
$$

To summarize the individual's problem, if one knows his initial savings or wealth $b_{s,t}$ and the time path of factor prices over his remaining lifetime, he can solve for all of his optimal savings levels $\{b_{s+1,t+s}\}_{s=1}^{S-1}$.

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates and wages $\{r_u, w_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Section 3.4, the equilibrium interest rate r_t and wage w_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$
\mathbf{\Gamma}_t \equiv \big\{b_{s,t}\big\}_{s=2}^{S} \quad \forall t \tag{3.14}
$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$
\mathbf{\Gamma}_{t+u}^{e} = \Omega^{u}\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}
$$

where the e superscript signifies that $\mathbf{\Gamma}_{t+u}^{e}$ is the expected distribution of wealth at time $t+u$

based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[1]

3.2 Firms

The production side of this economy is identical to the one in Section 2.2 with a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology,

$$Y_t = F(K_t, L_t) \equiv AK_t^{\alpha}L_t^{1-\alpha} \quad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad A > 0. \tag{2.18}$$

The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t,L_t} AK_t^{\alpha}L_t^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{2.19}$$

where $\delta \in [0,1]$ is the rate of capital depreciation, and the two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

[1] In Section 3.4 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 3.2.

3.3 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{S} n_s \tag{3.15}$$

$$K_t = \sum_{i=2}^{S} b_{s,t} \tag{3.16}$$

$$\begin{aligned} &Y_t = C_t + I_t \\ &\qquad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{aligned} \tag{2.24}$$

The goods market clearing equation (2.24) is redundant by Walras' Law.

3.4 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points

i. Households optimize according to equations (3.12).

ii. Firms optimize according to (2.20) and (2.21).

iii. Markets clear according to (3.15) and (3.16).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (3.15) and (3.16) into the firm's optimal conditions (2.20) and (2.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t(\mathbf{\Gamma}_t): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t}}{\sum_{s=1}^S n_s}\right)^\alpha \quad \forall t \tag{3.17}$$

$$r_t(\mathbf{\Gamma}_t): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^S n_s}{\sum_{s=2}^S b_{s,t}}\right)^{1-\alpha} - \delta \quad \forall t \tag{3.18}$$

Now (3.17) and (3.18) can be substituted into household Euler equations (3.12) to get the following $(S-1)$-equation system that completely characterizes the equilibrium.

$$\begin{aligned} &u'\Bigl(w_t(\mathbf{\Gamma}_t)n_s + \Bigl[1 + r_t(\mathbf{\Gamma}_t)\Bigr]b_{s,t} - b_{s+1,t+1}\Bigr) = \\ &\quad \beta\Bigl[1 + r_{t+1}(\mathbf{\Gamma}_{t+1})\Bigr]u'\Bigl(w_{t+1}(\mathbf{\Gamma}_{t+1})n_{s+1} + \Bigl[1 + r_{t+1}(\mathbf{\Gamma}_{t+1})\Bigr]b_{s+1,t+1} - b_{s+2,t+2}\Bigr) \\ &\qquad \forall t, \quad \text{and} \quad 1 \leq s \leq S-1 \end{aligned} \tag{3.19}$$

The system of $S-1$ nonlinear dynamic equations (3.19) characterizing the the lifetime savings decisions for each household $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ for all agents alive next period, and for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 3.1 (**Steady-state equilibrium**). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, capital $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (3.12),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (3.15) and (3.16).

As we saw earlier in this section, the characterizing equations in Definition 3.1 reduce to (3.19). These $S-1$ equations are exactly identified in the steady state. That is, they are $S-1$ equations and $S-1$ unknowns $\{\bar{b}_s\}_{s=2}^S$.

$$
\begin{aligned}
&u'\Bigl(\bar{w}(\bar{\mathbf{\Gamma}})n_s + \Bigl[1+\bar{r}(\bar{\mathbf{\Gamma}})\Bigr]\bar{b}_s - \bar{b}_{s+1}\Bigr) = \\
&\qquad \beta\Bigl[1+\bar{r}(\bar{\mathbf{\Gamma}})\Bigr]u'\Bigl(\bar{w}(\bar{\mathbf{\Gamma}})n_{s+1} + \Bigl[1+\bar{r}(\bar{\mathbf{\Gamma}})\Bigr]\bar{b}_{s+1} - \bar{b}_{s+2}\Bigr) \\
&\qquad\quad \text{for} \quad 1\leq s\leq S-1
\end{aligned} \tag{3.20}
$$

We can solve for steady-state $\{\bar{b}_s\}_{s=2}^S$ by using an unconstrained optimization solver. Then we solve for $\bar{w}$, $\bar{r}$, and $\{\bar{c}_s\}_{s=1}^S$ by substituting $\{\bar{b}_s\}_{s=2}^S$ into the equilibrium firm first order conditions and into the household budget constraints.

In the S-period-lived agent, perfect foresight, OG model described in this section, the state vector can be seen in the system of Euler equations (3.19). What is the smallest set of variables that completely summarize all the information necessary for the three generations of all three generations living at time t to make their consumption and saving decisions? What information do they have at time t that will allow them to make their savings decisions? The state vector of this model in each period is the distribution of capital $\mathbf{\Gamma_t}$.

Definition 3.2 (**Non-steady-state functional equilibrium**)**.** A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents is defined as stationary allocation functions of the state $\bigl\{b_{s+1,t+1} = \psi_s\bigl(\mathbf{\Gamma}_t\bigr)\bigr\}_{s=1}^{S-1}$ and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1$$

ii. households optimize according to (3.19),

iii. firms optimize according to (2.20) and (2.21),

iv. markets clear according to (3.15) and (3.16).

We have already shown how to boil down the characterizing equations in Definition 3.2 to $S-1$ equations (3.19) and $S-1$ unknowns. But we have also seen that those $S-1$ equations are not identified. So how do we solve for these equilibrium functions? The solution to the non-steady-state equilibrium in Definition 3.2 is a fixed point in function space. Choose $S-1$ functions $\{\psi_s\}_{s=1}^{S-1}$ and verify that they satisfy the Euler equations for all points in the state space (all possible values of the state).

3.5 Solution method: time path iteration (TPI)

The solution method is time path iteration (TPI) as described in Section 2.5. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\mathbf{\Gamma}}$ described in Definition 3.1 in a finite number of periods $T < \infty$ regardless of the initial state $\mathbf{\Gamma}_1$.

The first step is to assume a transition path for aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$ such that T is sufficiently large to ensure that $\mathbf{\Gamma}_T = \bar{\mathbf{\Gamma}}$. The superscript i is an index for the iteration number. The transition path for aggregate capital determines the transition path for both the real wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $\mathbf{\Gamma}_1$ can be arbitrarily chosen as long as it satisfies $K_1^i = \sum_{s=2}^S b_{s,1}$ according to market clearing condition (3.16). One could also first choose the initial distribution of capital $\mathbf{\Gamma}_1$ and then choose an initial aggregate capital stock K_1^i that corresponds to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $K_T^i = \bar{K} = \sum_{s=2}^S \bar{b}_s$ by period T. But the initial guess for the aggregate capital stocks K_t^j for periods $1 < t < T$ can be any level.

Given the initial capital distribution $\mathbf{\Gamma}_1$ and the transition paths of aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$, the real wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$, and the real return to investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal savings decision for the initial age $s = S-1$ individual for the last period of his life $b_{S,2}$ using his last intertemporal Euler equation similar to (3.5).

$$u'\Big([1+r_1^i]b_{S-1,1}+w_1^i n_{S-1}-b_{S,2}\Big)=\beta\Big(1+r_2^i\Big)u'\Big([1+r_2^i]b_{S,2}+w_2^i n_S\Big) \tag{3.21}$$

Notice that everything in equation (3.21) is known except for the savings decision $b_{S,2}$. This is one equation and one unknown.

The next step is to solve for the remaining lifetime savings decisions for the next oldest individual alive in period $t=1$. This individual is age $s=S-2$ and has two remaining savings decisions $b_{S-1,2}$ and $b_{S,3}$. From (3.12), we know that the two equations that characterize these two decisions are the following.

$$u'\Big([1+r_1^i]b_{S-2,1}+w_1^i n_{S-2}-b_{S-1,2}\Big)=\beta(1+r_2^i)u'\Big([1+r_2^i]b_{S-1,2}+w_2^i n_{S-1}-b_{S,3}\Big) \tag{3.22}$$

$$u'\Big([1+r_2^i]b_{S-1,2}+w_2^i n_{S-1}-b_{S,3}\Big)=\beta(1+r_3^i)u'\Big([1+r_3^i]b_{S,3}+w_3^i n_S\Big) \tag{3.23}$$

Euler equations (3.22) and (3.23) represent two equations and two unknowns $b_{S-1,2}$ and $b_{S,3}$. Everything else is known.

We continue solving the remaining lifetime decisions of each individual alive between periods 1 and T. This includes all the individuals who were already alive in period 1 and therefore have fewer than $S-1$ savings decisions to solve for. It also includes all the individuals born between periods 1 and T for whom we have the full set of $S-1$ lifetime decisions. Once we have solved for all the individual savings decisions for individuals alive between periods 1 and T, then we have the complete distribution of savings $\{\mathbf{\Gamma}_t\}_{t=1}^T$ for each period between 1 and T. We can use this to compute a new time path of the aggregate capital stock consistent with the individual savings decisions $K_t^{i'}=\sum_{s=2}^S b_{s,t}$ for all $1\leq t\leq T$. I put a "$'$" on this aggregate capital stock because, in general, $K_t^{i'}\neq K_t^i$. That is, the initial conjectured path of the aggregate capital stock from which the savings decisions were made is not necessarily equal to the path of the aggregate capital stock consistent with those savings decisions.[2]

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock. Common

[2]A check here for whether T is large enough is if $K_T^{i'}=\bar{K}$ as well as $K_{T+1}^{i'}$ and $K_{T+2}^{i'}$. If not, then T needs to be larger.

norms to use are the L^2 and the L^∞ norms. Then the fixed point necessary for the equilibrium transition path from Definition 3.2 has been found when the distance between $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$ is arbitrarily close to zero.

$$\left\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\right\| < \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{2.34}$$

If the fixed point has not been found $\left\|\boldsymbol{K}^{i'} - \boldsymbol{K}^i\right\| > \varepsilon$, then a new transition path for the aggregate capital stock is generated as a convex combination of $\boldsymbol{K}^{i'}$ and $\boldsymbol{K}^i$.

$$\boldsymbol{K}^{i+1} = \xi\boldsymbol{K}^{i'} + (1-\xi)\boldsymbol{K}^i \quad \text{for} \quad \xi \in (0,1) \tag{2.35}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization. TPI solves for the equilibrium transition path from Definition 3.2 by finding a fixed point in the time path of the economy.

3.6 Calibration

Use the following parameterization of the model for the problems below. Assume that agents are born at age 21 and die at age 100 (80 years of life). Now your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1-0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

3.7 Exercises

Exercise 3.1. Using the calibration from Section 3.6, write a Python function named `feasible()` that has the following form,

```
b_cnstr, c_cnstr, K_cnstr = feasible(f_params, bvec_guess)
```

where the inputs are a tuple `f_params = (nvec, A, alpha, delta)`, and a guess for the steady-state savings vector `bvec_guess = np.array([scalar`$_2$`, scalar`$_3$`,...scalar`$_S$`])`. The outputs should be Boolean (`True` or `False`, `1` or `0`) vectors of lengths $S-1$, S, and 1, respectively. `K_cnstr` should be a singleton Boolean that equals `True` if $K \leq 0$ for the given `f_params` and `bvec_guess`. The object `c_cnstr` should be a length-S Boolean vector in which the sth element equals `True` if $c_s \leq 0$ given `f_params` and `bvec_guess`. And `b_cnstr` is a length-$(S-1)$ Boolean vector that denotes which element of `bvec_guess` is likely responsible for any of the consumption nonnegativity constraint violations identified in `c_cnstr`. If the first element of `c_cnstr` is `True`, then the first element of `b_cnstr` is `True`. If the second element of `c_cnstr` is `True`, then both elements of `b_cnstr` are `True`. And if the last element of `c_cnstr` is `True`, then the last element of `b_cnstr` is `True`.

a. Which, if any, of the constraints is violated if you choose an initial guess for steady-state savings of `bvec_guess = np.ones(S-1)`?

b. Which, if any, of the constraints is violated if you choose the following initial guess for steady-state savings?

```
bvec_guess = \
  np.array([-0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
            -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2])
```

c. Which, if any, of the constraints is violated if you choose the following initial guess for steady-state savings?

```
bvec_guess = \
```

```
        np.array([-0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  -0.01, 0.1, 0.2, 0.23, 0.25, 0.23, 0.2, 0.1,
                  0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1,
                  0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1,
                  0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1])
```

d. What is a principle or a rule that might help you in this problem to choose a good initial guess? That is, what properties should a feasible initial guess have? [Hint: There are upper bounds and lower bounds on all the savings levels $\bar{b}_{s+1}$ that you cannot calculate *ex ante*.]

Exercise 3.2. Use the calibration from Section 3.6 and the steady-state equilibrium Definition 3.1. Write a function named `get_SS()` that has the following form,

```
ss_output = get_SS(params, bvec_guess, SS_graphs)
```

where the inputs are a tuple of the parameters for the model `params = (beta, sigma, nvec, L, A, alpha, delta, SS_tol)`, an initial guess of the steady-state savings `bvec_guess`, and a Boolean `SS_graphs` that generates a figure of the steady-state distribution of consumption and savings if it is set to `True`.

The output object `ss_output` is a Python dictionary with the steady-state solution values for the following endogenous objects.

```
ss_output = {
  'b_ss': b_ss, 'c_ss': c_ss, 'w_ss': w_ss, 'r_ss': r_ss,
  'K_ss': K_ss, 'Y_ss': Y_ss, 'C_ss': C_ss,
  'EulErr_ss': EulErr_ss, 'RCerr_ss': RCerr_ss,
  'ss_time': ss_time}
```

Let `ss_time` be the number of seconds it takes to run your steady-state program. You can time your program by importing the time library. And let the object `EulErr_ss` be a length-$(S-1)$ vector of the Euler errors from the resulting steady-state solution given in difference form $\beta(1+\bar{r})u'(\bar{c}_{s+1}) - u'(\bar{c}_s)$. The object `RCerr_ss` is a resource constraint error which should be close to zero. It is given by $\bar{Y} - \bar{C} - \delta\bar{K}$.

a. Solve numerically for the steady-state equilibrium values of $\{\bar{c}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$, $\bar{w}$, $\bar{r}$, $\bar{K}$, $\bar{Y}$, $\bar{C}$, the $S-1$ Euler errors and the resource constraint error. List those values. Time your function. How long did it take to compute the steady-state?

b. Generate a figure that shows the steady-state distribution of consumption and savings by age $\{\bar{c}_s\}_{s=1}^S$ and $\{\bar{b}_s\}_{s=2}^S$.

c. What happens to each of these steady-state values if all households retire sooner? That is, what happens if exogenous labor supply becomes the following?

$$n_s = \begin{cases} 1.0 & \text{if} \quad s \leq \text{round}\left(\frac{S}{2}\right) \\ 0.2 & \text{if} \quad s > \text{round}\left(\frac{S}{2}\right) \end{cases}$$

Specifically, how does this change affect each steady-state value $\{\bar{c}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$, $\bar{w}$, and $\bar{r}$? What is the intuition?

Exercise 3.3. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from Definition 3.2. Use the calibration from Section 3.6 and the steady-state solution computed in Exercise 3.2. Let the initial state of the economy be given by the following distribution of savings,

$$\{b_{s,1}\}_{s=2}^S = \{x(s)\bar{b}_s\}_{s=2}^S \quad \text{where} \quad x(s) = \frac{(1.5-0.87)}{78}(s-2) + 0.87$$

where the function of age $x(s)$ is simply a linear function of age s that equals 0.87 for $s=2$ and equals 1.5 for $s=S=80$. This gives an initial distribution where there is more inequality than in the steady state. The young have less than their steady-state values and the old have more than their steady-state values. You'll have to choose a guess for T and a time path updating parameter $\xi \in (0,1)$, but I can assure you that $T < 320$. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear initial guess for the time path of the aggregate capital stock from the initial state K_1^1 to the steady state K_T^1 at time T.

a. Plot the equilibrium time paths of the aggregate capital stock $\{K_t\}_{t=1}^{T+5}$, wage $\{w_t\}_{t=1}^{T+5}$, and interest rate $\{r_t\}_{t=1}^{T+5}$.

b. Also plot the equilibrium time path for savings of every person age $s = 15$ in every period $\{b_{15,t}\}$. Are there any periods t in which $b_{15,t}$ rises above its steady-state value $\bar{b}_{15}$?

c. How many periods did it take for the economy to get within 0.00001 of the steady-state aggregate capital stock $\bar{K}$? What is the period after which the aggregate capital stock never is again farther than 0.00001 away from the steady-state?

Part III

Endogenous Labor and Heterogeneous Ability

Chapter 4

S-period-lived Agents with Endogenous Labor

In this chapter, we take the S-period-lived agent model from Chapter 3, and add an endogenous labor decision in every period for every household. That is, now the household must choose in every period how much to work $n_{s,t}$ and how much to save $b_{s+1,t+1}$.

In adding the labor decision to the household optimization problem, we must add a utility of leisure or a disutility of labor to the period utility function. In this chapter, we will introduce a new functional form for the disutility of labor following Evans and Phillips (2017). This approach fits an ellipse to the standard constant Frisch elasticity disutility of labor specification. The elliptical disutility of labor functional form provides Inada conditions at both the upper and lower bounds of labor supply, which greatly simplifies the computation.

4.1 Disutility of labor

In previous chapters, the labor decision was exogenously imposed and was inelastic to changes in underlying parameters or other variables of the model. With endogenous labor supply $n_{s,t}$, we must specify how labor enters an agent's utility function and what are the constraints. Assume that each household is endowed with a measure of time $\tilde{l}$ each period that it can choose to spend as either labor $n_{s,t} \in [0,\tilde{l}]$ or leisure $l_{s,t} \in [0,\tilde{l}]$.

$$n_{s,t} + l_{s,t} = \tilde{l} \quad \forall s,t \tag{4.1}$$

In contrast to the CRRA period utility function (2.6) from the previous section, the endogenous labor of this version of the model requires that we add either a utility of leisure term to the period utility function or a disutility of labor term. This is usually done in one of three ways. One can add a multiplicative constant elasticity of substitution term for leisure in the utility function in which households can substitute between consumption and leisure. The other two options are additively separable terms. One can model the utility of leisure with constant relative risk aversion (CRRA), similar to our period utility function from (2.6). Or one can model the disutility of labor using a constant Frisch elasticity (CFE) functional form.

Our preferred specification in this chapter will be an approximation to the constant Frisch elasticity (CFE) functional form. The following equation is the period utility function with a CRRA utility of consumption as in (2.6) and an additively separable CFE disutility of labor,

$$u\big(c_{s,t},n_{s,t}\big) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma} - \chi^n_s\frac{\big(n_{s,t}\big)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}} \tag{4.2}$$

where $\sigma\geq 1$ is the coefficient of relative risk aversion on consumption and $\theta>0$ is the Frisch elasticity of labor supply. The constant $\chi^n_s>0$ for all s is a scale parameter influencing the relative disutility of labor to the utility of consumption that can potentially vary by age s.

The first term in the period utility function (4.2) represents the utility from consumption $c_{s,t}$. Consumption has a lower bound in that it cannot be negative $c_{s,t}\geq 0$. Notice that $c_{s,t}$ is included. You can imagine someone consuming nothing in some period. However, the CRRA functional form in (4.2) puts an extra restriction on consumption in that $(c_{s,t}^{1-\sigma}-1)/(1-\sigma)$ is not defined for $c_{s,t}=0$ as well as $c_{s,t}<0$. Furthermore, the marginal utility of consumption goes to ∞ as consumption gets close to 0.

$$\lim_{c\rightarrow 0^+} c^{-\sigma} = \infty \tag{4.3}$$

This infinite marginal utility as consumption declines from above toward zero is often called an Inada condition.[1] It is a natural condition in the theory that bounds solutions away from the corners and forces interior solutions. Consumption has no natural upper bound, so the one Inada condition on consumption is sufficient to avoid a needing an occasionally binding Lagrange multiplier on the lower bound of consumption $c \geq 0$. Occasionally binding constraints are a notoriously difficult problem in optimization science as noted by Guerrieri and Iacoviello (2015), Brumm and Grill (2014), Judd et al. (2003), and Christiano and Fisher (2000).

All three utility of leisure or disutility of labor specifications mentioned in this section have at most one Inada condition at either the upper or lower bound of household labor supply. CRRA utility of leisure has an Inada condition that bounds solutions away from nonpositive leisure $l_{s,t} \leq 0$, and therefore bounds solutions away from labor supply at or above its upper bound $n_{s,t} \geq \tilde{l}$. But CRRA utility of leisure has no Inada condition on the upper bound of leisure or the lower bound of labor. The CFE disutility of labor specification in (4.2) has an Inada condition for the lower bound of labor supply, but has no Inada condition for the upper bound of labor supply. For this reason, one must take care in computing solutions to make answers respect both the upper and lower bounds of labor supply.

Evans and Phillips (2017) propose a useful approximation to both the CRRA utility of leisure and the CFE disutility of labor specifications, which has some nice properties as an independent specification rather than just an approximation. Evans and Phillips propose using the upper-right quadrant of an ellipse as an approximation to the CFE functional form for the disutility of labor.[2] This elliptical disutility of labor provides Inada conditions at both the upper and lower bounds of labor supply.

The functional form for a general ellipse in x and y space is the following, where the centroid of the ellipse is at coordinates $(x_0, y_0) = (h, k)$, the horizontal radius is $a > 0$, the vertical radius is $b > 0$, and the curvature is controlled by $\mu > 1$.

$$\left(\frac{x-h}{a}\right)^{\upsilon} + \left(\frac{y-k}{b}\right)^{\upsilon} = 1, \quad a, b > 0 \quad \text{and} \quad \upsilon > 1 \tag{4.4}$$

[1]See Inada (1963).

[2]Evans and Phillips (2017) provide approximations for both CFE disutility of labor as well as CRRA utility of leisure.

Figure 4.1: Ellipse with $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$

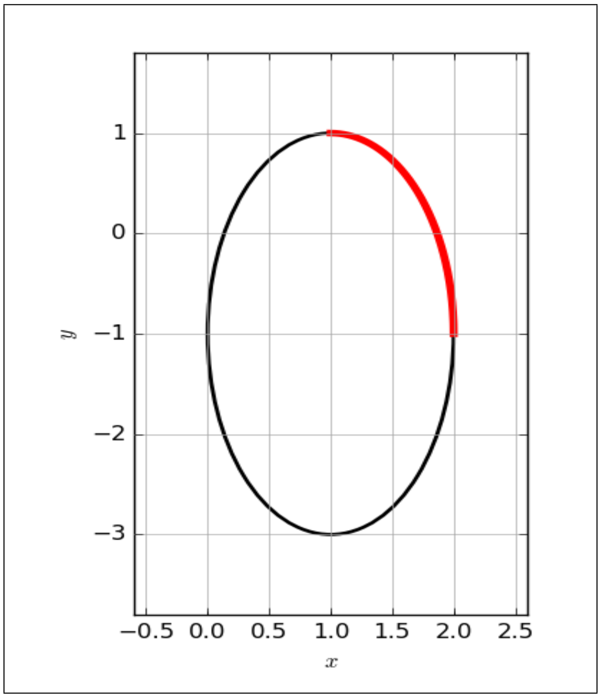


Figure 4.1 shows an ellipse with the parameterization $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$. The upper-right quadrant of the ellipse is highlighted because we focus on this portion of the function.

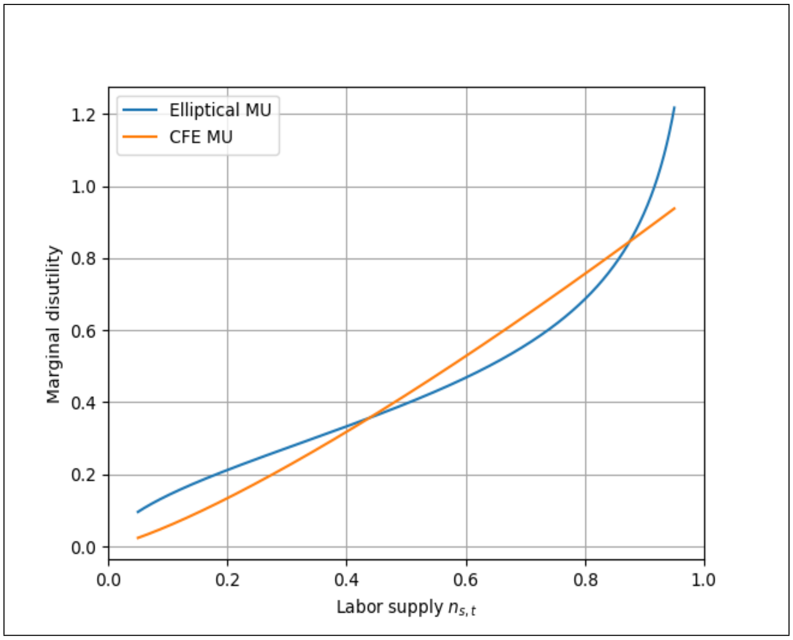
We want to rename the variables in (4.4) such that x is labor supply $n_{s,t}$ or $1 - l_{s,t}$ and y is the utility of labor $-g\big(n_{s,t}\big)$.[3] We want the x-coordinate of the centroid to be at zero $h = 0$ and the horizontal radius to be the labor endowment $a = \tilde{l}$ so that the ellipse is defined for $n_{s,t} \in [0, \tilde{l}]$ and has marginal benefit of zero at $n_{s,t} = 0$ and marginal benefit of $-\infty$ at $n_{s,t} = \tilde{l}$. We can normalize the centroid in the y dimension or $g(n)$ dimension to zero $k = 0$ because it will drop out of any marginal utility calculation. The vertical radius b and the curvature υ are free parameters that we can use to match another functional form. Using this specification and solving for $g(n)$, we get the following functional form for the elliptical

[3]The upper-right-quadrant of the ellipse represents the "utility" of labor because it is decreasing in n. With this interpretation, the functional forms we are matching are the negative of the CFE disutility of labor function $-g(n) = -\frac{(n)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$ to the upper-right-quadrant of the ellipse formula $b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}$. Another interpretation of this is to match the lower-right-quadrant of the ellipse (disutility of labor) to the CFE "disutility" of labor function.

disutility of labor representing the upper-right quadrant of the ellipse in Figure 4.1.

$$g\big(n_{s,t}\big) = -b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \quad \forall s,t \tag{4.5}$$

Figure 4.2: Comparison of CFE utility of leisure $\theta = 0.9$ to fitted elliptical utility



Peterman (2016) shows that in a macro-model—such as this overlapping generations model—that has only an intensive margin of labor supply and no extensive margin and represents a broad composition of individuals supplying labor, a Frisch elasticity of around 0.9 is probably appropriate.[4] In Exercise 4.1, you will estimate the parameters b and υ from (4.5) to match a CFE disutility of labor function (4.2) with a Frisch elasticity of $\theta = 0.9$. Figure 4.2 shows the CFE utility of leisure function plotted against an elliptical utility of leisure (disutility of labor) function that was estimated to closely approximate the CFE function.

[4]Peterman (2016) tests the implied macro elasticity when the assumed micro elasticities are small on the intensive margin but only macro aggregates—which include both extensive and intensive margin agents—are observed.

4.2 Households

A unit measure of identical individuals are born each period and live for S periods. The endogenous labor decision does not change the budget constraint from Section 2.

$$\begin{split} c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{split} \tag{2.1}$$

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^S \beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \tag{4.6}$$

$$\text{s.t.} \quad c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \tag{2.1}$$

$$\text{where} \quad u\bigl(c_{s,t},n_{s,t}\bigr) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{4.7}$$

where $u(c_{s,t},n_{s,t})$ is the period utility function with elliptical disutility of labor (4.7), and χ^n_s is a scale parameter that can potentially vary by age s influencing the relative disutility of labor to the utility of consumption. The household's lifetime problem (4.6) can be reduced to choosing S labor supplies $\{n_{s,t+s-1}\}_{s=1}^S$ and $S-1$ savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ by substituting the budget constraints (2.1) in for $c_{s,t}$ in each period utility function (4.7) of the lifetime utility function.

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (4.8), the following $S-1$ dynamic savings Euler equations (4.9), and a budget constraint that binds in all S periods (2.1),

$$\begin{split} w_t u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = -u_2\left(c_{s,t+s-1},n_{s,t+s-1}\right) \quad \text{for} \quad s\in\{1,2,...S\} \\ \Rightarrow \quad w_t\left(c_{s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \end{split} \tag{4.8}$$

$$u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = \beta(1+r_{t+1})u_1\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\}$$
$$\Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta(1+r_{t+1})\left(c_{s+1,t+1}\right)^{-\sigma} \tag{4.9}$$

$$c_{s,t}+b_{s+1,t+1} = (1+r_t)b_{s,t}+w_t n_{s,t} \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t},b_{S+1,t+S}=0 \tag{2.1}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$. As was demonstrated in detail in Section 3.1, the dynamic Euler equations (4.9) do not include marginal utilities of all future periods because of the principle of optimality and the envelope condition.

Note that these $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u,r_u\}_{u=t}^{t+S-1}$. As in section 3.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (3.14) and (2.17).

$$\mathbf{\Gamma}_t \equiv \left\{b_{s,t}\right\}_{s=2}^{S} \quad \forall t \tag{3.14}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{2.17}$$

4.3 Firms

Firms are characterized exactly as in Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (2.20) and its aggregate labor decision L_t governed by first order condition (2.21).

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

The per-period depreciation rate of capital is $\delta\in[0,1]$, the capital share of income is $\alpha\in(0,1)$, and total factor productivity is $A>0$.

4.4 Market Clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{S} n_{s,t} \quad \forall t \tag{4.10}$$

$$K_t = \sum_{i=2}^{S} b_{s,t} \quad \forall t \tag{3.16}$$

$$\begin{split} &Y_t = C_t + I_t \quad \forall t \\ &\quad\text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{split} \tag{2.24}$$

The goods market clearing equation (2.24) is redundant by Walras' Law.

The market clearing conditions for this version of the model are nearly equivalent to the three conditions described in Section 3.3. The exception is the labor market clearing condition (4.10) in which individual labor supply levels $n_{s,t}$ can vary endogenously with age s and time t.

4.5 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points.

i. Households optimize according to equations (4.8) and (4.9).

ii. Firms optimize according to (2.20) and (2.21).

iii. Markets clear according to (4.10) and (3.16).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (4.10) and (3.16) into the firm's optimal conditions (2.20) and (2.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t\big(\mathbf{\Gamma}_t\big): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t}}{\sum_{s=1}^S n_{s,t}}\right)^{\alpha} \quad \forall t \tag{4.11}$$

$$r_t\big(\mathbf{\Gamma}_t\big): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^S n_{s,t}}{\sum_{s=2}^S b_{s,t}}\right)^{1-\alpha} - \delta \quad \forall t \tag{4.12}$$

It is worth noting here that the equilibrium wage (4.11) and interest rate (4.12) are written as functions of the period-t distribution of savings (wealth) $\mathbf{\Gamma}_t$ from (3.14) and are not functions of the period-t distribution of labor supply, which labor distribution shows up in (4.11) and (4.12). This is because, similar to next period savings $b_{s+1,t+1}$, current period labor supply $n_{s,t}$ must be chosen in period t and is therefore a function of the current state $\mathbf{\Gamma}_t$ distribution of savings.

Now (4.11), (4.12), and the budget constraint (2.1) can be substituted into household Euler equations (4.8) and (4.9) to get the following $(2S-1)$-equation system. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{split} w_t\big(\mathbf{\Gamma}_t\big)\Big(w_t\big(\mathbf{\Gamma}_t\big)n_{s,t} + \Big[1 + r_t\big(\mathbf{\Gamma}_t\big)\Big]b_{s,t} - b_{s+1,t+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\ \text{for} \quad s \in \{1,2,...S\} \quad \text{and} \quad \forall t \end{split} \tag{4.13}$$

$$\begin{split} &\Big(w_t\big(\mathbf{\Gamma}_t\big)n_{s,t} + \Big[1 + r_t\big(\mathbf{\Gamma}_t\big)\Big]b_{s,t} - b_{s+1,t+1}\Big)^{-\sigma} = \\ &\quad \beta\Big[1 + r_{t+1}\big(\mathbf{\Gamma}_{t+1}\big)\Big]\Big(w_{t+1}\big(\mathbf{\Gamma}_{t+1}\big)n_{s+1,t+1} + \Big[1 + r_{t+1}\big(\mathbf{\Gamma}_{t+1}\big)\Big]b_{s+1,t+1} - b_{s+2,t+2}\Big)^{-\sigma} \\ &\quad \text{for} \quad s \in \{1,2,...S-1\} \quad \text{and} \quad \forall t \end{split} \tag{4.14}$$

The system of S nonlinear static equations (4.13) and $S-1$ nonlinear dynamic equations (4.14) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{s,t+s-1}\}_{s=1}^S$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 4.1 (**Steady-state equilibrium**). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, labor supply $\{\bar{n}_s\}_{s=1}^S$, and savings $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (4.8) and (4.9),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (4.10) and (3.16).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s(\mathbf{\Gamma}_t)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s(\mathbf{\Gamma}_t)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 4.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\big\{n_{s,t} = \phi_s\big(\mathbf{\Gamma}_t\big)\big\}_{s=1}^S$, $\big\{b_{s+1,t+1} = \psi_s\big(\mathbf{\Gamma}_t\big)\big\}_{s=1}^{S-1}$ and and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (4.8) and (4.9),

iii. firms optimize according to (2.20) and (2.21),

iv. markets clear according to (4.10) and (3.16).

4.6 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 4.1 and the transition path equilibrium from Definition 4.2.

4.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 4.1. The parameters needed for the steady-state solution of this model are $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta\right\}$, where S is the number of periods in an individual's life, $\left\{\beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S\right\}$ are household utility function parameters, and $\{A, \alpha, \delta\}$ are firm production function parameters. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$n_{s,t} = \bar{n}_s, \quad b_{s,t} = \bar{b}_s \quad \forall s, t \tag{4.15}$$

From the market clearing conditions (4.10) and (3.16) and the firms' first order equations (2.20) and (2.21), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K_t = \bar{K} \quad L_t = \bar{L} \quad \forall t \tag{4.16}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations (4.8) and (4.9) over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}\Bigl([1+\bar{r}]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Bigr)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\} \tag{4.17}$$

$$\Bigl([1+\bar{r}]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Bigr)^{-\sigma} = \beta(1+\bar{r})\Bigl([1+\bar{r}]\bar{b}_{s+1} + \bar{w}\bar{n}_{s+1} - \bar{b}_{s+2}\Bigr)^{-\sigma} \quad \text{for} \quad s=\{1,2,...S-1\} \tag{4.18}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distribution of labor supply and savings as shown in (4.11) and (4.12).

There are several approaches to solving this system of equations. We take an approach here that will continue to work for models with additional complexity.[5] This approach is to use a multivariate root finder that chooses the $2S-1$ steady-state variables $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ simultaneously to solve the zeros of the $2S-1$ Euler equations (4.17) and (4.18). The tradeoffs between labor supply and savings in each period can create difficulties for root finding algorithms and thus this method can be sensitive to starting values and model parameterizations.

The outer loop of the solution method is to iterate on guesses for the steady-state interest rate until the firm's capital demand equation is satisfied. One could also choose the wage $\bar{w}$ or the steady-state capital-labor ratio $\bar{K}/\bar{L}$ as the outer-loop choice variable. We prefer the interest rate $\bar{r}$ as the outer-loop choice variable because it is usually easier to make an initial guess that is closer to the true value given that it usually lies between 0 and 1. The steady-state solution method algorithm is detailed below.

[5]In particular, we are careful to choose a solution method here that will work for models with endogenous bequests for which the savings at the end of life are endogenous.

i. Make a guess for the steady-state interest rate $\bar{r}^i$.

(a) A guess for the steady-state interest rate $\bar{r}^i$ will imply a value for the steady-state wage $\bar{w}$ from (4.19), which is derived from solving equation (2.20) for the capital labor ratio K/L and substituting it into (2.21).

$$w_t = (1-\alpha)A\left(\frac{\alpha A}{r_t+\delta}\right)^{\frac{\alpha}{1-\alpha}} \quad \forall t \tag{4.19}$$

ii. Given steady-state prices $\bar{r}^i$ and $\bar{w}$, solve for the steady-state household's lifetime decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$.

(a) Given $\bar{r}^i$ and $\bar{w}$, use a root finder to solve for $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ from the $2S-1$ steady-state Euler equations (4.17) and (4.18).

(b) This solution can be sensitive the initial guess for $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ passed to the root finder.

iii. Given solution for optimal household decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ based on the guess for the interest rate $\bar{r}^i$ and the implied wage $\bar{w}$, solve for the aggregate capital $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household solutions and market clearing conditions.

$$\bar{K} = \sum_{s=2}^S \bar{b}_s \tag{4.20}$$

$$\bar{L} = \sum_{s=1}^S \bar{n}_s \tag{4.21}$$

iv. Compute a new value for the interest rate $\bar{r}^{i'}$ using the aggregate capital stock $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household optimization from equations (4.20) and (4.21).

$$\bar{r}^{i'} = \alpha A\left(\frac{\bar{L}}{\bar{K}}\right)^{1-\alpha} - \delta \tag{4.22}$$

v. Update the guess for the steady-state interest rate $\bar{r}^{i+1}$ until the interest rate implied by household optimization $\bar{r}^{i'}$ equals the initial guess for the interest rate $\bar{r}^i$.

(a) The bisection method characterizes the updated guess for the interest rate $\bar{r}^{i+1}$ as a convex combination of the initial guess $\bar{r}^i$ and the value implied by household and firm optimization $\bar{r}^{i'}$, where the weight put on the new value $\bar{r}^{i'}$ is given by $\xi \in (0,1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$\bar{r}^{i+1} = \xi\bar{r}^{i'} + (1-\xi)\bar{r}^i \quad \text{for} \quad \xi \in (0,1] \tag{4.23}$$

(b) Let $\|\cdot\|$ be a norm on the space of feasible interest rate values r. We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (4.24). If the distance is less than some tolerance `SS_toler` > 0, then the problem has converged. Otherwise, update the value of the interest rate according to (4.23) and repeat steps (ii) through (v).

$$\texttt{SS\_dist} \equiv \left\|\bar{r}^{i'} - \bar{r}^i\right\| \tag{4.24}$$

Figure 4.3: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$

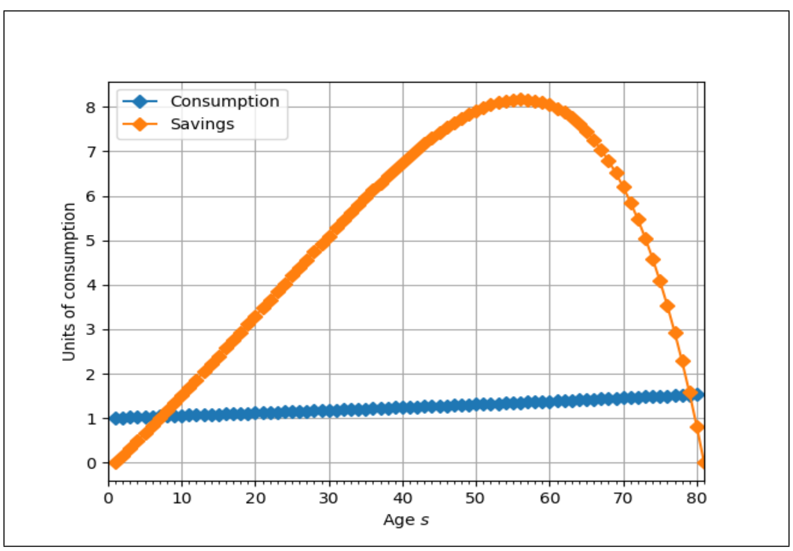


Figure 4.3 shows the steady-state distribution of individual consumption and savings in an 80-period-lived agent model with parameter values listed above the line in Table 4.3 in Section 4.7. Figure 4.4 shows the steady-state distribution of individual labor supply by age.

Figure 4.4: Steady-state distribution of labor supply $\bar{n}_s$

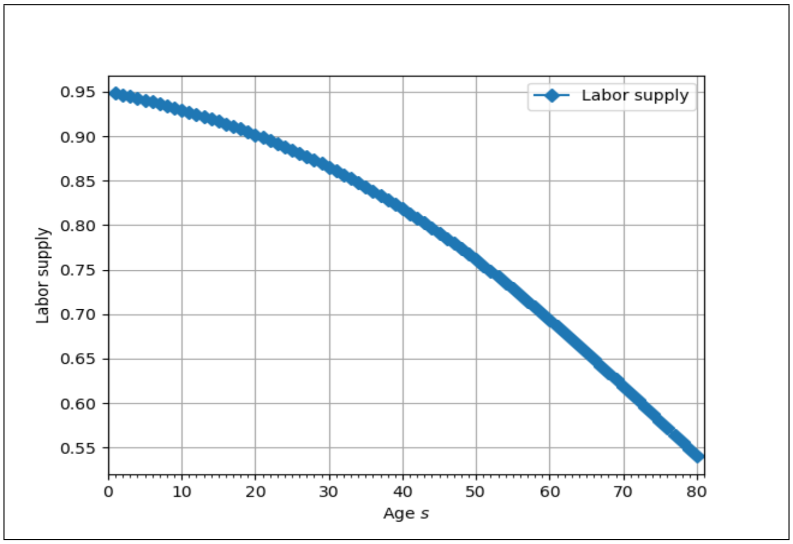


Table 4.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.055	Max. absolute savings Euler error	4.44e-16
$\bar{w}$	1.240	Max. absolute labor supply Euler error	4.44e-16
$\bar{K}$	399.875	Resource constraint error	9.13e-13
$\bar{L}$	63.186		
$\bar{Y}$	120.525		
$\bar{C}$	100.531	Serial computation time	10.15 sec.

The left side of Table 4.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $2S-1$ Euler errors from the labor supply and savings decisions, the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 4.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 4.5e-16 and the resource constraint error is less than 9.2e-13, we can be confident that we have successfully solved for the steady-state equilibrium.

4.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor is similar to the method described in Section 3.5 as well as to the steady-state solution method described in Section 4.6.1. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 4.1 in a finite number of periods $T < \infty$ regardless of the initial state $\boldsymbol{\Gamma}_1$.

To solve for the transition path (non-steady-state) equilibrium from Definition 4.2, we must know the parameters from the steady-state problem $\Big\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta\Big\}$, the steady-state interest rate $\bar{r}$, initial distribution of savings $\boldsymbol{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 4.3 and 4.1 show a particular calibration of the model and corresponding steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is similar to the steady-state algorithm described in Section 4.6.1.

i. Choose a period $T1$ in which the initial guess for the time path of interest rates $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_{T1}\}$ will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the entire economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Guess the initial time path for the interest rate $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_{T1}\}$.

   (a) The guess for the time path for interest rates $\boldsymbol{r}^i = \{r^i_1, r^i_2, ...r^i_{T1}\}$ implies a time path for wages $\boldsymbol{w} = \{w_1, w_2, ...w_{T1}\}$ using equation (4.19).

   (b) These time paths will have to be extended with their respective steady-state values so that they are each $T2 + S - 1$ elements long. This is the time-path length that enables one to solve the lifetime decisions of every individual alive from period $t = 1$ to $t = T2$.

iii. Given time paths $\boldsymbol{r}^i$ and $\boldsymbol{w}$, solve for the lifetime labor supply $n_{s,t}$, and savings $b_{s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.

   (a) The initial old $s = S$ cohort in period $t = 1$ only have one decision to make. They must choose how much to work in the last period of their life $n_{S,1}$. This decision is characterized by one unknown and one equation. The equation is the period-1 version of the labor supply Euler equation (4.8).

$$w_1\Bigl([1+r_1^i]b_{S,1}+w_1 n_{S,1}\Bigr)^{-\sigma} = \chi^n_S\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{S,1}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{S,1}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \tag{4.25}$$

(b) Each cohort who is age $1 < s < S$ has an incomplete lifetime of $p = S - s + 1$ periods remaining. Each of these individuals has $2p-1$ variables to solve for. Each has to choose p labor supply decisions $n_{s,t}$ and $p-1$ savings decisions $b_{s+1,t+1}$ using the appropriate system of $2p-1$ Euler equations (4.8) and (4.9), respectively, with the appropriate interest rates and wages from time paths $\boldsymbol{r}^i$ and $\boldsymbol{w}$.

(c) Each cohort born in periods 1 through $T2$ has a complete lifetime and has $2S-1$ variables to solve for. Each has to choose S labor supply decisions $n_{s,t}$ and $S-1$ savings decisions $b_{s+1,t+1}$ using the appropriate system of $2S-1$ Euler equations (4.8) and (4.9), respectively, with the appropriate interest rates and wages from time paths $\boldsymbol{r}^i$ and $\boldsymbol{w}$.

(d) One trick to doing this successfully is to use the previous cohort's solutions as the initial guess for the current cohort's root finder.

iv. Use the time paths of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households' optimal decisions given $\boldsymbol{r}^i$ and $\boldsymbol{w}$ to compute time paths for aggregate capital and aggregate labor $\boldsymbol{K} = \{K_1, K_2, ...K_{T2}\}$ and $\boldsymbol{L} = \{L_1, L_2, ...L_{T2}\}$ implied by capital and labor market clearing conditions (4.10) and (3.16).

v. Compute a new time path for interest rates $\boldsymbol{r}^{i'}$ using the time paths of the aggregate capital stock $\bar{K}$ and aggregate labor $\bar{L}$ implied by the household and firm optimization from part (iv) using equation (4.22).

vi. Compare the distance between the new time path new of interest rates implied by households and firms optimization $\boldsymbol{r}^{i'}$ versus the initial guess for the time path of the interest rate $\boldsymbol{r}^i$.

$$\texttt{TPI\_dist} = \left\|\boldsymbol{r}^{i'} - \boldsymbol{r}^i\right\| \geq 0 \tag{4.26}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the interest rate $\boldsymbol{r}^i$. Common norms to use are the L^2 and the L^∞ norms.

(a) Let the tolerance level `TPI_toler` > 0 be some strictly positive number close to zero. If the distance is less than or equal to some tolerance level `TPI_dist` $\leq$ `TPI_toler`, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level `TPI_dist` $>$ `TPI_toler`, then update the guess for a new interest rate time path to be a convex combination current initial time path and the implied time path.

$$\boldsymbol{r}^{i+1} = \xi \boldsymbol{r}^{i'} + (1-\xi)\boldsymbol{r}^i \quad \text{for} \quad \xi \in (0,1] \tag{4.27}$$

Table 4.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	4.31e-14
Maximum absolute savings Euler error	1.33e-14
Maximum absolute resource constraint error	3.98e-13
Serial computation time	25 min. 1.1 sec.

The six panels of Figure 4.5 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The three panels of Figure 4.6 show the transition paths of the distributions of consumption $c_{s,t}$, labor supply $n_{s,t}$ and savings $b_{s,t}$. Table 4.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than 4.0e-13 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

Figure 4.5: Equilibrium transition paths of prices and aggregate variables

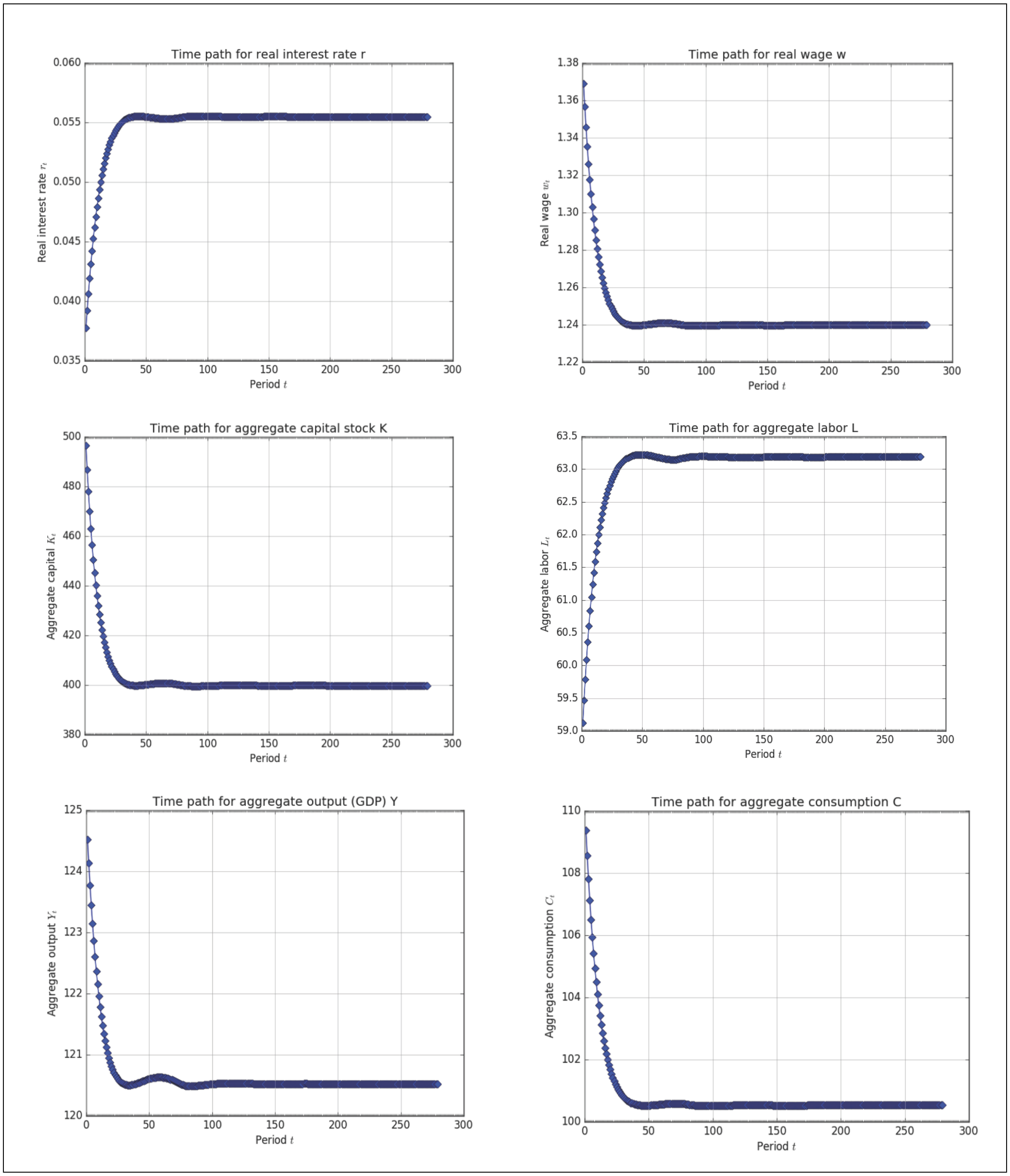
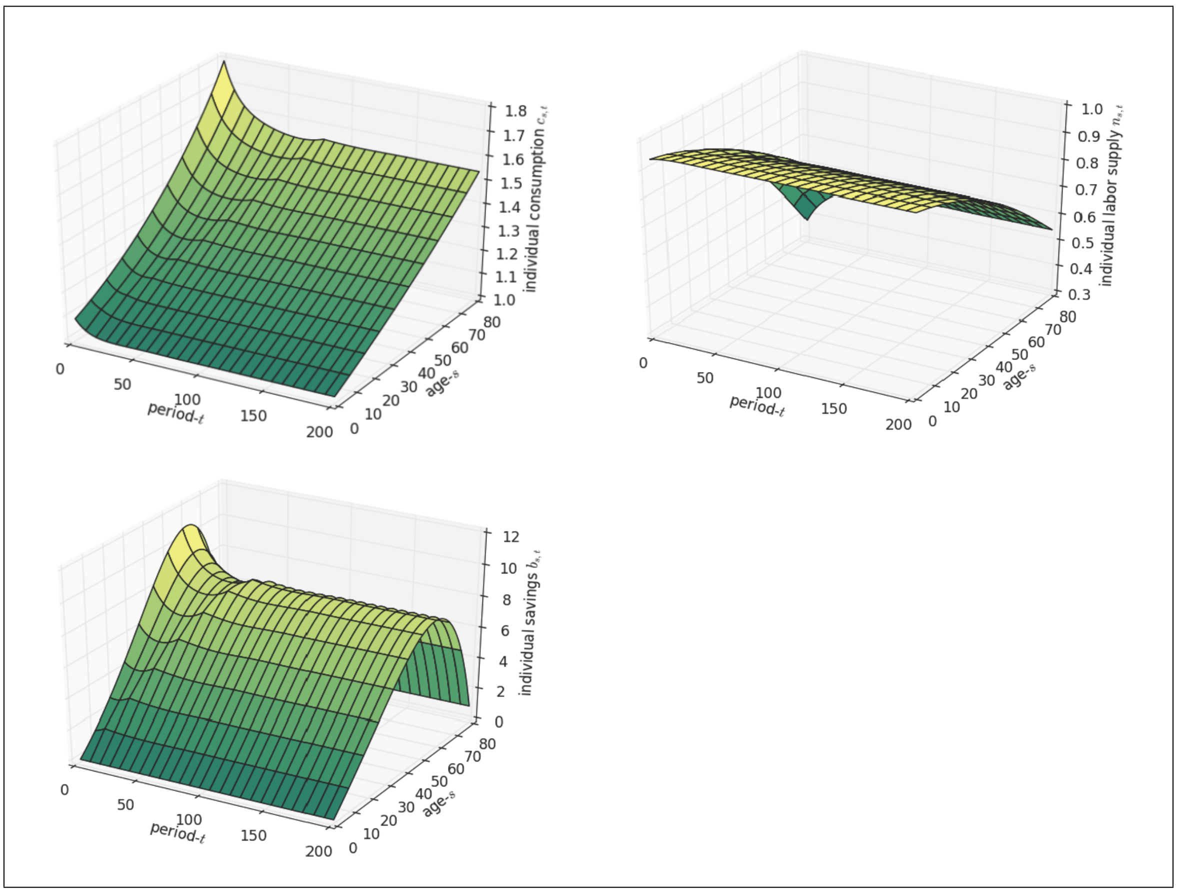


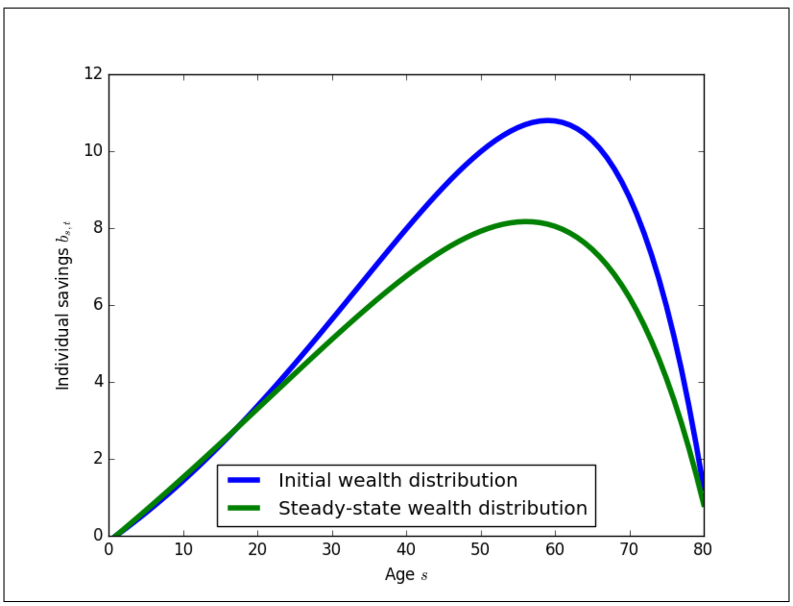
Figure 4.6: Equilibrium transition paths of distributions of consumption, labor supply, and savings



4.7 Calibration

Many of the parameters of the model can be calibrated by simply taking values from other studies or by setting them to intuitive values. Assume that agents are born at age 21 and die at age 100 (80 years of life). Your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1 - 0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$. Assume that each individual's time endowment in each period is $\tilde{l} = 1$.

Figure 4.7: Initial vs. steady-state distributions of wealth (savings) $b_{s,t}$



4.7.1 Calibrating χ^n_s and elliptical utility parameters

We might want to be more careful about our calibration of the parameters $\{\chi^n_s\}_{s=1}^S$ that can vary by age s and scale the disutility of labor on the right-hand-side of the household's Euler equation for labor supply (4.8). One approach is to choose values of $\{\chi^n_s\}_{s=1}^S$ that make the steady-state values of labor supply as a percent of total time endowment as close as possible

Table 4.3: Calibrated parameter values for simple endogenous labor model

Parameter	Description	Value
S	Number of periods in individual life	80
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
$\tilde{l}$	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501[a]
υ	Elliptical disutility of labor shape parameter	1.554[a]
$\{\chi^n_s\}_{s=1}^S$	Disutility of labor relative scale factor by age	(See Sec. 4.7.1)
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
$\boldsymbol{\Gamma}_1$	Initial distribution of savings (wealth)	(see Fig. 4.7)
$T1$	Time period in which initial path guess hits steady state	160
$T2$	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.3

[a] The calibration of b and υ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.9. See Evans and Phillips (2017) and Peterman (2016).

to their data analogues. This is a generalized method of moments (GMM) calibration in that we are choosing $\{\chi^n_s\}_{s=1}^S$ to match model moments to data moments.

Let the vector $\boldsymbol{\theta}$ represent all the parameters of the model, including $\{\chi^n_s\}_{s=1}^S$. Let $\tilde{\boldsymbol{x}}$ represent the endogenous variables of the model, and let $\boldsymbol{x}$ represent the data variables that are the real world data analogues of the model variables. The model moments $\boldsymbol{m}\left(\tilde{\boldsymbol{x}}|\boldsymbol{\theta}\right)$ are functions of model data $\tilde{\boldsymbol{x}}$ given parameters $\boldsymbol{\theta}$. Define the model moments as the steady-state labor supply by age as a percent of total time endowment.

$$\text{model moments:}\quad \boldsymbol{m}\left(\tilde{\boldsymbol{x}}|\boldsymbol{\theta}\right) = \left\{\frac{\bar{n}_s}{\tilde{l}}\right\}_{s=1}^S \tag{4.28}$$

The data moments ($\boldsymbol{x}$) that correspond to the model moments are the average hours of work as a percent of the total time endowment.

$$\text{data moments:}\quad \boldsymbol{m}\left(\boldsymbol{x}\right) = \left\{\frac{\text{avg. hours}_s}{\text{total hours available}}\right\}_{s=1}^S \tag{4.29}$$

To calculate average hours by age from the data for our data moments (4.29) we use the U.S. Current Population Survey (CPS) Basic Monthly Data.[6] We use three variables from the survey to calculate average weekly hours by age.[7] We then use the maximum hours worked in the data plus a small amount as our total hours available—the analogue of $\tilde{l}$.

We can now specify the GMM estimation we use to calibrate $\{\chi^n_s\}_{s=1}^S$ to minimize the distance between the model moments and the data moments,

$$\begin{aligned} &\min_{\{\chi^n_s\}_{s=1}^S} \boldsymbol{e}\big(\tilde{\boldsymbol{x}},\boldsymbol{x}|\boldsymbol{\theta}\big)^{\mathrm{T}}\,\boldsymbol{W}\,\boldsymbol{e}\big(\tilde{\boldsymbol{x}},\boldsymbol{x}|\boldsymbol{\theta}\big) \\ &\quad\text{where}\quad \boldsymbol{e}\big(\tilde{\boldsymbol{x}},\boldsymbol{x}|\boldsymbol{\theta}\big) \equiv \left(\frac{\boldsymbol{m}\left(\tilde{\boldsymbol{x}}|\boldsymbol{\theta}\right)-\boldsymbol{m}\left(\boldsymbol{x}\right)}{\boldsymbol{m}\left(\boldsymbol{x}\right)}\right) \end{aligned} \tag{4.30}$$

where T is the transpose operator and $\boldsymbol{W}$ is a weighting matrix.[8] This approach to estimating $\{\chi^n_s\}_{s=1}^S$ is exactly identified in that you are choosing S moments to estimate S parameters. This estimation is also nice inasmuch as each parameter value χ^n_s is most closely associated with one of the moments although each moment has some dependence on all the other parameters.

One problem with this approach to calibrating the parameters $\{\chi^n_s\}_{s=1}^S$ is that we are matching steady-state moments from the model with recent-period moments from the data. It is often unlikely that the recent periods of the economy are close to a steady-state. In other words, it is likely that the economy is usually in transition due to shocks it has experienced. For this reason, one might want to calibrate $\{\chi^n_s\}_{s=1}^S$ to some initial period model moments. Two problems that arise with this different approach are that initial-period endogenous variables require solving for the entire non-steady-state equilibrium and many initial period moments require scaling in order to match real-world data with model data. Given these two difficulties, we think the steady-state approach described above is acceptable.

You will solve for the elliptical disutility of labor supply parameters b and υ in Exercise 4.1 below. We do this by estimating b and υ such that the marginal disutilities of labor supply

[6] The CPS Basic Monthly Data are available through the National Bureau of Economic Research data portal at http://nber.org/data/cps_basic.html.

[7] See `README.md` at https://github.com/OpenSourceMacro/CPS_hrs_age for a description of how we calculated average weekly hours from the CPS data.

[8] See this GMM Jupyter notebook and Davidson and MacKinnon (2004, chap. 9) for a discussion of estimators of the optimal weighting matrix $\hat{\boldsymbol{W}}$. But using the identity matrix is an unbiased yet inefficient estimator of the optimal weighting matrix.

along the support of $n_{s,t}$ match the disutilities of labor supply implied by a constant Frisch elasticity (CFE) functional form for the disutility of labor supply with a Frisch elasticity of θ.

4.8 Exercises

Exercise 4.1. Assume that an individual's time endowment each period is one $\tilde{l} = 1$. Let the period utility of a household be an additively separable function of consumption and labor,

$$U(c,n) = u(c) - g(n)$$

where the disutility of labor function $g(n)$ is the constant Frisch elasticity (CFE) disutility of labor functional form.

$$g_{cfe}(n) = \frac{(n)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$$

Assume that an approximation to this disutility of labor function is the following elliptical disutility of labor function.

$$g_{elp}(n) = -b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \quad \text{for} \quad \tilde{l}, b > 0 \quad \text{and} \quad \upsilon \geq 1$$

a. The marginal disutility of labor $\frac{\partial g(n)}{\partial n}$ governs the household decision of how much to work. Give the expression for the marginal disutility of labor for the CFE specification $g_{cfe}(n)$ and for the elliptical specification $g_{elp}(n)$.

b. Write a function `fit_ellip()` that takes as inputs the Frisch elasticity of labor supply θ and the time endowment per period $\tilde{l}$ and returns the estimated values of the elliptical utility parameters b and υ.

```
b_ellip, upsilon = fit_ellip(elast_Frisch, l_tilde)
```

Assume that the Frisch elasticity of labor supply in υ_{cfe} is $\theta = 0.9$. Using 1,000 evenly spaced points from the support of leisure between 0.05 and 0.95, estimate the elliptical disutility of labor parameters b and υ that minimize the sum of squared deviations between the two marginal utility of leisure functions $g'_{cfe}(n)$ and $g'_{elp}(n)$ from part (i). Plot the two marginal disutility of labor functions.

Exercise 4.2. Optimizers (root finders and minimizers) can sometimes choose values that are outside the feasible set of solutions even if Inada conditions are present to theoretically constrain the values of a given iteration to the feasible set. This is often due to a step-size issue in the optimizer. One solution is to smoothly "stitch" to the original constrained function $f(x)$ a new function $h(x)$ that is defined over the entire direction of the real line. This stitched function $h(x)$ has the same value and slope at the stitching point x_0, and it should preserve the monotonicity of the function as x moves in that direction.

a. The marginal utility of consumption for the period utility function in this model is $c^{-\sigma}$, which is only defined for $c \geq 0$ and has an Inada condition at $c = 0$ because $\lim_{c\rightarrow 0} c^{-\sigma} = \infty$. Write a function `MU_c_stitch()` that takes as arguments a scalar or vector of values for c and a value for $\sigma \geq 1$ and returns a scalar or vector of marginal utilities $c^{-\sigma}$ associated with those values.

```
MU_c_vec = MU_c_stitch(cvec, sigma)
```

Create the function such that values of $c < 0.0001$ are a linear function that has the same slope and value as the marginal utility at $c = 0.0001$.

$$u'(c) = \begin{cases} c^{-\sigma} & \text{for} \quad c \geq 0.0001 \\ m_1 c + m_2 & \text{for} \quad c < 0.0001 \end{cases}$$
$$\text{s.t.} \quad m_1(0.0001) + m_2 = (0.0001)^{-\sigma} \quad \text{and} \quad m_1 = -\sigma(0.0001)^{-\sigma-1}$$

Report your marginal utility values $u'(c)$ for `cvec=np.array([-0.01,-0.004, 0.5, 2.6])` given $\sigma = 2.2$.

b. This model uses an elliptical disutility of labor function which has the following marginal disutility of labor function.

$$\texttt{MDU\_n} = g'(n) = \left(\frac{b}{\tilde{l}}\right)\left(\frac{n}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad b,\tilde{l} > 0 \quad \text{and} \quad \upsilon \geq 1$$

This function has Inada conditions at $n=0$ and $n=\tilde{l}$ because $\lim_{n\rightarrow 0} g'(n) = 0$ and $\lim_{n\rightarrow \tilde{l}} g'(n) = \infty$. Write a function `MU_n_stitch()` that takes as arguments a scalar or vector of values for n and values for $\tilde{l}, b > 0$ and $\upsilon \geq 1$ and returns a scalar or vector of marginal disutilities of labor $g'(n)$ associated with those values.

```
MDU_n_vec = MDU_n_stitch(nvec, ltilde, b_ellip, upsilon)
```

Create the function such that values of $n < 0.000001$ are a linear function that has the same slope and value as the marginal disutility at $n = 0.000001$ and such that values of $n > \tilde{l} - 0.000001$ are a linear function that has the same slope and value as the marginal disutility at $n = \tilde{l} - 0.000001$.

$$g'(n) = \begin{cases} m_1 n + m_2 \quad \text{for} \quad n < 0.000001 \\ \left(\frac{b}{\tilde{l}}\right)\left(\frac{n}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad 0.000001 \leq n \leq \tilde{l} - 0.000001 \\ q_1 n + q_2 \quad \text{for} \quad n > \tilde{l} - 0.000001 \end{cases}$$

$$\text{s.t.} \quad m_1(0.000001) + m_2 = g'(0.000001) \quad \text{and} \quad m_1 = g''(0.000001)$$
$$\text{and} \quad q_1(\tilde{l} - 0.000001) + q_2 = g'(\tilde{l} - 0.000001) \quad \text{and} \quad q_1 = g''(\tilde{l} - 0.000001)$$

Report your resulting marginal disutility values $g'(n)$ for `nvec=np.array([-0.013, -0.002, 0.42, 1.007, 1.011])` given $\tilde{l} = 1.0$, $b_{ellip} = 0.5$, and $\upsilon = 1.5$.

Exercise 4.3. Using the calibration from Section 4.7 and the steady-state equilibrium Definition 4.1, solve for the steady-state equilibrium values of $\{\bar{c}_s, \bar{n}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$, $\{\bar{r}, \bar{w}, \bar{K}, \bar{L}, \bar{Y}, \bar{C}\}$. Assume that $S = 10$, elliptical utility parameters are $b = 0.5$ and $\upsilon = 1.5$, and $\chi^n_s = 1.0$ for all s.

a. Plot the steady-state distributions of consumption, labor supply, and savings $\{\bar{c}_s, \bar{n}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$.

b. Show in a table the steady-state values for prices and aggregate variables $\{\bar{r}, \bar{w}, \bar{K}, \bar{L}, \bar{Y}, \bar{C}\}$. Display in the same table the maximum absolute errors in the savings Euler errors, labor supply Euler errors, final period savings $\bar{b}_{S+1}$, and resource constraint errors.

Exercise 4.4. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from $\mathbf{\Gamma}_1 = 1.08(\bar{\mathbf{\Gamma}})$ from the steady-state solution and calibration in Exercise 4.3 ($S = 10$, $b = 0.5$, $\upsilon = 1.5$, $\chi^n_s = 1.0$ for all s). You'll have to choose a guess for $T1$ and $T2$ and a time path updating parameter $\xi \in (0, 1]$, but it will be true that $T_1 \leq 60$ and $60 < T_2 \leq 90$. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of `TPI_tol` $= 10^{-12}$. Use a linear or quadratic initial guess for the time path of the aggregate capital stock from the initial state K^1_1 to the steady state $K^1_{T_1}$ at time T_1.

a. Make 3D surface plots of the equilibrium time path of the distribution of consumption $c_{s,t}$, labor supply $n_{s,t}$, and savings $b_{s,t}$

b. Make line plots of the equilibrium time paths for the prices and aggregate variables $\{r_t, w_t, K_t, L_t, Y_t, C_t\}$

c. Show in a table the maximum absolute errors across the equilibrium time path of the savings Euler errors, labor supply Euler errors, final period savings $\bar{b}_{S+1,t}$, and resource constraint errors.

d. How many periods did it take for the economy to get within 0.0001 of the steady-state aggregate capital stock $\bar{K}$? That is, what is T?

Exercise 4.5. [TODO: Need to update this calibration for the 80-period model.]Assume that $S = 20$. If we divide heads of household in the United States in to 20 age bins $\{21-24, 25-28, 29-32, ...93-96, 97-100\}$, the average hours for each of those age categories (as a percent of the maximum annual hours in the survey) is the following.

```
AnnHrs_US = np.array([0.5, 0.6, 0.7, 0.75, 0.77, 0.78, 0.785, 0.79, 0.8,
    0.81, 0.82, 0.82, 0.8, 0.77, 0.74, 0.7, 0.6, 0.45, 0.3, 0.2])
```

Calibrate χ^n_s for $s = 1, 2, ...20$ so that the steady-state labor supply produced by the model $\{\bar{n}_s\}_{s=1}^{20}$ is close to the vector of empirical hours above.

Chapter 5

S-period Lives, Endogenous Labor, and Heterogeneous Abilities

In this chapter, we augment the model from Section 4 with individuals of each age s having different abilities. The abilities will enter the labor income part of the budget constraint in a way that an individual will earn the average wage plus a premium or a discount based on their different abilities. Age and ability are two of the most important dimensions of individual heterogeneity to have in an economic model.

5.1 Heterogeneous lifetime ability paths

Differences among workers' productivity in terms of ability is one of the key dimensions of heterogeneity to model in a micro-founded macroeconomy. In this chapter, we model this heterogeneity as deterministic lifetime productivity paths to which new cohorts of agents in the model are randomly assigned.[1] In our model, agents' labor income comes from the equilibrium wage and the agent's endogenous quantity of labor supply. In this section, we augment the labor income expression with an individual productivity $e_{j,s}$, where j is the index of the ability type or path of the individual and s is the age of the individual with

[1] Stochastic and persistent idiosyncratic ability is another common way to model income heterogeneity. We treat this in Chapter 6.

that ability path.

$$\text{labor income:} \quad w_t e_{j,s} n_{j,s,t} \quad \forall j,s,t \tag{5.1}$$

In this specification, w_t is an equilibrium wage representing a portion of labor income that is common to all workers. Individual quantity of labor supply is $n_{j,s,t}$, and $e_{j,s}$ represents a labor productivity factor that augments or diminishes the productivity of a worker's labor supply relative to average productivity.

We calibrate deterministic ability paths such that each lifetime income group has a different life-cycle profile of earnings. The distribution on income and wealth are often focal components of macroeconomic models. As such, we use a calibration of deterministic lifetime ability paths from DeBacker et al. (2017b) that can represent U.S. earners in the top 1% of the distribution of lifetime income. Piketty and Saez (2003) show that income and wealth attributable to these households has shown the greatest growth in recent decades. The data come from the U.S. Internal Revenue Services's (IRS) Statistics of Income program (SOI) Continuous Work History Sample (CWHS). DeBacker et al. (2017b) match the SOI data with Social Security Administration (SSA) data on age and Current Population Survey (CPS) data on hours in order to generate a non-top-coded measure of hourly wage.

Figure 5.1: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$

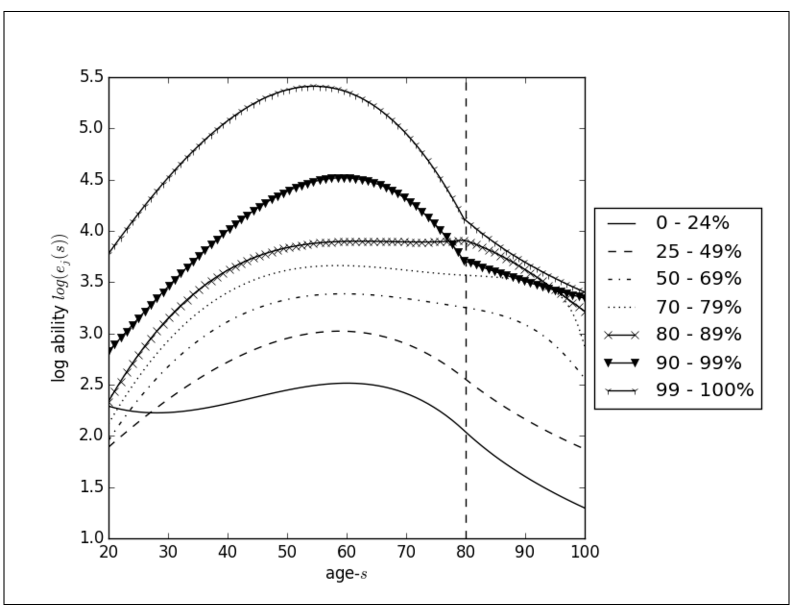


Figure 5.1 shows a calibration for $J = 7$ deterministic lifetime ability paths $e_{j,s}$ corresponding to labor income percentiles $\boldsymbol{\lambda} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Because there are few individuals above age 80 in the data, DeBacker et al. (2017b) extrapolate these estimates for model ages 80-100 using an arctan function.

5.2 Households

Individuals are born at age $s = 1$ and live to age $S \in [3, 80]$. We choose 80 as the upper bound of periods to live so that the minimum amount of time represented by a model period is one year. But this restriction is not important.

At birth, each individual age $s = 1$ is randomly assigned one of J ability groups, indexed by j. Let λ_j represent the fraction of individuals in each ability group, such that $\sum_j \lambda_j = 1$. Note that this implies that the distribution across ability types in each age is given by $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ...\lambda_J]$. Once an individual is born and assigned to an ability type, he remains that ability type for his entire lifetime. This is deterministic ability heterogeneity. Let $e_{j,s} > 0$ be a matrix of ability-levels such that an individual of ability type j will have lifetime abilities of $[e_{j,1}, e_{j,2}, ...e_{j,S}]$. The household budget constraint is now the following,

$$\begin{aligned} c_{j,s,t} + b_{j,s+1,t+1} = (1 + r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \quad \forall j, s, t \\ \text{where} \quad b_{j,1,t}, b_{j,S+1,t} = 0 \quad \forall j, t \end{aligned} \tag{5.2}$$

where many of the variables now have j subscripts. The variables with three subscripts (j, s, t) tell you to which ability type j and age s individual the variable belongs and in which period t.

The labor-leisure constraint and the period utility function are similar to (4.1) and (4.7), respectively, except the endogenous variables have an additional j subscript.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \tag{5.3}$$

$$u\big(c_{j,s,t},n_{j,s,t}\big) = \frac{c_{j,s,t}^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{5.4}$$

We describe the properties and derivation of the elliptical disutility of labor function in (5.4) in Section 4.1. It is an approximation of the standard constant Frisch elasticity disutility of labor supply function, but (5.4) has Inada conditions at both the upper and lower bounds of labor supply. This provides a significant increase in computational tractability with very little cost.

Households choose lifetime consumption $\{c_{j,s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{j,s,t+s-1}\}_{s=1}^S$, and savings $\{b_{j,s+1,t+s}\}_{s=1}^{S-1}$ to maximize discounted lifetime utility,

$$\max_{\{c_{j,s,t+s-1},n_{j,s,t+s-1}\}_{s=1}^S,\{b_{j,s+1,t+s}\}_{s=1}^{S-1}} \sum_{u=0}^{E+S-s}\beta^u u\left(c_{j,s+u,t+u},n_{j,s+u,t+u}\right) \quad \forall j,t \tag{5.5}$$

$$\text{s.t.}\quad c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \tag{5.2}$$

$$\text{where}\quad u\big(c_{j,s,t},n_{j,s,t}\big) = \frac{c_{j,s,t}^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{5.4}$$

where χ^n_s is a scale parameter that can potentially vary by age s influencing the relative disutility of labor to the utility of consumption. The household's lifetime problem (5.5) can be reduced to choosing S labor supplies $\{n_{j,s,t+s-1}\}_{s=1}^S$ and $S-1$ savings $\{b_{j,s+1,t+s}\}_{s=1}^{S-1}$ by substituting the budget constraints (5.2) in for $c_{j,s,t}$ in each period utility function (5.4) of the lifetime utility function.

The set of optimal lifetime choices for an agent of type j born in period t are characterized by the following S static labor supply Euler equations (5.6), the following $S-1$ dynamic savings Euler equations (5.7), and a budget constraint that binds in all S periods (5.2),

$$\begin{aligned}&w_t e_{j,s} u_1\left(c_{j,s,t+s-1},n_{j,s,t+s-1}\right) = -u_2\left(c_{j,s+1,t+s},n_{j,s+1,t+s}\right) \quad \forall j,t \quad \text{and} \quad s\in\{1,2,...S\} \\ &\Rightarrow \quad w_t e_{j,s}\left(c_{j,s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}}\end{aligned} \tag{5.6}$$

$$u_1\left(c_{j,s,t+s-1},n_{j,s,t+s-1}\right) = \beta(1+r_{t+1})u_1\left(c_{j,s+1,t+s},n_{j,s+1,t+s}\right) \quad \forall j,t \quad \text{and} \quad s\in\{1,2,...S-1\}$$
$$\Rightarrow \quad \left(c_{j,s,t}\right)^{-\sigma} = \beta(1+r_{t+1})\left(c_{j,s+1,t+1}\right)^{-\sigma} \tag{5.7}$$

$$c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \quad \forall j,t \quad \text{and} \quad s\in\{1,2,...S\}, \quad b_{1,t}, b_{S+1,t+S} = 0 \tag{5.2}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{j,s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{j,s,t}$. As was demonstrated in detail in Section 3.1, the dynamic Euler equations (5.7) do not include marginal utilities of all future periods because of the principle of optimality and the envelope condition.

Note that these $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u,r_u\}_{u=t}^{t+S-1}$. As in section 3.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (5.8) and (2.17).

$$\mathbf{\Gamma}_t \equiv \left\{b_{j,s,t}\right\}_{s=2}^{S} \quad \forall j,t \tag{5.8}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{2.17}$$

5.3 Firms

Firms are characterized exactly as in Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (2.20) and its aggregate labor decision L_t governed by first order condition (2.21).

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$
$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

The per-period depreciation rate of capital is $\delta\in[0,1]$, the capital share of income is $\alpha\in(0,1)$, and total factor productivity is $A>0$.

The only change to note, which is described more carefully in Section 5.4 in the labor market clearing condition (5.9), is that aggregate labor L_t in the production function is now in efficiency units. This means that the aggregate labor L_t used in production is made up of both labor hours $n_{j,s,t}$ and ability $e_{j,s}$.

5.4 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand. The market clearing conditions for this version of the model slightly different from those in the previous sections because we must sum not only over ages s but also over ability types j.

$$L_t = \sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j e_{j,s} n_{j,s,t} \quad \forall t \tag{5.9}$$

$$K_t = \sum_{s=2}^{S}\sum_{j=1}^{J}\lambda_j b_{j,s,t} \quad \forall t \tag{5.10}$$

$$\begin{aligned} &Y_t = C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ &\quad \text{where} \quad C_t \equiv \sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j c_{j,s,t} \end{aligned} \tag{5.11}$$

The goods market clearing equation (5.11) is redundant by Walras' Law. As noted in Section 5.3, the aggregate labor supply in (5.9) is in efficiency units (i.e., L_t sums over $e_{j,s}n_{j,s,t}$).

It is important to note that the distributional assumptions here are that each age-s cohort has a population of one, which means the total population is S and the population age-s and ability-j is λ_j. Each aggregate variable L_t, K_t, and C_t (and Y_t indirectly) sums over individual decisions multiplied by the individual population measure λ_j.

5.5 Equilibrium

As in previous sections, we give a rough sketch of the equilibrium before providing exact definitions of the functional equilibrium concepts so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points.

i. Households optimize according to equations (5.6) and (5.7).

ii. Firms optimize according to (2.20) and (2.21).

iii. Markets clear according to (5.9) and (5.10).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (5.9) and (5.10) into the firm's optimal conditions (2.20) and (2.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t\big(\boldsymbol{\Gamma}_t\big): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S\sum_{j=1}^J\lambda_j b_{j,s,t}}{\sum_{s=1}^S\sum_{j=1}^J\lambda_j e_{j,s}n_{j,s,t}}\right)^{\alpha} \quad \forall t \tag{5.12}$$

$$r_t\big(\boldsymbol{\Gamma}_t\big): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^S\sum_{j=1}^J\lambda_j e_{j,s}n_{j,s,t}}{\sum_{s=2}^S\sum_{j=1}^J\lambda_j b_{j,s,t}}\right)^{1-\alpha} - \delta \quad \forall t \tag{5.13}$$

It is worth noting here that the equilibrium wage (5.12) and interest rate (5.13) are written as functions of the period-t distribution of savings (wealth) $\boldsymbol{\Gamma}_t$ from (5.8) and are not functions of the period-t distribution of labor supply, which labor distribution shows up in (5.12) and (5.13). This is because, similar to next period savings $b_{j,s+1,t+1}$, current period labor supply $n_{j,s,t}$ must be chosen in period t and is therefore a function of the current state $\boldsymbol{\Gamma}_t$ distribution of savings.

Now (5.12), (5.13), and the budget constraint (5.2) can be substituted into household Euler equations (5.6) and (5.7) to get the following $(2S-1)$-equation system of lifetime characterizing equations for an individual of ability type j. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{aligned} &w_t\big(\boldsymbol{\Gamma}_t\big)e_{j,s}\left(\Big[1+r_t\big(\boldsymbol{\Gamma}_t\big)\Big]b_{j,s,t}+w_t\big(\boldsymbol{\Gamma}_t\big)e_{j,s}n_{j,s,t}-b_{j,s+1,t+1}\right)^{-\sigma} = ... \\ &\qquad \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s\in\{1,2,...S\} \quad \text{and} \quad \forall j,t \end{aligned} \tag{5.14}$$

$$
\begin{aligned}
&\left(\Bigl[1+r_t\bigl(\mathbf{\Gamma}_t\bigr)\Bigr]b_{j,s,t}+w_t\bigl(\mathbf{\Gamma}_t\bigr)e_{j,s}n_{j,s,t}-b_{j,s+1,t+1}\right)^{-\sigma} = \\
&\qquad \beta\Bigl[1+r_{t+1}\bigl(\mathbf{\Gamma}_{t+1}\bigr)\Bigr]\left(\Bigl[1+r_{t+1}\bigl(\mathbf{\Gamma}_{t+1}\bigr)\Bigr]b_{j,s+1,t+1}+w_{t+1}\bigl(\mathbf{\Gamma}_{t+1}\bigr)e_{j,s}n_{j,s+1,t+1}-b_{j,s+2,t+2}\right)^{-\sigma} \\
&\qquad\qquad \text{for} \quad s\in\{1,2,...S-1\} \quad \text{and} \quad \forall j,t
\end{aligned}
\tag{5.15}
$$

The system of S nonlinear static equations (5.14) and $S-1$ nonlinear dynamic equations (5.15) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{j,s,t+s-1}\}_{j,s=1}^{J,S}$ and $\{b_{j,s+1,t+s}\}_{j,s=1}^{J,S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1}=\{b_{j,s+1,t+1}\}_{j,s=1}^{J,S-1}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{j,s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1}=x_t=\bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 5.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents, endogenous labor supply, and deterministic heterogeneous lifetime ability paths is defined as constant allocations of consumption $\{\bar{c}_{j,s}\}_{j,s=1}^{J,S}$, labor supply $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$, and savings $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (5.6) and (5.7),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (5.9) and (5.10).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{j,s,t}=\phi_{j,s}\left(\mathbf{\Gamma}_t\right)$, and let the equilibrium policy functions for savings be represented by $b_{j,s+1,t+1}=\psi_{j,s}\left(\mathbf{\Gamma}_t\right)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 5.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents, endogenous labor supply, and deterministic heterogeneous lifetime ability paths is defined as stationary allocation functions of the state $\big\{n_{j,s,t} = \phi_{j,s}\big(\boldsymbol{\Gamma}_t\big)\big\}_{j,s=1}^{J,S}$, $\big\{b_{j,s+1,t+1} = \psi_{j,s}\big(\boldsymbol{\Gamma}_t\big)\big\}_{j,s=1}^{J,S-1}$ and and stationary price functions $w(\boldsymbol{\Gamma}_t)$ and $r(\boldsymbol{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\boldsymbol{\Gamma}_{t+u} = \boldsymbol{\Gamma}^e_{t+u} = \Omega^u\left(\boldsymbol{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (5.6) and (5.7),

iii. firms optimize according to (2.20) and (2.21),

iv. markets clear according to (5.9) and (5.10).

5.6 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 5.1 and the transition path equilibrium from Definition 5.2.

5.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 5.1. The parameters needed for the steady-state solution of this model are $\Big\{J, S, \{\lambda_j\}_{j=1}^J, \beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta\Big\}$, where S is the number of periods in an individual's life, J is the number of ability groups, $\{\lambda_j\}_{j=1}^J$ is the income percentiles of ability groups, $\Big\{\beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, \upsilon, \{\chi^n_s\}_{s=1}^S\Big\}$ are household utility function parameters, and $\{A, \alpha, \delta\}$ are firm production function parameters. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{j,s,t} = \bar{c}_{j,s}, \quad n_{j,s,t} = \bar{n}_{j,s}, \quad b_{j,s,t} = \bar{b}_{j,s} \quad \forall j, s, t \tag{5.16}$$

From the market clearing conditions (5.9) and (5.10) and the firms' first order equations (2.20) and (2.21), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K_t = \bar{K} \quad L_t = \bar{L} \quad \forall t \tag{4.16}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}e_{j,s}\Big([1+\bar{r}]\bar{b}_{j,s} + \bar{w}e_{j,s}\bar{n}_{j,s} - \bar{b}_{j,s+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j \quad \text{and} \quad s = \{1,2,...S\} \tag{5.17}$$

$$\Big([1+\bar{r}]\bar{b}_{j,s} + \bar{w}e_{j,s}\bar{n}_{j,s} - \bar{b}_{j,s+1}\Big)^{-\sigma} = \beta(1+\bar{r})\Big([1+\bar{r}]\bar{b}_{j,s+1} + \bar{w}e_{j,s_+1}\bar{n}_{j,s+1} - \bar{b}_{j,s+2}\Big)^{-\sigma} \quad \forall j \quad \text{and} \quad s = \{1,2,...S-1\} \tag{5.18}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distributions of labor supply and savings as shown in (5.12) and (5.13). This represents a system of $J(2S-1)$ nonlinear dynamic equations and $J(2S-1)$ unknowns.

One approach to solving this system would be to use a multivariate root finder that chooses the $J(2S-1)$ steady-state variables $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$ simultaneously to solve the zeros of the $J(2S-1)$ Euler equations (5.17) and (5.17). However, the trade off between labor supply and savings in each period create a series of saddle paths in the objective function that render the simultaneous equations root finder unreliable and somewhat intractable. We have found that this approach only works when your initial guess for the steady-state $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$ is close to the solution. It breaks down when the underlying parameters are changed.

Another method would be to perform an outer loop root finder on a guess for $\bar{r}$ and $\bar{w}$ (or equivalently $\bar{K}$ and $\bar{L}$), such that the household's problem is solved for the values of r and w in each iteration and the outer loop root finder uses the firm's first order condition as the error equations. This method also works well if one's initial guess is good. However, it can fail to find the equilibrium for a large range of initial guesses.

The most robust solution method that we have found takes significantly more time than the two methods described in the preceding paragraphs, but it successfully finds the steady-state equilibrium for most feasible initial guesses that we have constructed. This method is a bisection method in the outer loop guesses for steady-state equilibrium $\bar{K}$ and $\bar{L}$. In the inner loop of the household j's problem given r and w implied by K and L, we solve the problem by breaking the multivariate root finder problem with $2S-1$ equations and unknowns into a series of many univariate root finder problems and one bivariate root finder problem. The algorithm is the following.

i. Make a guess for the steady-state aggregate capital stock $\bar{K}^i$ and aggregate labor $\bar{L}^i$.

   (a) Values for $\bar{K}^i$ and $\bar{L}^i$ will imply values for the interest rate $\bar{r}^i$ and wage $\bar{w}^i$ from (2.20) and (2.21)

   (b) For the bisection method, we must use guesses for K and L because those uniquely determine r and w, whereas the converse is not true. From the firms' first order conditions (2.20) and (2.21), we see that r and w are functions of the same capital-labor ratio K/L. Infinitely many combinations of K and L determine a given r and w.

ii. Given $\bar{r}^i$ and $\bar{w}^i$, solve for the steady-state household's lifetime decisions $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$.

   (a) Given $\bar{r}^i$ and $\bar{w}^i$, guess an initial steady-state consumption $\bar{c}_{j,1}^m$, where m is the index of the inner-loop (household problem given $\bar{r}^i$, $\bar{w}^i$) iteration.

(b) Given $\bar{r}^i$, $\bar{w}^i$, and $\bar{c}^m_{j,1}$, use the sequence of $S-1$ dynamic savings Euler equations (5.7) to solve for the implied series of steady-state consumptions $\{\bar{c}^m_{j,s}\}_{j,s=1}^{J,S}$. This sequence has an analytical solution.

$$\bar{c}^m_{j,s+1} = \bar{c}^m_{j,s}\left[\beta(1+\bar{r}^i)\right]^{\frac{1}{\sigma}} \quad \forall j \quad \text{and} \quad s = \{1,2,...S-1\} \tag{5.19}$$

(c) Given $\bar{r}^i$, $\bar{w}^i$, and $\{\bar{c}^m_{j,s}\}_{j,s=1}^{J,S}$, solve for the series of steady-state labor supplies $\{\bar{n}^m_{j,s}\}_{j,s=1}^{J,S}$ using the S static labor supply Euler equations (5.6). This will require a series of S separate univariate root finders or one multivariate root finder.

$$\bar{w}^i e_{j,s}\left(\bar{c}^m_{j,s}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}^m_{j,s}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}^m_{j,s}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j \quad \text{and} \quad s = \{1,2,...S\} \tag{5.20}$$

It is this separation of the labor supply decisions from the consumption-savings decisions that gets rid of the saddle paths in the objective function that are so difficult for global optimization.

(d) Given $\bar{r}^i$, $\bar{w}^i$, and implied steady-state consumption $\{\bar{c}^m_{j,s}\}_{j,s=1}^{J,S}$ and labor supply $\{\bar{n}^m_{j,s}\}_{j,s=1}^{J,S}$, solve for implied time path of savings $\{\bar{b}^m_{j,s+1}\}_{j,s=1}^{J,S}$ across all ages of the representative lifetime using the household budget constraint (5.2).

$$\bar{b}^m_{j,s+1} = (1+\bar{r}^i)\bar{b}^m_{j,s} + \bar{w}^i e_{j,s}\bar{n}^m_{j,s} - \bar{c}^m_{j,s} \quad \forall j \quad \text{and} \quad s = \{1,2,...S\} \tag{5.21}$$

Note that this sequence of savings includes savings in the last period of life for the next period $\bar{b}_{j,S+1}$. This savings amount is zero in equilibrium, but is not zero for an arbitrary guess for $\bar{c}^m_{j,1}$ as in step (a).

(e) Update the initial guess for $\bar{c}^m_{j,1}$ to $\bar{c}^{m+1}_{j,1}$ until the implied savings in the last period equals zero $\bar{b}^{m+1}_{j,S+1} = 0$.

iii. Given solution for optimal household decisions $\{\bar{c}^m_{j,s}\}_{j,s=1}^{J,S}$, $\{\bar{n}^m_{j,s}\}_{j,s=1}^{J,S}$, and $\{\bar{b}^m_{j,s+1}\}_{j,s=1}^{J,S-1}$ based on the guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$, solve for the aggregate capital $\bar{K}^{i'}$ and aggregate labor $\bar{L}^{i'}$ implied by the household solutions and market clearing conditions.

$$\bar{K}^{i'} = \sum_{s=2}^{S}\sum_{j=1}^{J}\lambda_j \bar{b}^m_{j,s} \tag{5.22}$$

$$\bar{L}^{i'} = \sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j e_{j,s}\bar{n}^m_{j,s} \tag{5.23}$$

Update guesses for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ until the the aggregates implied by household optimization equal the initial guess for the aggregates $(\bar{K}^{i'+1}, \bar{L}^{i'+1}) = (\bar{K}^{i+1}, \bar{L}^{i+1})$.

(a) The bisection method characterizes the updated guess for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ as a convex combination of the initial guess $(\bar{K}^{i}, \bar{L}^{i})$ and the values implied by household and firm optimization $(\bar{K}^{i'}, \bar{L}^{i'})$, where the weight put on the new values $(\bar{K}^{i'}, \bar{L}^{i'})$ is given by $\xi \in (0,1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$(\bar{K}^{i+1}, \bar{L}^{i+1}) = \xi(\bar{K}^{i'}, \bar{L}^{i'}) + (1-\xi)(\bar{K}^{i}, \bar{L}^{i}) \quad \text{for} \quad \xi \in (0,1] \tag{??}$$

(b) Let $\|\cdot\|$ be a norm on the space of feasible aggregate capital and aggregate labor values (K, L). We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (4.24). If the distance is less than some tolerance `toler` > 0, then the problem has converged. Otherwise continue updating the values of aggregate capital and labor using (??).

$$\texttt{dist} \equiv \Big\|(\bar{K}^{i'}, \bar{L}^{i'}) - (\bar{K}^{i}, \bar{L}^{i})\Big\| \tag{4.24}$$

Define the updating of aggregate variable values $(\bar{K}^{i}, \bar{L}^{i})$ in step (iii) indexed by i as the "outer loop" of the fixed point solution. Although computationally intensive, the bisection method described above is often the most robust solution method.

Figure 5.2: Steady-state distribution of consumption $\bar{c}_{j,s}$

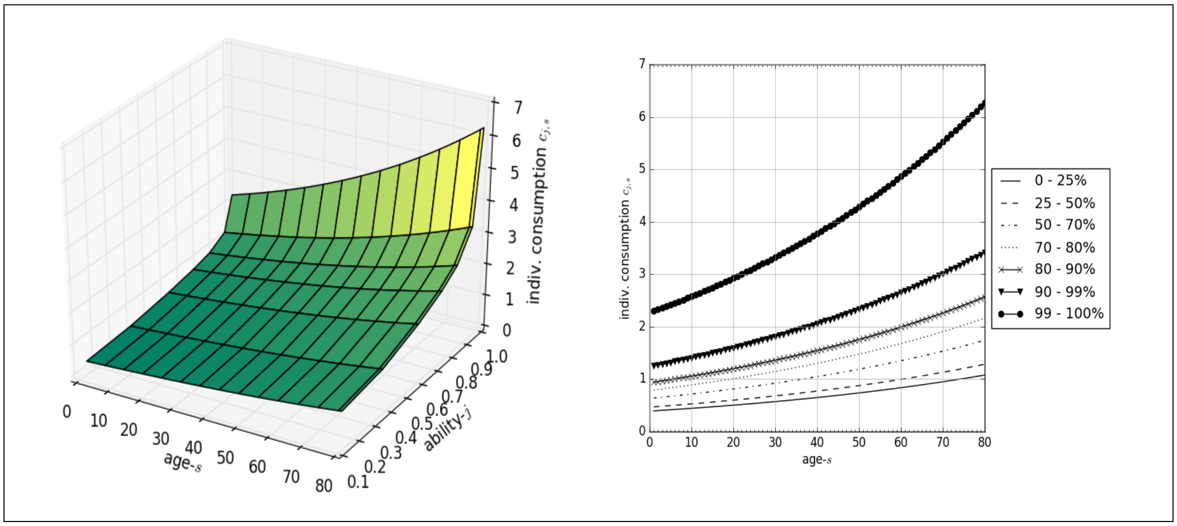


Figure 5.3: Steady-state distribution of labor supply $\bar{n}_{j,s}$

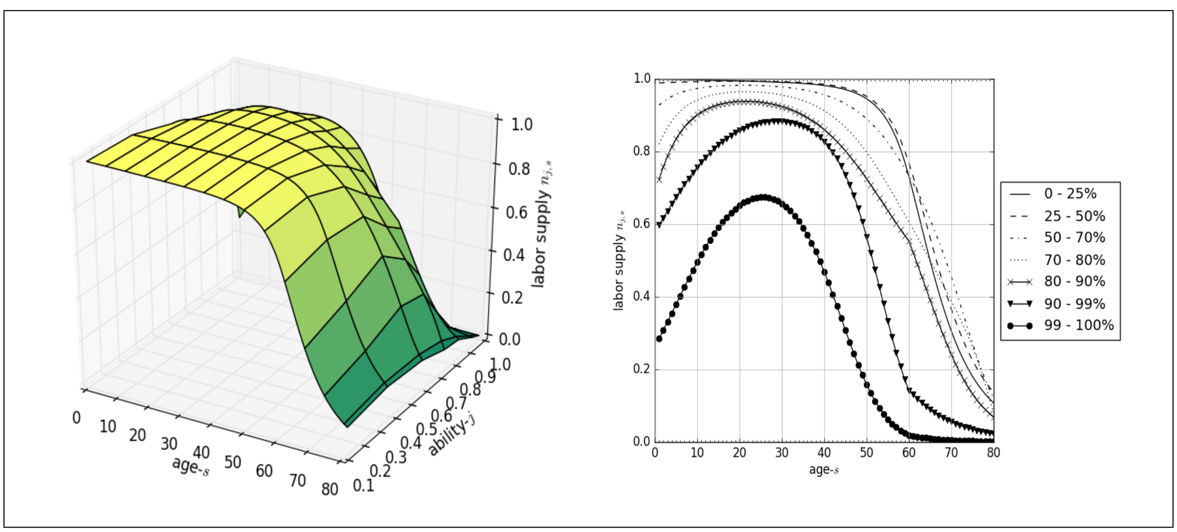


Figure 5.4: Steady-state distribution of savings/wealth $\bar{b}_{j,s}$

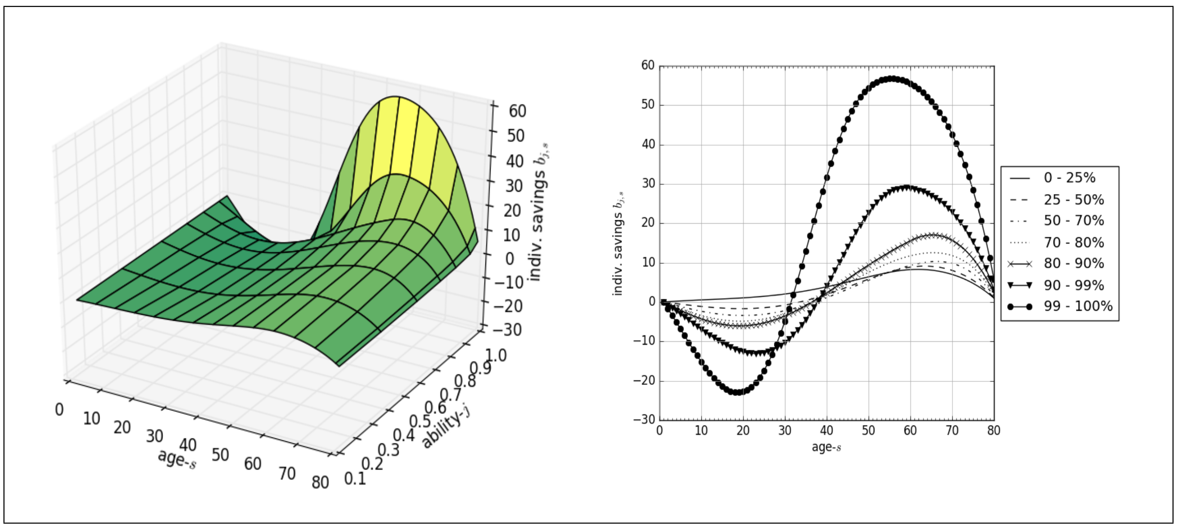


Table 5.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.075	Max. absolute savings Euler error	1.78e-15
$\bar{w}$	1.130	Max. absolute labor supply Euler error	7.02e-14
$\bar{K}$	290.632	Max. absolute final period savings $\bar{b}_{j,S+1}$	8.89e-12
$\bar{L}$	59.815	Resource constraint error	-0.576
$\bar{Y}$	104.015		
$\bar{C}$	90.060	Computation time	8 min. 4 sec.

Figures 5.2, 5.3, and 5.4 show the steady-state distributions of individual consumption, labor supply, and savings, respectively, by age s and ability j in an 80-period-lived agent model with parameter values listed above the line in Table 5.3 in Section 5.7. The left side of Table 5.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $J(2S-1)$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 5.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 7.0e-14, the final period individual savings is less than 8.9e-12, and the resource constraint error is smaller than -0.58, we can be confident that we have successfully solved for the steady-state.

5.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor and deterministic heterogeneous lifetime ability paths is similar to the method described in Section 4.6.2. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 5.1 in a finite number of periods $T < \infty$ regardless of the initial state $\boldsymbol{\Gamma}_1$.

To solve for the transition path (non-steady-state) equilibrium from Definition 5.2, we must know the parameters from the steady-state problem,

$$\Big\{J, S, \{\lambda_j\}_{j=1}^J, \beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta\Big\}$$

the steady-state solution values $\{\bar{K}, \bar{L}\}$, initial distribution of savings $\mathbf{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 5.1 and 5.3 show a particular calibration of the model and a steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

i. Choose a period $T1$ in which the initial guess for the time paths of aggregate capital and aggregate labor will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Given calibration for initial distribution of savings (wealth) $\mathbf{\Gamma}_1$, which implies an initial capital stock K_1, guess initial time paths for the aggregate capital stock $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_{T1}^i\}$ and aggregate labor $\boldsymbol{L}^i = \{L_1^i, L_2^i, ...L_{T1}^i\}$. Both of these time paths will have to be extended with their respective steady-state values so that they are $T2 + S - 1$ elements long. This is the time-path length that will allow you to solve the lifetime of every individual alive in period $T2$.

iii. Given time paths $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$, solve for the lifetime consumption $c_{j,s,t}$, labor supply $n_{j,s,t}$, and savings $b_{j,s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.

   (a) The initial paths for aggregate capital $\boldsymbol{K}^i$ and aggregate labor $\boldsymbol{L}^i$ imply time paths for the interest rate $\boldsymbol{r}^i = \big\{r_1^i, r_2^i, ...r_{T2+S-1}^i\big\}$ and wage $\boldsymbol{w}^i = \big\{w_1^i, w_2^i, ...w_{T2+S-1}^i\big\}$ using the firms' first order equations (2.20) and (2.21).

   (b) Given the time paths for the interest rate $\boldsymbol{r}^i$ and wage $\boldsymbol{w}^i$ and the period-1 distribution of savings (wealth) $\mathbf{\Gamma}_1$, solve for the lifetime decisions $c_{j,s,t}$, $n_{j,s,t}$, and $b_{j,s,t}$ of each household alive during periods 1 through $T2$. This is done using the method outlined in steps (ii)(a) through (ii)(e) of the steady-state computational algorithm outlined in Section 5.6.1.

iv. Use time path of the distribution of labor supply $n_{j,s,t}$ and savings $b_{j,s,t}$ from households optimal decisions given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ to compute new paths for aggregate capital and aggregate labor $\boldsymbol{K}^{i'}$ and $\boldsymbol{L}^{i'}$ implied by capital and labor market clearing conditions (5.9) and (5.10).

v. Compare the distance of the time paths of the new implied paths for the aggregate capital and labor $\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right)$ versus the initial aggregate capital and labor $\left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right)$.

$$\texttt{dist} = \left\|\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) - \left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right)\right\| \geq 0 \tag{4.26}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and aggregate labor $(\boldsymbol{K}^i, \boldsymbol{L}^i)$. Common norms to use are the L^2 and the L^∞ norms.

(a) If the distance is less than or equal to some tolerance level $\texttt{dist} \leq \texttt{TPI\_toler} > 0$, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level, then update the guess for a new set of initial time paths to be a convex combination current initial time paths and the implied time paths.

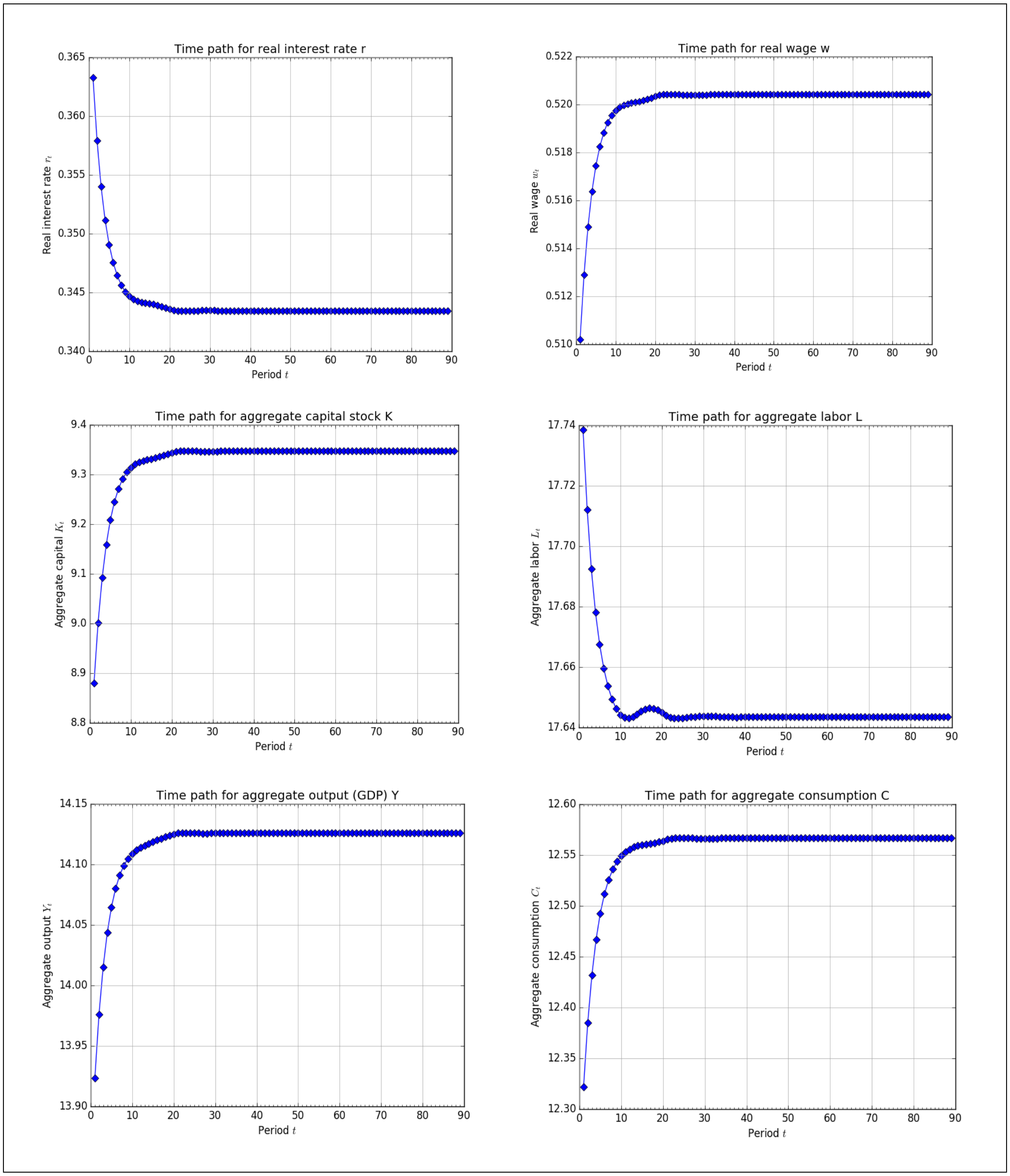
$$\left(\boldsymbol{K}^{i+1}, \boldsymbol{L}^{i+1}\right) = \xi\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) + (1-\xi)\left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right) \quad \text{for} \quad \xi \in (0,1] \tag{??}$$

Table 5.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	1.90e-12
Maximum absolute savings Euler error	2.13e-14
Maximum absolute final period savings $\bar{b}_{j,S+1,t}$	1.88e-13
Maximum absolute resource constraint error	1.74e-01

The 6 panels of Figure 5.5 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The calibration is the same as that of Section 5.6.1, except that we reduced the number of ages to $S = 20$ and the number of abilities to

Figure 5.5: Equilibrium transition paths of prices and aggregate variables



$J = 3$ with $\boldsymbol{\lambda} = [0.40, 0.35, 0.25]$. We do not show the time paths of individual consumption, labor supply, and savings because they are higher dimensional distributions that would be difficult to show over time. Table 5.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than 2.0e-12 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

5.7 Calibration

For the steady-state solution in Section 5.6.1 we used the calibration displayed in Table 5.3. We let agents live for $S = 80$ periods, which implies that each model period of life is one year. The annual discount factor is estimated to be 0.96, so the discount factor per period in the S-period model should be $\beta = 0.96^{80/S} = 0.96$. Let the annual depreciation rate of capital be 0.05. Then the depreciation rate in the S-period model should be $\delta = 1-(1-0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 2.5$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$. For the disutility of labor, let $\chi^n_s = 1$ for all s, and let $[b, \upsilon] = [.501, 1.554]$. We assume $J = 7$ ability types with percentages $\{\lambda_j\}_{j=1}^7 = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$, with a lifetime ability profiles $e_{j,s}$ given in Figure 5.1.

Because the computation of the non-steady-state equilibrium time path of the calibration used for the steady-state would take more than 24 hours, we use a simpler calibration for the solution in Section 5.6.2. We assume that $S = 20$, which implies that each model period equals 4 years. We also assume only three ability types $J = 3$ with $\{\lambda_j\}_{j=1}^3 = [0.40, 0.35, 0.25]$. This changes the value of the discount factor to $\beta = 0.96^{80/S} = 0.849$ and $\delta = 1 - (1 - 0.05)^{80/S} = 0.185$. We also use the method described in Exercise 5.1 to linearly interpolate a 20×3 ability matrix $e_{j,s}$ from the original 80×7 matrix shown in Figure 5.1. All the other parameters are the same as shown in Table 5.3.

Table 5.3: Calibrated parameter values for simple endogenous labor model

Parameter	Description	Value
J	Number of lifetime income (ability) groups	7
S	Number of periods in individual life	80
$\{\lambda_j\}_{j=1}^{J}$	Population distribution among ability types	(see Sec. 5.1)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Individual ability profile factor	(see Fig. 5.1)
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
$\tilde{l}$	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501[a]
υ	Elliptical disutility of labor shape parameter	1.554[a]
$\{\chi_s^n\}_{s=1}^{S}$	Disutility of labor relative scale factor by age	1.0
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
$\mathbf{\Gamma}_1$	Initial distribution of savings (wealth)	(see Fig. 4.7)
$T1$	Time period in which initial path guess hits steady state	160
$T2$	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.2

[a] The calibration of b and υ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See Evans and Phillips (2017).

5.8 Exercises

Exercise 5.1. Import the comma delimited data file `emat.txt` as a NumPy array `emat`. The file `emat.txt` is data for the 80×7 matrix of ability levels $e_{j,s}$. Each row represents $s = \{1, 2, ...80\}$, and each column represents ability levels $j = \{1, 2, ...7\}$ associated with income percentiles $\boldsymbol{\lambda} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Assume that the population distribution by age is uniform $1/S$. The data in `emat` are scaled such that the average ability is 1 so each value $e_{j,s}$ can be interpreted as the percent premium or discount of the average wage w_t.

$$\frac{1}{S}\sum_{s=1}^{S}\sum_{j=1}^{J}\lambda_j e_{j,s} = 1$$

a. Make a 2D plot of the ability matrix `emat` made up of seven lines showing ability $e_{j,s}$ over the life cycle s for each ability type j. That is, each line on the graph should be a column of the $e_{j,s}$ matrix. Include a legend on the graph that shows "0-24%" for the line for $j = 1$, "25-49%" for the line for $j = 2$, "50-69%" for the line for $j = 3$, "70-79%" for the line for $j = 4$, "80-89%" for the line for $j = 5$, "90-98%" for the line for $j = 6$, and "99-100%" for the line for $j = 7$.

b. Now assume that $S = 10$ (each model year is 8 years) and $J = 3$ with $\{\lambda_j\}_{j=1}^{3} = [0.4, 0.4, 0.3]$. Create a function that uses the original 80×7 `emat` matrix that you imported in the previous part of this exercise as the baseline, and creates a new matrix that is $S_{new} \times J_{new}$, which is 10×3 in this case. Solve for $e_{j,s}$ in the new matrix as a linear interpolation between points on the original matrix.

Exercise 5.2. Using a small version calibration similar to the one that was used for the non-steady-state equilibrium solution in Section 5.6.2, and the steady-state equilibrium Definition 5.1, solve for the steady-state equilibrium values of $\big\{\{\bar{c}_{j,s}\}_{s=1}^{S}\big\}_{j=1}^{J}$, $\big\{\{\bar{n}_{j,s}\}_{s=1}^{S}\big\}_{j=1}^{J}$, $\big\{\{\bar{b}_{j,s}\}_{s=2}^{S}\big\}_{j=1}^{J}$, $\bar{w}$, $\bar{r}$, $\bar{K}$, and $\bar{L}$ numerically. More specifically, let $S = 20$, $J = 2$, $\boldsymbol{\lambda} = [0.6, 0.4]$, $beta = 0.849$, and $\delta = 0.185$. Use the 20×2 $e_{j,s}$ matrix `emat20x2.txt`. And let all other values equal their values from Table 5.3.

Exercise 5.3. Use the calibration from Exercise 5.2 and time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from $\boldsymbol{\Gamma}_1 = 0.95\bar{\boldsymbol{\Gamma}}$ to the steady-state $\bar{\mathbf{\Gamma}}$. You must guess a value for T and a time path updating parameter $\xi \in (0,1)$, but it is likely that $T < 3 * S$ is sufficient. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear or quadratic initial guess for the time path of the aggregate capital stock from the initial state K_1^1 to the steady state K_T^1 at time T.

Exercise 5.4. Plot the equilibrium time paths of the aggregate capital stock $\{K_t\}_{t=1}^{T+10}$ and aggregate labor $\{L_t\}_{t=1}^{T+10}$. How many periods did it take for the economy to get within 0.0001 of the steady-state aggregate capital stock $\bar{K}$? That is, what is the true T?

Chapter 6

S-period-lived agents, exogenous labor, and stochastic ability

In this section, we take the model from Section 3 and add to it different ability types of agents as well as idiosyncratic shocks to individual's ability. In this chapter, we will solve a model in which household can choose from a continuum of savings amounts. However, a slightly simplified version of this model is found in Evans and Phillips (2014), in which the set of savings choices is discretized.

6.1 Households

A measure $1/S$ of individuals is born each period starting their lives at age $s = 1$ and live to age $S \in [3, 80]$. This implies a total population each period of 1. We choose $S = 80$ as the upper bound of periods to live so that the minimum amount of time represented by a model period is one year. But this restriction is not very important.

At birth, each individual age $s = 1$ is randomly assigned one of J discrete ability types $e_{j,t} \in \mathcal{E} = \{e_1, e_2, ...e_J\}$, indexed by j. Let the vector $\boldsymbol{\lambda}_t = [\lambda_{1,t}, \lambda_{2,t}, ...\lambda_{J,t}]$ with typical element $\lambda_{j,t}$ represent the fraction of individuals in each ability group j, such that $\sum_j \lambda_{j,t} = 1$ for all t. We are assuming that the distribution across ability types is the same within each age cohort and that the initial distribution is the ergodic distribution $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_t = \bar{\boldsymbol{\lambda}}$. We now define a Markov transition matrix $\Pi(e_{k,t+1}|e_{j,t})$ that gives the probability of being ability type $e_{k,t+1}$ next period given that you are type $e_{j,t}$ today with $k, j \in \{1, 2, ...J\}$. The Markov process Π is the following matrix of probabilities,

$$\Pi(e_{k,t+1}|e_{j,t}) = \begin{bmatrix} \pi_{1,1} & \pi_{1,2} & \dots & \pi_{1,J} \\ \pi_{2,1} & \pi_{2,2} & \dots & \pi_{2,J} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{J,1} & \pi_{J,2} & \dots & \pi_{J,J} \end{bmatrix} \tag{6.1}$$

where the typical element of $\Pi(e_{k,t+1}|e_{j,t})$ is $\pi_{j,k}$, each element is strictly positive $\pi_{j,k} > 0$ and each of the rows sums to one $\sum_{k=1}^{J} \pi_{j,k} = 1$. Π is a stochastic matrix.

We assume that the ability distribution $\boldsymbol{\lambda}_t = \bar{\boldsymbol{\lambda}}$ for all t is identical across ages s and across time periods t. This means that $\bar{\boldsymbol{\lambda}}$ is the ergodic distribution of the stochastic process Π. The population distribution across types $\boldsymbol{\lambda}_t$ evolves according to the following recursion.

$$\boldsymbol{\lambda}'_{t+1} = \Pi'\boldsymbol{\lambda}'_t \quad \Leftrightarrow \quad \begin{bmatrix} \lambda_{1,t+1} \\ \lambda_{2,t+1} \\ \vdots \\ \lambda_{J,t+1} \end{bmatrix} = \begin{bmatrix} \pi_{1,1} & \pi_{2,1} & \dots & \pi_{J,1} \\ \pi_{1,2} & \pi_{2,2} & \dots & \pi_{J,2} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{1,J} & \pi_{2,J} & \dots & \pi_{J,J} \end{bmatrix} \begin{bmatrix} \lambda_{1,t} \\ \lambda_{2,t} \\ \vdots \\ \lambda_{J,t} \end{bmatrix} \tag{6.2}$$

where the "$'$" represents the transpose of the vectors and matrices described previously. Let the ergodic distribution over ability types be $\bar{\boldsymbol{\lambda}} = [\bar{\lambda}_1, \bar{\lambda}_2, ...\bar{\lambda}_J]$. The ergodic ability distribution $\bar{\boldsymbol{\lambda}}$ is the eigenvector associated with the unit eigenvalue such that,

$$\bar{\boldsymbol{\lambda}}' = \Pi'\bar{\boldsymbol{\lambda}}' \tag{6.3}$$

With these stochastic abilities, individuals in our model are heterogeneous along three dimensions—ability $e_{j,t}$, age s, and wealth $b_{s,t}$. The household budget constraint is the following,

$$\begin{aligned} c_{s,t} + b_{s+1,t+1} = (1 + r_t)b_{s,t} + w_t e_{j,t} n_s \quad \forall j, s, t \\ \text{where} \quad b_{1,t}, b_{S+1,t} = 0 \quad \forall t \\ \text{and} \quad n_{s,t} = n_s \quad \forall j, t \end{aligned} \tag{6.4}$$

To simplify this model, we have assumed in (6.4) that labor is supplied inelastically. We assume that the individual supplies one unit of labor inelastically before retirement and then supplies 0.2 units of labor after retirement.

$$n_s = \begin{cases} 1 \quad \text{if} \quad s \leq \text{round}\left(\frac{2S}{3}\right) \\ 0.2 \quad \text{if} \quad s > \text{round}\left(\frac{2S}{3}\right) \end{cases} \tag{6.5}$$

We assume positive but reduced labor hours after retirement because the data reflect that and also because we want the ability transitions to still have some effect after retirement.

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$ and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility,

$$\begin{aligned} \max_{\{c_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} &E\big[U_{s,t}\big] = E\Big[\sum_{u=0}^{S-s}\beta^u u\left(c_{s+u,t+u}\right)\Big] \quad \forall s,t \\ \text{where} \quad &u\left(c_{s,t}\right) \equiv \frac{\left(c_{s,t}\right)^{1-\sigma}-1}{1-\sigma} \\ \text{and} \quad &c_{s,t} = (1+r_t)b_{s,t} + w_t e_{j,t} n_s - b_{s+1,t+1} \\ \text{and} \quad &b_{1,t}, b_{S+1,t} = 0 \end{aligned} \tag{6.6}$$

where the expectations operator $E[\cdot]$ is present because individuals do not know what their ability will $e_{k,t+u}$ be in future periods $u \geq 1$ of their lives.

The household problem as stated in (6.6) is often referred to as the sequence problem. We can restate the household's problem recursively using the concept of a value function $V_s : \mathbb{R}^4 \rightarrow \mathbb{R}$.

$$\begin{aligned} V_s\big(b_{s,t}, e_{j,t}, r_t, w_t\big) = &\max_{b_{s+1,t+1}} u\left(c_{s,t}\right) + \beta E\Big[V_{s+1}\big(b_{s+1,t+1}, e_{k,t+1}, r_{t+1}, w_{t+1}\big)\Big] \\ \text{where} \quad &u\left(c_{s,t}\right) \equiv \frac{\left(c_{s,t}\right)^{1-\sigma}-1}{1-\sigma} \\ \text{and} \quad &c_{s,t} = (1+r_t)b_{s,t} + w_t e_{j,t} n_s - b_{s+1,t+1} \\ \text{and} \quad &b_{1,t}, b_{S+1,t} = 0 \end{aligned} \tag{6.7}$$

The subscript s means that there is a different value function for each age. The value function $V_s(b,e,r,w)$ represents a household's remaining discounted expected lifetime utility of arriving at age s with wealth b, ability e, current interest rate r, and current wage w. The household's optimal solution is characterized by a value function $V_s\big(b,e,r,w\big)$ for each age $s=\{1,2,...S\}$ and a policy function for savings $b'=\psi_s\big(b,e,r,w\big)$ for each age $s=\{1,2,...S-1\}$, where the "$'$" notation signifies the value of a variable next period. Section 6.5.1 details how to solve for the household's problem given a path for prices $\{r_t,w_t\}_{t=1}^S$ over the lifetime of the household. This is done by solving for the value function and policy functions by backward induction starting from the last period of the household's life.

Each of the household's optimal policy functions $\psi_s(b,e,r,w)$ is characterized by an Euler equation of the following form,

$$\begin{split} u'\Bigl([1+r_t]b_{s,t}+w_t e_{j,t}n_s-\psi_s\Bigr) = \qquad\qquad\qquad\qquad \\ \beta E_{e_{k,t+1}|e_{j,t}}\Bigl[(1+r_{t+1})u'\Bigl([1+r_{t+1}]\psi_s+w_{t+1}e_{k,t+1}n_s-\psi_{s+1}\Bigr)\Bigr] \end{split} \tag{6.8}$$

where ψ_s and ψ_{s+1} are the respective policy functions for optimal $b_{s+1,t+1}$ and $b_{s+2,t+2}$.

$$b_{s+1,t+1}=\psi_s\big(b_{s,t},e_{j,t},r_t,w_t\big)\quad\forall t\quad\text{and}\quad 1\leq s\leq S-1 \tag{6.9}$$

$$\begin{split} b_{s+2,t+2}=\psi_{s+1}\big(b_{s+1,t+1},e_{k,t+1},r_{t+1},w_{t+1}\big)=\psi_{s+1}\big(\psi_s,e_{k,t+1},r_{t+1},w_{t+1}\big) \\ \forall t\quad\text{and}\quad 1\leq s\leq S-2 \end{split} \tag{6.10}$$

Here we must make an assumption about how the age-s household can forecast the time path of interest rates and wages $\{r_u,w_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Section 6.4, the equilibrium interest rate r_t and wage w_t will be functions of the state vector $\mathbf{\Gamma}_t$ in equilibrium, which turns out to be the population distribution across all the dimensions of heterogeneity in period t.

Define $\Gamma_t(b,e,s)$ as the distribution of the population for all possible wealth levels b, ability types e, and ages s at time t. With the total population normalized to 1 (each cohort has measure $1/S$), the population distribution $\Gamma_t(b,e,s)$ tells the percent of the population with wealth b, ability e_j, and age s, such that $\Gamma_t(b,e,s)\geq 0$ for all b, e, and s, and $\sum_s\sum_e\int_b\Gamma_t(b,e,s)\,db=1$ for all t. Note that the respective supports of age s and ability

$e_{j,t}$ are discrete, but the support of wealth $b_{s,t}$ is continuous.

Let general beliefs about the future population distribution in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[1] This assumption about beliefs pins down r_{t+u} and w_{t+u} in the household optimization problem. That is, households take prices as given and invariant to their individual decisions. And prices in every period r_t and w_t are just functions of the population distribution $\mathbf{\Gamma}_t$ in equilibrium. In addition, because the ability shocks are idiosyncratic, the household could conceivably perfectly forecast prices if they know the transition function of the state vector $\mathbf{\Gamma}_t$ over time.

6.2 Firms

The production side of this economy is identical to the one in Section 2.2 with a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology,

$$Y_t = F(K_t, L_t) \equiv A K_t^\alpha L_t^{1-\alpha} \quad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad A > 0. \tag{2.18}$$

The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t, L_t} A K_t^\alpha L_t^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{2.19}$$

[1] In Section 6.4 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 6.2.

where $\delta \in [0,1]$ is the rate of capital depreciation, and the two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

The only change to note, which is described more carefully in Section 6.3 in the labor market clearing condition (6.11), is that aggregate labor L_t in the production function is now in efficiency units. This means that the aggregate labor L_t used in production is made up of both labor hours n_s and ability $e_{j,t}$.

6.3 Market clearing

The market clearing conditions for this version of the model slightly different from those in the previous sections because we must sum not only over ages s but also over ability types j. The labor market, capital market, and goods market must clear. Labor market clearing is the following.

$$L_t = \frac{1}{S}\sum_{s=1}^{S}\sum_{j=1}^{J}\bar{\lambda}_j e_{j,t} n_s \quad \forall t \tag{6.11}$$

Capital market clearing is the following, where we integrate over the support of all possible savings b times the percent of the population with that amount of savings $\Gamma_t(s,e,b)$.

$$K_t = \sum_{s=1}^{S}\sum_{j=1}^{J}\int_b \Gamma_t(s,e,b)b\,db \quad \forall t \tag{6.12}$$

The goods market must also clear,

$$\begin{split} Y_t = C_t + K_{t+1} - (1-\delta)K_t \quad \forall t \\ \text{where} \quad C_t \equiv \frac{1}{S}\sum_{s=1}^{S}\sum_{j=1}^{J}\bar{\lambda}_j c_{s,t} \end{split} \tag{6.13}$$

where C_t is the population-weighted sum of all household consumption. But the goods market clearing equation (6.13) is redundant by Walras' Law. As noted in Section 6.2, the aggregate labor in the production function (2.18) and in (6.11) is in efficiency units (i.e., L_t sums over $e_{j,t}n_s$).

6.4 Equilibrium

The definition of steady-state equilibrium in this model with S-period lived agents, exogenous labor supply, and stochastic ability is quite different from Definition 3.1 in Section 3.4. Because individuals are changing ability types every period, the steady state is a constant population distribution across all dimensions of heterogeneity. And the non-steady-state equilibrium is a set of stationary policy functions and value functions that solve the equilibrium equations.

Definition 6.1 (Stochastic steady-state equilibrium). A non-autarkic stochastic steady-state rational expectations equilibrium in the overlapping generations model with S-period lived agents, exogenous labor supply, and stochastic ability is defined as a constant population distribution

$$\mathbf{\Gamma}_t = \mathbf{\Gamma}_{t+1} = \bar{\mathbf{\Gamma}} \quad \forall t,$$

a savings decision rule given beliefs $\bar{b}_{s+1} = \psi\big(s, e_j, \bar{b}_s|\Omega\big)$, constant prices $\bar{r}$ and $\bar{w}$, constant aggregate allocations $\bar{Y}$, $\bar{K}$, $\bar{L}$, and $\bar{C}$ such that:

i. households optimize according to (6.4) and (6.8),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (6.11) and (6.12).

Note that the steady-state rational expectations equilibrium definition 6.1 has no constraint that beliefs be correct $\Gamma_{t+1} = \Gamma^e_{t+1} = \Omega(\Gamma_t)$. The steady-state assumption $\Gamma_t = \Gamma_{t+1} = \bar{\Gamma}$ removes the need for beliefs about other households' actions because Γ_{t+1} is known. We describe in Section 6.5.3 how to solve for the steady-state equilibrium from Definition 6.1.

Definition 6.2 (Non-steady-state functional equilibrium). A non-steady-state functional equilibrium in the overlapping generations model with S-period lived agents, exogenous labor supply, and stochastic ability is defined as stationary allocation functions of the state for capital $\Big\{\big\{\psi_{j,s}(\boldsymbol{\Gamma}_t)\big\}_{s=1}^{S-1}\Big\}_{j=1}^{J}$ and stationary price functions $w(\boldsymbol{\Gamma}_t)$ and $r(\boldsymbol{\Gamma}_t)$ such that:

6.5 Solution Method

In this section, we describe how to compute the steady-state equilibrium of the model from Definition 6.1 and how to compute the non-steady-state equilibrium transition path from Definition 6.2. Because it is an essential piece of both the steady-state and transition path equilibria, we also show here how to solve a given household's lifetime problem given an arbitrary path of prices.

6.5.1 Solution to household problem given prices by backward induction

As is shown in equations (6.7) and (6.9), the household's problem each period is a function of its current wealth $b_{s,t}$, current ability $e_{j,t}$, and current prices r_t and w_t. Idiosyncratic ability $e_{j,t}$ evolves exogenously according to the Markov transition matrix Π in (6.1) and the aggregate prices r_t and w_t are functions of the distribution of wealth $\boldsymbol{\Gamma}_t$ and evolve according to some law of motion $\boldsymbol{\Gamma}_{t+u} = g\left(\boldsymbol{\Gamma}_t\right)$. Household take the prices r_t and w_t as given.

Although we will require that households' beliefs about the law of motion for the distribution of wealth be correct in equilibrium, it will be useful to solve the household's problem for an arbitrary path of prices over the household's lifetime. Specifically, we solve here for the household's set of policy functions and value functions $\Big\{\psi_s\big(b,e|r,w\big), V_s\big(b,e|r,w\big)\Big\}_{s=1}^{S}$ for a given path of prices $\{r_t, w_t\}_{t=1}^{S}$ for a household who is age $s=1$ (born) at time $t=1$. We do this by backward induction.

In the last period of a household's life $s,t=S$, the policy function for savings is trivially zero $b_{S+1} = \psi_S\left(b_S, e_j|r_S, w_S\right) = 0$ for all values of final period wealth b_S and ability e_j given final period prices r_S and w_S. This is because there is no benefit to positive savings and no one would lend any negative savings. The age-S policy function and resulting value function are the following.

$$\psi_S\big(b_S, e_j|r_S, w_S\big) = 0 \qquad \forall b_S, e_j, r_S, w_S \quad \text{and} \quad c_S > 0 \tag{6.14}$$

$$V_S\big(b_S, e_j|r_S, w_S\big) = u\Big([1+r_S]b_S + w_S e_j n_S - \psi_S\big(b_S, e_j|r_S, w_S\big)\Big) \tag{6.15}$$
$$\forall b_S, e_j, r_S, w_S \quad \text{and} \quad c_S > 0$$

We can approximate the functions ψ_S and V_S with a grid over values of b_S and e_j given assumed scalar values for prices r_S and w_S. Let our support of N points for b_S be in the ordered set $b_s \in \mathcal{B} = \{b_1, b_2, ...b_N\}$. Note that the set $\mathcal{B}$ can include negative savings values $b_n < 0$ (borrowing).[2] We have already assumed a discrete support of values for ability $e_{j,t} \in \mathcal{E} = \{e_1, e_2, ...e_J\}$. The approximating grid of points in the two-dimensional space is the tensor product of the two discrete supports $\mathcal{G} = \mathcal{B} \otimes \mathcal{E}$ with typical grid point element (b_n, e_j). Note that there are values of b and e given r and w for which $V_S(b, e|r, w)$ is not defined. The constraint is that $c_S > 0$, which means that $(1+r)b_S + we_j n_S - \psi_S(b_S, e_j) > 0$. For values of b_S and e_j for which the resulting $c_S \leq 0$, we place a `NaN` in the matrix approximating the policy function and value function represented in (6.14) and (6.15), respectively.

We can now solve for the second-to-last period of life policy function and value function for the agent over the same grid $\mathcal{G}$ and for the two period history of prices $(r_t, w_t)_{t=S-1}^{S}$. The household's problem in age $S-1$ is the following.

$$V_{S-1}\big(b_{S-1}, e_j|r_{S-1}, w_{S-1}\big) = \max_{b_S} u\big([1-r_{S-1}]b_{S-1} + w_{S-1}e_j n_{S-1} - b_S\big) + \beta E_{e_k|e_j}\Big[V_S\big(b_S, e_k|r_S, w_S\big)\Big] \tag{6.16}$$

Because we have already solved for $V_S(b, e|r, w)$ in (6.14) and (6.15), the policy function

[2] This model has the endogenous borrowing constraints which is the finite-lived analogue to the infinitely lived agent borrowing constraints described by Aiyagari (1994) and Zhang (1997). See Storesletten et al. (2007) for an overlapping generations example of endogenous borrowing constraints. And although individual savings can be negative the sum of all savings—the aggregate capital stock—must be strictly positive.

$\psi_{S-1}(b,e|r,w)$ is characterized by the following equation.

$$\begin{split}
b_S = \psi_{S-1}\left(b_{S-1},e_j|r_{S-1},w_{S-1}\right):\\
\Bigl[(1+r_{S-1})b_{S-1} + w_{S-1}e_j n_{S-1} - b_S\Bigr]^{-\sigma} = ...\\
\beta(1+r_S)\sum_{k=1}^J\left(\pi_{j,k}\Bigl[(1+r_S)b_S + w_S e_k n_S - \psi_S(b_S,e_k|r_S,w_S)\Bigr]^{-\sigma}\right)
\end{split} \tag{6.17}$$

The right-hand-side of the Euler error above is simplified in this case by the fact that the final period policy function is trivial $\psi_S = 0$.[3] However, this will not be the case for the solutions to age $s < S-1$. The age-$S-1$ value function is equation (6.16) with the optimal policy $b_S = \psi_{S-1}(b_{S-1},e_j|r_{S-1},w_{S-1})$ substituted in.

$$\begin{split}
V_{S-1}\bigl(b_{S-1},e_j|r_{S-1},w_{S-1}\bigr) = u\Bigl([1-r_{S-1}]b_{S-1} + w_{S-1}e_j n_{S-1} - \psi_{S-1}(b_{S-1},e_j|r_{S-1},w_{S-1})\Bigr) + ...\\
\beta\sum_{k=1}^J \pi_{j,k}\Bigl[V_S\Bigl(\psi_{S-1}(b_{S-1},e_j|r_{S-1},w_{S-1}),e_k|r_S,w_S\Bigr)\Bigr]
\end{split} \tag{6.18}$$

One works backward, solving for the lifetime policy functions and value functions of the household, in each step using the previous step's policy function and value function to solve the current period's policy function and value function. The general equation for the age-s

[3] Furthermore, the derivative of the two-periods ahead policy function ψ_S with respect to b_S is also true. And although this is only true in the age-$S-1$ problem, those non-zero derivatives cancel out for the $s < S-1$ problems by the envelope condition.

policy function and value function are the following.

$$
\begin{aligned}
&b_{s+1,t+1} = \psi_s\left(b_{s,t}, e_{j,t}|r_t, w_t\right) : \\
&\Bigl[(1+r_t)b_{s,t} + w_t e_{j,t} n_s - b_{s+1,t+1}\Bigr]^{-\sigma} = ... \\
&\beta(1+r_{t+1})\sum_{k=1}^{J}\left(\pi_{j,k}\Bigl[(1+r_{t+1})b_{s+1,t+1} + w_{t+1}e_{k,t+1}n_{s+1} - \psi_{s+1}(b_{s+1,t+1}, e_{k,t+1}|r_{t+1}, w_{t+1})\Bigr]^{-\sigma}\right)
\end{aligned} \tag{6.19}
$$

$$
\begin{aligned}
V_s\bigl(b_{s,t}, e_{j,t}|r_t, w_t\bigr) = u\Bigl(&[1-r_t]b_{s,t} + w_t e_{j,t} n_s - \psi_s(b_{s,t}, e_{j,t}|r_t, w_t)\Bigr) + ... \\
&\beta\sum_{k=1}^{J}\pi_{j,k}\Bigl[V_{s+1}\Bigl(\psi_s(b_{s,t}, e_{j,t}|r_t, w_t), e_{k,t+1}|r_{t+1}, w_{t+1}\Bigr)\Bigr]
\end{aligned} \tag{6.20}
$$

In solving for the policy functions $b_{s+1,t+1} = \psi_s(b_{s,t}, e_{j,t}|r_t, w_t)$ by backward induction, one difficulty is choosing a continuous value of $b_{s+1,t+1}$ for each discrete combination $(b_{s,t}, e_{j,t})$ in the grid $\mathcal{G}$ despite the fact that the right-hand-sides of both (6.19) and (6.20) are discrete in $b_{s+1,t+1}$. The best way to handle this issue is to interpolate the policy function (6.19) in the b_{s+1} dimension and to interpolate the value function (6.20) in the b_{s+1} dimension.

6.5.2 Calculating the distribution of wealth

In this section, we show how to compute the distribution of wealth $\mathbf{\Gamma}_t(b, e, s)$ in the economy given household savings decision rules $\psi_s(b_{s,t}, e_{j,t}|r_t, w_t)$ for all s and t.

6.5.3 Steady-state equilibrium

To compute the steady-state equilibrium described in Definition 6.1, we must first know the values of all the parameters of the model $\{S, J, A, \alpha, \beta, \delta, \sigma\}$, the inelastic labor supply amounts by age $\{n_s\}_{s=1}^S$ from (6.5), the support of ability levels $\boldsymbol{e} \equiv \{e_1, e_2, ...e_J\}$, and the income transition matrix Π. The key object for computing the steady-state equilibrium is to find the steady-state population distribution $\Gamma(s, e, b)$. Note that there are a discrete number of values for s and e. However, we are treating savings levels or wealth amounts $b_{s,t}$ as having continuous support.

The first step is to assume an initial distribution of the population $\Gamma_0(s, e, b)$. To do this, we know that the unconditional population by age in the model is $f(s) = \frac{1}{S}$ for all e_j and $\bar{b}_s$. We also know that the unconditional population distribution by ability is the ergodic distribution $f(e_j) = \bar{\lambda}_j$ for all s and $\bar{b}_s$. What we don't know is the conditional distribution of savings given age s and ability e_j. This distribution is difficult because there are infinitely many values that wealth can take.

In this first step we assume a flexible continuous distribution for savings that is the same for each age s and ability type e_j. Because the steady-state distribution of savings will likely be hump-shaped and can potentially have a fat right tail, we will use a distribution from the generalized beta (GB) family of positive continuous support probability distributions. Figure 6.1 is taken from McDonald and Xu (1995) and shows the hierarchy of distributions that are nested cases of the 5-parameter generalized beta distribution. McDonald (1984) and McDonald et al. (2013) have documented that the members of the generalized beta (GB) family of distributions with three parameters or more fit the U.S. distribution of income much better than the more commonly used two-parameter distributions in the GB family, such as the log normal distribution (LN) or gamma distribution (GA).

In our case, we start by assuming that savings $\bar{b}_s$ is distributed gamma (GA) for each age (s) and ability e_j category. The following equation is the probability density function for the gamma distribution.

$$\begin{aligned} (GA): \quad f(y;a,b) = &\frac{1}{b^a\Gamma(a)}y^{a-1}e^{-\left(\frac{y}{b}\right)} \quad y\in[0,\infty), \quad a,b>0 \\ &\text{and} \quad \Gamma(a)\equiv\int_0^\infty x^{a-1}e^{-x}dx \end{aligned} \tag{6.21}$$

Note that $\Gamma(a)$ in (6.21) is the gamma function, not to be confused with the population distribution over age, ability type, and savings $\Gamma_t(s,e,b)$. Figure 6.2 shows some different possible shapes of the gamma distribution (GA) for different values of the parameters a and b. [*Redo Figure 6.2 with values for a and b.*]

A nice property of the two-parameter gamma distribution (GA) is that it is a nested case of the three-parameter generalized gamma distribution (GG). The probability density

Figure 6.1: Generalized beta family of distributions [taken from McDonald and Xu (1995, Fig. 2)]

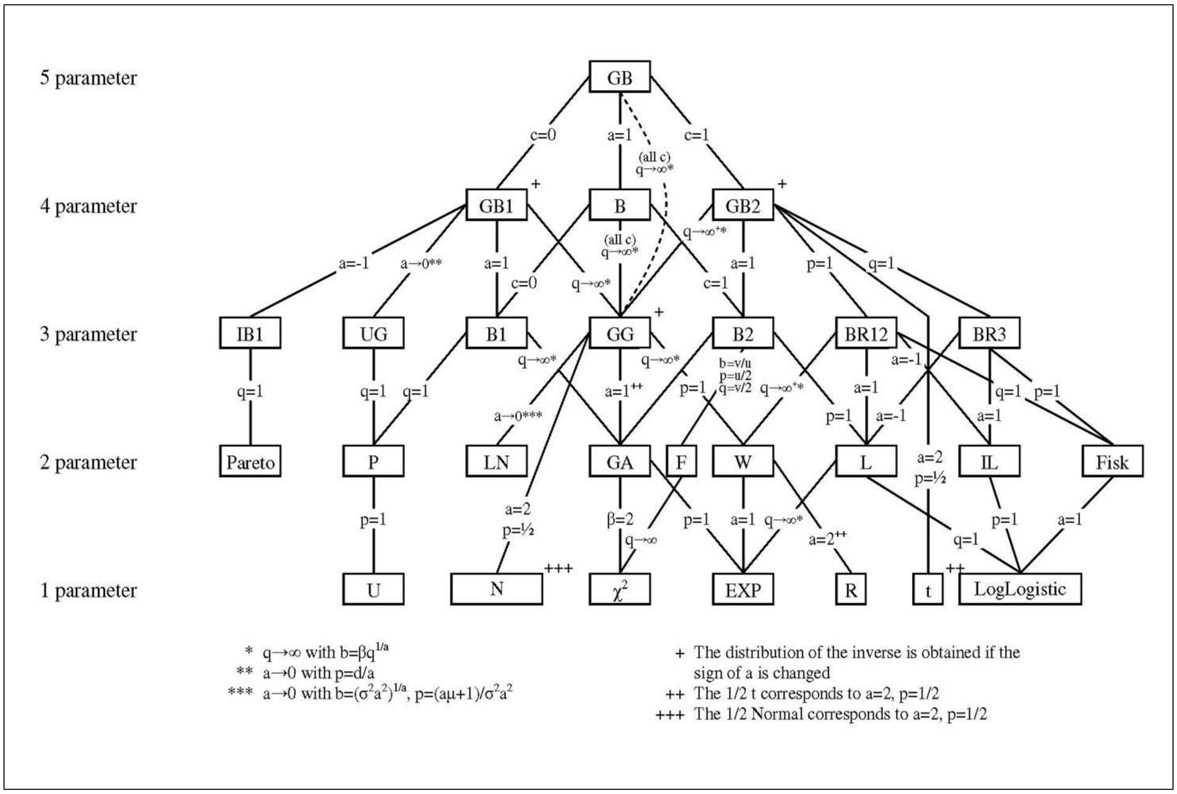
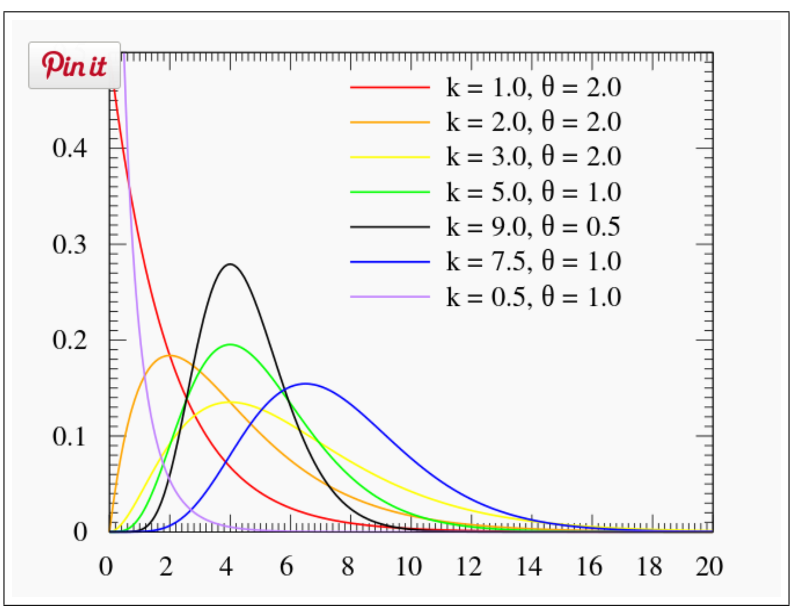


Figure 6.2: Shapes of the gamma (GA) distribution



function for the generalized gamma distribution (GG) is the following.

$$(GG): \quad f(y;a,b,m) = \frac{m}{b^a\Gamma\left(\frac{a}{m}\right)} y^{a-1}e^{-\left(\frac{y}{b}\right)^m} \quad y\in[0,\infty), \quad a,b,m>0 \tag{6.22}$$

The gamma (GA) distribution in (6.21) is a nested case of the generalized gamma (GG) distribution in (6.22) where $m=1$.

$$GA(y;a,b) = GG(y;a,b,m=1) \tag{6.23}$$

If we assume that savings in the steady-state is distributed gamma (GA) $f(b_s;a_0,b_0|s,e_j)$, which is really a specific version of the generalized gamma (GG) $f(b_s;a_0,b_0,m_0=1|s,e_j)$, then the implied initial guess for the steady-state population distribution $\mathbf{\Gamma}_0$ is the following,

$$\mathbf{\Gamma}_0\left(s,e_j,b_s\right) = \frac{\bar{\lambda}_j}{S} f(b_s;a_0,b_0,m_0=1|s,e_j) \quad \forall s,j,b_s \tag{6.24}$$

This initial guess for the population distribution $\mathbf{\Gamma}_0\left(s,e_j,b_s\right)$ implies steady-state values for the real wage $\bar{w}_0$ and the real interest rate $\bar{r}_0$

Because the distribution of savings b_s is continuous for every age s and ability type j, we must simulate that distribution by drawing a large number N of individuals from each distribution for each given ability type j. For each of these $N\times J$ individuals, we simulate a lifetime path of ability levels $e^n_{j,s}$, where n represents the particular individual from the entire sample of $N\times J$ individuals. These lifetime ability levels evolve according to the Markov transition matrix $\mathbf{\Pi}$ from (6.1). We do not need a sample for each age s because we will simulate the entire lifetime decisions of each of the N individuals for each ability type. This will give us their policy functions $\bar{b}_{s+1} = \psi\big(s,e_j,\bar{b}_s|\Omega\big)$.

In order to compute the policy functions, we must discretize the support of b_s in order to solve the policy functions by interpolation. We simply choose $B=100$ points between lower bound $b_{min}=-1$ and $b_{max}=10$. For some s and j, some values b may imply violations of the constraint that $c_s>0$ for all s, so the optimizer must appropriately disregard these points in the savings support b. The three values characterizing parameters of the support of b (B, b_{min}, and b_{max}) might need to be adjusted in order to ensure the accuracy of the approximated solutions to the continuous policy functions.

With the computed policy functions, the known initial wealth of each individual $b_1 = 0$ for all n, we can generate new implied values of the real wage $\bar{w}_{0'}$ and the real interest rate $\bar{r}_{0'}$ from the simulated population. We take our new guess of the wage rate $\bar{w}_1$ and interest rate $\bar{r}_1$ to be a convex combination of $(\bar{w}_0, \bar{r}_0)$ and $(\bar{w}_{0'}, \bar{r}_{0'})$ governed by the parameter $xi \in (0, 1]$,

$$\big(\bar{w}_{i+1}, \bar{r}_{i+1}\big) = \xi\big(\bar{w}_{i'}, \bar{r}_{i'}\big) + (1-\xi)\big(\bar{w}_i, \bar{r}_i\big) \tag{6.25}$$

where i represents the iteration number and ξ represents the weight placed on the new realization of the wage and interest rate $(\bar{w}_{i'}, \bar{r}_{i'})$. This process is repeated until the prices implied by the initial guess are arbitrarily close to equal.

Let Let $\|\cdot\|$ be a norm on the space of price vectors (w, r). Then the fixed point necessary for the equilibrium transition path from Definition 6.1 has been found when the distance between $(\bar{w}_{i'}, \bar{r}_{i'})$ and $(\bar{w}_i, \bar{r}_i)$ is arbitrarily close to zero.

$$\|(\bar{w}_{i'}, \bar{r}_{i'}) - (\bar{w}_i, \bar{r}_i)\| < \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{6.26}$$

Once the steady-state distribution is found, we estimate the three-parameter generalized gamma (GG) distributions $f(\bar{b}_s; a_{j,s}, b_{j,s}|s, e_j)$ for each age s and ability j. This gives us the steady-state population distribution $\bar{\mathbf{\Gamma}}$.

6.5.4 Time path iteration (TPI)

asdf

6.6 Exercises

Exercise 6.1. Solve for steady-state and time path for the following simple problem with $S = 3$ and $J = 2$, with the following transition matrix Π,

$$\Pi = \begin{bmatrix} \pi_{1,1} & \pi_{1,2} \\ \pi_{2,1} & \pi_{2,2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

with possible abilities being $e_{j,t} \in \{e_1, e_2\} = \{0.8, 1.2\}$.

Exercise 6.2. Solve for steady-state and time path for the following large problem with $S = 80$ and $J = 7$, with the following transition matrix Π,

$$\Pi = \begin{bmatrix} \pi_{1,1} & \pi_{1,2} & \dots & \pi_{1,7} \\ \pi_{2,1} & \pi_{2,2} & \dots & \pi_{2,7} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{7,1} & \pi_{7,2} & \dots & \pi_{7,7} \end{bmatrix} = \begin{bmatrix} 0.40 & 0.30 & 0.24 & 0.18 & 0.14 & 0.10 & 0.06 \\ 0.30 & 0.40 & 0.30 & 0.24 & 0.18 & 0.14 & 0.10 \\ 0.24 & 0.30 & 0.40 & 0.30 & 0.24 & 0.18 & 0.14 \\ 0.18 & 0.24 & 0.30 & 0.40 & 0.30 & 0.24 & 0.18 \\ 0.14 & 0.18 & 0.24 & 0.30 & 0.40 & 0.30 & 0.24 \\ 0.10 & 0.14 & 0.18 & 0.24 & 0.30 & 0.40 & 0.30 \\ 0.06 & 0.10 & 0.14 & 0.18 & 0.24 & 0.30 & 0.40 \end{bmatrix}$$

with possible abilities being $e_{j,t} \in \{e_1, e_2, ...e_7\} = \{0.6, 0.8, 1.0, 1.3, 1.6, 2.0, 2.5\}$.

Part IV

Population Dynamics, Productivity Growth, and Stationarization

Chapter 7

S-period-lived Agent Problem with Exogenous Labor Supply, Demographic Dynamics, and Productivity Growth

In this chapter, we extend the S-period-lived agent model with exogenous labor supply from Chapter 3 to include a realistic demographic transition process as well as productivity growth. Both of these sources of growth will render the model nonstationary. In order to solve the model, we will have to be careful to correctly stationarize all of the characterizing equations.

The addition of mortality rates to the model adds some uncertainty to the household's expectation as to whether it will survive to the next period. This creates a situation in which some fraction of age-s households will die every period and leave an unintended bequest. For this reason, we will incorporate some of the modeling structures from Chapter 8 into this chapter. However, for simplicity, we will not incorporate the warm-glow bequest motive of Chapter 8.

Nishiyama (2015) and DeBacker et al. (2017) are examples of overlapping generations papers that carefully model the demographics of the households in the respective models.

7.1 Population Dynamics

We define $\omega_{s,t}$ as the number of households of age s alive at time t. A measure $\omega_{1,t}$ of households is born in each period t and live for up to $E+S$ periods, with $S \geq 4$.[1] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$. We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned}
\omega_{1,t+1} &= (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t} + i_1\omega_{1,t} \quad \forall t \\
\omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} + i_{s+1}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1
\end{aligned} \tag{7.1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific net immigration rate, ρ_s is an age-specific mortality hazard rate, and ρ_0 is an infant mortality rate.[2] The total population in the economy N_t at any period is simply the sum of households in the economy, the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S}\omega_{s,t} \quad \forall t \tag{7.2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{7.3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S}\omega_{s,t} \quad \forall t \tag{7.4}$$

[1]Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), it is convenient for S to be at least 4.

[2]The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{7.5}$$

We discuss the approach to estimating fertility rates f_s, mortality rates ρ_s, and immigration rates i_s in Section 7.7.

7.2 Households

A measure $\omega_{1,t}$ of identical individuals are born each period, become economically relevant at age $s = E+1$ if they survive to that age, and live for up to $E+S$ periods (S economically active periods), with the population of age-s individuals in period t being $\omega_{s,t}$. Let the age of an individual be indexed by $s = \{1, 2, ...E+S\}$. An age-s individual faces a similar per-period budget constraint to that of (8.1) in the simple bequests chapter with the exception of a bequests term as the last term on the left-hand-side,

$$c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_s + \frac{BQ_t}{\tilde{N}_t} \quad \forall t \quad \text{and} \quad s \geq E+1 \tag{7.6}$$

where BQ_t represents total accidental bequests available in period t from households who died in period $t-1$. Dividing by the total economically relevant population $\tilde{N}_t$ implies that total bequests are equally distributed across the population. We will relax this assumption in Chapter 8. Total bequests are characterized by the following equation.

$$BQ_t = (1+r_t)\sum_{s=E+2}^{E+S} \rho_{s-1}\omega_{s-1,t-1}b_{s,t}\forall t \tag{7.7}$$

As in Chapter 3, we assume the individuals supply a unit of labor inelastically in the first two thirds of working life ($s \leq E + \text{round}(2S/3)$) and are retired during the last third of life ($s > E + \text{round}(2S/3)$).

$$n_{s,t} = \begin{cases} 1 & \text{if} \quad E+1 \leq s \leq E + \text{round}\left(\frac{2S}{3}\right) \\ 0.2 & \text{if} \quad s > E + \text{round}\left(\frac{2S}{3}\right) \end{cases} \quad \forall t \tag{7.8}$$

Because exogenous labor in (7.8) is not dependent on the time period, we drop the t subscript from labor n_s for the rest of this section. We also assume that households are born with no wealth from savings $b_{E+1,t} = 0$ and have no incentive to save anything in the last period of life $b_{E+S+1,t} = 0$ for all periods t. Assume that $c_{s,t} \geq 0$ because negative consumption neither has an intuitive interpretation nor is it household utility defined for it. It is the latter condition that will make $c_{s,t} > 0$ in equilibrium.

Because a fraction ρ_s of all age-s households dies every period t, we must account for where these households' savings $b_{s+1,t+1}$ go. We have already shown the the equation (7.7) for total bequests and in the household budget constraint (7.6) that these savings get bequeathed to the next generation. Because we add no warm-glow bequest motive as in Chapter 8, households simply save the amount they would otherwise save, and the fraction of that savings belonging to households that deceased becomes part of total bequests as in (7.7).

Let the utility of consumption in each period be defined by the constant relative risk aversion function (2.6) $u(c_{s,t})$, such that $u' > 0$, $u'' < 0$, and $\lim_{c\rightarrow 0} u(c) = -\infty$. Individuals choose lifetime consumption $\{c_{s,t+s-1}\}_{s=E+1}^{E+S}$, savings $\{b_{s+1,t+s}\}_{s=E+1}^{E+S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$\begin{aligned}
\max_{\{c_{s,t+v}\}_{s=E+1}^{E+S},\{b_{s+1,t+v+1}\}_{s=E+1}^{E+S-1}} \quad & \sum_{s=E+1}^{E+S} \beta^{s-E-1}\left[\Pi_{u=E}^{s-1}(1-\rho_u)\right] u(c_{s,t+s-E-1}) \quad \forall s,t \\
\text{s.t.} \quad c_{s,t} = (1+r_t)b_{s,t} + w_t n_s + \frac{BQ_t}{\tilde{N}_t} - b_{s+1,t+1} \quad \forall s,t \\
\text{and} \quad b_{E+1,t}, b_{E+S+1,t} = 0 \quad \forall t \quad \text{and} \quad c_{s,t} \geq 0 \quad \forall s,t
\end{aligned} \tag{7.9}$$

The product term in brackets $\Pi_{u=E}^{s-1}(1-\rho_u)$ multiplies each future period utility function by the cumulative probability of surviving to that period. For example, the third term in that sum $\beta^2(1-\rho_{E+1})(1-\rho_{E+2})u(c_{E+3,t+2})$ represents the discounted and mortality risk adjusted value of consumption utility in period $t+2$.

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (7.9). The optimal choice of how much to save in the each of the first $S-1$ periods of life $b_{s+1,t+1}$ is found by taking the derivative of the lifetime utility function with respect to each of the lifetime savings amounts $\{b_{s+1,t+s+1}\}_{s=E+1}^{E+S-1}$ and setting the derivatives equal to zero.

Similar to Chapter 3, the $S-1$ lifetime savings decisions $b_{s+1,t+1}$ are characterized by the following $S-1$ nonstationary dynamic Euler equations,

$$\begin{aligned} u'(c_{s,t}) = \beta(1+r_{t+1})(1-\rho_s)u'(c_{s+1,t+1}) \quad \forall t, \quad \text{and} \quad E+1 \leq s \leq S-1 \\ \text{and} \quad b_{E+1,t}, b_{E+S+1,t} = 0 \quad \forall t \end{aligned} \tag{7.10}$$

and the S consumption decisions are directly implied by the S budget constraints over the household's lifetime (7.6). The difference in the Euler equations (7.10) is the presence of the extra discounting due to the mortality risk ρ_s—the risk that someone alive at age-s will die at the end of that period and not be alive for age-$s+1$. The policy functions for each of the savings decisions is a function of the individual's wealth at the beginning of the period $b_{s,t}$ and the time path of wages and interest rates over the remaining periods of the individual's life.

$$b_{s+1,t+1} = \psi_s\Bigl(b_{s,t}, \{r_v\}_{u=t}^{t+E+S-s}, \{w_u\}_{u=t}^{t+E+S-s}\Bigr) \quad \forall t \quad \text{and} \quad E+1 \leq s \leq E+S-1 \tag{7.11}$$

To summarize the individual's problem, if one knows his initial savings or wealth $b_{s,t}$ and the time path of factor prices over his remaining lifetime, he can solve for all of his optimal savings levels $\{b_{s+1,t+s-E}\}_{s=E+1}^{E+S-1}$.

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates and wages $\{r_u, w_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Section 7.5, the equilibrium interest rate r_t and wage w_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \bigl\{b_{s,t}\bigr\}_{s=E+2}^{E+S} \quad \forall t \tag{7.12}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[3]

7.3 Firms

The production side of this economy is similar to the one in Chapter 2.2 with a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. A difference here is that we assume that the productivity of labor is growing at a constant rate g_y (labor augmenting technological change). Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology,

$$Y_t = F(K_t, L_t) \equiv AK_t^\alpha\big(e^{g_y t}L_t\big)^{1-\alpha} \quad \text{where} \quad \forall t \quad \alpha\in(0,1) \quad \text{and} \quad A>0 \tag{7.13}$$

The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t,L_t}\; AK_t^\alpha\big(e^{g_y t}L_t\big)^{1-\alpha} - (r_t+\delta)K_t - w_tL_t \tag{7.14}$$

where $\delta\in[0,1]$ is the rate of capital depreciation, and the two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha\left(\frac{Y_t}{K_t}\right) - \delta \tag{7.15}$$

$$w_t = (1-\alpha)\left(\frac{Y_t}{L_t}\right) \tag{7.16}$$

[3]In Section 7.5 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 7.2.

7.4 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. We will repeat the equation for total bequests (7.7) here, because it is similar to a market clearing condition in that it equates total bequests BQ_t with the accidental individual bequests that comprise it. Each of these equations amounts to a statement of supply equals demand. But in their adding up, the market clearing conditions must now account for the population of all the labor supplied, savings invested, and goods consumed. Furthermore, we must account for the fact that immigrants are bringing capital with them in to the country every period.

$$L_t = \sum_{s=E+1}^{E+S} \omega_{s,t} n_s \tag{7.17}$$

$$K_t = \sum_{s=E+2}^{E+S} \Bigl(\omega_{s-1,t-1} b_{s,t} + i_s \omega_{s,t-1} b_{s,t}\Bigr) \tag{7.18}$$

$$\begin{split} Y_t = C_t + I_t - \sum_{s=E+2}^{E+S} i_s \omega_{s,t} b_{s,t+1} \\ \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \\ \text{and} \quad C_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} c_{s,t} \end{split} \tag{7.19}$$

$$BQ_t = (1+r_t) \sum_{s=E+2}^{E+S} \rho_{s-1} \omega_{s-1,t-1} b_{s,t} \forall t \tag{7.7}$$

The capital market clearing equation (7.18) includes the capital saved by the previous period's agents adjusted by the inflow of capital from immigrants ($i_s > 0$) or the outflow of capital from immigrants ($i_s < 0$). This specification has two main implications. First, we are assuming that immigrants of age-s have the same wealth $b_{s,t}$ as their domestic counterparts. This assumption greatly simplifies the state vector of the model. This specification also implies that capital investment or savings goes into the production process of the country of destination for agents either moving in or moving out. That is, capital taken out of the country by emigrants becomes productive in the foreign country's gross domestic product, and capital brought into the country by immigrants becomes productive in the domestic country's gross domestic product Y_t.

Note that the last term in the goods market clearing condition (7.19) represents the capital account portion of net exports. The only international transaction included in this model is capital imports or exports through immigration. Another way to think about the last term in (7.19) is that we must subtract out the part of K_{t+1} that is coming from immigrants. The goods market clearing equation (7.19) is redundant by Walras' Law.

7.5 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points

a. Households optimize according to equations (7.10).

b. Firms optimize according to (7.15) and (7.16).

c. Markets clear according to (7.17) and (7.18).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

Table 7.1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{s,t} \equiv \frac{c_{s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	n_s
$\hat{b}_{s,t} \equiv \frac{b_{s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{C}_t \equiv \frac{C_t}{e^{g_y t}\tilde{N}_t}$	
		$\hat{BQ}_t \equiv \frac{BQ_t}{e^{g_y t}\tilde{N}_t}$	

[a] The interest rate r_t in (7.15) is already stationary because Y_t and K_t grow at the same rate. Household labor supply n_s are exogenous, constant, and therefore stationary.

Because the variables in the equations that will characterize the equilibrium are all growing due to the labor augmenting technological change g_y in (7.13) and because of the population growth $\tilde{g}_{n,t}$ specified in (7.5), we have to stationarize the model. Table 7.1 characterizes the stationary versions of the variables of the model in terms of the variables that grow because of labor augmenting technological change, population growth, both, or none. With the definitions in Table 7.1, we must stationarize all the characterizing equations of equilibrium. The equations (7.6), (7.10), (7.15), (7.16), (7.17), (7.18), and (7.7) characterize the equilibrium, but we must implement a change of variables such that all the growth components are removed.

We start with the market clearing conditions. Aggregate labor (7.17) is a function of population weights $\omega_{s,t}$, which are growing at the population growth rate, and stationary exogenous labor supply n_s. We can stationarize the aggregate labor market clearing condition by dividing both sides of (7.17) by the total economically relevant population in period t, $\tilde{N}_t$.

$$\hat{L}_t = \sum_{s=E+1}^{E+S} \hat{\omega}_{s,t} n_s \tag{7.20}$$

The capital market clearing equation (7.18) and total bequests equations can be stationarized by dividing both sides of the respective equations by $e^{g_y t}\tilde{N}_t$.

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}} \sum_{s=E+2}^{E+S} \Big(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t}\Big) \quad \forall t \tag{7.21}$$

$$\hat{BQ}_t = \left(\frac{1+r_t}{1+\tilde{g}_{n,t}}\right) \sum_{s=E+2}^{E+S} \rho_{s-1}\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} \quad \forall t \tag{7.22}$$

Because the goods market clearing equation ends up being relatively complicated to stationarize and it is an important check whether the model is solving correctly, we show its stationary version here despite its being redundant by Walras' Law.

$$\begin{aligned}
\hat{Y}_t &= \hat{C}_t + \hat{I}_t - e^{g_y}\sum_{s=E+2}^{E+S} i_s\hat{\omega}_{s,t}\hat{b}_{s,t+1} \\
&\text{where}\quad \hat{I}_t \equiv e^{g_y}(1+\tilde{g}_{n,t+1})\hat{K}_{t+1} - (1-\delta)\hat{K}_t \\
&\text{and}\quad \hat{C}_t \equiv \sum_{s=E+1}^{E+S}\hat{\omega}_{s,t}\hat{c}_{s,t}
\end{aligned} \tag{7.23}$$

The price levels in the model are functions of aggregate labor L_t and the aggregate capital stock K_t through the production function Y_t. We get a stationary version of the production function by dividing both sides of (7.13) by $e^{g_y t}\tilde{N}_t$.

$$\hat{Y}_t = A\hat{K}_t^\alpha\hat{L}_t^{1-\alpha} \quad\text{where}\quad \alpha\in(0,1) \quad\text{and}\quad A>0 \tag{7.24}$$

Note that there is no growth rate term in (7.24). The interest rate (7.15) is already stationary because aggregate output Y_t and the aggregate capital stock K_t grow at the same rate and appear in the function as a ratio.

$$r_t = \alpha\left(\frac{Y_t}{K_t}\right) - \delta = \alpha\left(\frac{\hat{Y}_t}{\hat{K}_t}\right) - \delta = \alpha A\left(\frac{\hat{L}_t}{\hat{K}_t}\right)^{1-\alpha} - \delta \tag{7.25}$$

The equilibrium wage (7.16), on the other hand, is a function of the ratio of aggregate output Y_t that grows with the population and with productivity and of aggregate labor L_t that grows only with the population. For this reason, we can see that the equilibrium wage grows only at the productivity growth rate because the population growth rates cancel out in the ratio of Y_t and L_t.

$$\hat{w}_t = (1-\alpha)\left(\frac{\hat{Y}_t}{\hat{L}_t}\right) = (1-\alpha)A\left(\frac{\hat{K}_t}{\hat{L}_t}\right)^\alpha \tag{7.26}$$

With these definitions, it can be shown that the Euler equations characterizing the equilibrium (7.10) can be written in stationary form by dividing both sides of the equation by

$e^{-\sigma g_y t}$ and $e^{-\sigma g_y (t+1)}$,

$$
\begin{aligned}
&u'(\hat{c}_{s,t}) = e^{-\sigma g_y}\beta(1+r_{t+1})(1-\rho_s)u'(\hat{c}_{s+1,t+1}) \quad \forall t, \quad \text{and} \quad E+1 \leq s \leq S-1 \\
&\qquad \text{and} \quad \hat{b}_{E+1,t}, \hat{b}_{E+S+1,t} = 0 \quad \forall t
\end{aligned} \tag{7.27}
$$

and the stationarized budget constraint is,

$$
\hat{c}_{s,t} = (1+r_t)\hat{b}_{s,t} + \hat{w}_t n_s + B\hat{Q}_t - e^{g_y}\hat{b}_{s+1,t+1} \quad \forall s,t \tag{7.28}
$$

The easiest way to understand the equilibrium solution is to substitute the stationary market clearing conditions (7.20) and (7.21) into the firm's stationary first order conditions (7.25) and (7.26) which characterize the equilibrium wage and interest rate as functions of the distribution of capital $\hat{\mathbf{\Gamma}}_t$.

$$
\hat{w}\big(\hat{\mathbf{\Gamma}}_t\big): \quad \hat{w}_t = (1-\alpha)A\left[\frac{\frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S}\big(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t}\big)}{\sum_{s=E+1}^{E+S}\hat{\omega}_{s,t}n_s}\right]^{\alpha} \quad \forall t \tag{7.29}
$$

$$
r\big(\hat{\mathbf{\Gamma}}_t\big): \quad r_t = \alpha A\left[\frac{\sum_{s=E+1}^{E+S}\hat{\omega}_{s,t}n_s}{\frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S}\big(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t}\big)}\right]^{1-\alpha} - \delta \quad \forall t \tag{7.30}
$$

Now (7.29) and (7.30) can be substituted into household stationary Euler equations (7.27) to get the following $(S-1)$-equation system that completely characterizes the equilibrium.

$$
\begin{aligned}
&\left(\Big[1+r\big(\hat{\mathbf{\Gamma}}_t\big)\Big]\hat{b}_{s,t} + \hat{w}\big(\hat{\mathbf{\Gamma}}_t\big)n_s + B\hat{Q}_t\big(\hat{\mathbf{\Gamma}}_t\big) - e^{g_y}\hat{b}_{s+1,t+1}\right)^{-\sigma} = \\
&e^{-\sigma g_y}\beta\Big[1+r\big(\hat{\mathbf{\Gamma}}_{t+1}\big)\Big](1-\rho_s)\left(\Big[1+r\big(\hat{\mathbf{\Gamma}}_{t+1}\big)\Big]\hat{b}_{s+1,t+1} + \hat{w}\big(\hat{\mathbf{\Gamma}}_{t+1}\big)n_{s+1} + B\hat{Q}_{t+1}\big(\hat{\mathbf{\Gamma}}_{t+1}\big) - e^{g_y}\hat{b}_{s+2,t+2}\right)^{-\sigma} \\
&\qquad \forall t, \quad \text{and} \quad E+1 \leq s \leq E+S-1
\end{aligned} \tag{7.31}
$$

The system of $S-1$ nonlinear dynamic equations (7.31) characterizing the the stationary lifetime savings decisions for each household $\{\hat{b}_{s+1,t+s}\}_{s=E+1}^{E+S-1}$ is not identified. Each individual knows the current distribution of stationary capital $\hat{\mathbf{\Gamma}}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\hat{\mathbf{\Gamma}}_{t+1} = \{\hat{b}_{s+1,t+1}\}_{s=E+1}^{E+S-1}$ for all agents alive next period, and for a policy function for the individual $\hat{b}_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of stationary endogenous variable $\hat{x}_t$ be characterized by $\hat{x}_{t+1} = \hat{x}_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the stationary steady-state equilibrium as follows.

Definition 7.1 (**Stationary steady-state equilibrium**)**.** A stationary non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents, exogenous labor supply, productivity growth, and population dynamics is defined as constant allocations of stationary consumption $\{\bar{c}_s\}_{s=E+1}^{E+S}$, savings $\{\bar{b}_s\}_{s=E+2}^{E+S}$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (7.27),

ii. firms optimize according to (7.25) and (7.26),

iii. markets clear according to (7.20), (7.21), and (7.22),

iv. The population has reached its stationary steady-state distribution $\{\bar{\omega}_s\}_{s=E+1}^{E+S}$ and steady-state population growth rate $\bar{g}_n$ as characterized in Section 7.7.5.

The characterizing equations in Definition 7.1 reduce to the system (7.31) in which all the variables are their steady-state versions. These $S-1$ equations are exactly identified in the steady state. That is, they are $S-1$ equations and $S-1$ unknowns $\{\bar{b}_s\}_{s=E+2}^{E+S}$.

$$\begin{split} &\left(\Big[1+r\big(\bar{\mathbf{\Gamma}}\big)\Big]\bar{b}_s + \hat{w}\big(\bar{\mathbf{\Gamma}}\big)n_s + \bar{BQ}\big(\bar{\mathbf{\Gamma}}\big) - e^{g_y}\bar{b}_{s+1}\right)^{-\sigma} = \\ &e^{-\sigma g_y}\beta\Big[1+r\big(\bar{\mathbf{\Gamma}}\big)\Big](1-\rho_s)\left(\Big[1+r\big(\bar{\mathbf{\Gamma}}\big)\Big]\bar{b}_{s+1} + \hat{w}\big(\bar{\mathbf{\Gamma}}\big)n_{s+1} + \bar{BQ}\big(\bar{\mathbf{\Gamma}}\big) - e^{g_y}\bar{b}_{s+2}\right)^{-\sigma} \\ &\qquad\qquad \text{for} \quad E+1 \leq s \leq E+S-1 \end{split} \tag{7.32}$$

We can solve for steady-state $\{\bar{b}_s\}_{s=E+2}^{E+S}$ by using an unconstrained root finder as we did in Chapter 3. Then we solve for $\bar{w}$, $\bar{r}$, and $\{\bar{c}_s\}_{s=E+1}^{E+S}$ by substituting $\{\bar{b}_s\}_{s=E+2}^{E+S}$ into the equilibrium firm first order conditions (7.29) and (7.30) and the substituting $\{\bar{b}_s\}_{s=E+2}^{E+S}$, $\bar{w}$, and $\bar{r}$ into the stationary household budget constraints (7.28).

Now we can describe the stationary non-steady-state functional equilibrium of the model in which each endogenous variable chosen in period t is a function of the state vector $\hat{\mathbf{\Gamma}}_t$, which is the distribution of stationary capital at time t.

Definition 7.2 (Stationary non-steady-state functional equilibrium). A stationary non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents, exogenous labor supply, productivity growth, and population dynamics is defined as stationary allocation functions of the state $\left\{\hat{b}_{s+1,t+1} = \psi_s\big(\hat{\mathbf{\Gamma}}_t\big)\right\}_{s=E+1}^{E+S-1}$ and stationary price functions $\hat{w}(\hat{\mathbf{\Gamma}}_t)$ and $r(\hat{\mathbf{\Gamma}}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of stationary savings equal the realized outcome (rational expectations),

$$\hat{\mathbf{\Gamma}}_{t+u} = \hat{\mathbf{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\mathbf{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (7.27),

iii. firms optimize according to (7.25) and (7.26),

iv. markets clear according to (7.20), (7.21), and (7.22).

We have already shown how to reduce the characterizing equations in Definition 7.2 to $S-1$ equations (7.31) and $S-1$ unknowns. But we have also seen that those $S-1$ equations are not identified. So how do we solve for these equilibrium functions? The solution to the non-steady-state equilibrium in Definition 7.2 is a fixed point in function space. Choose $S-1$ functions $\{\psi_s\}_{s=E+1}^{E+S-1}$ and verify that they satisfy the Euler equations for all points in the state space (all possible values of the state).

[TODO: Finish these figures and results.] Figure **??** shows the steady-state distributions of consumption $\bar{c}_s$ and savings $\bar{b}_{s+1}$. The steady-state capital stock for this calibration is $\bar{K}$ =?, and the steady-state interest rate and wage are $\bar{r}$ =? and $\bar{w}$ =?, respectively. Table **??** lists the steady-state values of the aggregate variables as well as the characterizing equation errors of our solution as evidence that we have found the steady-state equilibrium.

7.6 Solution method: time path iteration (TPI)

The solution method is time path iteration (TPI). The key assumption is that the economy will reach the steady-state equilibrium $\bar{\mathbf{\Gamma}}$ described in Definition 7.1 in a finite number of periods $T < \infty$ regardless of the initial state $\hat{\mathbf{\Gamma}}_1$.

In Chapter 3, we only had to guess a time path for the aggregate capital stock $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, which then determined the respective time paths of prices $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. However, in this model, these time paths are not enough to be able to solve all the households' respective problems. We must also guess the time path for total bequests $\hat{\boldsymbol{BQ}} = \left\{\hat{BQ}_1^i, \hat{BQ}_2^i, ...\hat{BQ}_T^i\right\}$.

The first step is to assume a transition path for stationary aggregate capital $\hat{\boldsymbol{K}}^i$ and total bequests $\hat{\boldsymbol{BQ}}^i$ such that T is sufficiently large to ensure that $\hat{\mathbf{\Gamma}}_T = \bar{\mathbf{\Gamma}}$. The superscript i is an index for the iteration number. The exogenously supplied population demographics $\hat{\omega}_{s,t}$ reach a steady-state in a known number of periods (see Section 7.7.5). T must be greater than that number of periods.

The transition path for stationary aggregate capital determines the transition path for both the stationary wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$ and the interest rate $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $\hat{\mathbf{\Gamma}}_1$ can be arbitrarily chosen as long as it satisfies $\hat{K}_1^i = \frac{1}{1+\tilde{g}_{n,1}}\sum_{s=E+2}^{E+S}\hat{\omega}_{s-1,0}\hat{b}_{s,1}$ according to market clearing condition (7.21) and satisfies $BQ_1^i = \left(\frac{1+r_t^i}{1+\tilde{g}_{n,1}}\right)\sum_{s=E+2}^{E+S}\rho_{s-1}\hat{\omega}_{s-1,0}\hat{b}_{s,1}$ according to market clearing condition (7.28). Note that both of these conditions require knowing the population growth rate from the period before the initial period to the initial period $\tilde{g}_{n,1}$ as well as the stationary population distribution from the period before the initial period $\{\hat{\omega}_{s,0}\}_{s=E+1}^{E+S-1}$. One could also first choose the initial stationary distribution of capital $\hat{\mathbf{\Gamma}}_1$ and then choose an initial stationary aggregate capital stock $\hat{K}_1^i$ and total bequests $\hat{BQ}_1^i$ that corresponds to that distribution. The only other restrictions on the initial transition paths for aggregate capital and total bequests is that they both equal the steady-state level $\hat{K}_T^i = \bar{K} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S}\left(\bar{\omega}_{s-1}\bar{b}_s + i_s\bar{\omega}_s\bar{b}_s\right)$ by period T and $\hat{BQ}_T^i = \left(\frac{1+\bar{r}}{1+\bar{g}_n}\right)\sum_{s=E+2}^{E+S}\rho_{s-1}\bar{\omega}_{s-1}\bar{b}_s$. But the initial guesses for the aggregate capital stocks $\hat{K}_t^i$ and total bequests $\hat{BQ}_t^i$ for periods $1 < t < T$ can be any level.

Given the initial capital distribution $\hat{\mathbf{\Gamma}}_1$ and the transition paths of aggregate capital $\hat{\boldsymbol{K}}^i = \left\{\hat{K}_1^i, \hat{K}_2^i, ...\hat{K}_T^i\right\}$, total bequests $\hat{\boldsymbol{BQ}}^i = \left\{\hat{BQ}_1^i, \hat{BQ}_2^i, ...\hat{BQ}_T^i\right\}$, the wage $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and the interest rate $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal savings decision for the initial age $s = E+S-1$ individual for the last period of his life $\hat{b}_{E+S,2}$

using his last stationary intertemporal Euler equation.

$$\begin{split}\Bigl([1+r_1^i]\hat{b}_{E+S-1,1} + \hat{w}_1^i n_{E+S-1} + \hat{BQ}_1^i - e^{g_y}\hat{b}_{E+S,2}\Bigr)^{-\sigma} = \\ e^{-\sigma g_y}\beta\bigl(1+r_2^i\bigr)\bigl(1-\rho_{E+S-1}\bigr)\Bigl([1+r_2^i]\hat{b}_{E+S,2} + \hat{w}_2^i n_{E+S} + \hat{BQ}_2^i\Bigr)^{-\sigma}\end{split} \tag{7.33}$$

Notice that everything in equation (7.33) is known except for the savings decision $\hat{b}_{E+S,2}$. This is one equation and one unknown.

The next step is to solve for the remaining lifetime savings decisions for the next oldest individual alive in period $t=1$. This individual is age $s=E+S-2$ and has two remaining savings decisions $b_{E+S-1,2}$ and $b_{E+S,3}$. From (7.31), we know that the two equations that characterize these two decisions are the following.

$$\begin{split}\Bigl([1+r_1^i]\hat{b}_{E+S-2,1} + \hat{w}_1^i n_{E+S-2} + \hat{BQ}_1^i - e^{g_y}\hat{b}_{E+S-1,2}\Bigr)^{-\sigma} = \\ e^{-\sigma g_y}\beta(1+r_2^i)(1-\rho_{E+S-2})\Bigl([1+r_2^i]\hat{b}_{E+S-1,2} + \hat{w}_2^i n_{E+S-1} + \hat{BQ}_2^i - e^{g_y}\hat{b}_{E+S,3}\Bigr)^{-\sigma}\end{split} \tag{7.34}$$

$$\begin{split}\Bigl([1+r_2^i]\hat{b}_{E+S-1,2} + \hat{w}_2^i n_{E+S-1} + \hat{BQ}_2^i - e^{g_y}\hat{b}_{E+S,3}\Bigr)^{-\sigma} = \\ e^{-\sigma g_y}\beta(1+r_3^i)(1-\rho_{E+S-1})\Bigl([1+r_3^i]\hat{b}_{E+S,3} + \hat{w}_3^i n_{E+S} + \hat{BQ}_3^i\Bigr)^{-\sigma}\end{split} \tag{7.35}$$

Euler equations (7.34) and (7.35) represent two equations and two unknowns $\hat{b}_{E+S-1,2}$ and $\hat{b}_{E+S,3}$. Everything else is known.

We continue solving the remaining lifetime decisions of each individual alive between periods 1 and T. This includes all the individuals who were already alive in period 1 and therefore have fewer than $S-1$ savings decisions for which to solve. It also includes all the individuals born between periods 1 and T for whom we have the full set of $S-1$ lifetime decisions. Once we have solved for all the individual savings decisions for individuals alive between periods 1 and T, then we have the complete distribution of savings $\{\hat{\mathbf{\Gamma}}_t\}_{t=1}^T$ for each period between 1 and T. We can use this to compute new time paths of the aggregate capital stock and total bequests consistent with the individual savings decisions $\hat{K}_t^{i'} = \frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S}\bigl(\hat{\omega}_{s-1,t-1}\hat{b}_{s,t} + i_s\hat{\omega}_{s,t-1}\hat{b}_{s,t}\bigr)$ and $\hat{BQ}_t^{i'} = \Bigl(\frac{1+r_t}{1+\tilde{g}_{n,t}}\Bigr)\sum_{s=E+2}^{E+S}\rho_{s-1}\hat{\omega}_{s-1,t-1}\hat{b}_{s,t}$ for all $1\leq t\leq T$. We place a "$'$" on the iteration index of these aggregate time series because, in general, $\hat{K}_t^{i'} \neq \hat{K}_t^i$ and $\hat{BQ}_t^{i'} \neq \hat{BQ}_t^{i}$. That is, the initial conjectured paths of the aggregate capital stock and total bequests from which the savings decisions were made is not necessarily equal to the paths of the aggregate capital stock and total bequests consistent with those savings decisions.[4]

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and total bequests. Common norms to use are the L^2 and the L^∞ norms. Then the fixed point necessary for the equilibrium transition path from Definition 7.2 has been found when the distance between $\left(\boldsymbol{\hat{K}}^{i'}, \boldsymbol{\hat{BQ}}^{i'}\right)$ and $\left(\boldsymbol{\hat{K}}^{i}, \boldsymbol{\hat{BQ}}^{i}\right)$ is arbitrarily close to zero.

$$\left\|\left(\boldsymbol{\hat{K}}^{i'}, \boldsymbol{\hat{BQ}}^{i'}\right) - \left(\boldsymbol{\hat{K}}^{i}, \boldsymbol{\hat{BQ}}^{i}\right)\right\| < \varepsilon \quad \text{for} \quad \varepsilon > 0 \tag{7.36}$$

If the fixed point has not been found $\left\|\left(\boldsymbol{\hat{K}}^{i'}, \boldsymbol{\hat{BQ}}^{i'}\right) - \left(\boldsymbol{\hat{K}}^{i}, \boldsymbol{\hat{BQ}}^{i}\right)\right\| > \varepsilon$, then new transition paths for the aggregate capital stock and total bequests are generated as a convex combination of $\left(\boldsymbol{\hat{K}}^{i'}, \boldsymbol{\hat{BQ}}^{i'}\right)$ and $\left(\boldsymbol{\hat{K}}^{i}, \boldsymbol{\hat{BQ}}^{i}\right)$.

$$\left(\boldsymbol{\hat{K}}^{i+1}, \boldsymbol{\hat{BQ}}^{i+1}\right) = \xi\left(\boldsymbol{\hat{K}}^{i'}, \boldsymbol{\hat{BQ}}^{i'}\right) + (1-\xi)\left(\boldsymbol{\hat{K}}^{i}, \boldsymbol{\hat{BQ}}^{i}\right) \quad \text{for} \quad \xi \in (0,1) \tag{7.37}$$

This process is repeated until the initial transition path for the aggregate capital stock is consistent with the transition path implied by those beliefs and household and firm optimization. TPI solves for the equilibrium transition path from Definition 7.2 by finding a fixed point in the time path of the economy.

[TODO: Finish these figures and results.] The five panels of Figure **??** show the equilibrium time paths of the stationary aggregate variables r_t, $\hat{w}_t$, $\hat{K}_t$, $\hat{Y}_t$, and $\hat{C}_t$. Figure **??** shows the equilibrium time path of stationary individual consumption $\hat{c}_{s,t}$ and savings $\hat{b}_{s+1,t+1}$. Table **??** lists the maximum absolute values across the time path of the characterizing equation errors of our solution as evidence that we have found the steady-state equilibrium.

---

[4] A check here for whether T is large enough is if $\hat{K}_T^{i'} = \bar{K}$ and $\hat{BQ}_T^{i'} = \bar{BQ}$ as well as those aggregate variables for the following periods. If not, then T needs to be larger.

7.7 Calibration

In this section, we discuss the calibration of the exogenous parameters of the model, with a special emphasis on estimating the parameters associated with the demographic dynamics. Assume that agents live for $E+S$ total periods, are not economically relevant for the first E periods, are born into economic relevance at age $s = E+1$ and live until a maximum age os $s = E+S$. This implies S economically relevant years. Let $E = 20$ and $S = 80$ such that each economically relevant model period represents a year from ages $s = E+1 = 21$ to $s = E+S = 100$.

The first subsection 7.7.1 discusses the calibration of the non-population related model parameters. Subsections 7.7.2, 7.7.3, and 7.7.4 deal with a general approach for calibrating fertility rates f_s, mortality rates ρ_s, and immigration rate i_s, which are the inputs to solving for the time-path and steady state of of the population distribution $\{\hat{\omega}_{s,t}\}_{s=1}^{E+S}$ and growth rate $\tilde{g}_{n,t}$, which is the topic of Section 7.7.5.

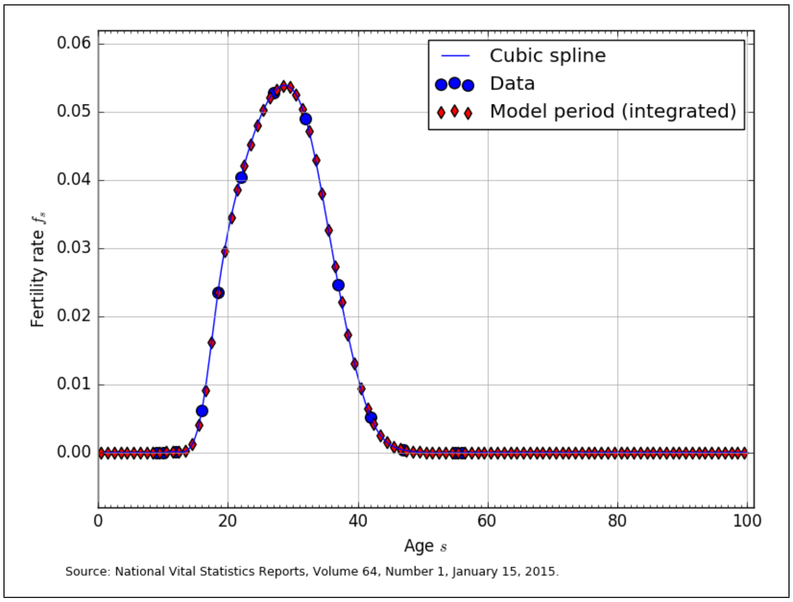
7.7.1 Non-population parameters

In general, the time-dependent parameters can be written as functions of total economically relevant lifetime model periods S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1-(1-0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 2.2$, let the productivity scale parameter of firms be $A = 1$, let the capital share of income be $\alpha = 0.35$, and let the growth rate of labor augmenting technological change be $g_y = 0.03$.

7.7.2 Fertility rates

In this model, we assume that the fertility rates for each age cohort f_s are constant across time. However, this assumption is conceptually straightforward to relax. Our data for U.S. fertility rates by age come from Martin et al. (2015, Table 3, p. 18) National Vital Statistics Report, which is final fertility rate data for 2013. Figure 7.1 shows the fertility-rate data and the estimated average fertility rates for $E+S=100$.

Figure 7.1: Fertility rates by age (f_s) for $E+S=$ 100



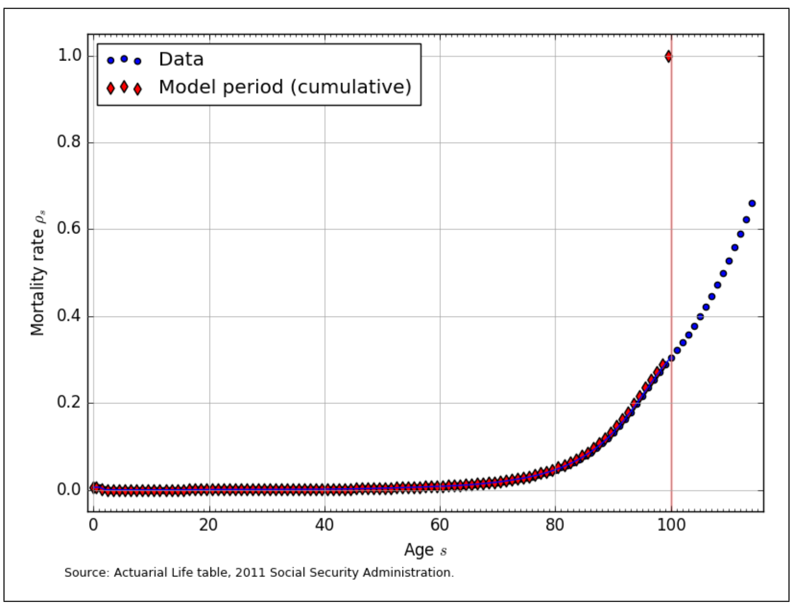
The large blue circles are the 2013 U.S. fertility rate data from Martin et al. (2015). These are 9 fertility rates $[0.3, 12.3, 47.1, 80.7, 105.5, 98.0, 49.3, 10.4, 0.8]$ that correspond to the midpoint ages of the following age (in years) bins $[10-14, 15-17, 18-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49]$. In order to get our cubic spline interpolating function to fit better at the endpoints we added to fertility rates of zero to ages 9 and 10, and we added two fertility rates of zero to ages 55 and 56. The blue line in Figure 7.1 shows the cubic spline interpolated function of the data.

The red diamonds in Figure 7.1 are the average fertility rate in age bins spanning households born at the beginning of period 1 (time $=0$) and dying at the end of their 100th year. Let the total number of model years that a household lives be $E+S\leq 100$. Then the span from the beginning of period 1 (the beginning of year 0) to the end of period 100 (the end of year 99) is divided up into $E+S$ bins of equal length. We calculate the average fertility rate in each of the $E+S$ model-period bins as the average population-weighted fertility rate in that span. The red diamonds in Figure 7.1 are the average fertility rates displayed at the midpoint in each of the $E+S$ model-period bins.

7.7.3 Mortality rates

The mortality rates in our model ρ_s are a one-period hazard rate and represent the probability of dying within one year, given that an household is alive at the beginning of period s. We assume that the mortality rates for each age cohort ρ_s are constant across time. The infant mortality rate of $\rho_0 = 0.00587$ comes from the 2015 U.S. CIA World Factbook. Our data for U.S. mortality rates by age come from the Actuarial Life Tables of the U.S. Social Security Administration (see Bell and Miller, 2015), from which the most recent mortality rate data is for 2011. Figure 7.2 shows the mortality rate data and the corresponding model-period mortality rates for $E + S = 100$.

Figure 7.2: Mortality rates by age (ρ_s) for $E+S = 100$



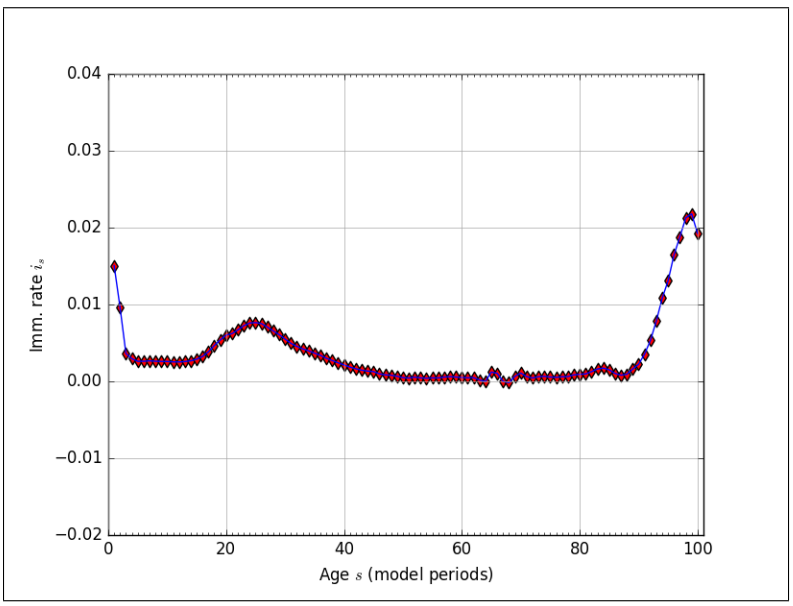
The mortality rates in Figure 7.2 are a population-weighted average of the male and female mortality rates reported in Bell and Miller (2015). Figure 7.2 also shows that the data provide mortality rates for ages up to 111-years-old. We truncate the maximum age in years in our model to 100-years old. In addition, we constrain the mortality rate to be 1.0 or 100 percent at the maximum age of 100.

7.7.4 Immigration rates

Because of the difficulty in getting accurate immigration rate data by age, we estimate the immigration rates by age in our model i_s as the average residual that reconciles the current-period population distribution with next period's population distribution given fertility rates f_s and mortality rates ρ_s. Solving equations (7.1) for the immigration rate i_s gives the following characterization of the immigration rates in given population levels in any two consecutive periods $\omega_{s,t}$ and $\omega_{s,t+1}$ and the fertility rates f_s and mortality rates ρ_s.

$$\begin{aligned} i_1 &= \frac{\omega_{1,t+1} - (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t}}{\omega_{1,t}} \quad \forall t \\ i_{s+1} &= \frac{\omega_{s+1,t+1} - (1-\rho_s)\omega_{s,t}}{\omega_{s+1,t}} \quad \forall t \quad \text{and} \quad 1\leq s\leq E+S-1 \end{aligned} \tag{7.38}$$

Figure 7.3: Immigration rates by age (i_s), residual, $E+S=100$



We calculate our immigration rates for three different consecutive-year-periods of population distribution data (2010 through 2013). Our four years of population distribution by age data come from Census Bureau (2015). The immigration rates i_s that we use in our model are the the residuals described in (7.38) averaged across the three periods. Figure 7.3

shows the estimated immigration rates for $E + S = 100$ and given the fertility rates from Section 7.7.2 and the mortality rates from Section 7.7.3.

At the end of Section 7.7.5, we describe a small adjustment that we make to the immigration rates after a certain number of periods in order to make computation of the transition path equilibrium of the model from Definition 7.2 compute more robustly.

7.7.5 Population steady-state and transition path

This model requires information about mortality rates ρ_s in order to solve for the household's problem each period. It also requires the steady-state stationary population distribution $\bar{\omega}_s$ and population growth rate $\bar{g}_n$ as well as the full transition path of the stationary population distribution $\hat{\omega}_{s,t}$ and population grow rate $\tilde{g}_{n,t}$ from the current state to the steady-state. To solve for the steady-state and the transition path of the stationary population distribution, we write the stationary population dynamic equations (7.39) and their matrix representation (7.40).

$$\begin{aligned}
\hat{\omega}_{1,t+1} &= \frac{(1-\rho_0)\sum_{s=1}^{E+S} f_s\hat{\omega}_{s,t} + i_1\hat{\omega}_{1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \\
\hat{\omega}_{s+1,t+1} &= \frac{(1-\rho_s)\hat{\omega}_{s,t} + i_{s+1}\hat{\omega}_{s+1,t}}{1+\tilde{g}_{n,t+1}} \qquad\qquad \forall t \quad \text{and} \quad 1\leq s\leq E+S-1
\end{aligned} \tag{7.39}$$

$$\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...$$

$$\begin{bmatrix} (1-\rho_0)f_1 + i_1 & (1-\rho_0)f_2 & (1-\rho_0)f_3 & \dots & (1-\rho_0)f_{E+S-1} & (1-\rho_0)f_{E+S} \\ 1-\rho_1 & i_2 & 0 & \dots & 0 & 0 \\ 0 & 1-\rho_2 & i_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & i_{E+S-1} & 0 \\ 0 & 0 & 0 & \dots & 1-\rho_{E+S-1} & i_{E+S} \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{7.40}$$

We can write system (7.40) more simply in the following way.

$$\hat{\boldsymbol{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}}\boldsymbol{\Omega}\hat{\boldsymbol{\omega}}_t \quad \forall t \tag{7.41}$$

The stationary steady-state population distribution $\bar{\boldsymbol{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (7.41).

$$(1+\bar{g}_n)\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega}\bar{\boldsymbol{\omega}} \tag{7.42}$$

Proposition 7.1. If the age $s=1$ immigration rate is $i_1 > -(1-\rho_0)f_1$ and the other immigration rates are strictly positive $i_s > 0$ for all $s \geq 2$ such that all elements of $\boldsymbol{\Omega}$ are nonnegative, then there exists a unique positive real eigenvector $\bar{\boldsymbol{\omega}}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible.

This can be easily shown. The matrix is of the form

$$\mathbf{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\mathbf{\Omega}^2 = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}; \quad \mathbf{\Omega}^{S+E-1} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

$$\mathbf{\Omega}^{S+E} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

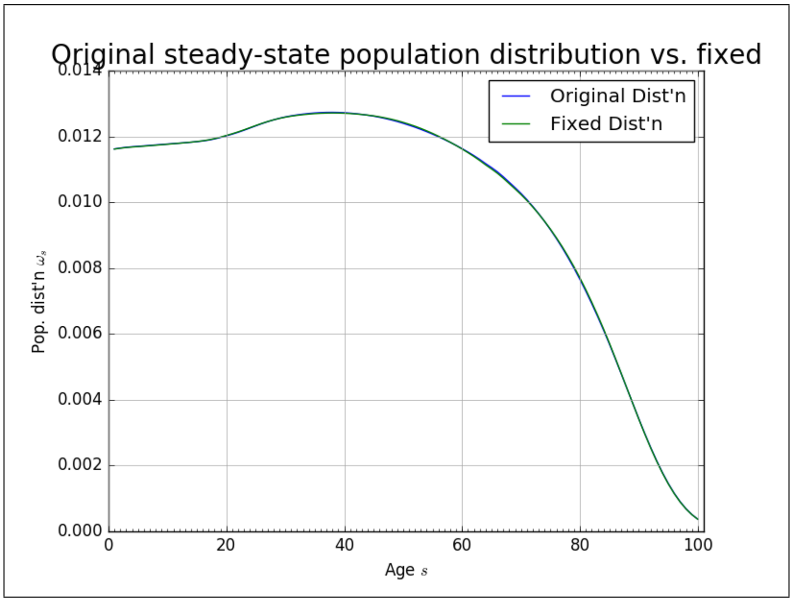
Existence of an $m \in \mathbb{N}$ such that $(\mathbf{\Omega^m})_{ij} \neq 0 \;\; (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\boldsymbol{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. □

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$.

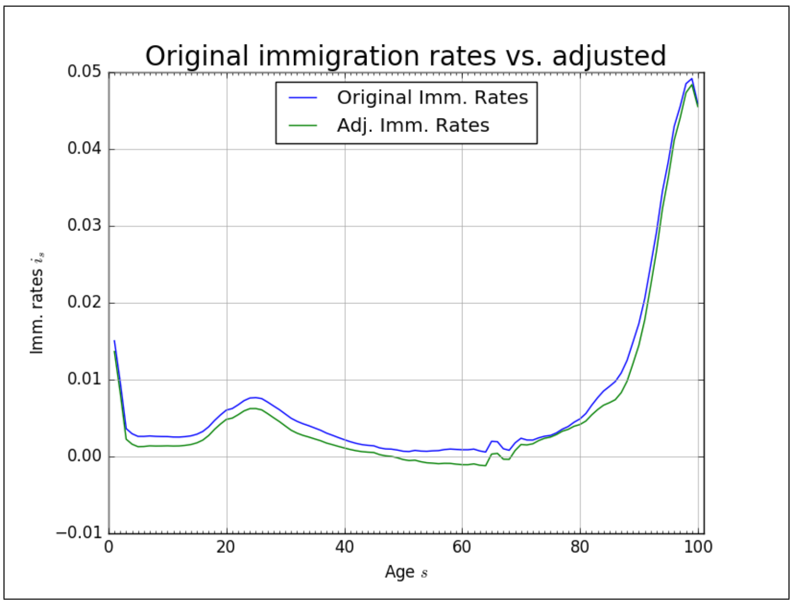
Figure 7.4 shows the steady-state population distribution $\bar{\boldsymbol{\omega}}$ and the population distribution after 120 periods $\hat{\boldsymbol{\omega}}_{120}$. Although the two distributions look very close to each other, they are not exactly the same.

Figure 7.4: Theoretical steady-state population distribution vs. population distribution at period $t = 120$



Further, we find that the maximum absolute difference between the population levels $\hat{\omega}_{s,t}$ and $\hat{\omega}_{s,t+1}$ was 1.3852×10^{-5} after 160 periods. That is to say, that after 160 periods, given the estimated mortality, fertility, and immigration rates, the population has not achieved its steady state. For convergence in our solution method over a reasonable time horizon, we want the population to reach a stationary distribution after T periods. To do this, we artificially impose that the population distribution in period $t = 120$ is the steady-state. As can be seen from Figure 7.4, this assumption is not very restrictive. Figure 7.5 shows the change in immigration rates that would make the period $t = 120$ population distribution equal be the steady-state. The maximum absolute difference between any two corresponding immigration rates in Figure 7.5 is 0.0028.

Figure 7.5: Original immigration rates vs. adjusted immigration rates to make fixed steady-state population distribution



The most recent year of population data come from Census Bureau (2015) population estimates for both sexes for 2013. We those data and use the population transition matrix (7.41) to age it to the current model year of 2015. We then use (7.41) to generate the transition path of the population distribution over the time period of the model. Figure 7.6 shows the progression from the 2013 population data to the fixed steady-state at period $t = 120$. The time path of the growth rate of the economically active population $\tilde{g}_{n,t}$ is shown in Figure 7.7.

Figure 7.6: Stationary population distribution at periods along transition path

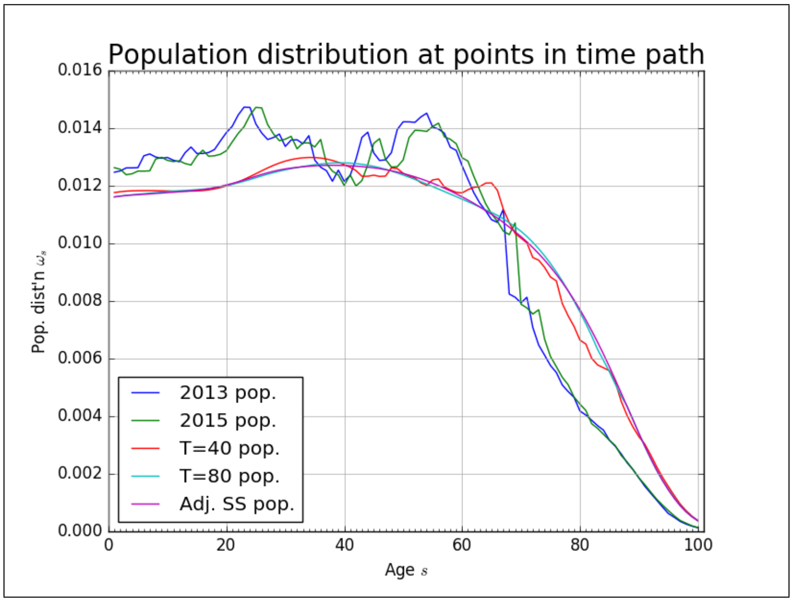
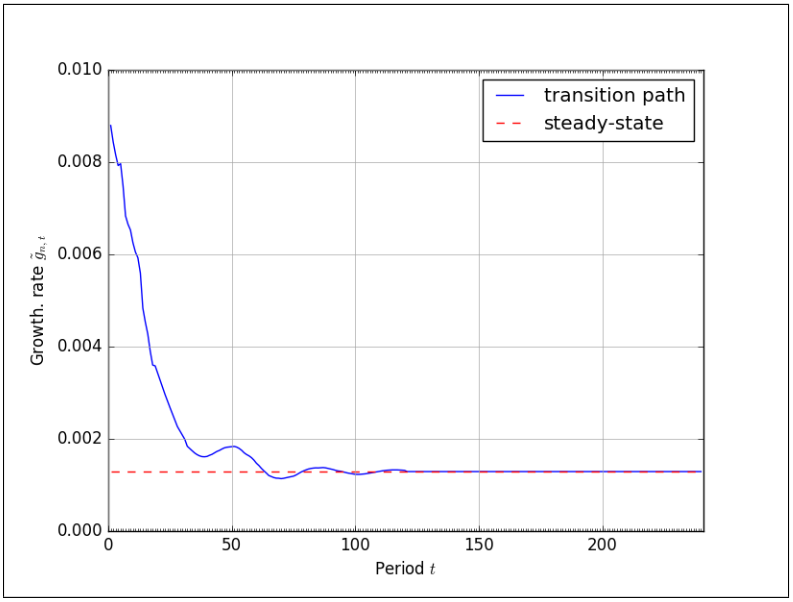


Figure 7.7: Time path of the population growth rate $\tilde{g}_{n,t}$



7.8 Exercises

Exercise 7.1. Fertility rates function. Get current U.S. fertility data (births per 1,000 women) from the National Vital Statistics Reports, volume 64, number 1. Use Table 3 from the appendix for fertility rates for the age categories 10-14, 15-17, 18-19, 20-24, 25-29, 30-34, 35-39, 40-44, and 45-49. Assume that the fertility value in each of those age bins is associated with the midpoint age in those bins. Convert the fertility rates to the form of births per person (both men and women) of a given age. For example, if the fertility value for women age 20-24 is 80.7 births per 1,000 women, then the fertility rate associated with the midpoint of age 22 is 0.04035 (80.7/2,000). Write a function `get_fert()` that has the following form,

```
fert_rates = get_fert(totpers, graph)
```

where `totpers` is the number of periods an agent lives ($E + S$), `fert_rates` is a vector of fertility rates of length `totpers` for each model period, and `graph` is a Boolean that equals `True` if you want to output a plot of the fertility rates. Inside the `get_fert()` function, use the function `scipy.interpolate.interp1d()` to fit a smoothly interpolated fertility rate function of age for 100 evenly spaced periods from 1 to 100. Then any number of periods greater than 100 or less than 100 evenly spans that period of years. Regardless of how many periods an individual's life is in the model (`totpers`), the number of years to which those total model periods correspond is 100. An agent is born at the beginning of year-1 and dies at the end of year-100. The total model periods (`totpers`) span that period of 100 years.

a. Plot the interpolated fertility rate function for `totpers`=100.

b. Plot the interpolated fertility rate function for `totpers`=80 as scatter points on top of the line you produced in part (a).

c. Plot the interpolated fertility rate function for `totpers`=20 as scatter points on top of the line you produced in part (a).

Exercise 7.2. Mortality rates function. Look up the infant mortality value (number of births per 1,000 births that die at birth or before they turn one year old). Transform that number into a mortality rate (percent of total births that die at birth or before they turn one year old). Get current U.S. mortality data by age from the Actuarial Life Table, 2011 from the U.S. Social Security Administration, which is available as `mort_rates2011.csv`. These data give mortality rates by age (percent of age-s individuals that died in that year) by gender. You need to calculate the total fertility rate (both men and women) by age using the population of men and women, respectively, in each of those age groups. Again, assume that agents in your model live for 100 years. Those years start at the beginning of year-1 and finish at the end of year-100. In the data, let the mortality rate from age=0 correspond to the mortality rate of year-1, and let the mortality rate in the data from age-98 correspond to the mortality rate of year-99. Set the mortality rate of year-100 to 1.0, such that everyone dies after 100 years. Write a function `get_mort()` that has the following form,

```
mort_rates, infmort_rate = get_mort(totpers, graph)
```

where `totpers` is the number of periods an agent lives ($E + S$), `mort_rates` is a vector of mortality rates of length `totpers` for each model period (percent of individuals age-s that die each period), and `graph` is a Boolean that equals `True` if you want to output a plot of the mortality rates. Note that when `totpers`< 100, each model period has a duration that is longer than a year. So mortality rates will not just be a simple interpolation along the mortality rate curve when `totpers`=100, as in part (a). You will have to calculate the cumulative mortality rate for the model age bin given the annual mortality rates and total population levels for the given ages from the data.

a. Plot the mortality rate function with the infant mortality rate at age ($s = 0$) for `totpers`=100.

b. Plot the mortality rate function with the infant mortality rate at age ($s = 0$) for `totpers`=80 as scatter points in the same plot of the line you produced in part (a).

c. Plot the mortality rate function with the infant mortality rate at age ($s = 0$) for `totpers`=20 as scatter points in the same plot of the line you produced in part (a).

Exercise 7.3. Immigration rates function. We will treat immigration rates as a residual based on two-consecutive periods of population data by age, mortality rates by age, and fertility rates by age. Get population level data by age from the comma-delimited file pop_data.csv taken from the U.S. Census Bureau Population Estimates. This file has U.S. population totals by age (ages 0 to 99) for years 2012 and 2013. For the model with $E+S=100$, treat age-0 in the data as model age 1 and age-99 in the data as model age 99. Let the immigration rates i_s be implicitly defined by the population dynamics equations (7.1) and explicitly defined in Section 7.7.4 as equations (7.38). Write a function get_imm_resid() that has the following form,

```
imm_rates = get_imm_resid(totpers, graph)
```

where totpers is the number of periods an agent lives $(E+S)$, imm_rates is a vector of immigration rates of length totpers for each model period (percent of individuals age-s immigrate into the country), and graph is a Boolean that equals True if you want to output a plot of the mortality rates. Note that negative immigration rates imply the people of age-s are moving out of the country. You will want to call your get_fert() and get_mort() functions inside of this get_imm_resid() function.

a. Plot the immigration rates for totpers=100.

b. Plot the immigration rates for totpers=80.

c. Plot the immigration rates for totpers=20.

Exercise 7.4. Time path of the population distribution. Let $E=20$ and $S=80$. For the following problems, use the fertility rates f_s, mortality rates ρ_s, and immigration rates i_s from Exercises 7.1, 7.2, and 7.3. You must use the stationary version of the population dynamics equations (7.39).

a. Use the instructions in Section 7.7.5 to solve for and plot the steady-state stationary population distribution $\bar{\omega}_s$.

b. Create an $80\times T+S-2$ matrix that contains the time path of the stationarized population distribution $\hat{\omega}_{s,t}$ for the economically relevant population $E+1\leq s\leq E+S$ over the time path $1\leq t\leq T+S-2$. Let the first column be $\hat{\omega}_{s,1}$. Calculate $\hat{\omega}_{s,1}$ from the 2013 population by age data from the file pop_data.csv. And every column for $t>T$ should be the steady-state stationary population distribution $\bar{\omega}_s$.

c. Make a single plot that shows the stationary population distribution at time $t = 1, 10, 30$, and T.

d. Plot the population growth rate of the economically relevant population $\tilde{g}_{n,t}$ for $1 \leq t \leq T + S - 2$.

Exercise 7.5. Solve for the steady-state equilibrium. Use the calibration from Section 7.7 and the steady-state equilibrium Definition 7.1. Write a function named `get_SS()` that has the following form,

```
ss_output = get_SS(params, bvec_guess, SS_graphs)
```

where the inputs are a tuple of the parameters and objects for the model,

```
params = (beta, sigma, nvec, L, A, alpha, delta, g_y, mort_rates,
    imm_rates, omega_SS, g_n_SS, SS_tol)
```

an initial guess of the steady-state savings `bvec_guess`, and a Boolean `SS_graphs` that generates a figure of the steady-state distribution of consumption and savings if it is set to `True`.

The output object `ss_output` is a Python dictionary with the steady-state solution values for the following endogenous objects.

```
ss_output = {
  'b_ss': b_ss, 'c_ss': c_ss, 'w_ss': w_ss, 'r_ss': r_ss,
  'K_ss': K_ss, 'Y_ss': Y_ss, 'C_ss': C_ss,
  'EulErr_ss': EulErr_ss, 'RCerr_ss': RCerr_ss,
  'ss_time': ss_time}
```

Let `ss_time` be the number of seconds it takes to run your steady-state program. You can time your program by importing the time library. And let the object `EulErr_ss` be a length-$(S-1)$ vector of the Euler errors from the resulting steady-state solution given in ratio form $\frac{e^{-\sigma g_y}\beta(1+\bar{r})(1-\rho_s)u'(\bar{c}_{s+1})}{u'(\bar{c}_s)} - 1$ or difference form $e^{-\sigma g_y}\beta(1+\bar{r})(1-\rho_s)u'(\bar{c}_{s+1}) - u'(\bar{c}_s)$. The object `RCerr_ss` is a resource constraint error which should be close to zero. It is given by,

$$\bar{Y} - \bar{C} - \Big[(1+\bar{g}_n)e^{g_y} - 1 + \delta\Big]\bar{K} + e^{g_y}\sum_{s=E+2}^{E+S} i_s\bar{\omega}_s\hat{b}_{s+1}$$

a. Solve numerically for the steady-state equilibrium values of $\{\bar{c}_s\}_{s=E+1}^{E+S}$, $\{\bar{b}_s\}_{s=E+2}^{E+S}$, $\bar{w}$, $\bar{r}$, $\bar{K}$, $\bar{Y}$, $\bar{C}$, the $S-1$ Euler errors, and the resource constraint error. List those values. Time your function. How long did it take to compute the steady-state?

b. Generate a figure that shows the steady-state distribution of consumption and savings by age $\{\bar{c}_s\}_{s=E+1}^{E+S}$ and $\{\bar{b}_s\}_{s=E+2}^{E+S}$.

Exercise 7.6. Solve for the non-steady-state equilibrium time path. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy. Let the initial state of the economy be given by the following distribution of savings,

$$\{b_{s,1}\}_{s=E+2}^{E+S} = \{x(s)\bar{b}_s\}_{s=E+2}^{E+S} \quad \text{where} \quad x(s) = \frac{(1.5 - 0.87)}{S+E-2}(s-2) + 0.87$$

where the function of age $x(s)$ is simply a linear function of age s that equals 0.87 for $s = E+2$ and equals 1.5 for $s = E+S$. This gives an initial distribution where there is more inequality than in the steady state. The young have less than their steady-state values and the old have more than their steady-state values. You'll have to choose a guess for T and a time path updating parameter $\xi \in (0,1)$, but you can be assured that $T < 320$. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear or parabolic initial guess for the time path of the aggregate capital stock from the initial state $\hat{K}_1^1$ to the steady state $\hat{K}_T^1$ at time T.

a. Plot the equilibrium time paths of the stationary aggregate capital stock $\{\hat{K}_t\}_{t=1}^{T+5}$, wage $\{\hat{w}_t\}_{t=1}^{T+5}$, and interest rate $\{r_t\}_{t=1}^{T+5}$.

b. How many periods did it take for the economy to get within 0.00001 of the steady-state aggregate capital stock $\bar{K}$? What is the period after which the aggregate capital stock never is again farther than 0.00001 away from the steady-state?

Part V

Bequests

Chapter 8

S-period Lives, Exogenous Labor, Intended Bequests

In this Chapter, we take the model from Chapter 3 and add to it a bequest motive in the last period of life. We are trying to make the model match the empirical regularity that, on average, individuals die with positive wealth that gets bequeathed to younger generations. This chapter's treatment of bequests will be very simple because we are assuming that labor is supplied exogenously and there is no mortality risk until the last period of life $s = S$. But this simple model provides the building blocks for a richer incorporation of these types of intergenerational transfers into an overlapping generations model.

8.1 Bequest Data

In reality, individuals give multiple types of bequests. Some wealth transfers between generations are given at death while some are given while the donor is still alive (inter vivos transfers). In addition to variance among the types of bequests, the reasons for why people transfer wealth to other generations are also varied. Bequest motives include altruism toward younger generations, precautionary savings, and tax incentives.

These motives for and types of wealth transfers might vary by the age and income of the potential donor. Unfortunately, data on who gives bequests is somewhat sparse. One reason why it is difficult to obtain good donor data is that wealth transfers are often executed in indirect and varied methods, often in response to the tax treatment of bequests. Wealth transfers can be reported on both individual tax filings and on separate entity pass-through filings. Further, there can often be a lag in the time wealth is transferred to a pass-through organization and when it is distributed to individuals of another generation. For these and other reasons, data on donors bequests is problematic.

Much cleaner is the data on who receives wealth transfers as this data is reported more uniformly. Wolff (2015, Ch. 3) describes the recipients of wealth transfers using two primary datasets—the Panel Study of Income Dynamics (PSID) and the Survey of Consumer Finances (SCF). Wolff (2015) shows that the percent of wealth transfers given to a particular demographic increases with that group's income and age. We will detail how to include that information in this modeling framework.

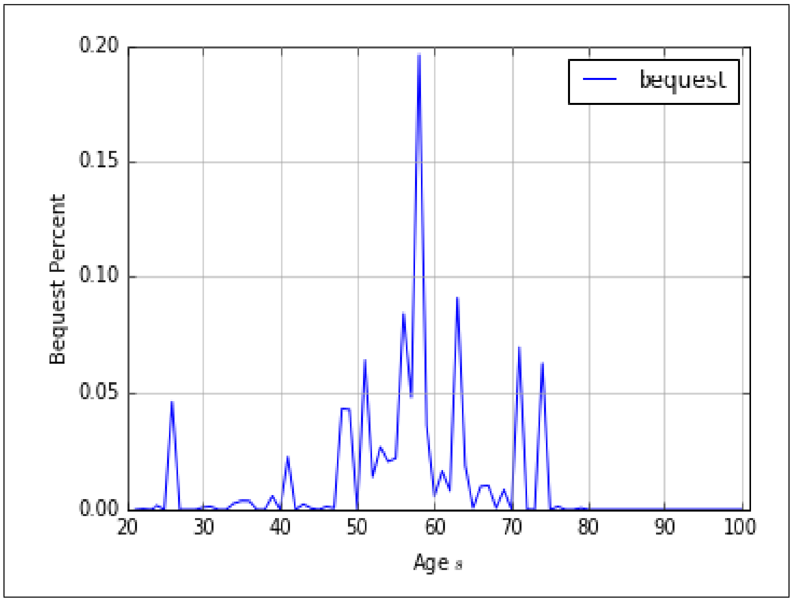
Using the Survey of Consumer Finances, we can construct a measure of total bequests per individual respondent by adding up the total of multiple types of wealth transfers that one can receive. Table 8.1 shows the variable names from the 2013 SCF and their descriptions, means, and variances.

Table 8.1: 2013 SCF bequest variables and description (? observations)

Variable name	Description	Mean	Standard deviation
X5804	What was its approximate value (of bequests) at the time it was received?		
X5805	In what year was it (X5804) received?		
X5809	(second instance of bequest reception)		
X5810	In what year was it (X5809) received?		
X5814	(third instance of bequest reception)		
X5815	In what year was it (X5814) received?		
X8022	Respondent's date of birth (for computing age)		

Figure 8.1 shows a histogram of the unconditional distribution of the percent of total bequests received by each age category, and Figure [TODO: include figure] shows a histogram of the unconditional distribution of the percent of total bequests received by portion of the income distribution. Ideally, we would want the joint distribution of bequests by both income and age. You will calculate this histogram in Exercise 8.1 at the end of the chapter.

Figure 8.1: Bequests received by age from SCF 2013



8.2 Households

A unit measure of identical individuals are born each period and live for S periods. Let the age of an individual be indexed by $s = \{1, 2, ...S\}$. The per-period budget constraint for an age-s individual changes from (2.1) of Chapter 3 because each agent now receives some amount of bequests each period,

$$c_{s,t} + b_{s+1,t+1} = (1 + r_t)b_{s,t} + w_t n_{s,t} + \zeta_s BQ_t \quad \forall s,t \tag{8.1}$$

where BQ_t is the total amount of bequests given in period t, and ζ_s is the percent of those bequests given to the age-s cohort in each period such that $\sum_s \zeta_s = 1$.[1]

We assume the individuals supply a unit of labor inelastically in the first two thirds of

[1] If there were other dimensions of heterogeneity among individuals, besides age s, the bequest distribution parameter could represent a joint distribution over those additional dimensions. For example, if we included the heterogeneous lifetime income process $e_{j,s}$ from Chapter 5, the bequest distribution parameter could be a function of both age s and ability j $\zeta_{j,s}$ such that $\sum_j \sum_s \zeta_{j,s} = 1$.

life ($s \leq \text{round}(2S/3)$) and are retired during the last third of life ($s > \text{round}(2S/3)$).

$$n_{s,t} = \begin{cases} 1 & \text{if} \quad s \leq \text{round}\left(\frac{2S}{3}\right) \\ 0.2 & \text{if} \quad s > \text{round}\left(\frac{2S}{3}\right) \end{cases} \quad \forall s,t \tag{3.1}$$

Because exogenous labor in (3.1) is not dependent on the time period, we drop the t subscript from labor n_s for the rest of this section. We also assume that households are born with no savings $b_{1,t} = 0$ for all periods t. Assume that $c_{s,t} \geq 0$ because negative consumption neither has an intuitive interpretation nor is it defined in the household utility function. It is the latter condition that will make $c_{s,t} > 0$ in equilibrium.

The period utility function changes in this section relative to Section 3.1. Here, we add a "warm glow" bequest motive in the last period of life. That is, we give households an incentive in the objective function to bequeath wealth to successive generations.

$$U(c_{s,t}, b_{S+1,t+1}) = \begin{cases} \frac{(c_{s,t})^{1-\sigma}-1}{1-\sigma} & \text{if} \quad s < S \\ \frac{(c_{S,t})^{1-\sigma}-1}{1-\sigma} + \chi^b\left[\frac{(b_{S+1,t+1})^{1-\sigma}-1}{1-\sigma}\right] & \text{if} \quad s = S \end{cases} \quad \forall t \tag{8.2}$$

The parameter χ^b changes the relative marginal utility of bequests in the last period of life $s = S$ to the marginal utility of consumption in the last period of life. Note that (8.2) is additively separable in $c_{s,t}$ and $b_{S+1,t+1}$ and is increasing and concave in both arguments with constant relative risk aversion parameterized by σ.

Because bequests are only given in the last period of life, these positive bequests take the form of positive savings in the last period of life $b_{S+1,t+1} > 0$. Because the unit measure of age-S individuals are no longer alive to receive those bequests, they are distributed among the remaining individuals according to ζ_s. Let the law of motion for total bequests be the following.

$$BQ_t = (1 + r_t)b_{S+1,t} \tag{8.3}$$

Individuals choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, savings $\{b_{s+1,t+s}\}_{s=1}^S$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$\begin{aligned}
&\max_{\{c_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^S} \sum_{u=0}^{S-s}\beta^u U(c_{s+u,t+u},b_{S+1,t+S}) \quad \forall s,t \\
&\text{s.t.} \quad c_{s,t} = (1+r_t)b_{s,t} + w_t n_s + \zeta_s BQ_t - b_{s+1,t+1} \quad \forall s,t \\
&\text{and} \quad b_{1,t}=0 \quad \forall t \quad \text{and} \quad c_{s,t}\geq 0 \quad \forall s,t
\end{aligned} \tag{8.4}$$

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints (8.1) into the optimization problem (8.4). The optimal choice of how much to save in the each of the S periods of life $b_{s+1,t+1}$ is found by taking the derivative of the lifetime utility function with respect to each of the lifetime savings amounts $\{b_{s+1,t+s+1}\}_{s=1}^S$ and setting the derivatives equal to zero.

In the last period of life, the household now optimally chooses positive savings $b_{S+1,t+1}>0$ for all t, which are intended bequests, because the period utility function (8.2) is not defined for $b_{S+1,t+1}<0$ and $\lim_{b_{S+1,t+1}\to 0}U(c_{s,t},b_{S+1,t+1}) = -\infty$. In the final period, the Euler equation is static, but trades off the marginal cost of savings with the marginal benefit of bequests.

$$\Big([1+r_t]b_{S,t} + w_t n_S + \zeta_S BQ_t - b_{S+1,t+1}\Big)^{-\sigma} = \chi^b\left(b_{S+1,t+1}\right)^{-\sigma} \quad \forall t \tag{8.5}$$

In this restricted case, we can solve first order condition (8.5) analytically for $b_{S+1,t+1}$ by raising both sides to the $-1/\sigma$ power.

$$b_{S+1,t+1} = \frac{1}{1+(\chi^b)^{-1/\sigma}}\Big([1+r_t]b_{S,t} + w_t n_S + \zeta_S BQ_t\Big) \quad \forall t \tag{8.6}$$

Note that $b_{S+1,t+1} = \psi_S\left(b_{S,t},r_t,w_t,BQ_t\right)$ is a function of wealth $b_{S,t}$, the interest rate r_t, wage w_t, and total bequests BQ_t.

In the second-to-last period of life $s=S-1$, the household has a savings decision to make. He enters the period with wealth $b_{S-1,t}$, he knows the current interest rate r_t and the current wage w_t, and he must know or be able to forecast next period's interest rate r_{t+1} and wage w_{t+1}. At age $s=S-1$, the individual's problem is to choose two variables to maximize lifetime utility.

$$\max_{b_{S,t+1},b_{S+1,t+2}} \quad U\big(c_{S-1,t}\big) + \beta U\big(c_{S,t+1}, b_{S+1,t+2}\big) \tag{8.7}$$

The first order condition, or dynamic Euler equation, for this second-to-last period of life savings decision is the following,

$$\begin{split} &\Big(w_t n_{S-1} + [1+r_t]b_{S-1,t} + \zeta_{S-1}BQ_t - b_{S,t+1}\Big)^{-\sigma} = ... \\ &\qquad \beta(1+r_{t+1})\Big([1+r_{t+1}]b_{S,t+1} + w_{t+1}n_S + \zeta_S BQ_{t+1} - b_{S+1,t+2}\Big)^{-\sigma} \end{split} \tag{8.8}$$

where $b_{S+1,t+2} = \psi_S\left(b_{S,t+1}, r_{t+1}, w_{t+1}, BQ_{t+1}\right)$ from (8.6). Because of the envelope condition using equation (8.5), all terms with $\partial\psi_S/\partial b_{S,t+1}$ drop out of (8.8). The solution for the savings policy function in the second-to-last period of life is the following.

$$b_{S,t+1} = \psi_{S-1}\left(b_{S-1,t}, r_t, w_t, BQ_t, r_{t+1}, w_{t+1}, BQ_{t+1}\right) \tag{8.9}$$

By backward induction, the partial equilibrium solution to the household's problem is to choose savings $\{b_{s+1,t+1}\}_{s=1}^{S}$ to maximize (8.4). The solution is a series of policy functions of the form,

$$b_{s+1,t+s} = \psi_s\Big(b_{s,t}, \{r_{t+v}\}_{v=0}^{S-s}, \{w_{t+v}\}_{v=0}^{S-s}, \{BQ_{t+v}\}_{v=0}^{S-s}\Big) \quad \forall t \quad \text{and} \quad 1 \leq s \leq S \tag{8.10}$$

that satisfy the S Euler equations that come from the maximization problem.

$$\begin{split} &\Big([1+r_t]b_{s,t} + w_t n_s + \zeta_s BQ_t - b_{s+1,t+1}\Big)^{-\sigma} = ... \\ &\qquad \beta(1+r_{t+1})\Big([1+r_{t+1}]b_{s+1,t+1} + w_{t+1}n_{s+1} + \zeta_{s+1}BQ_{t+1} - b_{s+2,t+2}\Big)^{-\sigma} \\ &\qquad\qquad \forall t \quad \text{and} \quad 1 \leq s \leq S-1 \end{split} \tag{8.11}$$

$$\Big([1+r_t]b_{S,t} + w_t n_S + \zeta_S BQ_t - b_{S+1,t+1}\Big)^{-\sigma} = \chi^b\left(b_{S+1,t+1}\right)^{-\sigma} \quad \forall t \tag{8.5}$$

To summarize the individual's problem, if one knows his initial savings or wealth $b_{s,t}$ and the time path of factor prices and total bequests received over his remaining lifetime, he can solve for all of his optimal savings levels $\{b_{s+1,t+s}\}_{s=1}^{S}$.

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates and wages $\{r_u, w_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Section 8.5, the equilibrium interest rate r_t and wage w_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \left\{b_{s,t}\right\}_{s=2}^{S+1} \quad \forall t \tag{8.12}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[2]

8.3 Firms

The production side of this economy is identical to the one in Section 2.2 with a unit measure of identical, perfectly competitive firms that rent investment capital from individuals for real return r_t and hire labor for real wage w_t. Firms use their total capital K_t and labor L_t to produce output Y_t every period according to a Cobb-Douglas production technology,

$$Y_t = F(K_t, L_t) \equiv AK_t^\alpha L_t^{1-\alpha} \quad \text{where} \quad \alpha \in (0,1) \quad \text{and} \quad A > 0. \tag{2.18}$$

[2]In Section 8.5 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 8.2.

The representative firm chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_t,L_t} AK_t^\alpha L_t^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \tag{2.19}$$

where $\delta \in [0,1]$ is the rate of capital depreciation, and the two first order conditions that characterize firm optimization are the following.

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

8.4 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{S} n_s \tag{3.15}$$

$$K_t = \sum_{s=2}^{S+1} b_{s,t} \tag{8.13}$$

$$\begin{split} &Y_t = C_t + I_t \\ &\quad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{split} \tag{2.24}$$

Note that the capital market clearing condition (8.13) differs from (3.16) in Chapter 3 because it includes intended bequests $b_{S+1,t}$. The goods market clearing equation (2.24) is redundant by Walras' Law.

8.5 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points

i. Households optimize according to equations (8.5) and (8.11).

ii. Firms optimize according to (2.20) and (2.21).

iii. Markets clear according to (3.15) and (8.13).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (3.15) and (8.13) into the firm's optimal conditions (2.20) and (2.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t\big(\boldsymbol{\Gamma}_t\big): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^{S+1} b_{s,t}}{\sum_{s=1}^{S} n_s}\right)^{\alpha} \tag{8.14}$$

$$r_t\big(\boldsymbol{\Gamma}_t\big): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^{S} n_s}{\sum_{s=2}^{S+1} b_{s,t}}\right)^{1-\alpha} - \delta \tag{8.15}$$

Now (8.14) and (8.15) can be substituted into household Euler equations (8.5) and (8.11) to get the following S-equation system that completely characterizes the equilibrium.

$$\begin{aligned}
&\left(\Big[1+r_t\big(\boldsymbol{\Gamma}_t\big)\Big]b_{s,t} + w_t\big(\boldsymbol{\Gamma}_t\big)n_s + \zeta_s BQ_t - b_{s+1,t+1}\right)^{-\sigma} = \\
&\quad \beta\Big[1+r_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)\Big]\left(\Big[1+r_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)\Big]b_{s+1,t+1} + w_{t+1}\big(\boldsymbol{\Gamma}_{t+1}\big)n_{s+1} + \zeta_{s+1}BQ_{t+1} - b_{s+2,t+2}\right)^{-\sigma} \\
&\qquad\qquad \forall t, \quad \text{and} \quad 1\leq s\leq S-1
\end{aligned} \tag{8.16}$$

$$\left(\Big[1+r_t\big(\boldsymbol{\Gamma}_t\big)\Big]b_{S,t} + w_t\big(\boldsymbol{\Gamma}_t\big)n_S + \zeta_S BQ_t - b_{S+1,t+1}\right)^{-\sigma} = \chi^b\left(b_{S+1,t+1}\right)^{-\sigma} \quad \forall t \tag{8.17}$$

The system of $S-1$ nonlinear dynamic equations (8.16) and one nonlinear static equation (8.17) characterizing the the lifetime savings decisions for each household $\{b_{s+1,t+s}\}_{s=1}^{S}$ is not identified. Each individual knows the current distribution of capital $\boldsymbol{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period

$\mathbf{\Gamma}_{t+1} = \{b_{s+1,t+s+1}\}_{s=1}^S$ for all agents alive next period, and for a policy function for the individual $b_{s+2,t+2}$ from these S equations. Even if we pile together all the sets of individual lifetime Euler equations, the system looks unidentified because today's decisions always depend upon tomorrow's decisions. This problem is characterized by a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 8.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and intended bequests is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, savings $\{\bar{b}_s\}_{s=2}^{S+1}$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (8.5) and (8.11),

ii. total bequests available $\bar{BQ}$ to all individuals is characterized by (8.3),

iii. firms optimize according to (2.20) and (2.21),

iv. markets clear according to (3.15) and (8.13).

As we saw earlier in this section, the characterizing equations in Definition 8.1 reduce to (8.16) and (8.17). These S characterizing equations are exactly identified in the steady state. That is, they are S equations and S unknowns $\{\bar{b}_s\}_{s=2}^{S+1}$.

$$\begin{split}\left(\Big[1+\bar{r}(\bar{\mathbf{\Gamma}})\Big]\bar{b}_s + \bar{w}(\bar{\mathbf{\Gamma}})n_s + \zeta_s\bar{BQ} - \bar{b}_{s+1}\right)^{-\sigma} = \\ \beta\Big[1+\bar{r}(\bar{\mathbf{\Gamma}})\Big]\left(\Big[1+\bar{r}(\bar{\mathbf{\Gamma}})\Big]\bar{b}_{s+1} + \bar{w}(\bar{\mathbf{\Gamma}})n_{s+1} + \zeta_{s+1}\bar{BQ} - \bar{b}_{s+2}\right)^{-\sigma} \quad \text{for} \quad 1\leq s\leq S-1\end{split} \tag{8.18}$$

$$\left(\Big[1+\bar{r}(\bar{\mathbf{\Gamma}})\Big]\bar{b}_S + \bar{w}(\bar{\mathbf{\Gamma}})n_S + \zeta_S\bar{BQ} - \bar{b}_{S+1}\right)^{-\sigma} = \chi^b\left(\bar{b}_{S+1}\right)^{-\sigma} \tag{8.19}$$

We can solve for steady-state $\{\bar{b}_s\}_{s=2}^{S+1}$ by using an unconstrained root finder. Then we solve for $\bar{w}$, $\bar{r}$, and $\{\bar{c}_s\}_{s=1}^S$ by substituting $\{\bar{b}_s\}_{s=2}^{S+1}$ into the equilibrium firm first order conditions and into the household budget constraints.

In the S-period-lived agent, perfect foresight, OG model with intentional bequests described in this section, the state vector can be seen in the system of Euler equations (8.16) and (8.17). What is the smallest set of variables that completely summarize all the information necessary for the three generations of all three generations living at time t to make their consumption and saving decisions? What information do they have at time t that will allow them to make their savings decisions? The state vector of this model in each period is the distribution of savings $\boldsymbol{\Gamma_t}$.

Definition 8.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and intended bequests is defined as stationary allocation functions of the state $\big\{b_{s+1,t+1} = \psi_s\big(\boldsymbol{\Gamma}_t\big)\big\}_{s=1}^{S}$ and stationary price functions $w(\boldsymbol{\Gamma}_t)$ and $r(\boldsymbol{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\boldsymbol{\Gamma}_{t+u} = \boldsymbol{\Gamma}^e_{t+u} = \Omega^u\left(\boldsymbol{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (8.5) and (8.11),

iii. total bequests available BQ_t to all individuals is characterized by (8.3),

iv. firms optimize according to (2.20) and (2.21),

v. markets clear according to (3.15) and (8.13).

We have already shown how to boil down the characterizing equations in Definition 8.2 to S equations (8.16) and (8.17) and S unknowns $\{b_{s,t}\}_{s=2}^{S+1}$, given the time paths for prices w_t and r_t and total bequests BQ_t. But we have also seen that those S equations are not identified. So how do we solve for these equilibrium functions? The solution to the non-steady-state equilibrium in Definition 8.2 is a fixed point in function space. Choose S functions of the state $\{\psi_s(\boldsymbol{\Gamma}_t)\}_{s=1}^{S}$ and verify that they satisfy the Euler equations for all points in the state space (all possible values of the state).

8.6 Solution method: time path iteration (TPI)

The solution method is time path iteration (TPI), analogous to the description in Section 3.5. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 8.1 in a finite number of periods $T < \infty$ regardless of the initial state $\mathbf{\Gamma}_1$.

The first step is to assume a transition path for aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$ and total bequests available each period $\boldsymbol{BQ}^i = \{BQ_1^i, BQ_2^i, ...BQ_T^i\}$ such that T is sufficiently large to ensure that $\mathbf{\Gamma}_T = \bar{\mathbf{\Gamma}}$. The superscript i is an index for the iteration number. The transition path for aggregate capital determines the transition path for both the real wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$ and the real return on investment $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$. The exact initial distribution of capital in the first period $\mathbf{\Gamma}_1$ can be arbitrarily chosen as long as it satisfies $K_1^i = \sum_{s=2}^{S+1} b_{s,1}$ according to market clearing condition (8.13) and $BQ_1^i = (1+r_1^i)b_{S+1,1}$ according to (8.3). One could also first choose the initial distribution of capital $\mathbf{\Gamma}_1$ and then choose an initial aggregate capital stock K_1^i and initial total bequests available BQ_1^i that correspond to that distribution. As mentioned earlier, the only other restriction on the initial transition path for aggregate capital is that it equal the steady-state level $K_T^i = \bar{K} = \sum_{s=2}^{S+1} \bar{b}_s$ and $BQ_T^i = \bar{BQ} = (1+\bar{r})\bar{b}_{S+1}$ by period T. But the initial guess for the aggregate capital stocks K_t^i and total bequests BQ_t^i for periods $1 < t < T$ can be any level.

Given the initial capital distribution $\mathbf{\Gamma}_1$ and the transition paths of aggregate capital $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$, total bequests $\boldsymbol{BQ}^i = \{BQ_1^i, BQ_2^i, ...BQ_T^i\}$, the wage $\boldsymbol{w}^i = \{w_1^i, w_2^i, ...w_T^i\}$, and the interest rate $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, one can solve for the optimal intended bequest decision $b_{S+1,2}$ of the age $s = S$ individual in the last period of his life in period $t = 1$ using his version of the static Euler equation (8.5).

$$\Bigl([1+r_1^i]b_{S,1} + w_1^i n_S + \zeta_S BQ_1^i - b_{S+1,2}\Bigr)^{-\sigma} = \chi^b \left(b_{S+1,2}\right)^{-\sigma} \tag{8.20}$$

The one equation (8.20) and one unknown $b_{S+1,2}$ represent the remaining lifetime decisions of that individual. The remaining lifetime decisions of the individual of age $s = S-1$ in period $t = 1$ are $b_{S,2}$ and $b_{S+1,3}$. These two decisions are characterized by the following two equations, analogous to (8.11) and (8.5).

$$\begin{aligned}\Bigl([1+r_1^i]b_{S-1,1} + w_1^i n_{S-1} + \zeta_{S-1} BQ_1^i - b_{S,2}\Bigr)^{-\sigma} &= ... \\ \beta\bigl(1+r_2^i\bigr)\Bigl([1+r_2^i]b_{S,2} + w_2^i n_S + \zeta_S BQ_2^i - b_{S+1,3}\Bigr)^{-\sigma}\end{aligned} \tag{8.21}$$

$$\Big([1+r^i_2]b_{S,2}+w^i_2n_S+\zeta_S BQ^i_2-b_{S+1,3}\Big)^{-\sigma}=\chi^b\left(b_{S+1,3}\right)^{-\sigma} \tag{8.22}$$

We can independently solve for the remaining lifetime decisions of each individual alive at period $t=1$ but not born in period $t=1$ (incomplete remaining lifetimes) as well as those born in periods $1\leq t\leq T$ with the appropriate set of the following equations.

$$\begin{aligned}&\Big([1+r^i_{t+s-1}]b_{s,t+s-1}+w^i_{t+s-1}n_s+\zeta_s BQ^i_{t+s-1}-b_{s+1,t+s}\Big)^{-\sigma}=...\\&\quad\beta\big(1+r^i_{t+s}\big)\Big([1+r^i_{t+s}]b_{s+1,t+s}+w^i_{t+s}n_{s+1}+\zeta_{s+1}BQ^i_{t+s}-b_{s+2,t+s+1}\Big)^{-\sigma}\\&\quad\text{for}\quad 1\leq s\leq S-1\end{aligned} \tag{8.23}$$

$$\Big([1+r^i_{t+S-1}]b_{S,t+S-1}+w^i_{t+S-1}n_S+\zeta_S BQ^i_{t+S-1}-b_{S+1,t+S}\Big)^{-\sigma}=\chi^b\left(b_{S+1,t+S}\right)^{-\sigma} \tag{8.24}$$

Once we have solved for all the individual savings decisions for individuals alive between periods 1 and T, then we have the complete distribution of savings $\{\boldsymbol{\Gamma}_t\}_{t=1}^T$ for each period between 1 and T. We can use this to compute a new time path of the aggregate capital stock and total bequests consistent with the individual savings decisions $K^{i'}_t=\sum_{s=2}^{S+1}b_{s,t}$ and $BQ^{i'}_t=(1+r^i_t)b_{S+1,t}$ for all $1\leq t\leq T$.[3] We use a "$'$" on this aggregate capital stock and total bequests because, in general, $K^{i'}_t\neq K^i_t$ and $BQ^{i'}_t\neq BQ^i_t$. That is, the initial conjectured path of the aggregate capital stock from which the savings decisions were made is not necessarily equal to the path of the aggregate capital stock consistent with those savings decisions.[4]

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and total bequests. Common norms to use are the L^2 and the L^∞ norms. Then the fixed point necessary for the equilibrium transition path from Definition 8.2 has been found when the distance between $[\boldsymbol{K}^{i'},\boldsymbol{BQ}^{i'}]$ and $[\boldsymbol{K}^i,\boldsymbol{BQ}^i]$ is arbitrarily close to zero.

$$\Big\|[\boldsymbol{K}^{i'},\boldsymbol{BQ}^{i'}]-[\boldsymbol{K}^i,\boldsymbol{BQ}^i]\Big\|<\varepsilon\quad\text{for}\quad\varepsilon>0 \tag{8.25}$$

[3]Note that we use the original or initial r^i_t that came from the initial K^i_t to calculate the new $BQ^{i'}_t$.

[4]A check here for whether T is large enough is if $K^{i'}_T=\bar{K}$ as well as $K^{i'}_{T+1}$ and $K^{i'}_{T+2}$. If not, then T needs to be larger.

If the fixed point has not been found $\left\|[\boldsymbol{K}^{i'}, \boldsymbol{BQ}^{i'}] - [\boldsymbol{K}^{i}, \boldsymbol{BQ}^{i}]\right\| > \varepsilon$, then a new transition path for the aggregate capital stock and total bequests is generated as a convex combination of the new implied paths $[\boldsymbol{K}^{i'}, \boldsymbol{BQ}^{i'}]$ and the initial paths $[\boldsymbol{K}^{i}, \boldsymbol{BQ}^{i}]$.

$$\boldsymbol{K}^{i+1} = \xi \boldsymbol{K}^{i'} + (1-\xi)\boldsymbol{K}^{i} \quad \text{for} \quad \xi \in (0,1) \tag{2.35}$$

$$\boldsymbol{BQ}^{i+1} = \xi \boldsymbol{BQ}^{i'} + (1-\xi)\boldsymbol{BQ}^{i} \quad \text{for} \quad \xi \in (0,1) \tag{8.26}$$

This process is repeated until the initial transition paths for the aggregate capital stock and total bequests are consistent with the transition paths implied by those beliefs and household and firm optimization. TPI solves for the equilibrium transition path from Definition 8.2 by finding a fixed point in the time path of the economy.

8.7 Calibration

Use the following parameterization of the model for the problems below. Assume that agents are born at age 21 and die at age 100 (80 years of life). Now your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$ and that the scale parameter on the utility of bequests is $\chi^b = 1$. You will estimate the distribution of bequests received ζ_s in part (a) of Exercise 8.1. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1-0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 2.2$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

8.8 Exercises

Exercise 8.1. Go to the 2013 Survey of Consumer Finances (SCF) page and download the 2013 main data file (`p13i6.dta`) and summary data file (`rscfp2013.dta`). I am recommending that you download the Stata format versions, but you could also download the ASCII versions. You can read the Stata versions into python as a pandas DataFrame using the following commands.

```
import pandas as pd
df_main = pd.read_stata('p13i6.dta')
df_summ = pd.read_stata('rscfp2013.dta')
```

SCF respondents can report up to three separate bequests over their lifetimes up to the date of the survey. These bequest amounts are located in variables `X5804`, `X5809`, and `X5814` with corresponding dates (year) of those bequests in variables `X5805`, `X5810`, and `X5815`. Age of respondent at the time of the survey is in variable `X8022`. Two variables in the summary file are also important. `Net Worth` gives the net worth in 2013 dollars of the respondent at the time of the survey, and `wgt` gives a population weight that allows the user of the data to reweight the respondent to make the survey results representative of the population.

a. Using the 2013 SCF main data (`p13i6.dta`) and summary data (`rscfp2013.dta`), caluculate the distribution of *total bequests* (`bq_tot`) by age ζ_s for ages 21 to 100. Plot ζ_s as a histogram. Make sure that you use the `wgt` variable to calculate the percentages. Note that your definition of *total bequests* will not include bequests in the data received by individuals younger than 21. And because the age variable is at the time of the survey, the recipient age must be adjusted by the year of the bequest in order to calculate the age distribution. Further, note that the bequest amounts must be inflation adjusted (use CPI adjustment factors of 2013=1.0000, 2012=0.9854, and 2011=0.9652). Lastly, only include bequests from the years 2011 to 2013.

b. Now assume that you care about both the distribution of bequests by age and by lifetime income. Suppose that the `net worth` variable in the summary data file (`rscfp2013.dta`) is a good proxy for lifetime income. Create four categories of net worth that represent the four quartiles (0-25%, 25-50%, 50-75%, and 75-100%). Calculate the distribution of *total bequests* (`bq_tot`) by net worth group j and by age s $\zeta_{j,s}$ for net worth groups $j = \{1,2,3,4\}$ and for ages $s = \{21, 22...100\}$. Plot $\zeta_{j,s}$ as a 3D histogram. Maks sure that you use the `wgt` variable to calculate the percentages.

Exercise 8.2. Use the calibration from Section 8.7 and the steady-state equilibrium Definition 8.1. Use your estimated ζ_s from part (a) of Exercise 8.1. Write a function named `get_SS()` that has the following form,

```
ss_output = get_SS(params, bvec_guess, SS_graphs)
```

where the inputs are a tuple of the parameters for the model `params = (beta, sigma, chi_b, zeta_s, A, alpha, delta, SS_tol, EulDiff)`, an initial guess of the steady-state savings `bvec_guess` that is S elements long with the last element being a guess for $\bar{b}_{S+1}$, and a Boolean `SS_graphs` that generates a figure of the steady-state distribution of consumption and savings if it is set to `True`.

The output object `ss_output` is a Python dictionary with the steady-state solution values for the following endogenous objects.

```
ss_output = {
  'b_ss': b_ss, 'c_ss': c_ss, 'w_ss': w_ss, 'r_ss': r_ss,
  'K_ss': K_ss, 'Y_ss': Y_ss, 'C_ss': C_ss, 'BQ_ss': BQ_ss,
  'b_err_ss': b_err_ss, 'RCerr_ss': RCerr_ss, 'ss_time': ss_time}
```

Let `ss_time` be the number of seconds it takes to run your steady-state program. You can time your program by importing the time library. And let the object `b_err_ss` be the steady-state vector of Euler errors for the savings decisions given in difference form $\beta(1+\bar{r})u(\bar{c}_{s+1}) - u'(\bar{c}_s)$. The object `RCerr_ss` is a resource constraint error which should be close to zero. It is given by $\bar{Y} - \bar{C} - \delta\bar{K}$.

a. Solve numerically for the steady-state equilibrium values of $\{\bar{c}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^{S+1}$, $\bar{w}$, $\bar{r}$, $\bar{K}$, $\bar{Y}$, $\bar{C}$, $\bar{BQ}$, the S Euler errors and the resource constraint error. List those values. Time your function. How long did it take to compute the steady-state?

b. Generate a figure that shows the steady-state distribution of consumption and savings by age $\{\bar{c}_s\}_{s=1}^S$ and $\{\bar{b}_s\}_{s=2}^{S+1}$.

Exercise 8.3. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy. Let the initial state of the economy be given by the following distribution of savings,

$$\{b_{s,1}\}_{s=2}^{S+1} = \{x(s)\bar{b}_s\}_{s=2}^{S+1} \quad \text{where} \quad x(s) = \frac{(1.5 - 0.87)}{79}(s-2) + 0.87$$

where the function of age $x(s)$ is simply a linear function of age s that equals 0.87 for $s = 2$ and equals 1.5 for $s = S + 1 = 81$. This gives an initial distribution where there is more inequality than in the steady state. The young have less than their steady-state values and the old have more than their steady-state values. You'll have to choose a guess for T and a time path updating parameter $\xi \in (0, 1)$, but know that $T < 300$ in the equilibrium. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-12}$. Use a linear or quadratic initial guess for the time path of the aggregate capital stock from the initial state K_1^1 to the steady state K_T^1 at time T.

a. Plot the equilibrium time paths of the aggregate capital stock $\{K_t\}_{t=1}^{T+5}$, total bequests $\{BQ_t\}_{t=1}^{T+5}$wage $\{w_t\}_{t=1}^{T+5}$, and interest rate $\{r_t\}_{t=1}^{T+5}$.

b. Also plot the equilibrium time path for savings $\{b_{25,t}\}$ of every person age $s = 25$ in every period.

Chapter 9

S-period Lives, Exogenous Labor, Bequests with Population Dynamics

Chapter 10

Bequests with Population Dynamics, Endogenous Labor, and Heterogeneous Abilities

Part VI

Household taxes and government budget constraint

Chapter 11

Household taxes

In this chapter, we focus on incorporating a government sector that levies taxes on households and firms and provides transfers to overlapping generations of households based off of the endogenous labor model from Chapter 4. These taxes will be distortionary to the household's decisions, and we will show a few options of how one might model realistic tax law. In this chapter, we will make the simplifying assumption that government revenue and government outlays are equal in every period. However, we show in Chapter 12 how to model a government budget constraint that is not balanced every period.

11.1 Incorporating Tax Functions into DGE Model

Many papers have incorporated tax functions of varying richness into dynamic general equilibrium (DGE) models. When taxes are included in a model with household optimization, the taxes are often distortionary to the household decisions. Further, the existence of taxes in a general equilibrium requires extra assumptions about what a government does with those taxes.

11.1.1 Some preliminary tax rate theory

In this chapter, we follow the simple and standard convention that tax liability $T^I_{s,t}$ is a function of some function of total household income. However, we use here a richer family of functions than most other studies by assuming that the total tax liability is a function of two types of income—labor income $x_{s,t}$ and capital income $y_{s,t}$.

$$T^I_{s,t} = T^I_{s,t}\big(x_{s,t}, y_{s,t}\big) \quad \forall s,t \tag{11.1}$$

$$x_{s,t} \equiv w_t n_{s,t} \quad \forall s,t \tag{11.2}$$

$$y_{s,t} \equiv r_t b_{s,t} \quad \forall s,t \tag{11.3}$$

Notice that the function in (11.1) is more general than a function of total income $x_{s,t} + y_{s,t}$ because a function of total income is a special case of (11.1). We leave the s and t subscripts on the function on the right-hand-side of (11.1) because one might want to estimate separate functions tax liability functions for different ages and time periods. Further, we assume that government transfers to households X_t are lump sum, equal, and are independent of household decision variables. The household budget constraint is the following,

$$\begin{split} c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{split} \tag{11.4}$$

and the two sets of household Euler equations for labor supply and for savings are the following.

$$\left(w_t - \frac{\partial T^I_{s,t}}{\partial n_{s,t}}\right) u_1\left(c_{s,t}, n_{s,t}\right) = -u_2\left(c_{s,t}, n_{s,t}\right) \quad \text{for} \quad \forall t, \quad s \in \{1,2,...S\} \tag{11.5}$$

$$u_1\left(c_{s,t}, n_{s,t}\right) = \beta\left(1 + r_{t+1} - \frac{\partial T^I_{s+1,t+1}}{\partial b_{s+1,t+1}}\right) u_1\left(c_{s+1,t+1}, n_{s+1,t+1}\right) \quad \forall t, \quad s \in \{1,2,...S-1\} \tag{11.6}$$

Note the distortionary effect of the household income tax on the Euler equations (11.5) and (11.6). We will derive these equations more carefully in Section 11.2.

We assume that the government's budget is balanced each period.

$$X_t = \frac{1}{S}\sum_{s=1}^{S} T^I_{s,t} \quad \forall t \tag{11.7}$$

Because most marginal tax rates for which we have data are based on some form of measurable income, such as labor income $x_{s,t}$ or capital income $y_{s,t}$, we need to write the marginal tax rates in our Euler equations (11.5) and (11.6) as functions of measurable income variables.

$$\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial n_{s,t}} = \frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial x_{s,t}}\frac{\partial x_{s,t}}{\partial n_{s,t}} = w_t\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial x_{s,t}} = w_t\tau^{MTRx}_{s,t}\bigl(x_{s,t},y_{s,t}\bigr) \quad \forall s,t \tag{11.8}$$

$$\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial b_{s,t}} = \frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial y_{s,t}}\frac{\partial y_{s,t}}{\partial b_{s,t}} = r_t\frac{\partial T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{\partial y_{s,t}} = r_t\tau^{MTRy}_{s,t}\bigl(x_{s,t},y_{s,t}\bigr) \quad \forall s,t \tag{11.9}$$

It is the marginal tax rates with respect to household choice variables on the left-hand-side of (11.8) and (11.9) that we need in our theory equations (11.5) and (11.6). But it is the marginal tax rates and prices on the right-hand-side of (11.8) and (11.9) for which we have data. We call $\frac{\partial T^I}{\partial x}$ on the right-hand-side of (11.8) the marginal tax rate of labor income τ^{MTRx}, and we call $\frac{\partial T^I}{\partial y}$ on the right-hand-side of (11.9) the marginal tax rate of capital income τ^{MTRy}. With these specifications, we can restate the household Euler equations (11.5) and (11.6) in terms of τ^{MTRx} and τ^{MTRy}.

$$w_t\left(1-\tau^{MTRx}_{s,t}\right)u_1\left(c_{s,t},n_{s,t}\right) = -u_2\left(c_{s,t},n_{s,t}\right) \quad \text{for} \quad \forall t, \quad s\in\{1,2,...S\} \tag{11.10}$$

$$u_1\left(c_{s,t},n_{s,t}\right) = \beta\Biggl(1+r_{t+1}\Bigl[1-\tau^{MTRy}_{s+1,t+1}\Bigr]\Biggr)u_1\left(c_{s+1,t+1},n_{s+1,t+1}\right) \quad \forall t, \quad s\in\{1,2,...S-1\} \tag{11.11}$$

The last tax rate that we need to estimate from data is the effective tax rate ($\tau^{ETR}_{s,t}$), which is sometimes called the average tax rate or, more confusingly, the average effective tax rate. This is defined as the total tax liability divided by total income and can also be represented as a function of labor income $x_{s,t}$.

$$\tau^{ETR}_{s,t}\bigl(x_{s,t},y_{s,t}\bigr) \equiv \frac{T^I_{s,t}\bigl(x_{s,t},y_{s,t}\bigr)}{x_{s,t}+y_{s,t}} \tag{11.12}$$

The effective tax rate is an important unit-free statistic in the tax literature and is widely reported. Because it is a rate (although it has no natural bounds), it will be easier to estimate than the tax liability function $T^I_{s,t}(x_{s,t},y_{s,t})$. For this reason, we restate the tax liability term in the budget constraint as the effective tax rate times total income.

$$\begin{split} c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - \tau^{ETR}_{s,t}(x_{s,t},y_{s,t})\bigl(x_{s,t}+y_{s,t}\bigr) \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{split} \tag{11.13}$$

Note that we are proposing in this chapter to estimate separately the marginal tax rates τ^{MTRx} and τ^{MTRy} from the effective tax rate τ^{ETR}. One could just as easily parameterize a function to fit the effective tax rate τ^{ETR} and then use the characterization of (11.12) to analytically derive the two marginal tax rates τ^{MTRx} and τ^{MTRy} using the expressions (11.8) and (11.9). Deriving the expressions for the marginal tax rates analytically from the effective tax rate would be fine if the function for the effective tax rate were the true function. However, because whatever function we estimate to fit the effective tax rate is simply an approximation, we might miss some features of the marginal tax rates if we do not estimate them separately.

11.1.2 Previous literature

Incorporating household income tax law into a DGE model is often a question of what functional forms and what data to use to represent the effective tax rate τ^{ETR} and two marginal tax rates τ^{MTRx} and τ^{MTRy} that appear in equations (11.13), (11.5), and (11.6). Krueger and Ludwig (2016) ask policy relevant questions regarding tax policy and have a rich model comprised of agents with heterogeneous skill-levels, assets, and age. But they model the tax code using linear tax functions. Even models at the frontier of the dynamic of analysis of fiscal policy, such as Nishiyama (2015), impose tax functions that are progressive but do not allow for marginal rates on a particular income source to be a function of other income.

Others have used parameterized tax functions to represent the tax code in general equilibrium models. Fullerton and Rogers (1993) estimate tax rate functions that vary by lifetime income group and age, but their marginal and average rates are not functions of realized income. Zodrow and Diamond (2013) follow a similar methodology. Many of these studies use micro data to estimate the tax functions. For example, Fullerton and Rogers (1993) use the Panel Study for Income Dynamics to estimate ordinary least squares models that identify the parameters of their tax functions. Guner et al. (2014) use data from the Statistics of Income (SOI) Public Use File to calibrate average and marginal tax rate functions for various definitions of household income, separately for those with different household structures. Other examples of the estimations of flexible tax functions on labor or household income (in the U.S. and across other countries) come from Gouveia and Strauss (1994), Guvenen et al. (2014), and Holter et al. (2014). Nishiyama (2015) uses a version of the Gouveia and Strauss (1994) tax function, but does not condition tax functions on age nor does he allow marginal tax rates to be multivariate functions of the agents' different income sources. Rather, the marginal tax rate on labor income is only a function of labor income and the marginal tax rate on capital income is constant. Nishiyama (2015) uses ordinary least squares to estimate the parameters of his proposed tax functions from data produced by the Congressional Budget Office's microsimulation model.

The approach to estimating marginal tax rate functions $\frac{\partial T^I}{\partial n}$ and $\frac{\partial T^I}{\partial b}$ follows the method of DeBacker et al. (2017). We fit a parameterized functional form to microsimulation data in order to have a smooth and well-behaved, yet rich, representation of the real world tax code that can be used in a DGE model. This method for combining incorporating tax data from rich microsimulation models into a DGE model is a recent innovation.

As a comparison, we describe more deeply the functional form proposed by Gouveia and Strauss (1994), which remains one of the more flexible specifications in the literature, and one of the more widely used functional forms.[1] The Gouveia and Strauss (1994) tax function is given by:

$$T^{I,GS} = \varphi_0[I - (I^{-\varphi_1} + \varphi_2)^{\frac{-1}{\varphi_1}}], \tag{11.14}$$

where $T^{I,GS}$ is total income tax liability and $I = x + y$ is total income. We transform this function to put it in terms of an effective tax rate.

$$\tau^{ETR,GS} = \varphi_0[I - (I^{-\varphi_1} + \varphi_2)^{\frac{-1}{\varphi_1}}]/I. \tag{11.15}$$

[1] See Guvenen et al. (2014) and Nishiyama (2015).

In addition, we use our microdata from from the`Tax-Calculator` microsimulation model described in Section 11.1.4 to estimate the above ETR specification separately by tax year and age. This is not done in others' work who use the Gouveia and Strauss (1994) functional form, but this will give that specification the best chance to fit the data as closely as our preferred functional form proposed in this paper. We use the same nonlinear least squares estimator to estimate the Gouveia and Strauss (1994) functions.

In Figure 11.1 we compare the Gouveia and Strauss (1994) specification to our specification described in Section 11.1.5. In orange, we show our specification when total income is entirely made up of labor income and in green we show our specification when total income is comprised of 70% labor income and 30% capital income. The Gouveia and Strauss (1994) specification is in purple. We can note a number of differences between these specifications from this picture. First, our specification shows more ability to capture the negative average tax rates at the lower end of the income distribution. Second, given the ability to have this negative intercept, our functional form can show more curvature over lower income ranges, allowing for a better fit to the steep gradient the data show over this range. Finally, by comparing the orange and green lines, one can see the ability of the share parameter to account for the role capital income plays in lowering effective tax rates as total income increases. In particular, our specification allows for the filers portfolio of income (i.e., the shares of total income derives from labor or capital income) at affect this average tax rate, which is a novel contribution of our functional form.

To more precisely test the fit of these specification against each other, Table 11.1 presents the standard errors of the estimates from these two specifications. The table shows the over all fit and also splits these out by age. What we find is that the ratio of polynomials function we propose shows a much better fit to the data, missing the effective tax rate by just under three percentage points on average, compared to an error of 5.5 percentage points in the estimated Gouveia and Strauss (1994) model.

Note that we compare only the effective tax rate functions since Gouveia and Strauss (1994) do not separately estimate marginal tax rate functions, as we do. Instead, they derive the marginal tax rates analytically from their total tax function. The implication, then, is that the marginal rates derives in this way will not fit the data as well as the effective tax

Figure 11.1: Plot of estimated ETR functions: $t = 2017$ and $s = 42$ under current law

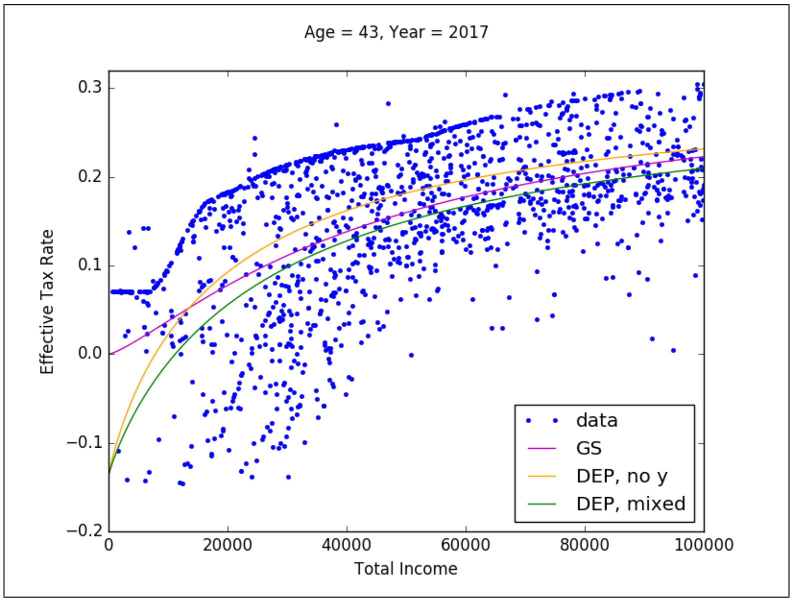


Table 11.1: Standard error of the estimates of ETR functions for age bins in period $t = 2017$

	Age ranges			
	All ages	21 to 54	55 to 65	66 to 80
Ratio of polynomials, ETR	2.98	3.42	1.67	2.22
Gouveia and Strauss (1994), ETR	5.50	6.54	2.57	3.11

rates, which were the target of the estimation. As we note in Section 11.1.1, we eschewed this approach of analytically deriving the marginal tax rates in favor of separately estimating the parameters of the effective and marginal tax rate functions. We have found this allows our model to better capture tax policy that differentially impacts average and marginal rates and to fit the data more closely.

11.1.3 Problem with actual marginal tax rates

Figure 11.2 shows the current implied marginal tax rate (MTR) schedule as a function of household income under U.S. law in October 2016 as well as the respective implied MTRs under the proposals from then-U.S. Presidential Candidates Hilary Clinton and Donald Trump. Note, in all cases, that the lumpiness of the MTR schedules creates a nonconvex function. If one were to incorporate functions with this level of real-life richness into Euler equations such as (11.5) and (11.6), it would be very difficult to find the root in just one of those equations. The difficulty is compounded exponentially when we must solve $2S-1$ or more of those equations.

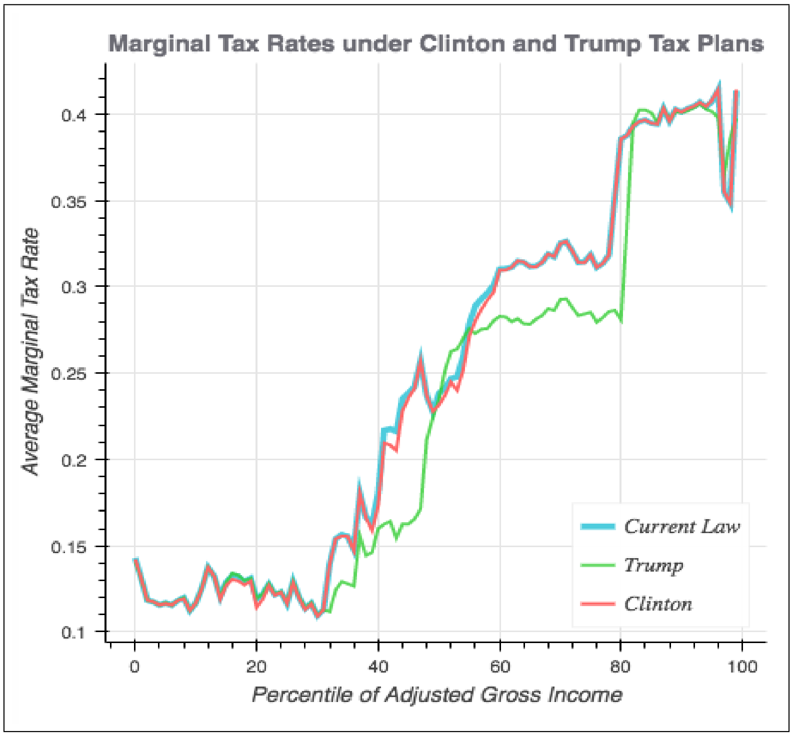
Even if we ignore the high-frequency ups and downs of the implied marginal tax rate schedules, there seems to be flat and even negatively sloped sections of the function for low incomes. This is due to tax credits for low-income individuals that go away as incomes rise. The high-frequency ups and downs come from the many dimensions of heterogeneity that exist in the real-world population on which the tax code may depend. Examples include marital status, filing status, number of children, deduction approach, and types of income.

11.1.4 Marginal tax rates from microsimulation models

The microsimulation model we use is called `Tax-Calculator` and is maintained a group of economists, software developers, and policy analysts.[2] Other than being completely open source, the `Tax-Calculator` is very similar to other tax calculators such as NBER's TaxSim and proprietary models used by think tanks and governmental organizations. For this reason,

[2] The documentation for using `Tax-Calculator` is available at http://taxcalc.readthedocs.org/en/latest/index.html A simple web application that provides an accessible user interface for `Tax-Calculator` is available from the Open Source Policy Center (OSPC) at http://www.ospc.org/taxbrain/. All the source code for the `Tax-Calculator` is freely available at https://github.com/open-source-economics/Tax-Calculator.

Figure 11.2: Implied marginal tax rate (MTR) schedules under Clinton and Trump 2016 proposals versus current law



much of what we say below generalizes if one were to use those other microsimulation models. In this section, we outline the main structure of the `Tax-Calculator` microsimulation model, but encourage the interested reader to follow the links for more detailed documentation.

`Tax-Calculator` uses microdata on a sample of tax filers from the tax year 2009 Public Use File (PUF) produced by the IRS.[3] These data contain detailed records from the tax returns of about 200,000 tax filers who were selected from the population of filers through a stratified random sample of tax returns. These data come from IRS Form 1040 and a set of the associated forms and schedules. The PUF data are then matched to the Current Population Survey (CPS) to get imputed values for filer demographics such as age, which are not included in the PUF, and to incorporate households from the population of non-filers. The PUF-CPS match includes 219,815 filers.

Since these data are for calendar year 2009, they must be "aged" to be representative of

[3]Technically, the `Tax-Calculator` could use other microdata as a source, but we choose to use the PUF for the relatively large sample size and the degree of detail provided for various income and deduction items.

the potential tax paying population in the years of interest (e.g. the current year through the end of the budget window). To do this, macroeconomic forecasts of wages, interest rates, GDP, and other variables are used to grow the 2009 values to be representative of the values one might see in subsequent years. Adjustments to the weights applied to each observation in the microdata are also made. More specifically, weights are adjusted to hit a number of targets in an optimization problem that sets out to minimize the distance between the extrapolated microdata values and the targets, with a penalty being applied for large changes in the weight individual observations from one year to the next. The targets are comprised of a number of aggregate totals of line items from Form 1040 (and related Schedules) produced by SOI for the years 2010-2013.[4]

Using these microdata, `Tax-Calculator` is able to determine total tax liability and marginal tax rates by computing the tax reporting that minimizes each filer's total tax liability given the filer's income and deductions items and the parameters describing tax law. The determination of total tax liability from the microsimulation model includes federal income taxes and payroll taxes but currently excludes state income taxes and estate taxes. The output of the microsimulation model includes forecasts of the total tax liability in each year, marginal tax rates in income sources, and items from the filers' tax returns for each of the 219,815 filers in the microdata. To calculate marginal tax rates on any given income source, the model adds one cent to the income source for each filing unit in the microdata and then computes the change in tax liability. The change in tax liability divided by the change in income (one cent) yields the marginal tax rate. Population sampling weights are determined through the extrapolation and targeting of the microsimulation model. These weights allow one to calculate population representative results from the model. One can determine changes in tax liability and marginal tax rates across different tax policy options by doing the same simulation where the parameters describing the tax policy are updated to reflect the proposed policy rather than the baseline policy. The baseline policy used by `Tax-Calculator` is a current-law baseline.

To map the output of the microsimulation model, which is based on income reported on

[4] For details on how these data are extrapolated, please see the `Tax Data` program and associated documentation.

tax returns, to the DGE model, where income is defined more broadly, we use the following definitions. In computing the effective tax rates from the microsimulation model, we divided total tax liability by a measure of "adjusted total income". Adjusted total income is defined as total income (Form 1040, line 22) plus tax-exempt interest income, IRA distributions, pension income, and Social Security benefits (Form 1040, lines 8b, 15a, 16a, and 20a, respectively). We consider adjusted total income from the microsimulation model to be the counterpart of total income in the DGE model. Total income in the DGE model is the sum of capital and labor income.

We define labor income as earned income, which is the sum of wages and salaries (Form 1040, line 7) and self-employment income (Form 1040 lines 12 and 18) from the microdata. Capital income is defined as a residual.[5]

To get the marginal tax rate on composite income amounts (e.g., labor income that is the sum of wage and self-employment income), we take a weighted average that accounts for negative income amounts. In particular, to we calculate the weighted average marginal tax rate on composite of n income sources as:

$$MTR_{composite} = \frac{\sum_{i=1}^{n} MTR_n * abs(Income_n)}{\sum_{i=1}^{n} abs(Income_n)} \tag{11.16}$$

When we look at the raw output from the microsimulation model, we find that there are several observations with extreme values for their effective tax rate. Since this is a ratio, such outliers are possible, for example when the denominator, adjusted total income, is very small. We omit such outliers by making the following restrictions on the raw output of the microsimulation model. First, we exclude observations with an effective tax rate greater than 1.5 times the highest statutory marginal tax rate. Second, we exclude observations where the effective tax rate is less than the lowest statutory marginal tax rate on income minus the maximum phase-in rate for the Earned Income Tax Credit (EITC). Third, we drop observations with marginal tax rates in excess of 99% or below the negative of the highest EITC rate (i.e., -45% under current law). These exclusions limit the influence of those with

[5]This is not an ideal definition of capital income, since it includes transfers between filers (e.g., alimony payments) and from the government (e.g., unemployment insurance), but we have chosen this definition in order to ensure that all of total income is classified as either capital or labor income.

extreme values for their marginal tax rate, which are few and usually result from the income of the filer being right at a kink in the tax schedule. Finally, since total income cannot be negative in the DGE model we use, we drop observations from the microsimulation model where adjusted total income is less than \$5.[6]

Because the tax rates are estimated as functions of income levels in the microdata, we have to adjust the model income units to match the units of the microdata. To do this, we find the $factor$ such that $factor$ times average steady-state model income equals the mean income in the final year of the microdata.

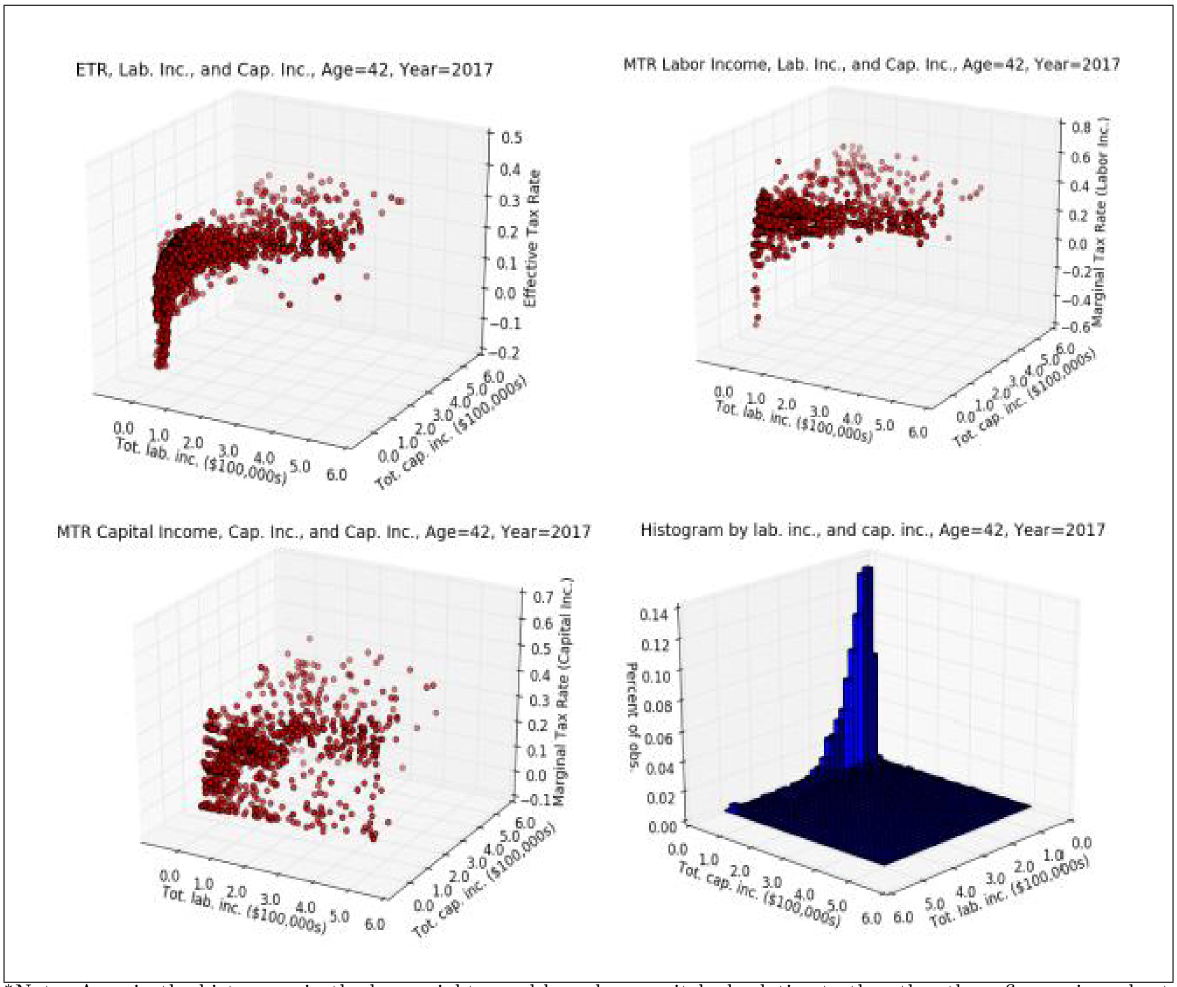
$$factor\sum_{s=1}^{S}\left(\bar{w}\bar{n}_s + \bar{r}\bar{b}_s\right) = (\text{data avg. income}) \tag{11.17}$$

To be precise, the income levels in the model, x and y, must be multiplied by this factor when they are used in the effective tax rate functions, marginal tax rate of labor income functions, and marginal tax rate of capital income functions of the form in Equation (11.18).

Figure 11.3 shows scatter plots of effective tax rates τ^{ETR}, marginal tax rates on labor income τ^{MTRx}, marginal tax rates on capital income τ^{MTRy}, and a histogram of the data points from the `Tax-Calculator` microsimulation model, each plotted as a function of labor income and capital income for all 42-year-olds in the year 2017. The data we use in the `Tax-Calculator` come from the 2009 IRS Public Use File and a statistical match of the Current Population Survey (CPS) demographic data. Labor and capital income are truncated at \$600,000 in order to more clearly see the shape of the data in spite of the long right tail of the income distribution. Although there is noise in the data, effective tax rates are generally increasing in both labor and capital income at a decreasing rate from some slightly negative level to an asymptote around 30 percent. This regular shape in effective tax rates is observed for all ages in all years of the budget window, 2017-2026.

[6]We choose \$5 rather than \$0 to provided additional assurance that small income values are not driving large ETRs.

Figure 11.3: Scatter plot of ETR, MTRx, MTRy, and histogram as functions of labor income and capital income from microsimulation model: $t = 2017$ and $s = 42$ under current law



*Note: Axes in the histogram in the lower-right panel have been switched relative to the other three figures in order to see the distribution more clearly.

11.1.5 Estimating ETR and MTR functions

Because of the regularity in the shape of the effective tax rates as shown in Figure 11.3, we choose to fit a smooth functional form to these data that is able to parsimoniously fit this shape while also being flexible enough to adjust to a wide range of tax policy changes. Our functional form, shown as a general tax rate function in (11.18), is a Cobb-Douglas aggregator of two ratios of polynomials in labor income x and capital income y. We use the same functional form for the effective and marginal tax rate functions. Important properties of this functional form are that it produces this bivariate negative exponential shape, is monotonically increasing in both labor income and capital income, and that it allows for negative tax rates. In order to capture variation in taxes by filer age and model year, we estimate functions for each model age and every year of the budget-window that the microsimulation model captures. In this way, we are able to map more of the heterogeneity from the microsimulation into the macro model than can be explicitly incorporated into a DGE model.

As an example, filing status is correlated with age and income. Thus, although the DGE model we use does not explicitly account for filing status, we are able to capture some of the effects of filing status on tax rates by having age and income dependent functions for effective and marginal tax rates. As another example, investment portfolio decisions differ over the life cycle and these are difficult to model in detail in a DGE model. By using age-dependent tax functions, we are able to capture some of the differentials in tax treatment across different assets (e.g. rates on dividends versus capital gains, tax-preferred retirement savings accounts, certain exemptions for interest income) even if the DGE model does not explicitly model these portfolio decisions.

Finally, consider that many macroeconomic models assume a single composite consumption good. Some of this composite good affects tax liability, such as the consumption of charitable contributions or housing. To the extent that the fraction of the composite good that comes from such consumption varies over a household's income and age, these tax functions will capture that, since they are fitted using microeconomic data that includes information on these tax-relevant forms of consumption.

Let x be total labor income, $x \equiv \hat{w}_t n_{s,t}$, and let y be total capital income, $y \equiv r_t \hat{b}_{s,t}$. We then write our tax rate functions as follows.

$$\begin{aligned}
\tau(x,y) = &\Bigl[\tau(x) + shift_x\Bigr]^{\phi}\Bigl[\tau(y) + shift_y\Bigr]^{1-\phi} + shift \\
&\text{where}\quad \tau(x) \equiv (max_x - min_x)\left(\frac{Ax^2 + Bx}{Ax^2 + Bx + 1}\right) + min_x \\
&\text{and}\quad \tau(y) \equiv (max_y - min_y)\left(\frac{Cy^2 + Dy}{Cy^2 + Dy + 1}\right) + min_y \\
&\text{where}\quad A, B, C, D, max_x, max_y, shift_x, shift_y > 0 \quad\text{and}\quad \phi \in [0,1] \\
&\text{and}\quad max_x > min_x \quad\text{and}\quad max_y > min_y
\end{aligned} \tag{11.18}$$

Note that we let $\tau(x,y)$ represent the effective and marginal rate functions, $\tau^{ETR}(x,y)$, $\tau^{MTRx}(x,y)$ and $\tau^{MTRy}(x,y)$. We assume the same functional form for each of these functions. The parameters values will, in general, differ across the different functions (effective and marginal rate functions) and by age, s, and tax year, t. We drop the subscripts for age and year from the above exposition for clarity.

By assuming each tax function takes the same form, we are breaking the analytical link between the the effective tax rate function and the marginal rate functions as we discussed at the end of Section 11.1.1. In particular, one could assume an effective tax rate function and then use the analytical derivative of that to find the marginal tax rate function. However, we've found it useful to separately estimate the marginal and average rate functions. One reason is that we want the tax functions to be able to capture policy changes that have differential effects on marginal and average rates. For example, and relevant to the policy experiment we present below, a change in the standard deduction for tax payers would have a direct effect on their average tax rates. But it will have secondary effect on marginal rates as well, as some filers will find themselves in different tax brackets after the policy change. These are smaller and second order effects. When tax functions are are fit to the new policy, in this case a lower standard deduction, we want them to be able to represent this differential impact on the marginal and average tax rates. The second reason is related to the first. As the additional flexibility allows us to model specific aspects of tax policy more closely, it also allows us to better fit the parameterized tax functions to the data.

The key building blocks of the functional form in equation (11.18) are the $\tau(x)$ and $\tau(y)$ univariate functions. The ratio of polynomials in the $\tau(x)$ function $\frac{Ax^2+Bx}{Ax^2+Bx+1}$ with positive coefficients $A, B > 0$ and positive support for labor income $x > 0$ creates a negative-exponential-shaped function that is bounded between 0 and 1, and the curvature is governed by the ratio of quadratic polynomials. The multiplicative scalar term $(max_x - min_x)$ on the ratio of polynomials and the addition of min_x at the end of $\tau(x)$ expands the range of the univariate negative-exponential-shaped function to $\tau(x) \in [min_x, max_x]$. The $\tau(y)$ function is an analogous univariate negative-exponential-shaped function in capital income y, such that $\tau(y) \in [min_y, max_y]$.

The respective $shift_x$ and $shift_y$ parameters in the functional form (11.18) are analogous to the additive constants in a Stone-Geary utility function. These constants ensure that the two sums $\tau(x) + shift_x$ and $\tau(y) + shift_y$ are both strictly positive. They allow for negative tax rates in the $\tau(\cdot)$ functions despite the requirement that the arguments inside the brackets be strictly positive. The general $shift$ parameter outside of the Cobb-Douglas brackets can then shift the tax rate function so that it can accommodate negative tax rates. The Cobb-Douglas share parameter $\phi \in [0,1]$ controls the shape of the function between the two univariate functions $\tau(x)$ and $\tau(y)$.

This functional form for tax rates delivers flexible parametric functions that can fit the tax rate data shown in Figure 11.3 as well as a wide variety of policy reforms. Further, these functional forms are monotonically increasing in both labor income x and capital income y. This characteristic of monotonicity in x and y is essential for guaranteeing convex budget sets and thus uniqueness of solutions to the household Euler equations. The assumption of monotonicity does not appear to be a strong one when viewing the the tax rate data shown in Figure 11.3. While it does limit the potential tax systems to which one could apply our methodology, tax policies that do not satisfy this assumption would result in non-convex budget sets and thus require non-standard DGE model solutions methods and would not guarantee a unique equilibrium. The 12 parameters of our tax rate functional form from (11.18) are summarized in Table 11.2.

We estimate a transformation of the τ^{ETR}, τ^{MTRx}, and τ^{MTRy} tax rate functions described in functional form equation (11.18) for each age s of the primary filer and time period t in our

Table 11.2: Description of tax rate function $\tau(x,y)$ parameters

Symbol	Description
A	Coefficient on squared labor income term x^2 in $\tau(x)$
B	Coefficient on labor income term x in $\tau(x)$
C	Coefficient on squared capital income term y^2 in $\tau(y)$
D	Coefficient on capital income term y in $\tau(y)$
max_x	Maximum tax rate on labor income x given $y = 0$
min_x	Minimum tax rate on labor income x given $y = 0$
max_y	Maximum tax rate on capital income y given $x = 0$
min_y	Minimum tax rate on capital income y given $x = 0$
$shift_x$	shifter $> \|min_x\|$ ensures that $\tau(x) + shift_x > 0$ despite potentially negative values for $\tau(x)$
$shift_y$	shifter $> \|min_y\|$ ensures that $\tau(y) + shift_y > 0$ despite potentially negative values for $\tau(y)$
$shift$	shifter (can be negative) allows for support of $\tau(x,y)$ to include negative tax rates
ϕ	Cobb-Douglas share parameter between 0 and 1

data and budget window, respectively (2,400 separate specifications). We transform these functions so that the labor income, x, and capital income, y, variables in the polynomials are transformed to percent deviations from their respective means. This helps with the scale of the variables in the optimization routine. The transformed ETR and MTR functions are estimated using a constrained, weighted, non-linear least squares estimator. The weighting in this estimator come from the weights assigned to the filers in the microsimulation model.

Let $\boldsymbol{\theta}_{s,t} = (A, B, C, D, max_x, min_x, max_y, min_y, shift_x, shift_y, shift, \phi)$ be the full vector of 12 parameters of the tax function for a particular age of filers in a particular year. We first directly specify min_x as the minimum tax rate in the data for age-s and period-t individuals for capital income close to 0 ($\$0 < y < \$3,000$) and min_y as the minimum tax rate for labor income close to 0 ($\$0 < x < \$3,000$). We then set $shift_x = |min_x| + \varepsilon$ and $shift_y = |min_y| + \varepsilon$ so that the respective arguments in the brackets of (11.18) are strictly positive. Let $\bar{\boldsymbol{\theta}}_{s,t} = \{min_x, min_y, shift_x, shift_y\}$ be the set of parameters we take directly from the data in this way.

We then estimate eight remaining parameters $\tilde{\boldsymbol{\theta}}_{s,t} = (A, B, C, D, max_x, max_y, shift, \phi)$

using the following nonlinear weighted least squares criterion,

$$
\begin{aligned}
\hat{\boldsymbol{\theta}}_{s,t} = \tilde{\boldsymbol{\theta}}_{s,t}: \quad &\min_{\tilde{\boldsymbol{\theta}}_{s,t}} \sum_{i=1}^{N}\Bigl[\tau_i - \tau_{s,t}\bigl(x_i, y_i|\tilde{\boldsymbol{\theta}}_{s,t}, \bar{\boldsymbol{\theta}}_{s,t}\bigr)\Bigr]^2 w_i, \\
&\text{subject to} \quad A, B, C, D, max_x, max_y > 0, \\
&\text{and} \quad max_x \geq min_x, \quad \text{and} \quad max_y \geq min_y \quad \text{and} \quad \phi \in [0,1]
\end{aligned} \tag{11.19}
$$

where τ_i is the effective (or marginal) tax rate for observation i from the microsimulation output, $\tau_{s,t}(x_i, y_i|\tilde{\boldsymbol{\theta}}_{s,t}, \bar{\boldsymbol{\theta}}_{s,t})$ is the predicted average (or marginal) tax rate for filing-unit i with x_i labor income and y_i capital income given parameters $\boldsymbol{\theta}_{s,t}$, and w_i is the CPS sampling weight of this observation. The number N is the total number of observations from the microsimulation output for age s and year t. Figure 11.4 shows the typical fit of an estimated tax function $\tau_{s,t}\bigl(x, y|\hat{\boldsymbol{\theta}}_{s,t}\bigr)$ to the data. The data in Figure 11.4 are the same age $s = 42$ and year $t = 2017$ as the data Figure 11.3.

The underlying data can limit the number of tax functions that can be estimated. For example, we use the age of the primary filer from the PUF-CPS match to be equivalent to the age of the DGE model household. The DGE model we use allows for individuals up to age 100, however the data contain few primary filers with age above age 80. Because we cannot reliably estimate tax functions for $s > 80$, we apply the tax function estimates for 80 year-olds to those with model ages 81 to 100. In the case certain ages below age 80 have too few observations to enable precise estimation of the model parameters, we use a linear interpolation method to find the values for those ages $21 \leq s < 80$ that cannot be precisely estimated.[7]

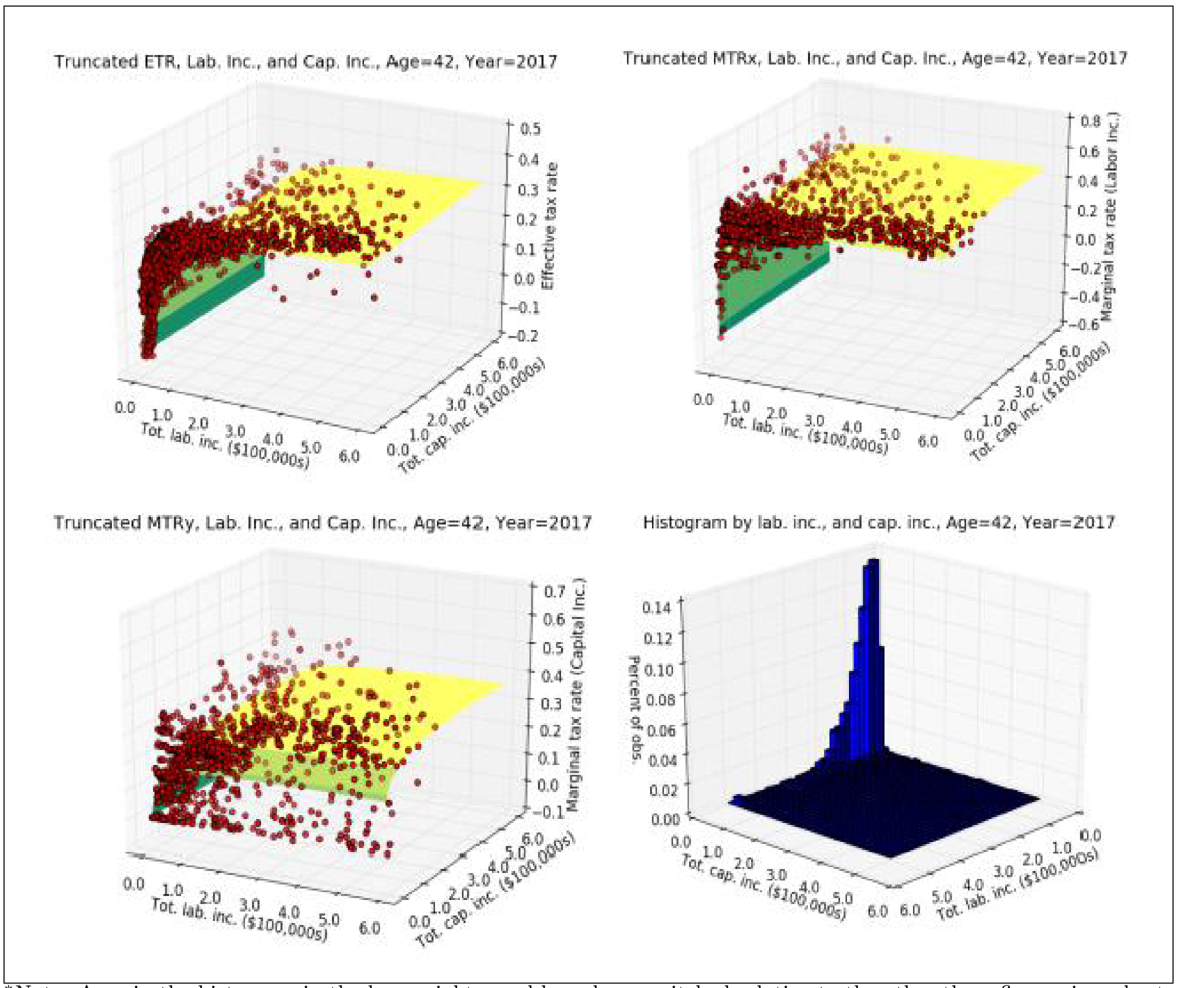
Figure 11.4 shows the estimated function surfaces for tax rate functions for the effective tax rate τ^{ETR}, marginal tax rate on labor income τ^{MTRx}, and marginal tax rate on capital income τ^{MTRy} data shown in Figure 11.3 for age $s = 42$ individuals in period $t = 2017$ under the current law. And the estimated parameters and the corresponding function surface change whenever any of the many policy levers in the microsimulation model that generate the tax rate data are adjusted. The total tax liability function is simply the effective tax

[7] We use two criterion to determine whether the function should be interpolated. First, we require a minimum number of observations of filers of that age and in that tax year. Second, we require that that sum of squared errors meet a pre-defined threshold.

rate function times total income $\tau(x,y)(x+y)$, which is simply a transformation of equation (11.12).

$$
\begin{aligned}
T^I_{s,t}(x,y) &\equiv ETR_{s,t}(x,y)\big(x+y\big) \\
&= \left(\Big[\tau_{s,t}(x) + shift_{x,s,t}\Big]^{\phi}_{s,t}\Big[\tau_{s,t}(y) + shift_{y,s,t}\Big]^{1-\phi_{s,t}} + shift_{s,t}\right)\big(x+y\big)
\end{aligned} \tag{11.20}
$$

Figure 11.4: Estimated tax rate functions of ETR, MTRx, MTRy, and histogram as functions of labor income and capital income from microsimulation model: $t = 2017$ and $s = 42$ under current law



*Note: Axes in the histogram in the lower-right panel have been switched relative to the other three figures in order to see the distribution.

As we describe above, each rate function $(\tau^{ETR}, \tau^{MTRx}, \tau^{MTRy})$ varies by age, s, and tax year t. This means a large number of parameters must be estimated. In particular, using our illustrative example with the model of DeBacker et al. (2017b), we will need to fit 12 parameters for each of three tax rate functions for each age (21 to 100) during each of the 10 years of the budget window with the estimated functions in the last year of the budget window assumed to be permanent. The microsimulation model we use, `Tax-Calculator` is able to provide marginal and average tax rates for 10-years forward from the present.[8] The DGE model is solved from the current period forward through the steady-state. The steady-state is generally arrived at well beyond a time horizon of 10 years. Thus we allow variation the in the rate functions only over this 10-year budget window and fix the parameters of the rate functions to the last year of the window for years $t \geq 10$. Thus, in our illustrative example, there are 2,400 tax rate functions comprised of 28,800 parameters.

Because we allow these many functions of labor income and capital income to be independently estimated for each tax rate type, age, and year, we can capture many of the characteristics and discrete variation in the tax code while still preserving the smoothness and monotonicity of the tax functions within each type, age, and year. This monotonicity and smoothness is sufficient to guarantee uniqueness and tractability of the computational solution of the household Euler equations. Allowing for different tax rate functions by age and time period also implicitly incorporates heterogeneity in the data in dimensions that we cannot model in the DGE model, such as broader income items, deductions items, credits, and filing unit structure. The effect of such heterogeneity on tax burdens will affect the effective tax rate functions we fit to the output of the microsimulation model.

Table 11.3: Average values of ϕ for ETR, MTRx, and MTRy for age bins in period $t = 2017$

	Age ranges			
	21 to 54	55 to 65	66 to 80	All years
ETR	0.66	0.28	0.38	0.44
$MTRx$	0.89	0.31	0.23	0.48
$MTRy$	0.77	0.25	0.14	0.43

* Note: Even though agents in the OG model live until age 100, the tax data was too sparse to estimate functions for ages greater than 80. For ages 81 to 100, we simply assumed the age 80 estimated tax functional forms.

It is difficult to show all the estimated tax functions for every age and period in the budget

[8] This is the standard timeframe considered by policy analysts analyzing the effects of tax policy on the federal budget.

window. But Table 11.3 gives a description of the estimated values of the ϕ parameter. This parameter ϕ in the tax function (11.18) governs how important the interaction is between labor income and capital income for determining tax rates. The further interior is ϕ (away from 0 or 1), the more important it is to model tax rates as functions of both labor income and capital income. And the closer ϕ is to 1, the more important is labor income for determining tax rates.

Two key results jump out from Table 11.3. First, it is clear that the interaction between labor income and capital income is significant at all ages for determining effective tax rates ETR, marginal tax rates on labor income $MTRx$, and marginal tax rates on capital income $MTRx$. The last column of Table 11.3 shows the average *phi* value for all ages in the data to be around 0.45 for all three tax rate types. This suggests that models that use univariate tax functions of any type of income miss important information and incentives present in the tax code.

A second result from Table 11.3 is that the relative importance of labor income in determining tax rates varies over the life cycle in similar ways for each tax rate type (τ^{ETR}, τ^{MTRx}, and τ^{MTRy}). The first three columns of each row of Table 11.3 show that labor income is most important for determining tax rates between the ages of 21 and 54 and that capital income is most important for determining tax rates between the ages of 55 and 65. For marginal tax rates capital income continues to be the most important determinant after age 65, but capital income and labor income are equally important determinants of the effective tax rate ETR after age 65. This suggests that models that use tax functions that do not vary with age also miss some important information and incentives present in the tax code.

11.2 Households

A unit measure of identical individuals are born each period and live for S periods. The budget constraint faced by these households in each period is similar to the on presented in Chapter 4. However, households in this model pay taxes on some function of their total income $T^I_{s,t}$ and receive transfers from the government X_t each period.

$$\begin{split}c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0\end{split} \tag{11.4}$$

We specify taxes and transfers in (11.4) such that if a household has a positive tax liability to the government, then T^I in positive ($T^I > 0$). And if the household receives a transfer from the government, then X_t is positive ($X_t > 0$).

For simplicity, we assume that transfers X_t are not a function of household labor supply $n_{s,t}$, wealth $b_{s,t}$, or savings $b_{s+1,t+1}$. We also assume that the tax levied on households $T^I_{s,t}$ is only a function of factors that influence current period income, namely $n_{s,t}$ and $b_{s,t}$, but is not a function of factors like savings $b_{s+1,t+1}$ that influence income in the next period. But these assumptions could be easily relaxed.

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^S \beta^{s-1} u(c_{s,t+s-1}, n_{s,t+s-1}) \tag{4.6}$$

$$\text{s.t.} \quad c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \tag{11.4}$$

$$\text{where} \quad u\bigl(c_{s,t}, n_{s,t}\bigr) = \frac{c_{s,t}^{1-\sigma} - 1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{4.7}$$

where $u(c_{s,t}, n_{s,t})$ is the period utility function with elliptical disutility of labor from Chapter 4, and χ^n_s is a scale parameter that can potentially vary by age s influencing the relative disutility of labor to the utility of consumption. The household's lifetime problem (4.6) can be reduced to choosing S labor supplies $\{n_{s,t+s-1}\}_{s=1}^S$ and $S-1$ savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ by substituting the budget constraints (11.4) in for $c_{s,t}$ in each period utility function (4.7) of the lifetime utility function.

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (4.8), the following $S-1$ dynamic savings

Euler equations (4.9), and a budget constraint that binds in all S periods (11.4),

$$\begin{aligned} &\left(w_t - \frac{\partial T^I_{s,t}}{\partial n_{s,t}}\right) u_1\left(c_{s,t+s-1}, n_{s,t+s-1}\right) = -u_2\left(c_{s+1,t+s}, n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S\} \\ &\Rightarrow \quad \left(w_t - \frac{\partial T^I_{s,t}}{\partial n_{s,t}}\right)\left(c_{s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \end{aligned} \tag{11.5}$$

$$\begin{aligned} &u_1\left(c_{s,t+s-1}, n_{s,t+s-1}\right) = \beta\left(1 + r_{t+1} - \frac{\partial T^I_{s+1,t+1}}{\partial b_{s+1,t+1}}\right) u_1\left(c_{s+1,t+s}, n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\} \\ &\Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta\left(1 + r_{t+1} - \frac{\partial T^I_{s+1,t+1}}{\partial b_{s+1,t+1}}\right)\left(c_{s+1,t+1}\right)^{-\sigma} \end{aligned} \tag{11.6}$$

$$c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + X_t - T^I_{s,t} \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t}, b_{S+1,t+S} = 0 \tag{11.4}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$. As was demonstrated in detail in Section 3.1, the dynamic Euler equations (11.6) do not include marginal utilities of all future periods because of the principle of optimality and the envelope condition.

The distortionary effect of taxation is evidenced in these two behavioral equations (11.5) and (11.6) by the new terms in parentheses which include the respective partial derivatives of $\frac{\partial T^I}{\partial n}$ and $\frac{\partial T^I}{\partial b}$. These are the marginal tax rates with respect to labor supply and with respect to savings, respectively. The simple transformation of multiplying each of these derivatives by the wage w_t and interest rate r_{t+1}, respectively, will transform them into the marginal tax rate with respect to labor income and the marginal tax rate with respect to capital income.

Note that these $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u, r_u\}_{u=t}^{t+S-1}$. As in Section 3.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (3.14) and (2.17).

$$\mathbf{\Gamma}_t \equiv \big\{b_{s,t}\big\}_{s=2}^{S} \quad \forall t \tag{3.14}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{2.17}$$

11.3 Firms

Firms are characterized exactly as in Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (2.20) and its aggregate labor decision L_t governed by first order condition (2.21).

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{2.20}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

The per-period depreciation rate of capital is $\delta \in [0,1]$, the capital share of income is $\alpha \in (0,1)$, and total factor productivity is $A > 0$.

11.4 Market Clearing

11.5 Equilibrium

11.6 Solution Method

11.6.1 Steady-state equilibrium

11.6.2 Transition path equilibrium

11.7 Calibration

11.8 Exercises

Chapter 12

Unbalanced government budget constraint

In this chapter, we build off of the S-period-lived agent model from Chapter 4 in order to understand how one may model a government that can run deficits for a finite number of periods. In order to understand such a model of the government's budget constraint, we add linear taxes on labor, capital, and corporate income as the source of the government's revenue. The government uses tax revenues and debt to finance spending on a government consumption good and lump sum transfers to households.

12.1 Households

The basic structure of the household's problem from Chapter 4 remains the same, but the addition of taxes and transfers alters the household's budget constraint and thus the necessary conditions describing optimal savings and labor supply.

Letting τ^l_t and τ^k_t represent the constant tax rate on labor and capital income, respectively, and letting $x_{s,t}$ represent government transfers to households of age s in period t, we can write the household's budget constraint as:

$$\begin{aligned} c_{s,t} + b_{s+1,t+1} = (1 + (1-\tau^k_t)r_t)b_{s,t} + (1-\tau^l_t)w_t n_{s,t} + x_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{aligned} \tag{2.1}$$

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^{S}$, labor supply $\{n_{s,t+s-1}\}_{s=1}^{S}$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^{S},\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^{S}\beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \tag{4.6}$$

$$\text{s.t.} \quad c_{s,t}+b_{s+1,t+1} \tag{12.1}$$

$$= (1+(1-\tau_t^k)r_t)b_{s,t}+(1-\tau_t^l)w_t n_{s,t}+x_t \tag{12.2}$$

$$\text{where} \quad u\big(c_{s,t},n_{s,t}\big) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma}+\chi_s^n b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \tag{4.7}$$

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (12.3), the following $S-1$ dynamic savings Euler equations (12.4), and a budget constraint that binds in all S periods (12.2),

$$\begin{aligned}&(1-\tau_t^l)w_t u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = -u_2\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S\}\\ &\Rightarrow \quad (1-\tau_t^l)w_t\left(c_{s,t}\right)^{-\sigma} = \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}\end{aligned} \tag{12.3}$$

$$\begin{aligned}&u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = \beta(1+(1-\tau_t^k)r_{t+1})u_1\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\}\\ &\Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta(1+(1-\tau_t^k)r_{t+1})\left(c_{s+1,t+1}\right)^{-\sigma}\end{aligned} \tag{12.4}$$

$$c_{s,t}+b_{s+1,t+1} = (1+(1-\tau_t^k)r_t)b_{s,t}+(1-\tau_t^l w_t n_{s,t} \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t},b_{S+1,t+S}=0 \tag{12.2}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$.

These $2S-1$ household decisions are perfectly identified if the household knows what prices and government transfers will be over its lifetime $\{w_u,r_u,x_u\}_{u=t}^{t+S-1}$. As in section 3.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (3.14) and (2.17).

$$\mathbf{\Gamma}_t \equiv \left\{b_{s,t}\right\}_{s=2}^{S} \quad \forall t \tag{3.14}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}$$

12.2 Firms

Firms are characterized similarly to Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (12.6) and its aggregate labor decision L_t governed by first order condition (12.7). However, the addition of the corporate income tax alters the firm's profit maximization problem slightly.

The firm seeks to maximize after-tax profits and thus solves,

$$\max_{K_t,L_t} \left(1-\tau^c_t\right)\left(Y_t - w_t L_t\right) - (r_t+\delta)K_t + \tau^c_t \delta K_t \tag{12.5}$$

Note that the corporate income tax is levied on accounting profits. Thus wage expenses and depreciation expenses are deductible, but payments to capital are not.

In the presence of the corprate income tax, the two first order conditions that characterize firm optimization are the following.

$$r_t = \left(1-\tau^c_t\right)\left(\alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta\right) \tag{12.6}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{12.7}$$

12.3 Government

The government takes in tax revenues and uses those revenues and borrowing to finance government purchases, G_t, and transfers, $X_t = \sum_{s=1}^{S} x_{s,t}$. In this model, we'll assume that these transfers are distributed lump-sum to all agents. Thus, $x_{s,t} = \frac{X_t}{S}\ \forall\ s$. We let D_t denote the stock of government debt at time t and R_t denote total government tax revenue. Thus we write the government budget constraint as:

$$D_{t+1} + R_t = (1 + r_t)D_t + G_t + X_t \tag{12.8}$$

Revenues are given by:

$$R_t = \underbrace{\tau_t^c \left(Y_t - w_t L_t\right) - \tau_t^c \delta K_t}_{\text{Corporate income tax revenue}} + \underbrace{\sum_{s=1}^{S} \tau_t^l w_t n_{s,t} + \sum_{s=2}^{S} \tau_t^k r_t b_{s,t}}_{\text{Individual income tax revenue}} \tag{12.9}$$

12.3.1 Budget Closure Rule

We've specified how government debt evolves (Equation 12.8) and how revenues are determined. We need to also specify how government purchase and transfers are determined. Before we do that, we point out that that government's budget must balance over the infinite horizon. In this simple model, if government debt grows in the steady-state, then at some point, payments of interest would exceed total economic output, which is not feasible. In a richer model, with economic growth, government debt can grow, but it cannot outpace the growth of total output for the same reason noted in the previous sentence. What we do to impose this fiscal condition is to alter the path of government spending in order to hit a target debt to GDP ratio in the steady-state.

We assume that transfers are a constant fraction of GDP in all periods:

$$X_t = \alpha_X * Y_t \tag{12.10}$$

Given this path for transfers and revenues, G_t adjusts to stabilize the debt to GDP ratio. Let α_D be the target debt to GDP ratio in the steady state. In the initial periods, we assume that the ratio of government spending to GDP remains constaint, $G_t = \alpha_G * Y_t$. At some future period, t_{G1}, government spending begins to adjust to move towards this target debt to GDP ratio. At priod t_{G2}, government spending is adjusted to hit the target debt to GDP ratio, if it has not already been reached. Letting ρ_G be the paramter that describes how quickly the convergences to the steady state debt to GDP ratio takes place and, we right the law of motion for government spending as:

$$
\begin{aligned}
&\text{For } t < t_{G1}: \\
&\qquad G_t = \alpha_G Y_t \\
&\text{For } t_{G1} \leq t < t_{G2}: \\
&\qquad G_t = [\rho_G \alpha_D Y_t + (1-\rho_G) D_t] - (1+r_t)D_t - X_t + R_t \\
&\text{For } t \geq t_{G2}: \\
&\qquad G_t = \alpha_D Y_t - (1+r_t)D_t - X_t + R_t
\end{aligned}
\tag{12.11}
$$

This law of motion for the fiscal variables, together with the amount of debt in the initial period of the time path, will allow us to solve for the values government spending and debt at each period. We set the intial debt level to match the debt to GDP ratio in the economy in the year we want the model to being. Let α_{D0} represent the debt to GDP ratio in the initial period. Thus $D_1 = \alpha_{D0} * Y_1$.

A few notes on this closure rule are in order. First, we chose to adjust government spending because of the simplicity of doing so. Since government spending does not enter into the household's utility function, it's level does not affect the solution of the household problem. This simplifies the model solution significantly. That said, one could chose to adjust taxes or transfers to close the budget (or a combination of all of the government fiscal policy levers). Second, since government spending is doing all of the lifting to hit the target debt to GDP ratio, it is possible that government spending is forced to be less than zero to make this happen. This would be the case if tax revenues bring in less than is needed to financed transfers and interest payments on the national debt. None of the equations we've specified above preclude that result, but it does raise conceptual difficulties. Namely, what does it mean for government spending to be negative? Is the government selling off pubic assets? We caution those using this budget closure rule to consider carefully how the budget is closed in the long run given their parameterization. We'll also note that such difficulties present themselves across all budget closure rules when analyzing tax or spending proposals that induce strucutral budget deficits.

12.4 Market Clearing

Four markets must clear in this model: the labor market, the capital market, the bond market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t^d = L_t^s \quad \forall t \tag{12.12}$$

$$K_t^d = K_t^s \quad \forall t \tag{12.13}$$

$$D_t^d = D_t^s \quad \forall t \tag{12.14}$$

$$\begin{split} &Y_t = C_t + I_t \quad \forall t \\ &\quad\text{where}\quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{split} \tag{12.15}$$

The goods market clearing equation (12.15) is redundant by Walras' Law.

Labor supply is straight forward and given by $L_t^s = \sum_{s=1}^S n_{s,t}$. In this model households hold two differnt assets; capital and government debt. Both are risk free and thus will yield the same rate of return in equilibrium.[1] Given this, we do not differentiate between the household's holdings of capital and government debt. Rather, $b_{s,t}$ represents total assets held by a household of age s at time t and is (potentially) a mix of capital and government debt. Thus, we have that

$$K_t^s + D_t^s = B_t = \sum_{i=2}^S b_{s,t}, \tag{12.16}$$

and thus by the capital and bond market clearing conditions we have:

$$B_t = \sum_{i=2}^S b_{s,t} = K_t^d + D_t^d \tag{12.17}$$

12.5 Equilibrium

An equilibrium is found when:

[1] Though in priciple this assumption could be relaxed and a wedge placed between the return on government debt and private capital.

i. Households optimize according to equations (12.3) and (12.4).

ii. Firms optimize according to (12.6) and (12.7).

iii. Government debt evolves according to (12.8) and (12.11).

iv. Markets clear according to (12.12) and (12.13).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

We can characterize the equilibrium solution by solving for the amount of government debt, D_t^d, as a function of the household's decisions and then substituting the market clearing conditions (12.12), (12.13), and (12.17) into the firm's optimal conditions (12.6) and (12.7) to solve for the equilibrium wage and interest rate as functions of the distribution of capital.

To find debt, we use the government budget constraint as characterized in (12.8) along with the exogenous laws of motion describing government spending nad transfers.

$$
\begin{aligned}
D_{t+1} &= (1+r_t)D_t + G_t + X_t - R_t \\
&= (1+r_t)D_t + \alpha_G Y_t + \alpha_X Y_t - \tau_t^c\left(Y_t - w_t L_t\right) - \tau_t^c \delta K_t^d - \sum_{s=1}^{S} \tau_t^l w_t n_{s,t} - \sum_{s=1}^{S} \tau_t^k r_t b_{s,t} \\
&= (1+r_t)D_t + (\alpha_G + \alpha_X)\left(A_t\left(\sum_{s=1}^{S} b_{s,t}\right)^{\alpha}\left(\sum_{s=1}^{S} n_{s,t}\right)^{1-\alpha}\right) - \tau_t^c\left(\left(A_t\left(\sum_{s=1}^{S} b_{s,t}\right)^{\alpha}\left(\sum_{s=1}^{S} n_{s,t}\right)^{1-\alpha}\right) - \right. \\
&\quad - \tau_t^c \delta \sum_{s=1}^{S} b_{s,t} - \sum_{s=1}^{S} \tau_t^l w_t n_{s,t} - \sum_{s=1}^{S} \tau_t^k r_t b_{s,t}
\end{aligned}
\tag{12.18}
$$

Note that the initial debt to GDP ratio is an exogenous initial condition and the steady-state target debt to GDP ratio is also exogenous. Thus the path for debt is entirely pinned down by the distribution of savings and labor supply, $D_t(\boldsymbol{\Gamma}_t)$.

We can then use the firm's first order conditions together with the market clearing conditons to show that the equilibrium interest rate and wage rates are functions of the distributions of savings and labor supply.

$$w_t(\mathbf{\Gamma}_t): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t} - D_t(\Gamma_t)}{\sum_{s=1}^S n_{s,t}}\right)^\alpha \quad \forall t \tag{12.19}$$

$$r_t(\mathbf{\Gamma}_t): \quad r_t = (1-\tau_t^c)\left(\alpha A\left(\frac{\sum_{s=1}^S n_{s,t}}{\sum_{s=2}^S b_{s,t} - D_t(\Gamma_t)}\right)^{1-\alpha} - \delta\right) \quad \forall t \tag{12.20}$$

Now (12.19), (12.20), and the budget constraint (12.2) can be substituted into household Euler equations (12.3) and (12.4) to get the following $(2S-1)$-equation system. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{aligned} &(1-\tau_t^l)w_t(\mathbf{\Gamma}_t)\Bigl((1-\tau_t^l)w_t(\mathbf{\Gamma}_t)n_{s,t} + \Bigl[1+(1-\tau_t^k)r_t(\mathbf{\Gamma}_t)\Bigr]b_{s,t} - b_{s+1,t+1}\Bigr)^{-\sigma} = \\ &\qquad \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \\ &\qquad \text{for} \quad s\in\{1,2,...S\} \quad \text{and} \quad \forall t \end{aligned} \tag{12.21}$$

$$\begin{aligned} &\Bigl((1-\tau_t^l)w_t(\mathbf{\Gamma}_t)n_{s,t} + \Bigl[1+(1-\tau_t^k)r_t(\mathbf{\Gamma}_t)\Bigr]b_{s,t} - b_{s+1,t+1}\Bigr)^{-\sigma} = \\ &\quad \beta\Bigl[1+(1-\tau_t^k)r_{t+1}(\mathbf{\Gamma}_{t+1})\Bigr]\Bigl((1-\tau_t^l)w_{t+1}(\mathbf{\Gamma}_{t+1})n_{s+1,t+1} + \Bigl[1+(1-\tau_t^k)r_{t+1}(\mathbf{\Gamma}_{t+1})\Bigr]b_{s+1,t+1} - b_{s+2,t+2}\Bigr)^{-\sigma} \\ &\quad \text{for} \quad s\in\{1,2,...S-1\} \quad \text{and} \quad \forall t \end{aligned} \tag{12.22}$$

The system of S nonlinear static equations (12.21) and $S-1$ nonlinear dynamic equations (12.22) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{s,t+s-1}\}_{s=1}^S$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 12.1 (**Steady-state equilibrium**)**.** A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, labor supply $\{\bar{n}_s\}_{s=1}^S$, and savings $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (12.3) and (12.4),

ii. firms optimize according to (12.6) and (12.7),

iii. government debt satisfies (12.8)

iv. markets clear according to (12.12) and (12.17).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s\left(\boldsymbol{\Gamma}_t\right)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s\left(\boldsymbol{\Gamma}_t\right)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 12.2 (**Non-steady-state functional equilibrium**)**.** A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\big\{n_{s,t} = \phi_s\big(\boldsymbol{\Gamma}_t\big)\big\}_{s=1}^S$, $\big\{b_{s+1,t+1} = \psi_s\big(\boldsymbol{\Gamma}_t\big)\big\}_{s=1}^{S-1}$ and and stationary price functions $w(\boldsymbol{\Gamma}_t)$ and $r(\boldsymbol{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\boldsymbol{\Gamma}_{t+u} = \boldsymbol{\Gamma}^e_{t+u} = \Omega^u\left(\boldsymbol{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (12.3) and (12.4),

iii. firms optimize according to (12.6) and (12.7),

iv. government debt evolves according to (12.8) and (12.11),

v. markets clear according to (12.12) and (12.17).

12.6 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 12.1 and the transition path equilibrium from Definition **??**.

12.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 12.1. The parameters needed for the steady-state solution of this model are $\Big\{S,\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S,A,\alpha,\delta,\bar{\tau}^l,\bar{\tau}^k,\bar{\tau}^c,\alpha_X,\alpha_D\Big\}$, where S is the number of periods in an individual's life, $\Big\{\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S\Big\}$ are household utility function parameters, $\{A,\alpha,\delta\}$ are firm production function parameters, and $\{\bar{\tau}^l,\bar{\tau}^k,\bar{\tau}^c,\alpha_X,\alpha_D\}$ are parameters that describe steady-state fiscal policies. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{s,t}=\bar{c}_s,\quad n_{s,t}=\bar{n}_s,\quad b_{s,t}=\bar{b}_s\quad\forall s,t \tag{12.23}$$

From the market clearing conditions (12.12) and (3.16) and the firms' first order equations (12.6) and (12.7), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t=\bar{r},\quad w_t=\bar{w},\quad B_t=\bar{B},\quad D_t=\bar{D},\quad K_t=\bar{K},\quad G_t=\bar{G},\quad L_t=\bar{L}\quad\forall t \tag{12.24}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^{S}$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$(1-\bar{\tau}^l)\bar{w}\Big([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_s + (1-\bar{\tau}^l)\bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\text{for}\quad s=\{1,2,...S\} \tag{12.25}$$

$$\Big([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_s + (1-\bar{\tau}^l)\bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \beta(1+(1-\bar{\tau}^k)\bar{r})\Big([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_{s+1} + (1-\bar{\tau}^l)\bar{w}\bar{n}_{s+1} - \bar{b}_{s+2}\Big)$$
$$\text{for}\quad s=\{1,2,...S-1\} \tag{12.26}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distribution of labor supply and savings as shown in (12.19) and (12.20). We solve this system of equations using the robust solution method proposed in Sectoin 4.6.1. Here, we need to update the method for the new specification of fiscal policy. In particular, we need to know government transfers to households when solving the household problem and we need to know government debt for the asset market clearing condition. This method is a bisection method in the outer loop guesses for steady-state equilibrium $\bar{K}$ and $\bar{L}$. In the inner loop of the household's problem given r and w implied by K and L, we solve the problem by breaking the multivariate root finder problem with $2S-1$ equations and unknowns into a series of many univariate root finder problems and one bivariate root finder problem. The algorithm is the following.

i. Make a guess for the steady-state aggregate capital stock $\bar{K}^i$ and aggregate labor $\bar{L}^i$.

  (a) Values for $\bar{K}^i$ and $\bar{L}^i$ will imply values for the interest rate $\bar{r}^i$ and wage $\bar{w}^i$ from (12.6) and (12.7). In addition, $\bar{K}^i$ and $\bar{L}^i$ imply values for $\bar{Y}^i$, which yields $\bar{X}^i = \alpha_X * \bar{Y}^i$. We can thus find $\bar{x}^i = \frac{\bar{X}^i}{S}$. All of the triple, $\{\bar{r}^i, \bar{w}^i, \bar{x}^i\}$ are necessary to solve the households' problems.

  (b) For the bisection method, we must use guesses for K and L because those uniquely determine r and w, whereas the converse is not true. From the firms' first order conditions (12.6) and (12.7), we see that r and w are functions of the same capital-labor ratio K/L. Infinitely many combinations of K and L determine a given r

and w.

ii. Given $\bar{r}^i$ and $\bar{w}^i$, solve for the steady-state household's lifetime decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$.

(a) Given $\bar{r}^i$, $\bar{w}^i$ and $\bar{x}^i$, guess an initial steady-state consumption $\bar{c}_1^m$, where m is the index of the inner-loop (household problem given $\bar{r}^i$, $\bar{w}^i$) iteration.

(b) Given $\bar{r}^i$, $\bar{w}^i$, $\bar{x}^i$, and $\bar{c}_1^m$, use the sequence of $S-1$ dynamic savings Euler equations (12.4) to solve for the implied series of steady-state consumptions $\{\bar{c}_s^m\}_{s=1}^S$. This sequence has an analytical solution.

$$\bar{c}_{s+1}^m = \bar{c}_s^m \left[\beta(1+(1-\bar{\tau}^k)\bar{r}^i)\right]^{\frac{1}{\sigma}} \quad \text{for} \quad s = \{1, 2, ...S-1\} \tag{12.27}$$

(c) Given $\bar{r}^i$, $\bar{w}^i$, and $\{\bar{c}_s^m\}_{s=1}^S$, solve for the series of steady-state labor supplies $\{\bar{n}_s^m\}_{s=1}^S$ using the S static labor supply Euler equations (12.3). This will require a series of S separate univariate root finders or one multivariate root finder.

$$(1-\bar{\tau}^l)\bar{w}^i\left(\bar{c}_s^m\right)^{-\sigma} = \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s = \{1, 2, ...S\} \tag{12.28}$$

It is this separation of the labor supply decisions from the consumption-savings decisions that gets rid of the saddle paths in the objective function that are so difficult for global optimization.

(d) Given $\bar{r}^i$, $\bar{w}^i$, and implied steady-state consumption $\{\bar{c}_s^m\}_{s=1}^S$ and labor supply $\{\bar{n}_s^m\}_{s=1}^S$, solve for implied time path of savings $\{\bar{b}_{s+1}^m\}_{s=1}^S$ across all ages of the representative lifetime using the household budget constraint (2.1).

$$\bar{b}_{s+1}^m = (1+(1-\bar{\tau}^k)\bar{r}^i)\bar{b}_s^m + (1-\bar{\tau}^l)\bar{w}^i\bar{n}_s^m + \bar{x}^i - \bar{c}_s^m \quad \text{for} \quad s = \{1, 2, ...S\} \tag{12.29}$$

Note that this sequence of savings includes savings in the last period of life for the next period $\bar{b}_{S+1}$. This savings amount is zero in equilibrium, but is not zero for an arbitrary guess for $\bar{c}_1^m$ as in step (a).

(e) Update the initial guess for $\bar{c}_1^m$ to $\bar{c}_1^{m+1}$ until the implied savings in the last period equals zero $\bar{b}_{S+1}^{m+1} = 0$.

iii. Given solution for optimal household decisions $\{\bar{c}_s^m\}_{s=1}^S$, $\{\bar{n}_s^m\}_{s=1}^S$, and $\{\bar{b}_s^m\}_{s=2}^S$ based on the guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$, we can solve for aggregate tax revenues using (12.9).

$$\bar{R}^i = \bar{\tau}^c \left(\bar{Y}^i - \bar{w}^i \bar{L}^i\right) - \bar{\tau}^c \delta \bar{K}^i + \sum_{s=1}^S \bar{\tau}^l \bar{w}^i \bar{n}_s + \sum_{s=2}^S \bar{\tau}^k \bar{r}^i \bar{b}_s \tag{12.30}$$

iv. The guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$ imply $\bar{Y}^i$. Our target steady-state debt to GDP ration, α_D then gives us steady-state govenment debt.

$$\bar{D}^i = \alpha_D \bar{Y}^i \tag{12.31}$$

v. With total tax revenues and the steady-state amount of debt determined, we can use the government budget constraint defined in (12.8) to find the steady state amount of government spending.

$$\bar{G}^i = \bar{R}^i - \bar{X}^i - \bar{r}^i \bar{D}^i \tag{12.32}$$

vi. Given solution for optimal household decisions $\{\bar{c}_s^m\}_{s=1}^S$, $\{\bar{n}_s^m\}_{s=1}^S$, and $\{\bar{b}_s^m\}_{s=2}^S$ based on the guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$, we can solve for the aggregate capital $\bar{K}^{i'}$ and aggregate labor $\bar{L}^{i'}$ implied by the household solutions and market clearing conditions.

$$\bar{B}^i = \sum_{s=2}^S \bar{b}_s^m \tag{12.33}$$

$$\bar{K}^{i'} = \bar{B}^i - \bar{D}^i \tag{12.34}$$

$$\bar{L}^{i'} = \sum_{s=1}^S \bar{n}_s^m \tag{12.35}$$

Update guesses for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ until the the aggregates implied by household optimization equal the initial guess for the aggregates $(\bar{K}^{i'+1}, \bar{L}^{i'+1}) = (\bar{K}^{i+1}, \bar{L}^{i+1})$.

(a) The bisection method characterizes the updated guess for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ as a convex combination of the initial guess $(\bar{K}^{i}, \bar{L}^{i})$ and the values implied by household and firm optimization $(\bar{K}^{i'}, \bar{L}^{i'})$, where the weight put on the new values $(\bar{K}^{i'}, \bar{L}^{i'})$ is given by $\xi \in (0,1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$(\bar{K}^{i+1}, \bar{L}^{i+1}) = \xi(\bar{K}^{i'}, \bar{L}^{i'}) + (1-\xi)(\bar{K}^{i}, \bar{L}^{i}) \quad \text{for} \quad \xi \in (0,1] \tag{12.36}$$

(b) Let $\|\cdot\|$ be a norm on the space of feasible aggregate capital and aggregate labor values (K, L). We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (12.37). If the distance is less than some tolerance `toler` > 0, then the problem has converged. Otherwise continue updating the values of aggregate capital and labor using (12.36).

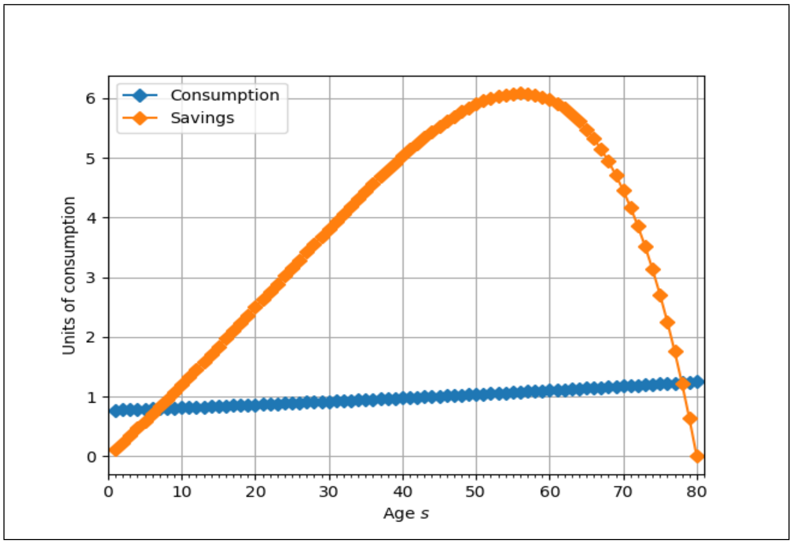
$$\texttt{dist} \equiv \left\|(\bar{K}^{i'}, \bar{L}^{i'}) - (\bar{K}^{i}, \bar{L}^{i})\right\| \tag{12.37}$$

Define the updating of aggregate variable values $(\bar{K}^{i}, \bar{L}^{i})$ in step (iii) indexed by i as the "outer loop" of the fixed point solution. Although computationally intensive, the bisection method described above is the most robust solution method we have found.

Figure 12.1 shows the steady-state distribution of individual consumption and savings in an 80-period-lived agent model with parameter values listed above the line in Table 12.3 in Section 12.7. Figure 12.2 shows the steady-state distribution of individual labor supply by age. The left side of Table 12.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $2S-1$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing

Figure 12.1: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$



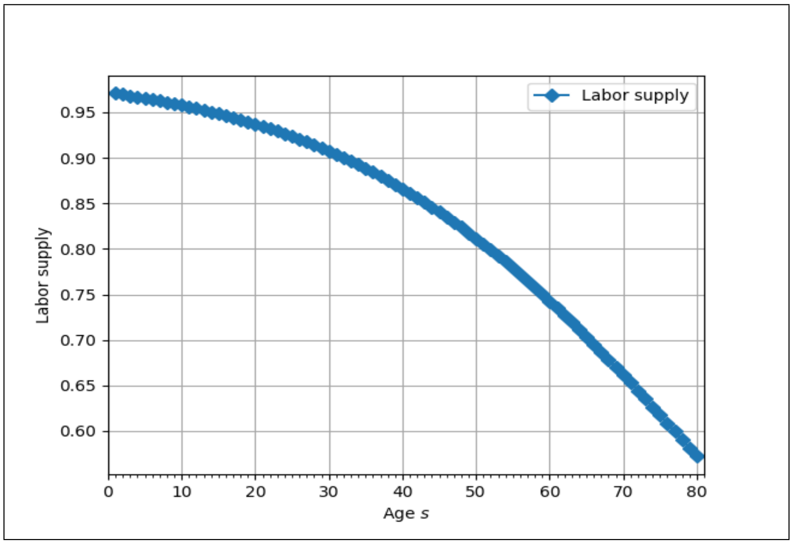
conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 12.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 4.5e-16, the final period individual savings is less than 6.1e-13, and the resource constraint error is less than 1.8e-06, we can be confident that we have successfully solved for the steady-state.

12.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor is similar to the method described in Section 4.6.2.

To solve for the transition path (non-steady-state) equilibrium from Definition **??**, we must know the parameters from the steady-state problem $\Big\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta, \bar{\tau}^l, \bar{\tau}^k, \bar{\tau}^c, \alpha_X, \alpha_D\Big\}$, the steady-state solution values $\{\bar{K}, \bar{L}\}$, initial distribution of savings $\boldsymbol{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 12.3 and 12.1 show a particular calibration of the model and a steady-state solution. In addition, we need parameters that describe fiscal policy over the time path:$\{\alpha_G, t_{g1}, t_{G2}, \rho_G, \alpha_{D0}, \boldsymbol{\tau}^l, \boldsymbol{\tau}^k, \boldsymbol{\tau}^c\}$. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

Figure 12.2: Steady-state distribution of labor supply $\bar{n}_s$



i. Choose a period $T1$ in which the initial guess for the time paths of aggregate capital and aggregate labor will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Given calibration for initial distribution of savings (wealth) $\boldsymbol{\Gamma}_1$, which implies an initial capital stock K_1, guess initial time paths for the aggregate capital stock $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_{T1}^i\}$ and aggregate labor $\boldsymbol{L}^i = \{L_1^i, L_2^i, ...L_{T1}^i\}$. Both of these time paths will have to be extended with their respective steady-state values so that they are $T2+S-1$ elements long. This is the time-path length that will allow you to solve the lifetime of every individual alive in period $T2$.

iii. Given time paths $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$, solve for the lifetime consumption $c_{s,t}$, labor supply $n_{s,t}$, and savings $b_{s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.

   (a) The initial paths for aggregate capital $\boldsymbol{K}^i$ and aggregate labor $\boldsymbol{L}^i$ imply time paths for the interest rate $\boldsymbol{r}^i = \left\{r_1^i, r_2^i, ...r_{T2+S-1}^i\right\}$ and wage $\boldsymbol{w}^i = \left\{w_1^i, w_2^i, ...w_{T2+S-1}^i\right\}$ using the firms' first order equations (12.6) and (12.7).

   (b) Aggregate capital and labor aslo imply the time path for aggregate output, $\boldsymbol{Y}^i$.

Table 12.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.082	Max. absolute savings Euler error	7.44e-11
$\bar{w}$	1.037	Max. absolute labor supply Euler error	1.47e-11
$\bar{K}$	252.648	Absolute final period savings $\bar{b}_{S+1}$	-1.16e-13
$\bar{L}$	66.423	Resource constraint error	4.20e-08
$\bar{Y}$	106.019		
$\bar{C}$	79.293		
$\bar{D}$	42.408		
$\bar{G}$	14.094		
$\bar{X}$	10.602		
$\bar{R}$	28.187		

Output, together with the fiscal rule for government transfers from (12.10) yield the path for government trantsfers, $\boldsymbol{X}^i$, which in turn gives the path for transfers per household: $\boldsymbol{x}^i = \frac{\boldsymbol{X}^i}{S}$.

(c) Given the time paths for the interest rate $\boldsymbol{r}^i$, wage $\boldsymbol{w}^i$, transfers $\boldsymbol{x}^i$ and the period-1 distribution of savings (wealth) $\boldsymbol{\Gamma}_1$, solve for the lifetime decisions $c_{s,t}$, $n_{s,t}$, and $b_{s,t}$ of each household alive during periods 1 and $T2$. This is done using the method outlined in steps (ii)(a) through (ii)(e) of the steady-state computational algorithm outlined in Section 12.6.1.

iv. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ to compute total tax revenue given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ using (12.9). Call this $\boldsymbol{R}^i$

v. By putting the time path for aggregate output, $\boldsymbol{Y}^i$, transfers, $\boldsymbol{X}^i$, and revenues, $\boldsymbol{R}^i$, into the budget closure rule from (12.11), we can find the time path for government debt, $\boldsymbol{D}^i_t$, and government spending, $\boldsymbol{G}^i$.

vi. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ to compute new paths for aggregate savings and aggregate labor $\boldsymbol{B}^i$ and $\boldsymbol{L}^{i'}$ implied by the labor market clearing condition (12.12).

vii. Using the asset market clearing condition, (12.17), find the new path for aggregate capital, $\boldsymbol{K}^{i'} = \boldsymbol{B}^i - \boldsymbol{D}^i$.

viii. Compare the distance of the time paths of the new implied paths for the aggregate capital and labor $\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right)$ versus the initial aggregate capital and labor $\left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right)$.

$$\texttt{dist} = \left\|\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) - \left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right)\right\| \geq 0 \tag{12.38}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and aggregate labor $(\boldsymbol{K}^{i}, \boldsymbol{L}^{i})$. Common norms to use are the L^2 and the L^∞ norms.

(a) If the distance is less than or equal to some tolerance level $\texttt{dist} \leq \texttt{TPI\_toler} > 0$, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level, then update the guess for a new set of initial time paths to be a convex combination current initial time paths and the implied time paths.

$$\left(\boldsymbol{K}^{i+1}, \boldsymbol{L}^{i+1}\right) = \xi\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) + (1-\xi)\left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right) \quad \text{for} \quad \xi \in (0,1] \tag{12.39}$$

Table 12.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	4.87e-13
Maximum absolute savings Euler error	8.07e-16
Maximum absolute final period savings $\bar{b}_{S+1,t}$	0.00
Maximum absolute resource constraint error	3.20e-08

The 6 panels of Figure 12.3 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The three panels of Figure 12.4 show the transition paths of the distributions of consumption $c_{s,t}$, labor supply $n_{s,t}$ and savings $b_{s,t}$. The time path of fiscal aggregates are shown in Figure 12.5. In all of the time paths, a sharp kink is evident at t_{G1}, when the budget closure rule begins. Table 12.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition

Figure 12.3: Equilibrium transition paths of prices and aggregate variables

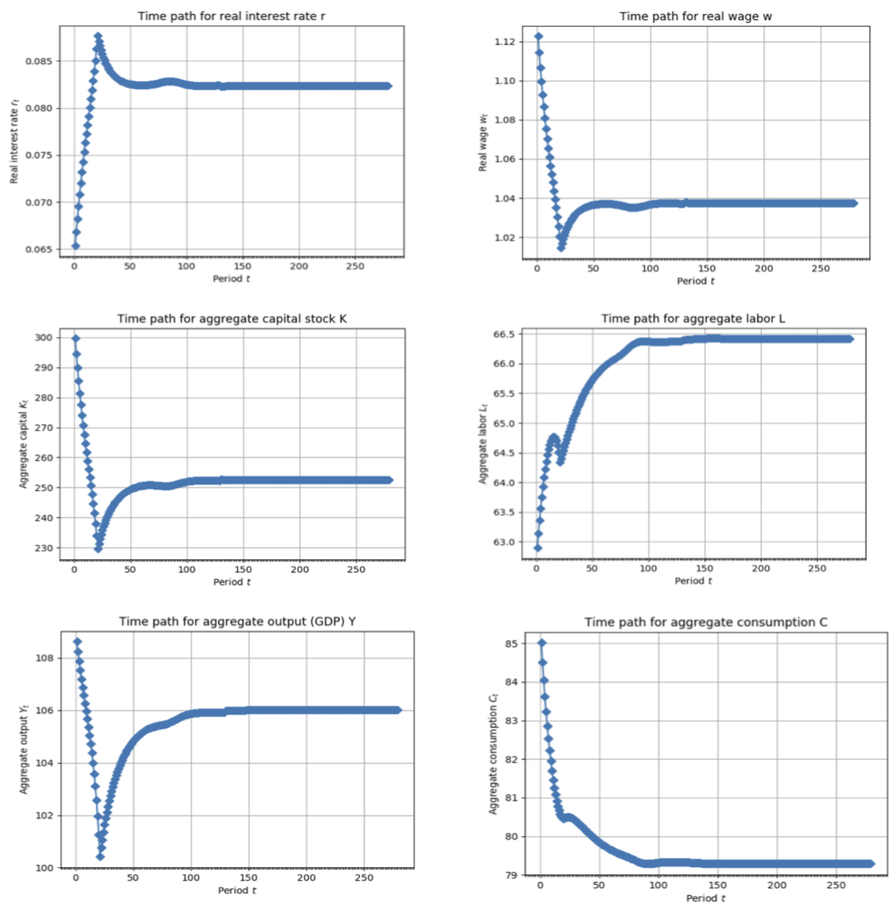


Figure 12.4: Equilibrium transition paths of distributions of consumption, labor supply, and savings

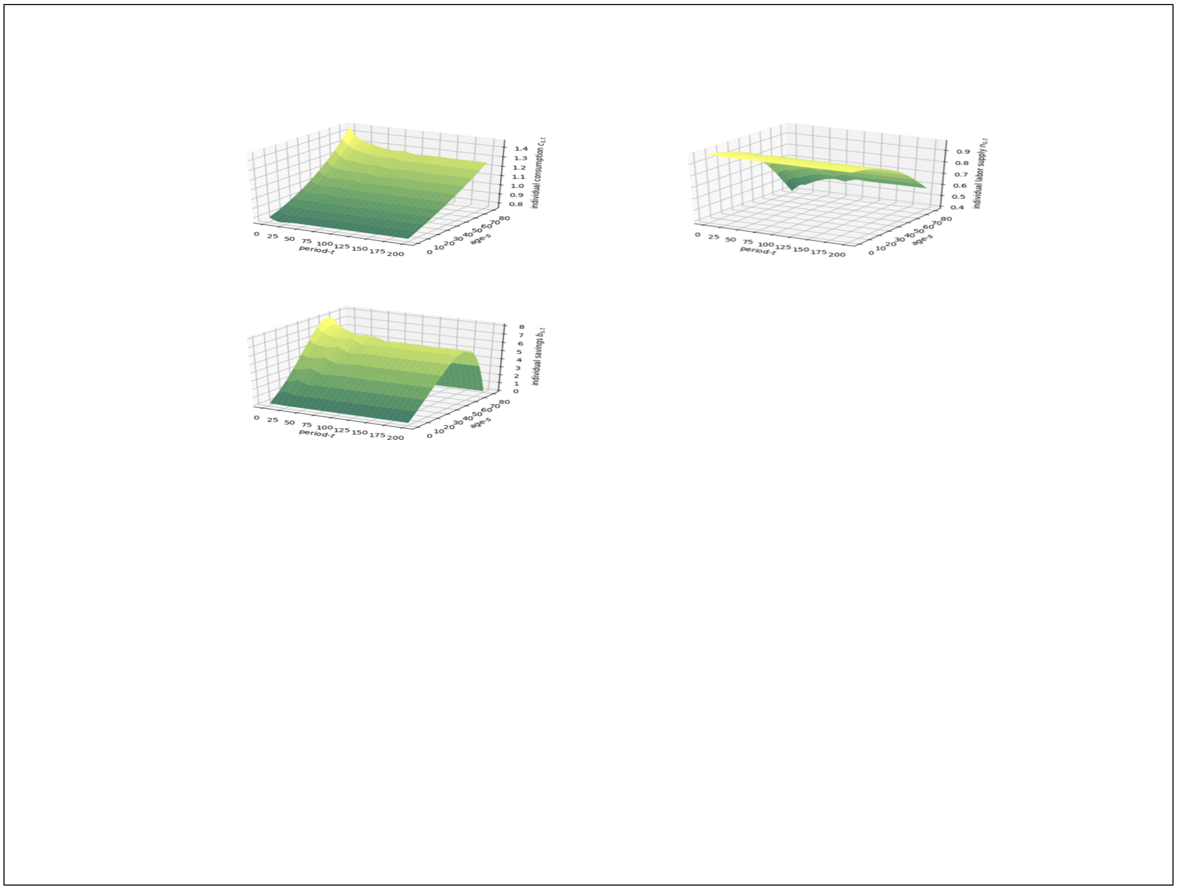
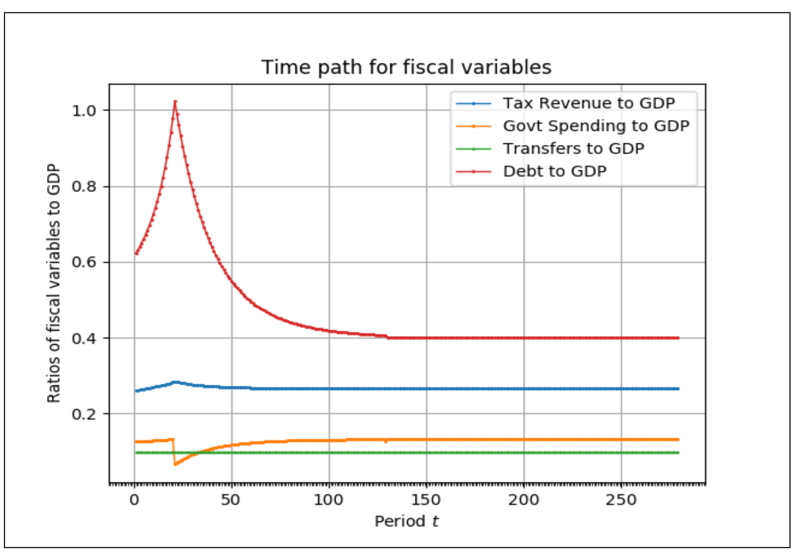


Figure 12.5: Equilibrium transition paths of government tax revenue, transfers, spending, and debt as percentages of output

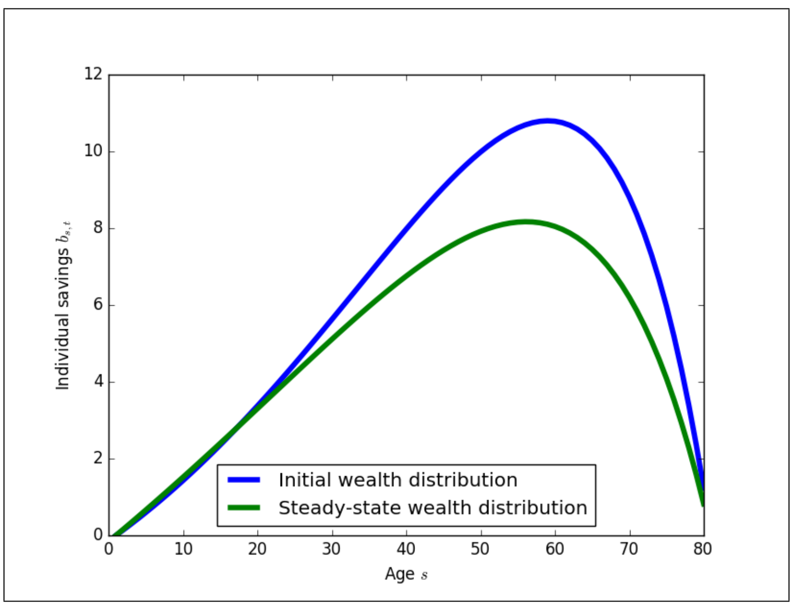


path. All of these should be zero in equilibrium. The fact that none of them is greater than 2.0e-12 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

12.7 Calibration

Use the following parameterization of the model for the problems below. Assume that agents are born at age 21 and die at age 100 (80 years of life). Your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1 - 0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

Figure 12.6: Initial vs. steady-state distributions of wealth (savings) $b_{s,t}$



Assume that each individual's time endowment in each period is $\tilde{l} = 1$. Initially, let $\chi^n_s = 1$ for all s. However, you could calibrate these values to match the empirical distribution of annual hours by age in the United states (see Exercise 4.5).

Table 12.3: Calibrated parameter values for simple endogenous labor model with government debt financing

Parameter	Description	Value
S	Number of periods in individual life	80
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
$\tilde{l}$	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501[a]
υ	Elliptical disutility of labor shape parameter	1.554[a]
$\{\chi^n_s\}_{s=1}^S$	Disutility of labor relative scale factor by age	1.0
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
α_X	Ratio of government transfers to GDP	0.10
α_G	Ratio of government spending to GDP up to t_{G1}	0.12
α_D	Steady-state ratio of government debt to GDP	0.40
$\bar{\tau}^l$	Steady-state marginal tax rate on labor income	0.25
$\bar{\tau}^k$	Steady-state marginal tax rate on capital income	0.30
$\bar{\tau}^c$	Steady-state marginal tax rate on corporate income	0.15
$\boldsymbol{\Gamma}_1$	Initial distribution of savings (wealth)	(see Fig. 4.7)
$T1$	Time period in which initial path guess hits steady state	160
$T2$	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.2
α_{D0}	Ratio of government spendign to GDP in $t = 1$	0.59
t_{G1}	Period when budget closure rule begins	20
t_{G2}	Period when budget closure rule ends	128
ρ_G	Rate at which debt to GDP adjusts to steady-state ratio	0.05
$\boldsymbol{\tau}^l$	Time path of tax rate on labor income	$0.25\ \forall\ t$
$\boldsymbol{\tau}^k$	Time path of tax rate on capital income	$0.30\ \forall\ t$
$\boldsymbol{\tau}^c$	Time path of tax rate on corporate income	$0.15\ \forall\ t$

[a] The calibration of b and υ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See Evans and Phillips (2017).

12.8 Exercises

TBD...

Some potential exercises:

- Write the algorithm to solve for the SS and time path if transfers are adjusted, rather than government spending, in order to hit the target debt to GDP ratio in the SS.
- Solve model with calibration given above. Plot paths of fiscal variables.
- Try to break the model. Can you make fiscal policies that are infeasible?

Exercise 12.1. Assume that an individual's time endowment each period is one $\tilde{l} = 1$. Let the period utility of a household be an additively separable function of consumption and labor,

$$U(c,n) = u(c) - g(n)$$

where the disutility of labor function $g(n)$ is the constant Frisch elasticity (CFE) disutility of labor functional form.

$$g_{cfe}(n) = \frac{(n)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$$

Assume that an approximation to this disutility of labor function is the following elliptical disutility of labor function.

$$g_{elp}(n) = -b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \quad \text{for} \quad \tilde{l}, b > 0 \quad \text{and} \quad \upsilon \geq 1$$

i. The marginal disutility of labor $\frac{\partial g(n)}{\partial n}$ governs the household decision of how much to work. Give the expression for the marginal disutility of labor for the CFE specification $g_{cfe}(n)$ and for the elliptical specification $g_{elp}(n)$.

ii. Write a function `fit_ellip()` that takes as inputs the Frisch elasticity of labor supply θ and the time endowment per period $\tilde{l}$ and returns the estimated values of the elliptical utility parameters b and υ.

```
b_ellip, upsilon = fit_ellip(elast_Frisch, l_tilde)
```

Assume that the Frisch elasticity of labor supply in v_{cfe} is $\theta = 2.0$. Using 1,000 evenly spaced points from the support of leisure between 0.15 and 0.95, estimate the elliptical disutility of labor parameters b and υ that minimize the sum of squared deviations between the two marginal utility of leisure functions $g'_{cfe}(n)$ and $g'_{elp}(n)$ from part (i). Plot the two marginal disutility of labor functions.

Exercise 12.2. Not all fiscal paths are achievable. For example, if tax revenue is low enough relative to outlays, then the debt burden may become unsustainable before the budget closure rule can kick in. Modify the fiscal parameters to see you can "break" the model. In particular:

i. Can you find a debt to GDP ratio (α_D) such that government spending is negative in the steady-state? How would you interpret this?

ii. Hold the rules for X, G, debt, and taxes constant, but push out the time until the closure rule kicks in (i.e., increase t_{G1}). Can you make this period far enough into the future that the debt burden becomes unsustainable and the time path equlibrium cannot be found?

iii. Once you've found broken the model by moving out t_{G1}, can you adjust α_G and α_X (holding constant at the new t_{G1}) so that the model no longer breaks?

Exercise 12.3. We've illustrated one of many ways to close the government's budget above. Try to implement another. Specifically, assume that government spending remains constant as a fraction of GDP for all time, but that government transfers, X_t, adjust to close the budget.

i. Outline the solution algorithm for the steady-state. How would this have to change?

ii. Now consider the time path. What would have to change from the algorithm outlined above? Think about the closure rule above and notice the time path for government spending. Would such a discete jump in transfers impose and problems computationally? Is it realistic? What would happen to the time paths of other economic aggregates if X did have such a discrete jump? Given your answers to these questions, sketch out a budget closure rule that closes the government's budget by adjusting transfers.

Harder Modify the code to implement this alternative budget closure rule.

Chapter 13

Modeling Social Security and pension systems

In this chapter, we want to incorporate the tax and transfer system associated with government pensions, such as the U.S. Social Security system.

Part VII

Multiple industries, multiple goods

Chapter 14

Two Industries, Two Goods, Three-period-lived agents

In this chapter, we take the 3-period-lived agent model from Chapter 2 with exogenous labor and add to it 2 distinct industries and 2 corresponding consumer goods.

14.1 Households

A unit measure of identical individuals are born each period and live for 3 periods. Let the age of an individual be indexed by $s = \{1, 2, 3\}$. Individuals in this economy choose between 2 different consumption goods every period $\{c_{1,s,t}, c_{2,s,t}\}$. However, we assume that this consumption can be aggregated into a composite consumption good $c_{s,t}$ every period in every individual's preferences according to the following Cobb-Douglas consumption aggregator,

$$c_{s,t} \equiv \left(c_{1,s,t} - \tilde{c}_1\right)^{\alpha} \left(c_{2,s,t} - \tilde{c}_2\right)^{1-\alpha} \quad \forall s, t \tag{14.1}$$

where $\tilde{c}_m$ represents a minimum consumption level for each consumption good $m = \{1, 2\}$. Note that the specification in (14.1) signifies that type-m consumption $c_{m,s,t}$ is consumption in addition to (excluding) the minimum consumption $\tilde{c}_m$. Then composite consumption $c_{s,t}$ must be interpreted as excluding some composite of the minimum consumption levels.

Assume that the price of each individual consumption is $p_{m,t}$. We can solve for the optimal type-m consumption demands $c_{m,s,t}$ as a function of composite consumption $c_{s,t}$ by minimizing the total expenditure on consumption given that individual consumption adds up to composite consumption according to (14.1). The Lagrangian for this expenditure minimization problem is the following.

$$\mathcal{L} = \sum_{m=1}^{2} p_{m,t} c_{m,s,t} + \lambda_{s,t}\Bigl[c_{s,t} - \left(c_{1,s,t} - \tilde{c}_1\right)^{\alpha}\left(c_{2,s,t} - \tilde{c}_2\right)^{1-\alpha}\Bigr] \quad \forall s,t \tag{14.2}$$

Because the Lagrangian multiplier on the constraint $\lambda_{s,t}$ represents the shadow price of an extra unit of composite consumption, we can relabel it as the price of composite consumption $p_{s,t}$.

$$\mathcal{L} = \sum_{m=1}^{2} p_{m,t} c_{m,s,t} + p_{s,t}\Bigl[c_{s,t} - \left(c_{1,s,t} - \tilde{c}_1\right)^{\alpha}\left(c_{2,s,t} - \tilde{c}_2\right)^{1-\alpha}\Bigr] \quad \forall s,t \tag{14.3}$$

Note that the price of composite consumption can be different for each age-s individual.

The three first order conditions of this constrained minimization problem are the following.

$$p_{1,t} = \alpha p_{s,t}\left(\frac{c_{2,s,t} - \tilde{c}_2}{c_{1,s,t} - \tilde{c}_1}\right)^{1-\alpha} \quad \forall s,t \tag{14.4}$$

$$p_{2,t} = (1-\alpha) p_{s,t}\left(\frac{c_{1,s,t} - \tilde{c}_1}{c_{2,s,t} - \tilde{c}_2}\right)^{\alpha} \quad \forall s,t \tag{14.5}$$

$$c_{s,t} = \left(c_{1,s,t} - \tilde{c}_1\right)^{\alpha}\left(c_{2,s,t} - \tilde{c}_2\right)^{1-\alpha} \quad \forall s,t \tag{14.1}$$

Multiplying both sides of equation (14.4) by $c_{1,s,t} - \tilde{c}_1$ and multiplying both sides of equation (14.5) by $c_{2,s,t} - \tilde{c}_2$ gives the following two equations.

$$p_{1,t}\left(c_{1,s,t} - \tilde{c}_1\right) = \alpha p_{s,t}\left(c_{1,s,t} - \tilde{c}_1\right)^{\alpha}\left(c_{2,s,t} - \tilde{c}_2\right)^{1-\alpha} \quad \forall s,t \tag{14.6}$$

$$p_{2,t}\left(c_{2,s,t} - \tilde{c}_2\right) = (1-\alpha) p_{s,t}\left(c_{1,s,t} - \tilde{c}_1\right)^{\alpha}\left(c_{2,s,t} - \tilde{c}_2\right)^{1-\alpha} \quad \forall s,t \tag{14.7}$$

The product term on the right-hand-side of both equations is simply the definition for composite consumption $c_{s,t}$. Substituting this in and solving for $c_{1,s,t}$ and $c_{2,s,t}$, respectively, gives the optimal demand functions for consumption of goods 1 and 2 by age-s individual in period t.

$$c_{1,s,t} = \alpha\left(\frac{p_{1,t}}{p_{s,t}}\right)^{-1} c_{s,t} + \tilde{c}_1 \qquad \forall s,t \tag{14.8}$$

$$c_{2,s,t} = (1-\alpha)\left(\frac{p_{2,t}}{p_{s,t}}\right)^{-1} c_{s,t} + \tilde{c}_2 \quad \forall s,t \tag{14.9}$$

Substituting the demand equations (14.8) and (14.9) back into the composite consumption definition (14.1) gives us the expression for the composite price $p_{s,t}$ as a function of each type-m good's price $p_{m,t}$.

$$p_{s,t} = \left(\frac{p_{1,t}}{\alpha}\right)^{\alpha}\left(\frac{p_{2,t}}{1-\alpha}\right)^{1-\alpha} \quad \forall s,t \tag{14.10}$$

However, in this case, because nothing on the right-hand-side of (14.10) is a function of s, then $p_{s,t} = p_t$ for all s.

$$p_t = \left(\frac{p_{1,t}}{\alpha}\right)^{\alpha}\left(\frac{p_{2,t}}{1-\alpha}\right)^{1-\alpha} \quad \forall t \tag{14.11}$$

With this composite price expression, we can rewrite the good-1 and good-2 demand functions.

$$c_{1,s,t} = \alpha\left(\frac{p_{1,t}}{p_t}\right)^{-1} c_{s,t} + \tilde{c}_1 \qquad \forall s,t \tag{14.12}$$

$$c_{2,s,t} = (1-\alpha)\left(\frac{p_{2,t}}{p_t}\right)^{-1} c_{s,t} + \tilde{c}_2 \quad \forall s,t \tag{14.13}$$

Because the individual-good problem from (14.12), (14.13), and (14.11) is determined by functions of composite consumption $c_{s,t}$, composite good price p_t, and individual good prices $p_{m,t}$, we can write the individual's utility maximization in terms of composite consumption $c_{s,t}$. An age-s individual faces the following per-period budget constraint.

$$p_t c_{s,t} + \sum_{m=1}^{2} p_{m,t}\tilde{c}_m + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \quad \forall s,t \tag{14.14}$$

We assume the individuals supply a unit of labor inelastically in the first two periods of life and are retired with reduced labor supply in the last period of life.

$$n_{s,t} = \begin{cases} 1 \quad \text{if} \quad s = 1,2 \\ 0.2 \quad \text{if} \quad s = 3 \end{cases} \qquad \forall s,t \tag{14.15}$$

Because exogenous labor in (14.15) is not dependent on the time period, we drop the t subscript from labor n_s for the rest of this section. We also assume that households are born with no savings $b_{1,t} = 0$ and that individuals save no income in the last period of their lives $b_{4,t} = 0$ for all periods t. Assume that $c_{s,t} \geq 0$ because individual good consumption $c_{m,s,t}$ less than the minimum $\tilde{c}_m$ is not defined. However, it will also be the case that period individual utility will not be defined for $c_{s,t} <= 0$.

Let the utility of composite consumption in each period be defined by the constant relative risk aversion function $u(c_{s,t})$, such that $u' > 0$, $u'' < 0$, and $\lim_{c\rightarrow 0} u(c) = -\infty$.

$$u(c_{s,t}) = \frac{(c_{s,t})^{1-\sigma} - 1}{1-\sigma} \quad \forall s,t \tag{14.16}$$

Individuals choose lifetime composite consumption $\{c_{s,t+s-1}\}_{s=1}^S$ and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$\begin{aligned} &\max_{\{c_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{u=0}^{S-s} \beta^u u(c_{s+u,t+u}) \quad \forall s,t \\ &\text{s.t.} \quad c_{s,t} = \frac{1}{p_t}\left[(1+r_t)b_{s,t} + w_t n_s - b_{s+1,t+1} - \sum_{m=1}^{2} p_{m,t}\tilde{c}_m\right] \quad \forall s,t \\ &\text{and} \quad b_{1,t}, b_{4,t} = 0 \quad \forall t \quad \text{and} \quad c_{s,t} \geq 0 \quad \forall s,t \end{aligned} \tag{14.17}$$

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (14.17). The optimal choice of how much to save in the each of the first 2 periods of life $\{b_{2,t+1}, b_{3,t+2}\}$ is found by taking the derivative of the lifetime utility function with respect to each of the two lifetime savings amounts and setting the derivatives equal to zero.

In the last period of life ($s = 3$), the household optimally chooses no savings $b_{4,t+1} = 0$ for all t because any positive savings only imposes a cost of reduced consumption, and negative savings (borrowing) would impose an automatic default on anyone lending to him. The final period $s = 3$ decision is simple. The individual enters the period with wealth $b_{3,t}$, he knows the interest rate r_t, the wage w_t, the composite good price p_t, and the individual goods prices $\{p_{m,t}\}_{m=1}^2$, and inelastically supplies labor n_S. Everything in the budget constraint (14.14) is determined except for $c_{3,t}$. In the final period, the individual simply consumes all his resources.

$$c_{3,t} = \frac{1}{p_t}\left[(1+r_t)b_{3,t} + w_t n_3 - \sum_{m=1}^{2} p_{m,t}\tilde{c}_m\right] \quad \forall t \tag{14.18}$$

In the second-to-last period of life ($s = 2$), the household has a savings decision to make. He enters the period with wealth $b_{2,t}$, he knows the current interest rate r_t, the current wage w_t, current composite price p_t, and current individual good prices $\{p_{m,t}\}_{m=1}^2$, and he must know or be able to forecast next period's prices r_{t+1}, w_{t+1}, p_{t+1}, and $\{p_{m,t+1}\}_{m=1}^2$. In this case, the household's lifetime utility function is one equation and one unknown.

$$\begin{split}\max_{b_{3,t+1}}\ u\left(\frac{1}{p_t}\left[(1+r_t)b_{2,t} + w_t n_2 - b_{3,t+1} - \sum_{m=1}^{2} p_{m,t}\tilde{c}_m\right]\right) + ... \\ \beta u\left(\frac{1}{p_{t+1}}\left[(1+r_{t+1})b_{3,t+1} + w_{t+1}n_3 - \sum_{m=1}^{2} p_{m,t+1}\tilde{c}_m\right]\right)\end{split} \tag{14.19}$$

The first order condition, or dynamic Euler equation, for this second-to-last period of life savings decision is the following.

$$\begin{split}\left(\frac{1}{p_t}\left[(1+r_t)b_{2,t} + w_t n_2 - b_{3,t+1} - \sum_{m=1}^{2} p_{m,t}\tilde{c}_m\right]\right)^{-\sigma} = ... \\ \beta(1+r_{t+1})\frac{p_t}{p_{t+1}}\left(\frac{1}{p_{t+1}}\left[(1+r_{t+1})b_{3,t+1} + w_{t+1}n_3 - \sum_{m=1}^{2} p_{m,t+1}\tilde{c}_m\right]\right)^{-\sigma}\end{split} \tag{14.20}$$

The solution for savings $b_{3,t+1}$ in the second-to-last period of life to be returned with interest in the last period of life is characterized by the nonlinear dynamic Euler equation (14.20) and is a function of individual wealth $b_{2,t}$, prices r_t, w_t, p_t, and $\{p_{m,t}\}_{m=1}^2$ at the beginning of the second-to-last period of life, as well as prices in the last period of life r_{t+1}, w_{t+1}, p_{t+1}, and $\{p_{m,t+1}\}_{m=1}^{2}$.

$$b_{3,t+1} = \psi_2\Big(b_{2,t}, r_t, w_t, p_t, \{p_{m,t}\}_{m=1}^{2}, r_{t+1}, w_{t+1}, p_{t+1}, \{p_{m,t+1}\}_{m=1}^{2}\Big) \quad \forall t \tag{14.21}$$

Call $\psi_2(\cdot)$ the policy function for savings $b_{3,t+1}$ in the second-to-last period of life.

It is straightforward to show that the $s=1$ savings decision is characterized by the following Euler equation,

$$\begin{aligned}&\left(\frac{1}{p_t}\left[w_t n_1 - b_{2,t+1} - \sum_{m=1}^{2} p_{m,t}\tilde{c}_m\right]\right)^{-\sigma} = \dots \\ &\beta(1+r_{t+1})\frac{p_t}{p_{t+1}}\left(\frac{1}{p_{t+1}}\left[(1+r_{t+1})b_{2,t+1} + w_{t+1}n_2 - b_{3,t+2} - \sum_{m=1}^{2} p_{m,t+1}\tilde{c}_m\right]\right)^{-\sigma}\end{aligned} \tag{14.22}$$

where the policy function characterizing the savings decision for $b_{2,t+1}$ is the following.

$$b_{2,t+1} = \psi_1\Big(0, \{r_u\}_{u=t}^{t+2}, \{w_u\}_{u=t}^{t+2}, \{p_u\}_{u=t}^{t+2}, \big\{\{p_{m,u}\}_{m=1}^{2}\big\}_{u=t}^{t+2}\Big) \quad \forall t \tag{14.23}$$

To summarize the individual's problem, if one knows his initial savings or wealth $b_{s,t}$ and the time path of factor prices over his remaining lifetime, he can solve for all of his optimal savings levels $\{b_{s+1,t+s}\}_{s=1}^{2}$.

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of prices $\big\{r_u, w_u, p_u, \{p_{m,u}\}_{m=1}^{2}\big\}_{u=t}^{t+3-s}$ over his remaining lifetime. As we will show in Section 14.4, the equilibrium prices in period t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \big\{b_{2,t}, b_{3,t}\big\} \quad \forall t \tag{14.24}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}_{t+u}^{e} = \Omega^{u}\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{2.17}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[1]

14.2 Firms

The production side of this economy is comprised of 2 industries indexed by $m \in \{1,2\}$, each of which has a unit measure of identical, perfectly competitive firms. These firms rent investment capital from individuals for real return r_t and hire labor for real wage w_t. The interest rate r_t and wage w_t are equal across industries because labor and capital are perfectly mobile. Firms in industry m use their total capital $K_{m,t}$ and labor $L_{m,t}$ to produce output $Y_{m,t}$ every period according to a constant elasticity of substitution (CES) production technology,

$$Y_{m,t} = F(K_{m,t}, L_{m,t}) \equiv A_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}}(K_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}(L_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m-1}} \quad \forall m,t \tag{14.25}$$

where $\gamma_m \in (0,1)$ is the capital share of income, $\varepsilon_m \in (0,1)$ is the elasticity of substitution between capital and labor, and total factor productivity $A_{m,t} > 0$ for all t.

The representative firm in each industry m chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_{m,t},L_{m,t}} \; p_{m,t}F\big(K_{m,t}, L_{m,t}\big) - (r_t + \delta_m)K_{m,t} - w_t L_{m,t} \quad \forall m,t \tag{14.26}$$

where $p_{m,t}$ is the price of output from industry m and $\delta_m \in [0,1]$ is the rate of capital depreciation. The two first order conditions that characterize firm optimization are the

[1] In Section 14.4 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 14.2.

following.

$$r_t = p_{m,t}(A_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}}\left(\gamma_m\frac{Y_{m,t}}{K_{m,t}}\right)^{\frac{1}{\varepsilon_m}} - \delta_m \quad \forall m,t \tag{14.27}$$

$$w_t = p_{m,t}(A_{m,t})^{\frac{\varepsilon_m-1}{\varepsilon_m}}\left([1-\gamma_m]\frac{Y_{m,t}}{L_{m,t}}\right)^{\frac{1}{\varepsilon_m}} \quad \forall m,t \tag{14.28}$$

Each firm in industry m can make its capital and labor demand decision if it knows the price of its good $p_{m,t}$, the interest rate r_t, and the wage w_t. For the entire supply side of the economy, all firms can optimize if r_t, w_t, and $\{p_{m,t}\}_{m=1}^2$ are known.

Although it is an equilibrium condition, we show the zero-profit condition for firms here. Because firms in each industry m are perfectly competitive, the price will increase or decrease until all profits have been claimed and there are none left. The zero profit condition is the following.

$$p_{m,t}Y_{m,t} = (r_t + \delta_m)K_{m,t} + w_t L_{m,t} \quad \forall m,t \tag{14.29}$$

However, this condition for the M industries ends up being redundant by Walra's Law.

Finally, it will be valuable in Section 14.4 to show some transformations of the four industry equations (14.25), (14.27), (14.28),and (14.29) in order to solve the model. We first solve for the firm's optimal capital demand $K_{m,t}$ from (14.27) as a function of r_t, $p_{m,t}$, and $Y_{m,t}$ and to show the firm's optimal labor demand $L_{m,t}$ from (14.28) as a function of w_t, $p_{m,t}$, and $Y_{m,t}$.

$$K_{m,t} = \frac{\gamma_m Y_{m,t}(p_{m,t})^{\varepsilon_m}}{(r_t+\delta_m)^{\varepsilon_m}(A_{m,t})^{1-\varepsilon_m}} \quad \forall m,t \tag{14.30}$$

$$L_{m,t} = \frac{(1-\gamma_m)Y_{m,t}(p_{m,t})^{\varepsilon_m}}{(w_t)^{\varepsilon_m}(A_{m,t})^{1-\varepsilon_m}} \quad \forall m,t \tag{14.31}$$

It is also helpful to show that substituting (14.30) and (14.31) into the production function (14.25), we get industry price $p_{m,t}$ as a function of the interest rate r_t and the real wage w_t.

$$p_{m,t} = \frac{1}{A_{m,t}}\Big[\gamma_m(r_t+\delta_m)^{1-\varepsilon_m} + (1-\gamma_m)(w_t)^{1-\varepsilon_m}\Big]^{\frac{1}{1-\varepsilon_m}} \quad \forall m,t \tag{14.32}$$

Lastly, we can divide equation (14.28) by (14.27) to get the solution to the labor-capital ratio as a function of r_t and w_t.

$$\frac{L_{m,t}}{K_{m,t}} = \left(\frac{1-\gamma_m}{\gamma_m}\right)\left(\frac{r_t+\delta_m}{w_t}\right)^{\varepsilon_m} \quad \forall m,t \tag{14.33}$$

We then substitute (14.33) into the production function to get another expression for capital as a function of r_t, w_t, and Y_t.

$$K_{m,t} = \frac{Y_{m,t}}{A_{m,t}}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{r_t+\delta_m}{w_t}\right)^{\varepsilon_m-1}\left(\frac{1-\gamma_m}{\gamma_m}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{1-\varepsilon_m}} \quad \forall m,t \tag{14.34}$$

14.3 Market clearing

Because we assume that capital and labor are perfectly mobile across industries in this economy, the market clearing conditions change significantly in this case of two industries and two goods. For labor market clearing, the sum of labor demand across each industry m must equal the sum of exogenous labor supplied by each household.

$$L_{1,t} + L_{2,t} = \sum_{s=1}^{3} n_s \quad \forall t \tag{14.35}$$

This implies that individuals are indifferent regarding to which industry they provide labor, and firms see household labor as perfectly substitutable across age cohorts.

The capital market clearing condition is similar. It equates total capital demand across industries with total capital supply from individuals.

$$K_{1,t} + K_{2,t} = \sum_{s=2}^{3} b_{s,t} \quad \forall t \tag{14.36}$$

This also implies that individuals are indifferent regarding to which industry they provide capital, and firms see household capital as perfectly substitutable across age cohorts.

In a one-industry, one-good model as in Chapter 2, the goods market clearing condition is a resource constraint requiring that $Y_t = C_t + K_{t+1} - (1+\delta)K_t$. It states that aggregate output can be used for aggregate consumption or aggregate investment. In the one-industry, one-good model, this goods market clearing condition is redundant by Walras' Law because aggregate capital in each period is already determined by household savings, aggregate consumption is determined by household consumption, and aggregate output is determined by household savings and labor supply.

With two goods and two industries, we need two resource constraints in order to account for total output from each industry being used for either consumption or investment. Define total consumption demand for good $m = 1$ from the same industry as $C_{1,t}$.

$$C_{1,t} \equiv \sum_{s=1}^{3} c_{1,s,t} \quad \forall t \tag{14.37}$$

$$C_{2,t} \equiv \sum_{s=1}^{3} c_{2,s,t} \quad \forall t \tag{14.38}$$

The two resource constraints in each industry are the following.

$$Y_{1,t} = C_{1,t} + K_{1,t+1} - (1-\delta_1)K_{1,t} \quad \forall t \tag{14.39}$$

$$Y_{2,t} = C_{2,t} + K_{2,t+1} - (1-\delta_2)K_{2,t} \quad \forall t \tag{14.40}$$

As a final comment in this section, it is important to note that we are implicitly assuming some type of financial intermediary in the capital market clearing equation (14.36) and in the two resource constraints (14.39) and (14.40). This is because everything in the budget constraint (14.14) is denominated in composite consumption units because we normalized $p_t = 1$. This means household savings $b_{s+1,t+1}$ is in composite consumption units. However, the capital market clearing condition (14.36) says that those composite consumption units of savings are transformed one-for-one into industry-specific capital stock. Lastly, the two resource constraints in (14.39) and (14.40) each have industry output $Y_{m,t}$ and aggregate industry consumption $C_{m,t}$ in industry consumption units, but have capital $K_{m,t}$ in composite consumption units. This implies that our invisible financial intermediaries can transfer the loaned capital with interest back to households in composite consumption units $(1 + r_t)b_{s,t}$.

This nuance will not matter in the more advanced versions of the model to which we are progressing because firms will own their own capital in those models and households will simply own equity shares of firms. However, one could make this current version of the model more realistic by explicitly modeling a financial intermediary that has a transformation function as described above.

14.4 Equilibrium

We first define the steady state equilibrium. This is the equilibrium in which all the household and firm optimal conditions are satisfied, markets clear, and endogenous variables are constant across time $x_{t+1} = x_t = \bar{x}$.

Definition 14.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with 3-period lived agents, two industries, and two goods is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$ and $\left\{\{\bar{c}_{m,s}\}_{m=1}^2\right\}_{s=1}^3$, capital $\{\bar{b}_s\}_{s=2}^3$, and prices $\bar{w}$, $\bar{r}$, $\bar{p}$, and $\{p_m\}_{m=1}^2$ such that:

i. households optimize each period according to 2 Euler equations (14.20) and (14.22), type-m consumption good demand equations (14.12) and (14.13), and one individual composite good price equation (14.11),

ii. firms in both industries choose capital and labor demand according to (14.27) and (14.28),

iii. perfect competition among firms implies that profits are zero in each industry according to (14.29),

iv. capital and labor markets clear according to (14.35) and (14.36), and the two goods markets clear according to (14.39) and (14.40).

This system of nonlinear dynamic difference equations is exactly identified. We discuss in detail how to solve for the steady state in Section 14.5.1.

The definition of non-steady-state equilibrium differs from the steady-state Definition 14.1 in a few important ways. First, the variables need not be constant across time. Also, we must make an assumption (rational expectations) about each household's ability to forecast the decisions of the other agents and, therefore, correctly forecast future prices. And lastly, the solution to the non-steady-state equilibrium is a set of functions that satisfy the conditions of the equilibrium.

Definition 14.2 (Non-steady-state functional equilibrium). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with 3-period lived agents, two industries and two goods is defined as stationary allocation functions of the state $\big\{b_{s+1,t+1} = \psi_s\big(\mathbf{\Gamma}_t\big)\big\}_{s=1}^2$ and stationary price functions $w(\mathbf{\Gamma}_t)$, $r(\mathbf{\Gamma}_t)$, $p(\mathbf{\Gamma}_t)$, and $p_m(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (2.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize each period according to 2 Euler equations (14.20) and (14.22), type-m consumption good demand equations (14.12) and (14.13), and one individual composite good price equation (14.11),

iii. firms in both industries choose capital and labor demand according to (14.27) and (14.28),

iv. perfect competition among firms implies that profits are zero in each industry according to (14.29),

v. capital and labor markets clear according to (14.35) and (14.36), and the two goods markets clear according to (14.39) and (14.40).

The difficulty in solving for the non-steady-state equilibrium is that the characterizing equations in Definition 14.2 are not identified. That is, because the dynamic equations are second-order difference equations. Further, we must use functional analysis to find the set of functions that solve all the conditions in Definition 14.2 in every possible time period t and for every possible state $\mathbf{\Gamma}_t$. We describe the technique for solving for a particular equilibrium non-steady-state time path in Section 14.5.2.

14.5 Solution method

This section describes the computational approach to solving for the steady-state of the model from Definition 14.1. We also describe here the time path iteration (TPI) approach to solving for a particular non-steady-state equilibrium described in Definition 14.1.

14.5.1 Solving for the steady-state

We solve for the steady-state from Definition 14.1 by choosing two variables $\bar{r}$ and $\bar{w}$ to clear the capital market (14.36) and labor market (14.35) given that all other equations from Definition 14.1 hold. We show in this section how the choice of steady-state $\bar{r}$ and $\bar{w}$ determines all the other endogenous variables in the model and how only the equilibrium $\bar{r}$ and $\bar{w}$ will clear both the capital market (14.36) and the labor market (14.35).

In the steady-state, $\bar{r}$ and $\bar{w}$ give us industry prices $\bar{p}_1$ and $\bar{p}_2$ from the steady-state version of (14.32).

$$\bar{p}_m = \frac{1}{A_m}\Big[\gamma_m(\bar{r}+\delta_m)^{1-\varepsilon_m} + (1-\gamma_m)(\bar{w})^{1-\varepsilon_m}\Big]^{\frac{1}{1-\varepsilon_m}} \quad \text{for} \quad m = 1,2 \tag{14.41}$$

And we can use the steady-state version of (14.11) to get the composite price $\bar{p}$.

$$\bar{p} = \left(\frac{\bar{p}_1}{\alpha}\right)^{\alpha}\left(\frac{\bar{p}_2}{1-\alpha}\right)^{1-\alpha} \tag{14.42}$$

With $\bar{r}$, $\bar{w}$, $\bar{p}$, $\bar{p}_1$, and $\bar{p}_2$, we can solve for all of the household's steady state savings decisions $\{\bar{b}_2, \bar{b}_3\}$ from the steady-state versions of the two lifetime Euler equations (14.20) and (14.22).

$$\begin{aligned} &\left(\frac{1}{\bar{p}}\left[\bar{w}n_1 - \bar{b}_2 - \sum_{m=1}^{2}\bar{p}_m\tilde{c}_m\right]\right)^{-\sigma} = \ldots \\ &\qquad \beta(1+\bar{r})\left(\frac{1}{\bar{p}}\left[(1+\bar{r})\bar{b}_2 + \bar{w}n_2 - \bar{b}_3 - \sum_{m=1}^{2}\bar{p}_m\tilde{c}_m\right]\right)^{-\sigma} \end{aligned} \tag{14.43}$$

$$\begin{aligned} &\left(\frac{1}{\bar{p}}\left[(1+\bar{r})\bar{b}_2 + \bar{w}n_2 - \bar{b}_3 - \sum_{m=1}^{2}\bar{p}_m\tilde{c}_m\right]\right)^{-\sigma} = \ldots \\ &\qquad \beta(1+\bar{r})\left(\frac{1}{\bar{p}}\left[(1+\bar{r})\bar{b}_3 + \bar{w}n_3 - \sum_{m=1}^{2}\bar{p}_m\tilde{c}_m\right]\right)^{-\sigma} \end{aligned} \tag{14.44}$$

With the full set of steady-state savings decisions $\{\bar{b}_2, \bar{b}_3\}$ and steady-state prices $\bar{r}$, $\bar{w}$, $\bar{p}$, $\bar{p}_1$, and $\bar{p}_2$, we can use the steady-state version of the household budget constraint (14.14) to solve for the three steady-state values of composite household consumption $\{\bar{c}_s\}_{s=1}^3$.

$$\bar{p}\bar{c}_s + \sum_{m=1}^{2}\bar{p}_m\tilde{c}_m + \bar{b}_{s+1} = (1+\bar{r})\bar{b}_s + \bar{w}n_s \quad \text{for} \quad s = 1,2,3 \tag{14.45}$$

With $\{\bar{c}_s\}_{s=1}^3$, $\bar{p}$, $\bar{p}_1$, and $\bar{p}_2$, we can solve for the household's steady-state consumption of good-1 and good-2 in each period of his life from the steady-state versions of (14.12) and (14.13).

$$\bar{c}_{1,s} = \alpha\frac{\bar{p}\bar{c}_s}{\bar{p}_1} + \tilde{c}_1 \qquad \text{for} \quad s = 1,2,3 \tag{14.46}$$

$$\bar{c}_{2,s} = (1-\alpha)\frac{\bar{p}\bar{c}_s}{\bar{p}_2} + \tilde{c}_2 \quad \text{for} \quad s = 1,2,3 \tag{14.47}$$

We can now solve for steady-state aggregate consumption in both industries $\bar{C}_1$ and $\bar{C}_2$ from the steady-state versions of (14.37) and (14.38) because we know the individual consumption demands $\bar{c}_{1,s}$ and $\bar{c}_{2,s}$ for $s = 1,2,3$.

$$\bar{C}_m = \sum_{s=1}^{3}\bar{c}_{m,s} \quad \text{for} \quad m = 1,2 \tag{14.48}$$

If we take the steady-state version of our firm's demand for capital in terms of output $Y_{m,t}$, interest rate r_t, and industry price $p_{m,t}$ from (14.34) and substitute it into the steady-state resource constraint (14.39), we can characterize steady-state industry output $\bar{Y}_m$ as a function of aggregate consumption from that industry $\bar{C}_m$, the interest rate $\bar{r}$, and the industry price

$\bar{p}_m$.

$$\begin{aligned}
\bar{Y}_m &= \bar{C}_m + \delta_m\left(\frac{\bar{Y}_m}{A_m}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{\bar{r}+\delta_m}{\bar{w}}\right)^{\varepsilon_m-1}\left(\frac{1-\gamma_m}{\gamma_m}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{1-\varepsilon_m}}\right) \\
\Rightarrow \quad \bar{Y}_m&\left(1 - \frac{\delta_m}{A_{m,t}}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{r_t+\delta_m}{w_t}\right)^{\varepsilon_m-1}\left(\frac{1-\gamma_m}{\gamma_m}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{1-\varepsilon_m}}\right) = \bar{C}_m \\
\Rightarrow \quad \bar{Y}_m &= \bar{C}_m\left(1 - \frac{\delta_m}{A_{m,t}}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{r_t+\delta_m}{w_t}\right)^{\varepsilon_m-1}\left(\frac{1-\gamma_m}{\gamma_m}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{1-\varepsilon_m}}\right)^{-1} \\
&\qquad \text{for} \quad m = 1,2
\end{aligned} \tag{14.49}$$

With $\bar{Y}_1$, $\bar{Y}_2$, $\bar{r}$, $\bar{w}$, $\bar{p}$, $\bar{p}_1$, and $\bar{p}_2$, we can solve for $\bar{K}_1$, $\bar{K}_2$, $\bar{L}_1$, and $\bar{L}_2$ using the steady-state versions of (14.34) and (14.33).

$$\bar{K}_m = \frac{\bar{Y}_m}{A_m}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}\left(\frac{\bar{r}+\delta_m}{\bar{w}}\right)^{\varepsilon_m-1}\left(\frac{1-\gamma_m}{\gamma_m}\right)^{\frac{\varepsilon_m-1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{1-\varepsilon_m}} \quad \text{for} \quad m = 1,2 \tag{14.50}$$

$$\bar{L}_m = \bar{K}_m\left(\frac{1-\gamma_m}{\gamma_m}\right)\left(\frac{\bar{r}+\delta_m}{\bar{w}}\right)^{\varepsilon_m} \quad \text{for} \quad m = 1,2 \tag{14.51}$$

At this point, all the steady-state variables are determined based on the initial guess of steady-state $\bar{r}$ and $\bar{w}$. The last step is to adjust the values of $\bar{r}$ and $\bar{w}$ until the steady-state version of the labor market clearing condition (14.35) and the steady-state version of the capital market clearing condition (14.36) are satisfied.

$$\bar{L}_1 + \bar{L}_2 = \sum_{s=1}^{3} n_s \tag{14.52}$$

$$\bar{K}_1 + \bar{K}_2 = \sum_{s=2}^{3} \bar{b}_s \tag{14.53}$$

14.5.2 Solving for the equilibrium time path (TPI)

The solution method for a particular non-steady-state equilibrium time path of the economy characterized in Definition 14.2 is time path iteration (TPI). The key assumption is that the economy will reach the steady-state equilibrium $\bar{\mathbf{\Gamma}}$ described in Definition 14.1 in a finite number of periods $T < \infty$ regardless of the initial state $\mathbf{\Gamma}_1$.

14.6 Calibration

14.7 Exercises

Chapter 15

Multiple industries, Multiple Goods, and S-period-lived individuals

In this chapter, we take the S-period-lived agent model from Chapter 3 with exogenous labor and add to it M distinct industries and I distinct consumer goods.

15.1 Households

A unit measure of identical individuals are born each period and live for S periods. Let the age of an individual be indexed by $s = \{1, 2, ...S\}$. Individuals in this economy choose among I different consumption $c_{i,s,t}$ goods every period. However, we assume that this consumption can be aggregated into a composite consumption good $c_{s,t}$ every period in every individual's preferences according to the following Cobb-Douglas consumption aggregator,

$$c_{s,t} \equiv \prod_{i=1}^{I} \left(c_{i,s,t} - \tilde{c}_i\right)^{\alpha_i} \quad \forall s,t \tag{15.1}$$

such that $\sum_{i=1}^{I} \alpha_i = 1$ and $\tilde{c}_i$ represents a minimum consumption level for each consumption good i. Note that the specification in (15.1) signifies that type-i consumption $c_{i,s,t}$ is consumption in addition to (excluding) the minimum consumption $\tilde{c}_i$. Then composite consumption $c_{s,t}$ must be interpreted as excluding some composite of the minimum consumption levels.

Assume that the price of each individual consumption is $p_{i,t}$. We can solve for the optimal type-i consumption demands $c_{i,s,t}$ as a function of composite consumption $c_{s,t}$ by minimizing the total expenditure on consumption given that individual consumption adds up to composite consumption according to (15.1). The Lagrangian for this expenditure minimization problem is the following.

$$\mathcal{L} = \sum_{i=1}^{I} p_{i,t}\left(c_{i,s,t} + \tilde{c}_i\right) + \lambda_{s,t}\Bigl[c_{s,t} - \prod_{i=1}^{I}\left(c_{i,s,t} - \tilde{c}_i\right)^{\alpha_i}\Bigr] \quad \forall s,t \tag{15.2}$$

Because the Lagrangian multiplier on the constraint $\lambda_{s,t}$ represents the shadow price of an extra unit of composite consumption, we can relabel it as the price of composite consumption $p_{s,t}$.

$$\mathcal{L} = \sum_{i=1}^{I} p_{i,t}\left(c_{i,s,t} + \tilde{c}_i\right) + p_{s,t}\Bigl[c_{s,t} - \prod_{i=1}^{I}\left(c_{i,s,t} - \tilde{c}_i\right)^{\alpha_i}\Bigr] \quad \forall s,t \tag{15.3}$$

Note that the price of composite consumption can be different for each age-s individual.

The $I+1$ first order conditions of this constrained minimization problem are the following.

$$p_{i,t} = p_{s,t}\alpha_i(c_{i,s,t} - \tilde{c}_i)^{\alpha_i - 1}\prod_{u\neq i}^{I}\left(c_{u,s,t} - \tilde{c}_u\right)^{\alpha_u} \quad \forall i,s,t \tag{15.4}$$

$$c_{s,t} = \prod_{i=1}^{I}\left(c_{i,s,t} - \tilde{c}_i\right)^{\alpha_i} \quad \forall s,t \tag{15.1}$$

Multiplying both sides of equation (15.4) by $c_{i,s,t} - \tilde{c}_i$ gives the following equation.

$$p_{i,t}\left(c_{i,s,t} - \tilde{c}_i\right) = p_{s,t}\alpha_i\prod_{i=1}^{I}\left(c_{i,s,t} - \tilde{c}_i\right)^{\alpha_i} \quad \forall i,s,t \tag{15.5}$$

The product term on the right-hand-side is simply the definition for composite consumption $c_{s,t}$. Substituting this in and solving for $c_{i,s,t}$ gives the optimal demand function for consumption of good i by age-s individual in period t.

$$c_{i,s,t} = \alpha_i\frac{p_{s,t}c_{s,t}}{p_{i,t}} + \tilde{c}_i \quad \forall i,s,t \tag{15.6}$$

Substituting the demand equations (15.6) back into the composite consumption definition

(15.1) gives us the expression for the composite price $p_{s,t}$ as a function of each type-i good's price $p_{i,t}$.

$$p_{s,t} = \prod_{i=1}^{I}\left(\frac{p_{i,t}}{\alpha_i}\right)^{\alpha_i} \quad \forall s,t \tag{15.7}$$

However, in this case, because nothing on the right-hand-side of (15.7) is a function of s, then $p_{s,t} = p_t$ for all s.

$$p_t = \prod_{i=1}^{I}\left(\frac{p_{i,t}}{\alpha_i}\right)^{\alpha_i} \quad \forall t \tag{15.8}$$

With this composite price expression, we can rewrite the type-i good demand function.

$$c_{i,s,t} = \alpha_i \frac{p_t c_{s,t}}{p_{i,t}} + \tilde{c}_i \quad \forall i,s,t \tag{15.9}$$

Because the individual-good problem from (15.8) and (15.9) is determined by functions of composite consumption $c_{s,t}$ and individual prices $p_{i,t}$, we can write the individual's utility maximization in terms of composite consumption $c_{s,t}$. An age-s individual faces the following per-period budget constraint.

$$p_t c_{s,t} + \sum_{i=1}^{I} p_{i,t}\tilde{c}_i + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \quad \forall s,t \tag{15.10}$$

We assume the individuals supply a unit of labor inelastically in the first two thirds of life ($s \leq \text{round}(2S/3)$) and are retired with reduced labor supply during the last third of life ($s > \text{round}(S/3)$).

$$n_{s,t} = \begin{cases} 1 \quad \text{if} \quad s \leq \text{round}\left(\frac{2S}{3}\right) \\ 0.2 \quad \text{if} \quad s > \text{round}\left(\frac{S}{3}\right) \end{cases} \quad \forall s,t \tag{3.1}$$

Because exogenous labor in (3.1) is not dependent on the time period, we drop the t subscript from labor n_s for the rest of this section. We also assume that households are born with no savings $b_{1,t} = 0$ and that individuals save no income in the last period of their lives $b_{S+1,t} = 0$ for all periods t. Assume that $c_{s,t} \geq 0$ because individual good consumption $c_{i,s,t}$ less than the minimum $\tilde{c}_i$ is not defined. However, it will also be the case that period individual utility will not be defined for $c_{s,t} <= 0$.

Let the utility of composite consumption in each period be defined by the constant relative risk aversion function $u(c_{s,t})$, such that $u' > 0$, $u'' < 0$, and $\lim_{c\to 0} u(c) = -\infty$.

$$u(c_{s,t}) = \frac{(c_{s_t})^{1-\sigma} - 1}{1-\sigma} \quad \forall s,t \tag{15.11}$$

Individuals choose lifetime composite consumption $\{c_{s,t+s-1}\}_{s=1}^{S}$ and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints.

$$\begin{aligned} &\max_{\{c_{s,t+s-1}\}_{s=1}^{S},\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{u=0}^{S-s} \beta^u u(c_{s+u,t+u}) \quad \forall s,t \\ &\text{s.t.} \quad c_{s,t} = \frac{1}{p_t}\Bigl[(1+r_t)b_{s,t} + w_t n_s - b_{s+1,t+1} - \sum_{i=1}^{I} p_{i,t}\tilde{c}_i\Bigr] \quad \forall s,t \\ &\text{and} \quad b_{1,t}, b_{S+1,t} = 0 \quad \forall t \quad \text{and} \quad c_{s,t} \geq 0 \quad \forall s,t \end{aligned} \tag{15.12}$$

The number of variables to choose in the household's optimization problem can be reduced by substituting the budget constraints into the optimization problem (15.12). The optimal choice of how much to save in the each of the first $S-1$ periods of life $b_{s+1,t+1}$ is found by taking the derivative of the lifetime utility function with respect to each of the lifetime savings amounts $\{b_{s+1,t+s+1}\}_{s=1}^{S-1}$ and setting the derivatives equal to zero.

In the last period of life, the household optimally chooses no savings $b_{S+1,t+1} = 0$ for all t because any positive savings only imposes a cost of reduced consumption, and negative savings (borrowing) would impose an automatic default on anyone lending to him. The final period S decision is simple. The individual enters the period with wealth $b_{S,t}$, he knows the interest rate r_t, the wage w_t, the composite goods price p_t, the individual goods prices $\{p_{i,t}\}_{i=1}^{I}$, and inelatically supplies labor $n_{S,t}$. Everything in the budget constraint (15.10) is determined except for $c_{S,t}$. In the final period, the individual simply consumes all his resources.

$$c_{S,t} = \frac{1}{p_t}\Bigl[(1+r_t)b_{S,t} + w_t n_S - \sum_{i=1}^{I} p_{i,t}\tilde{c}_i\Bigr] \quad \forall t \tag{15.13}$$

In the second-to-last period of life $s = S-1$, the household has a savings decision to make. He enters the period with wealth $b_{S-1,t}$, he knows the current interest rate r_t, the current wage w_t, current composite prices p_t, current individual good prices $\{p_{i,t}\}_{i=1}^{I}$, and he must know or be able to forecast next period's prices r_{t+1}, w_{t+1}, p_{t+1}, and $\{p_{i,t+1}\}_{i=1}^I$. In this case, the household's lifetime utility function is one equation and one unknown.

$$\begin{split}\max_{b_{S,t+1}}\; u\left(\frac{1}{p_t}\Bigl[(1+r_t)b_{S-1,t} + w_t n_{S-1,t} - b_{S,t+1} - \sum_{i=1}^I p_{i,t}\tilde{c}_i\Bigr]\right) + ... \\ \beta u\left(\frac{1}{p_{t+1}}\Bigl[(1+r_{t+1})b_{S,t+1} + w_{t+1}n_{S,t+1} - \sum_{i=1}^I p_{i,t+1}\tilde{c}_i\Bigr]\right)\end{split} \tag{15.14}$$

The first order condition, or dynamic Euler equation, for this second-to-last period of life savings decision is the following.

$$\begin{split}\left(\frac{1}{p_t}\Bigl[(1+r_t)b_{S-1,t} + w_t n_{S-1,t} - b_{S,t+1} - \sum_{i=1}^I p_{i,t}\tilde{c}_i\Bigr]\right)^{-\sigma} = ... \\ \beta(1+r_{t+1})\frac{p_t}{p_{t+1}}\left(\frac{1}{p_{t+1}}\Bigl[(1+r_{t+1})b_{S,t+1} + w_{t+1}n_{S,t+1} - \sum_{i=1}^I p_{i,t+1}\tilde{c}_i\Bigr]\right)^{-\sigma}\end{split} \tag{15.15}$$

The solution for savings $b_{S,t+1}$ in the second-to-last period of life to be returned with interest in the last period of life is characterized by the nonlinear dynamic Euler equation (15.15) and is a function of individual wealth $b_{S-1,t}$, prices r_t, w_t, p_t, and $\{p_{i,t}\}_{i=1}^I$ at the beginning of the second-to-last period of life, as well as prices in the last period of life r_{t+1}, w_{t+1}, p_{t+1}, and $\{p_{i,t+1}\}_{i=1}^I$.

$$b_{S,t+1} = \psi_{S-1}\Bigl(b_{S-1,t}, r_t, w_t, p_t, \{p_{i,t}\}_{i=1}^I, r_{t+1}, w_{t+1}, p_{t+1}, \{p_{i,t+1}\}_{i=1}^I\Bigr) \quad \forall t \tag{15.16}$$

Call $\psi_{S-1}(\cdot)$ the policy function for savings $b_{S,t+1}$ in the second-to-last period of life.

By backward induction, it is straightforward to show that the $S-1$ savings decisions over an individual's lifetime are characterized by $S-1$ nonlinear dynamic Euler equations of the form,

$$\begin{split}u'(c_{s,t}) = \beta(1+r_{t+1})\frac{p_t}{p_{t+1}}u'(c_{s+1,t+1}) \quad \forall t, \quad \text{and} \quad 1\leq s\leq S-1 \\ \text{and} \quad c_{s,t} = \frac{1}{p_t}\Bigl[(1+r_t)b_{s,t} + w_t n_{s,t} - b_{s+1,t+1} - \sum_{i=1}^I p_{i,t}\tilde{c}_i\Bigr] \quad \forall s,t \\ \text{and} \quad b_{1,t}, b_{S+1,t} = 0 \quad \forall t\end{split} \tag{15.17}$$

Following the pattern of (15.16), the policy functions for each of the savings decisions is a function of the individual's wealth at the beginning of the period $b_{s,t}$ and the time path of wages and interest rates over the remaining periods of the individual's life.

$$b_{s+1,t+1} = \psi_s\Big(b_{s,t}, \{r_u\}_{u=t}^{t+S-s}, \{w_u\}_{u=t}^{t+S-s}, \{p_u\}_{u=t}^{t+S-s}, \big\{\{p_{i,u}\}_{i=1}^{I}\big\}_{u=t}^{t+S-s}\Big) \quad \forall t \quad \text{and} \quad 1 \leq s \leq S-1 \tag{15.18}$$

To summarize the individual's problem, if one knows his initial savings or wealth $b_{s,t}$ and the time path of factor prices over his remaining lifetime, he can solve for all of his optimal savings levels $\{b_{s+1,t+s}\}_{s=1}^{S-1}$.

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of prices $\big\{r_u, w_u, p_u, \{p_{i,u}\}_{i=1}^{I}\big\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Section 15.4, the equilibrium prices in period t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \big\{b_{s,t}\big\}_{s=2}^{S} \quad \forall t \tag{3.14}$$

where the e superscript signifies that $\mathbf{\Gamma}_{t+u}^{e}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.

15.2 Firms

The production side of this economy is comprised of M industries indexed by $m \in \{1, 2, ...M\}$, each of which has a unit measure of identical, perfectly competitive firms. These firms rent investment capital from individuals for real return r_t and hire labor for real wage w_t. The interest rate r_t and wage w_t are equal across industries because labor and capital are perfectly mobile. Firms in industry m use their total capital $K_{m,t}$ and labor $L_{m,t}$ to produce output $Y_{m,t}$ every period according to a constant elasticity of substitution (CES) production technology,

$$Y_{m,t} = F(K_{m,t}, L_{m,t}) \equiv A_{m,t}\left[(\gamma_m)^{\frac{1}{\varepsilon_m}}(K_{m,t})^{\frac{\varepsilon_m - 1}{\varepsilon_m}} + (1-\gamma_m)^{\frac{1}{\varepsilon_m}}(L_{m,t})^{\frac{\varepsilon_m - 1}{\varepsilon_m}}\right]^{\frac{\varepsilon_m}{\varepsilon_m - 1}} \quad \forall m,t \tag{15.19}$$

where $\gamma_m \in (0,1)$ is the capital share of income, $\varepsilon_m \geq 1$ is the elasticity of substitution between capital and labor, and total factor productivity $A_{m,t} > 0$ for all t.

The representative firm in each industry m chooses how much capital to rent and how much labor to hire to maximize profits,

$$\max_{K_{m,t},L_{m,t}} \; p_{m,t}A_{m,t}F\big(K_{m,t}, L_{m,t}\big) - (r_t + \delta_m)K_{m,t} - w_t L_{m,t} \quad \forall m,t \tag{15.20}$$

where $p_{m,t}$ is the price of output from industry m, $\delta_m \in [0,1]$ is the rate of capital depreciation. The two first order conditions that characterize firm optimization are the following.

$$r_t = p_{m,t}(A_{m,t})^{\frac{\varepsilon_m - 1}{\varepsilon_m}}\left(\gamma_m \frac{X_{m,t}}{K_{m,t}}\right)^{\frac{1}{\varepsilon_m}} - \delta_m \quad \forall m,t \tag{15.21}$$

$$w_t = p_{m,t}(A_{m,t})^{\frac{\varepsilon_m - 1}{\varepsilon_m}}\left([1-\gamma_m]\frac{X_{m,t}}{L_{m,t}}\right)^{\frac{1}{\varepsilon_m}} \quad \forall m,t \tag{15.22}$$

Each firm in industry m can make its capital and labor demand decision if it knows the price of its good $p_{m,t}$, the interest rate r_t, and the wage w_t. For the entire supply side of the economy, all firms can optimize if r_t, w_t, and $\{p_{m,t}\}_{m=1}^M$ are known.

15.3 Market clearing

Because we assume that capital and labor are perfectly mobile across industries in this economy, the market clearing conditions change significantly in this case of multiple industries. For labor market clearing, the sum of labor demand across each industry m must equal the sum of exogenous labor supplied by each household.

$$\sum_{m=1}^M L_{m,t} = \sum_{s=1}^S n_{s,t} \quad \forall t \tag{15.23}$$

This implies that individuals are indifferent regarding to which industry they provide labor, and firms see household labor as perfectly substitutable across age cohorts.

The capital market clearing condition is similar. It equates total capital demand across industries with total capital supply from individuals.

$$\sum_{m=1}^{M} K_{m,t} = \sum_{s=2}^{S} b_{s,t} \quad \forall t \tag{15.24}$$

This also implies that individuals are indifferent regarding to which industry they provide capital, and firms see household capital as perfectly substitutable across age cohorts.

Goods market clearing is a little more involved because we are allowing the number of goods in the economy I to differ from the number of industries M. This means that each consumption good i in the economy is some composite of the output from each industry m. However, for reasons we will discuss in Section 15.4, the number of production industries must be as least as large as the number of consumption goods $M \geq I$.

Let $X_{i,t}$ be the total amount of consumption good i produced in period t. This total amount of consumption good $X_{i,t}$ is produced by some combination of production goods $Y_{m,t}$. Let $\iota_{i,m} \in [0,1]$ be the percent of total production good $Y_{m,t}$ used to make consumption good $X_{i,t}$. Then consumption good $X_{i,t}$ is the following function of all production goods $\{Y_{m,t}\}_{m=1}^{M}$,

$$X_{i,t} = \sum_{m=1}^{M} \iota_{i,m} Y_{m,t} \quad \forall i,t \quad \text{and} \quad \sum_{i=1}^{I} \iota_{i,m} = 1 \tag{15.25}$$

or in matrix form,

$$\underbrace{\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ \vdots \\ X_{I,t} \end{bmatrix}}_{I\times 1} = \underbrace{\begin{bmatrix} \iota_{1,1} & \iota_{1,2} & \dots & \iota_{1,M} \\ \iota_{2,1} & \iota_{2,2} & \dots & \iota_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \iota_{I,1} & \iota_{I,2} & \dots & \iota_{I,M} \end{bmatrix}}_{I\times M} \underbrace{\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \\ \vdots \\ Y_{M,t} \end{bmatrix}}_{M\times 1} \quad \forall t \tag{15.26}$$

where the columns of the $\iota_{i,m}$ matrix sum to 1. The problem in Chapter 14 implicitly has and $\iota_{i,m}$ matrix that is a two-by-two identity matrix.

We assume that the intermediary that transforms production goods $Y_{m,t}$ into consumption goods $X_{i,t}$ is frictionless and perfectly competitive such that the total cost of producing $X_{i,t}$ is just the sum of the costs of all the inputs $Y_{m,t}$,

$$p_{i,t}X_{i,t} = \sum_{m=1}^{M} \iota_{i,m}p_{m,t}Y_{m,t} \quad \forall i,t \tag{15.27}$$

or in matrix form.

$$\underbrace{\begin{bmatrix} p_{1,t}X_{1,t} \\ p_{2,t}X_{2,t} \\ \vdots \\ p_{I,t}X_{I,t} \end{bmatrix}}_{I\times 1} = \underbrace{\begin{bmatrix} \iota_{1,1} & \iota_{1,2} & \dots & \iota_{1,M} \\ \iota_{2,1} & \iota_{2,2} & \dots & \iota_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \iota_{I,1} & \iota_{I,2} & \dots & \iota_{I,M} \end{bmatrix}}_{I\times M} \underbrace{\begin{bmatrix} p_{1,t}Y_{1,t} \\ p_{2,t}Y_{2,t} \\ \vdots \\ p_{M,t}Y_{M,t} \end{bmatrix}}_{M\times 1} \quad \forall t \tag{15.28}$$

Now we can write the expression for goods market clearing. Total supply of consumption good i $X_{i,t}$ must equal total household demand for that good. Equation (15.25) relates $X_{i,t}$ to all the production output $\{Y_{m,t}\}_{m=1}^{M}$ output. So the market clearing condition for each consumption good i is the following.

$$\sum_{m=1}^{M} \iota_{i,m}Y_{m,t} = \sum_{s=1}^{S} c_{i,s,t} \quad \forall i,t \tag{15.29}$$

15.4 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points

i. Households optimize according to $S-1$ Euler equations (15.17), $S\times I$ type-i consumption good demand equations (15.9), and one individual composite good price equation (15.8).

ii. Firms in each of the m industries choose capital and labor demand according to (15.21) and (15.22).

iii. Perfect competition among firms implies that profits are zero in each industry.

iv. Markets clear according to (15.23), (15.24), and I goods market clearing equations (15.29).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The first equilibrium condition we describe comes from perfect competition among all the firms in each of the m industries. This implies that profits will be zero in each industry. Otherwise, firms would enter, exit, or adjust their prices until this condition was met.

$$p_{m,t}F\big(K_{m,t},L_{m,t}\big) = (r_t + \delta_m)K_{m,t} + w_t L_{m,t} \quad \forall m,t \tag{15.30}$$

Chapter 16

Multiple dynamic industries, Multiple Goods, and S-period-lived individuals

In this chapter, we take the S-period-lived agent model from Chapter 3 with exogenous labor and add to it M distinct dynamic industries and I distinct consumer goods.

Part VIII

International markets

Chapter 17

Small Open Economy OG Model

In this chapter, we take the S-period-lived agent model from Chapter 4 and open up international markets by treating the country as a small open economy. As a small open economy, the interest rate will be taken as exogenous, being determined by capital markets in the rest of the world. Capital will flow into and out of this small open economy to equilibrate the interest rate in this economy to this world interest rate.

17.1 Households

The basic structure of the household's problem from Chapter 4 remains the same. We must only change our notation regarding the interest rate that enters the household problem to denote that it is exogenous. Let us denote the world interest rate in the period t as r_t^* and the steady-state world interest rate as $\bar{r}^*$. With this, we can write the household's budget constraint as:

$$\begin{aligned} c_{s,t} + b_{s+1,t+1} = (1+r_t^*)b_{s,t} + w_t n_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{aligned} \tag{17.1}$$

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^{S},\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^{S}\beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \tag{4.6}$$

$$\text{s.t.}\quad c_{s,t}+b_{s+1,t+1} \tag{17.2}$$

$$=(1+r_t^*)b_{s,t}+w_t n_{s,t} \tag{17.1}$$

$$\text{where}\quad u\big(c_{s,t},n_{s,t}\big)=\frac{c_{s,t}^{1-\sigma}-1}{1-\sigma}+\chi_s^n b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \tag{4.7}$$

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (17.3), the following $S-1$ dynamic savings Euler equations (17.4), and a budget constraint that binds in all S periods (17.1),

$$\begin{aligned} &w_t u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right)=-u_2\left(c_{s+1,t+s},n_{s+1,t+s}\right)\quad\text{for}\quad s\in\{1,2,...S\}\\ &\Rightarrow\quad w_t\left(c_{s,t}\right)^{-\sigma}=\chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \end{aligned} \tag{17.3}$$

$$\begin{aligned} &u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right)=\beta(1+r_{t+1}^*)u_1\left(c_{s+1,t+s},n_{s+1,t+s}\right)\quad\text{for}\quad s\in\{1,2,...S-1\}\\ &\Rightarrow\quad\left(c_{s,t}\right)^{-\sigma}=\beta(1+r_{t+1}^*)\left(c_{s+1,t+1}\right)^{-\sigma} \end{aligned} \tag{17.4}$$

$$c_{s,t}+b_{s+1,t+1}=(1+r_t^*)b_{s,t}+w_t n_{s,t}\quad\text{for}\quad s\in\{1,2,...S\},\quad b_{1,t},b_{S+1,t+S}=0 \tag{17.1}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$.

These $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u,r_u\}_{u=t}^{t+S-1}$.

17.2 Firms

Firms are characterized similarly to Section 2.2, with the firm's aggregate capital decision K_t governed by first order condition (2.20) and its aggregate labor decision L_t governed by first order condition (2.21). In a small open economy, the interest rate is exogenous. So the firm's first order condition for the choice of capital will now be used to find its capital demand as this exogenous interest rate.

The firm seeks to maximize profits and thus solves,

$$\max_{K_t,L_t} Y_t - w_t L_t - (r_t^* + \delta)K_t \tag{17.5}$$

The two first order conditions that characterize firm optimization are the following.

$$r_t^* = \left(\alpha A \left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta\right) \tag{17.6}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{2.21}$$

17.3 Market Clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t^d = L_t^s = \sum_{s=1}^{S} n_{s,t} \quad \forall t \tag{17.7}$$

$$K_t^d = K_t^s + K_t^f = \sum_{s=2}^{S} b_{s,t} + \text{net foreign capital} \quad \forall t \tag{17.8}$$

$$\begin{aligned} &Y_t = C_t + I_t \quad \forall t \\ &\qquad \text{where} \quad I_t \equiv K_{t+1}^d - (1-\delta)K_t^d \end{aligned} \tag{17.9}$$

The goods market clearing equation (17.9) is redundant by Walras' Law.

17.4 Equilibrium

An equilibrium is found when:

i. Households optimize according to equations (17.3) and (17.4).

ii. Firms optimize according to (17.6) and (2.21).

iii. Markets clear according to (17.7) and (17.8).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

Given the exogenous interest rate r^*, we can use (17.6) to solve for the capital-labor ratio in each period. Then, given the capital-labor ratio in each period, we can use (2.21) to find the wage rate in each period. Thus, given the exogenous interest rate, we can also pin down the wage rate in each period. So all factor prices are exogenous and determined by the world interest rate.

We can then use the firm's first order conditoins together with the market clearing conditons to show that the equilibrium interest rate and wage rates are functions of the distributions of savings and labor supply.

Now (17.14), (17.12), and the budget constraint (17.1) can be substituted into household Euler equations (17.3) and (17.4) to get the following $(2S-1)$-equation system. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{aligned} w_t\left(w_t n_{s,t} + \Big[1+r_t\Big]b_{s,t} - b_{s+1,t+1}\right)^{-\sigma} &= \\ \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} & \\ \text{for} \quad s\in\{1,2,...S\} \quad \text{and} \quad \forall t & \end{aligned} \tag{17.10}$$

$$\begin{aligned} \left(w_t n_{s,t} + \Big[1+r_t\Big]b_{s,t} - b_{s+1,t+1}\right)^{-\sigma} &= \\ \beta\Big[1+r_{t+1}\Big]\left(w_{t+1}n_{s+1,t+1} + \Big[1+r_{t+1}\Big]b_{s+1,t+1} - b_{s+2,t+2}\right)^{-\sigma} & \\ \text{for} \quad s\in\{1,2,\dots S-1\} \quad \text{and} \quad \forall t & \end{aligned} \tag{17.11}$$

The system of S nonlinear static equations (17.10) and $S-1$ nonlinear dynamic equations (17.11) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{s,t+s-1}\}_{s=1}^{S}$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is perfectly identified in this case. This is because the factor prices, r_t and w_t are both functions of the exogenous interest rate.

$$r_t = r^* = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} \quad \forall t \tag{17.12}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \quad \forall t \tag{17.13}$$

$$= (1-\alpha)A\left(\frac{\alpha A}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} \quad \forall t \tag{17.14}$$

We first define the steady-state equilibrium. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 17.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, labor supply $\{\bar{n}_s\}_{s=1}^S$, and savings $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and r^* such that:

i. households optimize according to (17.3) and (17.4),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (17.7) and (12.17).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s(r^*)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s(r^*)$. The function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 17.2 (Non-steady-state functional equilibrium). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\big\{n_{s,t} = \phi_s\big(r^*\big)\big\}_{s=1}^S$, $\big\{b_{s+1,t+1} = \psi_s\big(r^*\big)\big\}_{s=1}^{S-1}$ and and stationary price functions $w(r^*)$ and r^* such that:

i. households optimize according to (17.3) and (17.4),

ii. firms optimize according to (2.20) and (2.21),

iii. markets clear according to (17.7) and (12.17).

17.5 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 17.1 and the transition path equilibrium from Definition ??.

17.5.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 17.1. The parameters needed for the steady-state solution of this model are $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta, r^*\right\}$, where S is the number of periods in an individual's life, $\left\{\beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S\right\}$ are household utility function parameters, $\{A, \alpha, \delta\}$ are firm production function parameters, and r^* is the exogenous world interest rate. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{s,t} = \bar{c}_s, \quad n_{s,t} = \bar{n}_s, \quad b_{s,t} = \bar{b}_s \quad \forall s, t \tag{17.15}$$

From the market clearing conditions (17.7) and (3.16) and the firms' first order equations (2.20) and (2.21), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K^d_t = \bar{K}^d_t, \quad K^s_t = \bar{K}^d, \quad L_t = \bar{L} \quad \forall t \tag{17.16}$$

The steady-state is characterized by the steady-state versions of the set of $2S - 1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}\Bigl([1+r^*]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Bigr)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\} \tag{17.17}$$

$$\Bigl([1+r^*]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Bigr)^{-\sigma} = \beta(1+r^*)\Bigl([1+r^*]\bar{b}_{s+1} + \bar{w}\bar{n}_{s+1} - \bar{b}_{s+2}\Bigr)^{-\sigma} \quad \text{for} \quad s=\{1,2,...S-1\} \tag{17.18}$$

where $\bar{w}$ is a function of r^* as in equation (17.12). Given that both factor prices are determined through the exogenous world interest rate, we modify our solution method. Knowing these factor prices, we can solve the problem by breaking the multivariate root finder problem with $2S-1$ equations and unknowns into a series of many univariate root finder problems and one bivariate root finder problem. The algorithm is the following.

i. Given r^* and $\bar{w}$, solve for the steady-state household's lifetime decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$.

   (a) Given r^* and $\bar{w}$, guess an initial steady-state consumption $\bar{c}_1^m$, where m is the index of the inner-loop (household problem given r^*, $\bar{w}$) iteration.

   (b) Given $\bar{r}^i$, $\bar{w}^i$, $\bar{x}^i$, and $\bar{c}_1^m$, use the sequence of $S-1$ dynamic savings Euler equations (17.4) to solve for the implied series of steady-state consumptions $\{\bar{c}_s^m\}_{s=1}^S$. This sequence has an analytical solution.

$$\bar{c}_{s+1}^m = \bar{c}_s^m\left[\beta(1+r^*)\right]^{\frac{1}{\sigma}} \quad \text{for} \quad s=\{1,2,...S-1\} \tag{17.19}$$

   (c) Given r^*, $\bar{w}$, and $\{\bar{c}_s^m\}_{s=1}^S$, solve for the series of steady-state labor supplies $\{\bar{n}_s^m\}_{s=1}^S$ using the S static labor supply Euler equations (17.3). This will require a series of S separate univariate root finders or one multivariate root finder.

$$\bar{w}\left(\bar{c}_s^m\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\} \tag{17.20}$$

   It is this separation of the labor supply decisions from the consumption-savings decisions that gets rid of the saddle paths in the objective function that are so difficult for global optimization.

(d) Given r^*, $\bar{w}$, and implied steady-state consumption $\{\bar{c}_s^m\}_{s=1}^S$ and labor supply $\{\bar{n}_s^m\}_{s=1}^S$, solve for implied time path of savings $\{\bar{b}_{s+1}^m\}_{s=1}^S$ across all ages of the representative lifetime using the household budget constraint (2.1).

$$\bar{b}_{s+1}^m = (1+r^*)\bar{b}_s^m + \bar{w}\bar{n}_s^m - \bar{c}_s^m \quad \text{for} \quad s = \{1,2,...S\} \tag{17.21}$$

Note that this sequence of savings includes savings in the last period of life for the next period $\bar{b}_{S+1}$. This savings amount is zero in equilibrium, but is not zero for an arbitrary guess for $\bar{c}_1^m$ as in step (a).

(e) Update the initial guess for $\bar{c}_1^m$ to $\bar{c}_1^{m+1}$ until the implied savings in the last period equals zero $\bar{b}_{S+1}^{m+1} = 0$.

ii. Given solution for optimal household decisions $\{\bar{c}_s^m\}_{s=1}^S$, $\{\bar{n}_s^m\}_{s=1}^S$, and $\{\bar{b}_s^m\}_{s=2}^S$ we can sollve for the aggregate capital supplied $\bar{K}^s$ and aggregate labor $\bar{L}$ implied by the household solutions and market clearing conditions.

iii. With $\bar{L}$ in hand, we can use the firms' FOC for choice of labor to solve for capital demand, $\bar{K}^d$.

iv. Finally, the net foreign capital stock, $\bar{K}^f$ can be found as the difference between the capital demand from firms and the capital supply by households: $\bar{K}^f = \bar{K}^d - \bar{K}^s$.

Table 17.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.060	Max. absolute savings Euler error	4.44e-16
$\bar{w}$	1.212	Max. absolute labor supply Euler error	6.66e-16
$\bar{K}^d$	352.282	Absolute final period savings $\bar{b}_{S+1}$	-9.01e-14
$\bar{L}$	59.367	Resource constraint error	0.000
$\bar{Y}$	110.717		
$\bar{C}$	103.410		

Figure 17.1: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$

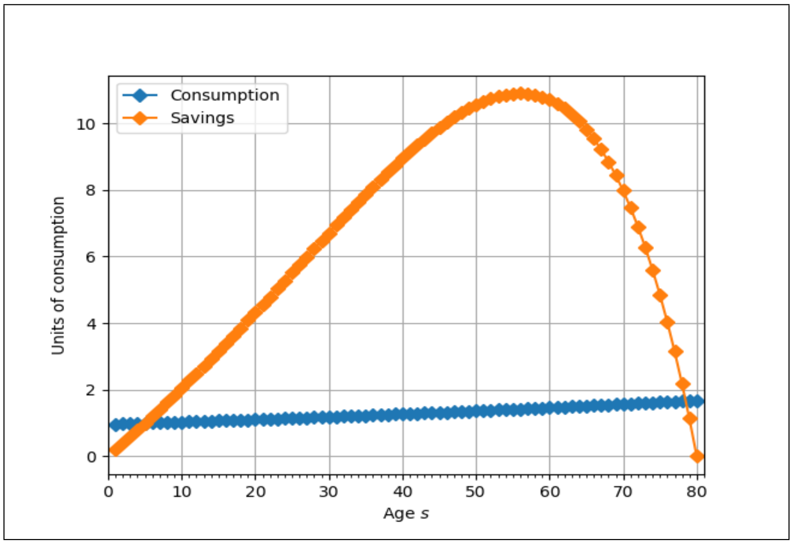


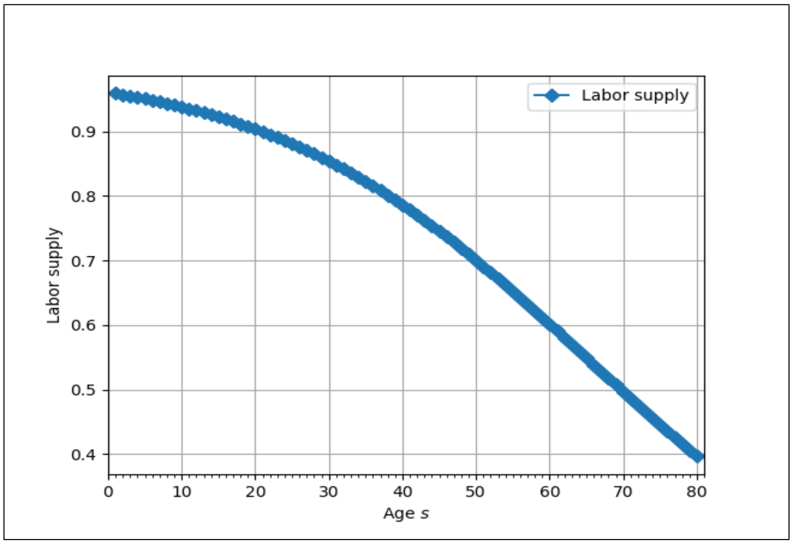
Figure 17.1 shows the steady-state distribution of individual consumption and savings in an 80-period-lived agent model with parameter values listed above the line in Table 17.3 in Section 17.6. Figure 17.2 shows the steady-state distribution of individual labor supply by age. The left side of Table 17.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $2S-1$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 17.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 4.0e-16, the final period individual savings is less than 8.0e-14, and the resource constraint error is 0.0, we can be confident that we have successfully solved for the steady-state.

17.5.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor and a small open economy can also be

Figure 17.2: Steady-state distribution of labor supply $\bar{n}_s$



simplified from method described in Section 4.6.2.

To solve for the transition path (non-steady-state) equilibrium from Definition **??**, we must know the parameters from the steady-state problem $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi_s^n\}_{s=1}^S, A, \alpha, \delta, r^*, \right\}$, the steady-state solution values $\{\bar{K}, \bar{L}\}$, initial distribution of savings $\mathbf{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 17.3 and 17.1 show a particular calibration of the model and a steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

i. Given calibration for initial distribution of savings (wealth) $\mathbf{\Gamma}_1$, which implies an initial capital stock K_1^s.

ii. Given the exogenous interest rate, r^*, we can solve for the entire time path of the wage rate, using the firms' first order consitions.

iii. Given the time paths for the interest rate $\boldsymbol{r}^i$, wage $\boldsymbol{w}^i$, transfers $\boldsymbol{x}^i$ and the period-1 distribution of savings (wealth) $\mathbf{\Gamma}_1$, solve for the lifetime decisions $c_{s,t}$, $n_{s,t}$, and $b_{s,t}$ of each household alive during periods 1 and $T2$. This is done using the method outlined in steps (ii)(a) through (ii)(e) of the steady-state computational algorithm outlined in Section 17.5.1.

iv. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions we can compute the paths of aggregate capital supply and aggregate labor supply, K_t^s and L_t^s.

v. With the time path of aggregate labor supply and the labor market clearing condition, we can use the firms' first order condition for capital to solve for capital demand, K_t^d.

vi. The time paths for aggregate capital supply and demand imply the path for net foreign capital invested domestically: $K^f = K_t^s - K_t^d$.

Table 17.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	8.88e-16
Maximum absolute savings Euler error	5.55e-16
Maximum absolute final period savings $\bar{b}_{S+1,t}$	8.09e-13
Maximum absolute resource constraint error	0.000

The 6 panels of Figure 17.3 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The three panels of Figure 17.4 show the transition paths of the distributions of consumption $c_{s,t}$, labor supply $n_{s,t}$ and savings $b_{s,t}$. Table 17.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than 7.0e-13 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

17.6 Calibration

Use the following parameterization of the model for the problems below. Assume that agents are born at age 21 and die at age 100 (80 years of life). Your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the

Figure 17.3: Equilibrium transition paths of prices and aggregate variables

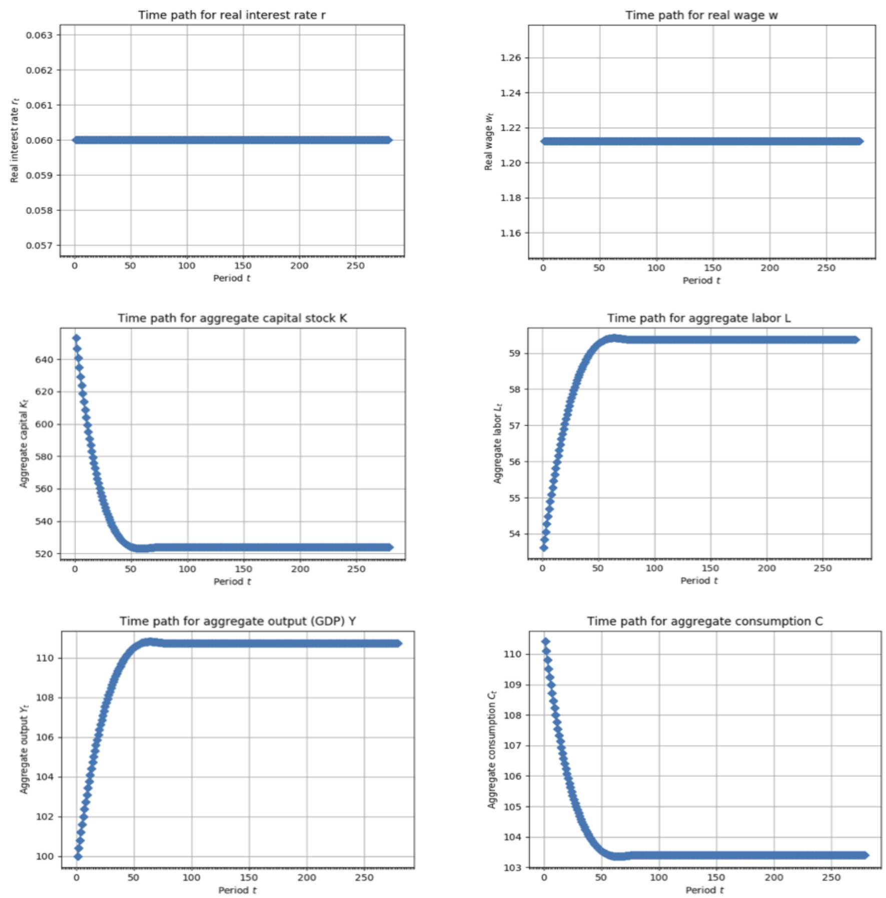
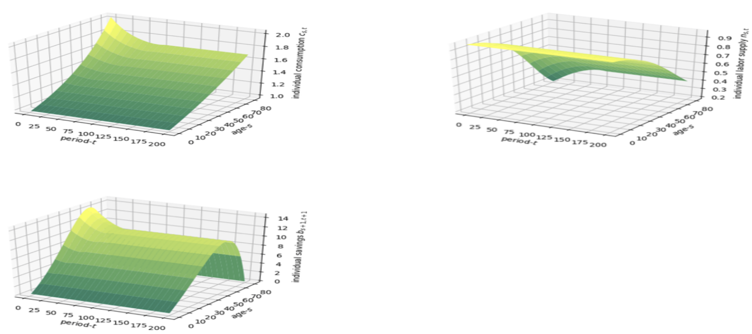


Figure 17.4: Equilibrium transition paths of distributions of consumption, labor supply, and savings



model period depreciation rate is $\delta = 1-(1-0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 2.5$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

Table 17.3: Calibrated parameter values for simple endogenous labor model with government debt financing

Parameter	Description	Value
S	Number of periods in individual life	80
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
$\tilde{l}$	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501[a]
υ	Elliptical disutility of labor shape parameter	1.554[a]
$\{\chi^n_s\}_{s=1}^S$	Disutility of labor relative scale factor by age	1.0
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
r^*	World interest rate	0.06
$\mathbf{\Gamma}_1$	Initial distribution of savings (wealth)	(see Fig. XX)
$T1$	Time period in which initial path guess hits steady state	160
$T2$	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.2

[a] The calibration of b and υ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See Evans and Phillips (2017).

Assume that each individual's time endowment in each period is $\tilde{l} = 1$. Initially, let $\chi^n_s = 1$ for all s. However, you could calibrate these values to match the empirical distribution of annual hours by age in the United states (see Exercise 4.5).

17.7 Exercises

Exercise 17.1. Consider the case where the distribution of capital in the initial period is $(t = 1)$ is the steady-state distribution of capital.

i. Without solving the model, what does your intuition tell you about the time path of the economic aggregates in this case?

ii. Modify the code for the model to implement this (i.e., make the initial distribution equal to the steady state distribution). What do you find?

iii. Explain what you find and why the results are what they are.

Exercise 17.2. In the above exposition, it is assumed that the exogenous world interest rate is constant. Consider the case where it is not constant over time. Modify the code to solve such a model. What do you have to be careful about with respect to the variation in r_t^* over time?

Exercise 17.3. In a more realistic setting, the interes rate in a small open economy may deviate from the risk free world interest rate due to some risk premium associated with holding assets in that country. Although this model abstracts from aggregate shocks that would generate such a risk premium, how could you incorporate a wedge between the world interest rate and the rate of return on capital in the home country? Such a wedge would proxy for the risk premium on investments in the home country. Modify the code to allow for such a wedge.

Chapter 18

Multi-country (Large Open Economy) OG Model

In this chapter, we take model an economy in which two countries that each look like the the S-period-lived agent economies from Chapter 4. In this large-open-economy environment, changes within the country can affect equilibrium interest rates. (probably make labor immobile)

[TODO: write this chapter]

Part IX

Aggregate Shocks, Asset Pricing, and Adaptive Expectations

Chapter 19

Aggregate Shocks and Business Cycles

In this chapter we add aggregate shocks to the model. Show two-period version similar to what did in Evans, Kotlikoff, and Phillips. Show Scheidegger method for $S \geq 3$.

[TODO: write this chapter]

Chapter 20

Asset Pricing

In this chapter, use aggregate shock model to do asset pricing. Show some results with $S = 2$. Then show how do do it with Scheidegger method. Use Evans, Kotlikoff, and Phillips.

[TODO: write this chapter]

Chapter 21

Adaptive Expectations

In this chapter, show how to relax the rational expectations assumption, how that simplifies solution method and how it can closely approximate rational expectations if individuals update beliefs every period. Evans and Phillips (2014).

[TODO: write this chapter]

Part X

Monetary Policy

Chapter 22

Monetary Policy

In this chapter, go through various examples of OG models of monetary policy. Samuelson (1958) was an early example, and Evans (2012) is a more recent example. Start with closed economy example, move to multi-country as in Evans (2012).

[TODO: write this chapter]

Appendices

Appendix A

Using Python

Python is a powerful and elegant open-source programming language that has become widely used in scientific computing and the new field of open data science. Since its introduction in 1991, Python's internal features, libraries, and user network have expanded in ways that have made it a go-to language for prototyping processes. Its ease of integrating with compiled languages such as C and Fortran make it ideal as the master scripting language in hierarchical programs and modules. Python also has great structures and syntax that are amenable to flexible functional programming, object oriented programming, and parallel processing.

It is true that Python's "interpreted" language status means that it is often slower than Fortran or C. But most of the Python functions and libraries that could leverage compiled language speedups–such as root finders, optimizers, matrix inversion, and matrix decomposition—are already implemented in Python as wrappers of the optimized C and Fortran code.

A.1 Installing Python

A basic Python kernel can be installed from https://www.python.org/downloads/. However, this is not the recommendation of this book. Because Python has many important libraries that can be added to the distribution, this book recommends downloading the Anaconda distribution of Python (https://www.continuum.io/downloads), which is curated by Continuum Analytics. In previous years, many users remained with version 2.x despite the existence of 3.x. However, Python 3.x has now been extensively tested and widely adopted by the Python community and it is advisable to download the latest version (currently 3.x). The Anaconda distribution of Python is easy to install on Mac OS X, Windows, and Linux machines.

A.2 Learning Python

Many resources exist for learning Python. Most widely used are the online tutorials. Below are some favorite examples.

- The official Python 3 tutorial site (https://docs.python.org/3/tutorial/)
- Code Academy's Python learning module (https://www.codecademy.com/learn/python)
- Quant-Econ.net tutorial of how to use Python for Economics applications (http://lectures.quantecon.c
- Applied and Computational Mathematics Emphasis at Brigham Young University open source Python training labs (http://www.acme.byu.edu/?page_id=2067)

In addition, a number of excellent textbooks and reference manuals are very helpful and may be available in your local library. Or you may just want to have these in your own library. Lutz (2013) is a giant 1,500-page reference manual that has an expansive collection of materials targeted at beginners. Beazley (2009) is a more concise reference but is targeted at readers with some experience using Python. Despite its focus on a particular set of tools in the Python programming language, McKinney (2013) has a great introductory section that can serve as a good starting tutorial. Further, its focus on Python's data analysis capabilities is truly one of the important features of Python. Rounding out the list is Langtangen (2010). This book's focus on scientists and engineers makes it a unique reference for optimization, wrapping C and Fortran and other scientific computing topics using Python.

As with any programming language, the long-run internalized learning comes from using the language in relevant applications. This book will provide many economic applications of a broad set of computational tools using Python. These tools are applicable in many other fields. And they can be implemented in other programming languages. But Python currently occupies a strong niche of being a programming language with broad capabilities, having an active and expanding user group, having a syntax that is efficient and relatively easy to write, and being open source. From a general standpoint, it is likely that Python's use will continue expanding in economics.

A.3 Principles of Writing Good Code

- Python has its own preferred style of writing good code. These best practices are listed in PEP 8 – Style Guide for Python Code (https://www.python.org/dev/peps/pep-0008/).[1]
- Work with 1-dimensional arrays or numpy vectors as much as possible. For example, a vector x with n elements should have shape `x.shape = (n,)`. In addition, some operations on matrix $\boldsymbol{A}$ with m rows and n columns can be more efficiently executed by vectorizing $\boldsymbol{A}$ or transforming it into a one-dimensional array a with $m \times n$ elements.
- Write functions for particular lines of code that get reused and represent a clear concept from the theory. This is an art of understanding efficient coding. It also means that if you change a concept or augment your code, you might only have to make the change in one place even though that concept gets used in the code in multiple places.
- Avoid using global variables across functions.
- Use Python tuples to pass arguments between functions because the data types can vary across tuple elements (e.g., you can pack integer elements with float elements).
- Learn how to use the `*args` and `**kwargs` constructs in passing tuples of varying lengths into functions.
- Store sets of functions in intuitive groups in separate python script (`file.py`) files. These scripts are called modules. Import these functions when needed by importing the module using `import file as fl`. Then you can call those functions in your current script using `fl.funcname(...)`.

[1] PEP stands for "Python Enhancement Proposals".

- Place comments and meta data extensively throughout your code. Every function should have explanations of what the function does, what are the inputs, what functions are called, what objects are created inside the function, and what is the output of the function. This will feel like overkill at the beginning, but it will save you and everyone else who every uses your code lots of time over the long run. PEP 257 (https://www.python.org/dev/peps/pep-0257/) provides a description of Python docstring and meta data conventions.

- Use the theory to map out what the optimal structure of your code should be. A universal principle in scientific computing is that insights, efficiency, and discovery can happen going both from theory to computation as well as from computation to theory. Use them both extensively. If you have a problem in the code, it might be highlighting an issue in the theory. If you have a problem in the theory, you might be able to discover the reason by computing different scenarios and special cases.

- Use parallelization where possible. Python's `multiprocessing` library makes this very easy on your own machine. For more advanced and flexible parallel processing, the `mpi4py` library imports tools for implementing MPI (message passing interface) operations.

A.4 Running Python

asdf

A.4.1 Text editor and terminal

Talk about Vim, Emacs, Sublime Text 3.

A.4.2 IDE

Spyder is like MATLAB. ipython

A.4.3 Jupyter Notebook

Jupyter is a general platform that can display "notebooks" in your browser that can interactively run code in a number of languages and can display other surrounding rich text elements such as text, equations, figures, and links. Jupyter notebooks are ideal for teaching programming and giving examples because they can be executed and manipulated in real time and they can be saved as a record for future use. I will use Jupyter notebooks in class to teach some of the computational techniques you will be using in your problem sets. I will save these notebooks to the class GitHub repository.

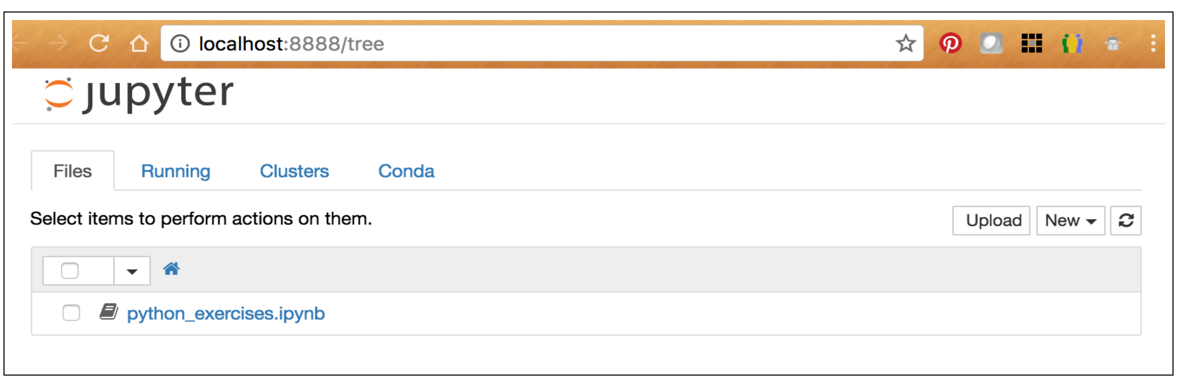
If you have installed the Anaconda distribution of Python on your computer, you can open the Jupyter notebook server dashboard by going to your terminal and typing `jupyter notebook`. If you do not have the Anaconda distribution of Python, you can install Jupyter by following the instructions at http://jupyter.readthedocs.io/en/latest/install.html. The Jupyter/IPython Notebook Quick Start Guide (https://jupyter-notebook-beginner-guide.readthedocs.io/en provides a nice introduction to installing and using the Jupyter notebook system.

You can open a Jupyter notebook by navigating in your terminal to the directory on your local computer in which a Jupyter notebook is saved. Type the following command in your terminal to open the Jupyter server dashboard.

```
>>> jupyter notebook
```

As shown in Figure A.1, the Jupyter dashboard interface will show all of the files available in that directory. To open one of the Jupyter notebooks, simply double click on the desired file with the ".ipynb" file-type suffix.

Figure A.1: Screenshot of Jupyter startup dashboard



A.5 Debugging

- Debugging is unavoidable.
- Develop an endurance to stick with the problem until it is solved.
- Coding errors go down with experience, holding ability constant.
  - Coding ability usually rises over time, which means that it is not clear that debugging ever goes away.
- Stackoverflow is your friend.

A.6 Python cheat sheet

In this section, I list a number of useful Python commands that are not as frequently used.

- How to set IPython to automatically reload modules that had previously been imported. Load the IPython `autoreload` extension by typing the following two Python magic function commands.

```
>>> %load_ext autoreload
>>> %autoreload 2
```

A.7 Optimization: Root Finders and Minimizers

Root finders and minimizers are the two key tools for solving numerical optimization problems. We will define the general formulation of an optimization problem as a minimization problem,

$$\min_{\boldsymbol{x}} \; f\left(\boldsymbol{x},\boldsymbol{z}|\boldsymbol{\theta}\right) \quad \text{s.t} \quad \boldsymbol{g}(\boldsymbol{x},\boldsymbol{z}|\boldsymbol{\theta}) \geq \boldsymbol{0} \quad \text{and} \quad \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0} \tag{A.1}$$

where $\boldsymbol{x} = \{x_1, x_2, ...x_N\}$ is a vector of N endogenous variables, $\boldsymbol{z}$ is a vector of exogenous variables, $\boldsymbol{\theta}$ is a vector of model parameters, f is a scalar-valued potentially nonlinear function of $(\boldsymbol{x},\boldsymbol{z})$ given parameters $\boldsymbol{\theta}$, $\boldsymbol{g}$ is a system of K potentially nonlinear inequality constraints, and $\boldsymbol{h}$ is a system of J potentially nonlinear equality constraints.

An example of a minimization problem of the form (A.1) is the 3-period lived agent OG model with exogenous labor supply in Chapter 2. Each household's optimization problem is the following.

$$
\begin{aligned}
\max_{b_{2,t+1},b_{3,t+2}} \quad & u\bigl(c_{1,t}\bigr) + \beta u\bigl(c_{2,t+1}\bigr) + \beta^2 u\bigl(c_{3,t+2}\bigr) \\
\text{s.t.} \quad & c_{s,t} > 0 \quad \text{for} \quad s = \{1,2,3\} \quad \text{and} \quad K_t > 0 \quad \forall t \\
\text{and} \quad & c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \quad \text{for} \quad s = \{1,2,3\}
\end{aligned}
\tag{A.2}
$$

Mapping the 3-period-lived OG model example from (A.2) to the general minimization problem formulation in (A.1) is instructive. First, notice that (A.2) is posed as a maximization problem while (A.1) is posed as a minimization problem. This is not an issue because any maximization problem of the form $\max_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x})$ has an isomorphic minimization problem formulation $\min_{\boldsymbol{x}} -\boldsymbol{f}(\boldsymbol{x})$.

The vector of choice variables $\boldsymbol{x}$ in (A.2) is the two-element vector of savings choices $(b_{2,t+1}, b_{3,t+2})$. The objective functions f to be minimized is the discounted lifetime utility $u(c_{1,t}) + \beta u(c_{2,t+1}) + \beta^2 u(c_{3,t+2})$ and is a function of.

A.7.1 Root finders

Press et al. (2007, p.442-443) highlight that with nonlinear multidimensional root finding problems, " you can never be sure that the root is there at all until you have found it.... It cannot be overemphasized, however, how crucially success depends on having a good first guess for the solution,...." Press et al. (2007, p. 473) also state, "We make an extreme, but wholly defensible statement: There are no good, general methods for solving systems of more than one nonlinear equation. Furthermore, it is not hard to see why (very likely) there never will be any good, general methods".

Because initial values are so important to root finders and minimization problems are often more robust to compute, Judd (1998, p.172) suggests running a minimization problem with a loose stopping rule on the vector of squared errors. Then one can use that solution as the initial guess for the root finder.

Appendix B

Using Git and GitHub.com

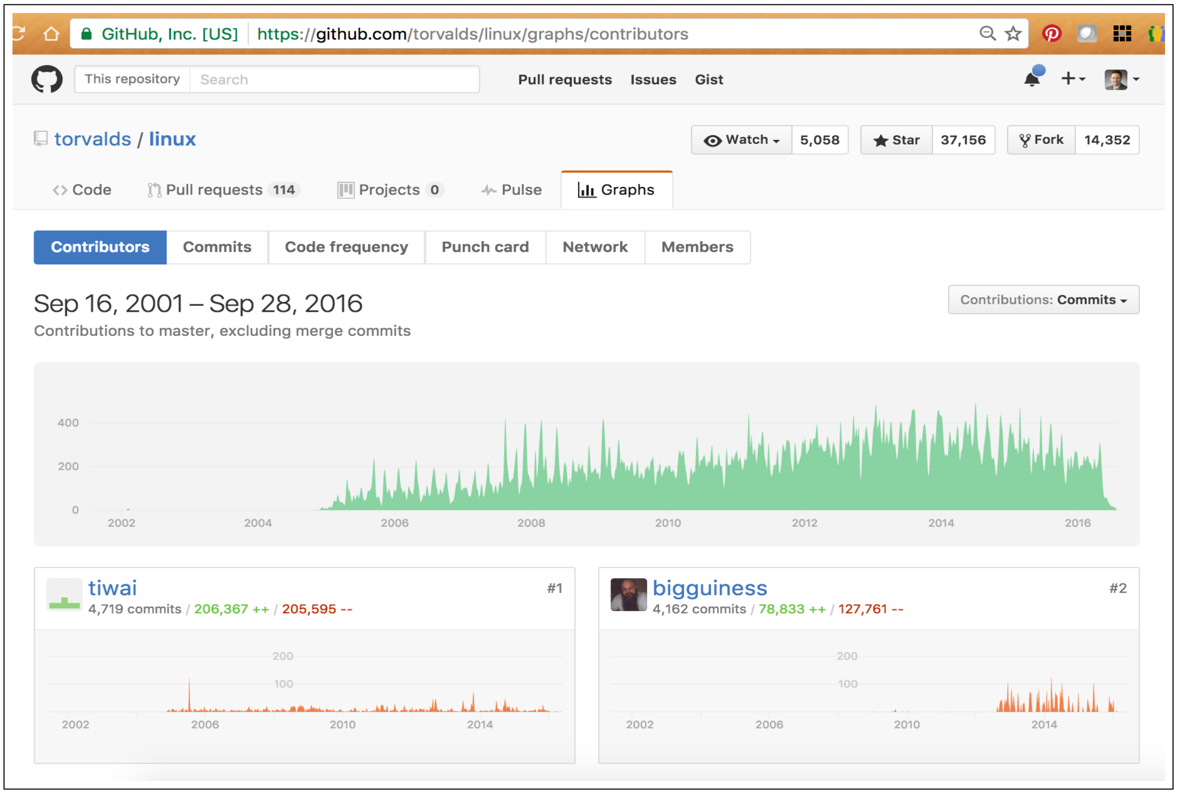
In technical terms, Git is a distributed version control system (DVCS). This means that users have complete copies of a source repository on their local hard drive. Git is valuable as a local version control system (LVCS) in that it can allow you to track changes in files and directories on your local computer without any connectedness to the internet or to other collaborators. However, Git 's most powerful characteristics come from its ability to carefully allow multiple users to collaborate on the same files and record the changes in an ordered, structured, hierarchical way.

Git is the software on your local machine that executes the commands and takes the snapshots that track the changes in marked files on your local machine and integrates those changes with remote repositories. The remote repositories are hosted by companies like GitHub.com or Bitbucket.org. This chapter will focus on GitHub.com, but Bitbucket.org is a good alternative with slightly different strengths and weaknesses.

Git software was born out of a dispute between the Linux kernel developers and the original version control provider for that group. The developers ended up creating their own free distributed version control sytem, which is Git .[1] GitHub.com currently hosts thousands of open source repositories with virtually unlimited numbers of contributors to each repository. The GitHub repository for the Linux kernel has over 200 contributors. Figure B.1 shows a snapshot of the contributors page of the Linux repository on GitHub. This

[1]See a short history of Git in Chacon and Straub (2014, p.5), which is also freely available online at https://git-scm.com/book/en/v2/.

Figure B.1: Screenshot of Linux Contributors



This snapshot was taken on September 28, 2016.

page shows exactly who has contributed, when they contributed, and what they contributed. Some employers are more interested in a potential employee's GitHub page than they are in that person's resume.

B.1 Why Not Use Dropbox or Google Docs?

Easy and commonly used alternatives to `Git` as your version control and collaboration platform are Google Docs and Dropbox. These two `Git` alternatives are file storage systems that sync changes to files across multiple storage locations of a single user or across many users. For simple file sharing, storage, and syncing, Google Docs and Dropbox are often preferred to `Git` . But for projects in which hierarchical permissions of who can edit, careful tracking of contribution attribution, and version history are important, `Git` is preferred.

Dropbox is nice because changes to a shared document on one person's machine are automatically updated on another person's machine. Dropbox offers some storage of previous versions of files. But it does not have detailed description and does not go back very far. Furthermore, Dropbox has trouble merging changes to a document that happen simultaneously. Suppose that you and your collaborator open a shared document simultaneously on your respective machines, and you both make changes to that document. Dropbox does not know whose changes dominate, so it updates the main document with the changes of whoever saves first and then makes a "conflicted copy" from the saved changes of whoever saves last. It is then up to the user to figure out how to manually merge those two files.

Google Docs have no merging problem because the document is automatically updated in real time on each user's computer, regardless of whether the document has been opened or not. This is made possible because a Google Doc resides primarily on remote Google servers. Despite this remote predominance, Google Docs do allow users to store copies of the files on their local drives to be able to use the documents while off-line. To a slightly greater degree than Dropbox, Google Docs allow some version history of who made changes, as well as a nice chat and comment interface for collaboration. But in Google Docs, everybody often has the same level of permission on making changes.

`Git` requires more deliberate decisions and effort about what gets merged, what does not get merged. And git has more specific rules about who decides what gets incorporated into the code and what does not. But with this extra complexity comes extra order, which is essential for large projects with lots of contributors. Additionally, `Git` provides a more specific version history with more refined ability to revert your code to a particular point in that history.

`Git` , Dropbox, and Google Docs each have different strengths and weaknesses. But `Git` is the standard for large projects with many contributors and a need for careful version control, changelog history, and contribution attribution.

B.2 Installing `Git` and Settings

A good set of instructions for installing `Git` is available on the `Git` website.[2] This `Git` site states, "Even if it's already installed, its probably a good idea to update to the latest version."

[2]See https://git-scm.com/book/en/v2/Getting-Started-Installing-Git.

This textbook recommends that you follow this instruction and update `Git` on your local machine. It is worth noting that `Git` comes installed on every Mac OSX.

Once `Git` is installed on your machine, you should update the settings in the `git config` tool. The `git config` tool controls how `Git` looks and operates and customizes `Git` with your information. The obvious starting place is to enter your user name with which you contributions will be associated as well as your e-mail address at which collaborators can contact you.

```
>>> git config --global user.name "FirstName LastName"
>>> git config --global user.email youremail@example.com
```

The `--global` option tells `Git` that these values are the default values that only need to be entered once. You can see all of the `--global` settings in `git config` by typing the `--list` command.

```
>>> git config --list
```

B.3 `Git` and GitHub Structure, Workflow

A number of different `Git` workflows are used in open source projects, but most recommended flows include some form of *fork→branch→pull request*. This textbook suggests the workflow displayed in Figure B.2. At first glance, this workflow looks very complicated and might make the user wish for the ease of Dropbox or a Google Doc. But the workflow depicted in Figure B.2 exhibits some important principles and rules that protect code integrity and allow for many organized contributors.

The first characteristic to note from the workflow displayed in Figure B.2 is the protected sanctity of the main code repository, labeled Ⓐ. There is only one arrow ⑩ leading into the main repository. Submitting a pull request is the only way for foreign code to be incorporated into the main repository. Related to this point is the characteristic that all work on the main repository Ⓐ is performed in separate and separated repositories, both remote and local. This is highlighted by the horizontal dotted line in Figure B.2 that separates the main repository from everything else in the figure.

Figure B.2: Flow diagram of Git and GitHub workflow

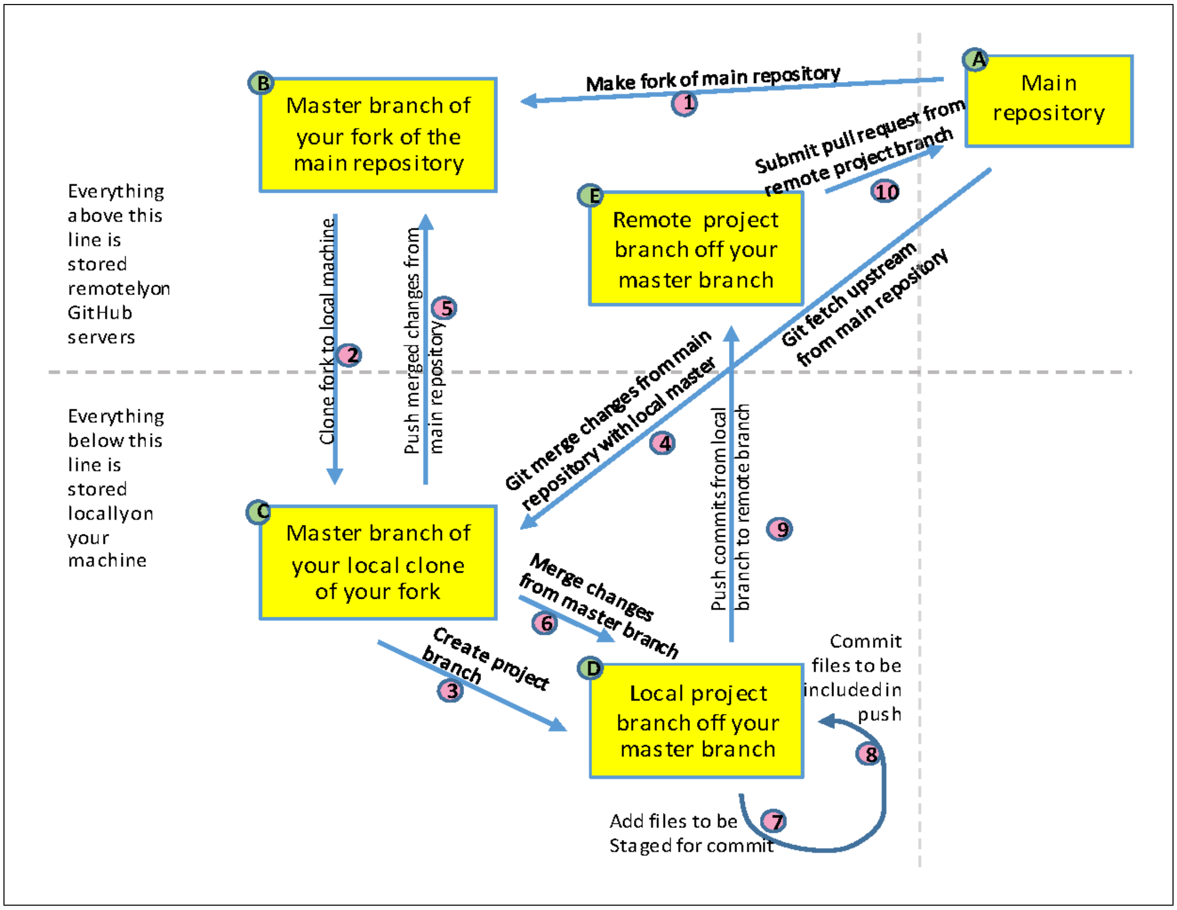


Table B.1: List of common Git commands

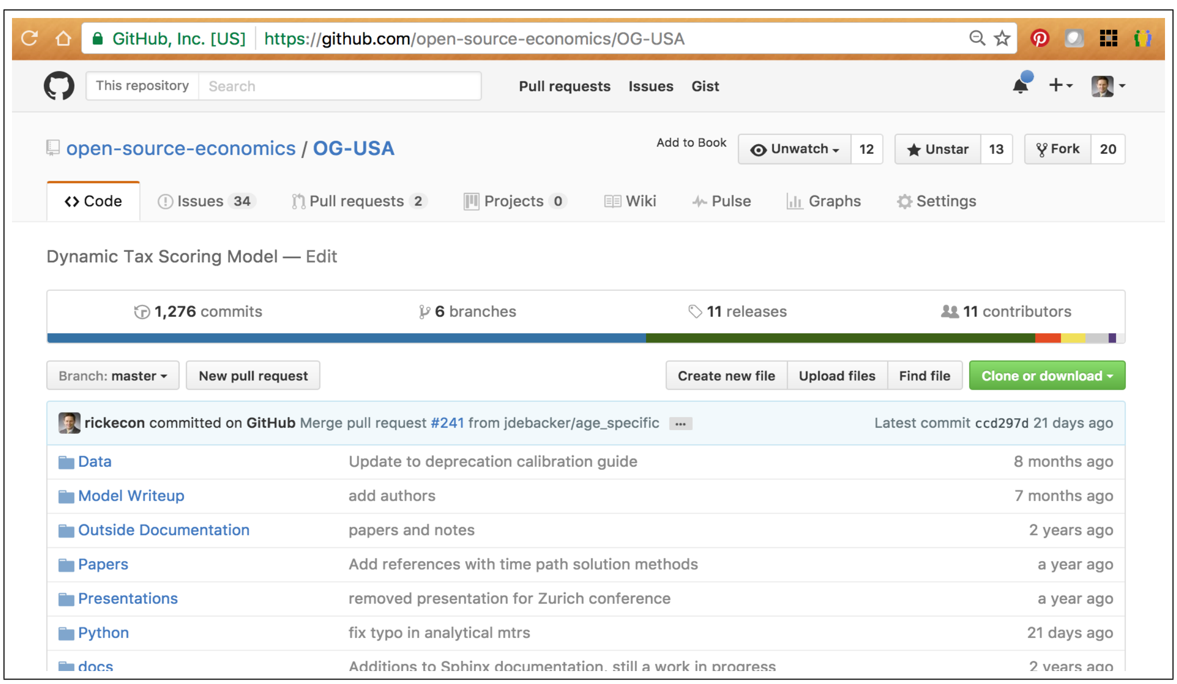
Fig. B.2 reference	Git command
①	Click "Fork" button at `https://github.com/main_acct/main_repo_name`
②	`git clone https://github.com/fork_acct/main_repo_name.git`
③	`git checkout -b [BranchName]`
④	`git fetch upstream` and `git merge upstream/master`
⑤	`git push origin master`
⑥	`git merge master/[BranchName]`
⑦	`git add [FileName]` or `git add -A`
⑧	`git commit -m "[descriptive commit message]"`
⑨	`git push origin [BranchName]`
⑩	Click "New pull request" button at `https://github.com/fork_acct/main_repo_name`

One last introductory distinction to make with `Git` is the difference between remote repositories and local repositories. The horizontal line in Figure B.2 separates these two concepts. Every object above the line in the figure is *remote* and is located on a GitHub server somewhere in the cloud. Of the remote repositories (Ⓐ, Ⓑ, and Ⓔ above the horizontal line), the two branches of your fork of the main repository (Ⓑ and Ⓔ) to the left of the horizontal line will be called *origin* and the main repository Ⓐ will be called *upstream*. Section B.3.3 discusses the significance of the *upstream* reference.

B.3.1 Create a fork and clone it

Assume that the main repository is a GitHub repository. The first step one takes in the `Git` workflow when joining a project is to "fork" the main repository. This is shown in step ① in Figure B.2. This is done by going to the URL of the main repository on GitHub.com and clicking on the "Fork" button toward the upper-right corner of the screen as shown in Figure B.3. GitHub will then give you the option to choose a GitHub account in which to place your fork of the main repository.

Figure B.3: Main repository GitHub main page



A fork is a copy of the main repository that is placed in your remote GitHub account.

The name of your fork is the same as the name of the main repository. The remote forked repository is labeled Ⓑ in Figure B.2. The only difference at this point is that the main repository is in a different account than your fork. It is important in the `Git` workflow that all changes that are made to the code from the main repository are made is a completely quarantined copy—the user's fork.

Once you have successfully forked the main repository, you want to *clone* your fork. This action is represented by ② in Figure B.2. Cloning is the `Git` terminology for making a copy of a remote repository on your local machine, which local repository is tracked and related to the remote repository by the `Git` software. The local clone of your remote fork is represented by Ⓒ in Figure B.2. You clone the remote fork opening your terminal on your local machine, navigating to the directory in which you want to place the cloned repository and typing the following command,

```
>>> git clone [remote fork Git URL]
```

where `[remote fork Git URL]` is the address that you copy when you click on the green "Clone or Download" button, which is below the "Fork" button on the main page of your remote fork Ⓑ (not the main repository Ⓐ).

B.3.2 Branching, making changes, updating your remotes

Branching is one of the most powerful functions of `Git` . Once you are ready to start modifying the code, this textbook recommends that you always make those changes in a new branch of your local fork. Each repository can have multiple branches. Branches represent copies of the repository that need not be identical. Think of each branch as representing a different project on the code repository.

The main branch of a repository is called *master*. It is automatically created with each newly forked or cloned repository. The master branch's purpose in this `Git` work flow is to be the baseline or fundamental reference, which is kept in sync with the main repository Ⓐ (see Section B.3.3). Branches Ⓑ and Ⓒ in Figure B.2 are the master branches of the remote fork and local fork, respectively.

You can always check what branch of your repository you are in by typing the following command.

```
>>> git branch
```

This will list all the branches of your repository, and it will highlight with an "*" the branch you are currently in. It is very important to always be sure you are working in the correct branch.

All changes to the master branch and new work should be in a new branch. You create a new branch off the master branch by the following command, which is action ③ in Figure B.2.

```
>>> git checkout -b [NewBranchName]
```

This command both creates the new branch and changes your directory to the new branch, no longer in the *master* branch. If you have multiple branches, you can change between them by typing:

```
>>> git checkout [BranchName]
```

It is important to note that the files in your local `Git` directory change when you change branches to the files associated with that branch. It is, therefore, important to make sure you are making changes in the branch with which you intend those changes to be associated.

As you work on your code and change the files in your repository, there are three steps you need to follow. You must (i) *add*, (ii) *commit*, and (iii) *push* changes to your branch in your remote fork Ⓔ from your local branch Ⓓ.

You can check the status of the branch of your local repository by typing:

```
>>> git status
```

This will show you if any files or folders have been modified, added, or deleted. You choose which of those files to track or stage for a future commit by adding them to the "staging area" as shown in action ⑦ in Figure B.2. You can add a particular file by using the following command,

```
>>> git add [FileName]
```

or you can add all the modified, added, or deleted files with the following command.

```
>>> git add -A
```

If you type `git status` after adding files to the "staging area" to be tracked by `Git` , you will see that the files you added are now shown with a different status.

Once you have completed an intuitive well defined set of changes is a good time to commit those *added* files. A *commit* is a bundled group of changed files that can be summarized in one or two short sentences. You commit all files in the staging area that have been previously added by typing the following command. This is action ⑧ in Figure B.2.

```
>>> git commit -m "[descriptive commit summary message]"
```

As mentioned above, a commit message should be no more than two sentences, but is probably better as one sentence. This implies that you should commit your work often and not wait until you have completed whatever change your branch was created for. Never go too long without committing. And you should always commit at the end of a coding session or when switching branches.

A *push*, shown as action ⑨ in Figure B.2, is an action that takes all the commits that have not already been *pushed* and copies them to the remote origin repository. This textbook recommends that commits from a project branch Ⓓ be pushed to a similar remote origin branch Ⓔ. A push is done by typing the following command.[3]

```
>>> git push origin [BranchName]
```

Each *push* will likely contain multiple *commits*. Notice that your master branch and project branch—both in your remote fork and in your local repository—will be different from each other.

Once you feel that your changes are done and you have pushed them to your remote branch so that Ⓓ and Ⓔ are identical, you are ready to incorporate them into the main repository Ⓐ. This is done through a *pull request*, as shown in action ⑩ in Figure B.2. A pull request is made through the GitHub website. You go to the main page of the repository, make sure you are in the project branch on the website by clicking on the "Branch:[BranchName]"

[3] If you leave off the `BranchName` in the command, your changes will default to the remote master branch of your fork. We do not recommend this.

button in the upper-left area as shown in Figure B.3. Then click on the "New pull request button", which is next to the "Branch:[BranchName]" button.

Before submitting this pull request, make sure it has an intuitive, descriptive, and concise title. Then make sure in the box below the title, that you put a detailed description of what is in the pull request. In the end, the pull request will be the cumulative changes from all the commits from all of the pushes since the creation of that branch. In the description, you may want to give context to the changes, and you may even want to point out areas on which you need an extra set of eyes.

Notice the different naming of this process. Rather than being called a *push* in which the energy is coming from the source of the changes, it is called a *pull request*. This name signifies that the energy comes from the destination of the change. You can think of a pull request as an invitation for the collaborators who run the main repository to *merge* your changes into the main repository. It is for this reason that this open source work flow allows for the full democratization of coding. Anyone can take the code and make any changes they want. But only the code that is accepted by those who manage the main repository is incorporated.

Once you make a pull request and before someone chooses to merge that pull request, the status of your branch is linked to the pull request. That is, you can continue to make changes to your local branch, add/commit/push those changes to your remote branch, and those commits will be automatically added to the pull request.

If the changes in your code are accepted, those who manage the main repository Ⓐ will *merge* in your changes. They may also open a dialog in the pull request in which the community can respond to and discuss the changes. In the end, the managers of the main repository have the option to reject the pull request.

Once your pull request is accepted and merged into the main repository. We recommend that you `git fetch upstream` the changes from the main repository Ⓐ to your local master branch Ⓒ, `git merge upstream/master` those changes, and `git push origin master` those changes to your remote master branch Ⓑ. You should then delete the local project branch Ⓓ by typing,

```
>>> git branch -d [BranchName]
```

and then delete that remote project branch Ⓔ from your remote repository by typing the following.

```
>>> git push origin --delete [BranchName]
```

B.3.3 Set the upstream remote, fetch, merge, and push

Once you have cloned your fork of the main repository, you will need a way to keep your fork updated—both your local cloned repository Ⓒ and your remote fork Ⓑ—with any changes that are made in the main repository Ⓐ. You will first want to designate a remote repository from which to draw code changes. `Git` designates your fork Ⓑ of the main repository as *origin*. Designate the main repository as the remote for your fork by opening your terminal in your local machine, navigate to the main directory of your local clone, and type the following code,

```
>>> git remote add upstream [main repo Git URL]
```

where `[main repo Git URL]` is the address that you copy when you go to the main repo main page and click on the green "Clone or Download" button, shown in Figure B.2 under the "Fork" button.

Naming the main repository Ⓐ "upstream" in your local clone Ⓒ makes the commands easier to write that execute the updating step displayed as labeled ④ and ⑤ in Figure B.2. Each time you come back to your local fork of the repository, you will want to check the status of your fork with respect to the remote upstream main repository Ⓐ and with respect to the remote origin fork of the repository Ⓑ. This section focuses on updating your local fork Ⓒ with new changes in the remote upstream main repository Ⓐ.

You can tell `Git` to go get any changes to the remote upstream main repository by opening your terminal, navigating to the directory of the master branch of your local fork Ⓒ, and typing the following.

```
>>> git fetch upstream
```

This command *fetches* all the changes from the upstream repository and stages them for potentially being *merged* into your local master branch Ⓒ. Note here that you are not staging this to be added to your new project branch Ⓓ of your local repository. The purpose of your local master branch Ⓒ is to remain up-to-date with the remote master repository Ⓐ.

Once these changes are staged with the `git fetch upstream` command, you can merge those changes from the remote master repo Ⓐ into your local master branch Ⓒ using the following command.

```
>>> git merge upstream/master
```

Because of the work flow that this textbook advocates in Figure B.2, you should have no merge conflicts with this action. Your local master branch of the forked repository Ⓒ is meant simply as a local source that receives updates from the remote main repository Ⓐ. The only other time that your local master Ⓒ is updated from another source is when it was created by cloning ② the remote master Ⓑ, which action should happen only once.

Once the changes from the remote main repository Ⓐ have been fetched and merged into your local main branch Ⓒ, you just need to *push* ⑤ those changes up to your remote master branch of your fork Ⓑ. This is done by being in your master branch and typing the following.

```
>>> git push origin master
```

The `push` command copies the changes from your local master branch Ⓒ into your remote master branch Ⓑ. The term *origin* refers to the set of branches, including the master, in your remote fork of the main repository. In Figure B.2, repositories Ⓑ and Ⓔ are branches of the *origin* remotes. At this point, your remote master branch of your fork Ⓑ, your local master branch of your fork Ⓒ, and the main repository Ⓐ are all syncronized.

The last function we detail here is the action of *merging* changes from your local master branch Ⓒ that came from the main repository Ⓐ into your local project branch Ⓓ. This is action ⑥ in Figure B.2. Go to the branch that will be the destination of the merge or the branch where you want to incorporate the changes. In this case, that is the local project branch Ⓓ.

```
>>> git checkout [ProjectBranch]
```

Now merge the master branch into the project branch.

```
>>> git merge master/[ProjectBranch]
```

If the merge is nontrivial, then you will get a conflict message like the following.

```
Auto-merging master.txt
CONFLICT (content): Merge conflict in master.txt
Automatic merge failed; fix conflicts and then commit the result.
```

Now type the following to open the globally set `Git` mergetool (see Section B.2 for mergetool global setting).

```
>>> git mergetool
```

Finally, you can commit the changed files and be done.

```
>>> git commit -m "Description of merge commit"
```

B.4 `Git` Cheat Sheet Commands

In this section, I list a number of useful `Git` commands that are not as frequently used.

- List the last commit for each branch in a local repository.

  ```
  >>> git branch -v
  ```

- List branches that you have or have not yet merged into your current branch.

  ```
  >>> git branch --merged
  >>> git branch --no-merged
  ```

- Undo erroneous commits in your local branch. Let `[commit#]` be the commit number to which you want to rewind. This will usually be a reference like `f2f7281451364c29c75e07ddb3be1d8d7` Type the following.

  ```
  >>> git reset [commit#]
  ```

- Undo erroneous commits merged into your *upstream* repository. Note that it is usually thought of as bad form to erase `Git` history.[4] Let "upstream" be the name of the repository and "BranchName" be the branch of that repository with the offending commits. First, pull the branch with the bad commits to your local repo:

```
>>> git pull upstream [BranchName]
```

  Let `[commit#]` be the commit number to which you want to rewind. Rewrite the commit history on your local repo using the following command:

```
>>> git reset --hard [commit#]
```

  Now push this back up to the remote repository.

```
>>> git push -f upstream [BranchName]
```

- Create a local branch that is a copy of someone's pull request branch.

```
>>> git checkout -b [NewBranchName]
>>> git pull [PR sender branch git URL] [NewBranchName]
```

B.5 Using GitHub for Collaborative Issue Tracking

GitHub repositories have an "Issues" section that is a powerful place for collaboratively discussing and resolving issues with the code. The issues interface of a GitHub repository is accessed via the "Issues" tab in the upper-right area of the main page of the repository as shown in Figure B.3. GitHub issues create a central remote location for resolving questions with your code. An issue creates a permanent record of what the question was, the path to resolving it, and what was the resolution. GitHub issues also serve as a non-email method of communicating about a project. This is valuable because resolution of issues can often span weeks and even months.

A good example of an effective GitHub issue is issue # 237 of the `OG-USA` repository. One can tag GitHub collaborators in these issues, add images and equations, and reference

[4] See `Git` Koan, "Only the Gods".

other issues and pull requests. This link and this link has some of the markdown options for augmenting your discussion in GitHub issues.

Appendix C

Using LaTeX

LaTeXis a document preparation system that produces high-quality typesetting and is ideal for technical, scientific, and computational subjects. The mathematical equation engine in LaTeXis the standard in technical and mathematical typesetting. Further, the broader philosophy of LaTeXdocuments is to provide document-type and reference-type formatting functions separate from the content of the document. This allows the LaTeXuser to focus on content, and let separate commands determine the formatting. This can also allow for particular content to be quickly reformatted to another style.

An encylopedic reference for most LaTeXfunctionality is Mittelbach and Goossens (2004). But it valuable to search the resources available online because they often describe the typesetting innovations that come from writing your own source LaTeXcode.

Programs like Scientific WorkPlace provide a GUI interface for LaTeX. But this program is expensive, and it is not as flexible as writing your own LaTeXdocument code. This book recommends writing your own LaTeXdocument source code. Very soon, you will begin to think in terms of the the compiled LaTeXdocument even though you are typing source code.

C.1 Installing LaTeX

A great summary page for installing LaTeXon Mac OS X, Windows, or Linux is available from this link (https://www.latex-project.org/get/). Following these instructions will install on your local machine a number of libraries, packages, and tools to help you use LaTeXfor your typesetting.

C.2 Running LaTeX

The traditional use and workflow of creating a document in LaTeXis to write a plain text source document (.tex subscript) in any text editor and then have the LaTeXengine compile that document into a format you want to use (typically a PDF). You can use any text editor for creating LaTeXsource documents (.tex files), but there are advantages to using text editors that "work well" with LaTeX.

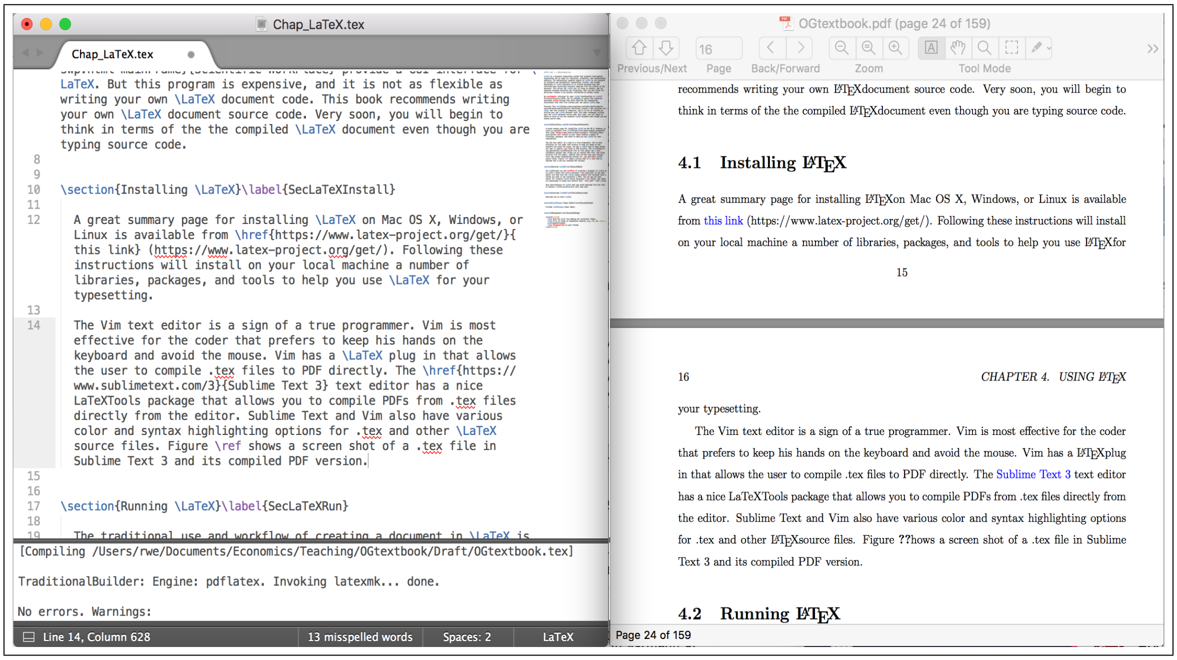
Most distributions of LaTeXthat you would download from the link in Section C.1 will come with a text editor, such as TeXShop for Mac, WinEdt for Windows, and TeXMaker for Linux. The Vim text editor is a sign of a true programmer. Vim is most effective for the coder that prefers to keep his hands on the keyboard and avoid the mouse. Vim has a LaTeXplug in that allows the user to compile .tex files to PDF directly. The Sublime Text 3 is a more versatile text editor that has a nice LaTeXTools package that allows you to compile PDFs from .tex files directly from the editor. On Mac OS X, Sublime Text also requires the Skim PDF viewer. Sublime Text and Vim also have various color and syntax highlighting options for .tex and other LaTeXsource files. Figure C.1 shows a screen shot of a .tex file in Sublime Text 3 and its compiled PDF version.

C.3 Learning LaTeX

Many tutorials are available online for learning LaTeX. However, most of those tutorials focus on functionality specific to a particular field. The best way to start using LaTeXis to get a template of a source .tex document and then use online resources to learn more specific functionality. Figure C.2 is a snapshot of a template.tex file.

Stackoverflow.com is a great resource for answers to particular LaTeXquestions. Anyone can publicly browse the questions and answers stored at Stackoverflow. If you register for an account, you will be able to ask questions through Stackoverflow and even give answers. In most cases, you will find that someone else has already asked a question similar to yours

Figure C.1: Screenshot Sublime Text 3 .tex document and compiled PDF in Skim



and that a series of answers have been given.

C.4 LATEX Cheat Sheet

This section provides a list of some of the key LATEXcommands and structures that you will likely use.

C.4.1 Math symbols and Greek Letters

A good list of the LATEXcommands for math symbols and Greek letters is available at this link.[1] Math symbols and Greek letters are included in the text by using dollar signs to bracket the commands that are to be rendered in math mode. Typing the text, "The Greek letter `$\theta$` represents `$x + y$`" will be rendered as "The Greek letter θ represents $x + y$".

Equations. The `amsmath` package (American Mathematical Society) allows for a number of higher math display options that you will use often. The standard construct for equations

[1]See http://web.ift.uib.no/Teori/KURS/WRK/TeX/symALL.html.

Figure C.2: Screenshot of LATEXtemplate .tex file

is the `\begin{equation}` environment. The following text will render the following equation.

```
\usepackage{amsmath}
\begin{equation}
  u\left(c_{s,t}\right) = \frac{(c_{s,t})^{1-\gamma} - 1}{1 - \gamma}
\end{equation}
```

$$u\left(c_{s,t}\right) = \frac{(c_{s,t})^{1-\gamma} - 1}{1 - \gamma} \tag{1}$$

Table C.1 shows various display environment options for the `amsmath` package.

Table C.1: Display environment options in the `amsmath` package

Numbered	Not numbered	Description
`equation`	`equation*`	One line, one equation
`multline`	`multline*`	One unaligned multiple-line equation, one equation number
`gather`	`gather*`	Several equations without alignment
`align`	`align*`	Several equations with multiple alignments
`flalign`	`flalign*`	Several equations: horizontally spread form of `align`
`split`		A simple alignment within a multiple-line equation
`gathered`		A "mini-page" with unaligned equations
`aligned`		A "mini-page" with multiple alignments

* See Table 8.1 in Mittelbach and Goossens (2004).

C.4.2 Brackets and parentheses in equations

TODO: Add content.

C.4.3 Matrices in equations

TODO: Add content.

C.4.4 Tables

TODO: Add content.

C.4.5 Figures

A good package and set up for producing figures is the `graphicx` package. These figures are called "float" objects because LATEXintelligently chooses their placement in the document based on natural breaks in the text. The code that produced Figure C.1 is the following.

```
\usepackage{graphicx}
\usepackage[format=hang,font=normalsize,labelfont=bf]{caption}

\begin{figure}[htbp]\centering\captionsetup{width=4.0in}
  \caption{\textbf{Screenshot of \LaTeX template .tex
      file}}\label{FigLaTeXtemplate}
  \fbox{\resizebox{4.0in}{5.0in}{\includegraphics{./images/LaTeXtemplate.png}}}
\end{figure}
```

C.4.6 BibTeX and references

TODO: Add content.

C.5 Debugging

At some point, you will have created a LATEXsource .tex file and hit compile and you will receive an error. The PDF will not compile. LATEXproduces an error log file in this case. The first time you read an error log, it may look incomprehensible. But as you experience more of these error logs, you find where the key information is located to help you solve your problem most efficiently.

The error log often displays at the bottom of your text editor. In Sublime Text, the error log is displayed in the window at the bottom of the screen, pictured in Figure C.1 at the bottom of the left-hand-side. Key information is the line number of the source code where the error occured. And the error log will also usually give you some kind of helpful description as to what the problem was.

In many cases, the error will be an incomplete closure, such as \$ \$, (), [], {}, `\begin{}\end{}`.

Other common errors include misspelled function names and missing source packages in the preamble.

One last comment with debugging LaTeXcode is that Stackoverflow.com is your friend. As was mentioned in Section C.2, Stackoverflow is a great resource for answers to particular LaTeXquestions. Anyone can publicly browse the questions and answers stored at Stackoverflow. This book recommends that you register for an account. This will allow you to ask questions through Stackoverflow and even give answers. In most cases, you will find that someone else has already asked a question similar to yours and that a series of answers have been given.

Appendix D

Calibration

Many of the chapters of this textbook have respective calibration sections toward the end of each chapter. However, we found that the methods for extensive "strong" calibration warranted an appendix chapter of their own. We will use as our pedagogical example the S-period lived agent model with endogenous labor supply from Chapter 4. And we will use the exercise of calibrating the χ^n_s parameters as our example.

There are two methods to calibrate the χ^n_s parameters of the OG model with endogenous labor. The first is to use the S initial-period labor supply Euler equations. The second is to use GMM to match average labor supply moments from the model steady-state to average labor supply moments from the data. The first method is orders of magnitude more simple, more intuitive, and more tractable.

D.1 Initial period labor supply Euler equations

In the model from Chapter 4 of the textbook, the general form of the labor supply Euler equation is the following.

$$w_t\left(c_{s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall s,t \tag{D.1}$$

We have already estimated the elliptic utility parameter values of b and υ, and we have calibrated the value for σ from other studies.

All the consumption $c_{s,t}$ and wage w_t values in the set of equations represented by (D.1) are given in consumption units (consumption is the numeraire good). So w_t represents the units of consumption that a worker is paid for each unit of labor supplied. And the implicit price of one unit of consumption is one consumption unit. This can be seen from the budget constraint.

$$c_{s,t} + b_{s+1,t+1} = (1 + r_t)b_{s,t} + w_t n_{s,t} \tag{D.2}$$

For this reason, $c_{s,t}$ also represents the total individual consumption expenditure of age-s household at in period t.

We have assumed that $\tilde{l} = 1$. Regardless of the value, the appearance of $\frac{n_{s,t}}{\tilde{l}}$ on the right-hand-side of (D.1) is a unit-free percent of total time endowment. However, both consumption $c_{s,t}$ and wages w_t on the left-hand-side of (D.1) have model specific units. The first step toward overcoming this issue of using real world data in this model specific equation is to divide both sides of (D.1) by some single aggregate or average variable from the model that will create unit-free versions of w_t and $c_{s,t}$. We choose average individual income here $\bar{y}_t$ because that is a value we end up using when we incorporate tax functions into the model. Dividing both sides of (D.1) by $(\bar{y}_t)^{1-\sigma}$ gives the following expression.

$$\begin{gathered} \left(\frac{w_t}{\bar{y}_t}\right)\left(\frac{c_{s,t}}{\bar{y}_t}\right)^{-\sigma} = \left[\frac{\chi^n_s}{(\bar{y}_t)^{1-\sigma}}\right]\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall s,t \\ \text{where} \quad \bar{y}_t \equiv \frac{1}{S}\sum_{s=1}^{S}\left(r_t b_{s,t} + w_t n_{s,t}\right) \quad \forall t \end{gathered} \tag{D.3}$$

Note that w_t and $c_{s,t}$ in consumption units from the left-hand-side of (D.1) are transformed into unit-free values relative to average individual income on the left-hand-side of (D.3).

Solving (D.3) for χ^n_s gives the following expression.

$$\begin{gathered} \chi^n_s = \frac{(\bar{y}_t)^{1-\sigma}\left(\frac{w_t}{\bar{y}_t}\right)\left(\frac{c_{s,t}}{\bar{y}_t}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s,t \\ \text{where} \quad \bar{y}_t \equiv \frac{1}{S}\sum_{s=1}^{S}\left(r_t b_{s,t} + w_t n_{s,t}\right) \quad \forall t \end{gathered} \tag{D.4}$$

All the terms on the right-hand-side of (D.4) are unit-free except for the lone instance of average individual income $\bar{y}_t$ in the numerator. All other terms with endogenous variables are in percent of average income or percent of time endowment.

With real world data on average income $\bar{y}_t$, consumption expenditures by each age group $\{c_{s,t}\}_{s=1}^S$, labor supply by each age group $\{n_{s,t}\}_{s=1}^S$, and parameter values for the total time endowment of each individual $\tilde{l}$, the coefficient of relative risk aversion σ, and the elliptical utility parameters b and υ, we can simply solve each equation for χ^n_s up to a constant factor that is the same for each χ^n_s.

The constant factor is related to the term $(\bar{y}_t)^{1-\sigma}$ in the numerator of (D.4), which is unavoidably in model units of consumption. But we do know that average individual income in the data is proportional to average individual model income.

$$\bar{y}_t = \bar{y}_t^{data} \times factor_t \quad \text{or} \quad factor_t = \frac{\bar{y}_t}{\bar{y}_t^{data}} \quad \forall t \tag{D.5}$$

For tractability reasons, we calibrate χ^n_s to the steady-state values of average individual income rather than the more theoretically appropriate initial-period average individual income. Because we have to solve the model in order to know what the model average income is in the steady-state, which solution is a function of the parameter values χ^n_s, we cannot know ex ante what the factor is to transform real world average income into model average income. But we can just include equation (D.5) as one of our outer-loop equations and $factor_t$ as one of our outer-loop variables along with $\bar{r}$ in the steady-state computational approach.

D.1.1 Summary and recap

In summary, in the steady-state, for a given guess of $\bar{r}$ and $\overline{factor}$, we can use real world data to solve exactly for all the χ^n_s values,

$$\chi^n_s = \frac{\left(\bar{y}_t^{data} \times \overline{factor}\right)^{1-\sigma}\left(\frac{w_t}{\bar{y}_t}\right)\left(\frac{c_{s,t}}{\bar{y}_t}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s \tag{D.6}$$

where the ratios $\frac{w_t}{\bar{y}_t}$, $\frac{c_{s,t}}{\bar{y}_t}$, and $\frac{n_{s,t}}{\tilde{l}}$ are values taken from the data. Because of these age-specific data, the values of the χ^n_s parameters will have the correct relative shape, but all of their levels will be off by the same factor.

Given the guess for $\bar{r}$, $\overline{factor}$, and the corresponding χ^n_s values, we can solve for the steady-state household decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_s\}_{s=2}^S$. From these decisions, we can compute the corresponding steady-state interest rate $\bar{r}$ and average household income in the model $\bar{y}$. We update $\bar{r}$ and $\overline{factor}$ until the interest rate and factor implied by household optimization equal the initial guess.

Because the difference between model units and real world units might be multiple orders of magnitude, it is helpful to get the initial guess for $\overline{factor}$ near its true value. A good strategy is to compute the steady-state of the model assuming that $\chi^n_s = 1$ for all s. This means that only $\bar{r}$ is in the outer loop. Use the resulting $\bar{y}$ to derive an initial guess for $\overline{factor}$ according to (D.5). This should get your initial guess in the neighborhood of the final value.

D.2 Alternative Explanation/Derivation

D.2.1 Initial period labor supply Euler equations

In the model from Chapter 4 of the textbook, the general form of the labor supply Euler equation is the following.

$$w_t\left(c_{s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall s,t \tag{D.1}$$

We have already estimated the elliptic utility parameter values of b and υ, and we have calibrated the value for σ from other studies.

All the consumption $c_{s,t}$ and wage w_t values in the set of equations represented by (D.1) are given in consumption units (consumption is the numeraire good). So w_t represents the units of consumption that a worker is paid for each unit of labor supplied. And the implicit price of one unit of consumption is one consumption unit. This can be seen from the budget constraint.

$$c_{s,t} + b_{s+1,t+1} = (1 + r_t)b_{s,t} + w_t n_{s,t} \tag{D.2}$$

For this reason, $c_{s,t}$ also represents the total individual consumption expenditure of age-s household at in period t.

One may identify the χ^n_s via a method of moments estimation that uses Equation D.1 as the set of moment conditions and data on consumption, wages, and labor supply. However, the estimates of χ^n_s will depend upon the units of measurement for wages and consumption. Thus we could only identify the χ^n_s from the model up to a scale. This is clearly seen if we rearrange Equation D.1 to isolate χ^n_s on the left hand side:

$$\chi^n_s = \frac{w_t \left(c_{s,t}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s, t \tag{D.7}$$

D.2.2 Scaling

We have assumed that $\tilde{l} = 1$. Regardless of the value, the appearance of $\frac{n_{s,t}}{\tilde{l}}$ denominator on the right-hand-side of (**??**) is a unit-free percent of total time endowment. However, both consumption $c_{s,t}$ and wages w_t on the right-hand-side of (D.7) have specific units.

To overcome this identification issue, consider a scaling that relates model units to data units. Call this parameter $factor$ and define it as:

$$factor_t = \frac{\bar{y}^{data}_t}{\bar{y}^{model}_t} \quad \forall t \quad \Longrightarrow \quad \bar{y}^{data}_t = \bar{y}^{model}_t \times factor_t \tag{D.8}$$

In other words, $factor$ scales model units to data units for those variables measured in consumption units in the model and nominal amounts in the data. Thus we also have the relations

$$\begin{aligned} w^{model}_t \times factor_t &= w^{data}_t \\ c^{model}_{s,t} \times factor_t &= c^{data}_{s,t} \end{aligned} \tag{D.9}$$

With this, we can return to Equation D.7. Let's write two versions of this equation. One that identifies the χ_s^n in the model and one that identifies it's data counterpart. To be clear, let χ_s^n be the model version and $\hat{\chi}_s^n$ represent the χ_s^n to be identified from the data. Thus we have:

$$\chi_s^n = \frac{w_t^{model}\left(c_{s,t}^{model}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s,t \tag{D.10}$$

and

$$\hat{\chi}_s^n = \frac{w_t^{data}\left(c_{s,t}^{data}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s,t \tag{D.11}$$

Note that for brevity, we do not have *data* or *model* superscripts on the labor supply terms. This is because, as noted above, labor supply is always divided by labor endowment and so the ratio is in percentages both in the data and the model.

Now let's do some algebra with Equation D.10:

$$
\begin{aligned}
\chi^n_s &= \frac{w^{model}_t \left(c^{model}_{s,t}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s,t \\
\implies factor_t^{1-\sigma}\chi^n_s &= \frac{factor_t^{1-\sigma} w^{model}_t \left(c^{model}_{s,t}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s,t \\
\implies factor_t^{1-\sigma}\chi^n_s &= \frac{\left(factor_t w^{model}_t\right) \left(factor_t c^{model}_{s,t}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}} \quad \forall s,t \\
\implies factor_t^{1-\sigma}\chi^n_s &= \underbrace{\frac{w^{data}_t \left(c^{data}_{s,t}\right)^{-\sigma}}{\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}}}_{=\hat{\chi}^n_s} \quad \forall s,t \\
\implies factor_t^{1-\sigma}\chi^n_s &= \hat{\chi}^n_s \quad \forall s,t \\
\implies \hat{\chi}^n_s &= factor_t^{\sigma-1}\chi^n_s \quad \forall s,t
\end{aligned}
\tag{D.12}
$$

Thus, by estimating $\hat{\chi^n_s}$ using the data on wages, consumption, and labor supply, one finds the model parameters up to a scale. That scale is function of the model scale parameter, $factor_t$.

D.2.3 Changes to Model Solution Algorithm

In theory, one wants to use the factor from model period t, where model period t corresponds to they year of your data (e.g., if the data are from 2017 and your initial period in the time path of your model is 2017, then you'd want determine the factor as $factor_0 = \frac{\bar{y}^{data}_{2017}}{\bar{y}^{model}_0}$). Because it depends on mean model income, the factor is endogenous and depends upon χ^n_s. Therefore, there is the need for some fixed point algorithm: guess a $factor$, use that to determine χ^n_s, solve the model and see if mean income in the data divided by mean income in the model returns the factor you guess, if not, update and do again. To compute the time path at each step in this fixed point algorithm would be very expensive. We therefore make a simplifying assumption. In particular, we assume that the factor is determined as the ratio of income in data from year t and from the model's steady state. That is,

$$factor_t = factor = \frac{\bar{y}^{data}_{2017}}{\bar{y}^{model}_{SS}} \ \forall t \tag{D.13}$$

While not a perfect mapping, this means that at each iteration of the fixed point algorithm that solves for the model $factor$ only the steady state needs to be computed. Also note that the model income represents real, stationarized income. So growth and inflation are not affecting this measurement, which helps this approximation be more accurate.

The modification to the algorithm to solve the steady state in Chapter 7 need only be modified to include the guess of $factor$ as one of our outer-loop variables along with $\bar{r}$ in the steady-state computational approach. Given the guess for $\bar{r}$ and $factor$ one can use the relation in Equation D.12 to transform the $\hat{\chi}^n_s$ into the model-scaled parameter. With these χ^n_s values, we can solve for the steady-state household decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_s\}_{s=2}^S$. From these decisions, we can compute the corresponding steady-state interest rate $\bar{r}$ and average household income in the model $\bar{y}$. We update $\bar{r}$ and $factor$ until the interest rate and factor implied by steady state equilibrium equal the guesses of $\bar{r}$ and $factor$ at that iteration. Once the steady state is solved and $factor$ is determined, then this same factor is applied over the time path, so no adjustment is needed for that solution method.

A Note About Initial the Initial Guess for $factor$

Because the difference between model units and real world units might be multiple orders of magnitude, it is helpful to get the initial guess for $factor$ near its true value. If one has trouble finding an initial guess that will not break the model, a good strategy is to compute the steady-state of the model assuming that $\chi^n_s = 1$ for all s. This means that only $\bar{r}$ is in the outer loop. Use the resulting $\bar{y}$ to derive an initial guess for $factor$ according to (D.8). This should get your initial guess in the neighborhood of the final value.

Appendix E

Tax Data with Tax-Calculator

In chapter 11 we studied how to incorporate realistic tax function data into an overlapping generations model. An extremely useful way to do this is to use the `Tax-Calculator` open-source microsimulation model.[1]

[TODO: Put description of what Tax-Calculator is and different ways to use it. Also describe other microsimulation models.]

E.1 TaxBrain Web Application

Describe TaxBrain web app, what it does, how to use it.

E.2 Command Line Interface (CLI)

Describe Tax-Calculator CLI, what it does, how to use it.

E.3 Python Application Program Interface (API)

Describe Tax-Calculator Python API, what it does, how to use it.

[1]Describe here with references.

E.4 Exercises

Exercise E.1. Generate Effective tax rates (ETR) for all the individuals. Create a scatterplot of effective tax rate by expanded income measure. Given the mean, max, min, and quartile values (`.describe()` method) for the ETR data. Use the plotting functions to create a skyline chart of ETR by income.

Exercise E.2. Generate marginal tax rates of expanded income.

Exercise E.3. Generate marginal tax rates of labor income (wage income plus self employment income)

Exercise E.4. Generate marginal tax rates of capital income

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