Quaternion

In [mathematics](#), the **quaternion** [number system](#) extends the [complex numbers](#). Quaternions were first described by the Irish mathematician [William Rowan Hamilton](#) in 1843[1][2] and applied to [mechanics](#) in [three-dimensional space](#). Hamilton defined a quaternion as the [quotient](#) of two *directed lines* in a three-dimensional space,[3] or, equivalently, as the quotient of two [vectors](#).[4] Multiplication of quaternions is [noncommutative](#).

Quaternions are generally represented in the form

$$a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k},$$

where a, b, c, and d are [real numbers](#); and 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the *basis vectors* or *basis elements*.[5]

Quaternions are used in [pure mathematics](#), but also have practical uses in [applied mathematics](#), particularly for [calculations involving three-dimensional rotations](#), such as in [three-dimensional computer graphics](#), [computer vision](#), and [crystallographic texture](#) analysis.[6] They can be used alongside other methods of rotation, such as [Euler angles](#) and [rotation matrices](#), or as an alternative to them, depending on the application.

In modern [mathematical language](#), quaternions form a four-dimensional [associative](#) [normed division algebra](#) over the real numbers, and therefore a ring, being both a [division ring](#) and a [domain](#). The algebra of quaternions is often denoted by $\mathbf{H}$ (for *Hamilton*), or in [blackboard bold](#) by $\mathbb{H}$. It can also be given by the [Clifford algebra](#) [classifications](#) $\mathrm{Cl}_{0,2}(\mathbb{R}) \cong \mathrm{Cl}_{3,0}^{+}(\mathbb{R})$. In fact, it was the first noncommutative [division algebra](#) to be discovered.

According to the [Frobenius theorem](#), the algebra $\mathbb{H}$ is one of only two finite-dimensional [division rings](#) containing a proper [subring](#) [isomorphic](#) to the real numbers; the other being the complex numbers. These rings are also [Euclidean Hurwitz algebras](#), of which the quaternions are the largest [associative algebra](#) (and hence the largest ring). Further extending the quaternions yields the [non-associative](#) [octonions](#), which is the last [normed division algebra](#) over the real numbers. (The [sedenions](#), the extension of the octonions, have [zero divisors](#) and so cannot be a normed division algebra.)[7]

Quaternion multiplication table

↓ × →	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

Note: This is a multiplication table, not a [Cayley table](#), because inverses do not appear in the row or column headings. (Left column shows the premultiplier. Top row shows the post-multiplier.)
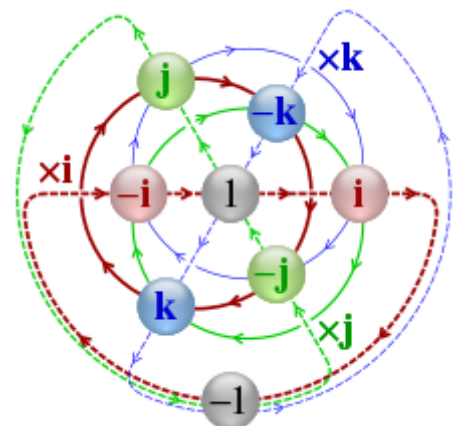
[Cayley Q8 graph](#) showing the six cycles of multiplication by **i**, **j** and **k**. (If the image is opened in the [Wikipedia commons](#) by clicking twice on it, cycles can be highlighted by hovering over or clicking on them.)

The unit quaternions can be thought of as a choice of a group structure on the 3-sphere S^3 that gives the group Spin(3), which is isomorphic to SU(2) and also to the universal cover of SO(3).

History

Quaternions were introduced by Hamilton in 1843.[8] Important precursors to this work included Euler's four-square identity (1748) and Olinde Rodrigues' parameterization of general rotations by four parameters (1840), but neither of these writers treated the four-parameter rotations as an algebra.[9][10] Carl Friedrich Gauss had also discovered quaternions in 1819, but this work was not published until 1900.[11][12]

Hamilton knew that the complex numbers could be interpreted as points in a plane, and he was looking for a way to do the same for points in three-dimensional space. Points in space can be represented by their coordinates, which are triples of numbers, and for many years he had known how to add and subtract triples of numbers. However, for a long time, he had been stuck on the problem of multiplication and division. He could not figure out how to calculate the quotient of the coordinates of two points in space. In fact, Ferdinand Georg Frobenius later proved in 1877 that for a division algebra over the real numbers to be finite-dimensional and associative, it cannot be three-dimensional, and there are only three such division algebras: $\mathbb{R}, \mathbb{C}$ (complex numbers) and $\mathbb{H}$ (quaternions) which have dimension 1, 2, and 4 respectively.
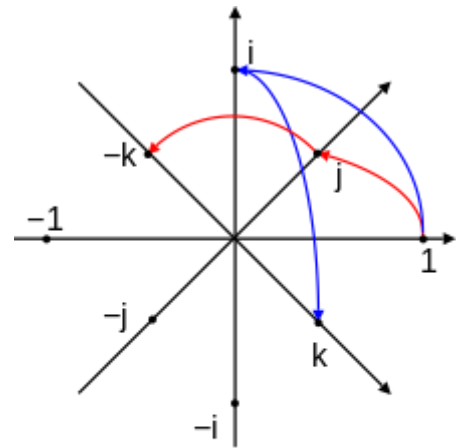
The great breakthrough in quaternions finally came on Monday 16 October 1843 in Dublin, when Hamilton was on his way to the Royal Irish Academy to preside at a council meeting. As he walked along the towpath of the Royal Canal with his wife, the concepts behind quaternions were taking shape in his mind. When the answer dawned on him, Hamilton could not resist the urge to carve the formula for the quaternions,

ij = k
ji = −k
ij = −ji

Graphical representation of products of quaternion units as 90° rotations in the planes of 4-dimensional space spanned by two of $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The left factor can be viewed as being rotated by the right factor to arrive at the product. Visually $\mathbf{i} \cdot \mathbf{j} = -(\mathbf{j} \cdot \mathbf{i})$.

- In *blue*:
  - $1 \cdot \mathbf{i} = \mathbf{i}$ (1/**i** plane)
  - $\mathbf{i} \cdot \mathbf{j} = \mathbf{k}$ (**i**/**k** plane)
- In *red*:
  - $1 \cdot \mathbf{j} = \mathbf{j}$ (1/**j** plane)
  - $\mathbf{j} \cdot \mathbf{i} = -\mathbf{k}$ (**j**/**k** plane)

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

into the stone of Brougham Bridge as he paused on it. Although the carving has since faded away, there has been an annual pilgrimage since 1989 called the Hamilton Walk for scientists and mathematicians who walk from Dunsink Observatory to the Royal Canal bridge in remembrance of Hamilton's discovery.

On the following day, Hamilton wrote a letter to his friend and fellow mathematician, John T. Graves, describing the train of thought that led to his discovery. This letter was later published in a letter to the *London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*;[13] Hamilton states:

> And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples ... An electric circuit seemed to close, and a spark flashed forth.[13]

Hamilton called a quadruple with these rules of multiplication a *quaternion*, and he devoted most of the remainder of his life to studying and teaching them. Hamilton's treatment is more geometric than the modern approach, which emphasizes quaternions' algebraic properties. He founded a school of "quaternionists", and he tried to popularize quaternions in several books. The last and longest of his books, *Elements of Quaternions*,[14] was 800 pages long; it was edited by his son and published shortly after his death.

After Hamilton's death, the Scottish mathematical physicist Peter Tait became the chief exponent of quaternions. At this time, quaternions were a mandatory examination topic in Dublin. Topics in physics and geometry that would now be described using vectors, such as kinematics in space and Maxwell's equations, were described entirely in terms of quaternions. There was even a professional research association, the Quaternion Society, devoted to the study of quaternions and other hypercomplex number systems.

From the mid-1880s, quaternions began to be displaced by vector analysis, which had been developed by Josiah Willard Gibbs, Oliver Heaviside, and Hermann von Helmholtz. Vector analysis described the same phenomena as quaternions, so it borrowed some ideas and terminology liberally from the literature on quaternions. However, vector analysis was conceptually simpler and notationally cleaner, and eventually quaternions were relegated to a minor role in mathematics and physics. A side-effect of this transition is that Hamilton's work is difficult to comprehend for many modern readers. Hamilton's original definitions are unfamiliar and his writing style was wordy and difficult to follow.

Quaternion plaque on Brougham (Broom) Bridge, Dublin, which reads:

Here as he walked by on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication

$i^2 = j^2 = k^2 = i\,j\,k = -1$
& cut it on a stone of this bridge

However, quaternions have had a revival since the late 20th century, primarily due to their utility in describing spatial rotations. The representations of rotations by quaternions are more compact and quicker to compute than the representations by matrices. In addition, unlike Euler angles, they are not susceptible to "gimbal lock". For this reason, quaternions are used in computer graphics,[15][16] computer vision, robotics,[17] control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, and orbital mechanics. For example, it is common for the attitude control systems of spacecraft to be commanded in terms of quaternions. Quaternions have received another boost from number theory because of their relationships with the quadratic forms.[18]

Quaternions in physics

P.R. Girard's 1984 essay *The quaternion group and modern physics*[19] discusses some roles of quaternions in physics. The essay shows how various physical covariance groups, namely $SO(3)$, the Lorentz group, the general theory of relativity group, the Clifford algebra $SU(2)$ and the conformal group, can easily be related to the quaternion group in modern algebra. Girard began by discussing group representations and by representing some space groups of crystallography. He proceeded to kinematics of rigid body motion. Next he used complex quaternions (biquaternions) to represent the Lorentz group of special relativity, including the Thomas precession. He cited five authors, beginning with Ludwik Silberstein, who used a potential function of one quaternion variable to express Maxwell's equations in a single differential equation. Concerning general relativity, he expressed the Runge–Lenz vector. He mentioned the Clifford

biquaternions (split-biquaternions) as an instance of Clifford algebra. Finally, invoking the reciprocal of a biquaternion, Girard described conformal maps on spacetime. Among the fifty references, Girard included Alexander Macfarlane and his *Bulletin* of the Quaternion Society. In 1999 he showed how Einstein's equations of general relativity could be formulated within a Clifford algebra that is directly linked to quaternions.[20]

The finding of 1924 that in quantum mechanics the spin of an electron and other matter particles (known as spinors) can be described using quaternions (in the form of the famous Pauli spin matrices) furthered their interest; quaternions helped to understand how rotations of electrons by 360° can be discerned from those by 720° (the "Plate trick").[21][22] As of 2018, their use has not overtaken rotation groups.[a]

Definition

A *quaternion* is an expression of the form

$$a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}\,,$$

where a, b, c, d, are real numbers, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, are symbols that can be interpreted as unit-vectors pointing along the three spatial axes. In practice, if one of a, b, c, d is 0, the corresponding term is omitted; if a, b, c, d are all zero, the quaternion is the *zero quaternion*, denoted 0; if one of b, c, d equals 1, the corresponding term is written simply $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$.

Hamilton describes a quaternion $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, as consisting of a scalar part and a vector part. The quaternion $b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ is called the *vector part* (sometimes *imaginary part*) of q, and a is the *scalar part* (sometimes *real part*) of q. A quaternion that equals its real part (that is, its vector part is zero) is called a *scalar* or *real quaternion*, and is identified with the corresponding real number. That is, the real numbers are *embedded* in the quaternions. (More properly, the field of real numbers is isomorphic to a subset of the quaternions. The field of complex numbers is also isomorphic to three subsets of quaternions.)[23] A quaternion that equals its vector part is called a *vector quaternion*.

The set of quaternions is a 4-dimensional vector space over the real numbers, with $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as a basis, by the component-wise addition

$$\begin{aligned}(a_1 + b_1\,\mathbf{i} + c_1\,\mathbf{j} + d_1\,\mathbf{k}) &+ (a_2 + b_2\,\mathbf{i} + c_2\,\mathbf{j} + d_2\,\mathbf{k}) \\ &= (a_1 + a_2) + (b_1 + b_2)\,\mathbf{i} + (c_1 + c_2)\,\mathbf{j} + (d_1 + d_2)\,\mathbf{k},\end{aligned}$$

and the component-wise scalar multiplication

$$\lambda(a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}) = \lambda a + (\lambda b)\,\mathbf{i} + (\lambda c)\,\mathbf{j} + (\lambda d)\,\mathbf{k}.$$

A multiplicative group structure, called the *Hamilton product*, denoted by juxtaposition, can be defined on the quaternions in the following way:

- The real quaternion $\mathbf{1}$ is the identity element.

- The **real** quaternions commute with all other quaternions, that is $aq = qa$ for every quaternion q and every real quaternion a. In algebraic terminology this is to say that the field of real quaternions are the *center* of this quaternion algebra.
- The product is first given for the basis elements (see next subsection), and then extended to all quaternions by using the distributive property and the center property of the real quaternions. The Hamilton product is not commutative, but is associative, thus the quaternions form an associative algebra over the real numbers.
- Additionally, every nonzero quaternion has an inverse with respect to the Hamilton product:

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}).$$

Thus the quaternions form a division algebra.

Multiplication of basis elements

The multiplication with 1 of the basis elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is defined by the fact that 1 is a multiplicative identity, that is,

$$\mathbf{i}\,1 = 1\,\mathbf{i} = \mathbf{i}, \qquad \mathbf{j}\,1 = 1\,\mathbf{j} = \mathbf{j}, \qquad \mathbf{k}\,1 = 1\,\mathbf{k} = \mathbf{k}.$$

The products of other basis elements are

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$
$$\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k}, \qquad \mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j} = \mathbf{i}, \qquad \mathbf{k}\mathbf{i} = -\mathbf{i}\mathbf{k} = \mathbf{j}.$$

Multiplication table

Non commutativity is emphasized by colored squares

×	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

Combining these rules,

$$\mathbf{i}\mathbf{j}\mathbf{k} = -1.$$

Center

The *center* of a noncommutative ring is the subring of elements c such that $cx = xc$ for every x. The center of the quaternion algebra is the subfield of real quaternions. In fact, it is a part of the definition that the real quaternions belong to the center. Conversely, if $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ belongs to the center, then

$$0 = \mathbf{i}q - q\mathbf{i} = 2c\mathbf{i}\mathbf{j} + 2d\mathbf{i}\mathbf{k} = 2c\mathbf{k} - 2d\mathbf{j},$$

and $c = d = 0$. A similar computation with $\mathbf{j}$ instead of $\mathbf{i}$ shows that one has also $b = 0$. Thus $q = a$ is a *real* quaternion.

The quaternions form a division algebra. This means that the non-commutativity of multiplication is the only property that makes quaternions different from a field. This non-commutativity has some unexpected consequences, among them that a polynomial equation over the quaternions can have more distinct solutions than the degree of the polynomial. For example, the equation $z^2 + 1 = 0$, has infinitely many quaternion solutions, which are the quaternions $z = b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ such that $b^2 + c^2 + d^2 = 1$. Thus these "roots of -1" form a unit sphere in the three-dimensional space of vector quaternions.

Hamilton product

For two elements $a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$, their product, called the **Hamilton product** $(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k})\,(a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})$, is determined by the products of the basis elements and the distributive law. The distributive law makes it possible to expand the product so that it is a sum of products of basis elements. This gives the following expression:

$$
\begin{array}{llll}
a_1 a_2 & +\,a_1 b_2\mathbf{i} & +\,a_1 c_2\mathbf{j} & +\,a_1 d_2\mathbf{k} \\
+\,b_1 a_2\mathbf{i} & +\,b_1 b_2\mathbf{i}^2 & +\,b_1 c_2\mathbf{ij} & +\,b_1 d_2\mathbf{ik} \\
+\,c_1 a_2\mathbf{j} & +\,c_1 b_2\mathbf{ji} & +\,c_1 c_2\mathbf{j}^2 & +\,c_1 d_2\mathbf{jk} \\
+\,d_1 a_2\mathbf{k} & +\,d_1 b_2\mathbf{ki} & +\,d_1 c_2\mathbf{kj} & +\,d_1 d_2\mathbf{k}^2
\end{array}
$$

Now the basis elements can be multiplied using the rules given above to get:[8]

$$
\begin{aligned}
& a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\
& +\,(a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)\mathbf{i} \\
& +\,(a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)\mathbf{j} \\
& +\,(a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)\mathbf{k}
\end{aligned}
$$

The product of two rotation quaternions[24] will be equivalent to the rotation $a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ followed by the rotation $a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$.

Scalar and vector parts

A quaternion of the form $a + 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$, where a is a real number, is called **scalar**, and a quaternion of the form $0 + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, where b, c, and d are real numbers, and at least one of b, c or d is nonzero, is called a **vector quaternion**. If $a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ is any quaternion, then a is called its **scalar part** and $b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ is called its **vector part**. Even though every quaternion can be viewed as a vector in a four-dimensional vector space, it is common to refer to the **vector** part as vectors in three-dimensional space. With this convention, a vector is the same as an element of the vector space $\mathbb{R}^3$.[b]

Hamilton also called vector quaternions **right quaternions**[26][27] and real numbers (considered as quaternions with zero vector part) **scalar quaternions**.

If a quaternion is divided up into a scalar part and a vector part, that is,

$$\mathbf{q} = (r, \ \vec{v}), \ \ \mathbf{q} \in \mathbb{H}, \ \ r \in \mathbb{R}, \ \ \vec{v} \in \mathbb{R}^3,$$

then the formulas for addition, multiplication, and multiplicative inverse are

$$(r_1, \ \vec{v}_1) + (r_2, \ \vec{v}_2) = (r_1 + r_2, \ \vec{v}_1 + \vec{v}_2),$$
$$(r_1, \ \vec{v}_1)(r_2, \ \vec{v}_2) = (r_1 r_2 - \vec{v}_1 \cdot \vec{v}_2, \ r_1 \vec{v}_2 + r_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2),$$
$$(r, \vec{v})^{-1} = \left(\frac{r}{r^2 + \vec{v} \cdot \vec{v}}, \frac{-\vec{v}}{r^2 + \vec{v} \cdot \vec{v}} \right),$$

where "·" and "×" denote respectively the dot product and the cross product.

Conjugation, the norm, and reciprocal

Conjugation of quaternions is analogous to conjugation of complex numbers and to transposition (also known as reversal) of elements of Clifford algebras. To define it, let $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ be a quaternion. The **conjugate** of q is the quaternion $q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. It is denoted by q^*, q^t, $\tilde{q}$, or $\bar{q}$.[8] Conjugation is an involution, meaning that it is its own inverse, so conjugating an element twice returns the original element. The conjugate of a product of two quaternions is the product of the conjugates *in the reverse order*. That is, if p and q are quaternions, then $(pq)^* = q^* p^*$, not $p^* q^*$.

The conjugation of a quaternion, in stark contrast to the complex setting, can be expressed with multiplication and addition of quaternions:

$$q^* = -\frac{1}{2}(q + \mathbf{i} q \mathbf{i} + \mathbf{j} q \mathbf{j} + \mathbf{k} q \mathbf{k}).$$

Conjugation can be used to extract the scalar and vector parts of a quaternion. The scalar part of p is $\frac{1}{2}(p + p^*)$, and the vector part of p is $\frac{1}{2}(p - p^*)$.

The square root of the product of a quaternion with its conjugate is called its *norm* and is denoted $\|q\|$ (Hamilton called this quantity the *tensor* of q, but this conflicts with the modern meaning of "tensor"). In formulas, this is expressed as follows:

$$\|q\| = \sqrt{qq^*} = \sqrt{q^* q} = \sqrt{a^2 + b^2 + c^2 + d^2}$$

This is always a non-negative real number, and it is the same as the Euclidean norm on $\mathbb{H}$ considered as the vector space $\mathbb{R}^4$. Multiplying a quaternion by a real number scales its norm by the absolute value of the number. That is, if α is real, then

$$\|\alpha q\| = |\alpha| \, \|q\| \, .$$

This is a special case of the fact that the norm is *multiplicative*, meaning that

$$\|pq\| = \|p\| \, \|q\|$$

for any two quaternions p and q. Multiplicativity is a consequence of the formula for the conjugate of a product. Alternatively it follows from the identity

$$\det \begin{pmatrix} a + ib & id + c \\ id - c & a - ib \end{pmatrix} = a^2 + b^2 + c^2 + d^2,$$

(where i denotes the usual imaginary unit) and hence from the multiplicative property of determinants of square matrices.

This norm makes it possible to define the **distance** $d(p, \ q)$ between p and q as the norm of their difference:

$$d(p, q) = \|p - q\| \, .$$

This makes $\mathbb{H}$ a metric space. Addition and multiplication are continuous in regard to the associated metric topology. This follows with exactly the same proof as for the real numbers $\mathbb{R}$ from the fact that $\mathbb{H}$ is a normed algebra.

Unit quaternion

A **unit quaternion** is a quaternion of norm one. Dividing a nonzero quaternion q by its norm produces a unit quaternion $\mathbf{U}q$ called the *versor* of q:

$$\mathbf{U}q = \frac{q}{\|q\|}.$$

Every nonzero quaternion has a unique polar decomposition $q = \|q\| \cdot \mathbf{U}q$, while the zero quarternion can be formed from any unit quarternion.

Using conjugation and the norm makes it possible to define the reciprocal of a nonzero quaternion. The product of a quaternion with its reciprocal should equal 1, and the considerations above imply that the product of q and $q^* / \|q\|^2$ is 1 (for either order of multiplication). So the *reciprocal* of q is defined to be

$$q^{-1} = \frac{q^*}{\lVert q \rVert^2}.$$

This makes it possible to divide two quaternions p and q in two different ways (when q is nonzero). That is, their quotient can be either $p\,q^{-1}$ or $q^{-1}p$; in general, those products are different, depending on the order of multiplication, except for the special case that p and q are scalar multiples of each other (which includes the case where $p = 0$). Hence, the notation $\frac{p}{q}$ is ambiguous because it does not specify whether q divides on the left or the right (whether q^{-1} multiplies p on its left or its right).
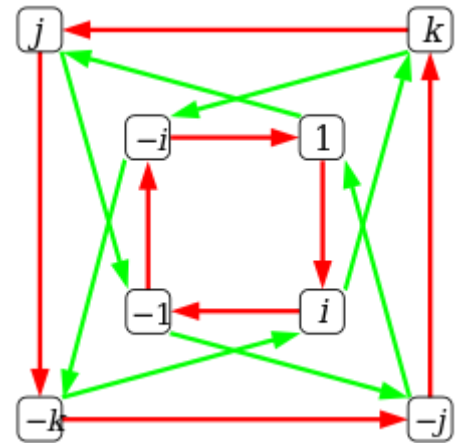
Algebraic properties

The set $\mathbb{H}$ of all quaternions is a vector space over the real numbers with dimension 4.[c] Multiplication of quaternions is associative and distributes over vector addition, but with the exception of the scalar subset, it is not commutative. Therefore, the quaternions $\mathbb{H}$ are a non-commutative, associative algebra over the real numbers. Even though $\mathbb{H}$ contains copies of the complex numbers, it is not an associative algebra over the complex numbers.

Cayley graph of Q_8. The red arrows represent multiplication on the right by $\mathbf{i}$, and the green arrows represent multiplication on the right by $\mathbf{j}$.

Because it is possible to divide quaternions, they form a division algebra. This is a structure similar to a field except for the non-commutativity of multiplication. Finite-dimensional associative division algebras over the real numbers are very rare. The Frobenius theorem states that there are exactly three: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. The norm makes the quaternions into a normed algebra, and normed division algebras over the real numbers are also very rare: Hurwitz's theorem says that there are only four: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ (the octonions). The quaternions are also an example of a composition algebra and of a unital Banach algebra.
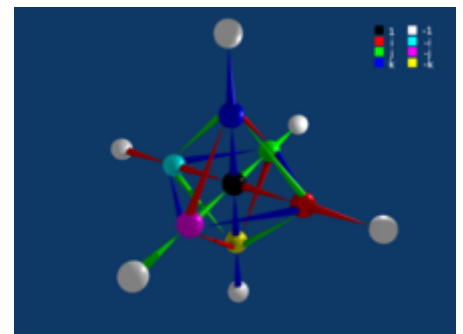
Because the product of any two basis vectors is plus or minus another basis vector, the set $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ forms a group under multiplication. This non-abelian group is called the quaternion group and is denoted Q_8.[28] The real group ring of Q_8 is a ring $\mathbb{R}[Q_8]$ which is also an eight-dimensional vector space over $\mathbb{R}$. It has one basis vector for each element of Q_8. The quaternions are isomorphic to the quotient ring of $\mathbb{R}[Q_8]$ by the ideal generated by the elements $1 + (-1)$, $\mathbf{i} + (-\mathbf{i})$, $\mathbf{j} + (-\mathbf{j})$, and $\mathbf{k} + (-\mathbf{k})$. Here the first term in each of the differences is one of the basis elements 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and the second term is one of basis elements -1, $-\mathbf{i}$, $-\mathbf{j}$, and $-\mathbf{k}$, not the additive inverses of 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Three-dimensional graph of Q_8. Red, green and blue arrows represent multiplication by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. Multiplication by negative numbers are omitted for clarity.

Quaternions and the space geometry

The vector part of a quaternion can be interpreted as a coordinate vector in $\mathbb{R}^3$; therefore, the algebraic operations of the quaternions reflect the geometry of $\mathbb{R}^3$. Operations such as the vector dot and cross products can be defined in terms of

quaternions, and this makes it possible to apply quaternion techniques wherever spatial vectors arise. A useful application of quaternions has been to interpolate the orientations of key-frames in computer graphics.[15]

For the remainder of this section, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ will denote both the three imaginary[29] basis vectors of $\mathbb{H}$ and a basis for $\mathbb{R}^3$. Replacing $\mathbf{i}$ by $-\mathbf{i}$, $\mathbf{j}$ by $-\mathbf{j}$, and $\mathbf{k}$ by $-\mathbf{k}$ sends a vector to its additive inverse, so the additive inverse of a vector is the same as its conjugate as a quaternion. For this reason, conjugation is sometimes called the *spatial inverse*.

For two vector quaternions $p = b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and $q = b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ their dot product, by analogy to vectors in $\mathbb{R}^3$, is

$$p \cdot q = b_1 b_2 + c_1 c_2 + d_1 d_2 .$$

It can also be expressed in a component-free manner as

$$p \cdot q = \tfrac{1}{2}(p^* q + q^* p) = \tfrac{1}{2}(pq^* + qp^*).$$

This is equal to the scalar parts of the products pq^*, qp^*, p^*q, and q^*p. Note that their vector parts are different.

The cross product of p and q relative to the orientation determined by the ordered basis $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is

$$p \times q = (c_1 d_2 - d_1 c_2)\mathbf{i} + (d_1 b_2 - b_1 d_2)\mathbf{j} + (b_1 c_2 - c_1 b_2)\mathbf{k} .$$

(Recall that the orientation is necessary to determine the sign.) This is equal to the vector part of the product pq (as quaternions), as well as the vector part of $-q^*p^*$. It also has the formula

$$p \times q = \tfrac{1}{2}(pq - qp).$$

For the commutator, $[p, q] = pq - qp$, of two vector quaternions one obtains

$$[p, q] = 2p \times q.$$

In general, let p and q be quaternions and write

$$p = p_s + p_v,$$
$$q = q_s + q_v,$$

where p_s and q_s are the scalar parts, and p_v and q_v are the vector parts of p and q. Then we have the formula

$$pq = (pq)_s + (pq)_v = (p_s q_s - p_v \cdot q_v) + (p_s q_v + q_s p_v + p_v \times q_v).$$

This shows that the noncommutativity of quaternion multiplication comes from the multiplication of vector quaternions. It also shows that two quaternions commute if and only if their vector parts are collinear. Hamilton[30] showed that this product computes the third vertex of a spherical triangle from two given vertices and their associated arc-lengths, which is also an algebra of points in Elliptic geometry.

Unit quaternions can be identified with rotations in $\mathbb{R}^3$ and were called versors by Hamilton.[30] Also see Quaternions and spatial rotation for more information about modeling three-dimensional rotations using quaternions.

See Hanson (2005)[31] for visualization of quaternions.

Matrix representations

Just as complex numbers can be represented as matrices, so can quaternions. There are at least two ways of representing quaternions as matrices in such a way that quaternion addition and multiplication correspond to matrix addition and matrix multiplication. One is to use 2 × 2 complex matrices, and the other is to use 4 × 4 real matrices. In each case, the representation given is one of a family of linearly related representations. In the terminology of abstract algebra, these are injective homomorphisms from $\mathbb{H}$ to the matrix rings $M(2,\mathbf{C})$ and $M(4,\mathbf{R})$, respectively.

Using 2 × 2 complex matrices, the quaternion $a + bi + cj + dk$ can be represented as

$$\begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}.$$

Note that the "i" of the complex numbers is distinct from the "i" of the quaternions.

This representation has the following properties:

- Constraining any two of b, c and d to zero produces a representation of complex numbers. For example, setting $c = d = 0$ produces a diagonal complex matrix representation of complex numbers, and setting $b = d = 0$ produces a real matrix representation.
- The norm of a quaternion (the square root of the product with its conjugate, as with complex numbers) is the square root of the determinant of the corresponding matrix.[32]
- The conjugate of a quaternion corresponds to the conjugate transpose of the matrix.
- By restriction this representation yields an isomorphism between the subgroup of unit quaternions and their image SU(2). Topologically, the unit quaternions are the 3-sphere, so

the underlying space of SU(2) is also a 3-sphere. The group $\mathrm{SU}(2)$ is important for describing spin in quantum mechanics; see Pauli matrices.

- There is a strong relation between quaternion units and Pauli matrices. Obtain the eight quaternion unit matrices by taking a, b, c and d, set three of them at zero and the fourth at 1 or −1. Multiplying any two Pauli matrices always yields a quaternion unit matrix, all of them except for −1. One obtains −1 via $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$; e.g. the last equality is

$$ijk = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = -1.$$

Using 4 × 4 real matrices, that same quaternion can be written as

$$
\begin{bmatrix}
a & -b & -c & -d \\
b & a & -d & c \\
c & d & a & -b \\
d & -c & b & a
\end{bmatrix}
= a
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
+ b
\begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

$$
+ c
\begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
+ d
\begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}.
$$

However, the representation of quaternions in $\mathrm{M}(4,\mathbf{R})$ is not unique. For example, the same quaternion can also be represented as

$$
\begin{bmatrix}
a & d & -b & -c \\
-d & a & c & -b \\
b & -c & a & -d \\
c & b & d & a
\end{bmatrix}
= a
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
+ b
\begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
$$

$$
+ c
\begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
+ d
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{bmatrix}.
$$

There exist 48 distinct matrix representations of this form in which one of the matrices represents the scalar part and the other three are all skew-symmetric. More precisely, there are 48 sets of quadruples of matrices with these symmetry constraints such that a function sending 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to the matrices in the quadruple is a homomorphism, that is, it sends sums and products of quaternions to sums and products of matrices.[33] In this representation, the conjugate of a quaternion corresponds to the transpose of the matrix. The fourth power of the norm of a quaternion is the determinant of the corresponding matrix. As with the 2 × 2 complex representation above, complex numbers can again be produced by constraining the coefficients suitably; for example, as block diagonal matrices with two 2 × 2 blocks by setting $c = d = 0$.

Each 4×4 matrix representation of quaternions corresponds to a multiplication table of unit quaternions. For example, the last matrix representation given above corresponds to the multiplication table

×	a	d	−b	−c
a	a	d	−b	−c
−d	−d	a	c	−b
b	b	−c	a	−d
c	c	b	d	a

which is isomorphic — through $\{a \mapsto 1, b \mapsto i, c \mapsto j, d \mapsto k\}$ — to

×	1	k	−i	−j
1	1	k	−i	−j
−k	−k	1	j	−i
i	i	−j	1	−k
j	j	i	k	1

Constraining any such multiplication table to have the identity in the first row and column and for the signs of the row headers to be opposite to those of the column headers, then there are 3 possible choices for the second column (ignoring sign), 2 possible choices for the third column (ignoring sign), and 1 possible choice for the fourth column (ignoring sign); that makes 6 possibilities. Then, the second column can be chosen to be either positive or negative, the third column can be chosen to be positive or negative, and the fourth column can be chosen to be positive or negative, giving 8 possibilities for the sign. Multiplying the possibilities for the letter positions and for their signs yields 48. Then replacing 1 with a, $\mathbf{i}$ with b, $\mathbf{j}$ with c, and $\mathbf{k}$ with d and removing the row and column headers yields a matrix representation of $a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$.

Lagrange's four-square theorem

Quaternions are also used in one of the proofs of Lagrange's four-square theorem in number theory, which states that every nonnegative integer is the sum of four integer squares. As well as being an elegant theorem in its own right, Lagrange's four square theorem has useful applications in areas of mathematics outside number theory, such as combinatorial design theory. The quaternion-based proof uses Hurwitz quaternions, a subring of the ring of all quaternions for which there is an analog of the Euclidean algorithm.

Quaternions as pairs of complex numbers

Quaternions can be represented as pairs of complex numbers. From this perspective, quaternions are the result of applying the Cayley–Dickson construction to the complex numbers. This is a generalization of the construction of the complex numbers as pairs of real numbers.

Let $\mathbb{C}^2$ be a two-dimensional vector space over the complex numbers. Choose a basis consisting of two elements 1 and $\mathbf{j}$. A vector in $\mathbb{C}^2$ can be written in terms of the basis elements 1 and $\mathbf{j}$ as

$$(a + bi)1 + (c + di)\mathbf{j}\,.$$

If we define $\mathbf{j}^2 = -1$ and $i\,\mathbf{j} = -\mathbf{j}\,i$, then we can multiply two vectors using the distributive law. Using $\mathbf{k}$ as an abbreviated notation for the product $i\,\mathbf{j}$ leads to the same rules for multiplication as the usual quaternions. Therefore, the above vector of complex numbers corresponds to the quaternion $a + b\,i + c\,\mathbf{j} + d\,\mathbf{k}$. If we write the elements of $\mathbb{C}^2$ as ordered pairs and quaternions as quadruples, then the correspondence is

$$(a + bi,\ c + di) \leftrightarrow (a, b, c, d).$$

Square roots

Square roots of −1

In the complex numbers, $\mathbb{C}$, there are just two numbers, i and $-i$, whose square is -1. In $\mathbb{H}$ there are infinitely many square roots of minus one: the quaternion solution for the square root of -1 is the unit sphere in $\mathbb{R}^3$. To see this, let $q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ be a quaternion, and assume that its square is -1. In terms of a, b, c, and d, this means

$$a^2 - b^2 - c^2 - d^2 = -1,$$
$$2ab = 0,$$
$$2ac = 0,$$
$$2ad = 0.$$

To satisfy the last three equations, either $a = 0$ or b, c, and d are all 0. The latter is impossible because a is a real number and the first equation would imply that $a^2 = -1$. Therefore, $a = 0$ and $b^2 + c^2 + d^2 = 1$. In other words: A quaternion squares to -1 if and only if it is a vector quaternion with norm 1. By definition, the set of all such vectors forms the unit sphere.

Only negative real quaternions have infinitely many square roots. All others have just two (or one in the case of 0).[d]

As a union of complex planes

Each antipodal pair of square roots of -1 creates a distinct copy of the complex numbers inside the quaternions. If $q^2 = -1$, then the copy is determined by the function

$$a + b\sqrt{-1} \mapsto a + bq\,.$$

This is an injective ring homomorphism from $\mathbb{C}$ to $\mathbb{H}$, which defines a field isomorphism from $\mathbb{C}$ onto its image. The images of the embeddings corresponding to q and $-q$ are identical.

Every non-real quaternion generates a subalgebra of the quaternions that is isomorphic to $\mathbb{C}$, and is thus a planar subspace of $\mathbb{H}$: write q as the sum of its scalar part and its vector part:

$$q = q_s + \vec{q}_v.$$

Decompose the vector part further as the product of its norm and its versor:

$$q = q_s + \|\vec{q}_v\| \cdot \mathbf{U}\vec{q}_v = q_s + \frac{q_v}{\|q_v\|}.$$

(Note that this is not the same as $q_s + \|q\| \cdot \mathbf{U}q$.) The versor of the vector part of q, $\mathbf{U}\vec{q}_v$, is a right versor with -1 as its square. A straightforward verification shows that

$$a + b\sqrt{-1} \mapsto a + b\mathbf{U}\vec{q}_v$$

defines an injective homomorphism of normed algebras from $\mathbb{C}$ into the quaternions. Under this homomorphism, q is the image of the complex number $q_s + \|\vec{q}_v\|i$.

As $\mathbb{H}$ is the union of the images of all these homomorphisms, this allows viewing the quaternions as a union of complex planes intersecting on the real line. Each of these complex planes contains exactly one pair of antipodal points of the sphere of square roots of minus one.

Commutative subrings

The relationship of quaternions to each other within the complex subplanes of $\mathbb{H}$ can also be identified and expressed in terms of commutative subrings. Specifically, since two quaternions p and q commute (i.e., $p\,q = q\,p$) only if they lie in the same complex subplane of $\mathbb{H}$, the profile of $\mathbb{H}$ as a union of complex planes arises when one seeks to find all commutative subrings of the quaternion ring.

Square roots of arbitrary quaternions

Any quaternion $\mathbf{q} = (r, \vec{v})$ (represented here in scalar–vector representation) has at least one square root $\sqrt{\mathbf{q}} = (x, \vec{y})$ which solves the equation $\sqrt{\mathbf{q}}^2 = (x, \vec{y})^2 = \mathbf{q}$. Looking at the scalar and vector parts in this equation separately yields two equations, which when solved gives the solutions

$$\sqrt{\mathbf{q}} = \sqrt{(r, \vec{v})} = \pm \left(\sqrt{\frac{\|\mathbf{q}\| + r}{2}},\ \frac{\vec{v}}{\|\vec{v}\|} \sqrt{\frac{\|\mathbf{q}\| - r}{2}} \right),$$

where $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{-\vec{v}^2}$ is the norm of $\vec{v}$ and $\|\mathbf{q}\| = \sqrt{\mathbf{q^*q}} = r^2 + \|\vec{v}\|^2$ is the norm of $\mathbf{q}$. For any scalar quaternion $\mathbf{q}$, this equation provides the correct square roots if $\frac{\vec{v}}{\|\vec{v}\|}$ is interpreted as an arbitrary unit vector.

Therefore, nonzero, non-scalar quaternions, or positive scalar quaternions, have exactly two roots, while 0 has exactly one root (0), and negative scalar quaternions have infinitely many roots, which are the vector quaternions located on $\{0\} \times S^2\left(\sqrt{-r}\right)$, i.e., where the scalar part is zero and the vector part is located on the 2-sphere with radius $\sqrt{-r}$.

Functions of a quaternion variable

Like functions of a complex variable, functions of a quaternion variable suggest useful physical models. For example, the original electric and magnetic fields described by Maxwell were functions of a quaternion variable. Examples of other functions include the extension of the Mandelbrot set and Julia sets into 4-dimensional space.[35]

The Julia sets and Mandelbrot sets can be extended to the Quaternions, but they must use cross sections to be rendered visually in 3 dimensions. This Julia set is cross sectioned at the *x y* plane.

Exponential, logarithm, and power functions

Given a quaternion,

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a + \mathbf{v},$$

the exponential is computed as[36]

$$\exp(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!} = e^a \left(\cos \|\mathbf{v}\| + \frac{\mathbf{v}}{\|\mathbf{v}\|} \sin \|\mathbf{v}\| \right),$$

and the logarithm is[36]

$$\ln(q) = \ln \|q\| + \frac{\mathbf{v}}{\|\mathbf{v}\|} \arccos \frac{a}{\|q\|}.$$

It follows that the polar decomposition of a quaternion may be written

$$q = \|q\| e^{\hat{n}\varphi} = \|q\| \left(\cos(\varphi) + \hat{n} \sin(\varphi) \right),$$

where the angle φ[e]

$$a = \|q\| \cos(\varphi)$$

and the unit vector $\hat{n}$ is defined by:

$$\mathbf{v} = \hat{n}\|\mathbf{v}\| = \hat{n}\|q\| \sin(\varphi) \,.$$

Any unit quaternion may be expressed in polar form as:

$$q = \exp\left(\hat{n}\varphi\right).$$

The power of a quaternion raised to an arbitrary (real) exponent x is given by:

$$q^x = \|q\|^x e^{\hat{n}x\varphi} = \|q\|^x \left(\cos(x\varphi) + \hat{n} \sin(x\varphi)\right) \,.$$

Geodesic norm

The geodesic distance $d_{\mathrm{g}}(p, q)$ between unit quaternions p and q is defined as:[38]

$$d_{\mathrm{g}}(p, q) = \|\ln(p^{-1}q)\|.$$

and amounts to the absolute value of half the angle subtended by p and q along a great arc of the S^3 sphere. This angle can also be computed from the quaternion dot product without the logarithm as:

$$\arccos(2(p \cdot q)^2 - 1).$$

Three-dimensional and four-dimensional rotation groups

The word "conjugation", besides the meaning given above, can also mean taking an element a to $r\,a\,r^{-1}$ where r is some nonzero quaternion. All elements that are conjugate to a given element (in this sense of the word conjugate) have the same real part and the same norm of the vector part. (Thus the conjugate in the other sense is one of the conjugates in this sense.) [39]

Thus the multiplicative group of nonzero quaternions acts by conjugation on the copy of $\mathbb{R}^3$ consisting of quaternions with real part equal to zero. Conjugation by a unit quaternion (a quaternion of absolute value 1) with real part $\cos(\varphi)$ is a rotation by an angle 2φ, the axis of the rotation being the direction of the vector part. The advantages of quaternions are:[40]

- Avoiding gimbal lock, a problem with systems such as Euler angles.
- Faster and more compact than matrices.
- Nonsingular representation (compared with Euler angles for example).
- Pairs of unit quaternions represent a rotation in 4D space (see *Rotations in 4-dimensional Euclidean space: Algebra of 4D rotations*).

The set of all unit quaternions (versors) forms a 3-sphere S^3 and a group (a Lie group) under multiplication, double covering the group $\mathrm{SO}(3,\mathbb{R})$ of real orthogonal 3×3 matrices of determinant 1 since *two* unit quaternions correspond to every rotation under the above correspondence. See plate trick.

The image of a subgroup of versors is a point group, and conversely, the preimage of a point group is a subgroup of versors. The preimage of a finite point group is called by the same name, with the prefix **binary**. For instance, the preimage of the icosahedral group is the binary icosahedral group.

The versors' group is isomorphic to $\mathrm{SU}(2)$, the group of complex unitary 2×2 matrices of determinant 1.

Let A be the set of quaternions of the form $a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ where a, b, c, and d are either all integers or all half-integers. The set A is a ring (in fact a domain) and a lattice and is called the ring of Hurwitz quaternions. There are 24 unit quaternions in this ring, and they are the vertices of a regular 24 cell with Schläfli symbol $\{3,4,3\}$. They correspond to the double cover of the rotational symmetry group of the regular tetrahedron. Similarly, the vertices of a regular 600 cell with Schläfli symbol $\{3,3,5\}$ can be taken as the unit icosians, corresponding to the double cover of the rotational symmetry group of the regular icosahedron. The double cover of the rotational symmetry group of the regular octahedron corresponds to the quaternions that represent the vertices of the disphenoidal 288-cell.[41]

Quaternion algebras

The Quaternions can be generalized into further algebras called *quaternion algebras.* Take F to be any field with characteristic different from 2, and a and b to be elements of F; a four-dimensional unitary associative algebra can be defined over F with basis 1, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{i}\,\mathbf{j}$, where $\mathbf{i}^2 = a$, $\mathbf{j}^2 = b$ and $\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i}$ (so $(\mathbf{i}\,\mathbf{j})^2 = -a\,b$).

Quaternion algebras are isomorphic to the algebra of 2×2 matrices over F or form division algebras over F, depending on the choice of a and b.

Quaternions as the even part of $\mathrm{Cl}_{3,0}(\mathrm{R})$

The usefulness of quaternions for geometrical computations can be generalised to other dimensions by identifying the quaternions as the even part $\mathrm{Cl}_{3,0}^+(\mathbb{R})$ of the Clifford algebra $\mathrm{Cl}_{3,0}(\mathbb{R})$. This is an associative multivector algebra built up from fundamental basis elements $\sigma_1, \sigma_2, \sigma_3$ using the product rules

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1,$$

$$\sigma_i \sigma_j = -\sigma_j \sigma_i \qquad (j \neq i).$$

If these fundamental basis elements are taken to represent vectors in 3D space, then it turns out that the *reflection* of a vector r in a plane perpendicular to a unit vector w can be written:

$$r' = -w\,r\,w.$$

Two reflections make a rotation by an angle twice the angle between the two reflection planes, so

$$r'' = \sigma_2 \sigma_1\,r\,\sigma_1 \sigma_2$$

corresponds to a rotation of 180° in the plane containing σ_1 and σ_2. This is very similar to the corresponding quaternion formula,

$$r'' = -\mathbf{k}\,r\,\mathbf{k}.$$

Indeed, the two structures $\mathrm{Cl}_{3,0}^{+}(\mathbb{R})$ and $\mathbb{H}$ are isomorphic. One natural identification is

$$1 \mapsto 1, \quad \mathbf{k} \mapsto \sigma_2 \sigma_1, \quad \mathbf{i} \mapsto \sigma_3 \sigma_2, \quad \mathbf{j} \mapsto \sigma_1 \sigma_3,$$

and it is straightforward to confirm that this preserves the Hamilton relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\,.$$

In this picture, so-called "vector quaternions" (that is, pure imaginary quaternions) correspond not to vectors but to bivectors – quantities with magnitude and orientations associated with particular 2D *planes* rather than 1D *directions*. The relation to complex numbers becomes clearer, too: in 2D, with two vector directions σ_1 and σ_2, there is only one bivector basis element $\sigma_1 \sigma_2$, so only one imaginary. But in 3D, with three vector directions, there are three bivector basis elements $\sigma_1 \sigma_2$, $\sigma_2 \sigma_3$, $\sigma_3 \sigma_1$, so three imaginaries.

This reasoning extends further. In the Clifford algebra $\mathrm{Cl}_{4,0}(\mathbb{R}),$ there are six bivector basis elements, since with four different basic vector directions, six different pairs and therefore six different linearly independent planes can be defined. Rotations in such spaces using these generalisations of quaternions, called rotors, can be very useful for applications involving homogeneous coordinates. But it is only in 3D that the number of basis bivectors equals the number of basis vectors, and each bivector can be identified as a pseudovector.

There are several advantages for placing quaternions in this wider setting:[42]

- Rotors are a natural part of geometric algebra and easily understood as the encoding of a double reflection.
- In geometric algebra, a rotor and the objects it acts on live in the same space. This eliminates the need to change representations and to encode new data structures and methods, which is traditionally required when augmenting linear algebra with quaternions.
- Rotors are universally applicable to any element of the algebra, not just vectors and other quaternions, but also lines, planes, circles, spheres, rays, and so on.
- In the conformal model of Euclidean geometry, rotors allow the encoding of rotation, translation and scaling in a single element of the algebra, universally acting on any element. In particular, this means that rotors can represent rotations around an arbitrary axis, whereas quaternions are limited to an axis through the origin.
- Rotor-encoded transformations make interpolation particularly straightforward.
- Rotors carry over naturally to pseudo-Euclidean spaces, for example, the Minkowski space of special relativity. In such spaces rotors can be used to efficiently represent Lorentz boosts, and to interpret formulas involving the gamma matrices.

For further detail about the geometrical uses of Clifford algebras, see Geometric algebra.

Brauer group

The quaternions are "essentially" the only (non-trivial) central simple algebra (CSA) over the real numbers, in the sense that every CSA over the real numbers is Brauer equivalent to either the real numbers or the quaternions. Explicitly, the Brauer group of the real numbers consists of two classes, represented by the real numbers and the quaternions, where the Brauer group is the set of all CSAs, up to equivalence relation of one CSA being a matrix ring over another. By the Artin–Wedderburn theorem (specifically, Wedderburn's part), CSAs are all matrix algebras over a division algebra, and thus the quaternions are the only non-trivial division algebra over the real numbers.

CSAs – finite dimensional rings over a field, which are simple algebras (have no non-trivial 2-sided ideals, just as with fields) whose center is exactly the field – are a noncommutative analog of extension fields, and are more restrictive than general ring extensions. The fact that the quaternions are the only non-trivial CSA over the real numbers (up to equivalence) may be compared with the fact that the complex numbers are the only non-trivial finite field extension of the real numbers.

Quotations

I regard it as an inelegance, or imperfection, in quaternions, or rather in the state to which it has been hitherto unfolded, whenever it becomes or seems to become necessary to have recourse to x, y, z, etc.

—William Rowan Hamilton (circa 1848)[43]

Time is said to have only one dimension, and space to have three dimensions. ... The mathematical quaternion partakes of both these elements; in technical language it may be said to be "time plus space", or "space plus time": and in this sense it has, or at least involves a reference to, four dimensions. ... *And how the One of Time, of Space the Three, Might in the Chain of Symbols girdled be.*

—William Rowan Hamilton (circa 1853)[44]

Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell.

—W. Thompson, Lord Kelvin (1892)[45]

There was a time, indeed, when I, although recognizing the appropriateness of vector analysis in electromagnetic theory (and in mathematical physics generally), did think it was harder to understand and to work than the Cartesian analysis. But that was before I had thrown off the quaternionic old-man-of-the-sea who fastened himself about my shoulders when reading the only accessible treatise on the subject – Prof. Tait's Quaternions. But I came later to see that, so far as the vector analysis I required was concerned, the quaternion was not only not required, but was a positive evil of no inconsiderable magnitude; and that by its avoidance the establishment of vector analysis was made quite simple and its working also simplified, and that it could be conveniently harmonised with ordinary Cartesian work. There is not a ghost of a quaternion in any of my papers (except in one, for a special purpose). The vector analysis I use may be described either as a convenient and systematic abbreviation of Cartesian analysis; or else, as Quaternions without the quaternions, ... "Quaternion" was, I think, defined by an American schoolgirl to be "an ancient religious ceremony". This was, however, a complete mistake. The ancients – unlike Prof. Tait – knew not, and did not worship Quaternions.

—Oliver Heaviside (1893)[46]

Neither matrices nor quaternions and ordinary vectors were banished from these ten [additional] chapters. For, in spite of the uncontested power of the modern Tensor Calculus, those older mathematical languages continue, in my opinion, to offer conspicuous advantages in the restricted field of special relativity. Moreover, in science as well as in everyday life, the mastery of more than one language is also precious, as it broadens our views, is conducive to criticism with regard to, and guards against hypostasy [weak-foundation] of, the matter expressed by words or mathematical symbols.

—Ludwik Silberstein (1924)[47]

... quaternions appear to exude an air of nineteenth century decay, as a rather unsuccessful species in the struggle-for-life of mathematical ideas. Mathematicians, admittedly, still keep a warm place in their hearts for the remarkable algebraic properties of quaternions but, alas, such enthusiasm means little to the harder-headed physical scientist.

—Simon L. Altmann (1986)[48]

See also

- Conversion between quaternions and Euler angles – Mathematical strategy
- Dual quaternion – eight-dimensional algebra over the real numbers
- Dual-complex number – Four-dimensional algebra over the real numbers

- [Exterior algebra](#) – Algebra of exterior/ wedge products
- [Hurwitz quaternion order](#)
- [Hyperbolic quaternion](#) – Mutation of quaternions where unit vectors square to +1
- [Lénárt sphere](#) – educational model for spherical geometry
- [Pauli matrices](#) – Matrices important in quantum mechanics and the study of spin
- [Quaternionic manifold](#)
- [Quaternionic matrix](#) – matrix whose entries are quaternions
- [Quaternionic polytope](#)
- [Quaternionic projective space](#) – Concept in mathematics
- [Rotations in 4-dimensional Euclidean space](#) – Special orthogonal group
- [Slerp](#) – Spherical linear interpolation in computer graphics
- [Split-quaternion](#) – Four-dimensional associative algebra over the reals
- [Tesseract](#) – Four-dimensional analogue of the cube

Notes

a. A more personal view of quaternions was written by [Joachim Lambek](#) in 1995. He wrote in his essay *If Hamilton had prevailed: quaternions in physics*: "My own interest as a graduate student was raised by the inspiring book by Silberstein". He concluded by stating "I firmly believe that quaternions can supply a shortcut for pure mathematicians who wish to familiarize themselves with certain aspects of theoretical physics." Lambek, J. (1995). "If Hamilton had prevailed: Quaternions in physics". *Math. Intelligencer*. Vol. 17, no. 4. pp. 7–15. [doi](#):[10.1007/BF03024783 (https://doi.org/10.1007%2FBF03024783)](https://doi.org/10.1007%2FBF03024783).

b. It is important to note that the vector part of a quaternion is, in truth, an "axial" vector or "[pseudovector](#)", *not* an ordinary or "polar" vector, as was formally proven by Altmann (1986).[25] A polar vector can be represented in calculations (for example, for rotation by a quaternion "similarity transform") by a pure imaginary quaternion, with no loss of information, but the two should not be confused. The axis of a "binary" (180°) rotation quaternion corresponds to the direction of the represented polar vector in such a case.

c. In comparison, the real numbers $\mathbb{R}$ have dimension 1, the complex numbers $\mathbb{C}$ have dimension 2, and the octonions $\mathbb{O}$ have dimension 8.

d. The identification of the square roots of minus one in $\mathbb{H}$ was given by Hamilton[34] but was frequently omitted in other texts. By 1971 the sphere was included by Sam Perlis in his three-page exposition included in *Historical Topics in Algebra* (page 39) published by the [National Council of Teachers of Mathematics](#). More recently, the sphere of square roots of minus one is described in [Ian R. Porteous](#)'s book *Clifford Algebras and the Classical Groups* (Cambridge, 1995) in proposition 8.13 on page 60.

e. Books on applied mathematics, such as Corke (2017)[37] often use different notation with $\varphi := \frac{1}{2}\theta$ — that is, [another variable](#) $\theta = 2\varphi$.

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  Ghiloni, R.; Moretti, V.; Perotti, A. (2017). "Spectral representations of normal operators via Intertwining Quaternionic Projection Valued Measures". *Rev. Math. Phys*. **29**: 1750034. arXiv:1602.02661 (https://arxiv.org/abs/1602.02661). doi:10.1142/S0129055X17500349 (https://doi.org/10.1142%2FS0129055X17500349). S2CID 124709652 (https://api.semanticscholar.org/CorpusID:124709652). two expository papers about continuous functional calculus and spectral theory in quanternionic Hilbert spaces useful in rigorous quaternionic quantum mechanics.
- Quaternions (https://play.google.com/store/apps/details?id=com.MoritzWillProduction.Quaternions) the Android app shows the quaternion corresponding to the orientation of the device.
- Rotating Objects Using Quaternions (https://www.gamasutra.com/view/feature/131686/rotating_objects_using_quaternions.php) article speaking to the use of Quaternions for rotation in video games/computer graphics.

External links

- Media related to Quaternions at Wikimedia Commons
- Paulson, Lawrence C. Quaternions (Formal proof development in Isabelle/HOL, Archive of Formal Proofs) (https://www.isa-afp.org/entries/Quaternions.html)
- Quaternions – Visualisation (https://quaternions.online/)