Finals Revision Guide

30.102 Electromagnetics & Applications, Term 5 2020

Wei Min Cher

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Contents

1 W6: Vector Algebra

1.1 Orthogonal Coordinate Systems

1.1.1 Cartesian Coordinates

- Coordinate variables: x, y, z

1.1.2 Cylindrical Coordinates

- Coordinate variables:
    - r: radial distance in x-y plane, $0 \le r \le \infty$
    - ϕ: azimuth angle from positive x-axis, $0 \le \phi \le 2\pi$
    - z: defined in Cartesian coordinate system, $-\infty \le z \le \infty$

1.1.3 Spherical Coordinates

- Coordinate variables:
    - R: range coordinate, $0 \le R \le \infty$
    - θ: zenith angle from positive z-axis, $0 \le \theta \le \pi$
    - ϕ: azimuth angle from positive x-axis, $0 \le \phi \le 2\pi$

1.1.4 Summary for Vectors in Coordinate Systems

	Cartesian	**Cylindrical**	**Spherical**
Coordinate variables	$x,\ y,\ z$	$r,\ \phi,\ z$	$R,\ \theta,\ \phi$
Position vector, $\overrightarrow{OP_1}$	$\hat{x}x_1 + \hat{y}y_1 + \hat{z}z_1$ for P$(x_1,\ y_1,\ z_1)$	$\hat{r}r_1 + \hat{z}z_1$ for P$(r_1,\ \phi_1,\ z_1)$	$\hat{R}R_1$ for P$(R_1,\ \theta_1,\ \phi_1)$
Unit vector operations	$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$ $\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0$ $\hat{x}\times\hat{y} = \hat{z}$ $\hat{y}\times\hat{z} = \hat{x}$ $\hat{z}\times\hat{x} = \hat{y}$	$\hat{r}\cdot\hat{r} = \hat{\phi}\cdot\hat{\phi} = \hat{z}\cdot\hat{z} = 1$ $\hat{r}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{z} = \hat{z}\cdot\hat{r} = 0$ $\hat{r}\times\hat{\phi} = \hat{z}$ $\hat{\phi}\times\hat{z} = \hat{r}$ $\hat{z}\times\hat{r} = \hat{\phi}$	$\hat{R}\cdot\hat{R} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1$ $\hat{R}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{R} = 0$ $\hat{R}\times\hat{\theta} = \hat{\phi}$ $\hat{\theta}\times\hat{\phi} = \hat{R}$ $\hat{\phi}\times\hat{R} = \hat{\theta}$
Dot product, $\vec{A}\cdot\vec{B}$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\vec{A}\times\vec{B}$	$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{R} & \hat{\theta} & \hat{\phi} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\vec{l}$	$\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$	$\hat{r}\,dr + \hat{\phi}r\,d\phi + \hat{z}\,dz$	$\hat{R}\,dR + \hat{\theta}R\,d\theta + \hat{\phi}R\sin\theta\,d\phi$
Differential areas	$d_{\vec{s}_x} = \hat{x}\,dy\,dz$ $d_{\vec{s}_y} = \hat{y}\,dx\,dz$ $d_{\vec{s}_z} = \hat{z}\,dx\,dy$	$d_{\vec{s}_r} = \hat{r}r\,d\phi\,dz$ $d_{\vec{s}_\phi} = \hat{r}r\,d\phi\,dz$ $d_{\vec{s}_z} = \hat{z}r\,dr\,d\phi$	$d_{\vec{s}_R} = \hat{R}R^2\sin\theta\,d\theta\,d\phi$ $d_{\vec{s}_\theta} = \hat{\theta}R\sin\theta\,dR\,d\phi$ $d_{\vec{s}_\phi} = \hat{\phi}R\,dR\,d\theta$
Differential volume, $d\vec{V}$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

1.2 Transformation between Coordinate Systems

Transformation	Coordinate Variables	Unit Vectors
Cartesian → cylindrical	$r = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$	$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$
Cylindrical → Cartesian	$x = r\cos\phi$ $y = r\sin\phi$	$\hat{x} = \hat{r}\cos\phi - \hat{\phi}\sin\phi$ $\hat{y} = \hat{r}\sin\phi + \hat{\phi}\cos\phi$
Cartesian → spherical	$R = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$	$\hat{R} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$ $\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$ $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$
Spherical → Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{x} = \hat{R}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$ $\hat{y} = \hat{R}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\sin\phi$ $\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$
Cylindrical → spherical	$R = \sqrt{r^2 + z^2}$ $\theta = \tan^{-1}\left(\frac{r}{z}\right)$	$\hat{R} = \hat{r}\sin\theta + \hat{z}\cos\theta$ $\hat{\theta} = \hat{r}\cos\theta - \hat{z}\sin\theta$
Spherical → cylindrical	$r = R\sin\theta$ $z = R\cos\theta$	$\hat{r} = \hat{R}\sin\theta + \hat{\theta}\cos\theta$ $\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$

1.3 Distance between Two Points

- Cartesian coordinates:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Cylindrical coordinates:

$$d = \sqrt{r_2^2 + r_1^2 - 2r_1r_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$$

- Spherical coordinates:

$$d = \sqrt{R_2^2 + R_1^2 - 2R_1R_2\left[\cos\theta_2\cos\theta_1 + \sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1)\right]}$$

1.4 Gradient of Scalar Field

- Gradient of T, ∇T

$$\nabla T = \hat{u}_1\frac{\partial T}{h_1\partial u_1} + \hat{u}_2\frac{\partial T}{h_2\partial u_2} + \hat{u}_3\frac{\partial T}{h_3\partial u_3}$$

- Directional directive of T along $\hat{a}_l$, $\frac{dT}{dl}$

$$\frac{dT}{dl} = \nabla T \cdot \hat{a}_l$$

1.5 Divergence of Vector Field

- Divergence of $\overrightarrow{E}$, $\nabla \cdot \overrightarrow{E}$

$$\nabla \cdot \overrightarrow{E} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 E_1) + \frac{\partial}{\partial u_2}(h_3 h_1 E_2) + \frac{\partial}{\partial u_3}(h_1 h_2 E_3)\right]$$

- Divergence Theorem

$$\int_{\mathcal{V}} \nabla \cdot \overrightarrow{E}\, d\mathcal{V} = \oint_{S} \overrightarrow{E} \cdot d\vec{s}$$

1.6 Curl of Vector Field

- Curl of $\overrightarrow{B}$, $\nabla \times \overrightarrow{B}$

$$\nabla \times \overrightarrow{B} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 H_1 & h_2 H_2 & h_3 H_3 \end{vmatrix}$$

- Stokes' Theorem

$$\int_{S} (\nabla \times \overrightarrow{B}) \cdot d\vec{s} = \oint_{C} \overrightarrow{B} \cdot d\vec{l}$$

1.7 Laplacian Operator

- Laplacian of scalar field, V

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1} h_2 h_3 \frac{\partial V}{h_1 \partial u_1} + \frac{\partial}{\partial u_2} h_3 h_1 \frac{\partial V}{h_2 \partial u_2} + \frac{\partial}{\partial u_3} h_1 h_2 \frac{\partial V}{h_3 \partial u_3}\right]$$

- Laplacian of vector field, $\overrightarrow{E}$

$$\nabla^2 \overrightarrow{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\overrightarrow{E}$$

- Relation between dot and cross products

$$\nabla^2 \overrightarrow{E} = \nabla(\nabla \cdot \overrightarrow{E}) - \nabla \times (\nabla \times \overrightarrow{E})$$

1.8 Summary for Advanced Vector Operations

Orthogonal Coordinate System	Cartesian	Cylindrical	Spherical
Base Vectors ($\hat{u}_1,\ \hat{u}_2,\ \hat{u}_3$)	$\hat{x},\ \hat{y},\ \hat{z}$	$\hat{r},\ \hat{\phi},\ \hat{z}$	$\hat{R},\ \hat{\theta},\ \hat{\phi}$
Metric Coefficients ($h_1,\ h_2,\ h_3$)	1, 1, 1	$1,\ r,\ 1$	$1,\ R,\ R\sin\theta$
Differential Volume ($h_1 h_2 h_3\ du_1\ du_2\ du_3$)	$dx\ dy\ dz$	$r\ dr\ d\phi\ dz$	$R^2 \sin\theta\ dR\ d\theta\ d\phi$

2 W8: Physics II Review

2.1 Epsilon & Mu

- Electric flux density, $\overrightarrow{D} = \epsilon \overrightarrow{E}$, measured in C/m^2
- Magnetic flux density, $\overrightarrow{B} = \mu \overrightarrow{H}$, measured in T

2.2 Static & Dynamic Fields

- Standard conditions:
  - Electric and magnetic fields are independent
- Dynamic conditions:
  - Electric and magnetic fields are coupled

2.3 Faraday's Law

$$\oint_C \overrightarrow{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \overrightarrow{B} \cdot d\vec{s}$$

- 3 scenarios:
  1. Time-varying magnetic field linking stationary loop

     $$\text{Induced transformer emf, } V_{emf}^{tr} = -N \int_S \frac{d\overrightarrow{B}}{dt} \cdot d\vec{s} \text{ (V)}$$

  2. Moving loop with time-varying surface area relative to the normal of $\overrightarrow{B}$

     $$\text{Induced motional emf, } V_{emf}^{m} = \oint_C (\vec{u} \times \overrightarrow{B}) \cdot d\vec{l} \text{ (V)}$$

     - where $\vec{u}$ is the velocity of the moving particle
  3. Moving loop in a time-varying field $\overrightarrow{B}$

     $$\text{Total emf, } V_{emf} = V_{emf}^{tr} + V_{emf}^{m}$$

- Using Stoke's Theorem, it can be expressed in differential form as $\boxed{\nabla \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}}$.

2.4 Maxwell-Ampere's Law

$$\begin{aligned}\oint_C \overrightarrow{B} \cdot d\vec{l} &= \mu_0 \iint_S \overrightarrow{J} \cdot \hat{n}\, d\vec{s} + \epsilon_0 \mu_0 \frac{d}{dt} \iint_S \overrightarrow{E} \cdot \hat{n}\, d\vec{s} \\ &= \mu_0 (I_c + I_d)\end{aligned}$$

- Enclosed current, $I_c = \iint_S \overrightarrow{J} \cdot \hat{n}\, d\vec{s}$

- Displacement current, $I_d = \epsilon_0 \frac{d}{dt} \iint\limits_S \overrightarrow{E} \cdot \hat{n}\, d\vec{s}$

- Using Stoke's Theorem, it can be expressed in differential form as $\boxed{\nabla \times \overrightarrow{H} = \overrightarrow{J} + \frac{d\vec{D}}{dt}}$.

2.5 Gauss' Law

$$\oint\limits_S \overrightarrow{D} \cdot d\vec{s} = \int\limits_V \rho d\vec{V}$$

- Using Divergence Theorem, it can be expressed in differential form as $\boxed{\nabla \cdot \overrightarrow{D} = \rho}$.

2.6 Gauss' Law for Magnetism

$$\oint\limits_S \overrightarrow{B} \cdot d\vec{s} = 0$$

- Using Divergence Theorem, it can be expressed in differential form as $\boxed{\nabla \cdot \overrightarrow{B} = 0}$.

3 W10 - W11: Plane Waves

3.1 Complex Permittivity

- Complex permittivity, $\epsilon_c = \epsilon - j\dfrac{\sigma}{\omega}$
  - Real part, $\epsilon' = \epsilon$
  - Complex part, $\epsilon'' = \dfrac{\sigma}{\omega}$
  - If lossless, $\epsilon'' = 0,\ \epsilon_c = \epsilon.$
- Loss tangent, $\tan\delta = \dfrac{\epsilon''}{\epsilon'} = \dfrac{\sigma}{\omega\epsilon}$

3.2 Wave Equations

$$\boxed{\begin{aligned} \nabla^2\widetilde{E} - \gamma^2\widetilde{E} = 0 \quad &\text{for } \widetilde{E} \\ \nabla^2\widetilde{H} - \gamma^2\widetilde{H} = 0 \quad &\text{for } \widetilde{H} \end{aligned}}$$

3.3 Wave Equations in Lossless Media

- Wave number, $k = \omega\sqrt{\mu\epsilon}$
  - In lossless medium, $\gamma^2 = -k^2$.

$$\boxed{\begin{aligned} \nabla^2\widetilde{E} + k^2\widetilde{E} = 0 \\ \nabla^2\widetilde{B} + k^2\widetilde{B} = 0 \end{aligned}}$$

  - Known as β in lossy medium.

3.4 Intrinsic Impedance of Lossless Medium

- Intrinsic impedance of lossless medium, η

$$\boxed{\eta = \frac{E^+_{x0}}{H^+_{y0}} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}\ (\Omega)}$$

3.5 Phase Velocity

$$\text{Phase velocity, } u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu\epsilon}}\ \text{(m/s)}$$

- In vacuum: $\epsilon = \epsilon_0,\ \mu = \mu_0$

$$\Rightarrow u_p = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3\times10^8\ \text{m/s}$$

3.6 Intrinsic Impedance of Free Space

$$\text{Intrinsic impedance of free space, } \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\ \Omega \approx 120\pi\ \Omega$$

3.7 Transverse Electromagnetic Waves (TEM)

- Generally, $\widetilde{E}$ and $\widetilde{H}$ can be described in terms of $\widetilde{k}$:

$$\boxed{\begin{aligned}\widetilde{H} &= \frac{1}{\eta}\hat{k} \times \widetilde{E} \\ \widetilde{E} &= -\eta\hat{k} \times \widetilde{H}\end{aligned}}$$

- If $\widetilde{E}$ has a component in positive x-direction and is travelling in the z-direction, then

$$\boxed{\begin{aligned}\widetilde{E}(z) &= \hat{x}\widetilde{E}_x^+(z), \quad & \widetilde{H}(z) &= \hat{y}\frac{\widetilde{E}_x^+(z)}{\eta} \\ \widetilde{H}_x^+ &= -\frac{\widetilde{E}_y^+(z)}{\eta}, \quad & \widetilde{H}_y^+ &= \frac{\widetilde{E}_x^+(z)}{\eta}\end{aligned}}.$$

3.8 Instantaneous Electric and Magnetic Fields

- E_{x0}^+ is a complex quantity with magnitude $|E_{x0}^+|$ and phase angle ϕ^+.

$$E_{x0}^+ = |E_{x0}^+|e^{j\phi^+}$$

- Instantaneous electric field $\overrightarrow{E}(z,t)$

$$\overrightarrow{E}(z,t) = \text{Re}\left[\widetilde{E}(z)e^{j\omega t}\right] = \hat{x}|E_{x0}^+|\cos(\omega t - kz + \phi^+) \ \text{(V/m)}$$

- Instantaneous magnetic field $\overrightarrow{H}(z,t)$

$$\overrightarrow{E}(z,t) = \text{Re}\left[\widetilde{H}(z)e^{j\omega t}\right] = \hat{x}\frac{|E_{x0}^+|}{\eta}\cos(\omega t - kz + \phi^+) \ \text{(A/m)}$$

3.9 Polarization

- Polarization: describes locus traced by tip of $\overrightarrow{E}$ at given point in space as function of time

- Types: linear, circular, elliptical

- For $\widetilde{E}$ propagating in z-direction, we can express it in terms of $\widetilde{E}_x$ and $\widetilde{E}_y$.

$$\widetilde{E}(z) = \hat{x}\widetilde{E}_x(z) + \hat{y}\widetilde{E}_y(z) = (\hat{x}a_x + \hat{y}a_y e^{j\delta})e^{-jkz}$$

  - where $\widetilde{E}_x(z) = E_{x0}e^{-jkz} = a_x e^{-jkz}$,
  - and $\widetilde{E}_y(z) = E_{y0}e^{-jkz} = a_x e^{j\delta}e^{-jkz}$.

- Instantaneous electric field $\overrightarrow{E}(z,t)$

$$\begin{aligned}\overrightarrow{E}(z,t) &= \text{Re}\left[\widetilde{E}(z)e^{j\omega t}\right] \\ &= \hat{x}a_x\cos(\omega t - kz) + \hat{y}a_y\cos(\omega t - kz + \delta)\end{aligned}$$

- Magnitude of electric field $|\overrightarrow{E}(z,t)|$

$$
\begin{aligned}
|\overrightarrow{E}(z,t)| &= \sqrt{E_x^2(z,t) + E_y^2(z,t)} \\
&= \sqrt{a_x^2 \cos^2(\omega t - kz) + a_y^2 \cos^2(\omega - kz + \delta)}
\end{aligned}
$$

- Inclination angle $\psi(z,t)$

$$
\psi(z,t) = \tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)
$$

3.9.1 Linear Polarization

- When $\delta = 0$ (in phase):

$$
\begin{aligned}
\overrightarrow{E}(z,t) &= (\hat{x}a_x + \hat{y}a_y)\cos(\omega t - kz) \\
|\overrightarrow{E}(z,t) &= \sqrt{a_x^2 + a_y^2}\cos(\omega t - kz) \\
\psi(z,t) &= \tan^{-1}\left(\frac{a_y}{a_x}\right)
\end{aligned}
$$

- When $\delta = \pi$ (out of phase):

$$
\begin{aligned}
\overrightarrow{E}(z,t) &= (\hat{x}a_x - \hat{y}a_y)\cos(\omega t - kz) \\
|\overrightarrow{E}(z,t) &= \sqrt{a_x^2 + a_y^2}\cos(\omega t - kz) \\
\psi(z,t) &= \tan^{-1}\left(\frac{-a_y}{a_x}\right)
\end{aligned}
$$

- $|\overrightarrow{E}(z,t)|$ is changing over time
- ψ is independent of z and $t \Rightarrow$ inclination angle is fixed
- When $a_y = 0,\ \psi = 0°$ or $180°$, the wave is x-polarized.
- When $a_x = 0,\ \psi = \pm 90°$, the wave is y-polarized.

3.9.2 Circular Polarization

- When $\delta = \frac{\pi}{2}$ (left-hand circular polarization):

$$
\begin{aligned}
\overrightarrow{E}(z,t) &= \hat{x}\,a\cos(\omega t - kz) - \hat{y}\,a\sin(\omega t - kz) \\
|\overrightarrow{E}(z,t)| &= a \\
\psi(z,t) &= -(\omega t - kz)
\end{aligned}
$$

- When $\delta = -\frac{\pi}{2}$ (right-hand circular polarization):

$$
\begin{aligned}
\overrightarrow{E}(z,t) &= \hat{x}\,a\cos(\omega t - kz) + \hat{y}\,a\sin(\omega t - kz) \\
|\overrightarrow{E}(z,t)| &= a \\
\psi(z,t) &= \omega t - kz
\end{aligned}
$$

- $\overrightarrow{E}(z,t)$ is independent of z and $t \Rightarrow$ magnitude is fixed
- ψ is changing over time

3.10 Plane Waves in Lossy Media

- General equations for α and β:

$$\boxed{\begin{aligned}\alpha &= \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]\right\}^{1/2} \quad \text{(Np/m)}\\ \beta &= \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}+1\right]\right\}^{1/2} \quad \text{(rad/m)}\end{aligned}}$$

- Intrinsic impedance of lossy medium, η_c

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon}}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{-1/2} \quad (\Omega)$$

- Approximations can be made:
  - If $\dfrac{\epsilon''}{\epsilon'} \ll 1$, material is a low-loss dielectric.
  - If $\dfrac{\epsilon''}{\epsilon'} \gg 1$, material is a good conductor.

3.10.1 Low-Loss Dielectric

$$\boxed{\begin{gathered}\alpha \approx \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \quad \text{(Np/m)}\\ \beta \approx \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu\epsilon} \quad \text{(rad/m)}\\ \eta_c \approx \sqrt{\frac{\mu}{\epsilon}} \quad (\Omega)\end{gathered}}$$

3.10.2 Good Conductor

$$\boxed{\begin{gathered}\alpha \approx \omega\sqrt{\frac{\mu\epsilon''}{2}} = \omega\sqrt{\frac{\mu\sigma}{2\omega}} = \sqrt{\pi f\mu\sigma} \quad \text{(Np/m)}\\ \beta \approx \alpha = \sqrt{\pi f\mu\sigma} \quad \text{(rad/m)}\\ \eta_c \approx \sqrt{j\frac{\mu}{\epsilon''}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma} \quad (\Omega)\\ u_p = \sqrt{\frac{4\pi f}{\mu\sigma}} \quad \text{(m/s)}\end{gathered}}$$

3.11 Skin Depth

- Skin depth, δ_s: propagation distance when magnitude of field becomes $\frac{1}{e}$ of the maximum value

$$\delta_s = \frac{1}{\alpha} \text{ (m)}$$

- Approximations:
  - At $z = \delta_s$, $e^{-1} \approx 0.37$.
  - At $z = 3\delta_s$, $e^{-3} \approx 0.05$.
  - At $z = 5\delta_s$, $e^{-3} \approx 0.01$.

3.12 Current Flow in Good Conductors

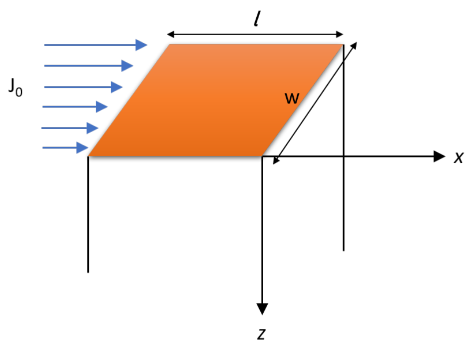
- In perfect conductors, current flows entirely on the surface of the wire.
- Assuming a current I flows in the x-direction:

$$\begin{aligned} \widetilde{J}_x(z) &= \sigma E_0 e^{-\alpha z} e^{-j\beta z} \\ &= J_0 e^{-\alpha z} e^{-j\beta z} \end{aligned}$$

  - where $J_0 = \sigma E_0$ is the amplitude of current density at the surface.

- For a good conductor, $\alpha = \beta$ and $\delta_s = \frac{1}{\alpha}$.

$$\Rightarrow \widetilde{J}_x(z) = J_0 e^{-\frac{(1+j)z}{\delta_s}} \text{ (A/m}^2\text{)}$$



Current crossing yz plane, $\widetilde{I} = w \int_0^\infty \widetilde{J}_x(z)\,dz$

$$\begin{aligned} &= w \int_0^\infty J_0 e^{-\frac{(1+j)z}{\delta_s}}\,dz \\ &= \frac{J_0 w \delta_s}{1+j} \text{ (A)} \end{aligned}$$

- Voltage across length ℓ at the surface, $\widetilde{V} = E_0 \ell = \frac{J_0}{\sigma}\ell$

$$\begin{aligned} Z = \frac{\widetilde{V}}{\widetilde{I}} &= \frac{1+j}{\sigma\delta_s}\frac{\ell}{w} \\ &= Z_s \frac{\ell}{w} \ (\Omega) \end{aligned}$$

- Internal/surface impedance of conductor, Z_s

$$\boxed{Z_s = \frac{1+j}{\sigma\delta_s}\ (\Omega)}$$

  - Defined as impedance Z for a 1 m length ℓ and 1 m width w
  - A complex quantity, can be expressed in terms of R_s and L_s, where $Z_s = R_s + j\omega L_s$.

$$\boxed{\begin{aligned} R_s &= \frac{1}{\sigma\delta_s} = \sqrt{\frac{\pi f\mu}{\sigma}}\ (\Omega) \\ L_s &= \frac{1}{\omega\sigma\delta_s} = \frac{1}{2}\sqrt{\frac{\mu}{\pi f\sigma}}\ (\mathrm{H}) \end{aligned}}$$

  - Conductor equivalent to resistor in series with inductor

- AC resistance of slab of width w and length ℓ, R

$$\boxed{R = R_s\frac{\ell}{w} = \frac{1}{\sigma\delta_s w}\ (\Omega)}$$

3.13 Power Density

- For any wave with $\overrightarrow{E}$ and $\overrightarrow{H}$, Poynting vector $\overrightarrow{S} = \overrightarrow{E} \times \overrightarrow{H}$ (W/m^2).
  - Direction of $\overrightarrow{S}$ along direction of wave propagation
  - $\overrightarrow{S}$: power density (power per unit area) carried by wave

- If wave is incident upon aperture of area with outward surface unit vector $\hat{n}$,

$$\text{Power intercepted by aperture, } P = \int_A \overrightarrow{S} \cdot \hat{n}\, dA\ (\mathrm{W})$$

  - For a plane wave propagating in direction $\hat{k}$ that makes an angle θ with $\hat{n}$, $P = SA\cos\theta$, where $S = |\overrightarrow{S}|$.

- Average power density of wave, $\overrightarrow{S_{\text{av}}}$

$$\overrightarrow{S_{\text{av}}} = \frac{1}{2}\mathrm{Re}\left[\widetilde{E} \times \widetilde{H}^*\right]\ (\mathrm{W/m^2})$$

3.14 Power Density in Lossless Medium

- Average power density carried by wave, $\overrightarrow{S_{\text{av}}}$

$$\boxed{\begin{aligned} \overrightarrow{S_{\text{av}}} &= \hat{z}\frac{1}{2\eta}(|E_{x0}|^2 + |E_{y0}|^2) \\ &= \hat{z}\frac{|\widetilde{E}|^2}{2\eta}\ (\mathrm{W/m^2}) \end{aligned}}$$

3.15 Power Density in Lossy Media

- Average power density carried by wave, $\overrightarrow{S}_{\text{av}}$

$$\boxed{\begin{aligned}\overrightarrow{S}_{\text{av}} &= \frac{1}{2}\,\text{Re}\left[\widetilde{E}\times\widetilde{H}^*\right]\\ &= \frac{\hat{z}(|E_{x0}|^2+|E_{y0}|^2)}{2}e^{-2\alpha z}\,\text{Re}\left(\frac{1}{\eta_c^*}\right)\ (\text{W/m}^2)\end{aligned}}$$

  - Express η_c in polar form, where $\eta_c = |\eta_c|e^{j\theta_\eta}$:

$$\boxed{\overrightarrow{S}_{\text{av}}(z) = \hat{z}\frac{|\widetilde{E}(0)|^2}{2|\eta_c|}e^{-2\alpha z}\cos\theta_\eta\ (\text{W/m}^2)}$$

  - While $\widetilde{E}(z)$ and $\widetilde{H}(z)$ decay with z as $e^{-\alpha z}$, power density $\overrightarrow{S}_{\text{av}}$ decreases as $e^{-2\alpha z}$.

3.16 Attenuation Rate

- Attenuation rate, A: rate of decrease of magnitude of $\overrightarrow{S}_{\text{av}}(z)$ as a function of propagation distance

$$\begin{aligned}A &= 10\log\left[\frac{S_{\text{av}}(z)}{S_{\text{av}}(0)}\right]\\ &= 10\log(e^{-2\alpha z})\\ &= -20\alpha z\log e\\ &= -8.68\alpha z\end{aligned}$$

  - where α [dB/m] = 8.68α [Np/m]

4 W11: Boundary Conditions

- Wave reflection and transmission can be divided into two types: normal and oblique incidences

4.1 Waves at Normal Incidence

- Wave crosses planar boundaries at an angle of 90°
- Assuming:
  - Planar boundary at $z = 0$,
  - Electric field propagates in positive x-direction,
  - Magnetic field propagates in positive z-direction.
- Incident waves
  - Incident electric field, $\widetilde{E}^i(z) = \hat{x}E_0^i e^{-jk_1 z}$
  - Incident magnetic field, $\widetilde{H}^i(z) = \hat{z} \times \dfrac{\widetilde{E}_0^i(z)}{\eta_1} = \hat{y}\dfrac{\widetilde{E}^i(z)}{\eta_1} = \hat{y}\dfrac{E_0^i}{\eta_1}e^{-jk_1 z}$
- Reflected waves
  - Reflected electric field, $\widetilde{E}^r(z) = \hat{x}E_0^r e^{jk_1 z}$
  - Reflected magnetic field, $\widetilde{H}^r(z) = \hat{z} \times \dfrac{\widetilde{E}_r(z)}{\eta_1} = -\hat{y}\dfrac{E_0^r}{\eta_1}e^{jk_1 z}$
- Transmitted waves
  - Transmitted electric field, $\widetilde{E}^t(z) = \hat{x}E_0^t e^{-jk_2 z}$
  - Transmitted magnetic field, $\widetilde{H}^t(z) = \hat{z} \times \dfrac{\widetilde{E}^t(z)}{\eta_2} = \hat{y}\dfrac{E_0^t}{\eta_2}e^{-jk_2 z}$
- Waves in Medium 1
  - Total electric field in Medium 1, $\widetilde{E}_1(z) = \widetilde{E}^i(z) + \widetilde{E}^r(z) = \hat{x}(E_0^i e^{-jk_1 z} + E_0^r e^{jk_1 z})$
  - Total magnetic field in Medium 1, $\widetilde{H}_1(z) = \widetilde{H}^i(z) + \widetilde{H}^r(z) = \hat{y}\dfrac{1}{\eta_1}(E_0^i e^{-jk_1 z} - E_0^r e^{jk_1 z})$
- Waves in Medium 2
  - Total electric field in Medium 2, $\widetilde{E}_2(z) = \hat{x}E_0^t e^{-jk_2 z}$
  - Total magnetic field in Medium 2, $\widetilde{H}_2(z) = \hat{y}\dfrac{E_0^t}{\eta_2}e^{-jk_2 z}$

4.2 Boundary Conditions

- At the boundary $z = 0$:

$$\widetilde{E}_1(0) = \widetilde{E}_2(0) \quad \text{or} \quad E_0^i + E_0^r = E_0^t$$
$$\widetilde{H}_1(0) = \widetilde{H}_2(0) \quad \text{or} \quad \frac{E_0^i}{\eta_1} - \frac{E_0^r}{\eta_1} = \frac{E_0^t}{\eta_2}$$

- Solving the above set of equations gives:

$$E_0^r = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_0^i = \Gamma E_0^i$$
$$E_0^t = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right) E_0^i = \tau E_0^i$$

- Reflection coefficient, $\Gamma = \dfrac{E_0^r}{E_0^i} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ (normal incidence)
- Transmission coefficient, $\tau = \dfrac{E_0^t}{E_0^i} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$ (normal incidence)
- Γ and τ are related: $\tau = 1 + \Gamma$ (normal incidence)
- For non-magnetic media, $\eta_1 = \dfrac{\eta_0}{\sqrt{\epsilon_{r1}}}$, $\eta_2 = \dfrac{\eta_0}{\sqrt{\epsilon_{r2}}}$ where η_0 is the intrinsic impedance of free space.

$$\Rightarrow \Gamma = \frac{\sqrt{\epsilon_{r1}} - \sqrt{\epsilon_{r2}}}{\sqrt{\epsilon_{r1}} + \sqrt{\epsilon_{r2}}} \quad \text{(non-magnetic media)}$$

4.3 Snell's Law

- Angles measured with respect to normal of planar boundaries:
  - θ_i: Angle of incidence
  - θ_r: Angle of reflection
  - θ_t: Angle of transmission

Snell's Law of Reflection	$\theta_i = \theta_r$
Snell's Law of Refraction	$\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{u_{p1}}{u_{p2}} = \sqrt{\dfrac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}$

- For non-magnetic materials, $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{n_1}{n_2} = \sqrt{\dfrac{\epsilon_{r1}}{\epsilon_{r2}}}$.

4.4 Refraction Index

- Refraction index: ratio of phase velocity in free space to phase velocity in a particular medium

$$n = \frac{c}{u_p} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r}$$

- A material is referred as more dense than another material if it has a greater index of refraction.

4.5 Critical Angle & Total Internal Reflection

- Critical angle: angle of incidence when $\theta_t = 90°$

$$\sin\theta_c = \frac{n_2}{n_1}\sin\theta_t\bigg|_{\theta_t=90°} = \frac{n_2}{n_1}$$

- For non-magnetic materials, $\mu_1 = \mu_2$:

$$\Rightarrow \sin\theta_c = \sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}$$

- When $\theta_i > \theta_c$, total internal reflection occurs.

4.6 Optical Fibre & Modal Dispersion

- Acceptance angle, θ_a: maximum value of θ_i for which total internal reflection remains satisfied

$$\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2}$$

  - where n_f is the refraction index of the fibre core,
  - n_c is the refraction index of cladding,
  - and n_0 is the refraction index of air.

- Optical fibres have a property called **modal dispersion**, which causes the distortion of the shape of transmitted pulses of digital data.

$$\text{Highest data rate, } f_p = \frac{1}{T} = \frac{1}{2\tau} = \frac{cn_c}{2\ell n_f(n_f - n_c)} \quad \text{(bits/s)}$$

4.7 Plane of Incidence

- Plane of incidence: plane which contains normal to boundary and direction of propagation of incident wave

4.8 Perpendicular Polarization (TE Waves)

- Incident electric field $\widetilde{E}^i$ is perpendicular to plane of incidence
- Also known as transverse electric waves (TE waves)
- Incident waves
  - Incident electric field, $\widetilde{E}^i_\perp = \hat{y}E^i_\perp e^{-jk_1x_i} = \hat{y}E^i_{\perp 0}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$

- Incident magnetic field, $\widetilde{H}^i_\perp = (-\hat{x}\cos\theta_i + \hat{z}\cos\theta_i) \times \dfrac{E^i_{\perp 0}}{\eta_1} e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$

- Reflected waves
  - Reflected electric field, $\widetilde{E}^r_\perp = \hat{y}E^r_{\perp 0}e^{-jk_1 x_r} = \hat{y}E^r_{\perp 0}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$
  - Reflected magnetic field, $\widetilde{H}^r_\perp = (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r) \times \dfrac{E^r_{\perp 0}}{\eta_1} e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$

- Transmitted waves
  - Transmitted electric field, $\widetilde{E}^t_\perp = \hat{y}E^t_{\perp 0}e^{-jk_2 x_t} = \hat{y}E^t_{\perp 0}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$
  - Transmitted magnetic field, $\widetilde{H}^t_\perp = (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t) \times \dfrac{E^t_{\perp 0}}{\eta_2} e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$

- Tangential electric field continuous boundary conditions

$$
\begin{aligned}
(\widetilde{E}^i_{\perp y} + \widetilde{E}^r_{\perp y})\Big|_{z=0} &= \widetilde{E}^t_{\perp y}\Big|_{z=0} \\
E^i_{\perp 0}e^{-jk_1 x\sin\theta_i} + E^r_{\perp 0}e^{-jk_1 x\sin\theta_r} &= E^t_{\perp 0}e^{-jk_2\sin\theta_t}
\end{aligned}
$$

- Tangential magnetic field continuous boundary conditions

$$
\begin{aligned}
(\widetilde{H}^i_{\perp x} + \widetilde{H}^r_{\perp x})\Big|_{z=0} &= \widetilde{H}^t_{\perp x}\Big|_{z=0} \\
-\frac{E^i_{\perp 0}}{\eta_1}\cos\theta_i e^{-jk_1 x\sin\theta_i} + \frac{E^r_{\perp 0}}{\eta_1}\cos\theta_r e^{-jk_1 x\sin\theta_r} &= -\frac{E^t_{\perp 0}}{\eta_2}\cos\theta_t e^{-jk_2\sin\theta_t}
\end{aligned}
$$

- Solutions of boundary equations
  1. Exponents are equal for all values of x.

$$
k_1\sin\theta_i = k_1\sin\theta_r = k_2\sin\theta_t
$$

  2. For the remaining terms:

$$
\begin{aligned}
E^i_{\perp 0} + E^r_{\perp 0} &= E^t_{\perp 0} \\
\Rightarrow \frac{\cos\theta_i}{\eta_1}(-E^i_{\perp 0} + E^r_{\perp 0}) &= -\frac{\cos\theta_t}{\eta_2}E^t_{\perp 0}
\end{aligned}
$$

4.9 Fresnel Reflection and Transmission Coefficients for TE Waves

- Reflection coefficient, $\Gamma_\perp = \dfrac{E^r_{\perp 0}}{E^i_{\perp 0}} = \dfrac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$
- Transmission coefficient, $\tau_\perp = \dfrac{E^t_{\perp 0}}{E^i_{\perp 0}} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$
- $\Gamma_\perp$ and $\tau_\perp$ are related: $\tau_\perp = 1 + \Gamma_\perp$
- For non-magnetic dielectrics, $\mu_1 = \mu_2 = \mu_0$:

$$
\Rightarrow \Gamma_\perp = \frac{\cos\theta_i - \sqrt{\dfrac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\dfrac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i}}
$$

4.10 Parallel Polarization (TM Waves)

- Incident electric field $\widetilde{E}^i$ is parallel to plane of incidence
- Also known as transverse magnetic waves (TM waves)
- Incident waves
  - Incident electric field, $\widetilde{E}^i_{\|}$

$$\begin{aligned}\widetilde{E}^i_{\|} &= (\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)E^i_{\|0}e^{-jk_1 x_i} \\ &= (\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)E^i_{\|0}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}\end{aligned}$$

  - Incident magnetic field, $\widetilde{H}^i_{\|} = \hat{y}\dfrac{E^i_{\|0}}{\eta_1}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$
- Reflected waves
  - Reflected electric field, $\widetilde{E}^r_{\|}$

$$\begin{aligned}\widetilde{E}^r_{\|} &= (\hat{x}\cos\theta_r + \hat{z}\sin\theta_i)E^r_{\|0}e^{-jk_1 x_r} \\ &= (\hat{x}\cos\theta_r + \hat{z}\sin\theta_i)E^r_{\|0}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}\end{aligned}$$

  - Reflected magnetic field, $\widetilde{H}^r_{\perp} = -\hat{y}\dfrac{E^r_{\|0}}{\eta_1}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$
- Transmitted waves
  - Transmitted electric field, $\widetilde{E}^t_{\|}$

$$\begin{aligned}\widetilde{E}^t_{\|} &= (\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)E^t_{\|0}e^{-jk_2 x_t} \\ &= (\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)E^t_{\|0}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}\end{aligned}$$

  - Transmitted magnetic field, $\widetilde{H}^t_{\|} = \hat{y}\dfrac{E^t_{\|0}}{\eta_2}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$

4.11 Fresnel Reflection and Transmission Coefficients for TM Waves

- Reflection coefficient, $\Gamma_{\|} = \dfrac{E^r_{\|0}}{E^i_{\|0}} = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_t}{\eta_2\cos\theta_t + \eta_1\cos\theta_t}$
- Transmission coefficient, $\tau_{\|} = \dfrac{E^t_{\|0}}{E^i_{\|0}} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_t}$
- $\Gamma_{\|}$ and $\tau_{\|}$ are related: $\tau_{\|} = (1 + \Gamma_{\|})\dfrac{\cos\theta_i}{\cos\theta_t}$
- For non-magnetic dielectrics, $\mu_1 = \mu_2 = \mu_0$:

$$\Rightarrow \Gamma_{\|} = \frac{-\dfrac{\epsilon_2}{\epsilon_1}\cos\theta_i + \sqrt{\dfrac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i}}{\dfrac{\epsilon_2}{\epsilon_1}\cos\theta_i + \sqrt{\dfrac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i}}$$

4.12 Brewster Angle

- The incidence angle θ_i at which the Fresnel reflection coefficient $\Gamma = 0$
- Does not exist for non-magnetic materials in perpendicular polarization (TE wave)
- In parallel polarization for $\mu_1 = \mu_2$:

$$\begin{aligned}\theta_{B\|} &= \sin^{-1}\sqrt{\frac{1}{1+\frac{\epsilon_1}{\epsilon_2}}} \\ &= \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}\end{aligned}$$

4.13 Reflectivity and Transmittivity

- Reflectivity: ratio of reflected power to incident power

$$\begin{aligned}R_\perp &= |\Gamma_\perp|^2 = \frac{P^r_\perp}{P^i_\perp} \\ R_\| &= |\Gamma_\||^2 = \frac{P^r_\|}{P^i_\|}\end{aligned}$$

- Transmittivity: ratio of transmitted power to incident power

$$\begin{aligned}T_\perp &= \frac{P^t_\perp}{P^i_\perp} = |\tau_\perp|^2\left(\frac{\eta_1\cos\theta_t}{\eta_2\cos\theta_i}\right) \\ T_\| &= \frac{P^t_\|}{P^i_\|} = |\tau_\||^2\left(\frac{\eta_1\cos\theta_t}{\eta_2\cos\theta_i}\right)\end{aligned}$$

- Reflectivity is related to transmittivity:

$$R_\perp + T_\perp = 1$$

$$R_\| + T_\| = 1$$

5 W12: Waveguides and Resonators

5.1 Waveguides

- Carries energy by non-TEM modes such as TE and TM modes or combination of both
  - TE mode: when $\widetilde{E}$ is transverse to $\hat{k}$ but $\widetilde{H}$ is not
  - TM mode: when $\widetilde{H}$ is transverse to $\hat{k}$ but $\widetilde{E}$ is not
- e.g. optical fiber, metal waveguides
- Inner conduction of coaxial cable can couple energy from and into waveguide

5.2 Transverse Magnetic (TM) Mode

- Boundary conditions:

$$k_x = \frac{m\pi}{a},\ m = 1, 2, 3, \ldots \quad \text{and} \quad k_y = \frac{n\pi}{b},\ n = 1, 2, 3, \ldots \quad \text{where } k_c^2 = k_x^2 + k_y^2$$

  - Cutoff wave number, $k_c = k^2 - \beta^2 = \omega^2\mu\epsilon - \beta^2$
- For a wave propagating in z-direction with $\widetilde{H}_z = 0$:
  - $\widetilde{E}_x = -\frac{j\beta}{k_c^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{E}_y = -\frac{j\beta}{k_c^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{E}_z = E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{H}_x = \frac{j\omega\epsilon}{k_c^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{H}_y = -\frac{j\omega\epsilon}{k_c^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
- m and n are positive integers
- Each combination of m and n represents a mode, denoted as $\mathrm{TM_{mn}}$.

5.3 Transverse Electric (TE) Mode

- Boundary conditions:

$$k_x = \frac{m\pi}{a},\ m = 1, 2, 3, \ldots \quad \text{and} \quad k_y = \frac{n\pi}{b},\ n = 1, 2, 3, \ldots \quad \text{where } k_c^2 = k_x^2 + k_y^2$$

  - Cutoff wave number, $k_c = k^2 - \beta^2 = \omega^2\mu\epsilon - \beta^2$

- For a wave propagating in z-direction with $\widetilde{E}_z = 0$:
  - $\widetilde{E}_x = \frac{j\omega\mu}{k_c^2}\left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{E}_y = -\frac{j\omega\mu}{k_c^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{H}_x = \frac{j\beta}{k_c^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{H}_y = \frac{j\beta}{k_c^2}\left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$
  - $\widetilde{H}_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$

- m and n are positive integers

- Each combination of m and n represents a mode, denoted as TE_{mn}.

5.4 Properties of TM and TE Modes

- Phase constant, β

$$\begin{aligned}\beta &= \sqrt{k^2 - k_c^2} \\ &= \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \text{(TE and TM)}\end{aligned}$$

- Cutoff frequency, f_{mn}

$$f_{mn} = \frac{u_{\text{p0}}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \text{(TE and TM)}$$

  - Frequency at each mode mn when $\beta = 0$
  - Value of 0 for m or n is NOT allowed for TM modes
  - A wave can propagate through waveguide if and only if $f > f_{mn}$

- Wave impedance in TM mode of waveguide, Z_{TM}

$$\begin{aligned}Z_{\text{TM}} &= \frac{\widetilde{E}_x}{\widetilde{H}_y} = -\frac{\widetilde{E}_y}{\widetilde{H}_x} = \frac{\beta\eta}{k} \\ &= \eta\sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}\end{aligned}$$

- Wave impedance in TE mode of waveguide, Z_{TE}

$$\begin{aligned}Z_{\text{TM}} &= \frac{\widetilde{E}_x}{\widetilde{H}_y} = -\frac{\widetilde{E}_y}{\widetilde{H}_x} \\ &= \frac{\eta}{\sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}}\end{aligned}$$

5.5 Propagation Velocities

- Phase velocity, u_p: velocity of sinusoidal pattern of wave

$$u_p = \frac{\omega}{\beta} = \frac{u_{\text{p0}}}{\sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}} \quad \text{(TE and TM)}$$

- Group velocity, u_g: velocity of envelope or wave group travelling through the medium

$$u_g = \frac{d\omega}{d\beta} = u_{\text{p0}} \sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}$$

5.6 ω-β Diagram

- Phase velocity u_p: ratio of ω to β
- Group velocity u_g: slope $\frac{d\omega}{d\beta}$ of the curve
- As $f \gg f_{mn}$, ω-β curve approaches TEM case where $u_p = u_g$.
- u_p and u_g are related by the following equation: $u_p u_g = u_{\text{p0}}^2$

5.7 Zigzag Reflections

- TE_{101} can be constructed as sum of two TEM waves

5.8 Resonant Cavities

- Cavities have metal walls on all 6 sides, unlike waveguides which only have 4 conducting sides
- Can be used as circuit elements in microwave oscillators, amplifiers and bandpass filters
- Resonant frequency, $f_{mnp} = \frac{u_{p0}}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$
  - For TE mode, m and n start at 0 while p starts at 1.
  - For TM mode, m and n start at 1 while p starts at 0.
- Quality factor, Q: approximately equivalent to normalized bandwidth

$$Q \approx \frac{f_{mnp}}{\Delta f}$$

  - In TE_{101} mode, $Q = \frac{1}{\delta_s} \frac{abd(a^2 + d^2)}{[a^3(d + 2b) + d^3(a + 2b)]}$,
    - where skin depth $\delta_s = \frac{1}{\sqrt{\pi f_{mnp} \mu_0 \sigma_c}}$,
    - and σ_c is the electrical conductivity of the conducting walls.

6 W13: Antennas II

6.1 Normalized Radiation Intensity

- Electric field, $\widetilde{E}_\theta = \frac{jI_0 \ell k \eta_0}{4\pi} \left(\frac{e^{-jkR}}{R} \right) \sin\theta \quad \text{(V/m)}$
- Magnetic field, $\widetilde{H}_\phi = \frac{\widetilde{E}_\theta}{\eta_0} \quad \text{(A/m)}$
- Average power density, $\overrightarrow{S_{\text{av}}} = \hat{R}\, S(R,\theta) = \frac{1}{2} \operatorname{Re}\left[\widetilde{E} \times \widetilde{H}^* \right] \quad \text{(W/m}^2\text{)}$
- Power density, $S(R,\theta)$

$$\begin{aligned} S(R,\theta) &= \left(\frac{\eta_0 k^2 I_0 \ell^2}{32\pi^2 R^2} \right) \sin^2\theta \\ &= S_0 \sin^2\theta \quad \text{(W/m}^2\text{)} \end{aligned}$$

- Maximum power density, $S_{\max}$

$$\begin{aligned} S_{\max} = S_0 &= \frac{\eta_0 k^2 I_0 \ell^2}{32\pi^2 R^2} \\ &= \frac{15\pi I_0^2}{R^2} \left(\frac{\ell}{\pi} \right)^2 \quad \text{(W/m}^2\text{)} \end{aligned}$$

- Normalized radiation intensity, $F(\theta,\phi)$:

$$F(\theta,\phi) = \frac{S(R,\theta,\phi)}{S_{\max}} \quad \text{(dimensionless)}$$

6.2 Reciprocity

- Receiving antenna has same directional antnna pattern as transmitting antenna.

6.3 Total Radiation Power

- Total radiation power, P_{rad}

$$\begin{aligned} P_r ad &= R^2 \int_{\phi=0^\circ}^{2\pi} \int_{\theta=0^\circ}^{\pi} S(R,\theta,\phi) \sin\theta \, d\theta \, d\phi \\ &= R^2 S_{\max} \int_0^{2\pi} \int_0^{\pi} F(\theta,\phi) \sin\theta \, d\theta \, d\phi \\ &= R^2 S_{\max} \iint_{4\pi} F(\theta,\phi) \, d\Omega \quad \text{(W)} \end{aligned}$$

6.4 Antenna Patterns

- Two principal planes:
    1. Elevation plane (xz and yz planes): constant value of ϕ
    2. Azimuth plane (xy plane): $\theta = 90°$
- 2D pattern:
    - Main lobe: where most energy is radiated
    - Side lobe: where some energy is radiated
    - Back lobe: opposite to main lobe

6.5 Pattern Solid Angle

- Pattern solid angle, Ω_p: integral of normalized radiation intensity $F(\theta, \phi)$ over a sphere
- Characterizes directional properties of 3D radiation pattern

$$\Omega_p = \iint_{4\pi} F(\theta, \phi)\, d\Omega \text{ (sr)}$$

- For isotropic antennas, $F(\theta, \phi) = 1 \Rightarrow \Omega_p = 4\pi$ (sr)

6.6 Half Power Beamwidth (HPBW)

- Half power beamwidth, β: angular width of main lobe between two angles at which magnitude of $F(\theta, \phi)$ is equal to half its peak value

$$\beta = \phi_2 - \phi_1$$

    - where θ_1 and θ_2 are the half-power angles at which $F(\theta, \phi) = 0.5F_{\text{max}}$.
- Characterizes directional properties of 2D radiation pattern
- Also known as the 3 dB beamwidth since 0.5 corresponds to -3 dB

6.7 Directivity

- Directivity: ratio of maximum normal radiation intensity F_{max} to average value of $F(\theta, \phi)$ over all directions

$$\begin{aligned} D &= \frac{F_{\text{max}}}{F_{\text{av}}} \\ &= \frac{1}{\dfrac{1}{4\pi} \displaystyle\iint_{4\pi} F(\theta, \phi)\, d\Omega} \\ &= \frac{4\pi}{\Omega_p} \text{ (dimensionless)} \end{aligned}$$

- Directivity $\uparrow$, Ω_p of antenna pattern $\downarrow$
- For isotropic antennas, $\Omega_p = 4\pi \Rightarrow D = \dfrac{4\pi}{4\pi} = 1$
- For antenna with single main lobe in z-direction, $\Omega_p \approx \beta_{xz}\, \beta_{yz}$
  - where β_{xz} and β_{yz} are the half power beamwidths in radians.

$$D = \frac{4\pi}{\Omega_p} \approx \frac{4\pi}{\beta_{xz}\, \beta_{yz}}$$

6.8 Radiation Efficiency

$$\text{Radiation efficiency, } \xi = \frac{P_{\text{rad}}}{P_{\text{t}}} \quad \text{(dimensionless)}$$

- where P_{rad} is the radiated power,
- and P_{t} is the transmitter power.

6.9 Antenna Gain

- Antenna gain, G: accounts for ohmic lossses in antenna material, unlike directivity

$$G = \xi D \quad \text{(dimensionless)}$$

- For a lossless antenna, $\xi = 1 \Rightarrow G = D$.

6.10 Effective Area of Antenna

- Effective area, A_e: ratio of intercepted power to power density of incident wave

$$A_e = \frac{P_{\text{int}}}{S_i} \quad (\text{m}^2)$$

- For short dipole antenna, $A_e = \dfrac{3\lambda^2}{8\pi} \quad (\text{m}^2)$.
- For any antenna, $A_e = \dfrac{\lambda^2 D}{4\pi} \quad (\text{m}^2)$.

6.11 Friis Transmission Formula

- Friis transmission formula:

$$\boxed{\frac{P_{\mathrm{rec}}}{P_{\mathrm{t}}} = \frac{\xi_t \xi_r A_t A_r}{\lambda^2 R^2} = G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2},$$

  - where $\frac{P_{\mathrm{rec}}}{P_{\mathrm{t}}}$ is called the power transfer ratio.

A Appendix

A.1 Derivatives

$$\frac{d}{dt}$$

$$\sin \to \cos, \quad \cos \to -\sin, \quad \tan \to \sec^2, \quad x^n \to nx^{n-1}, \quad e^{ax} \to ae^{ax}, \quad \ln x \to \frac{1}{x}$$

A.2 Integrals

$$\int$$

$$\sin \to -\cos, \quad \cos \to \sin, \quad \sec^2 \to \tan, \quad x^n \to \frac{1}{n}x^{n+1}, \quad e^{ax} \to \frac{1}{a}e^{ax}, \quad \frac{1}{x} \to \ln|x|$$

$$\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta = \left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{1}{3}$$

$$\int_0^{\pi} \sin^3\theta \, d\theta = \left[\frac{cos^3\theta}{3} - \cos\theta\right]_0^{\pi} = \frac{4}{3}$$

$$\int \sin\theta\cos\theta \, d\theta = \frac{\sin^2\theta}{2} + C$$

A.3 Integration by Parts

$$\int u \, dv = uv - \int v du$$

Priority of choosing u:

1. Logarithmic terms
2. Inverse trigonometric terms
3. Algebraic terms
4. Trigonometric terms
5. Exponential terms

A.4 Phasors

- Identities: $\sin x = \cos\left(\frac{\pi}{2} - x\right); \quad \cos(-x) = \cos x$
- Voltage across capacitor, $v_C(t)$:

$$\int i(t) = \text{Re}\left[\frac{\widetilde{I}e^{j\omega t}}{j\omega}\right] = \text{Re}\left[\frac{-j\widetilde{I}e^{j\omega t}}{\omega}\right] = \text{Re}\left[\frac{-\widetilde{I}j\cos\omega t + \widetilde{I}\sin\omega t}{\omega}\right] = \frac{\widetilde{I}\sin\omega t}{\omega}$$

$$\Rightarrow v_C(t) = \frac{\int i(t)}{C} = \frac{\widetilde{I}\sin\omega t}{\omega C}$$

- Voltage across inductor, $v_L(t)$:

$$\frac{di(t)}{dt} = \text{Re}\left[j\omega\widetilde{I}e^{j\omega t}\right] = \text{Re}\left[j\omega\widetilde{I}\cos\omega t - \widetilde{I}\omega\sin\omega t\right] = -\widetilde{I}\omega\sin\omega t$$

$$\Rightarrow v_L(t) = L\frac{di(t)}{dt} = -L\omega\widetilde{I}\sin\omega t$$