Midterms Revision Guide

30.102 Electromagnetics & Applications, Term 5 2020

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1 W1: Waves and Phasorss

1.1 Waves

- At high f, the phase plays an important role.
- Carry energy through vacuum.
  - $\checkmark$: EM waves, $\times$: mechanical waves
- Types of waves:
  - Transverse waves (displacement $\perp$ direction of wave travel) e.g. EM waves
  - Longitudinal waves (displacement || direction of wave travel) e.g. sound waves
  - Surface waves (circular motion) e.g. water, ocean waves

1.2 Time-varying Sinusoidal Waves

- General equation:

$$\boxed{y(x,t) = Ae^{-\alpha x} \cos(\omega t - \beta x + \phi_0)}$$

- Parameters:

1. Amplitude, A: maximum extent of vibration
2. Wavelength, λ: distance between 2 points with same displacement
3. Time period, T: amount of time for particle to travel back to same position
4. Frequency, f: number of periods in 1 second

$$f = \frac{1}{T} \quad \text{(Hz)}$$

5. Phase, ϕ:

$$\phi = \frac{2\pi t}{T} - \frac{2\pi x}{\lambda} + \phi_0, \text{ where } \phi_0 \text{ is the initial/reference phase}$$

6. Initial/reference phase, ϕ_0:

$$\phi_0 = \arccos\left(\frac{y(0,0)}{A}\right)$$

   - $\phi_0 < 0$: phase leading
   - $\phi_0 > 0$: phase lagging

7. Phase/propagation velocity, u_p:

$$u_p = f\lambda = \frac{\omega}{\beta} \quad \text{(m/s)}$$

8. Angular frequency/velocity, ω:

$$\omega = 2\pi f = \frac{2\pi}{T} \quad \text{(rad/s)}$$

9. Wavenumber, β:

$$\beta = \frac{2\pi}{\lambda} \quad \text{(rad/m)}$$

10. Direction of propagation:

    - Positive x-direction: $y(x,t) = Ae^{-\alpha x}\cos(\omega t - \beta x + \phi_0)$
        - Signs of ωt and βx are opposite
    - Negative x-direction: $y(x,t) = Ae^{-\alpha x}\cos(\omega t + \beta x + \phi_0)$
        - Signs of ωt and βx are the same

11. Attenuation factor, $e^{-\alpha x}$: factor that amplitude decreases by

    - Attenuation constant of medium, α
    - Units: Neper per meter (Np/m)

$$\boxed{\text{Solve for } \alpha \text{ using } \frac{Ae^{-\alpha x_1}}{Ae^{-\alpha x_1}} = \frac{y_1}{y_2}.}$$

1.3 Complex Numbers

- Rectangular form: $z = x + jy$ (easier to perform addition and subtraction)
    - where $x = \text{Re}(z)$; $y = \text{Im}(z)$
- Polar form: $z = |z|e^{j\theta} = |z|\underline{/\theta}$ (easier to perform multiplication and division)
    - $|z|$: magnitude of z; θ: phase angle; $\underline{/\theta}$: shorthand for $e^{j\theta}$
- **Euler's identity**: $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$

$$\text{Similarly, } \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}; \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

1.4 Phasors

- Any cosinusoidally time-varying function $z(t)$ can be expressed as

$$z(t) = \text{Re}\left[\widetilde{Z}e^{j\omega t}\right], \text{ where } \widetilde{Z} \text{ is the phasor of the instantaneous function } z(t).$$

- Time domain voltage across resistors, inductors and capacitors
    - Resistors: $v_R(t) = Ri(t) \quad \Leftrightarrow \quad v_R(t) = R \cdot \text{Re}\left[\widetilde{I}e^{j\omega t}\right]$
    - Inductors: $v_L(t) = L\frac{di(t)}{dt} \quad \Leftrightarrow \quad v_L(t) = L \cdot \text{Re}\left[j\omega\widetilde{I}e^{j\omega t}\right]$
    - Capacitors: $v_C(t) = \frac{1}{C}\int i(t)\,dt \quad \Leftrightarrow \quad v_C(t) = \frac{1}{C} \cdot \text{Re}\left[\frac{\widetilde{I}}{j\omega}e^{j\omega t}\right]$

1.5 Phasor Analysis

1. Adopt a cosine reference.
   - e.g. $v_S(t) = V_0 \sin(\omega t + \phi_0) = V_0 \cos\left(\frac{\pi}{2} - \omega t + \phi_0\right) = V_0 \cos\left(\omega t + \phi_0 - \frac{\pi}{2}\right)$
2. Express time-dependent variables as phasors.
3. Write equation in phasor form.
4. Solve phasor domain equation.
5. Find instantaneous value.

$$i(t) = \mathrm{Re}\left[\widetilde{I} e^{j\omega t}\right]$$

1.6 Impedance

- Ratio of phasor voltage across element to phasor current through element, $Z = \dfrac{\widetilde{V}}{\widetilde{I}}$
- Impedance of resistors, inductors and capacitors
  - Resistor: $Z_R = R$
  - Inductor: $Z_L = j\omega L$
  - Capacitor: $Z_C = \dfrac{1}{j\omega C}$

2 W2: Antennas I

Definition of antenna: a transducer that converts guided wave on TL ⇔ EM wave in free space

2.1 Properties

1. Reciprocity: Same radiation pattern for reception and transmission on 3 conditions:

    (a) Materials used for the antennas are linear.

    (b) Wave propogation medium is linear.

    (c) Transmit and receive modes of antenna are polarization matched.

2. Transmission Line Equivalent Circuit

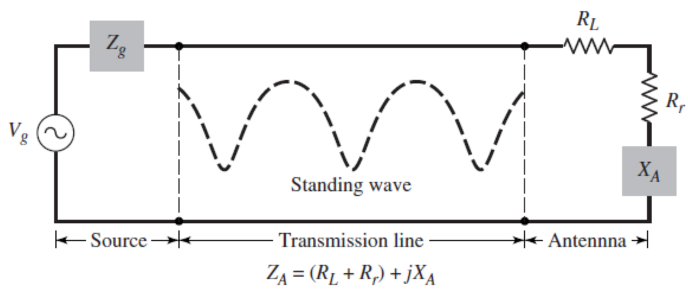


Figure 1: Transmission-line equivalent circuit of antenna from Balanis' *Antenna Theory* (2016)

3. Radiation efficiency, ξ

$$\xi = \frac{P_{\text{rad}}}{P_{\text{t}}}$$

    ◦ where P_{rad} is the radiated power,

    ◦ and P_{t} is the transmitter power.

2.2 Antenna Field Regions

Letting D be the largest dimension of the antenna and λ be the wavelength, they are:

1. Reactive near-field region: at a distance R, where $0 < R < 0.62\sqrt{\frac{D^3}{\lambda}}$

2. Radiating near-field (Fresnel) region: at distance R, where $0.63\sqrt{\frac{D^3}{\lambda}} < R < \frac{2D^2}{\lambda}$

3. Far-field (Fraunhofer) region: at a distance R, where $R > \frac{2D^2}{\lambda}$

2.3 Far Field Approximation

- Near radiation source: spherical wavefronts

- Far field: approximated as plane waves

2.4 Radiation Mechanism

- To create radiation, there must be:
  - either a time-varying current,
  - or an acceleration/deceleration of charge.
- Electric charges needed to excite fields but not sustain them.

2.5 Hertzian Dipole

- A thin, linear conductor with a length ℓ, where $\ell < \dfrac{\lambda}{50}$
- Current $i(t)$ is constant along the wire
- Current $i(t) = 0$ at the ends of the wire

3 W3: Transmission Lines

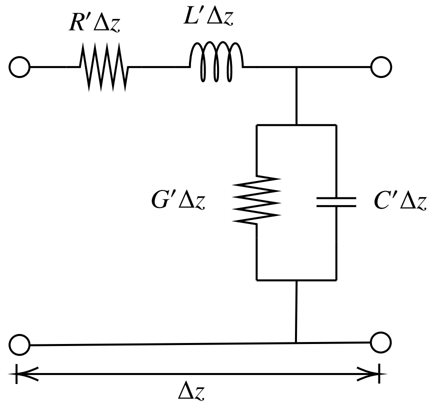
3.1 Unbounded & Guided Waves

- Unbounded waves
  - Propagate in a homogeneous medium
  - No obstacles
  - No material interface
- Guided waves
  - Propagate along a material surface/structure
  - e.g. coax cable <30 GHz, waveguide 5-100 GHz

3.2 Transmission Lines

- Can be classified into 2 types:
  1. Transverse electromagnetic (TEM) transmission lines
     - Electric and magnetic fields transverse to direction of propagation
     - Non-transverse fields negligible
     - Common feature: 2 || conducting surfaces
     - e.g. coaxial line, two-wire line, parallel-plate line, strip line, microstrip line, coplanar waveguide, etc.
  2. Higher-order transmission lines
     - ≥ 1 significant field component in direction of propagation
     - e.g. rectangular waveguide, optical fibre, etc.

3.3 Distributed Transmission Line Model



- 4 transmission line parameters:
  1. *R'*: Resistance per unit length in Ω/m
  2. *L'*: Inductance per unit length in H/m
  3. *G'*: Conductance per unit length in S/m
  4. *C'*: Capacitance per unit length in F/m
- Geometric parameters
  - Coaxial line
    - a: Outer radius of inner conductor in m
    - b: Inner radius of outer conductor in m
  - Two-wire line
    - d: Diameter of each line in m
    - D: Spacing between the centers of the wires in m
  - Parallel-plate line
    - w: Width of each plate in m
    - h: Thickness of insulation between plates in m
- Constitutive parameters
  - Conductors
    - μ_c: Magnetic permeability of conductors
    - σ_c: Electrical conductivity of conductors
  - Insulators
    - ϵ: Electrical permittivity of insulating material
    - μ: Magnetic permeability of insulating material
    - σ: Electrical conductivity of insulating material

3.3.1 Surface Resistance and Other Useful Relations

- Surface Resistance, $R_s = \sqrt{\frac{\pi f \mu_c}{\sigma_c}}$
- Perfect conductor: $\sigma_c = \infty, \Rightarrow R_s$ and R' = 0.
- Perfect dielectric: $\sigma = 0, \Rightarrow G' = 0$.
- Air line: $\epsilon = \epsilon_0, \mu = \mu_0, \sigma = 0$, G' = 0.
- All TEM transmission lines have the following relationships:

$$L'C' = \mu\epsilon$$
$$\frac{G'}{C'} = \frac{\sigma}{\epsilon}$$

3.3.2 Summary for Distributed Transmission Line Model

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln\frac{b}{a}$	$\frac{\mu}{\pi}\ln\left[\frac{D}{d}+\sqrt{\left(\frac{D}{d}\right)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln\left(\frac{b}{a}\right)}$	$\frac{\pi\sigma}{\ln\left[\frac{D}{d}+\sqrt{\left(\frac{D}{d}\right)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)}$	$\frac{\pi\epsilon}{\ln\left[\frac{D}{d}+\sqrt{\left(\frac{D}{d}\right)^2-1}\right]}$	$\frac{\epsilon w}{h}$	F/m

3.4 Transmission Line Equations

$$\boxed{\begin{aligned} -\frac{\partial v(z,t)}{\partial z} &= R'\, i(z,t) + L'\,\frac{\partial i(z,t)}{\partial t} \quad \Leftrightarrow \quad -\frac{\partial \widetilde{V}(z)}{\partial z} = (R' + j\omega L')\widetilde{I}(z) \\ -\frac{\partial i(z,t)}{\partial z} &= G'\, v(z,t) + C'\,\frac{\partial v(z,t)}{\partial t} \quad \Leftrightarrow \quad -\frac{\partial \widetilde{I}(z)}{\partial z} = (G' + j\omega C')\widetilde{V}(z) \end{aligned}}$$

3.5 Characteristic Parameters

3.5.1 Propagation Constant

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta$$

- γ: complex propagation constant
- α: attenuation constant in Np/m
- β: phase constant in rad/m

3.5.2 Characteristic Impedance

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \text{ in units of } \Omega$$

3.5.3 Summary for Characteristic Parameters

	Propagation constant $\gamma = \alpha + j\beta$	**Phase velocity** u_p	**Characteristic impedance** Z_0
General case	$\sqrt{(R' + j\omega L')(G' + j\omega C')}$	$\frac{\omega}{\beta}$	$\sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$
Lossless (R' = G' = 0)	$\alpha = 0,\ \beta = \frac{\omega\sqrt{\epsilon_r}}{c}$	$\frac{c}{\sqrt{\epsilon_r}}$	$\sqrt{\frac{L'}{C'}}$
Lossless coaxial			$\frac{60}{\sqrt{\epsilon_r}} \ln \frac{b}{a}$
Lossless two wire			$\frac{120}{\sqrt{\epsilon_r}} \ln\left[\frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1}\right]$ If D $\gg$ d, $\approx \frac{120}{\sqrt{\epsilon_r}} \ln \frac{2D}{d}$
Lossless || plate			$\frac{120\pi}{\sqrt{\epsilon_r}} \frac{h}{w}$

3.6 Dispersion & Guided Wavelength

- Dispersion: phase velocity of wave depends on its frequency
  - ✓: Dispersive media, ×: non-dispersive media
  - Degree of distortion $\propto$ length of dispersive line
- Guided wavelength, λ_g: distance between two equal phase planes along the transmission line

$$\lambda_g = \frac{c}{f\sqrt{\epsilon_r}} = \frac{\lambda_0}{\sqrt{\epsilon_r}}$$

3.7 Standing Waves & Standing Wave Pattern

- Standing wave: formed when two waves on transmission line propagating in opposite directions
- Standing wave patterns: sinusoidal patterns caused by interference of two travelling waves

$$\widetilde{V}(z) = \widetilde{V}^{+}(z) + \widetilde{V}^{-}(z) = V_0^{+} e^{-\gamma z} + V_0^{-} e^{\gamma z}$$
$$\widetilde{I}(z) = \widetilde{I}^{+}(z) + \widetilde{I}^{-}(z) = I_0^{+} e^{-\gamma z} + I_0^{-} e^{\gamma z}$$

  - Using either one of the two axes:
    - z-axis: load at $z = 0$, generator at $z = -l$
    - d-axis: load at $d = 0$, generator at $d = l$
  - Affected by:
    - Relation of $\widetilde{V}^{-}(z)$ and $\widetilde{V}^{+}(z)$ at $z = 0$
    - Relation of V_0^{+} and V_0^{-}
- Magnitude of voltage, $|\widetilde{V}(d)| = |V_0^{+}| \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d - \theta_r)\right]^{1/2}$
- Magnitude of current, $|\widetilde{I}(d)| = \dfrac{|V_0^{+}|}{Z_0} \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d - \theta_r)\right]^{1/2}$
- Voltage maximum, d_{max}: distance from load where $|\widetilde{V}(d)|$ is a maximum

$$d_{max} = \frac{\theta_r}{4\pi} + \frac{n\lambda}{2}, \begin{cases} n = 1, 2, \ldots & \text{if } \theta_r < 0 \\ n = 0, 1, 2, \ldots & \text{if } \theta_r \geq 0 \end{cases}$$

- Voltage minimum, d_{min}: distance from load where $|\widetilde{V}(d)|$ is a minimum

$$d_{min} = \begin{cases} d_{max} + \dfrac{\lambda}{4}, & \text{if } d_{max} < \dfrac{\lambda}{4} \\ d_{max} - \dfrac{\lambda}{4}, & \text{if } d_{max} \geq \dfrac{\lambda}{4} \end{cases}$$

- Voltage standing wave ratio (VSWR), $S = \dfrac{|\widetilde{V}|_{max}}{|\widetilde{V}|_{min}} = \dfrac{1 + |\Gamma|}{1 - |\Gamma|}$ (dimensionless)

3.8 Reflection Coefficient

- Ratio of reflected and incident voltages at the load. $\dfrac{I_0^-}{I_0^+} = -\dfrac{V_0^-}{V_0^+} = -\Gamma_L$

- A complex quantity. $\Gamma_L = |\Gamma_L| e^{j\theta_r} = \dfrac{V_0^-}{V_0^+} = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{z_L - 1}{z_L + 1}$ (dimensionless)

- Normalized load impedance, $z_L = \dfrac{Z_L}{Z_0}$ (dimensionless)

Load	$\lvert\Gamma\rvert$	θ_r
$Z_L = (r + jx)Z_0$	$\sqrt{\dfrac{(r-1)^2 + x^2}{(r+1)^2 + x^2}}$	$\tan^{-1}\left(\dfrac{x}{r-1}\right) - \tan^{-1}\left(\dfrac{x}{r+1}\right)$
$Z_L = Z_0$	0	NA
$Z_L =$ short-circuit	1	$\pm 180^\circ$
$Z_L =$ open-circuit	1	0
$Z_L = jX = j\omega L$ (capacitor)	1	$\pm 180^\circ - 2\tan^{-1}(X)$
$Z_L = jX = -\dfrac{j}{\omega C}$	1	$\pm 180^\circ + 2\tan^{-1}(X)$

3.9 Wave Impedance

$$Z(d) = \frac{\widetilde{V}(d)}{\widetilde{I}(d)} = Z_0 \left(\frac{1 + \Gamma_d}{1 - \Gamma_d}\right)$$

- Phase-shifted voltage reflection coefficient $\Gamma_d = |\Gamma| e^{j(\theta_r - 2\beta d)}$

3.10 Input Impedance

$$\begin{aligned} Z_{\text{in}} &= Z_0 \left(\frac{z_L \cos\beta l + j \sin\beta l}{\cos\beta l + j z_L \sin\beta l}\right) \\ &= Z_0 \left(\frac{z_L + j\tan\beta l}{1 + j z_L \tan\beta l}\right) \\ V_0^+ &= \left(\frac{\widetilde{V}_g Z_{\text{in}}}{Z_g + Z_{\text{in}}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \end{aligned}$$

3.11 Short-Circuited Line

- Γ = -1, $S = \infty$
- $\widetilde{V}_{\text{sc}}(d) = V_0^+ \left(e^{j\beta d} - e^{-j\beta d}\right) = 2jV_0^+ \sin\beta d$
- $\widetilde{I}_{\text{sc}}(d) = \dfrac{V_0^+}{Z_0} \left(e^{j\beta d} + e^{-j\beta d}\right) = \dfrac{2V_0^+}{Z_0} \cos\beta d$
- $Z_{\text{sc}}(d) = \dfrac{\widetilde{V}_{\text{sc}}(d)}{\widetilde{I}_{\text{sc}}(d)} = jZ_0 \tan\beta d$
- By choosing ℓ, can make L and C of any reactance.
  - If $\tan\beta l \geq 0,\ jZ_0 \tan\beta l = j\omega L_{\text{eq}}$.
  - If $\tan\beta l \leq 0,\ jZ_0 \tan\beta l = \dfrac{1}{j\omega C_{\text{eq}}}$.

3.12 Open-Circuited Line

- Γ = 1, $S = \infty$
- $\widetilde{V}_{\text{oc}}(d) = V_0^+ \left(e^{j\beta d} + e^{-j\beta d}\right) = 2V_0^+ \cos\beta d$
- $\widetilde{I}_{\text{oc}}(d) = \dfrac{V_0^+}{Z_0} \left(e^{j\beta d} - e^{-j\beta d}\right) = \dfrac{2jV_0^+}{Z_0} \sin\beta d$
- $Z_{\text{oc}}(d) = \dfrac{\widetilde{V}_{\text{oc}}(d)}{\widetilde{I}_{\text{oc}}(d)} = -jZ_0 \cot\beta d$
- By choosing ℓ, can make L and C of any reactance.
  - If $Z_{in}^{\text{oc}} \geq 0, -jZ_0 \cot\beta l = j\omega L_{\text{eq}}$.
  - If $Z_{in}^{\text{oc}} \geq 0, -jZ_0 \cot\beta l = \dfrac{1}{j\omega C_{\text{eq}}}$.

3.13 Measuring Characteristic Impedance and Phase Constant

$$Z_0 = \sqrt{Z_{\text{in}}^{\text{sc}} Z_{\text{in}}^{\text{oc}}}$$

$$\tan\beta l = \sqrt{\frac{-Z_{\text{in}}^{\text{sc}}}{Z_{\text{in}}^{\text{oc}}}} \quad \text{can be used to find } \beta$$

3.14 Quarter and Half Wavelength TLs

- For a half wavelength TL, where $l = \dfrac{n\lambda}{2}$, $\boxed{Z_{\text{in}} = Z_{\text{L}}.}$
- For a quarter wavelength TL, where $l = \dfrac{\lambda}{4} + \dfrac{n\lambda}{2}$, $\boxed{Z_{\text{in}} = \dfrac{Z_0^2}{Z_{\text{L}}}.}$

3.15 Instantaneous Power of a Lossless TL

- $v(d,t) = |V_0^+| \left[\cos(\omega t + \beta d + \phi^+) + |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_r)\right]$
- $i(d,t) = \dfrac{|V_0^+|}{Z_0} \left[\cos(\omega t + \beta d + \phi^+) - |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_r)\right]$
- $P(d,t) = \dfrac{|V_0^+|^2}{Z_0} \left[\cos^2(\omega t + \beta d + \phi^+) - |\Gamma|^2 \cos^2(\omega t - \beta d + \phi^+ + \theta_r)\right]$
    - The instantaneous power oscillates at twice the rate of the voltage or current.
- Incident power $P^i(d,t) = \dfrac{|V_0^+|^2}{2Z_0} \left[1 + \cos(2\omega t + 2\beta d + 2\phi^+)\right]$
- Reflected power $P^r(d,t) = -|\Gamma|^2 \dfrac{|V_0^+|^2}{2Z_0} \left[1 + \cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_r)\right]$

3.16 Time-Average Power of a Lossless TL

- Time-average incident power, $P^{\mathrm{i}}_{\mathrm{av}} = \dfrac{|V_0^+|^2}{2Z_0}$ is measured in W.
- Time-average reflected power, $P^{\mathrm{r}}_{\mathrm{av}} = -|\Gamma|^2 \dfrac{|V_0^+|^2}{2Z_0} = -|\Gamma|^2 P^{\mathrm{i}}_{\mathrm{av}}$ is measured in W.
    - Average reflected power is average incident power diminished by $|\Gamma|^2$.

4 W5: Matching Networks

4.1 Concept of Matching Networks

- Eliminate reflections for waves incident from source
- All power goes to load
- May consist of lumped elements i.e. capacitors and inductors, or sections of TLs

4.2 Matching Networks in Series

1. In-series $\lambda/4$ transformer inserted in front of Z_L (if Z_L is real)
2. In-series $\lambda/4$ transformer inserted at $d = d_{\text{max}}$ or $d = d_{\text{min}}$ (if Z_L is complex)

4.3 Matching Networks in Parallel

1. In-parallel insertion of capacitor at distance d_1
2. In-parallel insertion of inductor at distance d_2
3. In-parallel insertion of a short-circuited stub

4.4 Lumped Element Matching

- To achieve matched condition, $y_{\text{in}} = y_{\text{d}} + y_{\text{s}} + g_{\text{d}} + j(b_{\text{d}} + b_{\text{s}}) = 1 + j0$. As such,

$$\boxed{\begin{aligned} g_{\text{d}} &= 1 && \text{(real condition)} \\ b_{\text{d}} &= -b_{\text{s}} && \text{(imaginary condition)} \end{aligned}}$$

5 W5: Smith Chart

1. Complex unit circle
2. Concentric r_{L} circles
3. x_{L} curves
4. Wavelengths toward generator (WTG) scale (clockwise)
5. Wavelengths toward load (WTL) scale (counterclockwise)

5.1 How to Use the Smith Chart

- Read off Γ_{L} from z_{L}: $\Gamma_{\mathrm{L}} = |\Gamma_{\mathrm{L}}|e^{j\theta_r}$
- Constant Γ_L/SWR circle, move towards WTG/WTL directions
- Read off to find distance d
- Find Y from Z
  - y opposite from z in SWR circle
  - $\Rightarrow Y_{\mathrm{L}} = y_{\mathrm{L}} \cdot Y_0$ in units of S

6 W6: Vector Algebra

6.1 Basic Vector Operations

- Unit vector $\hat{a} = \dfrac{\overrightarrow{A}}{|\overrightarrow{A}|} = \dfrac{\hat{x}A_x + \hat{y}A_y + \hat{z}A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$
- Position vector: $\overrightarrow{R_1} = \overrightarrow{OP_1}$
- Distance vector: $\overrightarrow{R_{12}} = \overrightarrow{P_1P_2} = \overrightarrow{R_2} - \overrightarrow{R_1}$
  - Distance $d = |\overrightarrow{R_{12}}|$
- Vector addition
  - Done graphically using parallelogram or head-to-tail method.
  - Commutative: $\overrightarrow{A} + \overrightarrow{B} = \overrightarrow{B} + \overrightarrow{A}$

6.2 Dot Product

$$\overrightarrow{A} \cdot \overrightarrow{B} = AB\cos\theta_{\mathrm{AB}}, \quad \theta_{\mathrm{AB}} = \cos^{-1}\left(\frac{\overrightarrow{A} \cdot \overrightarrow{B}}{\sqrt{\overrightarrow{A} \cdot \overrightarrow{A}} \cdot \sqrt{\overrightarrow{B} \cdot \overrightarrow{B}}}\right)$$

- Commutative and distributive.

$$\overrightarrow{A} \cdot \overrightarrow{B} = \overrightarrow{B} \cdot \overrightarrow{A} \text{ (commutative)}$$
$$\overrightarrow{A} \cdot (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{A} \cdot \overrightarrow{C} \text{ (distributive)}$$

- Can be used to find magnitude of vector.

$$A = |\overrightarrow{A}| = \sqrt{\overrightarrow{A} \cdot \overrightarrow{A}}$$

6.3 Cross Product

$$\overrightarrow{A} \times \overrightarrow{B} = \hat{n}\, AB\sin\theta_{\mathrm{AB}},\ \hat{n} \text{ in direction of right hand rule}$$

- Anti-commutative and distributive.

$$\overrightarrow{A} \times \overrightarrow{B} = -\overrightarrow{B} \times \overrightarrow{A} \text{ (anti-commutative)}$$
$$\overrightarrow{A} \times (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \times \overrightarrow{B} + \overrightarrow{A} \times \overrightarrow{C} \text{ (distributive)}$$

- Can be re-expressed as a determinant.

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

6.4 Scalar Triple Product

$$\overrightarrow{A} \cdot (\overrightarrow{B} \times \overrightarrow{C}) = \overrightarrow{B} \cdot (\overrightarrow{C} \times \overrightarrow{A}) = \overrightarrow{C} \cdot (\overrightarrow{A} \times \overrightarrow{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

6.5 Vector Triple Product

$$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C})$$

- Not associative.

$$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C}) \neq (\overrightarrow{A} \times \overrightarrow{B}) \times \overrightarrow{C}$$

- Follows the ”bac-cab” rule.

$$\overrightarrow{A} \times (\overrightarrow{B} \times \overrightarrow{C}) = \overrightarrow{B}(\overrightarrow{A} \cdot \overrightarrow{C}) - \overrightarrow{C}(\overrightarrow{A} \cdot \overrightarrow{B})$$

A Appendix

A.1 Derivatives

$$\frac{d}{dt}$$

$$\sin \to \cos, \quad \cos \to -\sin, \quad \tan \to \sec^2, \quad x^n \to nx^{n-1}, \quad e^{ax} \to ae^{ax}, \quad \ln x \to \frac{1}{x}$$

A.2 Integrals

$$\int$$

$$\sin \to -\cos, \quad \cos \to \sin, \quad \sec^2 \to \tan, \quad x^n \to \frac{1}{n}x^{n+1}, \quad e^{ax} \to \frac{1}{a}e^{ax}, \quad \frac{1}{x} \to \ln|x|$$

A.3 Integration by Parts

$$\int u\, dv = uv - \int v du$$

Priority of choosing u:

1. Logarithmic terms
2. Inverse trigonometric terms
3. Algebraic terms
4. Trigonometric terms
5. Exponential terms

A.4 Phasors

- Identities: $\sin x = \cos\left(\frac{\pi}{2} - x\right); \quad \cos(-x) = \cos x$
- Voltage across capacitor, $v_C(t)$:

$$\begin{aligned}\int i(t) &= \mathrm{Re}\left[\frac{\widetilde{I}e^{j\omega t}}{j\omega}\right] = \mathrm{Re}\left[\frac{-j\widetilde{I}e^{j\omega t}}{\omega}\right] = \mathrm{Re}\left[\frac{-\widetilde{I}j\cos\omega t + \widetilde{I}\sin\omega t}{\omega}\right] \\ &= \frac{\widetilde{I}\sin\omega t}{\omega} \\ \Rightarrow v_C(t) &= \frac{\int i(t)}{C} = \frac{\widetilde{I}\sin\omega t}{\omega C}\end{aligned}$$

- Voltage across inductor, $v_L(t)$:

$$\begin{aligned}\frac{di(t)}{dt} &= \mathrm{Re}\left[j\omega\widetilde{I}e^{j\omega t}\right] = \mathrm{Re}\left[j\omega\widetilde{I}\cos\omega t - \widetilde{I}\omega\sin\omega t\right] = -\widetilde{I}\omega\sin\omega t \\ \Rightarrow v_L(t) &= L\frac{di(t)}{dt} = -L\omega\widetilde{I}\sin\omega t\end{aligned}$$