Estimation Effects of Various Demographic Forecasting Techniques in Japan Using an Overlapping Generations Model [*]

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Abstract

First paragraph in the abstract

Second paragraph in the abstract

keywords: Overlapping Generations Model, Demographic Transition, Japan Calibration

1 Introduction

Research question: how does changing the population forecasting model alter future tax revenue through its effect on labor supply?

Literature review: primarily building off Kei and my previous paper. That paper uses as inspiration Muto et. al 2016, who carry out an overlapping generations analysis of the effects of these demographic shifts on long-run aggregate outcomes for Japan. They find that decreasing fertility rates contribute to decreasing aggregate output, while increasing longevity counteracts this by increasing aggregate output. However, this increase is not enough to create a net positive effect. Overall, their simulation predicts negative effects on aggregate outputs during the next few decades.

There are many other OLG models looking at the Japanese demographic transition but that is the most recently published paper. These papers almost entirely use the National Institute of Population and Social Security Research fertility/mortality/population forecasts. Some use UN estimates.

2 Model

Simpler OLG than that used in Kei and my previous paper.

3 Data

Same data as Kei and my previous paper. Fertility data from the Human Fertility Collection. Mortality and population data from the Japanese Mortality Database. Frisch elasticity of 0.5 from Muto et. al. Wage data by age from Basic Survey on Wage Structure. Labor force participation rates by age from Labour Force Survey. Tax rates from Tax Guide Book.

4 Results

No OLG results yet. See Appendix A for demographic transition forecasts.

5 Conclusion

Extensions: endogenous fertility.

References

DeBacker, Jason and Richard W. Evans, *OG-USA: Documentation for the Large-Scale Dynamic General Equilibrium Overlapping Generations Model for U.S. Policy Analysis* 2018.

Evans, Richard W., *Simple OG Model with Non-constant Demographic Dynamics and Productivity Growth* 2020.

Human Fertility Collection, "Japan - ASFR and CPFR, Standardized Age Scale," 2018. Data retrieved from Max Planck Institute for Demographic Research, https://www.fertilitydata.org/cgi-bin/country.php?code=jpn.

Irizawa, Kei and Adam Oppenheimer, "Macroeconomic Impact of Population Aging in Japan Using an Overlapping Generations Model," 2019.

Japanese Mortality Database, "Births and Deaths," 2018. Data retrieved from National Institute of Population and Social Security Research, http://www.ipss.go.jp/p-toukei/JMD/index-en.asp.

A Appendix

A.1 Overlapping Generations Model

We use the overlapping generations model described in Evans (2020). The model includes households and firms, endogenous labor, and exogenous productivity growth. Demographics are exogenous and fertility, mortality, and immigration rates can evolve over time. As a result of including mortality rates, the model includes unintended bequests. The model also incorporates a warm bequest motive to introduce an implicit borrowing constraint. A detailed description of the model follows.

A.1.1 Overlapping Generations Economic Model

A.1.1.1 Households

The consumption side of the economy is given by identical households. The measure of all households of age s in period t is denoted by $\omega_{s,t}$. Households live at most $E+S$ periods and become economically relevant at age $E+1$. Households are endowed with a measure of time $\tilde{l}$ to distribute between labor ($n_{s,t} \in [0,\tilde{l}]$) and leisure ($l_{s,t} \in [0,\tilde{l}]$), where $n_{s,t} + l_{s,t} = \tilde{l} \quad \forall s, t$. Households then choose consumption $c_{s,t}$, labor $n_{s,t}$, and savings $b_{s+1,t+1}$ to maximize the following utility function:

$$u(c_{s,t}, n_{s,t}, b_{s+1,t+1}) \;=\; \frac{(c_{s,t})^{1-\sigma}-1}{1-\sigma} + e^{g_y t(1-\sigma)}\chi_{n,s} b \left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)\right]^{\frac{1}{v}}$$
$$+\rho_{s,t}\chi_b \frac{(b_{s+1,t+1})^{1-\sigma}-1}{1-\sigma} \quad \forall s, t \tag{A.1 eqn. 1}$$

subject to the following constraint:

$$c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} + \frac{BQ_t}{\tilde{N}_t} \quad \forall t \quad \text{and} \quad s \geq E+1 \tag{A.1 eqn. 2}$$

In the utility function, the utility of consumption is CRRA and is given by the first term on the right hand side; labor is subject to elliptical disutility as described in DeBacker and Evans (2018) and is given by the second term on the right hand side; and the warm bequest motive is given by the third term on the right hand side. We assume that savings upon entering the workforce and in the last possible period of life are zero ($b_{E+1,t} = b_{E+S+1,t} = 0$) and that $c_{s,t} \geq 0$ (this results in $c_{s,t} > 0$ in equilibrium).

In the elliptical disutility of labor term, $e^{g_y t(1-\sigma)}$ scales the utility of labor to be in units comparable to the utility of consumption. This is necessary because labor is stationary but consumption is non-stationary, meaning that their unscaled utility measures are not comparable over time. The remaining variables in the elliptical disutility of labor term are defined in DeBacker and Evans (2018).

In the warm bequest motive term, $\rho_{s,t}$ gives the probability of dying at age s in time t and χ_b gives a warm bequest motive parameter necessary for fitting data moments. The warm bequest motive provides

utility for dying with positive savings. This implicitly introduces a borrowing constraint at zero, as the marginal utility of savings becomes infinity at zero savings by the Inada condition on this term.

In the budget constraint, BQ_t gives the total sum of unintended bequests left by the population that died in the previous period.[1] This is divided by $\tilde{N}_t$ to ensure bequests are evenly distributed among the economically relevant population. We define r_t and w_t as the interest rate and wage in period t, respectively.

Further detail about solving the household's problem, such as deriving the first order conditions, solving the Euler equations, and solving the optimal paths of labor supply and savings given initial savings and the time paths of factor prices and total bequests can be read in Evans (2020).

A.1.1.2 Firms

The production side of the economy is given by a unit measure of identical, perfectly competitive firms. Firms produce following a Cobb-Douglas production function as follows:

$$Y_t = F(K_t, L_t) \equiv A(K_t)^\alpha (e^{g_y t} L_t)^{1-\alpha} \forall t \quad \alpha \in (0,1) \quad \text{and} \quad A > 0 \tag{A.1 eqn. 3}$$

where A gives the level of technology, K_t gives capital, L_t gives labor, α gives the fraction of spending on capital relative to labor, and g_y gives the growth rate of labor productivity over time.

Firms maximize the following profit function:

$$PR_t = A(K_t)^\alpha (e^{g_y t} L_t)^{1-\alpha} - (r_t + \delta)K_t - w_t L_t \quad \forall t \tag{A.1 eqn. 4}$$

where firms choose K_t and L_t, and $\delta \in [0,1]$ gives the rate of capital depreciation. Solving the first order conditions by taking derivatives with respect to capital and labor allows us to characterize the firm's optimal behavior as the following:

$$r_t = \alpha \left(\frac{Y_t}{K_t} \right) - \delta \quad \forall t \tag{A.1 eqn. 5}$$

$$w_t = (1 - \alpha) \left(\frac{Y_t}{L_t} \right) \quad \forall t \tag{A.1 eqn. 6}$$

A.1.1.3 Market Clearance

The model we consider includes three markets that must clear: the labor market, the capital market, and the goods market. By Walras' Law, we need only consider the clearance of two of these markets in order for all three to clear. The three markets clearing conditions are as follows:

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[1]We formally define bequests as $BQ_t = (1 + r_t) \sum_{s=E+2}^{E+S+1} \rho_{s-1,t-1} \omega_{s-1,t-1} b_{s,t} \quad \forall t$

Labor Market:

$$L_t = \sum_{s=E+1}^{E+S} \omega_{s,t} n_{s,t} \qquad \text{(A.1 eqn. 7)}$$

Capital Market:

$$K_t = \sum_{s=E+2}^{E+S+1} \left(\omega_{s-1,t-1} b_{s,t} + i_{s,t} \omega_{s,t-1} b_{s,t} \right) \qquad \text{(A.1 eqn. 8)}$$

Goods Market:

$$Y_t = C_t + I_t - \sum_{s=E+2}^{E+S+1} i_{s,t+1} \omega_{s,t} b_{s,t+1}$$

$$\text{where} \quad I_t \equiv K_{t+1} - (1-\delta) K_t \qquad \text{(A.1 eqn. 9)}$$

$$\text{and} \quad C_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} c_{s,t}$$

It is important to note that immigration is included in the capital and goods market clearing equations. This implies that capital held by immigrants goes into production in Japan and the capital held by emigrants leaves production in Japan. The last term in the goods market clearing equation represents net exports in this model.[2] It should be noted that we assume that the capital held by immigrant households is equal to the capital held by equivalent households within Japan.

A.1.1.4 Stationarization and Equilibrium

Please refer to Evans (2020) for a detailed description of how to stationarize the model and solve the equilibrium.

A.1.2 Demographics

We consider four demographic models. The first model assumes that demographics are constant over time. The second, third, and fourth models use the following methods to solve the steady state level of demographics. The second model assumes that fertility, mortality, and immigration are at their 2019 levels. The third model uses a parametric method described in Section A.2 to forecast fertility, mortality, and immigration over time. The fourth model uses existing packages to forecast demographics over time. For the second and third models, population evolution follows A.2 eqn. 1, where the s subscript indicates age and the t subscript indicates period. $\omega_{s,t}, f_{s,t}, \rho_{s,t} \geq 0$ represent the age- and period-specific population, fertility rate, and mortality rate. $i_{s,t}$ represents the net immigration rate. Agents live for a maximum of $E + S$ periods, where $S \geq 3$ and $E + 1$ indicates the first working year of life. Agents age $\leq E$ are modeled in population dynamics but are considered outside the economy. Population and mortality rate data comes from Japanese Mortality Database (2018). Fertility rate data comes from Human Fertility Collection (2018).

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[2]In general, this would represent the capital account portion of net exports.

$$\omega_{1,t+1} = (1 - \rho_{0,t}) \sum_{s=1}^{E+S} f_{s,t}\omega_{s,t} + i_{1,t}\omega_{1,t} \quad \forall t \tag{A.1 eqn. 10}$$

$$\omega_{s+1,t+1} = (1 - \rho_{s,t})\omega_{s,t} + i_{s+1,t}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1 \le s \le E + S - 1$$

This is a slight variation on the model used by DeBacker and Evans (2018) and Irizawa and Oppenheimer (2019). The model used in both these papers assumes fertility, mortality and immigration rates are fixed at their 2014 levels. This paper attempts to forecast changes in those variables over time.

We now define some variables that are relevant to the remaining model sections. N_t gives the total population. $g_{n,t}$ gives the population growth rate from period $t-1$ to period t. A variable accented by $\sim$ indicates that is defined for only the working age population. The formal definitions of these variables follow.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{A.1 eqn. 11}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{A.1 eqn. 12}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{A.1 eqn. 13}$$

$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{A.1 eqn. 14}$$

A.1.2.1 Fertility, Mortality, and Immigration Rates

In order to allow for a range of model years, we fit fertility and mortality rates using cubic spline interpolation. The fit of the interpolated data to the real data can be seen in Figure 1. In this case, the fertility data is for all ages 14 to 50, and we impute fertility rates of 0 for all other ages; while the mortality data is for all ages from 0 to 99, and we set mortality to 1 at age 99. The interpolated function is fit at the midpoints of the data. It should exactly overlay the midpoint of the trend in the data, as it does. While in this case we consider only 1-year bins, the model allows for any positive bin size through the use of population-weighted fertility and mortality rates.

Figure 1: Fertility and Mortality Rates by Age
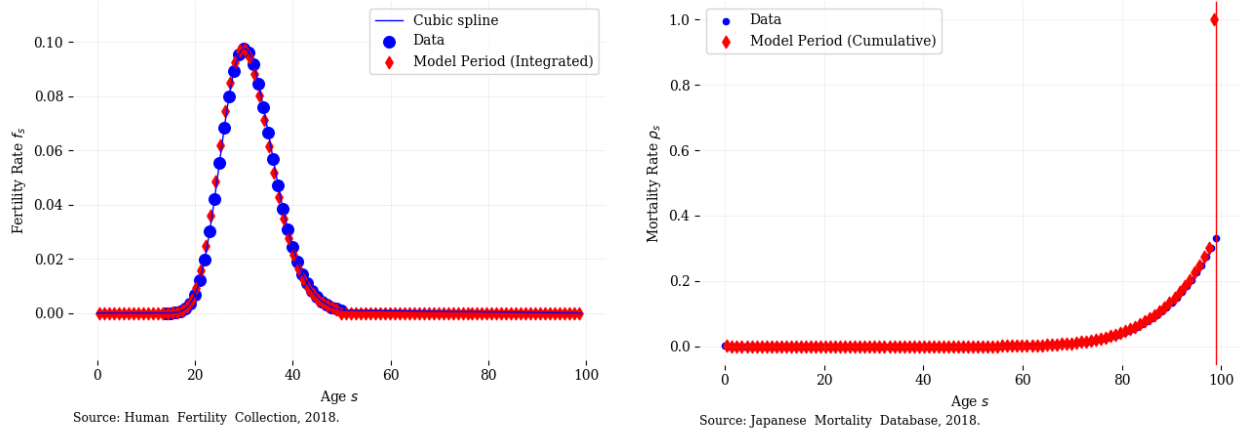
(a) Japanese Fertility Rates by Age (2020, Forecasted)   (b) Japanese Mortality Rates by Age (2020, Forecasted)

Because accurate age-specific immigration rate data is not available for Japan, we estimate it as the residual between predicted and true population from A.1 eqn. 1. The formal definition of immigration rates can be seen in ??.

$$
\begin{aligned}
i_{1,t+1} &= \frac{\omega_{1,t+1} - (1 - \rho_{0,t}) \sum_{s=1}^{E+S} f_{s,t}\omega_{s,t}}{\omega_{1,t}} \quad \forall t \\
i_{s+1,t+1} &= \frac{\omega_{s+1,t+1} - (1 - \rho_{0,t})\omega_{s,t}}{\omega_{s+1,t}} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E + S - 1
\end{aligned}
\tag{A.1 eqn. 15}
$$

A.1.2.2 Population Steady State and Transition Path

Evans (2020) proves that so long as the fertility, mortality, and immigration rates stabilize at some time T, we can solve for the unique eigenvalue and eigenvector of the matrix representation of the population evolution equation. This eigenvalue can be represented as $(1+\bar{g}_n)$, where $\bar{g}_n$ gives the steady state population growth rate. In order to ensure that the population reaches its steady state more quickly, we artificially force the population to achieve its steady state value at $t = 120$. This is achieved by a slight adjustment of immigration rates. Steady state population, adjusted immigration rates, and the population transition path for the four demographic models we consider can be seen in Figure 2.

A.2 Dynamic Demographics

The following sections discuss calibration and estimation for a parametric method to forecast fertility rates, mortality rates, immigration rates, and population.

Figure 2: Population Steady State, and Adjustments, and Transition Path

Static Demographics

Partial-Dynamic Demographics

Full-Dynamic Demographics

Alternate Full-Dynamic Demographics

(a)
Original vs. Adjusted
SS Population Distribution

(b)
Original vs. Adjusted
Immigration Rates

(c)
Population Distribution
Along Time Path
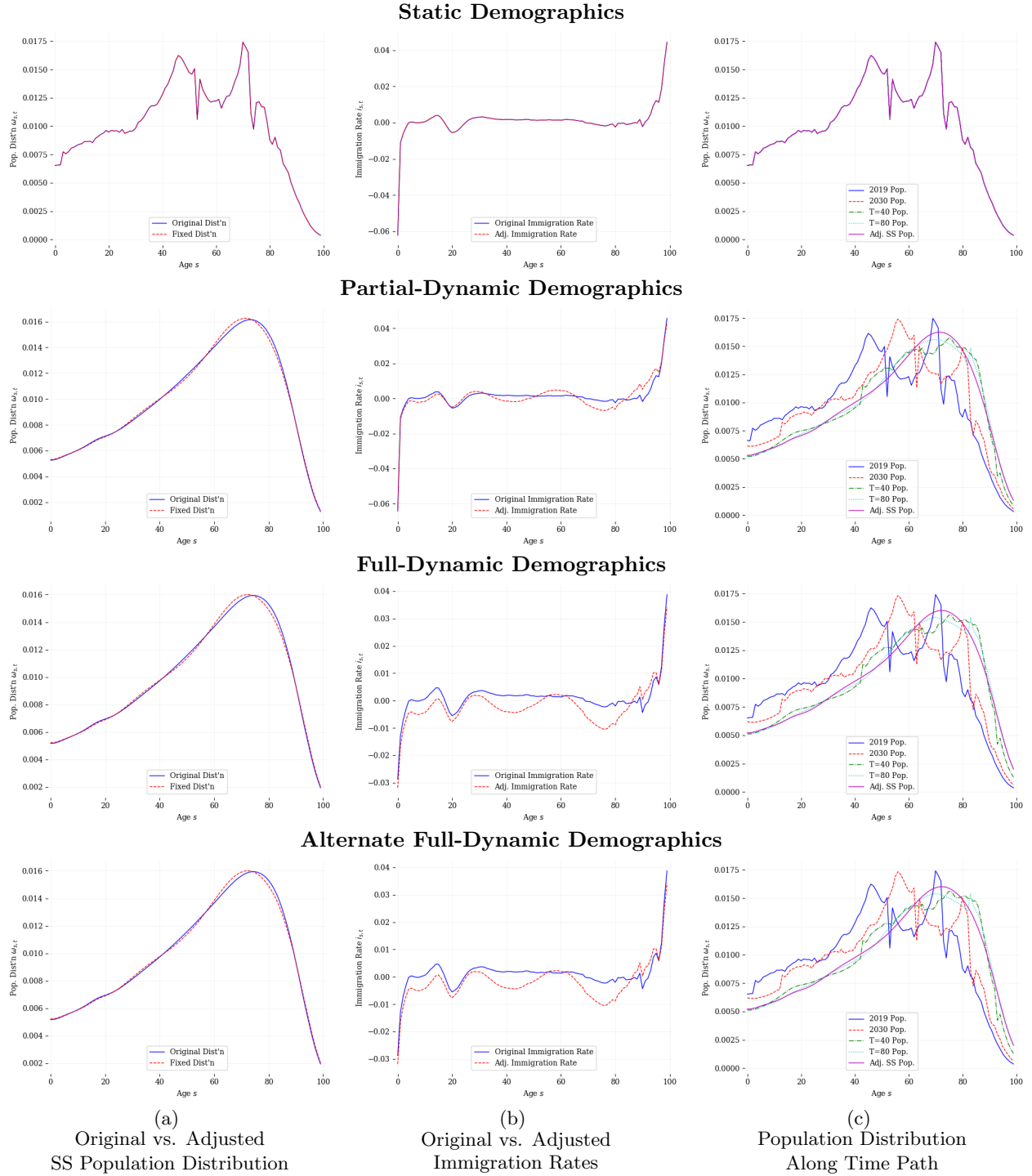
A.2.1 Fertility Rates

I begin the fertility rate estimation by fitting the yearly distributions using a generalized beta 2 distribution. The pdf for this distribution can be seen in A.1 eqn. 3:

$$f(x|a, b, p, q) = \frac{ax^{ap-1}}{b^{ap}B(p,q)\left(1 + \left(\frac{x}{b}\right)^a\right)^{p+q}}, \qquad x \in [0, \infty); a, b, p, q > 0 \qquad \text{(A.2 eqn. 1)}$$

where $B(v, w) = \int_0^1 t^{v-1}(1-t)^{w-1}dt$ is the beta function. Because the generalized beta 2 is a pdf but fertility rates are not a pdf, I also have to add in a scale parameter. I estimate fertility parameters from 1970 to 2014, the most recent year of data. I chose to begin in 1970 because the most recent trend in data seems to have started around then.

The fit of the model in selected years can be seen in Figure 3. The fit at the peak of the distribution is very close. The fit at the tails of the distribution is good at first but weakens over time - while it is possible to estimate parameters that better fit the tail for more recent data, the variance of the parameter estimates grows considerably and a trend in parameters disappears. Because of this, I chose to use parameters that had a slightly worse fit to the data in order to better estimate a trend.

Figure 3: Fertility Estimated by Generalized Beta 2
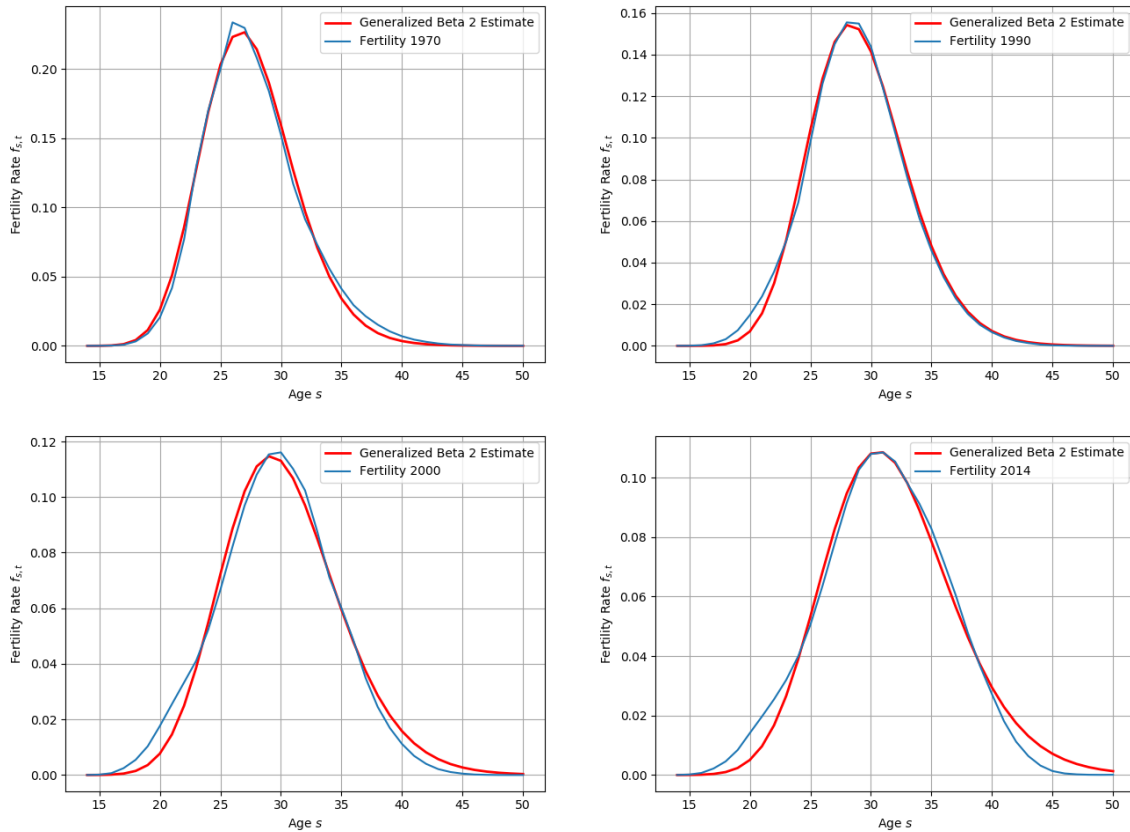
Parameter estimates and their fit over time can be seen in Figure 4. In order to ensure estimated trends converge over time, I fit these parameter estimates to logistic functions. The logistic function can be seen in A.1 eqn. 4:

$$f(x|L, k, x_0) = \frac{L}{1 + e^{-k(x-x_0)}} \qquad \text{(A.2 eqn. 2)}$$

where L represents the maximum value of the curve, k represents the steepness of the curve, and x_0 represents the midpoint of the curve.
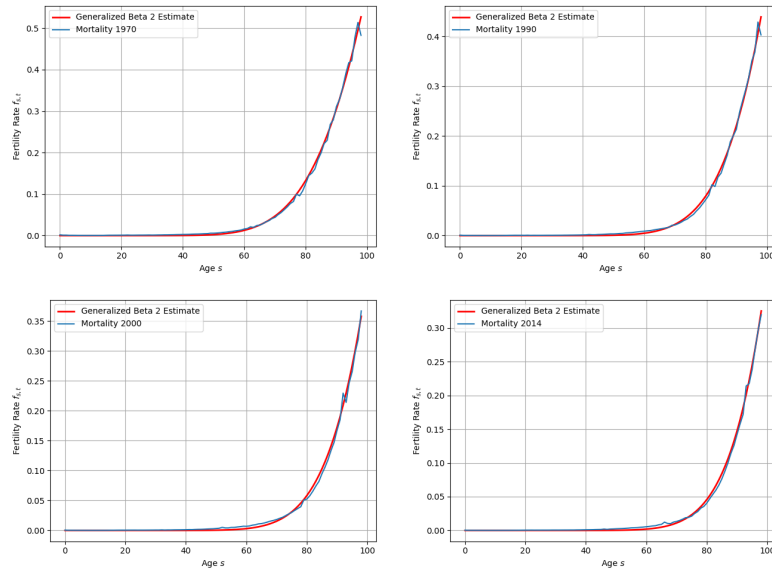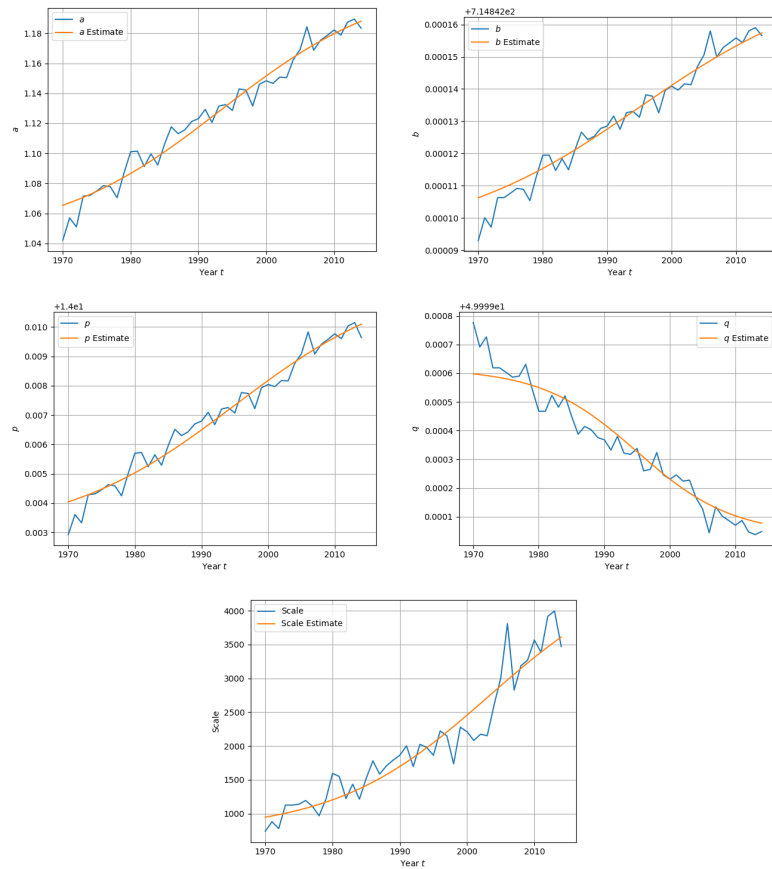
Figure 4: Fertility Generalized Beta 2 Parameter Estimates
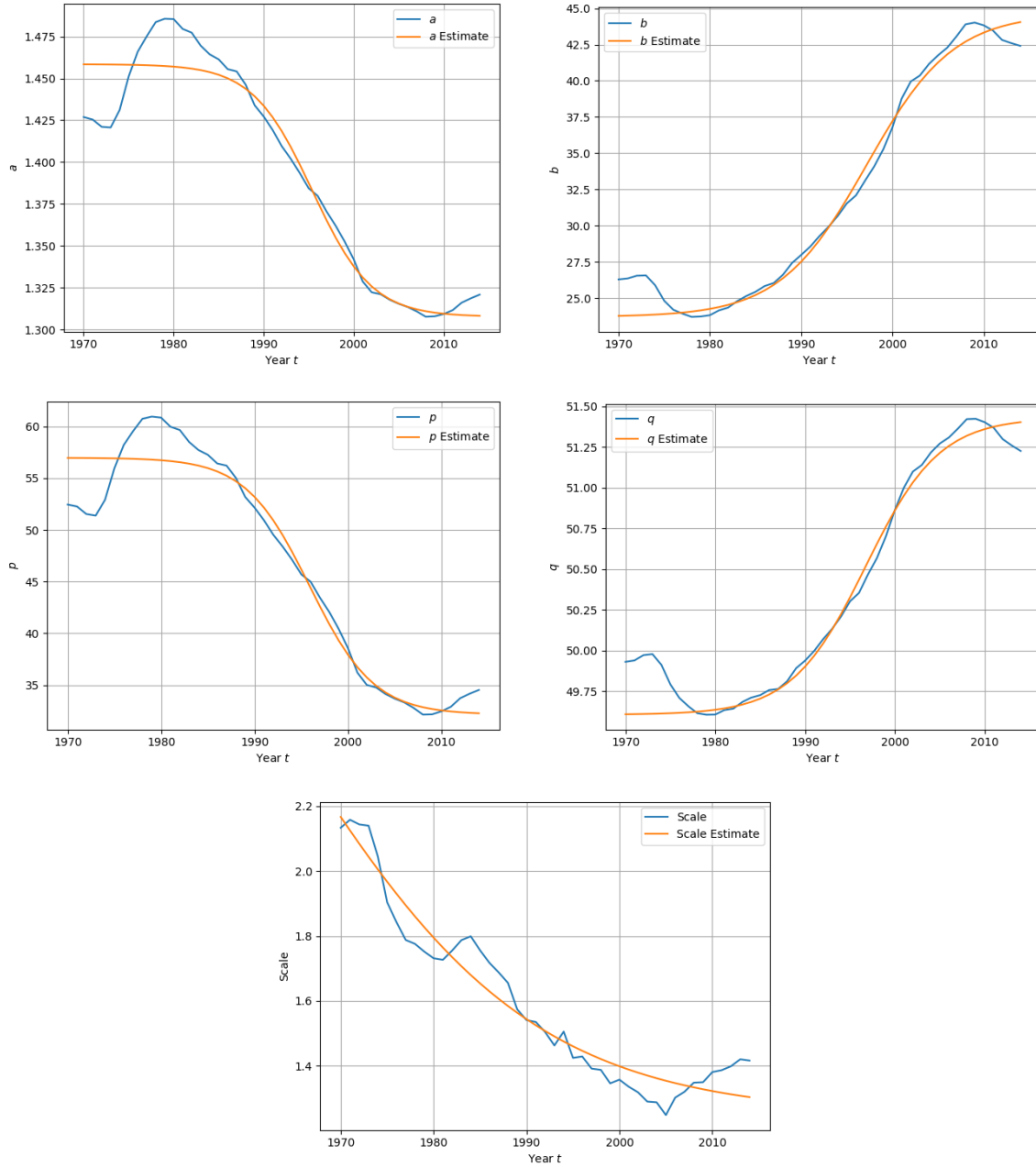
With the parameters from the logistic function estimated, we can now test the quality of our model fit. Comparisons of the models estimated with true data can be seen in Figure 5. The first panel shows the

trend in true fertility rates from 1970-2014; the second panel shows the trend in model fertility rates from 1970-2014 using the logistic estimates of the generalized beta 2 parameters; the third panel shows the first two trends overlaid, with true data in red and model data in blue; and the fourth panel shows select estimates from the model.

From the first three panels, we can see that the model represents the general trends in fertility rates properly. Just as in the true data, the model correctly increases the modal fertility age over time while decreasing overall fertility over time. As discussed before, we can see in the overlay that the model does a better job of fitting the center of the distribution than the tails of the distribution.

The fourth panel gives us an indication of how fertility will evolve over time. We can see the large drop in fertility from 1990 to 2000 to 2014, which follows the same trend as the true data. With our model we now also have the ability to forecast future fertility rates. Our model forecasts that by 2050 fertility rates will continue their trend downward with a concurrent increase in the modal fertility age. We also see that the logistic functional form leads to a convergence in fertility rates by 2050 - the estimate for 2100 is almost exactly overlaid with the estimate for 2050.

Figure 5: Fertility Generalized Beta 2 Model Fit
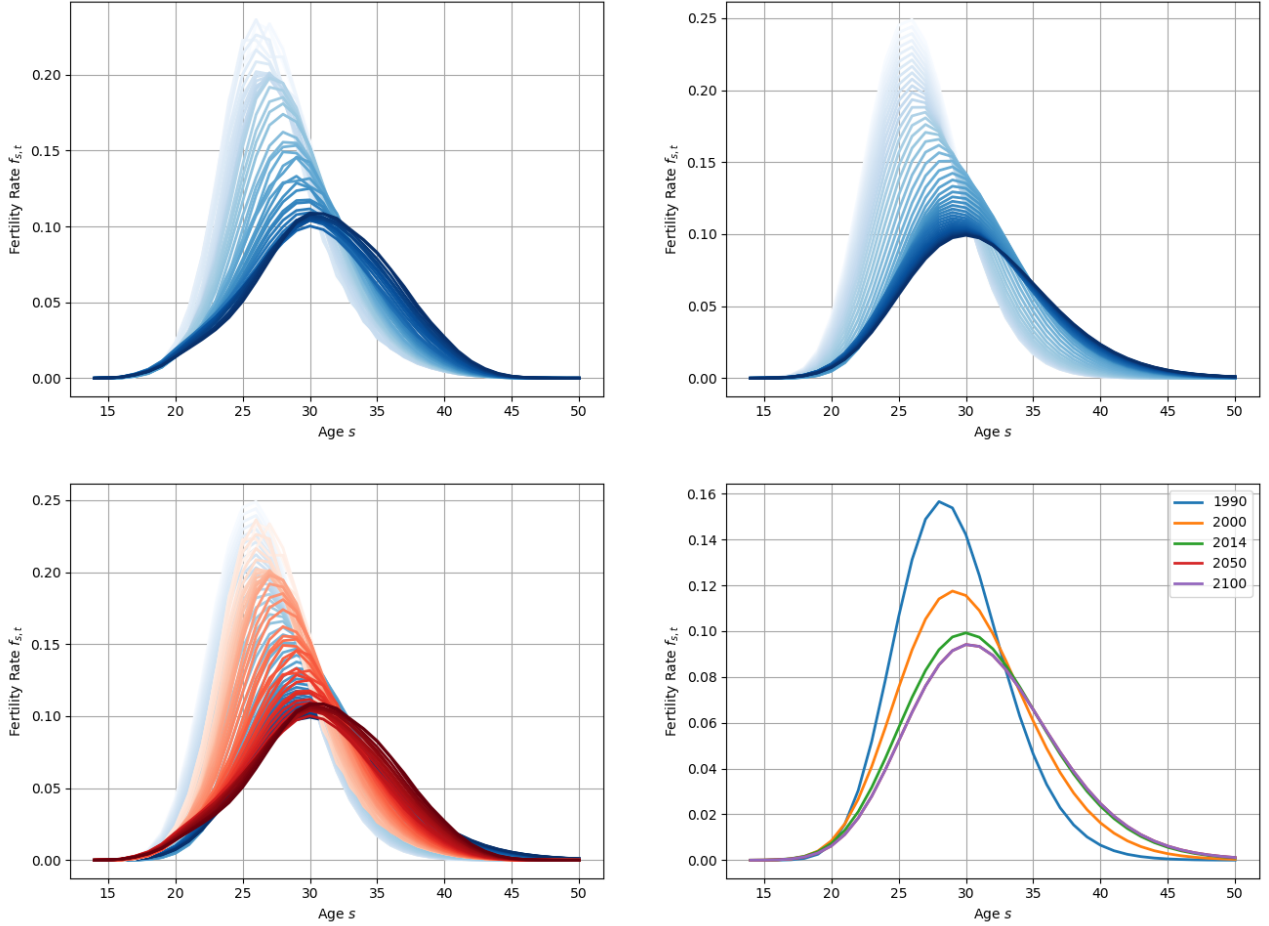
A.2.2 Mortality Rates

Mortality rate estimation requires working in two steps: first, estimating infant mortality, then estimating non-infant mortality. This is necessary because infant mortality rates over the past century have been decreasing at a much faster rate than mortality rates for other ages. It is therefore necessary to model them separately in order to accurately represent the varying trends.

The infant mortality rate estimation takes one step: I fit the trend to a generalized polynomial of the form seen in A.1 eqn. 5:

$$f(x|a, b, c, d, e) = a(e \cdot x - b)^{\frac{1}{c}} + d \tag{A.2 eqn. 3}$$

The fit of the model can be seen in Figure 6. Because of the steep drop in infant mortality rates over time, I chose to estimate it for the entire set of data from 1947-2016. However, to emphasize the importance of

fitting recent data, I add additional weight to data from the last 15 years of the sample when estimating parameters. I also penalize parameter estimates that forecast negative infant mortality rates by 2100 to ensure only positive future estimates before I assume convergence by 2050.

Figure 6: Infant Mortality Estimated by Polynomial
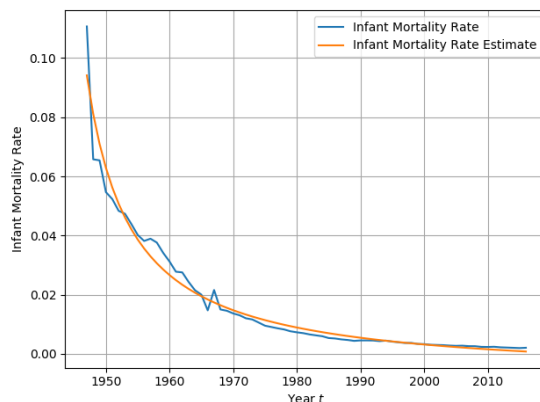
I begin the non-infant mortality rate estimation by fitting the yearly distributions using a generalized beta 2 distribution. The pdf for this distribution can be seen in A.1 eqn. 3. This is the same distribution used to fit fertility estimates. I choose to use a generalized beta 2 distribution rather than an exponential, which is what the data appears to follow, because the fit is almost identical but there is no discernable trend in parameter estimates for an exponential distribution but there is a very clear trend for the generalized beta 2 distribution.

As with fertility rates, because the generalized beta 2 is a pdf but mortality rates are not a pdf, I also have to add in a scale parameter. I estimate mortality parameters from 1970 to 2014, the same years as for fertility rates.

The fit of the model in selected years can be seen in Figure 7. The estimated distribution has a very close fit to the true data. This is true for all years, unlike the fertility estimates which do not fit the tails properly for recent data.

Parameter estimates and their fit over time can be seen in Figure 8. In order to ensure estimated trends converge over time, I fit these parameter estimates to logistic functions. The logistic function can be seen in A.1 eqn. 4. This is the same distribution used to fit fertility parameters.

Figure 7: Mortality Estimated by Generalized Beta 2

Figure 8: Mortality Generalized Beta 2 Parameter Estimates


With the parameters from the logistic function estimated, we can now test the quality of our model fit.

Comparisons of the models estimated with true data can be seen in Figure 9. The first panel shows the trend in true mortality rates from 1970-2014; the second panel shows the trend in model mortality rates from 1970-2014 using the logistic estimates of the generalized beta 2 parameters; the third panel shows the first two trends overlaid, with true data in red and model data in blue; and the fourth panel shows select estimates from the model.

From the first three panels, we can see that the model represents the general trends in mortality rates properly.

The fourth panel gives us an indication of how mortality will evolve over time. We can see consistent decline in mortality from 1990 to 2000 to 2014, which follows the same trend as the true data. With our model we now also have the ability to forecast future mortality rates. Our model forecasts that by 2050 mortality rates will continue their trend downward. We also see that the logistic functional form leads to a convergence in mortality rates by 2050 - the estimate for 2100 is almost exactly overlaid with the estimate for 2050.

Figure 9: Mortality Generalized Beta 2 Model Fit
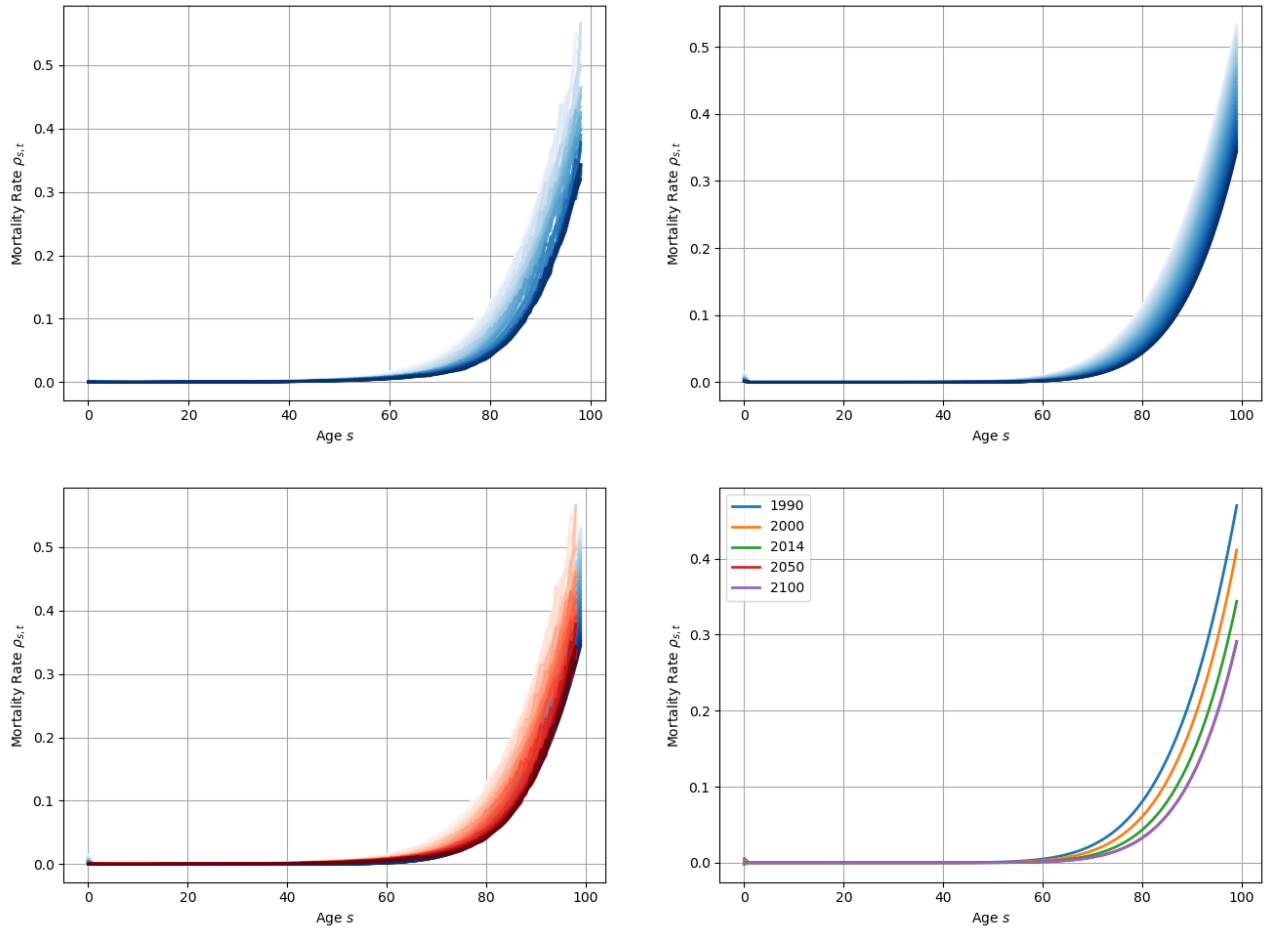
A.2.3 Immigration Rates

Estimating the evolution of immigration rates over time poses more of a challenge than fertility or mortality rates. While fertility and mortality rates can be estimated using a generalized beta 2 distribution, immigration rates following a distribution that is difficult to model without a mixture method. To get around this, I instead model the evolution in immigration rates for each age. While this no longer allows for the use of a simple distribution and clear parameter evolutions over time for the entire set of data, it seems like a more reasonable way to fit the data than to fit a mixture model with a large number of distributions.

To fit the evolution of immigration rates for each age, I estimate the trend from 1997-2014 using linear regression, then forecast using an exponential distribution that matches the slope of the linear regression in 2014 and is assumed to plateau in 15 years at a value 10% beyond the last value in the data. The exponential I fit can be seen in **??**:

$$f(x|a, b, c, d, p, s, \beta_0, \beta_1) = e^{a(x-s)^2 + b(x-s) + c} + p \qquad \text{(A.2 eqn. 4)}$$

$$s.t. \quad \left.\frac{\partial f}{\partial x}\right|_{x=s} = \beta_1 \qquad \text{(cond. 1)}$$

$$f(s) = \beta_0 + \beta_1 s \qquad \text{(cond. 2)}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=s+15} \approx 0 \qquad \text{(cond. 3)}$$

$$f(s+15) = d \cdot f(s) \qquad \text{(cond. 4)}$$

where a, b, and c are parameters to estimate, p shifts the curve so we can estimate c (this is explained in the derivation), s gives the last year of data, β_0 and β_1 are the OLS estimates from fitting the data, and we define d such that

$$d = \begin{cases} 0.9 & \text{if} \quad (\beta < 0 \text{ and } f(s) > 0) \text{ or } (\beta > 0 \text{ and } f(s) < 0) \\ 1.1 & \text{if} \quad (\beta < 0 \text{ and } f(s) < 0) \text{ or } (\beta > 0 \text{ and } f(s) > 0) \\ 1 & \text{if} \quad \beta = 0 \end{cases}$$

For notational simplicity, I will define $\beta = \beta_0 + \beta_1 s$. We therefore have from cond. 2 that $e^c + p = \beta \Leftrightarrow c = \log(\beta - p)$. In order to ensure that we can compute c, we set $p < \beta$.

Using $\frac{\partial f}{\partial x} = (2a(x-s) + b)e^{a(x-s)^2 + b(x-s) + c}$ and cond. 1, we have $be^c = \beta_1 \Leftrightarrow b = \frac{\beta_1}{\beta - p}$.

From cond. 3 we have $(2a(15) + b)(f(x + 15) - p) \approx 0$. Plugging in the value for $f(x + 15)$ from cond. 4, and recalling that $f(s) = \beta_0 + \beta_1 s = \beta$, we have $\left(2a(15) + \frac{\beta_1}{\beta - p}\right)(d \cdot \beta - g) \approx 0 \Leftrightarrow a \approx \frac{1}{30}\left(\frac{0}{d \cdot \beta - g} - \frac{\beta_1}{\beta - p}\right)$. Because the derivative cannot actually become 0, we can choose a value very close to 0 to estimate this curve numerically. At 2030, we assume that the estimate stays constant for future years.

We therefore have the following:

$$\left\{ p < \beta, a \approx \frac{1}{30}\left(\frac{0}{d \cdot \beta - g} - \frac{\beta_1}{\beta - p}\right), b = \frac{\beta_1}{\beta - p}, c = \log(\beta - p) \right\}$$

where $\beta = \beta_0 + \beta_1 s$.

The fit of the model for selected ages can be seen in Figure 10. Unlike fertility and mortality rates, fertility rates by age appear not to have as predictable a trend. However, the linear regression model with exponential forecasts seems sufficient to have reasonable forecasts for the short term.

Figure 10: Immigration Estimated by Linear Regression and Forecasted by Exponential
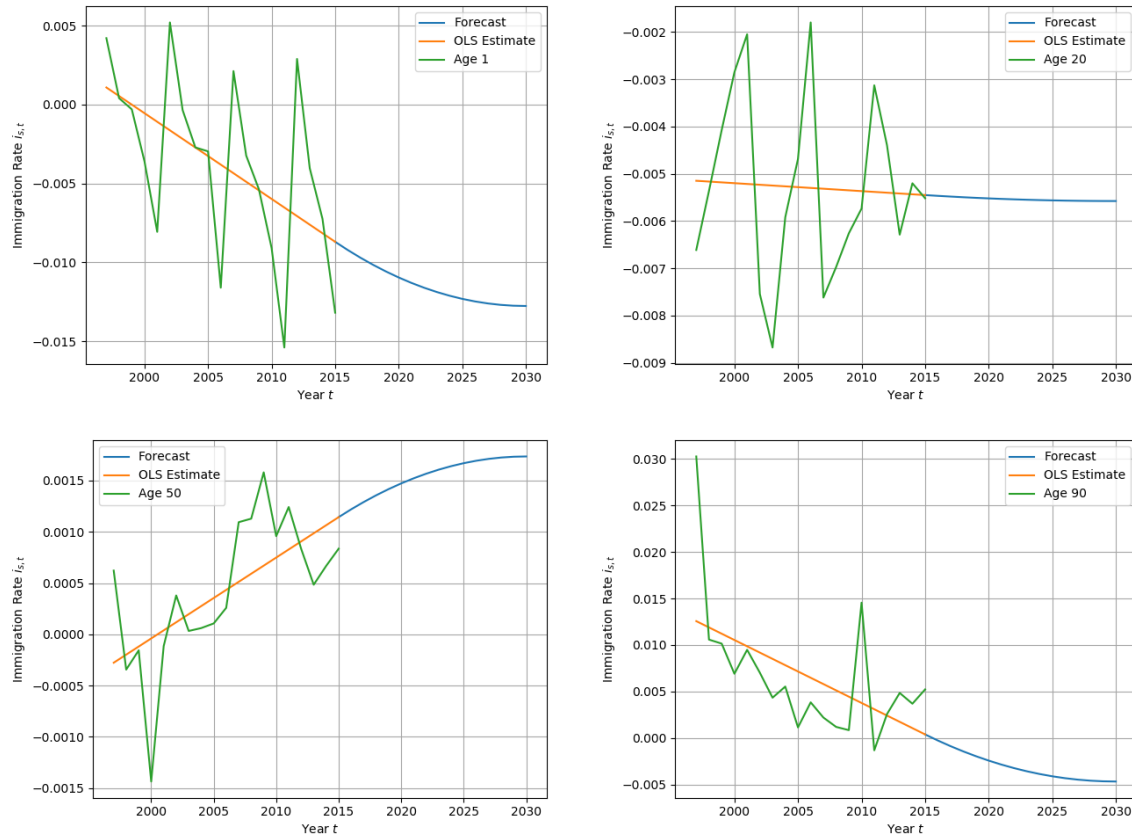
With the parameters from the linear regression and exponential estimated, we can now test the quality of our model fit. Comparisons of the models estimated with true data can be seen in Figure 11. The first panel shows the trend in true immigration rates from 1997-2014; the second panel shows the trend in model immigration rates from 1997-2014; the third panel shows the first two trends overlaid, with true data in red and model data in blue; and the fourth panel shows select estimates from the model.

From the first three panels, we can see that the model represents the general trends in immigration rates properly, although with reduced variance.

The fourth panel gives us an indication of how immigration will evolve over time. We can see consistent decline in immigration from 1990 to 2000 to 2014 for the young and the old, with little change in immigration rates for other ages, which follows the same trend as the true data. With our model we now also have the ability to forecast future immigration rates. Our model forecasts that by 2050 immigration rates will continue their trend downward for the old. In order to prevent population estimates from declining too rapidly I choose to increase immigration rates for the 0 year old population rather than let it continue to decline. Other ages seem to have relatively unchanged immigration rates over time. We can also see that

following our construction, we have a convergence in immigration rates by 2030.

Figure 11: Immigration Estimated by Linear Regression and Forecasted by Exponential
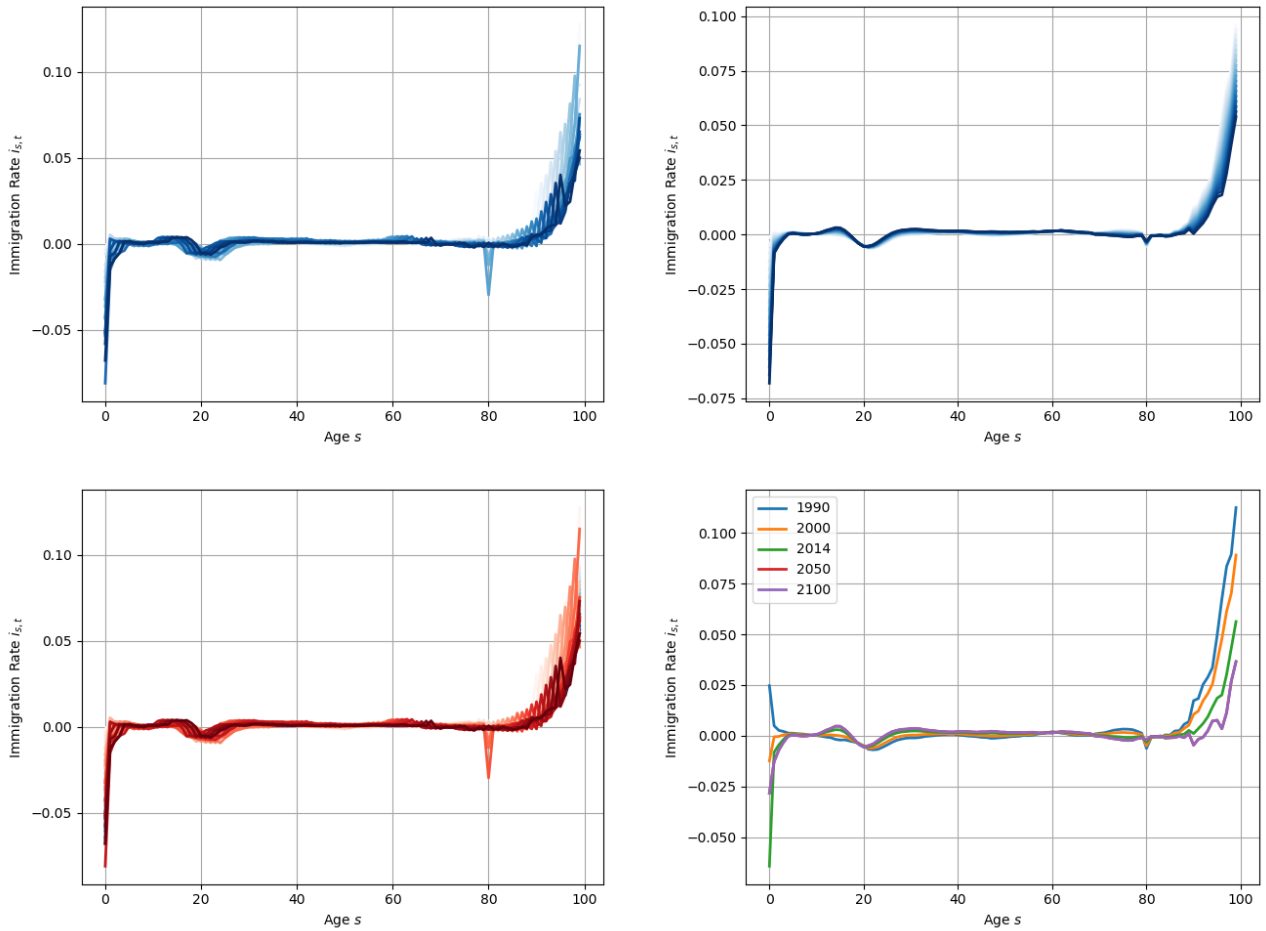
A.2.4 Population

Using our fertility, mortality, and immigration estimates, we can now use the population evolution described in A.1 eqn. 1 and A.1 eqn. 2 to forecast future population given an initial year of population data.

We will first analyze quality of fit on past data. I test by beginning with 1970 population data and comparing the true population evolution against the population evolution forecasted by the model. Selected years of the transition from 1970 to 2017, the most recent population data, can be seen in Figure 12. We can see in the first year of the model that there is a slight overestimate of births. This persists over time for this particular cohort, as the mortality rate should not be compensating for this. However, this slight overestimate is actually a rise in population that is is seen to occur just a few years later in the true data, when looking at the figure from 1985. The model does an accurate job of representing the inflection point in population that occurs around 1974 in the real data, and 1972 in the forecast. Looking at the figure from

2000, we can see that the model also accurately fits the inflection point in population around 1997. However, we begin to see that there a few ages where the model is overestimating the population and other ages where it is underestimating the population. The final figure is for 2017. This is 47 years after the start. The model appears to fit the data well and reflects the inflection points in the data.

Figure 12: Forecasted vs. True Population
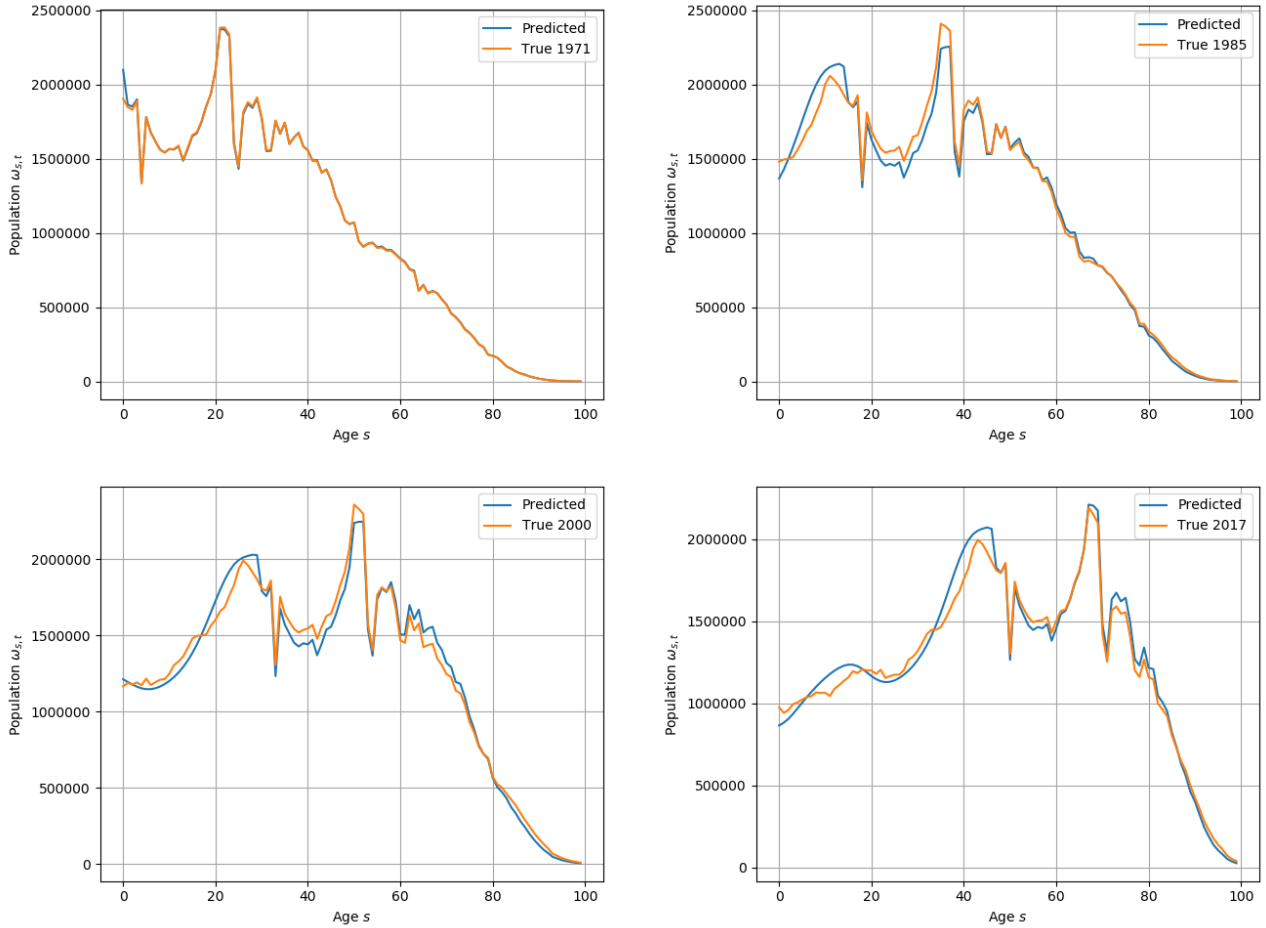
Given the model does a good job of fitting historical data, we now move on to forecasting future populations. We can see how population will evolve over time in Figure 13. The first panel shows the population at select years in the future. These results are rather surprising. We can see that given the forecasts for fertility, mortality, and immigration rates, the population in Japan is not sustainable. According to this model, by the year 2500 the population will have effectively disappeared. This result should caution the user of this model - it can lead to counterintuitive and unreasonable results. However, without another model to use in its place, this model is used for the estimation in this paper.

The second panel shows the distribution of the population at select years in the future. We can see that

over time, the modal age of the population increases until it peaks at around 75 years old. While this result indicates Japan faces a massive hurdle with its aging population, it should be taken with a grain of salt considering that the model estimates that by the time the demographic distribution reaches this point, the population will be effectively 0.

Figure 13: Forecasted Population and Population Distribution
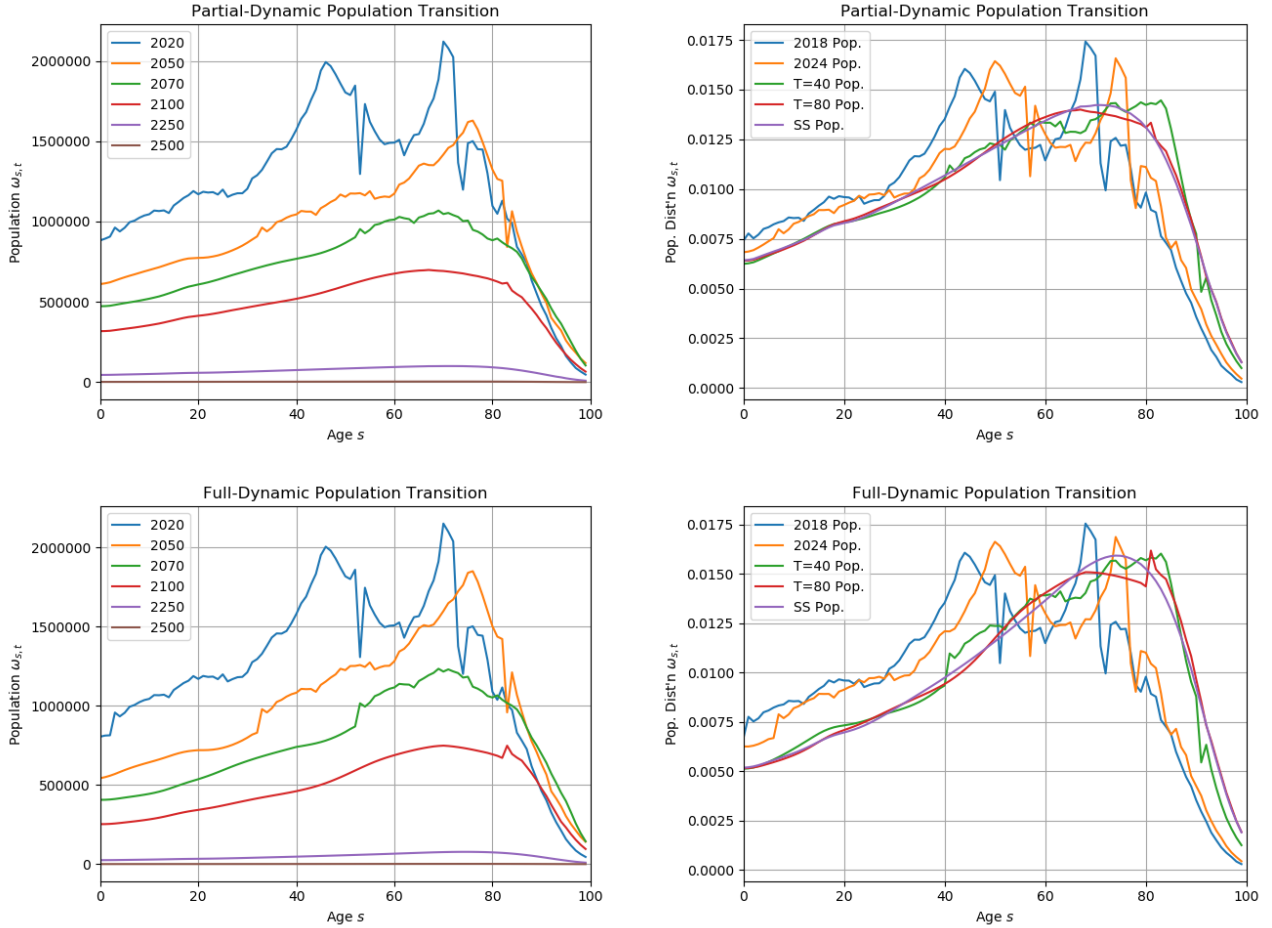
A.3 Results

A.3.1 Steady State Results

Figure 14: Steady State Consumption and Savings

(a) Static Demographics

(b) Partial-Dynamic Demographics

(c) Full-Dynamic Demographics
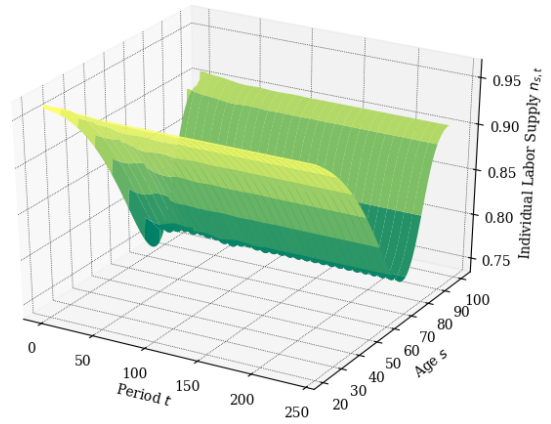

(d) Alternate Full-Dynamic Demographics (AO:FIX)
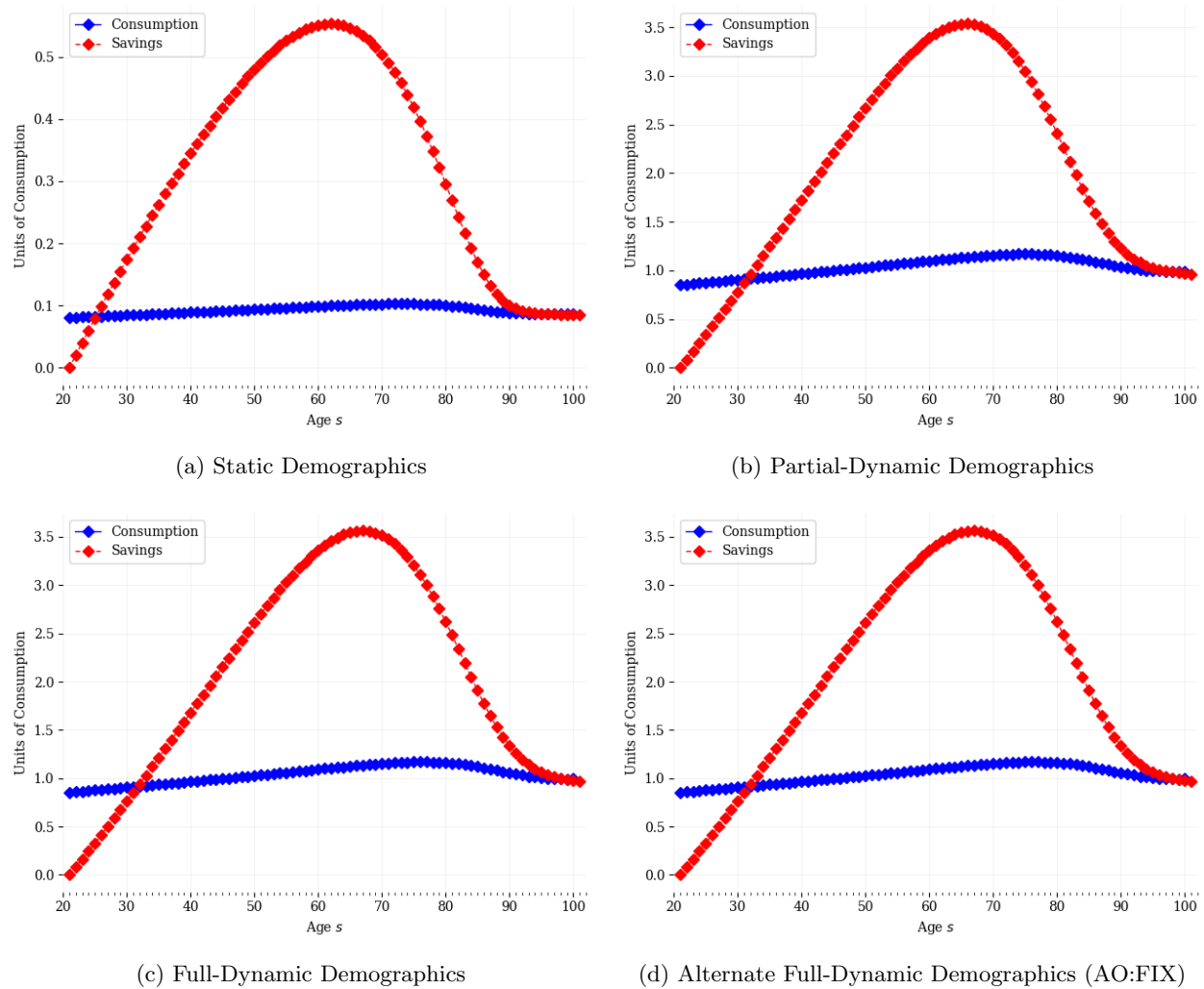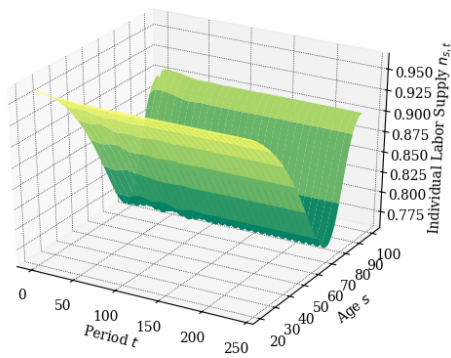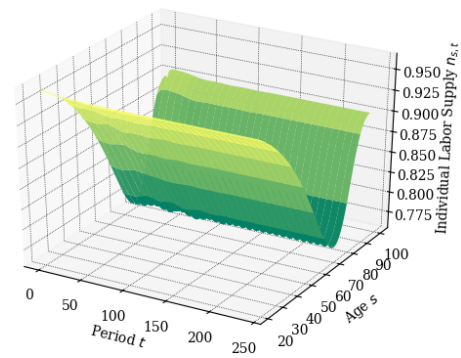
Figure 15: Steady State Labor Supply
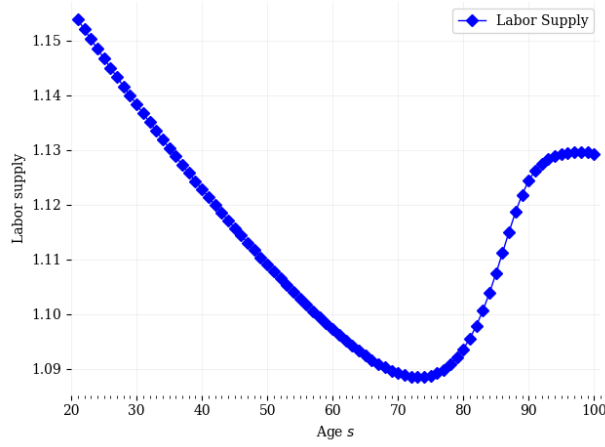
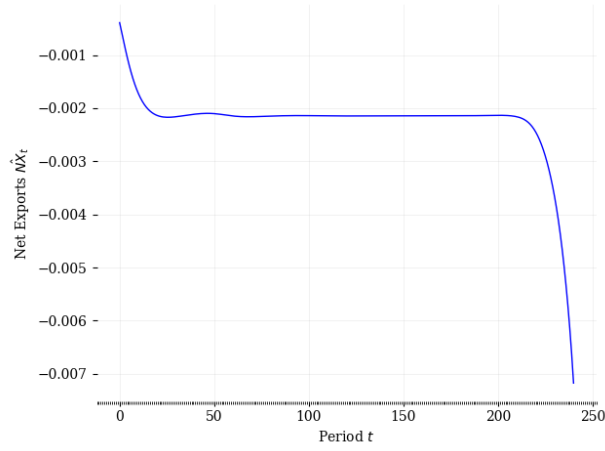(a) Static Demographics
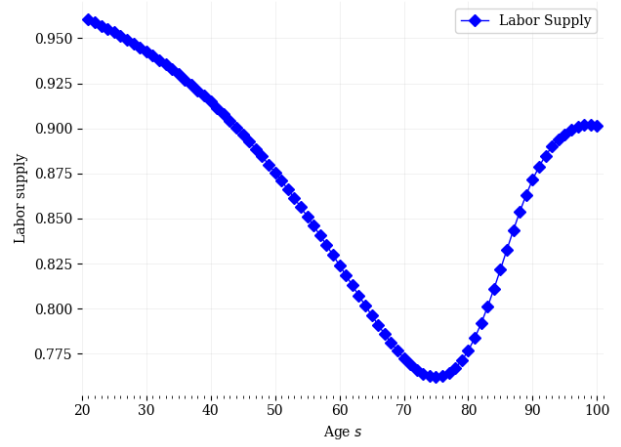
(b) Partial-Dynamic Demographics
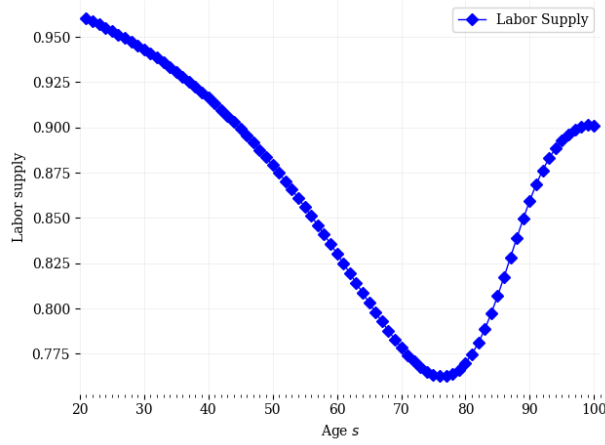
(c) Full-Dynamic Demographics
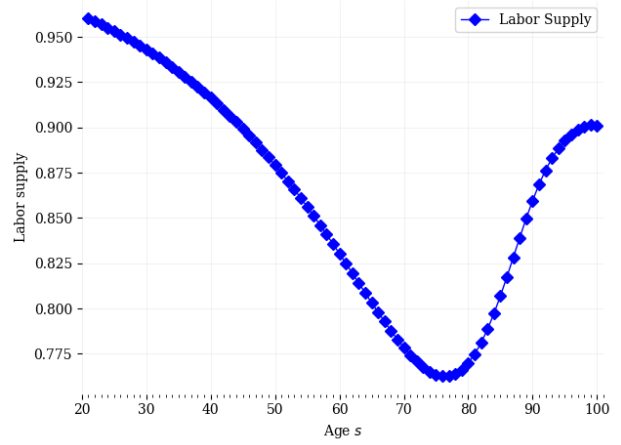
(d) Alternate Full-Dynamic Demographics (AO:FIX)

A.3.2 Time Path Results

Figure 16: Time Path of Total Bequests $\hat{BQ}_t$

(a) Static Demographics

(b) Partial-Dynamic Demographics

(c) Full-Dynamic Demographics

(d) Alternate Full-Dynamic Demographics (AO:FIX)
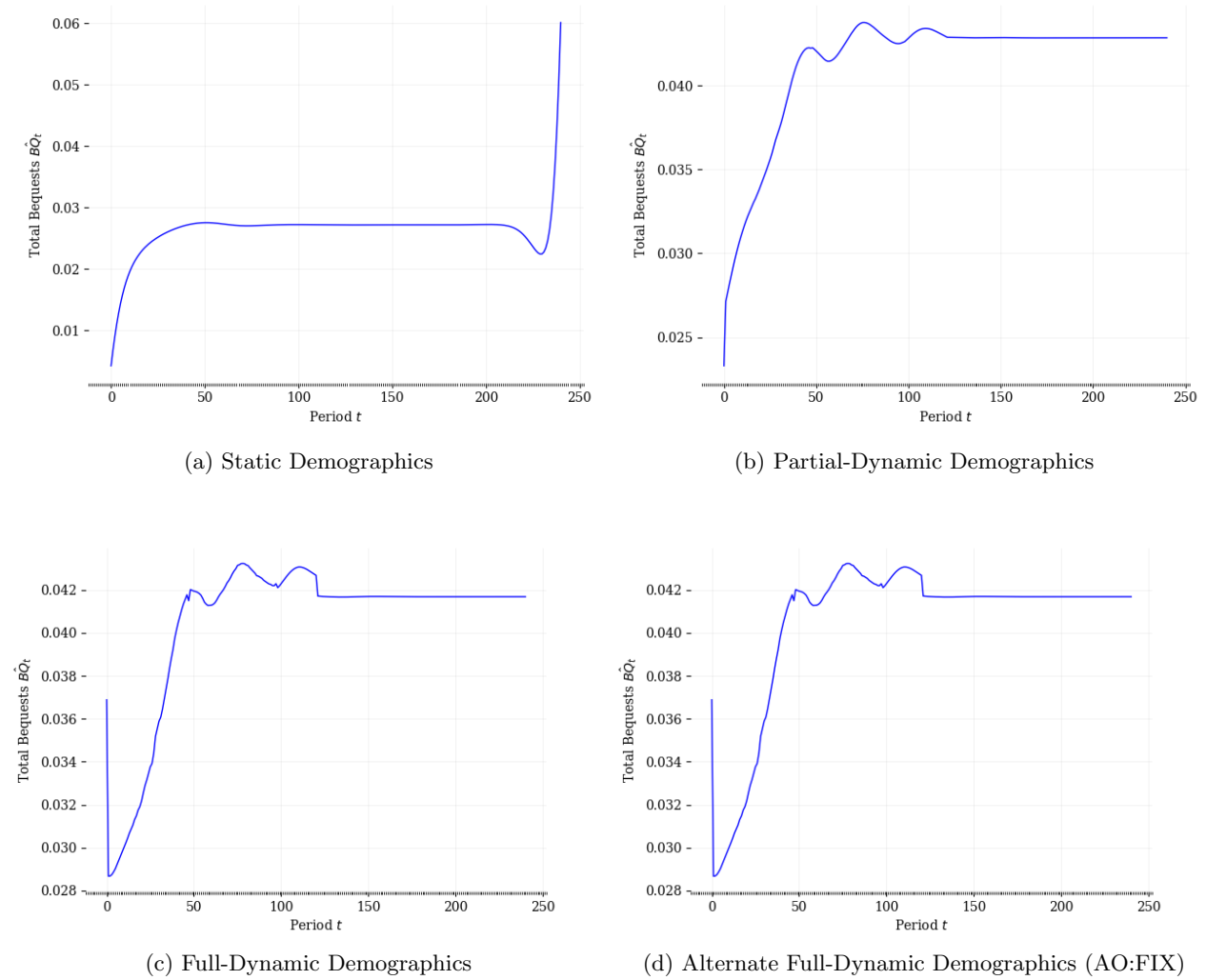
Figure 17: Time Path of Individual Savings $b_{s,t}$
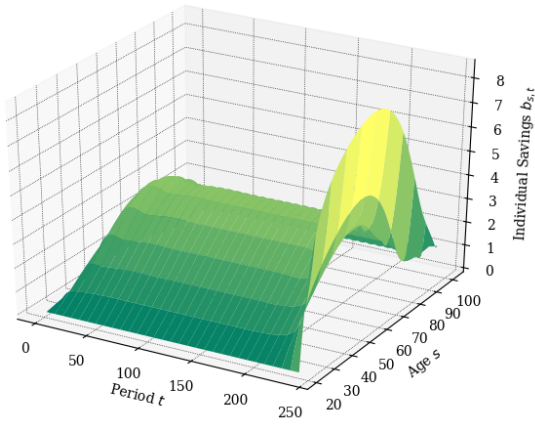
(a) Static Demographics
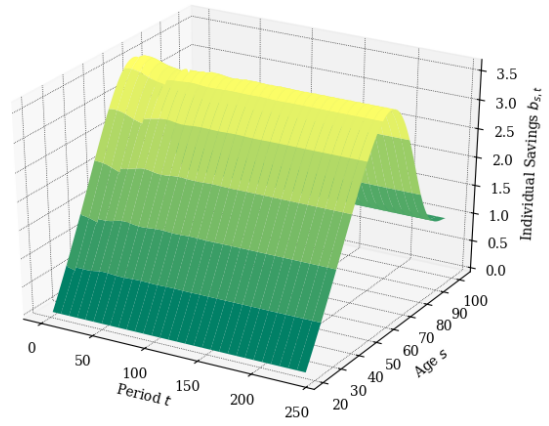
(b) Partial-Dynamic Demographics
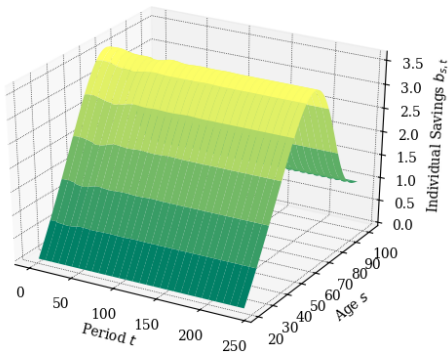
(c) Full-Dynamic Demographics
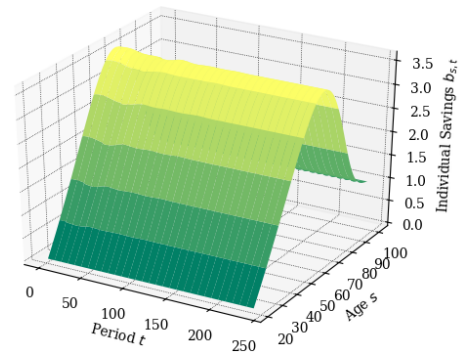
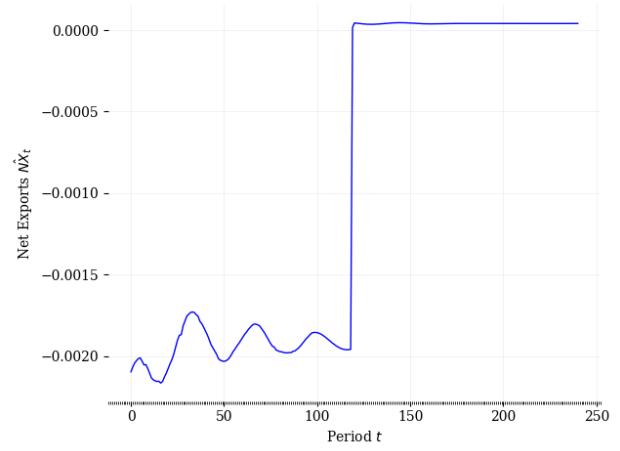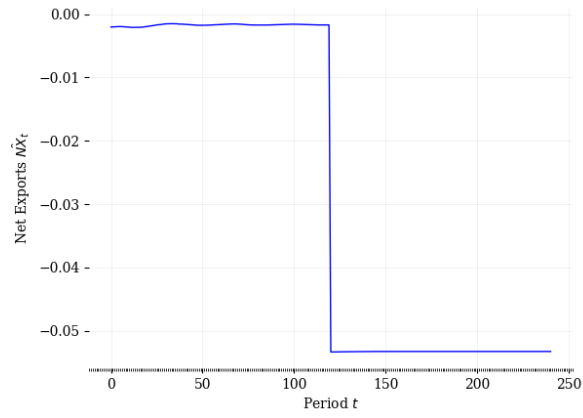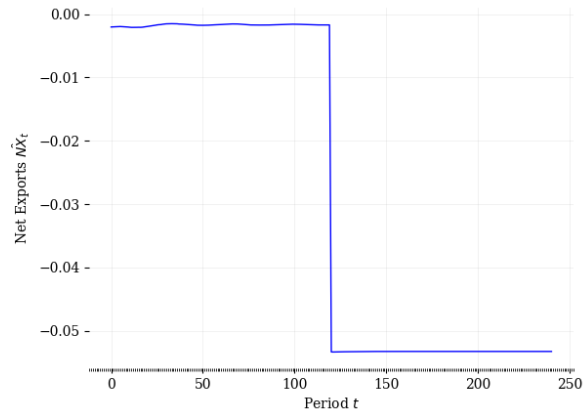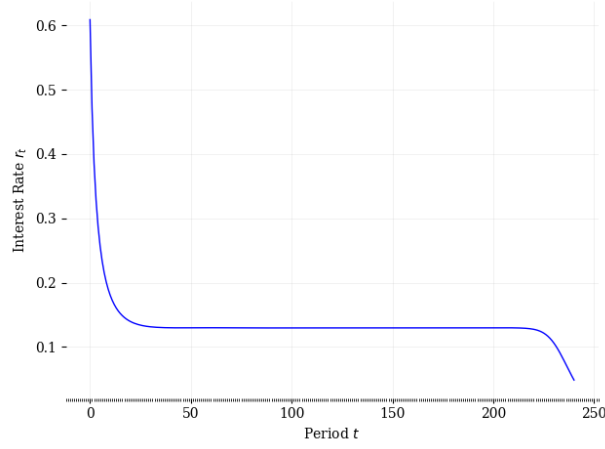(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 18: Time Path of Aggregate Consumption $\hat{C}_t$
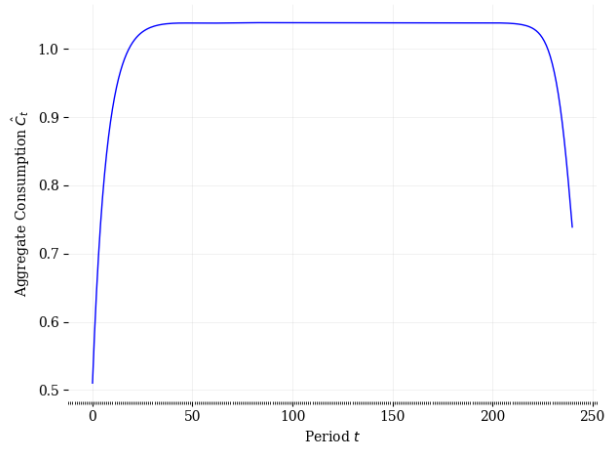
(a) Static Demographics
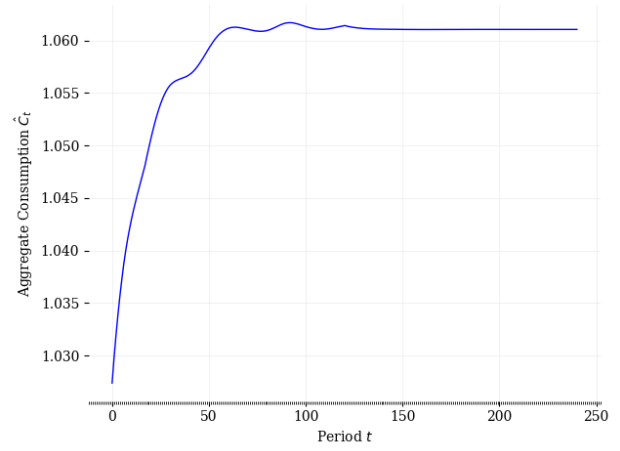
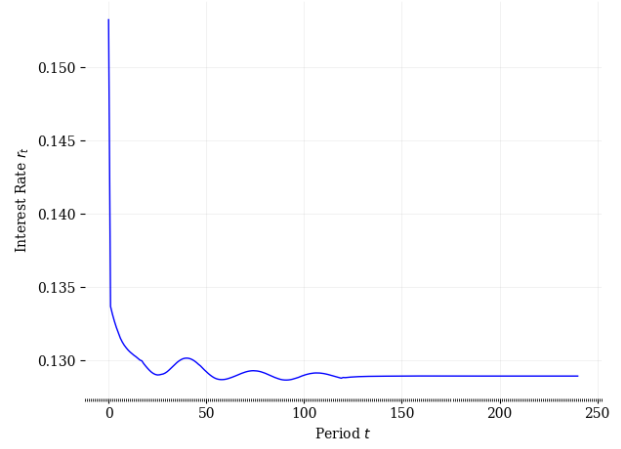(b) Partial-Dynamic Demographics
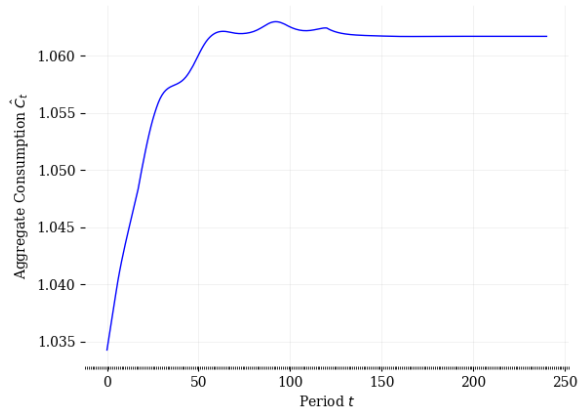
(c) Full-Dynamic Demographics
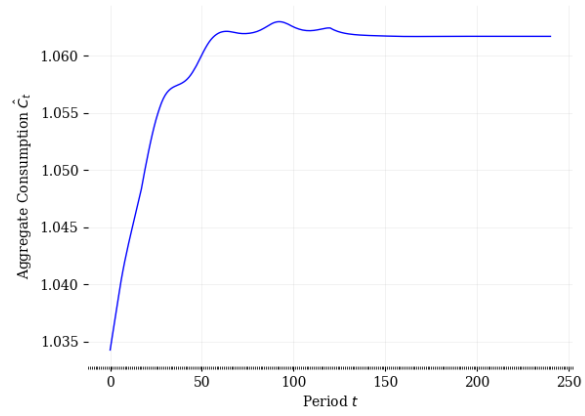
(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 19: Time Path of Individual Consumption $\hat{c}_{s,t}$
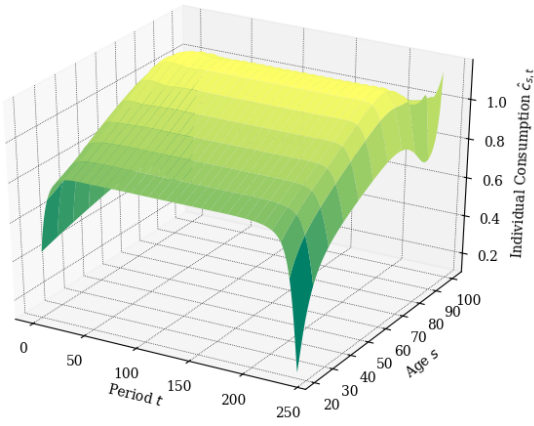
(a) Static Demographics
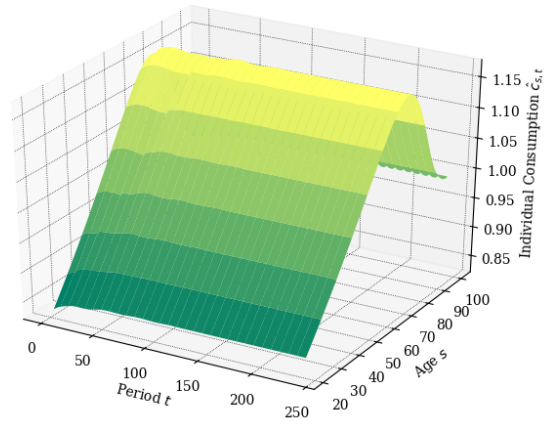
(b) Partial-Dynamic Demographics
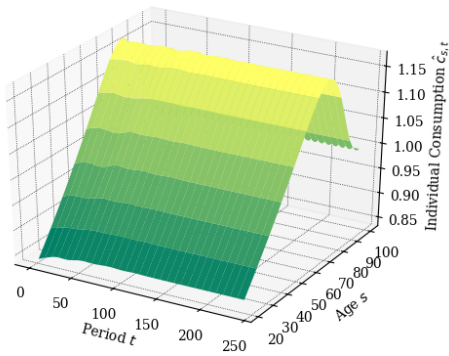
(c) Full-Dynamic Demographics
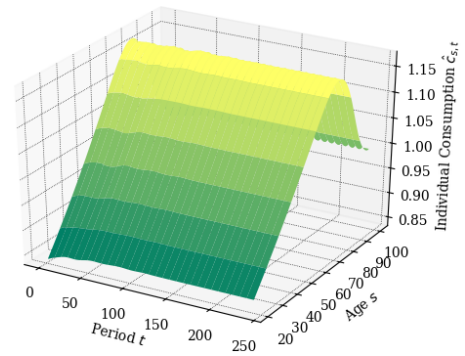
(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 20: Time Path of Aggregate Investment $\hat{I}_t$
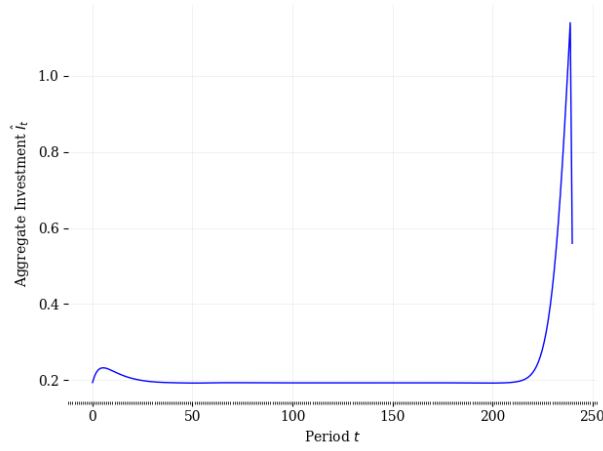
(a) Static Demographics
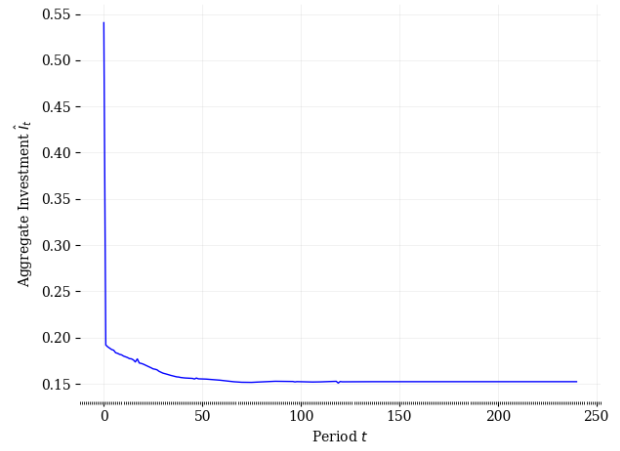
(b) Partial-Dynamic Demographics
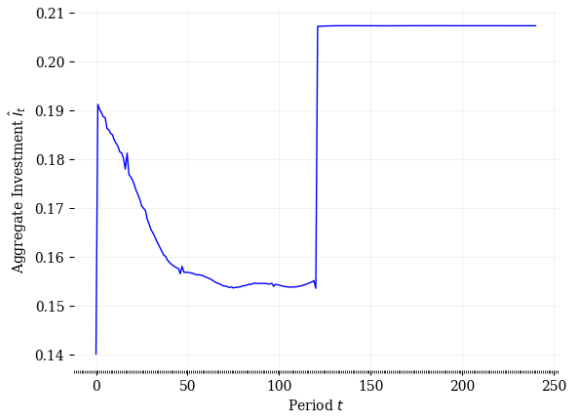
(c) Full-Dynamic Demographics
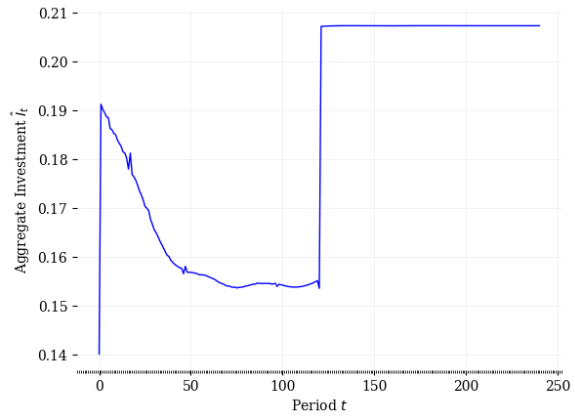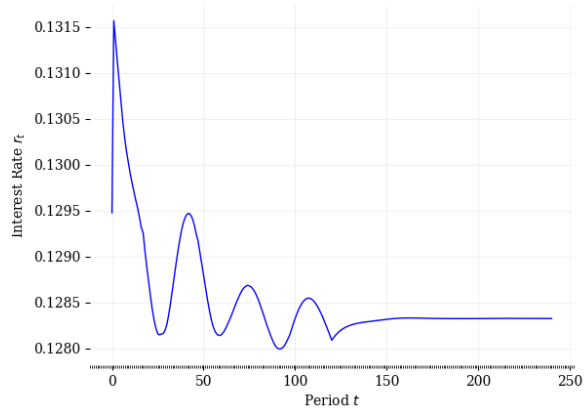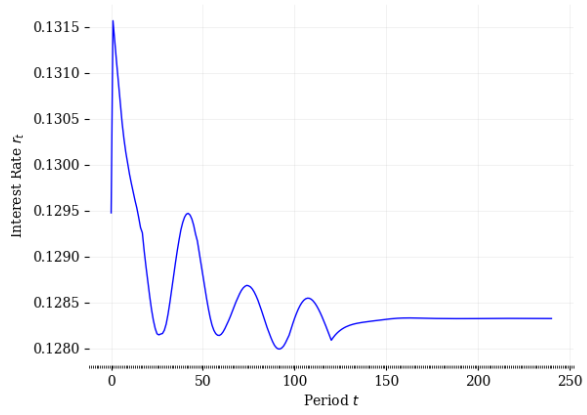
(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 21: Time Path of Aggregate Capital $\hat{K}_t$
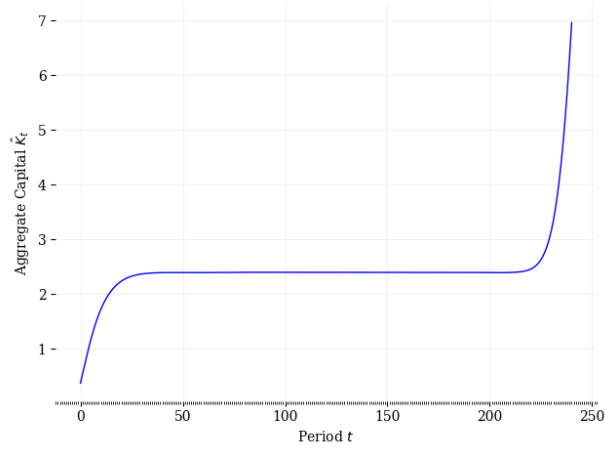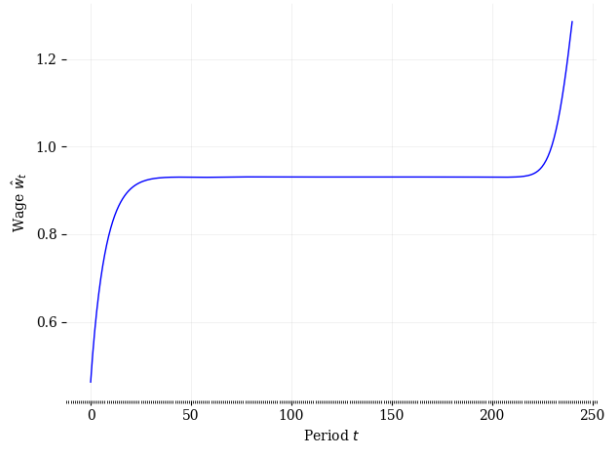
(a) Static Demographics
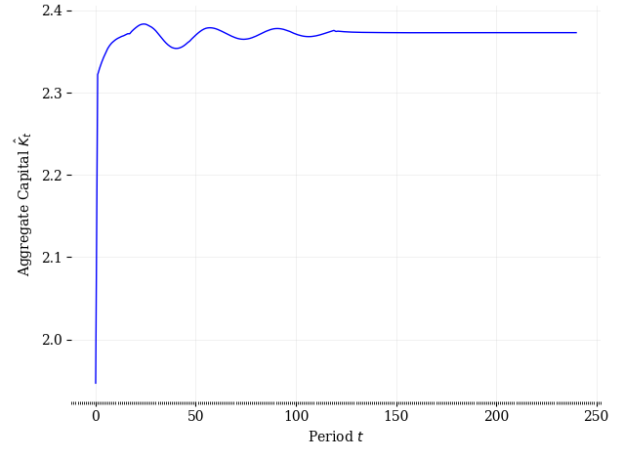
(b) Partial-Dynamic Demographics
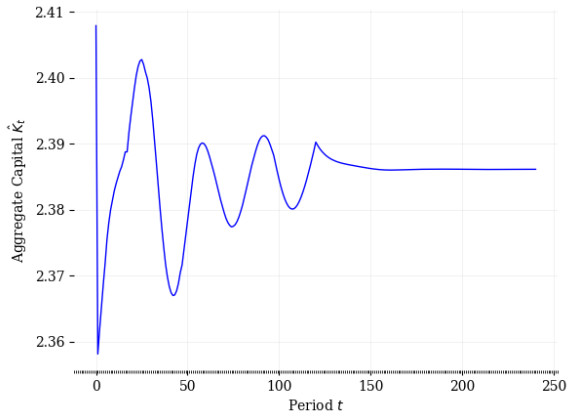
(c) Full-Dynamic Demographics
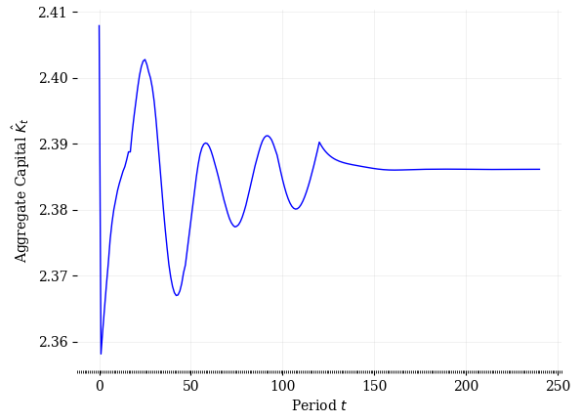
(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 22: Time Path of Aggregate Labor Supply $\hat{L}_t$
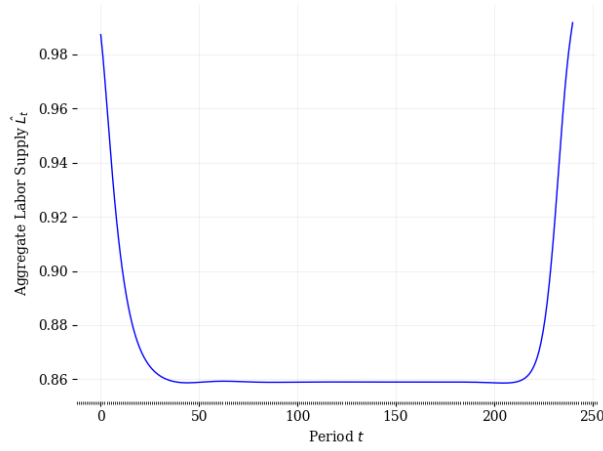
(a) Static Demographics
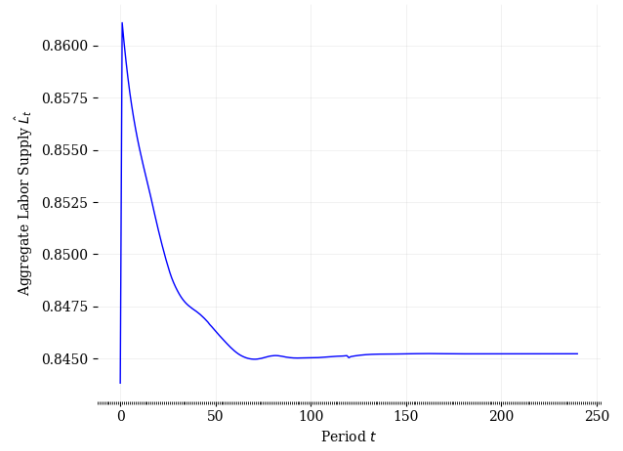
(b) Partial-Dynamic Demographics
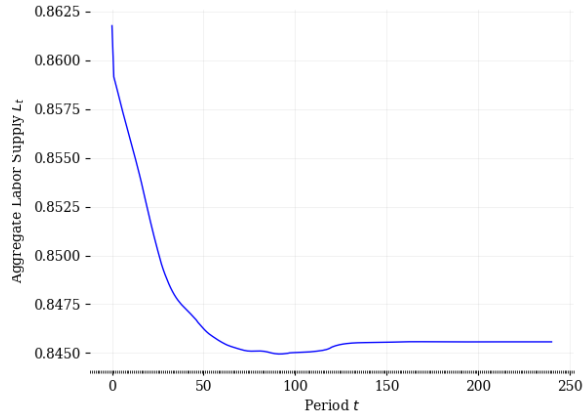
(c) Full-Dynamic Demographics
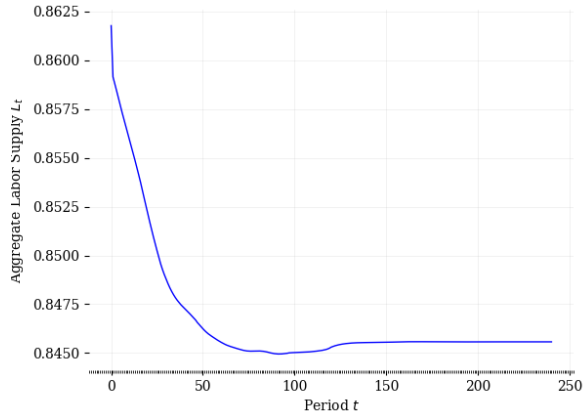
(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 23: Time Path of Individual Labor Supply $n_{s,t}$
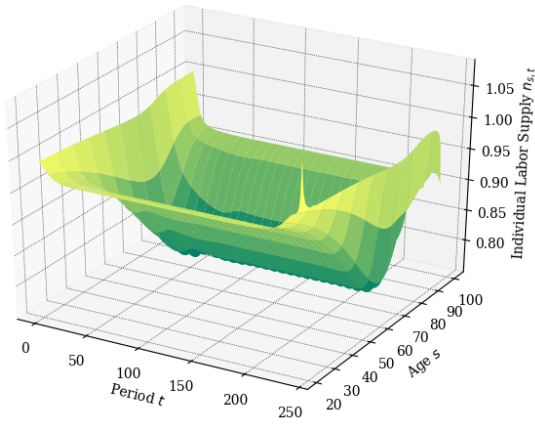
(a) Static Demographics

(b) Partial-Dynamic Demographics

(c) Full-Dynamic Demographics
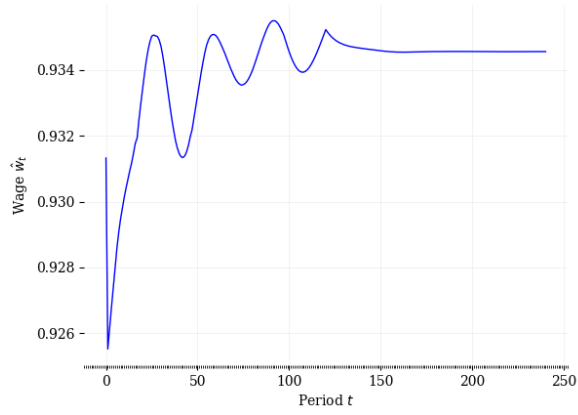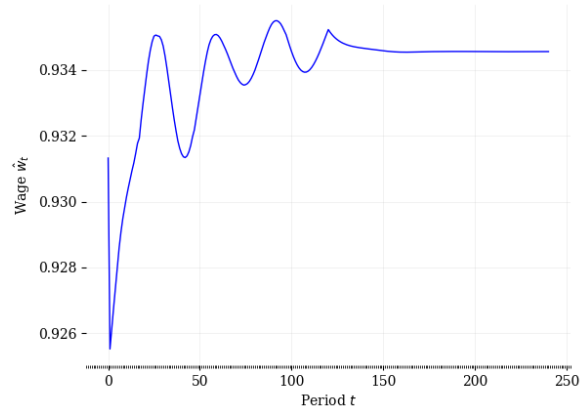

(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 24: Time Path of Net Exports $\hat{NX}_t$

(a) Static Demographics

(b) Partial-Dynamic Demographics

(c) Full-Dynamic Demographics

(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 25: Time Path of Interest Rate r_t

(a) Static Demographics

(b) Partial-Dynamic Demographics

(c) Full-Dynamic Demographics

(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 26: Time Path of Wage $\hat{w}_t$

(a) Static Demographics
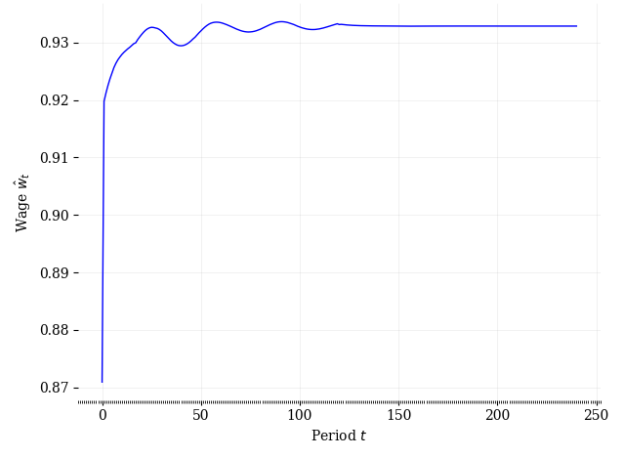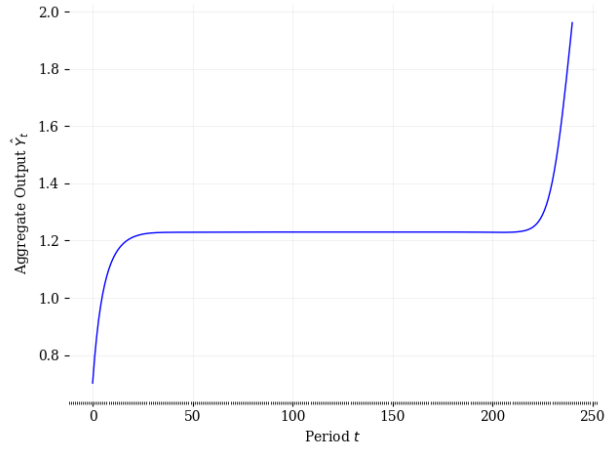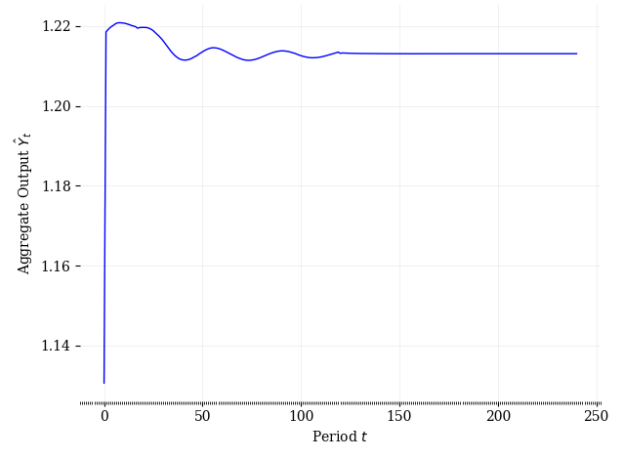
(b) Partial-Dynamic Demographics

(c) Full-Dynamic Demographics

(d) Alternate Full-Dynamic Demographics (AO:FIX)

Figure 27: Time Path of Aggregate Output $\hat{Y}_t$

(a) Static Demographics

(b) Partial-Dynamic Demographics

(c) Full-Dynamic Demographics

(d) Alternate Full-Dynamic Demographics (AO:FIX)