Step-by-Step Model Buildup

1 Simple S-period I-Country OLG Model

Consider a model with no complicated demographics. Every period a unit measure of labor is born in each of I countries and these workers live for exactly S period.

Each country produces the same single good which is mobile across country borders.

Households of age s in country i in period t solve the following maximization problem.

$$\max_{\{c_{i,s+j,t+j}\}_{j=0}^{S-s}} U_{ist} = \sum_{j=0}^{S-s} \beta^j \frac{1}{1-\sigma} c_{i,s+j,t+j}^{1-\sigma}$$

subject to the following budget constraint each period.

$$c_{ist} = w_{it} e_{st} + (1 + r_{it} - \delta) a_{ist} - a_{i,s+1,t+1} \tag{1.1}$$

with $a_{i1t} = a_{i,S+1,t} = 0$

The solution to this problem yields the following intertemporal Euler equation.

$$c_{ist}^{-\sigma} = \beta c_{i,s+1,t+1}^{-\sigma} (1 + r_{i,t+1} - \delta) \tag{1.2}$$

The representative firm in each country maximizes profits subject to a Cobb-Douglas production function.

$$\max_{n_{it}, k_{it}} \Pi_{it} = k_{it}^{\alpha} \left(A_i n_{it}\right)^{1-\alpha} - w_{it} n_{it} - ritk_{it}$$

This yields the following factor price equations.

$$y_{it} = k_{it}^{\alpha} \left(A_i n_{it}\right)^{1-\alpha} \tag{1.3}$$

$$r_{it} = \alpha \frac{y_{it}}{k_{it}} \tag{1.4}$$

$$w_{it} = (1-\alpha) \frac{y_{it}}{n_{it}} \tag{1.5}$$

Labor and capital market clearing conditions in each country give the following.

$$k_{it} = \sum_{s=1}^{S} a_{ist} - k_{it}^{f} \tag{1.6}$$

$$n_{it} = \sum_{s=1}^{S} e_{is} \tag{1.7}$$

where k_{it}^{f} denotes domestically-owned capital located in foreign countries.

Capital is allocated across countries so that the rate of return is equalized.

$$r_{it} = r_t \tag{1.8}$$

And the net sum of all foreign-located capital across countries must be zero.

$$\sum_{i=1}^{I} k_{it}^{f} = 0 \tag{1.9}$$

Given a set of state variables today $X_t = \{a_{ist}, k^f it\} \forall i, s$ we can calculate the following.

$$k_{it} = \sum_{s=1}^{S} a_{ist} - k_{it}^{f}; \forall i \tag{1.10}$$

$$n_{it} = \sum_{s=1}^{S} e_{is}; \forall i \tag{1.11}$$

$$y_{it} = k_{it}^{\alpha} \left(A_i n_{it}\right)^{1-\alpha}; \forall i \tag{1.12}$$

$$r_{it} = \alpha \frac{y_{it}}{k_{it}}; \forall i \tag{1.13}$$

$$w_{it} = (1-\alpha) \frac{y_{it}}{n_{it}}; \forall i \tag{1.14}$$

$$c_{ist} = w_{it} e_{st} + (1 + r_{1t} - \delta) a_{ist} - a_{i,s+1,t+1}; \forall i, s \tag{1.15}$$

The IS values of X_t in every period must satisfy the following equtions IS.

$$c_{ist}^{-\sigma} - \beta c_{i,s+1,t+1}^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i, s \tag{1.16}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{1.17}$$

$$\sum_{i=1}^{I} k_{it}^{f} = 0 \tag{1.18}$$

To find the steady state we impose $a_{ist} = \bar{a}_{is}$ and $k_{it}^{f} = \bar{k}_{i}^{f}$ for all t and using equations (1.10) - (1.15) search over these values to find the ones that set the values of equations (1.16) - (1.18) to zero. This involves using fsolve.

To find the transition path we make an initial guess of a history of $\{r_t^0\}$ and $\{w_{it}^0\}$ values for $t = 1$ to T with the values of $\bar{r}$ and $\bar{w}_i$ for $t > T$. We also have known initial value for a_{is1} and k_{i1}^{f}. With this information we search over the value of c_{is1} for each household to find the value that sets

$a_{iS,S-s}$ (their final savings) to zero when we chain equations (1.16) and (1.15). That is, we solve for each households optimal time path of consumption and savings given the assumed history of factor prices. We can then use equations (1.10) through (1.14) to find histories $\{r_t^{new}\}$ and $\{w_{it}^{new}\}$. If these are not sufficiently different from the previous guess, we are done and have the transition path. If they are different we use a new guess at the histories give by the equations below which are a convex combination of the old guess and the implied new histories.

$$r_t^{j+1} = \chi r_t^j + (1-\chi) r_t^{new}; \forall t \leq T \tag{1.19}$$

$$w_t^{j+1} = \chi w_t^j + (1-\chi) w_t^{new}; \forall t \leq T \tag{1.20}$$

2 Add Demographics and Growth

Denote the population of age s people in country i in period t as N_{ist}. Denote the fraction of children born to people of age s in country i in period t as f_{ist} and the immigration rate as m_{ist}. The mortality hazard is denoted ρ_{ist}. We assume that $\rho_{ist} = 0$ if $s < 68$ and $\rho_{iSt} = 1$. We also assume that $f_{ist} = 0$ for $s < 23$ and $s > 45$.

The population dynamics are given by:

$$N_{i,1,t+1} = \sum_{s=1}^{S} N_{ist} f_{ist} \tag{2.1}$$

$$N_{i,s+1,t+1} = N_{ist}(1 + m_{ist} - \rho_{ist}); 1 < s \leq S \tag{2.2}$$

We define the total world population as:

$$N_t = \sum_{i=1}^{I} \sum_{s=1}^{S} N_{ist} \tag{2.3}$$

The populations are, strictly speaking, state variables, but their steady state can be found independent of the other variables. For the model to be stationarizable, it must be that in the long run, by period T_1, that the fertility, immigration, and mortality rates are the same for all countries.

$$f_{ist} = \bar{f}_s; t \geq T_1 \tag{2.4}$$

$$m_{ist} = \bar{m}_s; t \geq T_1 \tag{2.5}$$

$$\rho_{ist} = \bar{\rho}_s; t \geq T_1 \tag{2.6}$$

Inheiritances are partially intended bequests and are the assets of agents that die at the end of period t to the survivors in that period. Let BQ_{it} denote total bequests in period t for country i and bq_{ist} denote the bequest received by a person s years old in that same country and time period.

$$BQ_{it} = \sum_{s=67}^{S} a_{ist} \rho_{ist} N_{ist} \tag{2.7}$$

$$= \sum_{s=23}^{67} bq_{ist}(1 - \rho_{ist}) N_{ist} \tag{2.8}$$

While relative populations stabilize in the steady state, the overall population may be growing or shrinking. Hence, all aggregate variables will need to be stationarized by dividing by the total population of a country. To help with this, we define the growth rate of the world population as $g_t^N = \frac{N_{it}}{N_{t-1}} - 1$.

We also allow for technical progress by having the endowment of labor per household rise by a constant amount each period. We denote this endowment as $\bar{\ell}_t$ and denote its growth rate as g^A.

The aggregate labor for a country is now given by $n_{it} = \sum_{s=1}^{S} N_{ist} e_{is} \bar{\ell}_t$. Output, y_t, is defined as before.

Aggregate variables will grow at the rate $g^A + g_t^N$ and per capita variables will grow at rate g^A. The stationarized versions of our behavioral equations are given below, where a carat denotes the stationarized variable. Note that $\hat{N}_{ist}$ is interpretable as the fraction of the world population in country i of age s at time t so that $\sum_{i=1}^{I} \sum_{s=1}^{S} \hat{N}_{ist} = 1$. Note that both k_{it} and n_{it} are growing at rate $g^A + g_t^N$, so r_{it} and w_{it} are stationary.

$$g_t^N = \sum_{i=1}^{I} \sum_{s=1}^{S} \hat{N}_{ist}(f_{ist} + m_{ist} - \rho_{ist}); \forall i \tag{2.9}$$

$$\hat{N}_{i,1,t+1} = e^{-g_t^N} \sum_{s=23}^{45} \hat{N}_{ist} f_{ist}; \forall i \tag{2.10}$$

$$\hat{N}_{i,s+1,t+1} = e^{-g_t^N} \hat{N}_{ist}(1 + m_{ist} - \rho_{ist}); \forall i, 1 < s \leq S \tag{2.11}$$

$$\hat{k}_{it} = \sum_{s=1}^{S} \hat{a}_{ist} \hat{N}_{ist} - \hat{k}_{it}^f; \forall i \tag{2.12}$$

$$\hat{n}_{it} = \sum_{s=1}^{S} e_{is} \hat{N}_{ist}; \forall i \tag{2.13}$$

$$\hat{y}_{it} = \hat{k}_{it}^{\alpha} \left(A_i \hat{n}_{it}\right)^{1-\alpha}; \forall i \tag{2.14}$$

$$r_{it} = \alpha \frac{\hat{y}_{it}}{\hat{k}_{it}}; \forall i \tag{2.15}$$

$$w_{it} = (1-\alpha) \frac{\hat{y}_{it}}{\hat{n}_{it}}; \forall i \tag{2.16}$$

$$\hat{B}Q_{it} = \sum_{s=67}^{S} \hat{a}_{ist} \rho_{ist} \hat{N}_{ist}; \forall i \tag{2.17}$$

$$\hat{B}Q_{it} = \sum_{s=23}^{67} \hat{b}q_{ist}(1 - \rho_{ist}) \hat{N}_{ist}; \forall i \tag{2.18}$$

$$\hat{c}_{ist} = w_{it} e_{st} + (1 + r_{1t} - \delta) \hat{a}_{ist} + \hat{bq}_{ist} - \hat{a}_{i,s+1,t+1} e^{g^A}; \forall i, s \tag{2.19}$$

$$\hat{c}_{ist}^{-\sigma} - \beta \left(\hat{c}_{i,s+1,t+1} e^{g^A}\right)^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i, s \tag{2.20}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{2.21}$$

$$\sum_{i=1}^{I} \hat{k}_{it}^f \left(\sum_{s=1}^{S} \hat{N}_{ist} \right) = 0 \tag{2.22}$$

Note that the first three equations of the above system can be solved for steady state values independent of the rest of the system as long as the steady state values of the fertility, immigration, and mortality rates are known.

Once we have these we can find the steady state values for the rest of the system as we did above.

Also note that for the transition path we can find the time paths for the $\hat{N}$ varaibles indepentdent of the time-path iteration used to find the other variables.

3 Add Children and Leisure Decision

Denote the number of children of age r in houshold of age s in country i in period t as KID_{irst}. Wer assume that mortality hazard rates are zero up until age 68 and that fertility rates are non zero only for ages 23 through 45. This means that we avoid the situation where a child lives, but the parents die, and vice versa.

The evolution of theses numbers is given by:

$$KID_{i,r+1,s+1,t+1} = \begin{cases} f_{ist} & r = 1 \\ KID_{irst} & 1 < r < 21 \end{cases}$$

Iterative substitution gives $KID_{i,r+1,s+1,t+1} = f_{i,s-r,t-r}$. If we then define the number of children of all ages to a household of age s as KID_{ist} we have the following formula.

$$KID_{i,s+1,t+1} = \sum_{r=1}^{20} f_{i,s-r,t-r} \tag{3.1}$$

Since this is a per household number of children it is already stationary and does not need to be turned into a "hat" variable.

The household's problem is now given by:

$$\max_{\{c_{i,s+j,t+j},\ell_{i,s+j,t+j},c^K{}_{i,s+j,t+j},a_{i,s+j,t+j}\}_{j=0}^{S-s}} U_{ist} =$$
$$\sum_{j=0}^{S-s} \beta^j \frac{1}{1-\sigma} \left[\left(c_{i,s+j,t+j}{}^{1-1/\rho} + \chi \ell_{i,s+j,t+j}{}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}} + KID_{i,s+j,t+j} c^K_{i,s+j,t+j}{}^{1-\sigma} \right]$$

subject to the following budget constraint each period.

$$c_{ist} + KID_{ist} c^K_{ist} = w_{it} e_{st} (\bar{\ell}_t - \ell_{ist}) + (1 + r_{it} - \delta) a_{ist} - a_{i,s+1,t+1} \tag{3.2}$$

with $a_{i1t} = a_{i,S+1,t} = 0$. ℓ denotes leisure consumption and c^K denotes consumption by children.

The first-order conditions are:

$$\beta^j \left(c_{i,s+j,t+j}{}^{1-1/\rho} + \chi \ell_{i,s+j,t+j}{}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}-1} \left(1 - \frac{1}{\rho}\right)^{-1} c_{i,s+j,t+j}{}^{-1/\rho}$$
$$= - \lambda_{i,s+j,t+j} \tag{3.3}$$
$$\beta^j \left(c_{i,s+j,t+j}{}^{1-1/\rho} + \chi \ell_{i,s+j,t+j}{}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}-1} \left(1 - \frac{1}{\rho}\right)^{-1} \chi \ell_{i,s+j,t+j}{}^{-1/\rho}$$
$$= - \lambda_{i,s+j,t+j} w_{i,t+j} e_{s+j,t+j} \tag{3.4}$$
$$\beta^j KID_{i,s+j,t+j} c^K_{i,s+j,t+j}{}^{-\sigma} = -\lambda_{i,s+j,t+j} KID_{i,s+j,t+j} \tag{3.5}$$
$$(1 + r_{i,t+j} - \delta) \lambda_{i,s+j,t+j} = \lambda_{i,s+j-1,t+j-1} \tag{3.6}$$

along with (3.2).

Solve the (3.5) for the LaGrange multiplier.

$$\lambda_{i,s+j,t+j} = -\beta^j {c^K_{i,s+j,t+j}}^{-\sigma} \tag{3.7}$$

Then substitute this into (3.6) to get an intertemporal Euler equation.

$${c^K_{i,s+j-1,t+j-1}}^{-\sigma} = \beta(1 + r_{i,t+j} - \delta){c^K_{i,s+j,t+j}}^{-\sigma} \tag{3.8}$$

Next, take the ratio of (3.3) and (3.4) and substitute in (3.7).

$$\begin{aligned} \frac{{c_{i,s+j,t+j}}^{1-1/\rho}}{\chi {\ell_{i,s+j,t+j}}^{1-1/\rho}} &= \frac{1}{w_{i,t+j} e_{s+j,t+j}} \\ \ell_{i,s+j,t+j} &= c_{i,s+j,t+j} \left(\frac{\chi}{w_{i,t+j} e_{s+j,t+j}} \right)^{\rho} \end{aligned} \tag{3.9}$$

Substitute (3.7) and (3.9) into (3.3).

$$\beta^j \left\{ {c_{i,s+j,t+j}}^{1-1/\rho} + \chi \left[c_{i,s+j,t+j} \left(\frac{\chi}{w_{i,t+j} e_{s+j,t+j}} \right)^{\rho} \right]^{1-1/\rho} \right\}^{\frac{1-\sigma}{1-1/\rho}-1} \left(1 - \tfrac{1}{\rho}\right)^{-1} {c_{i,s+j,t+j}}^{-1/\rho}$$

$$= \beta^j {c^K_{i,s+j,t+j}}^{-\sigma}$$

$$\left\{ {c_{i,s+j,t+j}}^{1-1/\rho} \left[1 + \chi \left(\frac{\chi}{w_{i,t+j} e_{s+j,t+j}} \right)^{\rho} \right]^{1-1/\rho} \right\}^{\frac{1-\rho\sigma}{\rho-1}} {c_{i,s+j,t+j}}^{-1/\rho}$$

$$= \left(1 - \tfrac{1}{\rho}\right) {c^K_{i,s+j,t+j}}^{-\sigma}$$

$$\left[1 + \chi \left(\frac{\chi}{w_{i,t+j} e_{s+j,t+j}} \right)^{\rho} \right]^{\left[(1-1/\rho)\frac{1-\rho\sigma}{\rho-1}\right]} {c_{i,s+j,t+j}}^{\left[(1-1/\rho)\frac{1-\rho\sigma}{\rho-1}-1/\rho\right]}$$

$$= \left(1 - \tfrac{1}{\rho}\right) {c^K_{i,s+j,t+j}}^{-\sigma}$$

$$\left[1 + \chi \left(\frac{\chi}{w_{i,t+j} e_{s+j,t+j}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} {c_{i,s+j,t+j}}^{-\sigma} = \left(1 - \tfrac{1}{\rho}\right) {c^K_{i,s+j,t+j}}^{-\sigma}$$

$$c^K_{i,s+j,t+j} = c_{i,s+j,t+j} \left\{ \left[1 + \chi \left(\frac{\chi}{w_{i,t+j} e_{s+j,t+j}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho - 1} \right\}^{-\frac{1}{\sigma}}$$

We rewrite this, more conveniently, as:

$$c^K_{i,s+j,t+j} = c_{i,s+j,t+j} \Gamma_{i,t+j} \tag{3.10}$$

$$\Gamma_{i,t+j} \equiv \left\{ \left[1 + \chi \left(\frac{\chi}{w_{i,t+j} e_{s+j,t+j}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho - 1} \right\}^{-\frac{1}{\sigma}} \tag{3.11}$$

Substituting (3.9), (3.10) and (3.11), into the budget constraint, (3.2), gives:

$$\begin{aligned}
c_{ist} + KID_{ist}\Gamma_{it}c_{ist} + \left(\frac{\chi}{w_{it}e_{st}}\right)^{\rho} c_{ist} &= w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} - a_{i,s+1,t+1} \\
c_{ist}\left[1 + KID_{ist}\Gamma_{it} + \left(\frac{\chi}{w_{it}e_{st}}\right)^{\rho}\right] &= w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} - a_{i,s+1,t+1} \\
c_{ist} &= \frac{w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} - a_{i,s+1,t+1}}{1 + KID_{ist}\Gamma_{it} + \left(\frac{\chi}{w_{it}e_{st}}\right)^{\rho}}
\end{aligned} \tag{3.12}$$

If we stationarize (3.9), (3.10), (3.11) and (3.12), we can add them to the list of other behavioral equations from previous sections above to get the following.

$$g_t^N = \sum_{i=1}^{I}\sum_{s=1}^{S}\hat{N}_{ist}(f_{ist} + m_{ist} - \rho_{ist}); \forall i \tag{3.13}$$

$$\hat{N}_{i,1,t+1} = e^{-g_t^N}\sum_{s=23}^{45}\hat{N}_{ist}f_{ist}; \forall i \tag{3.14}$$

$$\hat{N}_{i,s+1,t+1} = e^{-g_t^N}\hat{N}_{ist}(1 + m_{ist} - \rho_{ist}); \forall i, 1 < s \leq S \tag{3.15}$$

$$KID_{i,s+1,t+1} = \sum_{r=1}^{20} f_{i,s-r,t-r}; \forall i, 1 < s \leq S \tag{3.16}$$

$$\hat{k}_{it} = \sum_{s=1}^{S}\hat{a}_{ist}\hat{N}_{ist} - \hat{k}_{it}^{f}; \forall i \tag{3.17}$$

$$\hat{n}_{it} = \sum_{s=1}^{S} e_{is}\hat{N}_{ist}; \forall i \tag{3.18}$$

$$\hat{y}_{it} = \hat{k}_{it}^{\alpha}\left(A_i\hat{n}_{it}\right)^{1-\alpha}; \forall i \tag{3.19}$$

$$r_{it} = \alpha \frac{\hat{y}_{it}}{\hat{k}_{it}}; \forall i \tag{3.20}$$

$$w_{it} = (1-\alpha) \frac{\hat{y}_{it}}{\hat{n}_{it}}; \forall i \tag{3.21}$$

$$\hat{B}Q_{it} = \sum_{s=67}^{S} \hat{a}_{ist} \rho_{ist} \hat{N}_{ist}; \forall i \tag{3.22}$$

$$\hat{B}Q_{it} = \sum_{s=23}^{67} \hat{b}q_{ist} (1-\rho_{ist}) \hat{N}_{ist}; \forall i \tag{3.23}$$

$$\Gamma_{it} = \left\{ \left[1 + \chi \left(\frac{\chi}{w_{it} e_{st}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho - 1} \right\}^{-\frac{1}{\sigma}}; \forall i \tag{3.24}$$

$$\hat{c}_{ist} = \frac{w_{it} e_{st} \bar{\ell}_t + (1 + r_{it} - \delta) \hat{a}_{ist} - \hat{a}_{i,s+1,t+1} e^{g^A}}{1 + KID_{ist} \Gamma_{it} + \left(\frac{\chi}{w_{it} e_{st}} \right)^{\rho}}; \forall i, s \tag{3.25}$$

$$\hat{\ell}_{ist} = \hat{c}_{ist} \left(\frac{\chi}{w_{it} e_{st}} \right)^{\rho}; \forall i, s \tag{3.26}$$

$$\hat{c}^{K}_{ist} = \hat{c}_{ist} \left\{ \left[1 + \chi \left(\frac{\chi}{w_{it} e_{st}} \right)^{\rho - 1} \right]^{\frac{1-\rho\sigma}{\rho-1}} (1-\rho) \right\}^{\frac{1}{\sigma}} \tag{3.27}$$

$$\left(\hat{c}^{K}_{ist} \right)^{-\sigma} - \beta \left(\hat{c}^{K}_{i,s+1,t+1} e^{g^A} \right)^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i, s \tag{3.28}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{3.29}$$

$$\sum_{i=1}^{I} \hat{k}^{f}_{it} \left(\sum_{s=1}^{S} \hat{N}_{ist} \right) = 0 \tag{3.30}$$