Step-by-Step Model Buildup

1 Simple S-period I-Country OLG Model

Consider a model with no complicated demographics. Every period a unit measure of labor is born in each of I countries and these workers live for exactly S period.

Each country produces the same single good which is mobile across country borders.

Households of age s in country i in period t solve the following maximization problem.

$$\max_{\{c_{i,s+j,t+j}\}_{j=0}^{S-s}} U_{ist} = \sum_{j=0}^{S-s} \beta^j \frac{1}{1-\sigma} c_{i,s+j,t+j}^{1-\sigma}$$

subject to the following budget constraint each period.

$$c_{ist} = w_{it}e_{st} + (1 + r_{it} - \delta)a_{ist} - a_{i,s+1,t+1} \tag{1.1}$$

with $a_{i1t} = a_{i,S+1,t} = 0$

The solution to this problem yields the following intertemporal Euler equation.

$$c_{ist}^{-\sigma} = \beta c_{i,s+1,t+1}^{-\sigma}(1 + r_{i,t+1} - \delta) \tag{1.2}$$

The representative firm in each country maximizes profits subject to a Cobb-Douglas production function.

$$\max_{n_{it},k_{it}} \Pi_{it} = k_{it}^{\alpha}\left(A_i n_{it}\right)^{1-\alpha} - w_{it}n_{it} - r_{it}k_{it}$$

This yields the following factor price equations.

$$y_{it} = k_{it}^{\alpha} (A_i n_{it})^{1-\alpha} \tag{1.3}$$

$$r_{it} = \alpha \frac{y_{it}}{k_{it}} \tag{1.4}$$

$$w_{it} = (1 - \alpha) \frac{y_{it}}{n_{it}} \tag{1.5}$$

Labor and capital market clearing conditions in each country give the following.

$$k_{it} = \sum_{s=1}^{S} a_{ist} - k_{it}^f \tag{1.6}$$

$$n_{it} = \sum_{s=1}^{S} e_{is} \tag{1.7}$$

where k_{it}^f denotes domestically-owned capital located in foreign countries.

Capital is allocated across countries so that the rate of return is equalized.

$$r_{it} = r_t \tag{1.8}$$

And the net sum of all foreign-located capital across countries must be zero.

$$\sum_{i=1}^{I} k_{it}^f = 0 \tag{1.9}$$

Given a set of state variables today $X_t = \{a_{ist}, k^f it\} \forall i, s$ we can calculate

the following.

$$k_{it} = \sum_{s=1}^{S} a_{ist} - k_{it}^f; \forall i \tag{1.10}$$

$$n_{it} = \sum_{s=1}^{S} e_{is}; \forall i \tag{1.11}$$

$$y_{it} = k_{it}^{\alpha} \left(A_i n_{it} \right)^{1-\alpha}; \forall i \tag{1.12}$$

$$r_{it} = \alpha \frac{y_{it}}{k_{it}}; \forall i \tag{1.13}$$

$$w_{it} = (1 - \alpha) \frac{y_{it}}{n_{it}}; \forall i \tag{1.14}$$

$$c_{ist} = w_{it} e_{st} + (1 + r_{1t} - \delta) a_{ist} - a_{i,s+1,t+1}; \forall i, s \tag{1.15}$$

The IS values of X_t in every period must satisfy the following equations IS.

$$c_{ist}^{-\sigma} - \beta c_{i,s+1,t+1}^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i, s \tag{1.16}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{1.17}$$

$$\sum_{i=1}^{I} k_{it}^f = 0 \tag{1.18}$$

To find the steady state we impose $a_{ist} = \bar{a}_{is}$ and $k_{it}^f = \bar{k}_i^f$ for all t and using equations (1.10) - (1.15) search over these values to find the ones that set the values of equations (1.16) - (1.18) to zero. This involves using fsolve.

To find the transition path we make an initial guess of a history of $\{r_t^0\}$ and $\{w_{it}^0\}$ values for $t = 1$ to T with the values of $\bar{r}$ and $\bar{w}_i$ for $t > T$. We also have known initial value for a_{is1} and k_{i1}^f. With this information we search over the value of c_{is1} for each household to find the value that sets

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$a_{iS,S-s}$ (their final savings) to zero when we chain equations (1.16) and (1.15). That is, we solve for each households optimal time path of consumption and savings given the assumed history of factor prices. We can then use equations (1.10) through (1.14) to find histories $\{r_t^{new}\}$ and $\{w_{it}^{new}\}$. If these are not sufficiently different from the previous guess, we are done and have the transition path. If they are different we use a new guess at the histories give by the equations below which are a convex combination of the old guess and the implied new histories.

$$r_t^{j+1} = \chi r_t^j + (1-\chi)r_t^{new}; \forall t \leq T \tag{1.19}$$

$$w_t^{j+1} = \chi w_t^j + (1-\chi)w_t^{new}; \forall t \leq T \tag{1.20}$$

2 Add Demographics and Growth

Denote the population of age s people in country i in period t as N_{ist}. Denote the fraction of children born to people of age s in country i in period t as f_{ist} and the immigration rate as m_{ist}. The mortality hazard is denoted ρ_{ist}. We assume that $\rho_{ist} = 0$ if $s < 68$ and $\rho_{iSt} = 1$. We also assume that $f_{ist} = 0$ for $s < 23$ and $s > 45$.

The population dynamics are given by:

$$N_{i,1,t+1} = \sum_{s=1}^{S} N_{ist} f_{ist} \tag{2.1}$$

$$N_{i,s+1,t+1} = N_{ist}(1 + m_{ist} - \rho_{ist}); 1 < s \leq S \tag{2.2}$$

We define the total world population as:

$$N_t = \sum_{i=1}^{I} \sum_{s=1}^{S} N_{ist} \tag{2.3}$$

The populations are, strictly speaking, state variables, but their steady state can be found independent of the other variables. For the model to be stationarizable, it must be that in the long run, by period T_1, that the fertility, immigration, and mortality rates are the same for all countries.

$$f_{ist} = \bar{f}_s; t \geq T_1 \tag{2.4}$$

$$m_{ist} = \bar{m}_s; t \geq T_1 \tag{2.5}$$

$$\rho_{ist} = \bar{\rho}_s; t \geq T_1 \tag{2.6}$$

Inheritances are partially intended bequests and are the assets of agents that die at the end of period t to the survivors in that period. Let BQ_{it} denote total bequests in period t for country i and bq_{ist} denote the bequest received by a person s years old in that same country and time period.

$$BQ_{it} = \sum_{s=67}^{S} a_{ist} \rho_{ist} N_{ist} \tag{2.7}$$

$$= \sum_{s=23}^{67} bq_{ist} (1 - \rho_{ist}) N_{ist} \tag{2.8}$$

We specify a bequest distribution rule that tells what proportion of available bequest funds goes to each household within a country. One possible simple rule would be to divide the funds up equally over all survivors. This rule would be:

$$bq_{ist} = \frac{BQ_{it}}{\sum_{s=23}^{67} (1 - \rho_{ist}) N_{ist}}$$

While relative populations stabilize in the steady state, the overall population may be growing or shrinking. Hence, all aggregate variables will need to be stationarized by dividing by the total population of a country. To help with this, we define the growth rate of the world population as $g_t^N = \frac{N_{it}}{N_{t-1}} - 1$.

We also allow for technical progress by having the endowment of labor per household rise by a constant amount each period. We denote this endowment as $\bar{\ell}_t$ and denote its growth rate as g^A.

The aggregate labor for a country is now given by $n_{it} = \sum_{s=1}^{S} N_{ist} e_{is} \bar{\ell}_t$. Output, y_t, is defined as before.

Aggregate variables will grow at the rate $g^A + g_t^N$ and per capita variables

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will grow at rate g^A. The stationarized versions of our behavioral equations are given below, where a carat denotes the stationarized variable. Note that $\hat{N}_{ist}$ is interpretable as the fraction of the world population in country i of age s at time t so that $\sum_{i=1}^{I} \sum_{s=1}^{S} \hat{N}_{ist} = 1$. Note that both k_{it} and n_{it} are growing at rate $g^A + g_t^N$, so r_{it} and w_{it} are stationary.

$$g_t^N = \sum_{i=1}^{I} \sum_{s=1}^{S} \hat{N}_{ist}(f_{ist} + m_{ist} - \rho_{ist}); \forall i \tag{2.9}$$

$$\hat{N}_{i,1,t+1} = e^{-g_t^N} \sum_{s=23}^{45} \hat{N}_{ist} f_{ist}; \forall i \tag{2.10}$$

$$\hat{N}_{i,s+1,t+1} = e^{-g_t^N} \hat{N}_{ist}(1 + m_{ist} - \rho_{ist}); \forall i, 1 < s \leq S \tag{2.11}$$

$$\hat{k}_{it} = \sum_{s=1}^{S} \hat{a}_{ist} \hat{N}_{ist} - \hat{k}_{it}^f; \forall i \tag{2.12}$$

$$\hat{n}_{it} = \sum_{s=1}^{S} e_{is} \hat{N}_{ist}; \forall i \tag{2.13}$$

$$\hat{y}_{it} = \hat{k}_{it}^{\alpha} \left(A_i \hat{n}_{it}\right)^{1-\alpha}; \forall i \tag{2.14}$$

$$r_{it} = \alpha \frac{\hat{y}_{it}}{\hat{k}_{it}}; \forall i \tag{2.15}$$

$$w_{it} = (1 - \alpha) \frac{\hat{y}_{it}}{\hat{n}_{it}}; \forall i \tag{2.16}$$

$$\hat{BQ}_{it} = \sum_{s=67}^{S} \hat{a}_{ist} \rho_{ist} \hat{N}_{ist}; \forall i \tag{2.17}$$

$$\hat{BQ}_{it} = \sum_{s=23}^{67} \hat{bq}_{ist}(1 - \rho_{ist}) \hat{N}_{ist}; \forall i \tag{2.18}$$

$$\hat{c}_{ist} = w_{it} e_{st} + (1 + r_{1t} - \delta)\hat{a}_{ist} + \hat{bq}_{ist} - \hat{a}_{i,s+1,t+1} e^{g^A}; \forall i, s \tag{2.19}$$

$$\hat{c}_{ist}^{-\sigma} - \beta \left(\hat{c}_{i,s+1,t+1} e^{g^A} \right)^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i, s \tag{2.20}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{2.21}$$

$$\sum_{i=1}^{I} \hat{k}_{it}^{f} = 0 \tag{2.22}$$

Note that the first three equations of the above system can be solved for steady state values independent of the rest of the system as long as the steady state values of the fertility, immigration, and mortality rates are known.

Once we have these we can find the steady state values for the rest of the system as we did above.

Also note that for the transition path we can find the time paths for the $\hat{N}$ variables independent of the time-path iteration used to find the other variables.

3   Add a Leisure Decision

Assume the household chooses some fraction of its available time to consume as leisure and supplies the rest as labor. Denote is leisure consumption as ℓ_{ist}. The household's problem is now given by:

$$\max_{\{c_{i,s+j,t+j}, \ell_{i,s+j,t+j}\}_{j=0}^{S-s}} U_{ist} =$$

$$\sum_{j=0}^{S-s} \beta^j \frac{1}{1-\sigma} \left(c_{i,s+j,t+j}^{1-1/\rho} + \chi \ell_{i,s+j,t+j}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}}$$

subject to the following budget constraint each period.

$$c_{ist} = w_{it} e_{st} (\bar{\ell}_t - \ell_{ist}) + (1 + r_{it} - \delta) a_{ist} + b q_{ist} - a_{i,s+1,t+1} \qquad (3.1)$$

with $a_{i1t} = a_{i,S+1,t} = 0$.

We model technical progress by assuming that the time endowment per household $\bar{\ell}_t$ is growing at rate g^A each period.

The first-order conditions are:

$$\beta^j \left(c_{i,s+j,t+j}^{1-1/\rho} + \chi \ell_{i,s+j,t+j}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}-1} c_{i,s+j,t+j}^{-1/\rho}$$

$$= -\lambda_{i,s+j,t+j} \qquad (3.2)$$

$$\beta^j \left(c_{i,s+j,t+j}^{1-1/\rho} + \chi \ell_{i,s+j,t+j}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}-1} \chi \ell_{i,s+j,t+j}^{-1/\rho}$$

$$= -\lambda_{i,s+j,t+j} w_{i,t+j} e_{s+j,t+j} \qquad (3.3)$$

$$(1 + r_{i,t+j} - \delta) \lambda_{i,s+j,t+j} = \lambda_{i,s+j-1,t+j-1} \qquad (3.4)$$

along with (3.1).

Take the ratio of (3.2) and (3.3) and rearrange.

$$\frac{c_{i,s+j,t+j}^{-1/\rho}}{\chi \ell_{i,s+j,t+j}^{-1/\rho}} = \frac{1}{w_{i,t+j}e_{s+j,t+j}}$$

$$c_{i,s+j,t+j}^{-1/\rho}w_{i,t+j}e_{s+j,t+j} = \chi \ell_{i,s+j,t+j}^{-1/\rho}$$

$$\ell_{i,s+j,t+j} = \left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^\rho c_{i,s+j,t+j} \tag{3.5}$$

Substituting (3.5) into (3.2) gives:

$$\lambda_{i,s+j,t+j} =$$

$$- \beta^j \left(c_{i,s+j,t+j}^{\frac{\rho-1}{\rho}} + \chi \left[\left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^\rho c_{i,s+j,t+j}\right]^{\frac{\rho-1}{\rho}}\right)^{\frac{1-\rho\sigma}{\rho-1}} c_{i,s+j,t+j}^{-1/\rho}$$

$$- \beta^j \left(c_{i,s+j,t+j}^{\frac{\rho-1}{\rho}} \left[1 + \chi \left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^\rho\right]^{\frac{\rho-1}{\rho}}\right)^{\frac{1-\rho\sigma}{\rho-1}} c_{i,s+j,t+j}^{-1/\rho}$$

$$- \beta^j c_{i,s+j,t+j}^{\frac{1-\rho\sigma}{\rho}} \left[1 + \chi \left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^\rho\right]^{\frac{1-\rho\sigma}{\rho}} c_{i,s+j,t+j}^{-1/\rho}$$

$$- \beta^j \left[1 + \chi \left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^\rho\right]^{\frac{1-\rho\sigma}{\rho}} c_{i,s+j,t+j}^{-\sigma} \tag{3.6}$$

Define $\Psi_{i,s+j,t+j}$.

$$\Psi_{i,s+j,t+j} \equiv \left[1 + \chi \left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^\rho\right]^{\frac{1-\rho\sigma}{\rho}} \tag{3.7}$$

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Now substitute (3.6) and (3.7) into (3.4).

$$(1 + r_{i,t+j} - \delta)\beta^j \Psi_{i,s+j,t+j} c_{i,s+j,t+j}{}^{-\sigma} = \beta^{j-1} \Psi_{i,s+j-1,t+j-1} c_{i,s+j-1,t+j-1}{}^{-\sigma}$$

$$\Psi_{i,s+j-1,t+j-1} c_{i,s+j-1,t+j-1}{}^{-\sigma} = \beta \Psi_{i,s+j,t+j} c_{i,s+j,t+j}{}^{-\sigma}(1 + r_{i,t+j} - \delta)$$

$$(3.8)$$

Substituting (3.5) into (3.1):

$$c_{ist} = w_{it}e_{st}\bar{\ell}_t - w_{it}e_{st}\left(\frac{\chi}{w_{it}e_{st}}\right)^\rho c_{ist} + (1 + r_{it} - \delta)a_{ist} + bq_{ist} - a_{i,s+1,t+1}$$

$$c_{ist}\left[1 + w_{it}e_{st}\left(\frac{\chi}{w_{it}e_{st}}\right)^\rho\right] = w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} + bq_{ist} - a_{i,s+1,t+1}$$

$$c_{ist} = \frac{w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} + bq_{ist} - a_{i,s+1,t+1}}{1 + w_{it}e_{st}\left(\frac{\chi}{w_{it}e_{st}}\right)^\rho} \qquad (3.9)$$

If we stationarize (3.5), (3.7) and (3.9), we can add them to the list of other behavioral equations from previous sections above to get the following.

$$g_t^N = \sum_{i=1}^{I}\sum_{s=1}^{S} \hat{N}_{ist}(f_{ist} + m_{ist} - \rho_{ist}); \forall i \qquad (3.10)$$

$$\hat{N}_{i,1,t+1} = e^{-g_t^N}\sum_{s=23}^{45} \hat{N}_{ist}f_{ist}; \forall i \qquad (3.11)$$

$$\hat{N}_{i,s+1,t+1} = e^{-g_t^N}\hat{N}_{ist}(1 + m_{ist} - \rho_{ist}); \forall i, 1 < s \le S \qquad (3.12)$$

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$$\hat{k}_{it} = \sum_{s=1}^{S} \hat{a}_{ist} \hat{N}_{ist} - \hat{k}_{it}^{f}; \forall i \tag{3.13}$$

$$\hat{n}_{it} = \sum_{s=1}^{S} e_{is}(\bar{\ell}_t - \hat{\ell}_{ist}) \hat{N}_{ist}; \forall i \tag{3.14}$$

$$\hat{y}_{it} = \hat{k}_{it}^{\alpha} (A_i \hat{n}_{it})^{1-\alpha}; \forall i \tag{3.15}$$

$$r_{it} = \alpha \frac{\hat{y}_{it}}{\hat{k}_{it}}; \forall i \tag{3.16}$$

$$\hat{w}_{it} = (1-\alpha) \frac{\hat{y}_{it}}{\hat{n}_{it}}; \forall i \tag{3.17}$$

$$\hat{BQ}_{it} = \sum_{s=67}^{S} \hat{a}_{ist} \rho_{ist} \hat{N}_{ist}; \forall i \tag{3.18}$$

$$\hat{BQ}_{it} = \sum_{s=23}^{67} \hat{bq}_{ist}(1 - \rho_{ist}) \hat{N}_{ist}; \forall i \tag{3.19}$$

$$\hat{c}_{ist} = \frac{\hat{w}_{it} e_{st} + (1 + r_{it} - \delta)\hat{a}_{ist} + \hat{bq}_{ist} - \hat{a}_{i,s+1,t+1} e^{g^A}}{1 + \hat{w}_{it} e_{st} \left(\frac{\chi}{\hat{w}_{it} e_{st}}\right)^{\rho}}; \forall i, s \tag{3.20}$$

$$\hat{\ell}_{ist} = \hat{c}_{ist} \left(\frac{\chi}{\hat{w}_{it} e_{st}}\right)^{\rho}; \forall i, s \tag{3.21}$$

$$\Psi_{ist} = \left[1 + \chi \left(\frac{\chi}{\hat{w}_{it} e_{st}}\right)^{\rho}\right]^{\frac{1-\rho\sigma}{\rho}}; \forall i, s \tag{3.22}$$

$$\Psi_{ist} \hat{c}_{ist}^{-\sigma} = \beta(1 - \rho_{i,s+1,t+1}) \Psi_{i,s+1,t+1} \left(\hat{c}_e iste^{g^A}\right)^{-\sigma} (1 + r_{1,t+1} - \delta); \forall i, s \tag{3.23}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{3.24}$$

$$\sum_{i=1}^{I} \hat{k}_{it}^{f} = 0 \tag{3.25}$$

To find the transition path we use the following algorithm.

First, use equations (3.10) through (3.12) to get the time path for $\{N_{ist}\}$

for $t = 1$ to T.

Then declare the values of the initial state $\{a_{ist}$ for $t = 1$ to T.

Next, make an initial guess of a history of $\{r_t^0\}$ and $\{bq_{it}^0\}$ values for $t = 1$ to T with the values of $\bar{r}$ and $\bar{bq}_i$ for $t > T$.

We can rewrite equations (3.15) - (3.16) as follows.

$$w_{it} = (1 - \alpha)A_i k_{it}^{\alpha} n_{it}^{-\alpha}$$

$$r_t = \alpha A_i k_{it}^{\alpha-1} n_{it}^{1-\alpha}$$

Solving for k_{it} gives:

$$k_{it} = \left(\frac{w_{it}}{(1-\alpha)A_i}\right)^{\frac{1}{\alpha}} n_{it}$$

$$k_{it} = \left(\frac{\alpha A_i}{rt}\right)^{\frac{1}{1-\alpha}} n_{it}$$

This can be solved for w_{it}.

$$w_{it} = \left(\frac{\alpha A_i}{rt}\right)^{\frac{\alpha}{1-\alpha}} (1 - \alpha)A_i \tag{3.26}$$

With this information search over the value of c_{is1} for each household to find the value that sets $a_{iS,S-s}$ (their final savings) to zero when we chain equations (3.22), (3.23) and (3.20). That is, we solve for each households optimal time path of consumption and savings given the assumed history of factor prices. Note that we will use equations (3.18) and (3.19) in this step. This step requires the use of `fsolve`.

We can then use equations (3.21) and (3.14) to get individual labor supply

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and total labor input.

We then define the total domestically-owned capital stock.

$$\hat{k}^d_{it} = \sum_i \hat{a}_{ist} \hat{N}_{ist} \tag{3.27}$$

Using (3.13) and (3.16) gives $r_{it} = \alpha A_i (\hat{k}^d_{it} + \hat{k}^d_{it})^{\alpha-1} \hat{n}^{1-\alpha}_{it}$. Solving this for $\hat{k}^f_{it}$ noting (3.24) gives:

$$\hat{k}^f_{it} = \left(\frac{\alpha A_t}{r_t}\right)^{\frac{1}{1-\alpha}} \hat{n}_{it} - \hat{k}^d_{it} \tag{3.28}$$

We then use (3.25) and solve for k^f_{1t} as the negative of the sum of the k^f's from all the other countries.

$$\hat{k}^f_{1t} = -\sum_{i=2}^{I} \hat{k}^f_{it} \tag{3.29}$$

Finally, we use equations (3.16) for country 1 and (3.17) for all countries to generate updated values of $\{r_t\}$ and $\{w_{it}\}$. We then use the a convex combination of the original guesses and these new values as guesses for the next round of the iteration.

As an alternative, we can use `fsolve` to find the whole histories of $\{r_t\}$ and $\{bq_{it}\}$. that satisfy equations (3.14) and (3.30) for all time periods.

$$bq_{it} - \frac{\sum_{s=23}^{67} \hat{bq}_{ist}(1-\rho_{ist})\hat{N}_{ist}}{\sum_{s=67}^{S} \hat{a}_{ist}\rho_{ist}\hat{N}_{ist}} = 0 \tag{3.30}$$

With this approach we can define a function that chains equations (3.22), (3.23) and (3.20) to generate final assets as a function of initial consumption. We can write this as `afinal = g(cinitial,S,Omega)`, where `S` denotes the

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household's age, and `Omega` denotes the history of wages, interest rates and bequests over the household's lifetime.

For households alive in period one we can stack the initial consumptions into an $S \times 1$ vector denoted `Calive` and define the vector function `falive` as the following.

$$
\texttt{falive(Calive,Omega)} =
\begin{bmatrix}
\texttt{g(c1initial,1,Omega)} \\
\texttt{g(c2initial,2,Omega)} \\
\vdots \\
\texttt{g(cSinitial,S,Omega)}
\end{bmatrix}
$$

For households that will be born in the future we can stack all their age 1 consumptions into a $(T-1) \times 1$ vector denoted `Cfuture` and define the vector function `ffuture` as the following.

$$
\texttt{ffuture(Cfuture,BigOmega)} =
\begin{bmatrix}
\texttt{g(c1per2,S,BigOmega(2:S+1))} \\
\texttt{g(c2per3,S,BigOmega(3:S+2))} \\
\vdots \\
\texttt{g(cSperT,S,BigOmega(T-S+1,T))}
\end{bmatrix}
$$

where `BigOmega` is the history of ages, interest rates and bequests from period 2 to T.

We can use `fsolve` on these two functions separately to find the consumption times paths and asset holdings within the larger `fsolve`

Solving for the steady state is accomplished in a similar fashion only using steady state values.

4   Add Children

Denote the number of children of age r in household of age s in country i in period t as KID_{irst}. We assume that mortality hazard rates are zero up until age 68 and that fertility rates are non zero only for ages 23 through 45. This means that we avoid the situation where a child lives, but the parents die, and vice versa.

The evolution of theses numbers is given by:

$$KID_{i,r+1,s+1,t+1} = \begin{cases} f_{ist} & r = 1 \\ KID_{irst} & 1 < r < 21 \end{cases}$$

Iterative substitution gives $KID_{i,r+1,s+1,t+1} = f_{i,s-r,t-r}$. If we then define the number of children of all ages to a household of age s as KID_{ist} we have the following formula.

$$KID_{i,s+1,t+1} = \sum_{r=1}^{20} f_{i,s-r,t-r} \tag{4.1}$$

Since this is a per household number of children it is already stationary and does not need to be turned into a "hat" variable.

The household's problem is now given by:

$$\max_{\{c_{i,s+j,t+j},\ell_{i,s+j,t+j},c^K{}_{i,s+j,t+j},a_{i,s+j,t+j}\}_{j=0}^{S-s}} U_{ist} =$$

$$\sum_{j=0}^{S-s} \beta^j \frac{1}{1-\sigma} \left[\left(c_{i,s+j,t+j}^{1-1/\rho} + \chi \ell_{i,s+j,t+j}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}} + KID_{i,s+j,t+j} c_{i,s+j,t+j}^{K\ 1-\sigma} \right]$$

subject to the following budget constraint each period.

$$c_{ist} + KID_{ist}c_{ist}^K = w_{it}e_{st}(\bar{\ell}_t - \ell_{ist}) + (1 + r_{it} - \delta)a_{ist} + bq_{ist} - a_{i,s+1,t+1} \quad (4.2)$$

with $a_{i1t} = a_{i,S+1,t} = 0$. ℓ denotes leisure consumption and c^K denotes consumption by children.

The first-order conditions are:

$$\beta^j \left(c_{i,s+j,t+j}^{1-1/\rho} + \chi\ell_{i,s+j,t+j}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}-1} c_{i,s+j,t+j}^{-1/\rho}$$

$$= -\lambda_{i,s+j,t+j} \quad (4.3)$$

$$\beta^j \left(c_{i,s+j,t+j}^{1-1/\rho} + \chi\ell_{i,s+j,t+j}^{1-1/\rho} \right)^{\frac{1-\sigma}{1-1/\rho}-1} \chi\ell_{i,s+j,t+j}^{-1/\rho}$$

$$= -\lambda_{i,s+j,t+j}w_{i,t+j}e_{s+j,t+j} \quad (4.4)$$

$$\beta^j KID_{i,s+j,t+j}c_{i,s+j,t+j}^K{}^{-\sigma} = -\lambda_{i,s+j,t+j}KID_{i,s+j,t+j} \quad (4.5)$$

$$(1 + r_{i,t+j} - \delta)\lambda_{i,s+j,t+j} = \lambda_{i,s+j-1,t+j-1} \quad (4.6)$$

along with (4.2).

Solve the (4.5) for the LaGrange multiplier.

$$\lambda_{i,s+j,t+j} = -\beta^j c_{i,s+j,t+j}^K{}^{-\sigma} \quad (4.7)$$

Then substitute this into (4.6) to get an intertemporal Euler equation.

$$c_{i,s+j-1,t+j-1}^K{}^{-\sigma} = \beta(1 + r_{i,t+j} - \delta)c_{i,s+j,t+j}^K{}^{-\sigma} \quad (4.8)$$

Next, take the ratio of (4.3) and (4.4) and substitute in (4.7).

$$\frac{c_{i,s+j,t+j}^{1-1/\rho}}{\chi \ell_{i,s+j,t+j}^{1-1/\rho}} = \frac{1}{w_{i,t+j}e_{s+j,t+j}}$$

$$\ell_{i,s+j,t+j} = c_{i,s+j,t+j}\left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^{\rho} \tag{4.9}$$

Substitute (4.7) and (4.9) into (4.3).

$$\beta^j \left\{ c_{i,s+j,t+j}^{1-1/\rho} + \chi\left[c_{i,s+j,t+j}\left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^{\rho}\right]^{1-1/\rho} \right\}^{\frac{1-\sigma}{1-1/\rho}-1} \left(1-\frac{1}{\rho}\right)^{-1} c_{i,s+j,t+j}^{-1/\rho}$$

$$=\beta^j c_{i,s+j,t+j}^{K}{}^{-\sigma}$$

$$\left\{ c_{i,s+j,t+j}^{1-1/\rho}\left[1 + \chi\left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^{\rho}\right]^{1-1/\rho} \right\}^{\frac{1-\rho\sigma}{\rho-1}} c_{i,s+j,t+j}^{-1/\rho}$$

$$= \left(1-\frac{1}{\rho}\right) c_{i,s+j,t+j}^{K}{}^{-\sigma}$$

$$\left[1 + \chi\left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^{\rho}\right]^{\left[(1-1/\rho)\frac{1-\rho\sigma}{\rho-1}\right]} c_{i,s+j,t+j}^{\left[(1-1/\rho)\frac{1-\rho\sigma}{\rho-1}-1/\rho\right]}$$

$$= \left(1-\frac{1}{\rho}\right) c_{i,s+j,t+j}^{K}{}^{-\sigma}$$

$$\left[1 + \chi\left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^{\rho}\right]^{\frac{1-\rho\sigma}{\rho}} c_{i,s+j,t+j}^{-\sigma} = \left(1-\frac{1}{\rho}\right) c_{i,s+j,t+j}^{K}{}^{-\sigma}$$

$$c_{i,s+j,t+j}^{K} = c_{i,s+j,t+j}\left\{\left[1 + \chi\left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}}\right)^{\rho}\right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho-1}\right\}^{-\frac{1}{\sigma}}$$

We rewrite this, more conveniently, as:

$$c^K_{i,s+j,t+j} = c_{i,s+j,t+j}\Gamma_{i,t+j} \tag{4.10}$$

$$\Gamma_{i,t+j} \equiv \left\{ \left[1 + \chi \left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}} \right)^\rho \right]^{\frac{1-\rho\sigma}{\rho}} \right\}^{-\frac{1}{\sigma}} \tag{4.11}$$

$$c^K_{i,s+j,t+j} = c_{i,s+j,t+j}\Gamma_{i,t+j} \tag{4.12}$$

$$\Gamma_{i,t+j} \equiv \left\{ \left[1 + \chi \left(\frac{\chi}{w_{i,t+j}e_{s+j,t+j}} \right)^\rho \right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho-1} \right\}^{-\frac{1}{\sigma}} \tag{4.13}$$

Substituting (4.9), (4.10) and (4.11), into the budget constraint, (4.2), gives:

$$c_{ist} + KID_{ist}\Gamma_{it}c_{ist} + \left(\frac{\chi}{w_{it}e_{st}} \right)^\rho c_{ist} = w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} - a_{i,s+1,t+1}$$

$$c_{ist} \left[1 + KID_{ist}\Gamma_{it} + \left(\frac{\chi}{w_{it}e_{st}} \right)^\rho \right] = w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} - a_{i,s+1,t+1}$$

$$c_{ist} = \frac{w_{it}e_{st}\bar{\ell}_t + (1 + r_{it} - \delta)a_{ist} - a_{i,s+1,t+1}}{1 + KID_{ist}\Gamma_{it} + \left(\frac{\chi}{w_{it}e_{st}} \right)^\rho}$$

$$\tag{4.14}$$

If we stationarize (4.9), (4.10), (4.11) and (4.14), we can add them to the list of other behavioral equations from previous sections above to get the following.

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$$g_t^N = \sum_{i=1}^{I} \sum_{s=1}^{S} \hat{N}_{ist}(f_{ist} + m_{ist} - \rho_{ist}); \forall i \tag{4.15}$$

$$\hat{N}_{i,1,t+1} = e^{-g_t^N} \sum_{s=23}^{45} \hat{N}_{ist} f_{ist}; \forall i \tag{4.16}$$

$$\hat{N}_{i,s+1,t+1} = e^{-g_t^N} \hat{N}_{ist}(1 + m_{ist} - \rho_{ist}); \forall i, 1 < s \le S \tag{4.17}$$

$$g_t^N = \sum_{i=1}^{I} \sum_{s=1}^{S} \hat{N}_{ist}(f_{ist} + m_{ist} - \rho_{ist}); \forall i \tag{4.18}$$

$$\hat{N}_{i,1,t+1} = e^{-g_t^N} \sum_{s=23}^{45} \hat{N}_{ist} f_{ist}; \forall i \tag{4.19}$$

$$\hat{N}_{i,s+1,t+1} = e^{-g_t^N} \hat{N}_{ist}(1 + m_{ist} - \rho_{ist}); \forall i, 1 < s \le S \tag{4.20}$$

$$KID_{i,s+1,t+1} = \sum_{r=1}^{20} f_{i,s-r,t-r}; \forall i, 1 < s \le S \tag{4.21}$$

$$\hat{k}_{it} = \sum_{s=1}^{S} \hat{a}_{ist} \hat{N}_{ist} - \hat{k}_{it}^{f}; \forall i \tag{4.22}$$

$$\hat{n}_{it} = \sum_{s=1}^{S} e_{is}(\bar{\ell}_t - \hat{\ell}_{ist}) \hat{N}_{ist}; \forall i \tag{4.23}$$

$$\hat{y}_{it} = \hat{k}_{it}^{\alpha} \left(A_i \hat{n}_{it}\right)^{1-\alpha}; \forall i \tag{4.24}$$

$$r_{it} = \alpha \frac{\hat{y}_{it}}{\hat{k}_{it}}; \forall i \tag{4.25}$$

$$\hat{w}_{it} = (1 - \alpha) \frac{\hat{y}_{it}}{\hat{n}_{it}}; \forall i \tag{4.26}$$

$$\hat{BQ}_{it} = \sum_{s=67}^{S} \hat{a}_{ist} \rho_{ist} \hat{N}_{ist}; \forall i \tag{4.27}$$

$$\hat{BQ}_{it} = \sum_{s=23}^{67} \hat{bq}_{ist}(1 - \rho_{ist}) \hat{N}_{ist}; \forall i \tag{4.28}$$

$$\Gamma_{it} = \left\{ \left[1 + \chi \left(\frac{\chi}{\hat{w}_{it} e_{st}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \right\}^{-\frac{1}{\sigma}}; \forall i, s \tag{4.29}$$

$$\hat{c}_{ist} = \frac{\hat{w}_{it} e_{st} + (1 + r_{it} - \delta)\hat{a}_{ist} + \hat{bq}_{ist} - \hat{a}_{i,s+1,t+1} e^{g^A}}{1 + KID_{ist}\Gamma_{it} + \left(\frac{\chi}{\hat{w}_{it} e_{st}} \right)^{\rho}}; \forall i, s \tag{4.30}$$

$$\Gamma_{it} = \left\{ \left[1 + \chi \left(\frac{\chi}{\hat{w}_{it} e_{st}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho - 1} \right\}^{-\frac{1}{\sigma}}; \forall i, s \tag{4.31}$$

$$\hat{c}_{ist} = \frac{\hat{w}_{it} e_{st} \bar{\ell}_t + (1 + r_{it} - \delta)\hat{a}_{ist} - \hat{a}_{i,s+1,t+1} e^{g^A}}{1 + KID_{ist}\Gamma_{it} + \left(\frac{\chi}{\hat{w}_{it} e_{st}} \right)^{\rho}}; \forall i, s \tag{4.32}$$

$$\hat{\ell}_{ist} = \hat{c}_{ist} \left(\frac{\chi}{\hat{w}_{it} e_{st}} \right)^{\rho}; \forall i, s \tag{4.33}$$

$$\hat{c}_{ist}^{K} = \hat{c}_{ist}\Gamma_{it}; \forall i, s \tag{4.34}$$

$$\left(\hat{c}_{ist}^{K} \right)^{-\sigma} - \beta \left(\hat{c}_{i,s+1,t+1}^{K} e^{g^A} \right)^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i, s \tag{4.35}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{4.36}$$

$$\sum_{i=1}^{I} \hat{k}_{it}^{f} = 0 \tag{4.37}$$

Note that in the code we use the following methodology to implement equations (4.15) - (4.17).

First, use the current distribution the population, $\{\hat{N}_{ist}\}\forall i, s$ and calculate tomorrow's population using the non-stationary equations (2.1) and (2.2) to generate next period's population, $\{N_{i,s,t+1}\}\forall i, s$. Note that the current values will sum to one, but next period's will not.

Calculate the growth of the population using $g_t^N = \sum_s \sum_i N_{i,s,t+1} - 1$.

Then let $\hat{N}_{i,s,t+1} = N_{i,s,t+1}e^{-g_t^N}$.

5   Adding Different Types of Labor

Let us allow for workers to be divided into J types in each country. We mostly use $J = 2$, but we will keep the notation more general than this. We will index these types by j so that c_{ijst} denotes consumption by a type j household in country i that is s periods old in period t.

We will need to specify fertility, immigration and mortality rates over the J types in each country. However, one simplifying assumption we can use is that these are the same across all types, so that, for example, $f_{ijst} = f_{ist} \forall j$.

We also need to specify an ability matrix that varies by type as well as age and country now; i.e. use e_{ijs}.

Bequests can be distributed by type as well, so we need to specify a distribution rule that allows this. In the Kotlikoff papers the rule used specifies that all funds are distributed within types, but may vary by age.

$$BQ_{ijt} = \sum_{s=67}^{S} a_{ijst} \rho_{ijst} N_{ijst} \qquad (5.1)$$

$$= \sum_{s=23}^{67} bq_{ijst}(1 - \rho_{ijst}) N_{ijst} \qquad (5.2)$$

Child types do not matter for household consumption. Since types index earnings abilities, children's consumption is determined by their parent's type, not their own. This also simplifies the model.

In effect, all we need to do to add labor types is add an extra dimension to household heterogeneity, allows for J different wages in each country, and

respecify the production function and first-order conditions as below.

$$y_{it} = k_{it}^{\alpha} \prod_{j=1}^{J} (A_i n_{ijt})^{\alpha_j} \; ; \alpha + \sum_{j=1}^{J} \alpha_j = 1 \tag{5.3}$$

$$r_{it} = \alpha \frac{y_{it}}{k_{it}} \tag{5.4}$$

$$w_{ijt} = (1 - \alpha_j) \frac{y_{it}}{n_{ijt}} \tag{5.5}$$

Our behavioral equations become:

$$g_t^N = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{s=1}^{S} \hat{N}_{ijst} (f_{ijst} + m_{ijst} - \rho_{ijst}); \forall i \tag{5.6}$$

$$\hat{N}_{i,j,1,t+1} = e^{-g_t^N} \sum_{s=23}^{45} \hat{N}_{ijst} f_{ijst}; \forall i, j \tag{5.7}$$

$$\hat{N}_{i,j,s+1,t+1} = e^{-g_t^N} \hat{N}_{ijst} (1 + m_{ijst} - \rho_{ijst}); \forall i, 1 < s \le S \tag{5.8}$$

$$KID_{i,j,s+1,t+1} = \sum_{r=1}^{20} f_{i,j,s-r,t-r}; \forall i, 1 < s \le S \tag{5.9}$$

$$\hat{k}_{it} = \sum_{j=1}^{J} \sum_{s=1}^{S} \hat{a}_{ijst} \hat{N}_{ijst} - \hat{k}_{it}^f; \forall i \tag{5.10}$$

$$\hat{n}_{ijt} = \sum_{s=1}^{S} e_{ijs} (\bar{\ell}_t - \hat{\ell}_{ijst}) \hat{N}_{ijst}; \forall i, j \tag{5.11}$$

$$\hat{y}_{it} = \hat{k}_{it}^{\alpha} \prod_{j=1}^{J} (A_i \hat{n}_{ijt})^{\alpha_j} \; ; \alpha + \sum_{j=1}^{J} \alpha_j = 1; \forall i \tag{5.12}$$

$$r_{it} = \alpha \frac{\hat{y}_{it}}{\hat{k}_{it}}; \forall i \tag{5.13}$$

$$\hat{w}_{ijt} = (1 - \alpha_j) \frac{\hat{y}_{it}}{\hat{n}_{ijt}}; \forall i.j \tag{5.14}$$

$$\hat{BQ}_{ijt} = \sum_{s=67}^{S} \hat{a}_{ijst}\rho_{ijst}\hat{N}_{ijst}; \forall i,j \tag{5.15}$$

$$\hat{BQ}_{ijt} = \sum_{s=23}^{67} \hat{bq}_{ijst}(1-\rho_{ijst})\hat{N}_{ijst}; \forall i \tag{5.16}$$

$$\Gamma_{ijt} = \left\{ \left[1 + \chi \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \right\}^{-\frac{1}{\sigma}}; \forall i,j,s \tag{5.17}$$

$$\hat{c}_{ijst} = \frac{\hat{w}_{ijt}e_{jst} + (1+r_{it}-\delta)\hat{a}_{ijst} + \hat{bq}_{jist} - \hat{a}_{i,j,s+1,t+1}e^{g^A}}{1 + KID_{ist}\Gamma_{ijt} + \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho}}; \forall i,j,s \tag{5.18}$$

$$\Gamma_{ijt} = \left\{ \left[1 + \chi \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho-1} \right\}^{-\frac{1}{\sigma}}; \forall i,j,s \tag{5.19}$$

$$\hat{c}_{ijst} = \frac{\hat{w}_{ijt}e_{jst}\bar{\ell}_t + (1+r_{it}-\delta)\hat{a}_{ijst} - \hat{a}_{i,j,s+1,t+1}e^{g^A}}{1 + KID_{ist}\Gamma_{ijt} + \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho}}; \forall i,j,s \tag{5.20}$$

$$\hat{\ell}_{ijst} = \hat{c}_{ijst} \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho}; \forall i,j,s \tag{5.21}$$

$$\hat{c}_{ijst}^{K} = \hat{c}_{ijst}\Gamma_{ijt}; \forall i,j,s \tag{5.22}$$

$$\left(\hat{c}_{ijst}^{K} \right)^{-\sigma} - \beta \left(\hat{c}_{i,j,s+1,t+1}^{K} e^{g^A} \right)^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i,j,s \tag{5.23}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{5.24}$$

$$\sum_{i=1}^{I} \hat{k}_{it}^{f} = 0 \tag{5.25}$$

6 Adding Corporate Taxes

Corporate taxes T_{it}^K equal the corporate tax rate τ_{it}^K times output net of labor costs and depreciation.

$$T_{it}^K = \tau_{it}^K \left[k_{it}^\alpha \left(A_i n_{it} \right)^{1-\alpha} - w_{it} n_{it} - \delta k_{it} \right] \tag{6.1}$$

Hence, the firm's problem each period becomes:

$$\max_{n_{it},k_{it}} \Pi_{it} = k_{it}^\alpha \left(A_i n_{it} \right)^{1-\alpha} - w_{it} n_{it} - r_{it} k_{it} - T_{it}^K$$

$$= (1 - \tau_{it}^K) \left[k_{it}^\alpha \left(A_i n_{it} \right)^{1-\alpha} - w_{it} n_{it} \right] + (\tau_{it}^K \delta - r_{it}) k_{it}$$

And the first-order conditions now yield:

$$
\begin{aligned}
y_{it} &= k_{it}^\alpha \left(A_i n_{it} \right)^{1-\alpha} \\
r_{it} &= (1 - \tau_{it}^K) \alpha \frac{y_{it}}{k_{it}} + \delta \tau_{it}^K \\
w_{it} &= (1 - \alpha) \frac{y_{it}}{n_{it}}
\end{aligned}
\tag{6.2}
$$

Only (6.2) is different than what we have in previous sections.

We assume the government spends all of its tax revenues on lump-sum transfers to households, which we will denote T_{it}^L. Since these are non-distortionary, they do note affect the household's Euler equations, but they do alter the budget constraint.

$$c_{ijst} + KID_{ist} c_{ijst}^K = w_{ijt} e_{jst} (\bar{\ell}_t - \ell_{ijst}) + (1 + r_{it} - \delta) a_{ijst} - a_{i,j,s+1,t+1} + T_{it}^L$$

Finally we get a balanced budget constraint for the government of: budget constraint.

$$T_{it}^L = \tau_{it}^K (y_{it} - w_{it} n_{it} - \delta k_{it})$$

The new set of behavioral equations is given below.

$$g_t^N = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{s=1}^{S} \hat{N}_{ijst}(f_{ijst} + m_{ijst} - \rho_{ijst}); \forall i \tag{6.3}$$

$$\hat{N}_{i,j,1,t+1} = e^{-g_t^N}\sum_{s=23}^{45} \hat{N}_{ijst}f_{ijst}; \forall i,j \tag{6.4}$$

$$\hat{N}_{i,j,s+1,t+1} = e^{-g_t^N}\hat{N}_{ijst}(1 + m_{ijst} - \rho_{ijst}); \forall i, 1 < s \le S \tag{6.5}$$

$$KID_{i,j,s+1,t+1} = \sum_{r=1}^{20} f_{i,j,s-r,t-r}; \forall i, 1 < s \le S \tag{6.6}$$

$$\hat{k}_{it} = \sum_{j=1}^{J}\sum_{s=1}^{S} \hat{a}_{ijst}\hat{N}_{ijst} - \hat{k}_{it}^f; \forall i \tag{6.7}$$

$$\hat{n}_{ijt} = \sum_{s=1}^{S} e_{ijs}(\bar{\ell}_t - \hat{\ell}_{ijst})\hat{N}_{ijst}; \forall i,j \tag{6.8}$$

$$\hat{y}_{it} = \hat{k}_{it}^{\alpha}\prod_{j=1}^{J}(A_i\hat{n}_{ijt})^{\alpha_j}; \alpha + \sum_{j=1}^{J}\alpha_j = 1; \forall i \tag{6.9}$$

$$r_{it} = (1 - \tau_{it}^K)\alpha\frac{\hat{y}_{it}}{\hat{k}_{it}} + \delta\tau_{it}^K; \forall i \tag{6.10}$$

$$\hat{w}_{ijt} = (1 - \alpha_j)\frac{\hat{y}_{it}}{\hat{n}_{ijt}}; \forall i.j \tag{6.11}$$

$$\hat{BQ}_{ijt} = \sum_{s=67}^{S} \hat{a}_{ijst}\rho_{ijst}\hat{N}_{ijst}; \forall i,j \tag{6.12}$$

$$\hat{BQ}_{ijt} = \sum_{s=23}^{67} \hat{bq}_{ijst}(1 - \rho_{ijst})\hat{N}_{ijst}; \forall i \tag{6.13}$$

$$\tag{6.14}$$

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$$\Gamma_{ijt} = \left\{ \left[1 + \chi \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho} \right]^{\frac{1-\rho\sigma}{\rho}} \frac{\rho}{\rho - 1} \right\}^{-\frac{1}{\sigma}}; \forall i, j, s \tag{6.15}$$

$$\hat{T}_{it}^{L} = \tau_{it}^{K} (\hat{y}_{it} - \hat{w}_{it}\hat{n}_{it} - \delta\hat{k}_{it}); \forall i \tag{6.16}$$

$$\hat{c}_{ijst} = \frac{\hat{w}_{ijt}e_{jst} + (1 + r_{it} - \delta)\hat{a}_{ijst} + \hat{bq}_{ijst} - \hat{a}_{i,j,s+1,t+1}e^{g^A} + \hat{T}_{it}^{L}}{1 + KID_{ist}\Gamma_{ijt} + \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho}}; \forall i, s \tag{6.17}$$

$$\hat{c}_{ijst} = \frac{\hat{w}_{ijt}e_{jst}\bar{\ell}_t + (1 + r_{it} - \delta)\hat{a}_{ijst} - \hat{a}_{i,j,s+1,t+1}e^{g^A} + \hat{T}_{it}^{L}}{1 + KID_{ist}\Gamma_{ijt} + \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho}}; \forall i, s \tag{6.18}$$

$$\hat{\ell}_{ijst} = \hat{c}_{ijst} \left(\frac{\chi}{\hat{w}_{ijt}e_{jst}} \right)^{\rho}; \forall i, j, s \tag{6.19}$$

$$\hat{c}_{ijst}^{K} = \hat{c}_{ijst}\Gamma_{ijt}; \forall i, j, s \tag{6.20}$$

$$\left(\hat{c}_{ijst}^{K} \right)^{-\sigma} - \beta \left(\hat{c}_{i,j,s+1,t+1}^{K} e^{g^A} \right)^{-\sigma} (1 + r_{1,t+1} - \delta) = 0; \forall i, j, s \tag{6.21}$$

$$r_{it} - r_{1t} = 0; \forall i > 1 \tag{6.22}$$

$$\sum_{i=1}^{I} \hat{k}_{it}^{f} = 0 \tag{6.23}$$