第一章

习题1-1

1. $\tilde{A}=(A,b)$

其中

$$A=\begin{pmatrix}1&1&2&2\\2&1&3&-1\\1&-1&1&4\end{pmatrix},b=\begin{pmatrix}1\\3\\5\end{pmatrix}$$

因此有

$$\begin{pmatrix}1&1&2&2\\2&1&3&-1\\1&-1&1&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1\\3\\5\end{pmatrix}$$

写成线性方程组的形式

$$\begin{cases}x_1+x_2+2x_3+2x_4=1,\\2x_1+x_2+3x_3-x_4=3,\\x_1-x_2+x_3+4x_4=5.\end{cases}$$

2.

$$\begin{pmatrix}1&2\\a&b\end{pmatrix}+\begin{pmatrix}x&y\\3&4\end{pmatrix}=\begin{pmatrix}3&-4\\7&1\end{pmatrix}$$

$$\begin{pmatrix}1+x&2+y\\a+3&b+4\end{pmatrix}=\begin{pmatrix}3&-4\\7&1\end{pmatrix}$$

得到

$$\begin{cases}1+x=3\\2+y=-4\\a+3=7\\b+4=1\end{cases}$$

所以有$x=2,y=-6,a=4,b=-3.$

3.

(1)

$$A+2B=\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix}+2\begin{pmatrix} 1 & 5 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$
$$=\begin{pmatrix} 3+2 & -1+10 & 2+2 \\ 2-4 & 1-2 & -2 \end{pmatrix}$$
$$=\begin{pmatrix} 5 & 9 & 4 \\ -2 & -1 & -2 \end{pmatrix}$$

$$3A-B=3\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix}-\begin{pmatrix} 1 & 5 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$
$$=\begin{pmatrix} 9-1 & -3-5 & 6-1 \\ 6+2 & 3+1 & -6 \end{pmatrix}$$
$$=\begin{pmatrix} 8 & -8 & 5 \\ 8 & 4 & -6 \end{pmatrix}$$

(2)

$$AB^T=\begin{pmatrix} 3 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 5 & -1 \\ 1 & 0 \end{pmatrix}$$
$$=\begin{pmatrix} 3-5+2 & -6+1 \\ 2+5-2 & -4-1 \end{pmatrix}$$
$$=\begin{pmatrix} 0 & -5 \\ 5 & -5 \end{pmatrix}$$

$$A^TB=\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 5 & 1 \\ -2 & -1 & 0 \end{pmatrix}$$
$$=\begin{pmatrix} 3-4 & 15-2 & 3 \\ -1-2 & -5-1 & -1 \\ 2+4 & 10+2 & 2 \end{pmatrix}$$
$$=\begin{pmatrix} -1 & 13 & 3 \\ -3 & -6 & -1 \\ 6 & 12 & 2 \end{pmatrix}$$

4.

$$
\begin{aligned}
(A+B)(A-B) &= \left(\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 3 & -3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ -2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} -3+6 & -9-9 \\ -4-2 & -12+3 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -18 \\ -6 & 9 \end{pmatrix}
\end{aligned}
$$

5.

$$
\begin{aligned}
A^2+3A-2B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}^2 + 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 5 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 6 & 3 \end{pmatrix} + \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & -10 \\ 0 & -10 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 1+3-2 & 0 & 0 \\ 0 & 5+3-4 & 4+6-10 \\ 0 & 4+6-10 & 5+3-4 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$

6. (1)

$$
\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 1 \\ 6 & -4 & 2 \\ 9 & -6 & 3 \end{pmatrix}
$$

(2)

$$
\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 2-3+2 = 1
$$

(3)

$$\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y-z & x+2y+z & -x+y \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= (x^2+xy-xz)+(xy+2y^2+yz)+(-xz+yz)$$
$$= x^2+2y^2+2xy-2xz+2yz$$

(4)

设

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

则

$$A = I + B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

当$k > 2$时,$B^k = 0$

$$\begin{aligned} A^n &= (I+B)^n = I + nB + \frac{n(n-1)}{2}B^2 + 0 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & n & 0 \\ 0 & 0 & n \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{n(n-1)}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

7. 与A可交换的矩阵B满足$AB = BA$,

设

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

即有

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}=\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

因此

$$\begin{cases} a+c=a \\ b+d=a+b \\ c=c \\ d=c+d \end{cases} \Rightarrow \begin{cases} c=0 \\ a=d \end{cases}$$

与A可交换的矩阵为

$$B=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

其中a,b为任意数.

8.

证明：

$$(A^T+A)^T=A+A^T=A^T+A$$
$$(A^T-A)^T=A-A^T=-(A^T-A)$$

所以A^T+A是对称矩阵,A^T-A是反对称矩阵.

9.

$$A^n=\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}^n=\begin{pmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^n \end{pmatrix}$$

习题1-2

1.

$$A=\left(\begin{array}{cc|cc} 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 5 \end{array}\right)=\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}$$

$$B=\left(\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 3 & 0 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{array}\right)=\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

$$C=\left(\begin{array}{cc|cc} 2 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)=\begin{pmatrix} C_1 & O \\ O & C_2 \end{pmatrix}$$

$$AC=\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}\begin{pmatrix} C_1 & O \\ O & C_2 \end{pmatrix}=\begin{pmatrix} A_1C_1 & O \\ O & A_2C_2 \end{pmatrix}$$

而

$$A_1C_1=\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}=\begin{pmatrix} 7 & 15 \\ 5 & 11 \end{pmatrix}, A_2C_2=\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}=\begin{pmatrix} 3 & 9 \\ 6 & 12 \end{pmatrix}$$

所以

$$AC=\begin{pmatrix} 7 & 15 & 0 & 0 \\ 5 & 11 & 0 & 0 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 6 & 12 \end{pmatrix}$$

$$\begin{aligned} AB-B^TA &= \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}-\begin{pmatrix} B_1^T & B_3^T \\ B_2^T & B_4^T \end{pmatrix}\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_3 & A_2B_4 \end{pmatrix}-\begin{pmatrix} B_1^TA_1 & B_3^TA_2 \\ B_2^TA_1 & B_4^TA_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1B_1-B_1^TA_1 & A_1B_2-B_3^TA_2 \\ A_2B_3-B_2^TA_1 & A_2B_4-B_4^TA_2 \end{pmatrix} \end{aligned}$$

而

$$
\begin{aligned}
A_1B_1 - B_1^TA_1 &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -3 & -1 \\ -2 & -1 \end{pmatrix} - \begin{pmatrix} -3 & -1 \\ -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A_1B_2 - B_3^TA_2 &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 17 \\ -2 & -5 \end{pmatrix} \\
&= \begin{pmatrix} -2 & -16 \\ 4 & 6 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A_2B_3 - B_2^TA_1 &= \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -4 \\ 11 & -5 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 4 & -5 \\ 9 & -6 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A_2B_4 - B_4^TA_2 &= \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 6 & 9 \\ 9 & 12 \end{pmatrix} - \begin{pmatrix} 4 & 13 \\ 5 & 14 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -4 \\ 4 & -2 \end{pmatrix}
\end{aligned}
$$

所以

$$
AB - B^TA = \begin{pmatrix} 0 & 0 & -2 & -16 \\ 0 & 0 & 4 & 6 \\ 4 & -5 & 2 & -4 \\ 9 & -6 & 4 & -2 \end{pmatrix}
$$

2.

$$A = \left(\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right) = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix}$$

$$B = \left(\begin{array}{c|c|c} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{array}\right) = \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \end{pmatrix}$$

$$\begin{aligned} AB &= \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \end{pmatrix} \\ &= A_1\Lambda_1 + A_2\Lambda_2 + A_3\Lambda_3 \\ &= \begin{pmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} & \lambda_3 a_{13} \\ \lambda_1 a_{21} & \lambda_2 a_{22} & \lambda_3 a_{23} \\ \lambda_1 a_{31} & \lambda_2 a_{32} & \lambda_3 a_{33} \end{pmatrix} \end{aligned}$$

3. $A = \begin{pmatrix} e_n & e_1 & e_2 & \cdots & e_{n-1} \end{pmatrix}$

由于Ae_k等于A的第k列, 所以当$k \neq 1$时,$Ae_k = e_{k-1}$,并且$Ae_1 = e_n$.

当$i > k$时,$A^k e_i = A^{k-1} e_{i-1} = \cdots = Ae_{i-k+1} = e_{i-k}$,

当$i \leq k$时,$A^k e_i = A^{k-i+1} A^{i-1} e_i = A^{k-i+1} e_1 = A^{k-i} Ae_1 = A^{k-i} e_n = e_{n-k+i}$.

因此

$$\begin{aligned} A^k = A^k E &= A^k \begin{pmatrix} e_1 & \cdots & e_k & e_{k+1} & \cdots & e_n \end{pmatrix} \\ &= \begin{pmatrix} A^k e_1 & \cdots & A^k e_k & A^k e_{k+1} & \cdots & A^k e_n \end{pmatrix} \\ &= \begin{pmatrix} e_{n-k+1} & \cdots & e_n & e_1 & \cdots & e_{n-k} \end{pmatrix} \\ &= \begin{pmatrix} 0 & E_{n-k} \\ E_k & 0 \end{pmatrix} \end{aligned}$$

.

4.

$$D^k = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_s \end{pmatrix}^k = \begin{pmatrix} A_1^k & O & \cdots & O \\ O & A_2^k & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_s^k \end{pmatrix}$$

习题1-3

1. (1) 用初等行变换将矩阵化为行最简形矩阵：

$$\begin{pmatrix} 2 & 2 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \end{pmatrix} \xrightarrow[\substack{r_3+(-1)r_1 \\ r_4+(-2)r_1}]{\frac{1}{2}r_1} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 3 & -3 \end{pmatrix} \xrightarrow[r_4+(-1)r_2]{r_3+(-1)r_2} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{r_1+(-1)r_2} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(2) 用初等行变换将矩阵化为行最简形矩阵：

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 0 & 2 & -3 & 1 \\ 0 & 4 & -7 & -1 \\ 0 & 3 & -4 & 3 \end{pmatrix} \xrightarrow[\substack{r_3+(-4)r_1 \\ r_4+(-3)r_1}]{r_2+(-2)r_1} \begin{pmatrix} 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -11 & 3 \\ 0 & 0 & -7 & 6 \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_2 \\ r_3+11r_2 \\ r_4+7r_2}]{-\frac{1}{5}r_2} \begin{pmatrix} 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & -\frac{18}{5} \\ 0 & 0 & 0 & -\frac{9}{5} \end{pmatrix}$$

$$\xrightarrow[\substack{r_1+(-\frac{2}{5})r_3 \\ r_2+\frac{3}{5}r_3 \\ r_4+\frac{9}{5}r_3}]{-\frac{5}{18}r_3} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) 用初等行变换将矩阵化为行最简形矩阵：

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 3 \\ 0 & 2 & -4 \\ 2 & -1 & 4 \end{pmatrix} \xrightarrow[\substack{r_2+(-1)r_1 \\ r_4+(-2)r_1}]{\frac{1}{3}r_1} \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{4}{3} & \frac{8}{3} \\ 0 & 2 & -4 \\ 0 & -\frac{5}{3} & \frac{10}{3} \end{pmatrix} \xrightarrow[\substack{r_1+(-\frac{1}{3})r_2 \\ r_3+(-2)r_2 \\ r_4+\frac{5}{3}r_2}]{-\frac{3}{4}r_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2. (1)

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1 & 1 & -1 \\ 3 & 1 & 4 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow[r_3+(-1)r_1]{r_2+(-3)r_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 7 \\ 0 & -3 & 4 \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_2 \\ r_3+3r_2}]{-\frac{1}{2}r_2} \begin{pmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & -\frac{7}{2} \\ 0 & 0 & -\frac{13}{2} \end{pmatrix}$$

$$\xrightarrow[\substack{r_1+(-\frac{5}{2})r_3 \\ r_2+\frac{7}{2}r_3}]{\frac{2}{13}r_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

即为原方程的解

(2)

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1 & -5 & 2 & -3 \\ 2 & 4 & 2 & 1 \\ 5 & 3 & 6 & -1 \end{pmatrix} \xrightarrow[r_3+(-5)r_1]{r_2+(-2)r_1} \begin{pmatrix} 1 & -5 & 2 & -3 \\ 0 & 14 & -2 & 7 \\ 0 & 28 & -4 & 14 \end{pmatrix} \xrightarrow[\substack{r_1+5r_2 \\ r3+(-28)r_2}]{\frac{1}{14}r_2} \begin{pmatrix} 1 & 0 & \frac{9}{7} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{7} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + \frac{9}{7}x_3 - \frac{1}{2}x_4 = 0 \\ x_2 - \frac{1}{7}x_3 + \frac{1}{2}x_4 = 0 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -\frac{9}{7}C_1 + \frac{1}{2}C_2 \\ x_2 = \frac{1}{7}C_1 - \frac{1}{2}C_2 \\ x_3 = C_1 \\ x_4 = C_2 \end{cases},$$

其中C_1, C_2为任意常数

(3)

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 0 & -1 \\ 1 & 2 & -9 & -5 \\ -1 & -2 & 3 & 2 \end{pmatrix} \xrightarrow[\substack{r_3+(-1)r_1 \\ r_4+r_1}]{r_2+(-2)r_1} \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & -6 & -3 \\ 0 & 0 & -12 & -6 \\ 0 & 0 & 6 & 3 \end{pmatrix} \xrightarrow[\substack{r_1+(-3)r_2 \\ r_3+12r_2 \\ r_4+(-6)r_2}]{-\frac{1}{6}r_2} \begin{pmatrix} 1 & 2 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + 2x_2 - \frac{1}{2}x_4 = 0 \\ x_3 + \frac{1}{2}x_4 = 0 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -2C_1 + \frac{1}{2}C_2 \\ x_2 = C_1 \\ x_3 = -\frac{1}{2}C_2 \\ x_4 = C_2 \end{cases},$$

其中C_1, C_2为任意常数

3.(1)

对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 1 & -2 & 4 & -5 \\ 2 & 3 & 1 & 4 \\ 3 & 8 & -2 & 13 \end{pmatrix} \xrightarrow[r_3+(-3)r_1]{r_2+(-2)r_1} \begin{pmatrix} 1 & -2 & 4 & -5 \\ 0 & 7 & -7 & 14 \\ 0 & 14 & -14 & 28 \end{pmatrix} \xrightarrow[\substack{r_1+2r_2 \\ r_3+(-14)r_2}]{-\frac{1}{7}r_2} \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + 2x_3 = -1 \\ x_2 - x_3 = 2 \end{cases}$$

令$x_3 = c$,移项,得原方程的解为

$$\begin{cases} x_1 = -1 - 2c \\ x_2 = 2 + c \\ x_3 = c \end{cases},$$

其中c为任意常数

(2)

对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 2 & 3 & 0 & -1 & 0 \\ 3 & 1 & 5 & -4 & 2 \\ 0 & 7 & -10 & 5 & -4 \\ 3 & -6 & 15 & -9 & 1 \end{pmatrix} \xrightarrow[\substack{r_2+(-3)r_1 \\ r_4+(-3)r_1}]{\frac{1}{2}r_1} \begin{pmatrix} 1 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{7}{2} & 5 & -\frac{5}{2} & 2 \\ 0 & 7 & -10 & 5 & -4 \\ 0 & -\frac{21}{2} & 15 & -\frac{15}{2} & 1 \end{pmatrix} \xrightarrow[\substack{r_1+(-\frac{3}{2})r_2 \\ r_3+(-7)r_2 \\ r_4+\frac{21}{2}r_2}]{-\frac{2}{7}r_2}$$

$$\begin{pmatrix} 1 & 0 & \frac{15}{7} & -\frac{11}{7} & \frac{6}{7} \\ 0 & 1 & -\frac{10}{7} & \frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix} \xrightarrow{r_3 \leftrightarrow r_4} \begin{pmatrix} 1 & 0 & \frac{15}{7} & -\frac{11}{7} & \frac{6}{7} \\ 0 & 1 & -\frac{10}{7} & \frac{5}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{r_1+(-\frac{6}{7})r_3 \\ r_2+\frac{4}{7}r_3}]{-\frac{1}{5}r_3} \begin{pmatrix} 1 & 0 & \frac{15}{7} & -\frac{11}{7} & 0 \\ 0 & 1 & -\frac{10}{7} & \frac{5}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 + \frac{15}{7}x_3 - \frac{11}{7}x_4 = 0 \\ x_2 - \frac{10}{7}x_3 + \frac{5}{7}x_4 = 0 \\ 0 = 1 \end{cases},$$

所以该方程组无解.

(3) 对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & -2 & 1 & 4 \\ 1 & -1 & -1 & 6 \end{pmatrix} \xrightarrow[r_3+(-1)r_1]{r_2+(-3)r_1} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -8 & 4 & 4 \\ 0 & -3 & 0 & 6 \end{pmatrix} \xrightarrow[\substack{r_1+(-2)r_2 \\ r_3+3r_2}]{-\frac{1}{8}r_2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{3}{2} & \frac{9}{2} \end{pmatrix}$$

$$\xrightarrow[r_2+\frac{1}{2}r_3]{-\frac{2}{3}r_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = -3 \end{cases},$$

即为原方程的解.

(4) 对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 2 & -2 & 1 & 0 & 1 \\ 1 & 1 & -2 & -1 & -1 \\ 1 & -1 & 1 & 1 & 2 \end{pmatrix} \xrightarrow[\substack{r_3+(-1)r_1 \\ r_4+(-1)r_1}]{r_2+(-2)r_1} \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -3 & 2 & -1 \\ 0 & 2 & -4 & 0 & -2 \\ 0 & 0 & -1 & 2 & 1 \end{pmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 2 & -4 & 0 & -2 \\ 0 & 0 & -3 & 2 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[r_1+r_2]{\frac{1}{2}r_2} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & -3 & 2 & -1 \\ 0 & 0 & -1 & 2 & 1 \end{pmatrix} \xrightarrow[\substack{r_2+2r_3 \\ r_4+r_3}]{\frac{1}{3}r_3} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{4}{3} \end{pmatrix} \xrightarrow[\substack{r_1+r_4 \\ r_2+\frac{4}{3}r_4 \\ r_3+\frac{2}{3}r_4}]{\frac{3}{4}r_4} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

从而原方程等价于

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 1 \end{cases},$$

即为原方程的解.

4.

对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{pmatrix} \xrightarrow[r_3+(-\lambda)r_1]{r_2+(-1)r_1} \begin{pmatrix} 1 & 1 & \lambda \\ 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda^2 \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_2 \\ r_3-(1-\lambda)r_2}]{\frac{1}{\lambda-1}r_2} \begin{pmatrix} 1 & 0 & \lambda+1 \\ 0 & 1 & -1 \\ 0 & 0 & -\lambda^2-\lambda+2 \end{pmatrix}$$

$\lambda \neq -2$且$\lambda \neq 1$时只有零解, $\lambda = -2$或$\lambda = 1$时有非零解;

当$\lambda = -2$时,

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = c \\ x_2 = c \\ x_3 = c \end{cases},$$

其中c为任意常数.

当$\lambda=1$时,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = -c_1 - c_2 \\ x_2 = c_1 \\ x_3 = c_2 \end{cases},$$

其中c_1, c_2为任意常数.

5.

对该线性方程组的增广矩阵实施初等行变换

$$\begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & -6 & 4 & -1 \\ 3 & 2 & p & 7 & -1 \\ 1 & -1 & -6 & -1 & t \end{pmatrix} \xrightarrow[\substack{r_3+(-3)r_1 \\ r_4+(-1)r_1}]{r_2+(-2)r_1} \begin{pmatrix} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -2 & -2 & -1 \\ 0 & -1 & 6+p & -2 & -1 \\ 0 & -2 & -4 & -4 & t \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_2 \\ r_3+r_2 \\ r_4+2r_2}]{(-1)r_2} \begin{pmatrix} 1 & 0 & -4 & 1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 8+p & 0 & 0 \\ 0 & 0 & 0 & 0 & t+2 \end{pmatrix}$$

当$t \neq -2$时,方程组无解;

当$t = -2$时,方程有无穷多解;

当$t = -2, p \neq -8$时,原方程等价于

$$\begin{cases} x_1 - 4x_3 + x_4 = -1 \\ x_2 + 2x_3 + 2x_4 = 1 \\ (8+p)x_3 = 0 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -1 - c \\ x_2 = 1 - 2c \\ x_3 = 0 \\ x_4 = c \end{cases}$$

其中c为任意常数.

当$t = -2, p = -8$时,原方程等价于

$$\begin{cases} x_1 - 4x_3 + x_4 = -1 \\ x_2 + 2x_3 + 2x_4 = 1 \end{cases}$$

移项,得原方程的解为

$$\begin{cases} x_1 = -1 + 4c_1 - c_2 \\ x_2 = 1 - 2c_1 - 2c_2 \\ x_3 = c_1 \\ x_4 = c_2 \end{cases}$$

其中c_1, c_2为任意常数

习题1-4

1. 交换A的第1列和第3列得到矩阵B相当于在矩阵A右方乘以

$$Q_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

把B的第1列乘以非零数k加到B的第2列相当于在矩阵B右方乘以

$$Q_2 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

因此

$$\begin{aligned} Q = Q_1Q_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & k & 0 \end{pmatrix} \end{aligned}$$

2.

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \xrightarrow{r_2+(-1)r_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{r_1+(-2)r_2} \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

(1)

$$P^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

(2)

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 & -6 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

(3) 设$B = P^{-1}AP$,则$A = PBP^{-1}$

$$\begin{aligned} A^{10} &= (PBP^{-1})^{10} \\ &= PB^{10}P^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{10} & 2^{11} \\ 3^{10} & 3\cdot 2^{10} \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{11}-2^{11} & 2^{11}-2\cdot 3^{10} \\ 3^{11}-3\cdot 2^{10} & 3\cdot 2^{10}-2\cdot 3^{10} \end{pmatrix} \end{aligned}$$

3.(1)

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+r_1]{r_2+(-1)r_1} \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_2 \\ r_3+(-2)r_2}]{(-\frac{1}{2})r_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{pmatrix} \xrightarrow[r_2+r_3]{(\frac{1}{2})r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

所以逆矩阵为

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(2)

$$\begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{r_2+(-3)r_1 \\ r_3+(-1)r_1}]{\frac{1}{2}r_1} \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & \frac{5}{2} & -\frac{1}{2} & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-1)r_2]{r_1+(-1)r_2}$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix} \xrightarrow[r_2+(-\frac{3}{2})r_3]{r_1+r_3} \begin{pmatrix} 1 & 0 & 0 & 3 & -2 & 1 \\ 0 & 1 & 0 & -3 & \frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

所以逆矩阵为

$$\begin{pmatrix} 3 & -2 & 1 \\ -3 & \frac{5}{2} & -\frac{3}{2} \\ 1 & -1 & 1 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+2r_1]{r_2+r_1} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 4 & 3 & -2 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-4)r_2]{r_1+r_2}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -6 & -4 & 1 \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_3 \\ r_2+(-1)r_3}]{(-1)r_3} \begin{pmatrix} 1 & 0 & 0 & -4 & -3 & 1 \\ 0 & 1 & 0 & -5 & -3 & 1 \\ 0 & 0 & 1 & 6 & 4 & -1 \end{pmatrix}$$

所以逆矩阵为

$$\begin{pmatrix} -4 & -3 & 1 \\ -5 & -3 & 1 \\ 6 & 4 & -1 \end{pmatrix}$$

(4)

$$
\begin{pmatrix} 1 & 3 & 1 & 6 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 7 & 1 & 8 & 0 & 0 & 0 & 1 \end{pmatrix}
\xrightarrow[\substack{r_3+(-3)r_1 \\ r_4+(-5)r_1}]{r_2+(-2)r_1}
\begin{pmatrix} 1 & 3 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & -5 & -2 & -12 & -2 & 1 & 0 & 0 \\ 0 & -7 & -3 & -18 & -3 & 0 & 1 & 0 \\ 0 & -8 & -4 & -22 & -5 & 0 & 0 & 1 \end{pmatrix}
\xrightarrow[\substack{r_1+(-3)r_2 \\ r_3+7r_2 \\ r_4+8r_2}]{(-\frac{1}{5})r_2}
$$

$$
\begin{pmatrix} 1 & 0 & -\frac{1}{5} & -\frac{6}{5} & -\frac{1}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 1 & \frac{2}{5} & \frac{12}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{6}{5} & -\frac{1}{5} & -\frac{7}{5} & 1 & 0 \\ 0 & 0 & -\frac{4}{5} & -\frac{14}{5} & -\frac{9}{5} & -\frac{8}{5} & 0 & 1 \end{pmatrix}
\xrightarrow[\substack{r_1+5r_3 \\ r_2+(-\frac{2}{5})r_3 \\ r_4+\frac{4}{5}r_3}]{(-5)r_3}
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 6 & 1 & 7 & -5 & 0 \\ 0 & 0 & 0 & 2 & -1 & 4 & -4 & 1 \end{pmatrix}
$$

$$
\xrightarrow[r_3+(-6)r_4]{\frac{1}{2}r_4}
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 & 4 & -5 & 7 & -3 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 2 & -2 & \frac{1}{2} \end{pmatrix}
$$

所以逆矩阵为

$$
\begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 4 & -5 & 7 & -3 \\ -\frac{1}{2} & 2 & -2 & \frac{1}{2} \end{pmatrix}
$$

4. (1)

$$
\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{pmatrix}
\xrightarrow[r_3+(-2)r_1]{r_2+(-2)r_1}
\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & 2 & -3 & -2 & 0 & 1 \end{pmatrix}
\xrightarrow[\substack{r_1+(-1)r_2 \\ r_3+(-2)r_2}]{\frac{1}{3}r_2}
$$

$$
\begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{5}{3} & -\frac{2}{3} & -\frac{2}{3} & 1 \end{pmatrix}
\xrightarrow[\substack{r_1+\frac{1}{3}r_3 \\ r_2+\frac{2}{3}r_3}]{-\frac{3}{5}r_3}
\begin{pmatrix} 1 & 0 & 0 & \frac{9}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{pmatrix}
$$

因此逆矩阵为

$$
\begin{pmatrix} \frac{9}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{pmatrix}
$$

所以

$$X = \begin{pmatrix} \frac{9}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 4 & 8 \\ 1 & 9 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 2 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+(-2)r_1]{r_2+(-2)r_1} \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & 3 & -3 & -2 & 0 & 1 \end{pmatrix} \xrightarrow[\substack{r_1+r_2 \\ r_3+(-3)r_2}]{\frac{1}{3}r_2}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{pmatrix} \xrightarrow[\substack{r_1+\frac{1}{3}r_3 \\ r_2+\frac{2}{3}r_3}]{-r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix}$$

因此逆矩阵为

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 1 & -1 \end{pmatrix}$$

所以

$$X = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{2}{3} & 1 & -\frac{2}{3} \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & -2 \\ 0 & 5 & -4 \end{pmatrix}$$

(3)

$$
\begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}
\xrightarrow[\substack{r_2+(-1)r_1 \\ r_3+(-1)r_1}]{-1r_1}
\begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{pmatrix}
\xrightarrow{r_2\leftrightarrow r_3}
$$

$$
\begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{pmatrix}
\xrightarrow[r_1+r_2]{\frac{1}{2}r_2}
\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 2 & 1 & 1 & 0 \end{pmatrix}
\xrightarrow[r_1+r_3]{\frac{1}{2}r_3}
$$

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}
$$

因此逆矩阵为

$$
\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}
$$

所以

$$
\begin{aligned}
X &= \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 2 \\ -10 & 4 \\ -2 & 1 \end{pmatrix}
\end{aligned}
$$

测试题一

一、填空题

1.

$(A+B)(A-B)=A^2-AB+BA-B^2$,因此要使$(A+B)(A-B)=A^2-B^2$的充分必要条件是$AB=BA$.

2.

$$
\begin{aligned}
\alpha\beta^T - E &= \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 6 \\ 4 & 5 \end{pmatrix}
\end{aligned}
$$

3.

$$
\begin{aligned}
A &= \alpha\beta^T \\
&= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A \cdot A &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \\
&= 4 \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}
$$

$$
A^8 = 4^7 \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}
$$

4.

$$
(A - 2E) = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
$$

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2+(-1)r_1]{\frac{1}{2}r_1} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2}r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

因此

$$(A-2E)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

5.

矩阵P相当于初等变换：将第三行乘以（-2）加到第二行上,

因此

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

二、选择题

1.对于选项A,

取

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

此时$B \neq C$.

对于选项B,

取

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

.

对于选项C,要求A,B维数相同.

2.$(AB^T)^{-1} = (B^T)^{-1}A^{-1}$

4.

$$\begin{pmatrix} 1 & \lambda & 1 \\ -2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & \lambda & 1 \\ 0 & 1+2\lambda & 2 \end{pmatrix}$$

若线性方程组无解，取$\lambda = -\frac{1}{2}$.

5.

$$A^3 - E = (A-E)(A^2+A+E) = (A^2+A+E)(A-E) = -E$$

$$A^3+E=(A+E)(A^2-A+E)=(A^2-A+E)(A+E)=E$$

所以矩阵$A+E$与$A-E$都是可逆的.

三、解答题

1.

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+r_1]{r_2+(-1)r_1} \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_1+r_2]{\frac{1}{2}r_2}$$
$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[r_2+r_3]{\frac{1}{2}r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

因此

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

所以

$$X = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & -1 \\ 2 & \frac{1}{3} & 1 \\ -1 & -1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & -2 \\ 6 & 1 & 3 \\ -3 & -3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{9}{2} & \frac{7}{2} & 2 \\ -1 & -3 & -1 \end{pmatrix}$$

2. 对该线性方程组的系数矩阵实施初等行变换,得

$$\begin{pmatrix} 1+\lambda & 1 & 1 \\ 1 & 1+\lambda & 1 \\ 1 & 1 & 1+\lambda \end{pmatrix} \xrightarrow{r_3 \leftrightarrow r_1} \begin{pmatrix} 1 & 1 & 1+\lambda \\ 1 & 1+\lambda & 1 \\ 1+\lambda & 1 & 1 \end{pmatrix} \xrightarrow[r_3+(-\lambda-1)r_1]{r_2+(-1)r_1}$$

$$\begin{pmatrix} 1 & 1 & 1+\lambda \\ 0 & \lambda & -\lambda \\ 0 & -\lambda & -\lambda^2-2\lambda \end{pmatrix} \xrightarrow[\substack{r_1+(-1)r_2 \\ r_3+\lambda r_2}]{\frac{1}{\lambda}r_2} \begin{pmatrix} 1 & 0 & 2+\lambda \\ 0 & 1 & -1 \\ 0 & 0 & -\lambda^2-3\lambda \end{pmatrix}$$

$\lambda \neq 0$且$\lambda \neq 3$时只有零解;

$\lambda = 0$时有非零解，

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = -C_1 - C_2 \\ x_2 = C_1 \\ x_3 = C_2 \end{cases},$$

其中C_1, C_2为任意常数.

当$\lambda = -3$时有非零解,

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

非零解为

$$\begin{cases} x_1 = C \\ x_2 = C \\ x_3 = C \end{cases},$$

其中C为任意常数.

3.

$$
\begin{aligned}
A\cdot A &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} \\
&= 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

故

$$
A^n = 2^{n-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}
$$

因此

$$
\begin{aligned}
A^n - 2A^{n-1} &= 2^{n-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} - 2\cdot 2^{n-2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= 0
\end{aligned}
$$

四、证明题

因为A是所有元素均为1的n阶方阵,所以$A \cdot A = nA$.

$$
\begin{aligned}
(E-A)(E-\frac{1}{n-1}A) &= E - \frac{1}{n-1}A - A + \frac{1}{n-1}A^2 \\
&= \frac{1}{n-1}A^2 - \frac{n}{n-1}A + E \\
&= \frac{n}{n-1}A - \frac{n}{n-1}A + E \\
&= E
\end{aligned}
$$

所以

$$
(E-A)^{-1} = E - \frac{1}{n-1}A
$$

第二章

习题2-1

1.

(1) $\tau(634521) = 5+2+2+2+1 = 12$

(2) $\tau(53142) = 4+2+1 = 7$

(3) $\tau(123454321) = 1+2+3+4+3+2+1 = 16$

(4) $\tau(135\cdots(2n-1)(2n)(2n-2)\cdots 42) = 0+1+2+\cdots+(n-1)+(n-1)+\cdots+1+0 = n(n-1)$

2.

$$D_5 = (-1)^{\tau(34512)}a_{13}a_{24}a_{35}a_{41}a_{52} = (-1)^6 a_{13}a_{24}a_{35}a_{41}a_{52} = a_{13}a_{24}a_{35}a_{41}a_{52}$$

3.

$f(x) = \sum_{i_1,i_2,i_3,i_4\in S_4}(-1)^{\tau(i_1,i_2,i_3,i_4)}a_{1i_1}a_{2i_2}a_{3i_3}a_{4i_4}$,当$i_4 \neq 4$时，$deg(a_{1i_1}a_{2i_2}a_{3i_3}a_{4i_4}) \leq 2$，同样,当$i_2 \neq 2$时,$deg(a_{1i_1}a_{2i_2}a_{3i_3}a_{4i_4}) \leq 2$,所以要使$deg(a_{1i_1}a_{2i_2}a_{3i_3}a_{4i_4}) \geq 3$，只有$-a_{11}a_{23}a_{32}a_{44} = -6x^4$和$a_{21}a_{23}a_{31}a_{44} = 3x^4.x^3$ 系数是3,x^4系数是-6.

4.

(1)

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 2 & 3 & 2 \end{vmatrix} &= 1\times1\times2+2\times0\times2+1\times3\times3-1\times1\times2-0\times3\times1-2\times3\times2 \\ &= -3\end{aligned}$$

(2)

$$\begin{aligned}\begin{vmatrix} 2 & 5 & 3 \\ 0 & 4 & 7 \\ -2 & -2 & 3 \end{vmatrix} &= 2\times4\times3+5\times7\times(-2)+3\times0\times(-2)-3\times4\times(-2)-7\times(-2)\times2-3\times5\times0 \\ &= 24-70+24+28 \\ &= 6\end{aligned}$$

(3)

$$\begin{aligned}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} &= a\times c\times b+b\times a\times c+c\times b\times a-c\times c\times c-a\times a\times a-b\times b\times b \\ &= 3abc-a^3-b^3-c^3\end{aligned}$$

(4)

$$\begin{vmatrix} 1 & 1 & 1 \\ 2a & a+b & 2b \\ a^2 & ab & b^2 \end{vmatrix} = (a+b)b^2 + 2ba^2 + 2aab - (a+b)a^2 - 2ab^2 - 2bab$$

$$= b^3 - 3ab^2 + 3a^2b - a^3$$

$$= (b-a)^3$$

习题2-2

1. (1)

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 2 - 6 + 8 - 3 - 8 + 4 = -3$$

(2)

$$\begin{vmatrix} 2 & 1 & 4 & 3 \\ 4 & 2 & 3 & 11 \\ 3 & 0 & 9 & 2 \\ 1 & -1 & -1 & 4 \end{vmatrix} \overset{r_2+(-2)r_1}{\underset{\substack{r_3+(-\frac{3}{2})r_1 \\ r_4+(-\frac{1}{2}r_1}}{=\!=\!=\!=}} \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & 0 & -5 & 5 \\ 0 & -\frac{3}{2} & 3 & -\frac{5}{2} \\ 0 & 0 & -6 & 5 \end{vmatrix} \overset{r_2 \leftrightarrow r_3}{=\!=\!=} - \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & -\frac{3}{2} & 3 & -\frac{5}{2} \\ 0 & 0 & -5 & 5 \\ 0 & 0 & -6 & 5 \end{vmatrix}$$

$$\overset{r_4+(-\frac{6}{5})r_3}{=\!=\!=\!=} - \begin{vmatrix} 2 & 1 & 4 & 3 \\ 0 & -\frac{3}{2} & 3 & -\frac{5}{2} \\ 0 & 0 & -5 & 5 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 15$$

(3) 将第一行的-1倍加到其他行

$$\begin{vmatrix} a & b & b & b \\ a & a & b & b \\ a & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ b-a & 0 & 0 & a-b \end{vmatrix} \xlongequal{c_1+c_4}$$

$$\begin{vmatrix} a+b & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix} = (a+b)(a-b)^3$$

(4) 将第一行的-1倍加到其他行,再将各列加到第一列

$$\begin{vmatrix} 2 & a & a & \cdots & a \\ a & 2 & a & \cdots & a \\ a & a & 2 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & 2 \end{vmatrix} = \begin{vmatrix} 2 & a & a & \cdots & a \\ a-2 & 2-a & 0 & \cdots & 0 \\ a-2 & 0 & 2-a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a-2 & 0 & 0 & \cdots & 2-a \end{vmatrix}$$

$$= \begin{vmatrix} 2+(n-1)a & a & a & \cdots & a \\ 0 & 2-a & 0 & \cdots & 0 \\ 0 & 0 & 2-a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2-a \end{vmatrix} = (2-a)^{n-1}[2+(n-1)a]$$

(5) 将第一行的-i倍加到第i行,再将第i列的i倍加到第一列

$$\begin{vmatrix} 1+a & 1 & 1 & \cdots & 1 \\ 2 & 2+a & 2 & \cdots & 2 \\ 3 & 3 & 3+a & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n+a \end{vmatrix} = \begin{vmatrix} 1+a & 1 & 1 & \cdots & 1 \\ -2a & a & 0 & \cdots & 0 \\ -3a & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -na & 0 & 0 & \cdots & a \end{vmatrix}$$

$$= \begin{vmatrix} 1+a+2+\cdots+n & 1 & 1 & \cdots & 1 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{vmatrix} = [a+\frac{(1+n)n}{2}]a^{n-1}$$

2. (1)

将第一列的-1倍加到其他列

$$\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 \\ b^2 & (b+1)^2 & (b+2)^2 \\ c^2 & (c+1)^2 & (c+2)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 \\ b^2 & 2b+1 & 4b+4 \\ c^2 & 2c+1 & 4c+4 \end{vmatrix} \xlongequal{c_3+(-2)r_2} \begin{vmatrix} a^2 & 2a+1 & 2 \\ b^2 & 2b+1 & 2 \\ c^2 & 2c+1 & 2 \end{vmatrix} \xlongequal[r_3++(-1)r_1]{r_2+(-1)r_1}$$

$$\begin{vmatrix} a^2 & 2a+1 & 2 \\ b^2-a^2 & 2b-2a & 0 \\ c^2-a^2 & 2c-2a & 0 \end{vmatrix} \xlongequal{r_3+(-\frac{2c-2a}{2b-2a})r_2} \begin{vmatrix} a^2 & 2a+1 & 2 \\ b^2-a^2 & 2b-2a & 0 \\ (c-a)(c-b) & 0 & 0 \end{vmatrix} = 4(a-b)(a-c)(b-c)$$

(2) 将第i列的$-\frac{1}{a_i}$倍加到其他列

$$\begin{vmatrix} a_1 & 1 & 1 & \cdots & 1 \\ 1 & a_2 & 0 & \cdots & 0 \\ 1 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_1-\sum_{i=2}^n \frac{1}{a_i} & 1 & 1 & \cdots & 1 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = a_2a_3\cdots a_n(a_1-\sum_{i=2}^n \frac{1}{a_i})$$

(3) 先拆分,在把第一列加到其他列上

$$
\begin{aligned}
\begin{vmatrix} a_1-b_1 & a_1-b_2 & \cdots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \cdots & a_2-b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n-b_1 & a_n-b_2 & \cdots & a_n-b_n \end{vmatrix} &= \begin{vmatrix} a_1 & a_1-b_2 & \cdots & a_1-b_n \\ a_2 & a_2-b_2 & \cdots & a_2-b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n-b_2 & \cdots & a_n-b_n \end{vmatrix} - \begin{vmatrix} b_1 & a_1-b_2 & \cdots & a_1-b_n \\ b_1 & a_2-b_2 & \cdots & a_2-b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & a_n-b_2 & \cdots & a_n-b_n \end{vmatrix} \\
&= \begin{vmatrix} a_1 & -b_2 & \cdots & -b_n \\ a_2 & -b_2 & \cdots & -b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n & -b_2 & \cdots & -b_n \end{vmatrix} - \begin{vmatrix} b_1 & a_1 & \cdots & a_1 \\ b_1 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1 & a_n & \cdots & a_n \end{vmatrix} \\
&= 0
\end{aligned}
$$

3.

$$|3A^TB^2| = 3^3|A^T||B^2| = 3^3|A||B|^2 = 972$$

4.

由书上例6可知,若A、B均可逆，则M、D、N均可逆，设矩阵

$$X = \begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix},$$

依次求解满足$MX = XM = E$、$DX = XD = E$、$NX = XN = E$的X，

得到

$$M^{-1} = \begin{vmatrix} O & B^{-1} \\ A^{-1} & O \end{vmatrix}, D^{-1} = \begin{vmatrix} A^{-1} & O \\ -B^{-1}CA^{-1} & B^{-1} \end{vmatrix}, N^{-1} = \begin{vmatrix} -B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & O \end{vmatrix}.$$

5.

$$
\begin{aligned}
A^2 - A - 2E = 0 \\
A(A-E) - 2E = 0 \\
A\frac{A-E}{2} = E
\end{aligned}
$$

又因为

$$\frac{A-E}{2}A = E,$$

所以

$$A^{-1} = \frac{A-E}{2}.$$

$$(A+2E)(A+kE)=A^2+kA+2A+2kE$$

所以$k+2=-1$,$k=-3$,

所以有

$$(A+2E)(A-3E)=A^2-A-6E=-4E$$
$$-\frac{A-3E}{4}(A+2E)=E$$

又因为

$$(A+2E)(-\frac{A-3E}{4})=E,$$

所以

$$(A+2E)^{-1}=-\frac{A-3E}{4}.$$

习题2-3

1.

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & b & c \\ x^2 & a^2 & b^2 & c^2 \\ x^3 & a^3 & b^3 & c^3 \end{vmatrix}$$
$$=(a-x)(b-x)(c-x)(b-a)(c-a)(c-b)=0$$

$f(x)=0$的根为$x=a, x=b, x=c$.

2.（1）按第二行展开

$$\begin{vmatrix} 2 & 2 & 2 & 2 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 3 & 2 & 0 & 4 \end{vmatrix} = -3\begin{vmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 12$$

(2) 先将第一行其他元素化为0,在按第一行展开

$$\begin{vmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 3 & -3 & 3 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 3 & -3 & 3 \\ 1 & 1 & 2 \end{vmatrix} = 9$$

(3) 按第一行展开

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 1 \\ a_0 & a_1 & a_2 & a_3 \end{vmatrix} = x\begin{vmatrix} x & 1 & 0 \\ 0 & x & 1 \\ a_1 & a_2 & a_3 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & x & 1 \\ a_0 & a_2 & a_3 \end{vmatrix}$$
$$= x(a_3x^2 + a_1 - a_2x) - a_0$$
$$= a_3x^3 - a_2x^2 + a_1x - a_0$$

(4) 先做变换,再按第一行展开

$$\begin{vmatrix} 1 & -1 & 1 & x-1 \\ 1 & -1 & x+1 & -1 \\ 1 & x-1 & 1 & -1 \\ x+1 & -1 & 1 & -1 \end{vmatrix} \xlongequal[\substack{c_3+(-1)c_1 \\ c_4+c_1}]{c_2+c_1} \begin{vmatrix} 1 & 0 & 0 & x \\ 1 & 0 & x & 0 \\ 1 & x & 0 & 0 \\ x+1 & x & -x & x \end{vmatrix}$$
$$= \begin{vmatrix} 0 & x & 0 \\ x & 0 & 0 \\ x & -x & x \end{vmatrix} - x\begin{vmatrix} 1 & 0 & x \\ 1 & x & 0 \\ x+1 & x & -x \end{vmatrix}$$
$$= -x^3 - x[-x^2 + x^2 - x^2(x+1)]$$
$$= x^4$$

3. (1) 按第一列展开

$$\begin{vmatrix} x & y & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & y & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & y \\ y & 0 & 0 & 0 & \cdots & 0 & x \end{vmatrix} = x\begin{vmatrix} x & y & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ 0 & 0 & 0 & \cdots & 0 & x \end{vmatrix} + (-1)^{n+1}y\begin{vmatrix} y & 0 & 0 & \cdots & 0 & 0 \\ x & y & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \end{vmatrix}$$
$$= x^n + (-1)^{n+1}y^n$$

(2) 将第一行的-1倍加到其他行,再将各列加到第一列

$$
\begin{vmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{vmatrix} = \begin{vmatrix} a & b & b & \cdots & b \\ b-a & 0 & 0 & \cdots & 0 \\ b-a & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b-a & 0 & 0 & \cdots & 0 \end{vmatrix}
$$

$$
= \begin{vmatrix} a+(n-1)b & b & b & \cdots & b \\ 0 & a-b & 0 & \cdots & 0 \\ 0 & 0 & a-b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{vmatrix}
$$

$$
= [a+(n-1)b](a-b)^{n-1}
$$

(3) 将第一行的-1倍加到其他行,再将各列加到第一列

$$
\begin{vmatrix} a_1+b & a_2 & a_3 & \cdots & a_n \\ a_1 & a_2+b & a_3 & \cdots & a_n \\ a_1 & a_2 & a_3+b & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n+b \end{vmatrix} = \begin{vmatrix} a_1+b & a_2 & a_3 & \cdots & a_n \\ -b & b & 0 & \cdots & 0 \\ -b & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b & 0 & 0 & \cdots & b \end{vmatrix}
$$

$$
= \begin{vmatrix} a_1+a_2+\cdots+a_n+b & a_2 & a_3 & \cdots & a_n \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{vmatrix}
$$

$$
= b^{n-1}(\sum_{i=1}^{n} a_i + b)
$$

(4) 将第一行的-1倍加到其他行,再将第i列的$-\frac{1}{i}$倍加到第一列

$$
\begin{vmatrix}
x+1 & x & x & \cdots & x \\
x & x+2 & x & \cdots & x \\
x & x & x+3 & \cdots & x \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x & x & x & \cdots & x+n
\end{vmatrix}
=
\begin{vmatrix}
x+1 & x & x & \cdots & x \\
-1 & 2 & 0 & \cdots & 0 \\
-1 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & n
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
x+1+\frac{1}{2}x+\frac{1}{3}x+\cdots+\frac{1}{n}x & x & x & \cdots & x \\
0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & n
\end{vmatrix}
$$

$$
= n!x(1+\frac{1}{2}+\cdots+\frac{1}{n})
$$

(5) 将第一行的-1倍加到其他行,再将各列加到第一列

$$
\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & 1 & \cdots & 1 & 1 \\
1 & 1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{vmatrix}
=
\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & -1 & 0 & \cdots & 0 & 0 \\
1 & 0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & -1 & 0 \\
1 & 0 & 0 & \cdots & 0 & -1
\end{vmatrix}
$$

$$
=
\begin{vmatrix}
n-1 & 1 & 1 & \cdots & 1 & 1 \\
0 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1
\end{vmatrix}
$$

$$
= (-1)^{n-1}(n-1)
$$

(6)将第一行加到其他行

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & 3 & \cdots & n \\ -1 & -2 & 0 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 2 & 6 & \cdots & 2n \\ 0 & 0 & 3 & \cdots & 2n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{vmatrix}$$

$$= n!$$

习题2-4

1. (1)

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ -2 & 3 & 4 \\ 2 & -4 & 3 \end{vmatrix} = 1 \neq 0$$

A可逆

$$D_1 = \begin{vmatrix} -1 & -2 & 2 \\ 2 & 3 & 4 \\ 1 & -4 & 3 \end{vmatrix} = -43$$

$$D_2 = \begin{vmatrix} 1 & -1 & 2 \\ -2 & 2 & 4 \\ 2 & 1 & 3 \end{vmatrix} = -24$$

$$D_3 = \begin{vmatrix} 1 & -2 & -1 \\ -2 & 3 & 2 \\ 2 & 4 & 1 \end{vmatrix} = -3$$

因此

$$x_1 = \frac{D_1}{|A|} = -43, x_2 = \frac{D_2}{|A|} = -24, x_3 = \frac{D_3}{|A|} = -3.$$

(2)

$$|A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 2 \end{vmatrix} = 3 \neq 0$$

A可逆

$$D_1 = \begin{vmatrix} -2 & -2 & 1 \\ 4 & 1 & -2 \\ 1 & 1 & 2 \end{vmatrix} = 15$$

$$D_2 = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 4 & -2 \\ -2 & 1 & 2 \end{vmatrix} = 15$$

$$D_3 = \begin{vmatrix} 1 & -2 & -2 \\ 1 & 1 & 4 \\ -2 & 1 & 1 \end{vmatrix} = 9$$

因此

$$x_1 = \frac{D_1}{|A|} = 5, x_2 = \frac{D_2}{|A|} = 5, x_3 = \frac{D_3}{|A|} = 3.$$

2. 将点代入方程得到：

$$\begin{cases} a + b + c = 2 \\ a + 2b + 4c = 3 \\ a + 3b + 9c = -2 \end{cases}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2 \neq 0$$

A可逆

$$D_1 = \begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 4 \\ -2 & 3 & 9 \end{vmatrix} = -10$$

$$D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & -2 & 9 \end{vmatrix} = 20$$

$$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -2 \end{vmatrix} = -6$$

因此

$$a = \frac{D_1}{|A|} = -5, b = \frac{D_2}{|A|} = 10, c = \frac{D_3}{|A|} = -3.$$

3.

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1, A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1,$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2, A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1, A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -3,$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1, A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2,$$

所以

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -3 & 2 \end{pmatrix}$$

4.

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & \lambda \\ -1 & \lambda & 1 \end{vmatrix} = (-\lambda + 4)(\lambda + 1)$$

当$\lambda \neq 4$且$\lambda \not= -1$时有唯一解；

当$\lambda = -1$时

$$\begin{pmatrix} 1 & -1 & 2 & -4 \\ 1 & 1 & -1 & 4 \\ -1 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

此时方程无解；

当$\lambda = -4$时

$$\begin{pmatrix} 1 & -1 & 2 & -4 \\ 1 & 1 & 4 & 4 \\ -1 & 4 & 1 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

原方程组等价于

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 + x_3 = 4 \end{cases}$$

移项,得原方程组解为

$$\begin{cases} x_1 = -3C \\ x_2 = 4 - C \\ x_3 = C \end{cases},$$

其中C为任意常数.

5. (1)

如果$A = O$,则$|A^*| = 0$；

如果$A \neq O$,假设$|A^*| \neq 0$,由$A^*A = |A|E = O$, 所以$A = 0$, 矛盾,故$|A^*| = 0$.

(2)

当A可逆时,$A^* = |A|A^{-1}$,$|A^*| = ||A|A^{-1}| = |A|^n|A|^{-1} = |A|^{n-1}$.

当A不可逆时,由（1）,$|A^*| = |A|^{n-1} = 0$.

6. 由A可逆知$|A| \neq 0$,因为$AA^* = |A|E$,

所以

$$(\frac{A}{|A|})A^* = E,$$

又因为

$$A^*(\frac{A}{|A|}) = E,$$

所以

$$(A^*)^{-1} = \frac{A}{|A|}.$$

测试题二

一、填空题

1. $\tau(35214) + \tau(41253) = (2+3+1) + (3+1) = 10$

2.

$$\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 & a & a+b \\ 0 & 2a & 3a+2b \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 & a & a+b \\ 0 & 0 & a \end{vmatrix} = a^3$$

3.

$$\begin{aligned} -A_{12} - A_{22} + A_{32} + 3A_{42} &= -(-1)^{1+2}\begin{vmatrix} 1 & -4 & -1 \\ 0 & -1 & 2 \\ -2 & -3 & 1 \end{vmatrix} - (-1)^{2+2}\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ -2 & -3 & 1 \end{vmatrix} \\ &+ (-1)^{3+2}\begin{vmatrix} 1 & 2 & 3 \\ 1 & -4 & -1 \\ -2 & -3 & 1 \end{vmatrix} + 3\cdot(-1)^{4+2}\begin{vmatrix} 1 & 2 & 3 \\ 1 & -4 & -1 \\ 0 & -1 & 2 \end{vmatrix} \\ &= 23 + 9 + 38 - 48 \\ &= 22 \end{aligned}$$

4.将各列加到第一列再展开

$$
\begin{aligned}
D &= \begin{vmatrix} 1 & a & 0 & 0 & 0 \\ 0 & 1-a & a & 0 & 0 \\ 0 & -1 & 1-a & a & 0 \\ 0 & 0 & -1 & 1-a & a \\ -a & 0 & 0 & -1 & 1-a \end{vmatrix} \\
&= \begin{vmatrix} 1-a & a & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ 0 & 0 & -1 & 1-a \end{vmatrix} - a \begin{vmatrix} a & 0 & 0 & 0 \\ 1-a & a & 0 & 0 \\ -1 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \end{vmatrix} \\
&= \begin{vmatrix} 1 & a & 0 & 0 \\ 0 & 1-a & a & 0 \\ 0 & -1 & 1-a & a \\ -a & 0 & -1 & 1-a \end{vmatrix} - a^5 \\
&= \begin{vmatrix} 1-a & a & 0 \\ -1 & 1-a & a \\ 0 & -1 & 1-a \end{vmatrix} + (-a)(-1)^{4+1} \begin{vmatrix} a & 0 & 0 \\ 1-a & a & 0 \\ -1 & 1-a & a \end{vmatrix} - a^5 \\
&= \begin{vmatrix} 1 & a & 0 \\ 0 & 1-a & a \\ -a & -1 & 1-a \end{vmatrix} + a^4 - a^5 \\
&= \cdots \\
&= 1 - a + a^2 - a^3 + a^4 - a^5
\end{aligned}
$$

5. 因为$BA = B + 2E$,所以$B(A-E) = 2E$,所以$|B| \cdot |A-E| = 4$,又$|A-E| = 2$,所以$|B| = 2$.

二、选择题

1. 按第一行展开

$$\begin{vmatrix} a-1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix} = a_1 \begin{vmatrix} a_2 & b_2 & 0 \\ b_3 & a_3 & 0 \\ 0 & 0 & a_4 \end{vmatrix} - b_1 \begin{vmatrix} 0 & a_2 & b_2 \\ 0 & b_3 & a_3 \\ b_4 & 0 & 0 \end{vmatrix}$$
$$= a_1a_2a_3a_4 - a_1a_4b_2b_3 + b_1b_2b_3b_4 - b_1b_4a_2a_3$$
$$= (a_2a_3 - b_2b_3)(a_1a_4 - b_1b_4)$$

2. $|\alpha_3, \alpha_2, \alpha_1, \beta_1| = -m, |\alpha_3, \alpha_2, \alpha_1, \beta_2| = n,$ 所以$|\alpha_3, \alpha_2, \alpha_1, \beta_1 + \beta - 2| = n - m$

3.

$$|2A^*B^{-1}| = |2|A|A^{-1}B^{-1}| = |A|^{-1}|B|^{-1} = 2 \times (-\frac{1}{2}) = -1$$

4.

因为

$$|A^{-1} + B| = |A^{-1}(E + AB)| = 2,$$

所以

$$|AB + E| = 6$$

$$|A + B^{-1}| = |(AB + E)B^{-1}| = |AB + E||B|^{-1} = 6 \times \frac{1}{2} = 3$$

5.

$$\begin{aligned} |B| &= \begin{vmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_2 + 8\alpha_3 \end{vmatrix} \\ &= \begin{vmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + 3\alpha_3 & 2\alpha_3 \end{vmatrix} \\ &= \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} \\ &= 2 \end{aligned}$$

三、解答题

1.

$$\alpha^T\alpha = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$$

所以

$$A^n = 2^{n-1}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = 2^{n-1}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} |aE - A^n| &= \begin{vmatrix} a-2^{n-1} & 0 & 2^{n-1} \\ 0 & a & 0 \\ 2^{n-1} & 0 & a-2^{n-1} \end{vmatrix} \\ &= a(a-2^{n-1})^2 - a2^{2n-2} \\ &= a^2(a-2^n) \end{aligned}$$

2. $AA^* = AA^T = |A|E$,假设$|A| = 0$,则$AA^T = 0$,一个矩阵乘以其转置矩阵为零矩阵时,这个矩阵必为零矩阵,所以$A = O$,与题设矛盾,故$|A| \neq 0$.

3.

$$\begin{aligned} \begin{pmatrix} O & A \\ B & O \end{pmatrix}^* &= \begin{vmatrix} O & A \\ B & O \end{vmatrix}\begin{pmatrix} O & A \\ B & O \end{pmatrix}^{-1} \\ &= \begin{pmatrix} O & |A||B|B^{-1} \\ |B||A|A^{-1} & O \end{pmatrix} \\ &= \begin{pmatrix} O & |A|B^* \\ |B|A^* & O \end{pmatrix} \\ &= \begin{pmatrix} O & 3B^* \\ 2A^* & O \end{pmatrix} \end{aligned}$$

4.

$$\begin{aligned} (A^{-1}+B^{-1})^{-1} &= (B^{-1}BA^{-1} + B^{-1}AA^{-1})^{-1} \\ &= (B^{-1}(A+B)A^{-1})^{-1} \\ &= A(A+B)^{-1}B \end{aligned}$$

5. 在等式两端同时左乘矩阵A,

$$\begin{aligned} AA^*X &= AA^{-1} + 2AX \\ |A|X &= E + 2AX \end{aligned}$$

所以

$$X=(|A|E-2A)^{-1}=\begin{pmatrix}\frac{1}{4}&\frac{1}{4}&0\\0&\frac{1}{4}&\frac{1}{4}\\\frac{1}{4}&0&\frac{1}{4}\end{pmatrix}$$

6.

$$\begin{vmatrix}\lambda&1&1\\1&\lambda&1\\1&1&\lambda\end{vmatrix}=(\lambda-1)^2(\lambda+2)$$

当$\lambda\neq 1$且$\lambda\neq -2$时有唯一解;

当$\lambda=-2$时,简化阶梯阵为

$$\begin{pmatrix}1&0&-1&4\\0&1&-1&3\\0&0&0&-9\end{pmatrix}$$

所以方程组无解

当$\lambda=1$时,简化阶梯阵为

$$\begin{pmatrix}1&1&1&-2\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

方程组的解为

$$\begin{cases}x_1=-2-C_1-C_2\\x_2=C_1\\x_3=C_2\end{cases}$$

其中C_1,C_2为任意常数.

第三章

习题3-1

1.

$$2\alpha-\beta=2\begin{pmatrix}2\\1\\3\end{pmatrix}-\begin{pmatrix}3\\5\\7\end{pmatrix}=\begin{pmatrix}1\\-3\\-1\end{pmatrix}$$

同理，

$$\alpha-\beta+2\gamma=\begin{pmatrix}-5\\4\\-2\end{pmatrix}$$

2.

令

$$\alpha=k_1\beta_1+k_2\beta_2+k_3\beta_3$$

解得

$$k_1=2,k_2=-1,k_3=-1,$$

故

$$\alpha=2\beta_1-\beta_2-\beta_3$$

3.

$$(\beta_1,\beta_2,...,\beta_n)=(\alpha_1,\alpha_2,...,\alpha_n)\begin{pmatrix}0&1&1&\cdots&1\\1&0&1&\cdots&1\\1&1&0&\cdots&1\\\cdots&\cdots&\cdots&\cdots&\cdots\\1&1&1&\cdots&0\end{pmatrix}$$

将上式记为

$$B=AK,$$

其中

$$|k|=(-1)^{n-1}(n-1)\neq 0,$$

K可逆，故向量组可以相互表示。

4.

$$(\beta_1,\beta_2,\beta_3,\alpha_1,\alpha_2,\alpha_3)\to\begin{pmatrix}1&-2&-2&1&1&a\\0&a+2&a+2&0&a-1&0\\0&0&a-4&0&3(1-a)&1-a\end{pmatrix}$$

满秩，$a\neq-2,a\neq 4$

$$(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3)\to\begin{pmatrix}1&1&a&1&-2&-2\\0&a-1&1-a&0&a+2&a+2\\0&0&2-a-a^2&0&6+3a&4a+2\end{pmatrix}$$

不满秩$a-1=0$或$2-a-a^2=0$

综上$a=1$

5. 向量组A中的每个向量均可由向量组B中向量的线性组合表示，而每个B中的向量均可由C中向量的线性组合表示，故A中每个的向量均可由C中向量的线性组合表示，故向量组A可由向量组C线性表示。同理，向量组C也可由向量组A线性表示，得证。

习题3-2

1. （1）正确

证明：若$\alpha_1,\alpha_2,...,\alpha_m$线性相关，则存在一组不全为0的数$k_1,k_2,...,k_m$,

使

$$k_1\alpha_1{+}k_2\alpha_2{+}...{+}k_m\alpha_m=0,$$

不妨设$k_1\notin 0$,则

$$\alpha_1=-\frac{k_2}{k_1}\alpha_2+...-\frac{k_m}{k_1}\alpha_m,$$

与条件矛盾。所以$\alpha_1,\alpha_2,...,\alpha_m$线性无关。

（2）错误

反例：

若

$$\alpha_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\alpha_2=\begin{pmatrix}0\\1\\0\end{pmatrix}\alpha_3=\begin{pmatrix}0\\1\\0\end{pmatrix},$$

则$\alpha_1,\alpha_2,\alpha_3$线性相关，但$\alpha_1$不能由$\alpha_2$和$\alpha_3$表示。

(3)正确

证明：

$$k_1\alpha_1{+}k_2\alpha_2{+}k_3\alpha_3+k_4\left(\beta_1+\beta_2\right)=0$$

若$k_4\notin 0$,则

$$\beta_2=\frac{k_1}{k_4}\alpha_1+\frac{k_2}{k_4}\alpha_2+\frac{k_3}{k_4}\alpha_3-\beta_1,$$

则β_2可由$\alpha_1,\alpha_2,\alpha_3$线性表示，矛盾！那么$k_4=0$,再由$\alpha_1,\alpha_2,\alpha_3$线性无关，

得

$$k_1=k_2=k_3=0.$$

综上

$$k_1=k_2=k_3=k_4=0,$$

得证。

（4）错误

反例：

$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$\alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$\beta_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$\beta_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$\beta_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

(5) 错误

反例：

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \beta_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \beta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

若要使得线性相关，对于α，必有$k_1 = 0, k_2 \notin 0$,对于β，必有$k_2 = 0, k_1 \notin 0$,矛盾

（6）错误

反例：

取

$$\beta_1 = \alpha_1, \beta_2 = \alpha_2, ..., \beta_m = \alpha_m$$

2.

(1) $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$ 系数矩阵行列式

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{vmatrix} = -7 \neq 0$$

$k_1 = k_2 = k_3 = 0$线性无关

(2) $\beta_3 = 2\beta_2$,线性相关

(3) $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\gamma_1, \gamma_2, \gamma_3, \gamma_4$可由$e_1, e_2, e_3$线性表示。

由推论5知，$\gamma_1, \gamma_2, \gamma_3, \gamma_4$线性相关。

3. 由$\beta_4 = \alpha_1 + \alpha_4 = \beta_1 + \beta_3 - \beta_2$，知向量组线性相关

4.

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$$

$$\begin{vmatrix} 1 & a & 2 \\ -1 & 2 & a \\ 1 & 1 & 0 \end{vmatrix} = a^2 - a - 6$$

当$a=3$或$a=-2$时，方程有非零解，$\alpha_1,\alpha_2,\alpha_3$线性相关。

当$a\neq 3$且$a\neq -2$时，方程有唯一零解，$\alpha_1,\alpha_2,\alpha_3$线性无关。

5.

$$k_1\beta_1+k_2\beta_2+...+k_m\beta_m=0$$

$$(k_1+k_m)\alpha_1+(k_1+k_2)\alpha_2+...+(k_{m-1}+k_m)\alpha_m=0$$

由$\alpha_1,\alpha_2,...,\alpha_m$线性无关知

$$\begin{cases} k_m+k_1=0 \\ k_1+k_2=0 \\ k_2+k_3=0 \\ \vdots \\ k_{m-1}+k_m=0 \end{cases}$$

$$\begin{pmatrix} 1 & 0 & 0 & & 0 & 1 \\ 1 & 1 & 0 & & 0 & 0 \\ 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_m \end{pmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & 0 & & 0 & 1 \\ 1 & 1 & 0 & & 0 & 0 \\ 0 & 1 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 1 \end{vmatrix} = 1+(-1)^{\mathrm{m}-1}$$

m为偶数，行列式等于0，线性相关。m为奇数，行列式不等于0，线性无关。

6.

设

$$\alpha_1=\begin{pmatrix} \mathrm{a}_{11} \\ \mathrm{a}_{21} \\ \vdots \\ \mathrm{a}_{n1} \end{pmatrix} \alpha_2=\begin{pmatrix} \mathrm{a}_{12} \\ \mathrm{a}_{22} \\ \vdots \\ \mathrm{a}_{n2} \end{pmatrix} \alpha_n=\begin{pmatrix} \mathrm{a}_{1n} \\ \mathrm{a}_{2n} \\ \vdots \\ \mathrm{a}_{nn} \end{pmatrix}$$

$$A=(\alpha_1,\alpha_2,...,\alpha_m)$$

线性无关$\Leftrightarrow A$可逆

充分性.

$$e_1 = x_{11}\alpha_1 + x_{12}\alpha_2 + ... x_{n1}\alpha_n$$

$$e_2 = x_{12}\alpha_1 + x_{22}\alpha_2 + ... x_{n2}\alpha_n$$

$$\vdots$$

$$e_n = x_{1n}\alpha_1 + x_{2n}\alpha_2 + ... x_{nn}\alpha_n$$

$$E = (e_1, e_2, ..., e_n) = (\alpha_1, \alpha_2, ..., \alpha_n)\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

A可逆得证,充分性得证

必要性.

$\alpha_1, \alpha_2, ..., \alpha_n$线性无关，则$A$可逆。

$$\forall \beta, A^{-1}\beta = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\beta = A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\alpha_1 + x_1\alpha_2 + ... x_n\alpha_n$$

必要性得证

习题3-3

1. (1)

$$(\alpha_1, \alpha_2, ..., \alpha_5) \sim \begin{pmatrix} 1 & 0 & 3 & 6 & 3 \\ 0 & 1 & -6 & -7 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

秩是2，极大无关组为α_1, α_2

$$\alpha_3 = 3\alpha_1 - 6\alpha_2, \alpha_4 = 6\alpha_1 - 7\alpha_2, \alpha_5 = 3\alpha_1 - 2\alpha_2.$$

(2)

$$(\alpha_1,\alpha_2,...,\alpha_5)\sim\begin{pmatrix}1&0&0&-2&1\\0&1&0&1&1\\0&0&1&1&0\\0&0&0&0&0\end{pmatrix}$$

秩是3，极大无关组为$\alpha_1,\alpha_2,\alpha_3$

$$\alpha_4=-2\alpha_1+\alpha_2+\alpha_3,\alpha_5=\alpha_1+\alpha_2$$

2.

(1)

$$A\sim\begin{pmatrix}1&0&2&0&4\\0&1&-1&0&-6\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}$$

秩为3

(2)

$$B\sim\begin{pmatrix}1&0&0&\frac{5}{4}\\0&1&0&\frac{1}{4}\\0&0&1&0\end{pmatrix}$$

秩为3

3. P,Q可逆，则P,Q可写为初等矩阵之积$P=P_1P_2...P_S\ Q=Q_1Q_2..Q_t$

$$B=P_1P_2...P_SAQ_1Q_2..Q_t$$

$$B\sim A$$

B与A有相同的秩。

4. 设A的行向量组为$\alpha_1,\alpha_2,...,\alpha_m$,划去第$k$行，则$B$的行向量为

$$\alpha_1,\alpha_2,...\alpha_{k-1},\alpha_{k+1},...,\alpha_m$$

(1)若α_k可由$\alpha_1,\alpha_2,...\alpha_{k-1},\alpha_{k+1},...,\alpha_m$线性表示，则$A$，$B$行向量组等价,$r(A)=r(B)$

(2)若α_k不可由$\alpha_1,\alpha_2,...\alpha_{k-1},\alpha_{k+1},...,\alpha_m$线性表示，则$\alpha_1,\alpha_2,...\alpha_{k-1},\alpha_{k+1},...,\alpha_m$极大无关组加上$\alpha_k$构成$\alpha_1,\alpha_2,...,\alpha_m$的极大无关组，$r(A)=r(B)+1$

5. $\alpha_1,\alpha_2,\alpha_3,\alpha_5$秩为4，则向量组$B$线性无关。$\alpha_1,\alpha_2,\alpha_3$线性无关，是向量组$A$的一个极大无关组，则$\alpha_4$可由$\alpha_1,\alpha_2,\alpha_3$线性表示，所以$\alpha_1,\alpha_2,\alpha_3,\alpha_5$是向量组$C$的一个极大无关组，所以秩为4.

6. $A=(\alpha_1,\alpha_2,...,\alpha_r)$ $B=(\beta_1,\beta_2,...,\beta_r)$,

则

$$B=A\begin{pmatrix} 0 & 1 & 1 & & 1 \\ 1 & 0 & 1 & & 1 \\ 1 & 1 & 0 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & & 0 \end{pmatrix}=AQ$$

$$\begin{vmatrix} 0 & 1 & 1 & & 1 \\ 1 & 0 & 1 & & 1 \\ 1 & 1 & 0 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & & 0 \end{vmatrix}=(-1)^{r-1}(r-1)\neq 0$$

则矩阵Q可逆,由题3知$R(A)=R(B)$,得证

7.

(1) $R(A)=n$,则A可逆, $A^*=|A|\,A^{-1}$ $R\left(A^*\right)=n$

(2) $R(A)=n-1$, A至少有一个n阶子式不为0, $A^*\neq 0R\left(A^*\right)\geq 1$, $AA^*=|A|\,E=0$,

$R\left(A\right)+R\left(A^*\right)\leq n$, $R\left(A^*\right)\leq 1$, $R\left(A^*\right)=1$

(3) $R\left(A\right)\leq n-2$，A的每一个n-1阶子式为0，$A^*=0$ $R\left(A^*\right)=0$

习题3-4

1.

(1)

系数矩阵

$$A=\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 5 & -2 \end{pmatrix}\to\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

自由未知量为x_3,x_4

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

基础解系为

$$\eta_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} \eta_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

通解为$x = k_1\eta_1 + k_2\eta_2$

(2) 系数矩阵$A = \begin{pmatrix} 1 & -3 & 1 & 1 \\ 2 & -5 & 1 & 2 \\ 5 & -7 & -3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ 自由未知量为x_3,x_4

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

基础解系为

$$\eta_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \eta_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

通解为$x = k_1\eta_1 + k_2\eta_2$

2.

(1)增广矩阵

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

自由未知量为x_3,x_4

$$x_1=-x_3+2,x_2=-x_3+x_4-1$$

令

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得特解

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

得原方程一个特解

$$\eta=\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

导出组

$$x_1=-x_3,x_2=x_3-x_4$$

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\zeta_1=\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\zeta_2=\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

解为

$$x=k_1\zeta_1+k_2\zeta_2+\eta$$

(2)增广矩阵

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

自由未知量为x_3,x_4

$$x_1 = -2x_4 + 1$$

$$x_2 = 3x_3 + 3x_4$$

令

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得原方程一个特解

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

导出组

$$x_1 = -2x_4, x_2 = 3x_3 + 3x_4$$

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得

$$\zeta_1 = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\zeta_2 = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

解为

$$x = k_1\zeta_1 + k_2\zeta_2 + \eta$$

3.

$\zeta_1,\zeta_2,...,\zeta_{n-r}$是导出组的基础解系，则其线性无关。

下证η不可用$\zeta_1\zeta_2,...,\zeta_{n-r}$线性表示, 若可以,则$\eta=k_1\zeta_1+k_2\zeta_2+...+k_{n-r}\zeta_{n-r}$,

$A\eta=k_1A\zeta_1+k_2A\zeta_2+...+k_{n-r}A\zeta_{n-r}=0$与$Ax=\beta$的特解矛盾。

η不可用$\zeta_1\zeta_2,...,\zeta_{n-r}$线性表示,则$\zeta_1\zeta_2,...,\zeta_{n-r},\eta$线性无关

4.

$$A(2\eta_1-\eta_2-\eta_3)=2\beta-\beta-\beta=0$$

所以$2\eta_1-\eta_2-\eta_3$是Ax =0的解。

由$R(A)=3$知Ax =0的基础解系只有一个向量，即为

$$\zeta=2\eta_1-\eta_2-\eta_3=\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$$

通解为

$$\eta_1+k\zeta$$

5. 证明

$$A(\eta_i-\eta_1)=\beta-\beta=0\ i=2,3,...,n-r+1$$

所以$\eta_i-\eta_1$是Ax=0的解,$i=2,3,...,n-r+1$

$$\sum_{i=2}^{n-r+1}k_i(\eta_i-\eta_1)=0$$

$$k_2\eta_2+k_3\eta_3+...+k_{m-r+1}\eta_{n-r+1}-(k_2+k_2+...+k_{n-r+1})\eta_1=0$$

由η线性无关知，k_2=k_3=...=$k_{n-r+1}=0$所以$\eta_2-\eta_1,\eta_3-\eta_1,...,\eta_{n-r+1}-\eta_1$线性无关。

所以$\eta_2-\eta_1,\eta_3-\eta_1,...,\eta_{n-r+1}-\eta_1$是方程Ax =0的n-r个线性无关的解。

由$R(A)=r$知$Ax=0$有$n-r$个线性无关的解。

$\eta_2-\eta_1,\eta_3-\eta_1,...,\eta_{n-r+1}-\eta_1$是Ax=0的基础解系。

6.由第5题知，$\eta_2-\eta_1,\eta_3-\eta_1,...,\eta_{n-r+1}-\eta_1$是导出组Ax=0的基础解系。

$Ax=\beta$通解为

$$\begin{aligned}x&=\eta_1+a_2(\eta_2-\eta_1)+a_3(\eta_3-\eta_1)+...+a_{n-r+1}(\eta_{n-r+1}-\eta_1)\\&=(1-a_2-a_3-a_4-...-a_{n-r+1})\eta_1+a_2\eta_2+a_3\eta_3+...+a_{n-r+1}\eta_{n-r+1}\end{aligned}$$

令

$$k_1=1-a_2-a_3-a_4-...-a_{n-r+1}$$

$$k_2=a_2$$

$$k_{n-r+1}=a_{n-r+1}$$

即满足

$$k_1+k_2+...+k_{n-r+1}=1$$

解为

$$x=k_1\eta_1+k_2\eta_2+...+k_{n-r+1}\eta_{n-r+1}$$

7.
设A_{m*n}的秩为r，则存在初等方阵P
使

$$PA=\begin{pmatrix}\tilde{A}_{r\times n}\\ O_{\mathrm{m-r\times n}}\end{pmatrix}$$

$$PAB=\begin{pmatrix}\tilde{A}_{r\times n}\\ O_{\mathrm{m-r\times n}}\end{pmatrix}B=\begin{pmatrix}\tilde{A}_{r\times n}B\\ O\end{pmatrix}=PC$$

PC的秩小于等于r，即C的秩小于等于r, 即$R(C)\le R(A)$, 同理$R(C)\le R(B)$,
所以$R(C)\le\min\{R(A),R(B)\}$
8.

$$\beta_1=c_{11}\alpha_1+c_{21}\alpha_2+...+c_{t1}\alpha_t$$

$$\beta_2=c_{12}\alpha_1+c_{22}\alpha_2+...+c_{t2}\alpha_t$$

$$\vdots$$

$$\beta_s=c_{1s}\alpha_1+c_{2s}\alpha_2+...+c_{ts}\alpha_t$$

由7题结论，

$$R(\beta_1,\beta_2,...,\beta_s)\le R(\alpha_1,\alpha_2,...,\alpha_t)$$

习题3-5

1.（1）构成

$$\forall x = (x_1, x_2, ..., x_n)^T \in V_1$$
$$\forall y = (y_1, y_2, ..., y_n)^T \in V_1$$

有

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

所以$x + y \in V_1$

$$kx = (kx_1, kx_2, ..., kx_n)$$
$$kx_1 + kx_2 + ... + kx_n = 0$$
$$kx \in V_1$$

所以构成线性空间

(2) 不构成

$$\forall x = (x_1, x_2, ..., x_n)^T \in V_2$$
$$\forall y = (y_1, y_2, ..., y_n)^T \in V_2$$

有

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

所以$\mathrm{x} + \mathrm{y} \in V_1$

$$x_1 + y_1 + x_2 + y_2 + ... + x_n + y_n = 2$$
$$x + y \notin V_2$$

所以不构成线性空间

（3）构成

$$\forall x = (x_1, x_2, ..., x_n)^T \in V_3$$
$$\forall y = (y_1, y_2, ..., y_n)^T \in V_3$$

有

$$x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$kx = (kx_1, kx_2, ..., kx_n)$$

$$x_1 + y_1 = x_2 + y_2 = ... = x_n + y_n$$

$$kx_1 = kx_2 = ... = kx_n$$

$$x + y, kx \in V_3$$

所以构成线性空间

2.

$$\forall x \in L_1(\alpha_1, \alpha_2, ..., \alpha_s)$$

$$x = k_1\alpha_1 + k_2\alpha_2 + ... + k_s\alpha_s$$

由两个向量组等价，知α_i, $i = 1, 2, ..., s$可由$\beta_1, \beta_2, ..., \beta_t$表示

$$\alpha_i = l_{i1}\beta_1 + l_{i2}\beta_2 + ... + l_{it}\beta_t$$

$$\begin{aligned} x &= k_1\left(l_{i1}\beta_1 + l_{i2}\beta_2 + ... + l_{it}\beta_t\right) + k_2\left(l_{21}\beta_1 + l_{22}\beta_2 + ... + l_{2t}\beta_t\right) + ... + k_s\left(l_{s1}\beta_1 + l_{s2}\beta_2 + ... + l_{st}\beta_t\right) \\ &= (k_1l_{11} + k_2l_{21} + ... + k_sl_{s1})\beta_1 + (k_1l_{12} + k_2l_{22} + ... + k_sl_{s2})\beta_2 + ... + (k_1l_{1t} + k_2l_{2\mathrm{t}} + ... + k_sl_{st})\beta_t \end{aligned}$$

$$x \in L_2(\beta_1, \beta_2, ..., \beta_s)$$

同理可证$\forall x \in L_2$，有$x \in L_1$ 所以$L_1 = L_2$

3.

$$(\alpha_1, \alpha_2, ..., \alpha_5) = \begin{pmatrix} 1 & 2 & 2 & 3 & -1 \\ 2 & 2 & 3 & 2 & -1 \\ 3 & 2 & 1 & 1 & 2 \\ -1 & -1 & 1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

所以基为$\alpha_1, \alpha_2, \alpha_3$，维数为3

4（1）

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

所以$\alpha_1, \alpha_2, \alpha_3, \alpha_4$是一组基, $\alpha = -\alpha_2 + \alpha_3 + \alpha_4$

（2）

$$(\beta_1,\beta_2,\beta_3,\beta_4,\alpha)=\begin{pmatrix}1&1&1&0&1\\0&1&1&1&1\\1&0&1&1&2\\1&1&0&1&1\end{pmatrix}\to\begin{pmatrix}1&0&0&0&\frac{2}{3}\\0&1&0&0&-\frac{1}{3}\\0&0&1&0&\frac{2}{3}\\0&0&0&1&\frac{2}{3}\end{pmatrix}$$

所以$\beta_1,\beta_2,\beta_3,\beta_4$是一组基, $\alpha=\frac{2}{3}\beta_1-\frac{1}{3}\beta_2+\frac{2}{3}\beta_3+\frac{2}{3}\beta_4$

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4)=\begin{pmatrix}1&1&1&1&1&1&1&0\\0&1&1&1&0&1&1&1\\0&0&1&1&1&0&1&1\\0&0&0&1&1&1&0&1\end{pmatrix}\to\begin{pmatrix}1&0&0&0&1&0&0&-1\\0&1&0&0&-1&1&0&0\\0&0&1&0&0&-1&1&0\\0&0&0&1&1&1&0&1\end{pmatrix}$$

所以，过渡矩阵

$$P=\begin{pmatrix}1&0&0&-1\\-1&1&0&0\\0&-1&1&0\\1&1&0&1\end{pmatrix}$$

坐标为$\begin{pmatrix}\frac{2}{3}&-\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\end{pmatrix}^T$.

测试题三

一、填空题

1. (1)

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\to\begin{pmatrix}1&0&-\frac{1}{3}&\frac{5}{3}\\0&1&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$$

过渡矩阵为

$$\begin{pmatrix}-\frac{1}{3}&\frac{5}{3}\\\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$$

(2) β_1,β_2,β_3线性相关，

$$\begin{vmatrix}1&1&3\\1&2&4\\1&3&a\end{vmatrix}=0$$

则$a=5$

(3)

$$A\alpha=k\alpha$$

$$\begin{cases} a = ka \\ 2a+3=k \\ 3a+4=k \end{cases}$$

解得$a=-1$

(4) 由$|A|=0$ 解得$\lambda=1$或$\lambda=-1$

当$\lambda=1$时，

$$(A|b)=\begin{pmatrix} 1 & 1 & 1 & a \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

不可能成立

当$\lambda=-1$时，

$$(A|b)\to\begin{pmatrix} 1 & -1 & -1 & -a \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & a+2 \end{pmatrix},$$

所以$a=-2$

二、选择题

1.

由于$(\alpha_1-\alpha_2)+(\alpha_2-\alpha_3)+(\alpha_3-\alpha_1)=0$，选A

2.

当$r<s$时，取第一组为$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ 第二组为$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$知A、C错误

当$r>s$时，取第一组为$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ 第二组为$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$知B错误,选D

3.

取

$$A=\begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix} B=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

可知选D

4.

$$k_1A\alpha_1+k_2A\alpha_2+...+k_sA\alpha_s=A(k_1\alpha_1+k_2\alpha_2+...+k_s\alpha_s)=0$$

可知选A

5.

$$\alpha_1+\alpha_2=1*\alpha_1+2*\frac{1}{2}\alpha_2+0*\alpha_3$$

$$\alpha_2+\alpha_3=0*\alpha_1+2*\frac{1}{2}\alpha_2+3*\frac{1}{3}\alpha_3$$

$$\alpha_3+\alpha_1=1*\alpha_1+0*\frac{1}{2}\alpha_2+3*\frac{1}{3}\alpha_3$$

可知选A

三、解答题

1.

$$|\alpha_1,\alpha_2,\alpha_3,\alpha_4|=\begin{vmatrix}1+a & 2 & 3 & 4\\ 1 & 2+a & 3 & 4\\ 1 & 2 & 3+a & 4\\ 1 & 2 & 3 & 4+a\end{vmatrix}=a^4+10a^3$$

$a=0$或$a=-10$时，$\alpha_1,\alpha_2,\alpha_3,\alpha_4$线性相关

（1）$a=0$时，$\alpha_2=2\alpha_1$，$\alpha_3=3\alpha_1$，$\alpha_4=4\alpha_1$，α_1为一个极大无关组。

（2）$a=-10$时，

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\begin{pmatrix}-9 & 2 & 3 & 4\\ 1 & -8 & 3 & 4\\ 1 & 2 & -7 & 4\\ 1 & 2 & 3 & -6\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$\alpha_1,\alpha_2,\alpha_3,$为一个极大无关组$\alpha_4=-\alpha_1-\alpha_2-\alpha_3$

2.由a,b,c不全为0知，$R(A)\geq 1$ 由$AB=0$知，$R(A)<3$ 所以$R(A)=1$或2

（1）$R(A)=2$，则$Ax=0$的解空间是1维的。

由$A\begin{pmatrix}1\\2\\3\end{pmatrix}=0$知,$\begin{pmatrix}1\\2\\3\end{pmatrix}$是$Ax=0$的一个解, 所以齐次方程通解为

$$x=k_1\begin{pmatrix}1\\2\\3\end{pmatrix}$$

（2）$R(A)=1$，则$Ax=0$的解空间是二维。$Ax=0$可化为

$$(a,b,c)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0$$

若$c=0$,则$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$与$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$均为$Ax=0$的解。

通解为

$$x=k_1\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}+k_2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

若$c\neq 0$，则x_1,x_2可看作自由变量

取

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} \mathrm{c} \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ \mathrm{c} \end{pmatrix}$$

得

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} c \\ 0 \\ -a \end{pmatrix},\begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$$

通解为

$$x=k_1\begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}+k_2\begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$$

3.

（1）

$$(A|\,\xi_1)\rightarrow\begin{pmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3是自由变量，取$x_3=0$

得特解

$$\eta=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

齐次方程$\begin{cases}x_1-\frac{1}{2}x_3=0\\x_2+\frac{1}{2}x_3=0\end{cases}$的通解为

$$x=k\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

，

所以

$$\xi_2=\begin{pmatrix}0\\0\\1\end{pmatrix}+k\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\left(A^2\middle|\,\xi_1\right)\to\begin{pmatrix}1&1&0&-\frac{1}{2}\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

x_2,x_3是自由变量，取$\begin{pmatrix}x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$，

特解为

$$\eta=\begin{pmatrix}-\frac{1}{2}\\0\\0\end{pmatrix}$$

齐次方程$x_1+x_2=0$的通解为

$$x=k_1\begin{pmatrix}-1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

，

所以

$$\xi_3=\begin{pmatrix}-\frac{1}{2}\\0\\0\end{pmatrix}+k_1\begin{pmatrix}-1\\1\\0\end{pmatrix}+k_2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

（2）

$$\left(A \middle| \xi_1\right) \to \begin{pmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3是自由变量，取$x_3 = 0$ 得特解

$$\eta = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

齐次方程$\begin{cases} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \end{cases}$ 的通解为

$$x = k \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

所以

$$\xi_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\left(A^2 \middle| \xi_1\right) \to \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_3是自由变量，取$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$，特解为

$$\eta = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

齐次方程$x_1 + x_2 = 0$的通解为

$$x = k_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

所以

$$\xi_3=\begin{pmatrix}-\frac{1}{2}\\0\\0\end{pmatrix}+k_1\begin{pmatrix}0\\0\\1\end{pmatrix}+k_2\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

令

$$\alpha_1=\begin{pmatrix}1\\-1\\2\end{pmatrix}\alpha_2=\begin{pmatrix}1\\-1\\0\end{pmatrix}\alpha_3=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

易证$\alpha_1,\alpha_2,\alpha_3$线性无关。

观察有

$\xi_1=-\alpha_1$，$\xi_2=-\frac{1}{2}\alpha_2+k\alpha_1$，$\xi_3=-\frac{1}{2}\alpha_3+\frac{1}{2}k_1(\alpha_1-\alpha_2)+k_2\alpha_2=-\frac{1}{2}\alpha_3+\frac{1}{2}k_1\alpha_1+(k_2-\frac{1}{2}k_1)\alpha_2$

$$(\xi_1,\xi_2,\xi_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}-1&k&\frac{1}{2}k_1\\0&-\frac{1}{2}&k_2-\frac{1}{2}k_1\\0&0&-\frac{1}{2}\end{pmatrix}$$

$|P|\neq 0$且$\alpha_1,\alpha_2,\alpha_3$线性无关，所以$\xi_1,\xi_2,\xi_3$线性无关

4 (1)设三个线性无关解为ξ_1,ξ_2,ξ_3 则$\xi_2-\xi_1,\xi_3-\xi_1$是齐次方程Ax=b的解，则$R(A)\leq 2$；另一方面，A的第一行与第二行线性无关$R(A)\geq 2$。综上，$R(A)=2$

（2）由$R(A)=2$，知A的三阶子式为0，

即

$$\begin{vmatrix}1&1&1\\4&3&5\\a&1&3\end{vmatrix}=0,$$

得$a=2$

$$\begin{vmatrix}1&1&1\\3&5&-1\\1&3&b\end{vmatrix}=0,$$

得$b=-3$

$$(A|\,b)\to\begin{pmatrix}1&0&2&-4&2\\0&1&-1&5&-3\\0&0&0&0&0\end{pmatrix}$$

x_3, x_4为自由变量，取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

得特解为

$$\eta = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix}$$

齐次方程$\begin{cases} x_1 + 2x_3 - 4x_4 = 0 \\ x_2 - x_3 + 5x_4 = 0 \end{cases}$

取

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

得齐次方程通解为

$$x = k_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

原方程通解为

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} + k_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 4 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

5.

(1)（数学归纳法）$n = 2$时，$|A| = 4a^2 - a^2 = 3a^2$成立。

假设$n = j$时，$|A| = (j+1)a^j$成立，当$n = j+1$时，$|A|$按第一行展开得

$$|A| = 2a \begin{vmatrix} 2a & 1 & & & \\ a^2 & 2a & 1 & & \\ & & \ddots & & \\ & & & \ddots & 1 \\ & & & a^2 & 2a \end{vmatrix} - \begin{vmatrix} a^2 & 1 & & & \\ 0 & 2a & 1 & & \\ & a^2 & 2a & 1 & \\ & & & \ddots & \\ & & & & \ddots \end{vmatrix} = (j+2)a^{j+1}$$

综上所述，

$$|A| = (n+1)a^n$$

（2）$a \neq 0$有唯一解，由克莱姆法则，$x_1 = \frac{|A_1|}{|A|} = \frac{na^{n-1}}{(n+1)a^n} = \frac{n}{(n+1)a}$

（A_1为b替换A的第一列所成矩阵)

（3）$a = 0$ 有无穷解，此时

$$(A|\,b) = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

x_1是自由变量

取$x_1 = 0$,得特解

$$\eta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

取$x_1 = 1$,得齐次方程通解

$$x = k \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

通解为

$$x = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + k \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

6.$R(A)=3$ $R(A^*)=1$ 由$A\begin{pmatrix}1\\0\\1\\0\end{pmatrix}=0$ 知$\alpha_1+\alpha_3=0$ 所以$\alpha_1,\alpha_2,\alpha_4$或$\alpha_2,\alpha_3,\alpha_4$是A的一个极大无关组$A^*A=|A|E=0$知$A^*\alpha_1=0,A^*\alpha_2=0,A^*\alpha_3=0,A^*\alpha_4=0$ 再由$R(A^*)=1$知，$A^*x=0$的基础解系是$\alpha_1,\alpha_2,\alpha_4$或$\alpha_2,\alpha_3,\alpha_4$

第四章

习题4-1

1.欲求γ 与α 和β 均正交, 设$\gamma=(x,y,z)$, 则γ 满足

$$\begin{cases} x+y+2z=0, \\ -4x+2y+2z=0. \end{cases}$$

则$\gamma=(x,5x,-3x)$, 取$x=-1$, 则$\gamma=(-1,-5,3)$.

2. (1) 取

$$\beta_1=\alpha_1=\begin{pmatrix}1\\1\\2\end{pmatrix},$$

$$\beta_2=\alpha_2-\frac{[\beta_1,\alpha_2]}{[\beta_1,\beta_1]}\beta_1=\begin{pmatrix}1\\2\\3\end{pmatrix}-\frac{3}{2}\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix},$$

$$\beta_3=\alpha_3-\frac{[\beta_1,\alpha_3]}{[\beta_1,\beta_1]}\beta_1-\frac{[\beta_2,\alpha_3]}{[\beta_2,\beta_2]}\beta_2=\begin{pmatrix}-1\\3\\5\end{pmatrix}-2\begin{pmatrix}1\\1\\2\end{pmatrix}-4\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}=\begin{pmatrix}-1\\-1\\1\end{pmatrix}.$$

(2) 取

$$\beta_1=\alpha_1=\begin{pmatrix}1\\-1\\0\\0\end{pmatrix},$$

$$\beta_2=\alpha_2-\frac{[\beta_1,\alpha_2]}{[\beta_1,\beta_1]}\beta_1=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\\0\end{pmatrix},$$

$$\beta_3=\alpha_3-\frac{[\beta_1,\alpha_3]}{[\beta_1,\beta_1]}\beta_1-\frac{[\beta_2,\alpha_3]}{[\beta_2,\beta_2]}\beta_2=\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\-1\\0\end{pmatrix}=\begin{pmatrix}-\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\\1\end{pmatrix}.$$

3.(1)不是；(2)是.

4. $\because A$ 是正交阵 $\therefore A^TA=E \because A^T(A^T)^T=A^TA=E \therefore A^{-1}=A^T$ 也是正交阵

$\because |A^TA|=1 \therefore |A^T|\cdot|A|=1 \ \therefore |A|=1$ 或-1.

5. $\because A,B$ 都是正交阵 $\therefore A^TA=E, B^TB=E$

$\therefore (AB)^T(AB)=B^TA^TAB=E. \therefore AB$ 也是正交阵.

6. 先证对称性. $H^T=(E-2xx^T)^T=E-2xx^T=E$. 再证$H$ 是正交阵.

$\because x^Tx=1 \therefore H^TH=(E-2xx^T)^T(E-2xx^T)=E \ \therefore H$ 是对称的正交阵.

习题4-2

1. (1)矩阵$A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$ 的特征多项式为

$$|A-\lambda E|=\begin{vmatrix}-\lambda&1&1\\1&-\lambda&1\\1&1&-\lambda\end{vmatrix}=(2-\lambda)(\lambda+1)^2,$$

所以A 的全部特征值为$\lambda_1=\lambda_2=-1,\lambda_3=2$.

当$\lambda_1=\lambda_2=-1$ 时, 解方程$(A+E)x=0$, 由

$$A+E=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\underset{\sim}{r}\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$$

得基础解系
$$\alpha_1=\begin{pmatrix}-1\\1\\0\end{pmatrix},\ \alpha_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}$$
从而α_1,α_2 就是对应于$\lambda_1=\lambda_2=-1$ 的两个线性无关的特征向量,
并且对应于$\lambda_1=\lambda_2=-1$ 的全部特征向量为$k_1\alpha_1+k_2\alpha_2(k_1,k_2$ 不同时为零).

当$\lambda_3=2$ 时, 解方程$(A-2E)x=0$, 由
$$A-2E=\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}\underset{\sim}{r}\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}$$
得基础解系
$$\alpha_3=\begin{pmatrix}1\\1\\1\end{pmatrix},$$
从而α_3 就是对应于$\lambda_3=2$ 的特征向量,
并且对应于$\lambda_3=2$ 的全部特征向量为$k\alpha_3(k\neq 0)$.

（2）矩阵$A=\begin{pmatrix}2&-1&1\\0&1&1\\-1&1&1\end{pmatrix}$ 的特征多项式为
$$|A-\lambda E|=\begin{vmatrix}2-\lambda&-1&1\\0&1-\lambda&1\\-1&1&1-\lambda\end{vmatrix}=-(\lambda-1)^2(\lambda-2),$$
所以A 的全部特征值为$\lambda_1=\lambda_2=1,\lambda_3=2$.
当$\lambda_1=\lambda_2=1$ 时, 解方程$(A-E)x=0$, 由
$$A+E=\begin{pmatrix}1&-1&1\\0&0&1\\-1&1&0\end{pmatrix}\underset{\sim}{r}\begin{pmatrix}1&-1&1\\0&0&1\\0&0&0\end{pmatrix}$$
得基础解系
$$\alpha_1=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

从而α_1 就是对应于$\lambda_1 = \lambda_2 = 1$ 的特征向量,

并且对应于$\lambda_1 = \lambda_2 = 1$ 的全部特征向量为$k\alpha_1 (k \neq 0)$

当$\lambda_3 = 2$ 时, 解方程$(A - 2E)x = 0$, 由

$$A - 2E = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \underset{\sim}{r} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

得基础解系

$$\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

从而α_2 就是对应于$\lambda_3 = 2$ 的特征向量,并且对应于$\lambda_2 = 2$ 的全部特征向量为$k\alpha_2 (k \neq 0)$.

(3) 矩阵$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ 的特征多项式为

$$|A - \lambda E| = (\lambda - 1)(\lambda - 2)^2(\lambda - 3),$$

所以A 的全部特征值为$\lambda_1 = \lambda_2 = 2, \lambda_3 = 1, \lambda_4 = 3$.

当$\lambda_1 = \lambda_2 = 2$ 时, 解方程$(A - 2E)x = 0$, 由

得基础解系

$$\alpha_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

从而α_1 就是对应于$\lambda_1 = \lambda_2 = 2$ 的特征向量, 并且对应于$\lambda_1 = \lambda_2 = 2$ 的全部特征向量为$k\alpha_1$ $(k \neq 0)$.

当$\lambda_3 = 1$ 时, 解方程$(A - E)x = 0$

得基础解系

$$\alpha_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

从而α_2 就是对应于$\lambda_3 = 1$ 的特征向量,并且对应于$\lambda_3 = 1$ 的全部特征向量为$k\alpha_2(k \neq 0)$.

当$\lambda_4 = 3$ 时, 解方程$(A - 3E)x = 0$

得基础解系

$$\alpha_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

从而α_3 就是对应于$\lambda_4 = 3$ 的特征向量,并且对应于$\lambda_4 = 3$ 的全部特征向量为$k\alpha_3(k \neq 0)$.

2. $\because |A - \lambda E| = |(A - \lambda E)^T| = |A^T - \lambda E| \therefore A^T$与$A$ 的特征值相同.

3.取A 的任意一个特征值λ, 则存在非零向量ξ, 满足$A\xi = \lambda\xi$. 由$A^2 - 4A + 3E = 0$,

可得

$$(A^2 - 4A + 3E)\xi = (\lambda^2 - 4\lambda + 3)\xi = (\lambda - 1)(\lambda - 3)\xi = 0$$

$\because \xi \neq 0 \therefore \lambda = -1$ 或$\lambda = 3$.

4. 因A 的特征值全不为0, 知A可逆,故$A^* = |A|A^{-1}$. 而$|2A| = 2^3\lambda_1\lambda_2\lambda_3 = 16$, 记

$$\varphi(A) = (2A)^* + 3A - 2E,$$

这里,$\varphi(A)$ 虽不是矩阵多项式, 但也具有矩阵多项式的特性, 由

$$\varphi(\lambda) = (2\lambda)^* + 3\lambda - 2$$

得$\varphi(A)$的特征值为

$$\varphi(-1) = -8 - 3 - 2 = -13, \varphi(1) = 8 + 3 - 2 = 9, \varphi(-2) = -4 - 6 - 2 = -12.$$

所以,$|(2A)^* + 3A - 2E| = (-13) \times 9 \times (-12) = 1404.$

5. 由$|A| = 0, |A + 2E| = 0, |A - E| = 0$, 可知$A$ 的特征值为$\lambda_1 = 0, \lambda_2 = -2, \lambda_3 = 1$,从而$|A + E|$的特征值为$\mu_1 = 1, \mu_2 = -1, \mu_3 = 2$,故$|A + E| = -2$.

6. 设x 是AB 的对应于λ 的特征向量, 则$ABx = \lambda x \neq 0$, 因此, $Bx \neq 0$, 则有$B(ABx) = B(\lambda x) = \lambda Bx$, 根据特征值与特征向量的概念, λ 是BA 的特征值, 且Bx 是其对应的特征向量.

7. 设λ 为A 的特征值,则存在非零向量ξ, 使得$A\xi = \lambda\xi$. 由$A^2 = A$, 可得

$$(A^2 - A)\xi = A^2\xi - A\xi = (\lambda^2 - \lambda)\xi = 0$$

由$\xi \neq 0$, 得$\lambda = 0$ 或$\lambda = 1$, 又因为$R(A) = 2$, 故A 的全部特征值为$\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$.

8. 设$k_1\alpha_1 + \cdots + k_s\alpha_s + l_1\beta_1 + \cdots + l_t\beta_t = 0$, 则$k_1\alpha_1 + \cdots + k_s\alpha_s = -l_1\beta_1 - \cdots - l_t\beta_t$, 由于属于不同特征值的特征向量线性无关, 故$k_1\alpha_1 + \cdots + k_s\alpha_s = 0, l_1\beta_1 + \cdots + l_t\beta_t = 0$, 因为$\alpha_1 \cdots \alpha_s$ 线性无关,$\beta_1, \cdots \beta_t$ 线性无关,故$k_1 = \cdots k_s = l_1 = \cdots l_t = 0$, 所以$\alpha_1, \alpha_2, \cdots, \alpha_s, \beta_1, \beta_2, \cdots \beta_t$ 线性无关.

习题4-3

1. 由于A, B 相似, 存在可逆矩阵P 使得$B = P^{-1}AP$, 由P 可逆知,A, B 等价,故$R(A) = R(B)$. $|B = P^{-1}AP| = |P^{-1}||A||P| = |A|$.

2. 因为A 可逆, 故$BA = A^{-1}ABA = A^{-1}(AB)A$, 令$P = A$, 则有$AB$ 与BA 相似.

3. 显然, A 的特征值为$\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = 2$, 故$A$ 可相似对角化,只需λ 的重数等于其特征子空间的维数.

对$\lambda = 1$,

$$\dim(V_\lambda) = 4 - r(A - \lambda E) = 4 - r\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 2 & b & 1 & 0 \\ 2 & 3 & c & 1 \end{pmatrix} = 2.$$

故, $a = 0$.

对$\lambda = 2$,

$$\dim(V_\lambda) = 4 - r(A - \lambda E) = 4 - r\begin{pmatrix} -1 & 0 & 0 & 0 \\ a & -1 & 0 & 0 \\ 2 & b & 0 & 0 \\ 2 & 3 & c & 0 \end{pmatrix} = 2.$$

故,$c = 0$.

从而, $a = 0, c = 0$, b 可取任意值.

4. (1)

$$AP = \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2+a \\ 1+b \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = -1 \cdot P,$$

故$2+a=-1, 1+b=1$, 即$a=-3, b=0$, P 所对应的特征值为$\lambda=-1$.

(2)

$$|A-\lambda E| = \begin{vmatrix} 2-\lambda & -1 & 2 \\ 5 & -3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{vmatrix} = -(\lambda+1)^3$$

故$\lambda_{1,2,3}=-1$,从而

$$r(A-\lambda E) = r\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix} = 2,$$

$V_\lambda = 1 \neq \lambda$ 的重数. 故A 不能相似对角化.

5.

$$|\lambda E - A| = \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-1 & -1 \\ 0 & -1 & \lambda-1 \end{vmatrix} = \lambda(\lambda-1)(\lambda-2).$$

故A的特征值为$\lambda_1=0, \lambda_2=1, \lambda_3=2$.

解$(\lambda_1 E - A)x=0$, 得基础解系$\xi_1=(1,1,-1)^T$.

解$(\lambda_2 E - A)x=0$, 得基础解系$\xi_2=(1,0,0)^T$.

解$(\lambda_3 E - A)x=0$, 得基础解系$\xi_3=(1,1,1)^T$.

令$P=(\xi_1,\xi_2,\xi_3)^{-1}$, 则有

$$A = P^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} P,$$

从而,

$$A^{100} = P^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{pmatrix} P = \begin{pmatrix} 1 & 2^{99}-1 & 2^{99} \\ 0 & 2^{99} & 2^{99} \\ 0 & 2^{99} & 2^{99} \end{pmatrix}$$

6. 令$P=(p_1,p_2,p_3)$. 由题意知,

$$A(p_1,p_2,p_3)=(p_1,p_2,p_3)\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$$

故,

$$A=P\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}P^{-1}.$$

经过计算, 得

$$P^{-1}=\begin{pmatrix}-1 & 1 & 0\\ 1 & -1 & 1\\ 0 & 1 & -1\end{pmatrix}.$$

因此,

$$A=\begin{pmatrix}0 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 1 & 0\\ 1 & -1 & 1\\ 0 & 1 & -1\end{pmatrix}=\begin{pmatrix}-2 & 3 & -3\\ -4 & 5 & -3\\ -4 & 4 & -2\end{pmatrix}.$$

7.假设A与对角矩阵$\mathrm{diag}(\lambda_1,\lambda_2,\cdots\lambda_n)$ 相似, 从而

$$A=P^{-1}\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}P$$

则,

$$A^k=P^{-1}\begin{pmatrix}\lambda_1^k & 0 & 0\\ 0 & \lambda_2^k & 0\\ 0 & 0 & \lambda_3^k\end{pmatrix}P=0.$$

由P可逆,故$\lambda_1^k,\lambda_2^k,\cdots\lambda_n^k=0$, 所以$\lambda_1,\lambda_2,\cdots\lambda_n=0$,所以$A=0$, 这与$A$为非零矩阵矛盾,所以$A$不与对角阵相似.

8. 由于A与B相似,C与D相似, 所以存在可逆矩阵P,T, 使得$B=P^{-1}AP,D=T^{-1}CT$,

从而有

$$\begin{pmatrix} P & 0 \\ 0 & T \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & T \end{pmatrix} = \begin{pmatrix} P^{-1} & 0 \\ 0 & T^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & T \end{pmatrix}$$
$$= \begin{pmatrix} P^{-1}AP & 0 \\ 0 & T^{-1}CT \end{pmatrix}$$
$$= \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}.$$

因此, $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$与$\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$ 相似.

习题4-4

1. (1) 矩阵的特征多项式为$-\lambda(\lambda-2)(\lambda-3)$,

当$\lambda=0$ 时, 解方程$(A-0E)x=0$,

得到标准化特征向量为

$$\alpha_1 = \begin{pmatrix} -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{pmatrix},$$

当$\lambda=2$ 时, 解方程$(A-2E)x=0$,

得到标准化特征向量为

$$\alpha_2 = \begin{pmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix},$$

当$\lambda=3$ 时, 解方程$(A-3E)x=0$,

得到标准化特征向量为

$$\alpha_3 = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix},$$

$$P = \begin{pmatrix} -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(2) 矩阵的特征多项式为$-(\lambda-10)(\lambda-1)^2$,

当$\lambda=1$ 时, 解方程$(A-E)x=0$,

得到标准化特征向量为

$$\alpha_1=\begin{pmatrix}-\frac{2\sqrt{5}}{5}\\ \frac{\sqrt{5}}{5}\\ 0\end{pmatrix}$$

$$\alpha_2=\begin{pmatrix}\frac{2\sqrt{5}}{15}\\ \frac{4\sqrt{5}}{15}\\ \frac{\sqrt{5}}{3}\end{pmatrix}$$

当$\lambda=10$ 时, 解方程$(A-10E)x=0$,

得到标准化特征向量为

$$\alpha_3=\begin{pmatrix}-\frac{1}{3}\\ -\frac{2}{3}\\ \frac{2}{3}\end{pmatrix},$$

$$P=\begin{pmatrix}-\frac{2\sqrt{5}}{5} & \frac{2\sqrt{5}}{15} & -\frac{1}{3}\\ \frac{\sqrt{5}}{5} & \frac{4\sqrt{5}}{15} & -\frac{2}{3}\\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3}\end{pmatrix}$$

$$P^{-1}AP=\begin{pmatrix}1&0&0\\0&1&0\\0&0&10\end{pmatrix}$$

2. $\varphi(A)=\begin{pmatrix}2&2&-4\\2&2&-4\\-4&-4&8\end{pmatrix}$.

3. 设A的第三个特征向量单位化后为$p_3=(a,b,c)^T$, 则将p_1,p_2单位化后,得

$$A=\begin{pmatrix}\frac{1}{3}&\frac{2}{3}&a\\ \frac{2}{3}&\frac{1}{3}&b\\ \frac{2}{3}&-\frac{2}{3}&c\end{pmatrix}\begin{pmatrix}3&0&0\\0&-3&0\\0&0&0\end{pmatrix}\begin{pmatrix}\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\ \frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\\ a&b&c\end{pmatrix}=\begin{pmatrix}-1&0&2\\0&1&2\\2&2&0\end{pmatrix}.$$

4. 设$p_1=(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}})^T$, 则有$Ap_1=\lambda_1p_1$,得

$$\begin{pmatrix}0&-1&4\\-1&3&a\\4&a&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{pmatrix}=\lambda_1\begin{pmatrix}\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{pmatrix}$$

解得,$\lambda_1=2,a=-1.$

因此,

$$A=\begin{pmatrix}0&-1&4\\-1&3&-1\\4&-1&0\end{pmatrix}.$$

由

$$|\lambda E-A|=\begin{vmatrix}\lambda&1&-4\\1&\lambda-3&1\\-4&1&\lambda\end{vmatrix}=(\lambda-2)(\lambda+4)(\lambda-5)$$

所以,$\lambda_1=2,\lambda_2=-4,\lambda_3=5.$

当$\lambda=-4$ 时,解$(\lambda E-A)x=0$,得基础解系$p_2=(-1,0,1)^T$;

当$\lambda=5$ 时,解$(\lambda E-A)x=0$,得基础解系$p_3=(1,-1,1)^T$;

将p_1,p_2,p_3 单位正交化,得

$$p_1=\begin{pmatrix}\frac{1}{\sqrt6}\\\frac{2}{\sqrt6}\\\frac{1}{\sqrt6}\end{pmatrix},p_2=\begin{pmatrix}-\frac{1}{\sqrt2}\\0\\\frac{1}{\sqrt2}\end{pmatrix},p_2=\begin{pmatrix}\frac{1}{\sqrt3}\\-\frac{1}{\sqrt3}\\\frac{1}{\sqrt3}\end{pmatrix},$$

从而,

$$P=\begin{pmatrix}\frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\\\frac{2}{\sqrt6}&0&-\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\end{pmatrix}$$

5. (1) 由题意知, A 的所有特征值为$-1,1,0$, 对应的特征向量为

$$\alpha_1=(1,0,-1)^T,\alpha_2=(1,0,1)^T,\alpha_3=(0,1,0)^T.$$

(2)将向量$\alpha_1,\alpha_2,\alpha_3$ 单位化后,得

$$A=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\\0&0&1\\-\frac{1}{\sqrt2}&\frac{1}{\sqrt2}&0\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&1&0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}.$$

6.(1)设λ 是A的任意一个特征值, y 是A的对应于λ 的特征向量, 则有

$$Ay=\lambda y,\ \lambda^2 y=A^2 y=xx^{\mathrm{T}}xx^{\mathrm{T}}y=x^{\mathrm{T}}xAy=\lambda x^{\mathrm{T}}xy,$$

于是可得$\lambda^2=\lambda x^{\mathrm{T}}x$, 从而$\lambda=0$ 或$\lambda=x^{\mathrm{T}}x$.

设$\lambda_1,\lambda_2,\cdots\lambda_n$ 是A 的所有特征值, 因为$A=xx^{\mathrm{T}}$的主对角线上的元素为$x_1^2,x_2^2,\cdots,x_n^2$, 所以

$$x_1^2+x_2^2+\cdots+x_n^2=x^{\mathrm{T}}x=\lambda_1+\lambda_2+\cdots+\lambda_n,$$

这说明, 在$\lambda_1,\lambda_2,\cdots,\lambda_n$ 中, 有且只有一个等于$x^{\mathrm{T}}x$, 而其余$n-1$ 个全为0, 即$\lambda=0$ 是A的$n-1$ 重特征值.

(2)非零特征值是$x^{\mathrm{T}}x=x_1^2+x_2^2+\cdots x_n^2$, 对应的特征向量为$\boldsymbol{x}=(x_1,x_2,\cdots,x_n)^T$;

特征值$\lambda=0$ 对应的特征向量为

$$(-x_2,x_1,0,\cdots,0)^T,(-x_3,0,x_1,\cdots,0)^T,\cdots,(-x_n,0,0,\cdots,x_1)^T.$$

习题4-5

1. (1) $f=(x_1,x_2,x_3)\begin{pmatrix}2&-2&2\\-2&-2&3\\2&3&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$;

(2) $f=(x,y,z)\begin{pmatrix}-1&1&-3\\1&2&-2\\-3&-2&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$;

(3) $f=(x_1,x_2,x_3)\begin{pmatrix}1&-1&0\\-1&0&3\\0&3&-3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$.

2.(1) $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0&1&0\\\frac{1}{\sqrt2}&0&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$, $f=y_1^2+2y_2^2+3y_3^2$;

(2) $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&\frac{2}{\sqrt6}&\frac{1}{\sqrt3}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$, $f=y_1^2+y_2^2+2y_3^2$;

(3) $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}\frac{2}{\sqrt5}&-\frac{2}{3\sqrt5}&\frac13\\0&\frac{2}{3\sqrt5}&\frac23\\\frac{2}{\sqrt5}&\frac{4}{3\sqrt5}&-\frac23\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$, $f=y_1^2+y_2^2+10y_3^2$.

3. 设A 为实对称矩阵, 则有一正交矩阵P 使得

$$PAP^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) = \Lambda$$

成立, 其中$\lambda_1, \lambda_2, \cdots, \lambda_n$ 为A 的特征值, 不妨设λ_1 最大.
作正交变换$y = Px$, 即$x = P^{\mathrm{T}}y$, 注意到$P^{-1} = P^{\mathrm{T}}$, 有

$$f = x^{\mathrm{T}}Ax = y^{\mathrm{T}}PAP^{\mathrm{T}}y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2.$$

因$y = Px$ 正交变换, 所以当$\|x\| = 1$ 时, 有$\|y\| = \|x\| = 1$, 即$y_1^2 + y_2^2 + \cdots + y_n^2 = 1$, 因此,

$$f = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2 \leq \lambda_1.$$

又当$y_1 = 1, y_2 = y_3 = \cdots = y_n = 0$ 时$f = \lambda_1$, 所以$f_{max} = \lambda_1$.

4. (1)$f = y_1^2 - y_2^2 + y_3^2$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$;

(2) $f = y_1^2 + y_2^2 + y_3^2$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$.

5. (1) $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2+\frac{a}{2} & 2-\frac{a}{2} \\ 0 & 2-\frac{a}{2} & 2+\frac{a}{2} \end{pmatrix}$, 当$a = 0$ 时, $R(A) = 2$, $a \neq 0$ 时,$R(A) = 3$**;**

(2)$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

习题4-6

1. 二次型$f = x^{\mathrm{T}}Ax$, 其中$A = \begin{pmatrix} 1 & a & 1 \\ a & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, 通过计算使得$A$的各阶顺序主子式都为正,$a$的取值范围为$0 < a < 1$.

2.(1)此二次型的矩阵为

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -6 & 0 \\ 1 & 0 & -4 \end{pmatrix},$$

它的各阶顺序主子式为

$$a_{11}=-2<0,\ \begin{vmatrix}-2 & 1\\ 1 & -6\end{vmatrix}=11>0,\ \begin{vmatrix}-2 & 1 & 1\\ 1 & -6 & 0\\ 1 & 0 & -4\end{vmatrix}=-38<0,$$

所以,该二次型是负定的.

(2)此二次型的矩阵为

$$A=\begin{pmatrix}5 & 2 & -4\\ 2 & 1 & -2\\ -4 & -2 & 5\end{pmatrix},$$

它的各阶顺序主子式为

$$a_{11}=5>0,\ \begin{vmatrix}5 & 2\\ 2 & 1\end{vmatrix}=1>0,\ \begin{vmatrix}5 & 2 & -4\\ 2 & 1 & -2\\ -4 & -2 & 5\end{vmatrix}=1>0,$$

所以,该二次型是正定的.

(3)此二次型的矩阵为

$$A=\begin{pmatrix}2 & 0 & 0\\ 0 & 3 & 2\\ 0 & 2 & 3\end{pmatrix},$$

它的各阶顺序主子式为

$$a_{11}=2>0,\ \begin{vmatrix}2 & 0\\ 0 & 3\end{vmatrix}=6>0,\ \begin{vmatrix}2 & 0 & 0\\ 0 & 3 & 2\\ 0 & 2 & 3\end{vmatrix}=10>0,$$

所以,该二次型是正定的.

3. 由于D 是n 阶对角阵且对角元全非负,所以D是半正定的. 所以,

$$x^{\mathrm{T}}Ax>0, x^{\mathrm{T}}Dx\geq 0.$$

因此,$x^{\mathrm{T}}(A+D)x=x^{\mathrm{T}}Ax+x^{\mathrm{T}}Dx>0$, 于是$x^{\mathrm{T}}(A+D)x$ 必为正定二次型,从而$A+D$为正定矩阵.

4. 对称阵A 是正定阵,则A 的特征值$\lambda_1, \lambda_2, \cdots \lambda_n$ 都是正的,存在正交阵P,使得

$$\begin{aligned} A &= P^{\mathrm{T}} \cdot \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_n) \cdot P \\ &= P^{\mathrm{T}} \cdot \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}) \cdot \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}) \cdot P \end{aligned}$$

记$Q = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3})P$,则$A = Q^{\mathrm{T}}Q$,即$A$ 与单位阵E合同.

反之,A 与单位阵E合同,即存在可逆矩阵S,使得$A = S^{\mathrm{T}}S$,对任意非零向量x,有$x^{\mathrm{T}}Ax = x^{\mathrm{T}}S^{\mathrm{T}}Sx = (Sx)^{\mathrm{T}}(Sx) > 0$. 因此,$A$ 是正定的.

5.由于A 是正定矩阵, 对于任意$x \neq 0$, 则二次型$f(x) = x^{\mathrm{T}}Ax > 0$. 令$B = CAC^{\mathrm{T}}$, 则$A \simeq B$, 对于任意$x \neq 0$, 令$g(x) = x^{\mathrm{T}}Bx$, 则$g(x) = x^{\mathrm{T}}CAC^{\mathrm{T}}x = (C^{\mathrm{T}}x)^{\mathrm{T}}A(C^{\mathrm{T}}x) > 0$, 从而得到$CAC^{\mathrm{T}}$正定.

6.由于A 是正定阵,所以对于任意的$x \neq 0$ 都有$x^{\mathrm{T}}Ax > 0$, 又因为$k > 0$, 所以$x^{\mathrm{T}}(kA)x > 0$,所以kA也是正定阵.

7. 由于A 是正定阵,故存在可逆矩阵C, 使得$A = C^{\mathrm{T}}C$. 设$D = (C^{-1})^{\mathrm{T}}$, 则$D$是可逆的. 因此, $A^{-1} = (C^{\mathrm{T}}C)^{-1} = D^{\mathrm{T}}D$, 所以$A^{-1}$也是正定的. 由$A^* = |A|A^{-1}$ 及$|A| > 0$,A^{-1}是正定阵,可得A^*是正定矩阵.

8. 由于A, B 是正定矩阵,所以存在可逆矩阵P, Q,使得$A = P^{\mathrm{T}}P$, $B = Q^{\mathrm{T}}Q$, 则有

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P^{\mathrm{T}}P & 0 \\ 0 & Q^{\mathrm{T}}Q \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = C.$$

显然,$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$可逆,所以,$C$为正定阵.

测试题四

一、填空题

1. $\begin{vmatrix} 1 & 2 \\ 2 & x \end{vmatrix} > 0$,得$x > 4$

$\begin{vmatrix} 1 & 2 & 3 \\ 2 & x & 6 \\ 3 & 6 & x \end{vmatrix} > 0$,得$x < 4$或$x > 9$

综上，$x > 9$

2.

$$A=\begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix}$$

由题知，6和0是矩阵A的两个特征值

由$|A-6E|=0$,解得$a=2,8,8$

由$|A-0E|=0$,解得$a=-4,2,2$

综上$a=2$

3.

$$D=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A=PDP^T$$

$$A^3-3A=P\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}P^T=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

4.

$$A=P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}P^T$$

$$\left|A^3-5A^2+7A\right|=\left|P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^3P^T-5P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^2P^T+7P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}P^T\right|=\begin{vmatrix} 3 & & \\ & 2 & \\ & & 3 \end{vmatrix}=18$$

5.

$$A=P\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}P^T$$

$$|2A|=\begin{vmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2\lambda \end{vmatrix}=-48$$

解得$\lambda=-1$

二、选择题

1.

$$k_1+k_2A(\alpha_1+\alpha_2)=0$$

$$(k_1+k_2\lambda_1)\,\alpha_1+k_2\lambda_2\alpha_2=0$$

$$\begin{cases} k_1+k_2\lambda_1=0 \\ k_2\lambda_2=0 \end{cases}$$

只有零解，行列式不等于0

$$\begin{vmatrix} 1 & \lambda_1 \\ 0 & \lambda_2 \end{vmatrix} \neq 0$$

知$\lambda_2\neq 0$，选B

2.

$$P^TAP\to P^TA^2P\to P^T\frac{1}{3}A^2P\to P^T(\frac{1}{3}A^2)^{-1}P$$

$$(\frac{1}{3}*2^2)^{-1}=\frac{3}{4}$$

选B

3. 特征多项式相同

$$\begin{vmatrix} 0 & a & 1 \\ a & b-1 & a \\ 1 & a & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b-1 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

得到$a=0$,b任意

4. 由$A\alpha=-\alpha$，知一个特征值为-1，又由A的秩为3，选D

5. A与B秩相同，故合同;A与B迹不同，故不相似，选B

三、解答题

1.令$H=\begin{pmatrix} A \\ B \end{pmatrix}$, $R(H)\leq R(A)+R(B)<n$, 所以$Hx=0$ 有非零解$\alpha\neq 0$, 所以$A\alpha=0, B\alpha=0, \alpha\neq 0$, 故$\alpha$ 是A,B 属于特征值0 的公共特征向量. 所以, A 和B 有公共的特征值, 有公共的特征向量.

2. 由于A 为正交阵,所以$A^{\mathrm{T}}A=E$.设λ为A的特征值,P为对应的特征向量,则有$AP=\lambda P$, 进而有,$P^{\mathrm{T}}A^{\mathrm{T}}AP=\lambda^2P^{\mathrm{T}}P$, 从而$\lambda^2=1$, $\lambda=1$ 或者$\lambda=-1$. 而$|A|=\lambda_1\cdots\lambda_n=-1$, 故$\lambda=-1$ 是A 的特征值.

3. 设A 的两两互异的特征值为$\lambda_1, \lambda_2, \cdots \lambda_n$, 其对应的线性无关的特征向量为$\xi_1, \xi_2, \cdots \xi_n$. 则有$A\xi_i = \lambda_i \xi_i,\ i = 1, 2, \cdots, n$. 又$\xi_i$也是$B$的特征向量,故有$B\xi_i = \mu_i \xi_i,\ i = 1, 2, \cdots, n$. 从而我们有$BA\xi_i = \lambda_i B\xi_i = \lambda_i \mu_i \xi_i (i = 1, 2, \cdots, n)$, $AB\xi_i = \mu_i A\xi_i = \lambda_i \mu_i \xi_i (i = 1, 2, \cdots, n)$. 从而,$(BA - AB)\xi_i = 0,\ i = 1, 2, \cdots, n$,由$\xi_i,\ i = 1, 2, \cdots, n$ 线性无关,故有$AB = BA$.

4. 由题意A 可逆,且A 与B 相似,故存在可逆矩阵P,使得$P^{-1}AP = B$, 因此我们有$P^{-1}A^{-1}P = B^{-1}$, 进而,有$P^{-1}(|B|A^{-1})P = |B|B^{-1}$, 由于$A$ 与B 相似,所以$|A| = |B|$. 故$P^{-1}(|A|A^{-1})P = |B|B^{-1}$, 因此$A^*$与$B^*$相似.

5. 由于$\lambda = 2$ 是A 的二重特征值,故$r(2E - A) = 1$. 从而,由$A = \begin{pmatrix} 1 & -1 & 1 \\ x & 4 & y \\ -3 & -3 & 5 \end{pmatrix}$ 可知,$x = 2,\ y = -2$, 计算得$|A| = 24, \lambda_{1,2} = 2, \lambda_3 = 6$.

当$\lambda = 2$ 时,解方程组$(2E - A)x = 0$, 得到基础解系为$\alpha_1 = (-1, 1, 0)^{\mathrm{T}}, \alpha_2 = (1, 0, 1)^{\mathrm{T}}$.

当$\lambda = 6$ 时,解方程组$(6E - A)x = 0$, 得到基础解系为$\alpha_3 = (1, -2, 3)^{\mathrm{T}}$.

令

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

则有

$$P^{\mathrm{T}}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

6. 显然A 的特征值分别为$-1, -1, 2$, 设其对应的特征向量分别为$p_1 = (a_1, b_1, c_1)^T,\ p_2 = (a_2, b_2, c_2)^T$, 已知$p_3 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$, 则令$P = (p_1, p_2, p_3)$, P 为正交阵,由$P^{-1}P = E$及$A = P \cdot \mathrm{diag}(-1, -1, 2) \cdot P^{\mathrm{T}}$, 得到

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

解方程$(-E - A)x = 0$,得到基础解系$\xi_1 = (-1, 1, 0)^{\mathrm{T}}, \xi_2 = (1, 0, 1)^{\mathrm{T}}$, 正交化得$p_1 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), p_2 = (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$ 故

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

7. 由题目知,矩阵$A=\begin{pmatrix}1&1&1\\1&3&1\\1&1&1\end{pmatrix}$.

由

$$|\lambda E-A|=\begin{vmatrix}\lambda-1&-1&-1\\-1&\lambda-3&-1\\-1&-1&\lambda-1\end{vmatrix}=\lambda(\lambda-1)(\lambda-4),$$

得A的特征值为$\lambda_1=0,\lambda_2=1,\lambda_3=4$.

当$\lambda=0$ 时,解得基础解系为$\xi_1=(-1,0,1)^{\mathrm{T}}$;

当$\lambda=1$ 时,解得基础解系为$\xi_2=(1,-1,1)^{\mathrm{T}}$;

当$\lambda=4$ 时,解得基础解系为$\xi_3=(1,2,1)^{\mathrm{T}}$.

单位化,得$p_1=(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})^T,p_2=(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})^T,p_3=(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}})^T$.

因此,

$$P=\begin{pmatrix}-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&-\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\end{pmatrix}.$$

标准形方程为$v^2+4w^2=4$该二次曲面是椭圆柱面.

第五章

习题5-1

1. (1)

$$S_1=\begin{pmatrix}s_{11}&s_{12}\\s_{21}&s_{22}\end{pmatrix}$$

其中$s_{11}=s_{22}$.

(2)

$$S_2=\begin{pmatrix}s_{11}&s_{12}&s_{13}\\s_{21}&s_{22}&s_{23}\\s_{31}&s_{32}&s_{33}\end{pmatrix}$$

其中$s_{11}=s_{21},s_{13}=s_{31},s_{23}=s_{32}$.

(3)

$$S_2=\begin{pmatrix} 0 & s_{12} & s_{13} \\ s_{21} & 0 & s_{23} \\ s_{31} & s_{32} & 0 \end{pmatrix}$$

其中$s_{11}=-s_{21},s_{13}=-s_{31},s_{23}=-s_{32}$.

容易验证(1)(2)(3)均满足定义1中的8条性质，构成线性空间.

2. 容易验证，

$$S[x]=\{s=A\sin(x+B)\,|A,B\in R\}$$

满足定义1中的8条性质，构成线性空间.

3.

取

$$a_1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},a_2=\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix},a_1+a_2=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\notin V,$$

故V 不构成线性空间.

4.

(1) $(a_1,b_1)\oplus(a_2,b_2)=(a_2,b_2)\oplus(a_1,b_1)$

(2) $[(a_1,b_1)\oplus(a_2,b_2)]\oplus(a_3,b_3)=(a_1,b_1)\oplus[(a_2,b_2)\oplus(a_3,b_3)]$

(3) $\forall\,(a_1,b_1)\in V,(a_1,b_1)\oplus(0,0)=(a_1,b_1)$

(4) $\forall\,(a_1,b_1)\in V,(a_1,b_1)\oplus(-a_1,a_1^2-b_1)=(0,0)$

(5) $1\cdot(a_1,b_1)=(a_1,b_1)$

(6) $\lambda(\mu\,(a_1,b_1))=\mu(\lambda\,(a_1,b_1))$

(7) $(\lambda+\mu)\,(a_1,b_1)=\lambda\,(a_1,b_1)+\mu\,(a_1,b_1)$

(8) $\lambda\,[(a_1,b_1)\oplus(a_2,b_2)]=\lambda\,(a_1,b_1)\oplus\lambda\,(a_2,b_2)$

5. 取$\alpha,\beta\in W$, $\alpha=k_1\alpha_1+...+k_t\alpha_t$ $\beta=l_1\alpha_1+...+l_t\alpha_t$ $k\alpha+l\beta=(kk_1+ll_1)\alpha_1+...+(kk_t+ll_t)\alpha_t\in R$ 故W是V的一个子空间.

习题5-2

1. 对$(\alpha_1,\alpha_2,\alpha_3,\alpha)$作如下初等行变换

$$\begin{pmatrix} 1 & 6 & 3 & 3 \\ 3 & 3 & 1 & 7 \\ 5 & 2 & 0 & 1 \end{pmatrix}\rightarrow\begin{pmatrix} 1 & 0 & 0 & 33 \\ 0 & 1 & 0 & -82 \\ 0 & 0 & 1 & 154 \end{pmatrix}$$

故

$$\alpha = 33\alpha_1 - 82\alpha_2 + 154\alpha_3.$$

2. S_1的基$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

S_2的基$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

S_3的基$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

3.

(1) 令$A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $B = (\beta_1, \beta_2, \beta_3, \beta_4)$, $B = AP$,

故

$$P = A^{-1}B = \begin{pmatrix} \frac{16}{13} & 1 & 1 & 1 \\ \frac{19}{13} & 0 & 0 & 0 \\ \frac{20}{13} & 1 & 0 & 1 \\ -\frac{9}{13} & 0 & 1 & 1 \end{pmatrix}.$$

(2)

$$B^{-1}A \begin{pmatrix} 1 \\ 19 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ -23 \\ 5 \\ 3 \end{pmatrix}.$$

4. (1)

$$\left(1, 1+x, 1+x+x^2, 1+x+x^2+x^3\right) = \left(1, x, x^2, x^3\right) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

故

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(2)

$$f(x)=\left(1,1+x,1+x+x^2,1+x+x^2+x^3\right)P^{-1}\begin{pmatrix}1\\0\\-2\\5\end{pmatrix},$$

$$g(x)=\left(1,x,x^2,x^3\right)P\begin{pmatrix}7\\0\\8\\2\end{pmatrix}.$$

$f(x)+g(x)$在基一的坐标为

$$\begin{pmatrix}1\\0\\-2\\5\end{pmatrix}+P\begin{pmatrix}7\\0\\8\\2\end{pmatrix}=\begin{pmatrix}18\\10\\8\\7\end{pmatrix}.$$

在基二的坐标为

$$P^{-1}\begin{pmatrix}1\\0\\-2\\5\end{pmatrix}+\begin{pmatrix}7\\0\\8\\2\end{pmatrix}=\begin{pmatrix}8\\2\\1\\7\end{pmatrix}.$$

5. 证明，显然两个向量组可以互相线性表示，故$2\alpha_2,3\alpha_3,...,n\alpha_n,\alpha_1$ 也是V_n 的一个基.

$$(2\alpha_2,3\alpha_3,...,n\alpha_n,\alpha_1)=(\alpha_1,\alpha_2,...,\alpha_n)\begin{pmatrix}0&0&\cdots&0&1\\2&0&\cdots&0&0\\0&3&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&n&0\end{pmatrix},$$

故过渡矩阵为

$$P=\begin{pmatrix}0&0&\cdots&0&1\\2&0&\cdots&0&0\\0&3&\cdots&0&0\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&n&0\end{pmatrix}.$$

习题5-3

2. 经验证$\forall x, y \in Mn\left(R\right), \forall k, l, \lambda \in R$,

$$T(kx+ly) = A(kx+ly) - (kx+ly)A = kT(x) + lT(y),$$

$$T\left(\lambda x\right) = A\left(\lambda x\right) - \left(\lambda x\right)A = \lambda T\left(x\right),$$

故T是$Mn\left(R\right)$ 上的线性变换.

3.

$$D\left(e^x, xe^x, x^2e^x\right) = \left(e^x, e^x + xe^x, 2xe^x + x^2e^x\right) = \left(e^x, xe^x, x^2e^x\right)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

故D在这个基下的矩阵为

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. 对$(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$作初等行变换，可得

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 \end{pmatrix},$$

故过渡矩阵

$$P = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 2 & 3 & 4 \end{pmatrix}.$$

5.

$$\left(\beta_1, \beta_2, \beta_3\right) = \left(e_1, e_2, e_3\right)P,$$

$$P = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix},$$

$$B = PAP^{-1} = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

测试题五

一.选择题

1.A与B相似，有相同的特征值和行列式，但是不一定有相同的特征向量，故选C

二.填空题

1.

$$kx = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_1 + ka_2 + kb \end{pmatrix} \in V$$

故$kb = b$,k任意,得$b = 0$

2.

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = k_1\beta_1 + k_2\beta_2 + k_3\beta_3$$

得$k_1 = -k_3$，$k_2 = -k_3$

取$k_3 = -k$

得

$$\eta = k \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, k \in R$$

三.解答题

1.

$$T(1, x, x^2, x^3) = (0, -1, -2x-1, -3x^2-3x-1) = \left(1, x, x^2, x^3\right) \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

线性变换T的矩阵为

$$\begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2.

(1) $\forall A, B \in V, \forall k, l, \lambda \in R,$

$T(kA + lB) = P^T(kA + lB)P = kT(A) + lT(B),$

$T(A) = P^T(\lambda A)P = \lambda T(A)$.

故T是V上的线性变换.

(2) $T(A_1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $T(A_2) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ $T(A_3) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

$$T(A_1, A_2, A_3) = (A_1, A_2, A_3)\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix},$$

故T在V下的矩阵为

$$\begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

3.易证满足线性空间的8条性质.

4. 对$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$作初等行变换，可得

$$\begin{pmatrix} 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

故α_1, α_2 是V的一组基，dim（V）=2，且

$$\alpha_3 = 2\alpha_1 - \alpha_2, \alpha_4 = \alpha_1 + 3\alpha_2, \alpha_5 = -2\alpha_1 - \alpha_2.$$

5.

(1)

$$\begin{aligned} &\forall A, B \in V, \forall k, l, \lambda \in R, \\ &P(kA + lB) = \tfrac{1}{2}((kA + lB) - (kA + lB)^T) = kP(A) + lP(B) \ , \\ &P(\lambda A) = \tfrac{1}{2}((\lambda A) - (\lambda A)^T) = \lambda P(A) \end{aligned}$$

故P是线性变换.

(2) $P(E_{11}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $P(E_{12}) = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}$ $P(E_{21}) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ $P(B) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & \end{pmatrix}$,

$$P(E_{11}, E_{12}, E_{21}, B) = (E_{11}, E_{12}, E_{21}, B)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

故P在基下的矩阵为

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$