# Span equivalence between weak n-categories

Yuya Nishimura, Hiroyuki Miyoshi

# 1 Background

- In this talk, we suppose that a weak n-category is defined as an algebra for "special" n-globular operad.
- An n-globular operad is an n-globular set whose elements are operations that has a (globular) pasting diagram as its arity.
- An n-globular operad induces a monad. the algebra for n-globular operad is the algebra for the monad.

## 2 Motivation

• It is interesting question what equivalence is "good" for weak n-categories.

DEFINITION. "Cottrell's equivalence" Let K be an n-globular operad. K-algebras  $KX \to X$  and  $KY \to Y$  are equivalent if there exists a map of K-algebras  $u: X \to Y$  or  $u:Y\to X$  such that u is surjective on 0-cells, full on m-cells for all  $1 \le m \le n$ , and faithful on n-cells.

- Cottrell's equivalence is defined for his first coherence theorem (every free K-algebra is equivalent to a free strict n-caregory).
- and he says

"this definition of equivalence is much more strict (and thus much less general) than it ought to be. ... If we required a more general definition of equivalence of K-algebra, ... Another option is to replace the map u with a span of maps of K-algebra, ..." [Cottrell 2015]

DEFINITION. "Span equivalence" Let K be an n-globular operad. K-algebras  $KX \to X$  and  $KY \to Y$  are span equivalent if there exists a triple  $\langle \psi, u, v \rangle$  such that  $\psi:KZ o Z$  is K-algebra,u:Z o X and  $v:Z\to Y$  are maps of K-algebras, surjective on 0-cells, full on m-cells for all  $1 \le m \le n$ , and faithful on n-cells.

• What properties span equivalence satisfies.

## 3 Main Theorems

THEOREM.1

Span equivalence is equivalence relation on K-algebras

THEOREM.2

ordinary equivalent  $\iff$  span equivalent in  $\mathbf{Cat}$ 

## 4 Another form for Theorem.2

THEOREM.2

ordinary equivalent  $\iff$  span equivalent in  $\mathbf{Cat}$ 

DEFINITION Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories. We say that  $\mathcal{A}$  and  $\mathcal{B}$  are span equivalent if there exists a triple  $\langle \mathcal{C}, u, v \rangle$  such that  $\mathcal{C}$  is a category,  $u: \mathcal{C} \to \mathcal{A}$  and  $v: \mathcal{C} \to \mathcal{B}$  are functors, surjective on objects, full and faithful.

Proposition Let A be presheaf category. The forgetful functor

$$U: \mathcal{A}\text{-}\mathbf{Cat} \longrightarrow \mathcal{A}\text{-}\mathbf{Gph}$$

is monadic. (prop F.1.1 in [Leinster 2004a])

Let  $A = \mathbf{Set}$ , the induced monad  $T_1$  is the free strict 1-category monad,  $\mathbf{Set}\text{-}\mathbf{Gph}$  is the category of 1-globular sets. Hence

$$T_1$$
-Alg  $\cong$  Cat.

Morever, the category of weak 1-category  $\mathbf{Wk1}$ - $\mathbf{Cat}$  is isomorphic to  $T_1$ - $\mathbf{Alg}$ . Thus

Wk1-Cat  $\cong T_1$ -Alg  $\cong$  Cat.

Let  $K: \mathbf{Cat} \to \mathbf{Wk1}\text{-}\mathbf{Cat}$  be the isomorphism, C and D be categories. C is span equivalent to D if and only if K(C) is span equivalent to K(D).

Theorem.2 can be stated in another way

THEOREM.2' Let C and D be categories. C is ordinary equivalent to D if and only if K(C) is span equivalent to K(D).

# 5 Preparation of Theorem.1

#### THEOREM.1

Span equivalence is equivalence relation on K-algebras

- It is trivial that span equivalence is reflexive and symmetric relation.
- In order to show that span equivalence is transitive relation, We show that some properties of map of n-globular sets are stable under pullback.

Let  $f: X \to Y$  be a map of n-globular sets.

- f:surjective (resp. injective) on k-cells : $\iff f_k: X_k \to Y_k$ :surjective (resp. injective)
- f:full on k-cells

$$:\iff \forall x, x' \in X_{k-1}, g \in \mathbf{Hom}_Y(f(x), f(x')),$$
$$\exists h \in \mathbf{Hom}_X(x, x') \text{s.t.} f(h) = g$$

• f:faithful on k-cells

$$:\iff \forall x, x' \in X_{k-1}, g, g' \in \mathbf{Hom}_X(x, x'),$$

$$"g \neq g' \Rightarrow f(g) \neq f(g')"$$

PROPOSITION. the pullback diagram in  $[\mathbf{G}_n^{\mathrm{op}}, \mathbf{Set}]$ 

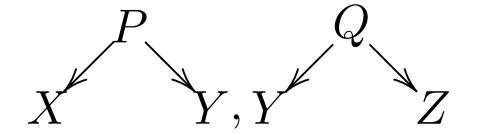
$$P \xrightarrow{j} Y$$

$$\downarrow i \qquad \qquad \downarrow g$$

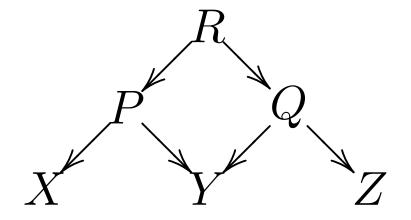
$$X \xrightarrow{f} S$$

- f:surjective on 0-cell, full on k-cell for  $1 \le m \le n \Rightarrow j$ :surjective on 0-cell, full on m-cell for 1 < m < n.
- f:faithful on k-cell  $\Rightarrow j$ :faithful on k-cell.

#### COROLLARY. let



be span equivalences, then



is span equivalence.

# 6 Proof of Theorem.1

THEOREM.1

Span equivalence is equivalence relation on K-algebras

**Proof.** By definition, it is trivial that span equivalence is reflexive and symmetric relation. By corollary, span equivalence is transitive relation.

# 7 Preparation of Theorem.2

THEOREM.2

ordinary equivalent  $\iff$  span equivalent in  $\mathbf{Cat}$ 

- In order to prove theorem.2. We construct a triple  $\langle C, u, v \rangle$  of adjoint equivalence. (C is span equivalence, u and v are its projections.)
- We show that the triple  $\langle C, u, v \rangle$  is span equivalence in **Cat**.

# 8 Equivalence Fusion

DEFINITION. Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories, and  $\langle S, T, \eta, \epsilon \rangle$  be adjoint equivalence between  $\mathcal{A}$  and  $\mathcal{B}$ . We define a category  $\mathcal{A} \mu \mathcal{B}$ , equivalence fusion, as follows:

object-set

$$\mathbf{Ob}(\mathcal{A} \ \ \mathcal{B}) := \mathbf{Ob}(\mathcal{A}) \ \ \mathbf{Ob}(\mathcal{B}) \ \ (disjoint)$$

hom-set

$$\mathbf{Hom}(x,y) := \begin{cases} \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x,y)\} & (x,y) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x,y)\} & (x,y) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x,y)\} & (x \in \mathcal{B}(x,y))\} \end{cases}$$

identities

$$id_x := \begin{cases} \langle id_x, x, x \rangle & (x \in \mathcal{A}, id_x \in \mathcal{A}(x, x)) \\ \langle id_x, x, x \rangle & (x \in \mathcal{B}, id_x \in \mathcal{B}(x, x)) \end{cases}$$

#### composition

$$\langle \langle g, y, z \rangle, \langle f, x, y \rangle \rangle \mapsto \langle g \tilde{\circ} f, x, z \rangle$$

$$g \circ_{\mathcal{B}} f \qquad (x, y, z \in \mathcal{A}) \qquad (x, y, z \in \mathcal{B}) \qquad (x, y, z \in \mathcal{B}) \qquad (x, y \in \mathcal{A}, z \in \mathcal{A}, y, z \in \mathcal{B}) \qquad (x, y \in \mathcal{A}, z \in \mathcal{A}, y, z \in \mathcal{A}) \qquad (x, y \in \mathcal{B}, z \in \mathcal{A}, y \in \mathcal{B}) \qquad (x, y \in \mathcal{B}, z \in \mathcal{A}, y \in \mathcal{B}) \qquad (x, z \in \mathcal{B}, y \in \mathcal{B}, y \in \mathcal{B}) \qquad (x, z \in \mathcal{B}, y \in \mathcal{B}) \qquad (x, z \in \mathcal{B}, y \in \mathcal{B}, y \in \mathcal{B}) \qquad (x, z \in \mathcal{B}, y \in \mathcal{B}, y \in \mathcal{B}) \qquad (x, z \in \mathcal{B}, y \in \mathcal{B}, y \in \mathcal{B}) \qquad (x, z \in \mathcal{B}, y \in \mathcal{B}, y \in \mathcal{B}, y \in \mathcal{$$

DEFINITION. Let  $A \mu B$  be equivalence fusion, the projection u, v are defined by followings.

$$x \longmapsto ux := \begin{cases} x & (x \in \mathcal{A}) \\ Tx & (x \in \mathcal{B}) \end{cases}$$

hom-functions  $u: \mathbf{Hom}(x,y) \longrightarrow \mathcal{A}(ux,uy)$ 

$$\langle f, x, y \rangle \longmapsto uf := \begin{cases} f & (x, y \in \mathcal{A}) \\ Tf & (x, y \in \mathcal{B}) \\ Tf \circ_{\mathcal{A}} \eta_{x} & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \eta_{y}^{-1} \circ_{\mathcal{A}} Tf & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

•  $u: \mathcal{A} \ \ \mathcal{B} \longrightarrow \mathcal{B}$ object-function  $v: \mathbf{Ob}(\mathcal{A} \ \ \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{B})$ 

$$x \longmapsto vx := \begin{cases} Sx & (x \in \mathcal{A}) \\ x & (x \in \mathcal{B}) \end{cases}$$

hom-functions  $v: \mathbf{Hom}(x,y) \longrightarrow \mathcal{B}(ux,uy)$ 

$$\langle f, x, y \rangle \longmapsto vf := \begin{cases} Sf & (x, y \in \mathcal{A}) \\ f & (\text{others}) \end{cases}$$

PROPOSITION.

An equivalence fusion forms a category.

PROPOSITION.

u,v are functors, surjective on objects, full and faithful.

# 9 Proof of Theorem.2

THEOREM.2

ordinary equivalent  $\iff$  span equivalent in  $\mathbf{Cat}$ 

**Proof.**  $(\Leftarrow)$  Let  $\mathcal{A}$  and  $\mathcal{B}$  are span equivalent,  $\langle \mathcal{C}, u, v \rangle$  be the span equivalence. Then u, v are equivalence. So the categories are equivalent.  $(\Rightarrow)$  Let  $\mathcal{A}$  and  $\mathcal{B}$  are equivalent, thus are adjoint equivalent. Then we can construct equivalence fusion. By propositions,  ${\cal A}$  and  ${\cal B}$ are span equivalent.

## 10 Reference

[Cottrell 2015] Thomas Cottrell. OPERADIC DEFINITIONS OF WEAK N-CATEGORY: COHERENCE AND COMPARISONS. Theory and Applications of Categories, Vol.30, No13, 2015, pp. 433-488.

[Leinster 2004a] Tom Leinster. *Higher operads, higher categories.* Volume 298 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2004.