Span equivalence between algebras for n-globular operads

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Résumé. Nous définissons une nouvelle équivalence entre algèbres pour les opérades n-globulaires suggérée dans [Cottrell 2015], et montrons qu'il s'agit d'une généralisation de l'équivalence ordinaire entre les catégories.
Abstract. We define a new equivalence between algebras for n-globular operads which is suggested in [Cottrell 2015], and show that it is a generalization of ordinary equivalence between categories.
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1. Introduction

In [Cottrell 2015], Thomas Cottrell defined an equivalence of K-algebras on an n-globular set to show the following coherence theorem:

Theorem 1.1. *Let K be an n-globular operad with unbiased contraction γ, and let X be n-globular set. Then the free K-algebra on X is equivalent to the free strict n-category on X.*

His equivalence in this theorem is as follows:

Definition 1.2. *Let K be an n-globular operad. K-algebras $KX \to X$ and $KY \to Y$ are equivalent if there exists a map of K-algebras $u : X \to Y$ or $u : Y \to X$ such that u is surjective on 0-cells, full on m-cells for all $1 \leq m \leq n$, and faithful on n.*

But, as he said, this equivalence is not the best one: "This definition of equivalence is much more (and thus much less general) than ought to be." To improve it, he suggested two approaches. The one of them is to replace the map u with a span of maps of K-algebras. In this paper, we adopt this approach and prove two theorem. The first is that we define an adequate equivalence using spans and prove this is indeed an equivalence relation. The second is that our equivalence is a generalization of ordinary equivalence between categories.

In Section 2 we recall the preliminary definitions (globular sets, thier maps, operads, algebras for a operad). In Section 3 we define the notion of *span equivalence* and prove the first theorem. In Section 4, for ordinary categories, we define span equivalence. Then we show that two categories are ordinary equivalent if and only if they are span equivalent in category of ordinary categories. To prove this, we use a combinatorial construction named *equivalence fusion*. Futhermore, we show the second theorem.

2. Preliminary

The contents of the section is in [Cottrell 2015].

Definition 2.1. *Let* $n \in \mathbb{N}$*. An* n-globular set *is a diagram*

$$X = (X_n \underset{t_n^X}{\overset{s_n^X}{\rightrightarrows}} X_{n-1} \underset{t_{n-1}^X}{\overset{s_{n-1}^X}{\rightrightarrows}} \dots \underset{t_1^X}{\overset{s_1^X}{\rightrightarrows}} X_0)$$

of sets and maps such that

$$s_{k-1}^X s_k^X(x) = s_{k-1}^X t_k^X(x), \quad t_{k-1}^X s_k^X(x) = t_{k-1}^X t_k^X(x)$$

for all $k \in \{2, ..., n\}$ *and* $x \in X_k$*.*
Elements of X_k *are called* k-cells *of* X*. We defined* hom-sets *of* X *as follows:*

$$\mathbf{Hom}_X(x, y) := \{\alpha \in X_k \mid s_k^X(\alpha) = x, t_k^X(\alpha) = y\}$$

for all $k \in \{1, ...n\}$ *and* $x, y \in X_{k-1}$*.*
Let X, Y *be* n*-globular sets, A* map *of* n*-globular sets from* X *to* Y *is a collection* $f = \{f_k : X_k \to Y_k\}_{k \in \{1,...,n\}}$ *of maps of sets such that*

$$s_k^Y f_k(x) = f_{k-1} s_k^X(x), \quad t_k^Y f_k(x) = f_{k-1} t_k^X(x)$$

for all $k \in \{1, ..., n\}$ *and* $x \in X_k$*.*
The category of n*-globular sets and their maps is denoted by* n-$\mathbf{GSet}$*.*

Definition 2.2. *A category is* cartesian *if it has all pullbacks. A functor is* cartesian *if it preserves pullbacks. A natural transformation is* cartesian *if it all of its naturality squares are pullbacks squares. A monad is* cartesian *if its functor part, unit and counit are cartesian. A map of monad is* cartesian *if its underlying natural transformation is cartesian.*

Definition 2.3. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1. and T be a cartesian monad on $\mathcal{C}$. The category of T*-collections *is the slice category $\mathcal{C}/T1$. The category has a monoidal structure: let $k : K \to T1, k' : K' \to T1$ be collections; then their tensor product is defined to be the composite along the top of the diagram*

$$\begin{tikzcd} K \otimes K' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TK' \arrow[r, "Tk'"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu_1^T"] & T1 \\ K \arrow[r, "k"'] & T1 & & \end{tikzcd}$$

where $!$ is the unique map $K' \to 1$. the unit for this tensor product is the collection

$$\begin{tikzcd} 1 \arrow[d, "\mu_1^T"] \\ T1 \end{tikzcd}$$

The monoidal category is denoted by T-$\mathbf{Coll}$.

Definition 2.4. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1, and T be a cartesian monad on $\mathcal{C}$. A T*-operad *is a monoid in the monoidal category T-$\mathbf{Coll}$. In the case in which T is the free strict n-category monad on n-$\mathbf{GSet}$, a T-operad is called an n*-globular operad.

Definition 2.5. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1, T be a cartesian monad on $\mathcal{C}$ and K be a T-operad. Then there is an* induced monad *on $\mathcal{C}$, which by abuse of notation we denote (K, η^K, μ^K): The endfunctor*

$$K : \mathcal{C} \to \mathcal{C}$$

is defined as follows; The object part of the functor, for $X \in \mathcal{C}$, KX is defined by the pullback:

$$\begin{tikzcd}
KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\
TX \arrow[r, "T!"'] & T1
\end{tikzcd}$$

The arrow part of the functor, for $Y \in \mathcal{C}, u : X \to Y$, Ku is defined by the unique property of the pullback:

$$\begin{tikzcd}
KX \arrow[r, dotted, "Ku"] \arrow[rr, bend left, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & KY \arrow[r, "K!"] \arrow[d, "K_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\
TX \arrow[r, "Tu"'] \arrow[rr, bend right, "T!"'] & TY \arrow[r, "T!"'] & T1
\end{tikzcd}$$

Components $\eta^K_X, \mu^K_X$of the unit map $\eta^K : 1 \Rightarrow K$ and $\mu^K : K^2 \Rightarrow K$ are defined by the following diagrams:

$$\begin{tikzcd}
X \arrow[rr, "!"] \arrow[dr, dotted, "\eta^K_X"] \arrow[ddr, bend right, "\eta^T_X"'] & & 1 \arrow[d, "\epsilon"] \arrow[dd, bend left, "\eta^T_1"] \\
& KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\
& TX \arrow[r, "T!"'] & T1
\end{tikzcd}$$

$$
\begin{tikzcd}
 & & K^2X \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", pos=0.25] \arrow[dddd, dotted, "\mu^K_X"] & & \\
 & K\otimes K \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", pos=0.25] \arrow[dddd, bend right=60, "\mu^K"'] & & TKX \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", pos=0.25] & \\
K \arrow[dr, "k"'] & & TK \arrow[dl, "T!"] \arrow[dr, "Tk"'] & & T^2X \arrow[dl, "T^2!"] \arrow[dddl, bend left, "\mu^T_X"] \\
 & T1 & & T^21 & \\
 & & KX \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee", pos=0.25] & & \\
 & K \arrow[dr, "k"'] & & TX \arrow[dl, "T!"] & \\
 & & T1 & &
\end{tikzcd}
$$

Definition 2.6. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1, T be a cartesian monad on $\mathcal{C}$ and K be a T-operad. We define a K-*algebra *as an algebra for the induced monad (K, η^K, μ^K). Similarly, a* map *of algebras for T-operad K is a map of algebras for the induced monad. The category of K-algebras and thier maps is denoted by K-$\mathbf{Alg}$.*

Leinster's weak n-category is an algebra for specific operad. See section 9 and 10 in [Leinster 2004] for details.

3. Span equivalence

Definition 3.1. *Let $f : X \to Y$ be a map of n-globular sets.*

- *f is* surjective on k-cells $:\Leftrightarrow$ *$f_k : X_k \to Y_k$ is surjective*
- *f is* injective on k-cells $:\Leftrightarrow$ *$f_k : X_k \to Y_k$ is injective*
- *f is* full on k-cells $:\Leftrightarrow \left\{ \begin{array}{l} \forall x, x' \in X_{k-1}, \beta \in \mathbf{Hom}_Y(f_{k-1}(x), f_{k-1}(x')), \\ \exists \alpha \in \mathbf{Hom}_X(x, x') \text{ s.t. } f_k(\alpha) = \beta \end{array} \right.$
- *f is* faithful on k-cell $:\Leftrightarrow \left\{ \begin{array}{l} \forall x, x' \in X_{k-1}, \alpha, \alpha' \in \mathbf{Hom}_X(f_{k-1}(x), f_{k-1}(x')), \\ \alpha \neq \alpha' \Rightarrow f_k(\alpha) \neq f_k(\alpha') \end{array} \right.$

Let f be a map of K-algebras. f is surjective (respectively, injective, full, faithful) on k-cells if the underlying map is surjective (respectively, injective, full, faithful) on k-cells.

Definition 3.2. *Let K be an n-globular operad. K-algebras $KX \to X$ and $KY \to Y$ are* span equivalent *if there exists a triple $\langle \psi, u, v \rangle$ such that $\psi : KZ \to Z$ is an K-algebra, $u : Z \to X$ and $v : Z \to Y$ are maps of K-algebras, surjective on 0-cells, full on m-cells for all $1 \le m \le n$, and faithful on n-cells. The triple $\langle \psi, u, v \rangle$ is referred to as an span equivalence of K-algebras.*

Trivially, under the same situation as Theorem 1.1, the free K-algebra on X is span equivalent to the free strict n-category on X.

Proposition 3.3. *In the pullback diagram in n-$\mathbf{GSet}$*

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

- *f is surjective on 0-cells $\Rightarrow$ j is surjective on 0-cells*
- *f is full on k-cells $\Rightarrow$ j is full on k-cells*
- *f is faithful on k-cells $\Rightarrow$ j is faithful on k-cells*

Proof. We define an n-globular set P as follows:

$$P_k := \{(x, y) \in X_k \times Y_k \mid f_k(x) = g_k(y)\}$$

$$s_l^P := (P_l \ni (x, y) \mapsto (s_l^X(x), s_l^Y(y)) \in P_{l-1})$$
$$t_l^P := (P_l \ni (x, y) \mapsto (t_l^X(x), t_l^Y(y)) \in P_{l-1})$$

for all $k \in \{0, ..., n\}, l \in \{1, ..., n\}$, and maps of n-globular sets i, j as follows:

$$i_k := (P_k \ni (x, y) \mapsto x \in X_k), \quad j_k := (P_k \ni (x, y) \mapsto y \in Y_k)$$

for all $k \in \{0, ..., n\}$. Then (P, i, j) is a pullback of X and Y over S. It is enough to prove the proposition that we check the claims for (P, i, j). Firstly, we prove surjectivity on 0-cells. For $y \in Y_0$, there exists $x \in X_0$ such that $f_0(x) = g_0(y)$, So $(x, y) \in P_0$ and $j_0((x, y)) = y$. which is the condition of surjectivity. To show fullness, we suppose $(x, y), (x', y') \in P_{k-1}, \phi \in \mathbf{Hom}(y, y')$, we can see $s_k g_k(\phi) = g_{k-1}(y) = f_{k-1}, t_k g_k(\phi) = g_{k-1}(y') = f_{k-1}(x')$. Thus $g_k(\phi) \in \mathbf{Hom}(f_{k-1}(x), f_{k-1}(x'))$. For fullness, there exists $\psi \in \mathbf{Hom}(x, x')$ such that $f_k(\psi) = g_k(\phi)$. Then $(\psi, \phi) \in \mathbf{Hom}((x, y), (x', y'))$ and $j_k(\psi, \phi) = \phi$. Therefore j is full on k-cells. Lastly, let f be faithful on k-cells. let $(x, y), (x', y') \in P_{k-1}, \psi, \phi \in \mathbf{Hom}((x, y), (x', y'))$ such that $j_k(\psi) = j_k(\phi)$. Then $f_k i_k(\psi) = g_k j_k(\psi) = g_k j_k(\phi) = f_k i_k(\phi)$. From faithfulness, $i_k(\psi) = i_k(\phi)$, and $\psi = (i_k(\psi), j_k(\psi)) = (i_k(\phi), j_k(\phi)) = \phi$. Therefore j is faithful on k-cells. □

Remark 3.4. Let T be a monad in $\mathcal{C}$. Then the forgetful functor $U : T\text{-}\mathbf{Alg} \to \mathcal{C}$ creats limits. Hence any monadic functor reflects limits. (Theorem 3.4.2. in [TTT])

By the remark, we can get similar facts in category of K-algebras to proposition 3.3.

Proposition 3.5. *Let*

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & Q \arrow[dl, "h"'] \arrow[dr, "i"] & \\ Y & & Z \end{tikzcd}$$

be span equivalences, then

$$\begin{tikzcd} & & R \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & P \arrow[dl, "f"'] \arrow[dr, "g"] & & Q \arrow[dl, "h"'] \arrow[dr, "i"] & \\ X & & Y & & Z \end{tikzcd}$$

is span equivalence.

Proof. By the fact, p, q are are surjective on 0-cells, full on k-cells for $1 \leq k \leq n$ and faithful on n-cells. Therefore $f \circ p, i \circ q$ are surjective on 0-cells, full on k-cells for $1 \leq k \leq n$ and faithful on n-cells. So the span is span equivalence. □

Theorem 3.6. *Span equivalence is equivalence relation on K-algebras.*

Proof. It is straightforward from the definition and previous proposition that span equivalence is equivalence relation. □

4. Characterizing equivalence of categories via spans

Definition 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. We say that $\mathcal{A}$ and $\mathcal{B}$ are* span equivalent *if there exists a triple $\langle \mathcal{A}, u, v \rangle$ such that $\mathcal{C}$ is a category, $u : \mathcal{C} \to \mathcal{A}$ and $v : \mathcal{C} \to \mathcal{B}$ are functors, surjective on objects, full and faithful.*

Definition 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, let $\langle S : \mathcal{A} \to \mathcal{B}, T : \mathcal{B} \to \mathcal{A}, \eta : I_{\mathcal{A}} \to TS, \epsilon : ST \to I_{\mathcal{B}} \rangle$ be an adjoint equivalence between $\mathcal{A}$ and $\mathcal{B}$. We define a category,* equivalence fusion $\mathcal{A} \uplus \mathcal{B}$ *, as follows:*

- *object-set*

$$\mathbf{Ob}(\mathcal{A} \uplus \mathcal{B}) := \mathbf{Ob}(\mathcal{A}) \bigsqcup \mathbf{Ob}(\mathcal{B}) \qquad \text{(disjoint)}$$

- *hom-set*

$$\mathbf{Hom}(x, y) := \begin{cases} \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x, y)\} & (x, y \in \mathcal{A}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, y)\} & (x, y \in \mathcal{B}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y)\} & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy)\} & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- *composition*

$$\begin{aligned} \tilde{\circ} : \mathbf{Hom}(y, z) \times \mathbf{Hom}(x, y) &\longrightarrow \mathbf{Hom}(x, z) \\ \langle \langle g, y, z \rangle, \langle f, x, y \rangle \rangle &\longmapsto \langle g, y, z \rangle \tilde{\circ} \langle f, x, y \rangle := \langle g \circ f, x, z \rangle \end{aligned}$$

$$g \circ f := \begin{cases} g \circ_{\mathcal{A}} f & (x, y, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x, y, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} Sf & (x, y \in \mathcal{A}, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{A}, y, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x, y \in \mathcal{B}, z \in \mathcal{A}) \\ Sg \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y, z \in \mathcal{A}) \\ \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}) \end{cases}$$

- *identities*

$$\mathrm{id}_x := \begin{cases} \langle \mathrm{id}_x, x, x \rangle & (x \in \mathcal{A}, \mathrm{id}_x \in \mathcal{A}(x, x)) \\ \langle \mathrm{id}_x, x, x \rangle & (x \in \mathcal{B}, \mathrm{id}_x \in \mathcal{B}(x, x)) \end{cases}$$

Proposition 4.3. *The equivalence fusion $\mathcal{A} \Cup \mathcal{B}$ forms a category.*

Proof. It is easy to check that the composition $\tilde{\circ}$ is map from $\mathbf{Hom}(x, y) \times \mathbf{Hom}(y, z)$ to $\mathbf{Hom}(x, z)$. Now, we prove that the composition $\tilde{\circ}$ satisfies associative law and identity law by case analysis.

- associative law
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$,
    $h \circ (g \circ f) = h \circ_{\mathcal{A}} (g \circ_{\mathcal{A}} f)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{A}} g) \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$,
    $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} S(g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} Sf)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} Sf$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$,
    $$\begin{aligned} h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x \\ &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\ &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y \circ_{\mathcal{A}} f \end{aligned}$$
    $(h \circ g) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$,
    $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} Sf)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} Sf$

- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$,
  $h \circ (g \circ f) = h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) = h \circ_{\mathcal{A}} \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
  $= \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
  $$\begin{aligned}(h \circ g) \circ f &= (Sh \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &= \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x\end{aligned}$$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,
  $$\begin{aligned}h \circ (g \circ f) &= h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} (\epsilon_{Sz} \circ_{\mathcal{B}} S\eta_z) \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} \epsilon_{Sz} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \circ_{\mathcal{B}} \epsilon_{Sx} \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f\end{aligned}$$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$,
  $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} f) = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(h \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$,
  $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$,
  $h \circ (g \circ f) = h \circ (Sg \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{A}} g) \circ f = S(h \circ_{\mathcal{A}} g) \circ_{\mathcal{B}} f = (Sh \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$,
  $h \circ (g \circ f) = h \circ (Sg \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$,
  $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $$\begin{aligned}(h \circ g) \circ f &= (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f \\ &= S(\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{B}} f \\ &= S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= (\epsilon_{Sw} \circ_{\mathcal{B}} S\eta_w) \circ_{\mathcal{B}} S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f\end{aligned}$$

$$
\begin{aligned}
&= \epsilon_{Sw} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\
&= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} \epsilon_{Sy} \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\
&= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f
\end{aligned}
$$

- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$,
  $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$,
  $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,
  $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$,
  $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$,
  $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$

- identity law
  - $x \in \mathcal{A}, y \in \mathcal{A}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{A}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{A}} f = f$
  - $x \in \mathcal{A}, y \in \mathcal{B}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} S\mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_{Sx} = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{B}} f = f$
  - $x \in \mathcal{B}, y \in \mathcal{A}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = S\mathrm{id}_y \circ_{\mathcal{B}} f = \mathrm{id}_{Sy} \circ_{\mathcal{B}} f = f$
  - $x \in \mathcal{B}, y \in \mathcal{B}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{B}} f = f$

$\square$

Definition 4.4. *Let* $\langle S : \mathcal{A} \to \mathcal{B}, T : \mathcal{B} \to \mathcal{A}, \eta : I_\mathcal{A} \to TS, \epsilon : ST \to I_\mathcal{B}\rangle$ *be an adjoint equivalence, let* $\mathcal{A} \Cup \mathcal{B}$ *be the equivalence fusion. We define the* projections u, v *as follows:*

- $u : \mathcal{A} \Cup \mathcal{B} \longrightarrow A$
  object-function $u : \mathbf{Ob}(\mathcal{A} \Cup \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{A})$
  $$x \longmapsto ux := \begin{cases} x & (x \in \mathcal{A}) \\ Tx & (x \in \mathcal{B}) \end{cases}$$
  hom-functions $u : \mathbf{Hom}(x, y) \longrightarrow \mathcal{A}(ux, uy)$
  $$\langle f, x, y\rangle \longmapsto uf := \begin{cases} f & (x, y \in \mathcal{A}) \\ Tf & (x, y \in \mathcal{B}) \\ Tf \circ_\mathcal{A} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \eta_y^{-1} \circ_\mathcal{A} Tf & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- $v : \mathcal{A} \Cup \mathcal{B} \longrightarrow B$
  object-function $v : \mathbf{Ob}(\mathcal{A} \Cup \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{B})$
  $$x \longmapsto vx := \begin{cases} Sx & (x \in \mathcal{A}) \\ x & (x \in \mathcal{B}) \end{cases}$$
  hom-functions $v : \mathbf{Hom}(x, y) \longrightarrow \mathcal{B}(ux, uy)$
  $$\langle f, x, y\rangle \longmapsto vf := \begin{cases} Sf & (x, y \in \mathcal{A}) \\ f & (\text{others}) \end{cases}$$

Proposition 4.5. *The projections* u, v *are functors.*

Proof. We show that u, v preserve composition of morphisms and identity morphism by case analysis.

- u preserves composition of morphisms
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}$,
    $u(g \circ f) = u(g \circ_\mathcal{A} f) = g \circ_\mathcal{A} f$
    $ug \circ_\mathcal{A} uf = g \circ_\mathcal{A} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_\mathcal{B} Sf) = T(g \circ_\mathcal{B} Sf) \circ_\mathcal{A} \eta_x = Tg \circ_\mathcal{A} TSf \circ_\mathcal{A} \eta_x$
    $ug \circ_\mathcal{A} uf = (Tg \circ_\mathcal{A} \eta_y) \circ_\mathcal{A} f = Tg \circ_\mathcal{A} TSf \circ_\mathcal{A} \eta_x$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}$,
    $u(g \circ f) = u(\eta_z^{-1} \circ_\mathcal{A} Tg \circ_\mathcal{A} Tf \circ_\mathcal{A} \eta_x) = \eta_z^{-1} \circ_\mathcal{A} Tg \circ_\mathcal{A} Tf \circ_\mathcal{A} \eta_x$
    $ug \circ_\mathcal{A} uf = (\eta_z^{-1} \circ_\mathcal{A} Tg) \circ_\mathcal{A} (Tf \circ_\mathcal{A} \eta_x)$

- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}$,
  $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
  $ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x)$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}$,
  $u(g \circ f) = u(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{B}} Tf$
  $ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{A}} Tf$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}$,
  $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf$
  $ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = Tg \circ_{\mathcal{A}} Tf$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}$,
  $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf$
  $ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} Tf$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}$,
  $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf$
  $ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} Tf$

• u preserves identity morphisms

- $x \in \mathcal{A}$,
  $u(\mathrm{id}_x) = \mathrm{id}_x = \mathrm{id}_{ux}$
- $x \in \mathcal{B}$,
  $u(\mathrm{id}_x) = T\mathrm{id}_x = \mathrm{id}_{Tx} = \mathrm{id}_{ux}$

• v preserves composition of morphisms

- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}$,
  $v(g \circ f) = v(g \circ_{\mathcal{A}} f) = S(g \circ_{\mathcal{A}} f) = Sg \circ_{\mathcal{B}} Sf$
  $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{A}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}$,
  $v(g \circ f) = v(g \circ_{\mathcal{B}} Sf) = g \circ_{\mathcal{B}} Sf$
  $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}$,
  $$\begin{aligned} v(g \circ f) &= v(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \end{aligned}$$

$$= g \circ_{\mathcal{B}} f$$
$$vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$$

  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}$,
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}$,
    $v(g \circ f) = v(Sg \circ_{\mathcal{B}} f) = Sg \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}$,
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}$,
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}$,
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$

- v preserves identity morphisms
  - $x \in \mathcal{A}$
    $v(\mathrm{id}_x) = S\mathrm{id}_x = \mathrm{id}_{Sx} = \mathrm{id}_{vx}$
  - $x \in \mathcal{B}$
    $v(\mathrm{id}_x) = \mathrm{id}_x\ \mathrm{id}_{vx}$

□

Proposition 4.6. *The projections u, v are surjective on objects, full and faithful.*

Proof. It's trivial by definitions that u, v are surjective on objects. So we check fullness and faithfulness.

- u is full and faithful
  - $x, y \in \mathcal{A}$,
    $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{A}(x, y)$ is bijective.

- $x, y \in \mathcal{B}$,
  $T : \mathcal{B}(x, y) \to \mathcal{A}(Tx, Ty)$ is bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{A}(x, y) \ni \langle f, x, y\rangle \mapsto Tf \in \mathcal{A}(Tx, Ty) = \mathcal{A}(ux, uy)$ is bijective.
- $x \in \mathcal{A}, y \in \mathcal{B}$,
  $\mathcal{B}(Sx, y) \ni f \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty)$ is the right adjunct of each f, and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y\rangle \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty) = \mathcal{A}(ux, uy)$ is bijective.
- $x \in \mathcal{B}, y \in \mathcal{A}$,
  $\mathcal{B}(x, Sy) \ni f \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y)$ is the left adjunct of each f, and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y\rangle \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y) = \mathcal{A}(ux, uy)$

- v is full and faithful
  - $x, y \in \mathcal{A}$,
    $S : \mathcal{A}(x, y) \to \mathcal{B}(Sx, Sy)$ is bijective. Therefore $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y\rangle \mapsto Sf \in \mathcal{B}(Sx, Sy) = \mathcal{B}(vx, vy)$ is bijective.
  - $x, y \in \mathcal{B}$,
    $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(x, y) = \mathcal{B}(vx, vy)$ is bijective.
  - $x \in \mathcal{A}, y \in \mathcal{B}$,
    $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(Sx, y) = \mathcal{B}(vx, vy)$ is bijective.
  - $x \in \mathcal{B}, y \in \mathcal{A}$,
    $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(x, Sy) = \mathcal{B}(vx, vy)$ is bijective.

$\square$

Theorem 4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. $\mathcal{A}$ is ordinary equivalent to $\mathcal{B}$ if and only if $\mathcal{A}$ is span equivalent to $\mathcal{B}$.*

Proof. Let $\mathcal{A}$ be ordinary equivalent to $\mathcal{B}$, then $\mathcal{A}$ is adjoint equivalent to $\mathcal{B}$. Thus there exists a adjoint equivalence between $\mathcal{A}$ and $\mathcal{B}$. So we can construct the equivalence fusion and the projections. By Propositions, they are span equivalence. Therefore $\mathcal{A}$ is span equivalent to $\mathcal{B}$.
On the other hand, let $\mathcal{A}$ be span equivalent to $\mathcal{B}$. Then there exists a span equivalence $\langle \mathcal{C}, u, v\rangle$ between $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{C}$ is ordinary equivalent to both $\mathcal{A}$ and $\mathcal{B}$. Therefore $\mathcal{A}$ is ordinary equivalent to $\mathcal{B}$. $\square$

Remark 4.8. Let $\mathcal{A}$ be presheaf category. The forgetful functor

$$U : \mathcal{A}\text{-}\mathbf{Cat} \longrightarrow \mathcal{A}\text{-}\mathbf{Gph}$$

is monadic. (Proposition F 1.1 in [Leinster 2004])

Let $\mathcal{A} = \mathbf{Set}$, we can see $\mathbf{Set}\text{-}\mathbf{Cat} = \mathbf{Cat}$, $\mathbf{Set}\text{-}\mathbf{Grp} = 1\text{-}\mathbf{GSet}$, and the induced monad T_1 is the free strict 1-category monad on $1\text{-}\mathbf{GSet}$. By the remark, the comparison functor

$$K : \mathbf{Cat} \longrightarrow T_1\text{-}\mathbf{Alg}$$

is isomorphic and arrow part of the functor is

$$K : f \longmapsto Uf.$$

Moreover, the category $\mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$ of Leinster's weak 1 categories is the category $T_1\text{-}\mathbf{Alg}$ of algebras for the monad. (As for details, refer to the proof of Theorem 9.1.4 in [Leinster 2004].) So the isomorphism $K : \mathbf{Cat} \to \mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$ preserve surjectivity, fullness and faithfullness. Hence,

Theorem 4.9. *Let $K : \mathbf{Cat} \to \mathbf{Wk\text{-}1\text{-}Cat}$ be the isomorphism above. let $\mathcal{A}$ and $\mathcal{B}$ be categories. $\mathcal{A}$ is span equivalent to $\mathcal{B}$ in $\mathbf{Cat}$ if and only if $K(\mathcal{A})$ is span equivalent to $K(\mathcal{B})$ in $\mathbf{Wk\text{-}1\text{-}Cat}$.*

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