

SPAN EQUIVALENCE BETWEEN WEAK N -CATEGORIES

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ABSTRACT.

1. Introduction

2. Span equivalence

2.1. DEFINITION. Let $n \in \mathbb{N}$, An n -globular set is a diagram

$$X = (X_n \xrightleftharpoons[t_n^X]{s_n^X} X_{n-1} \xrightleftharpoons[t_{n-1}^X]{s_{n-1}^X} \dots \xrightleftharpoons[t_1^X]{s_1^X} X_0)$$

of sets and maps such that

$$s_{k-1}^X s_k^X(x) = s_{k-1}^X t_k^X(x), \quad t_{k-1}^X s_k^X(x) = t_{k-1}^X t_k^X(x)$$

for all $k \in \{2, \dots, n\}$ and $x \in X_k$.

Elements of X_k are called k -cells of X . And we defined hom-sets of X as follows:

$$\mathbf{Hom}(x, y) := \{\alpha \in X_k \mid s_k^X(\alpha) = x, t_k^X(\alpha) = y\}$$

for all $k \in \{1, \dots, n\}$ and $x, y \in X_{k-1}$.

Let X, Y be n -globular sets, A map of n -globular sets from X to Y is a collection $f = \{f_k : X_k \rightarrow Y_k\}_{k \in \{1, \dots, n\}}$ of maps of sets such that

$$s_k^Y f_k(x) = f_{k-1} s_k^X(x), \quad t_k^Y f_k(x) = f_{k-1} t_k^X(x)$$

for all $k \in \{1, \dots, n\}$ and $x \in X_k$.

The category of n -globular sets and maps is written $n\text{-}\mathbf{GSet}$.

2.2. DEFINITION. Let $f : X \rightarrow Y$ be a map of n -globular sets.

- f :surjective on k -cells $:\Leftrightarrow f_k : X_k \rightarrow Y_k$:surjective
- f :injective on k -cells $:\Leftrightarrow f_k : X_k \rightarrow Y_k$:injective
- f :full on k -cells $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g \in \mathbf{Hom}_Y(f(x), f(x')), \\ \exists h \in \mathbf{Hom}_X(x, x') \text{ s.t. } f(h) = g \end{cases}$
- f :faithful on k -cell $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g, g' \in \mathbf{Hom}_X(f(x), f(x')), \\ g \neq g' \Rightarrow f(g) \neq f(g') \end{cases}$

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2.3. DEFINITION. Let K be an n -globular operad. K -algebras $KX \rightarrow X$ and $KY \rightarrow Y$ are span equivalent if there exists a triple $\langle \psi, u, v \rangle$ such that $\psi : KZ \rightarrow Z$ is a K -algebra, $u : Z \rightarrow X$ and $v : Z \rightarrow Y$ are maps of K -algebras, surjective on 0-cells, full on m -cells for all $1 \leq m \leq n$, and faithful on n -cells. The triple $\langle \psi, u, v \rangle$ is referred to as a span equivalence of K -algebras.

2.4. PROPOSITION. In the pullback diagram in $n\text{-GSet}$

$$\begin{array}{ccc} P & \xrightarrow{j} & Y \\ \downarrow i & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

- f :surjective on 0-cells $\Rightarrow j$:surjective on 0-cells
- f :full on k -cells $\Rightarrow j$:full on k -cells
- f :faithful on k -cells $\Rightarrow j$:faithful on k -cells

PROOF. We define n -globular set P as follows:

$$P_k := \{(x, y) \in X_k \times Y_k \mid f_k(x) = g_k(y)\}$$

$$s_l^P := (P_l \ni (x, y) \mapsto (s_l^X(x), s_l^Y(y)) \in P_{l-1})$$

$$t_l^P := (P_l \ni (x, y) \mapsto (t_l^X(x), t_l^Y(y)) \in P_{l-1})$$

for all $k \in \{0, \dots, n\}$, $l \in \{1, \dots, n\}$, and maps of n -globular sets i, j as follows:

$$i_k := (P_k \ni (x, y) \mapsto x \in X_k), \quad j_k := (P_k \ni (x, y) \mapsto y \in Y_k)$$

for all $k \in \{0, \dots, n\}$. Then (P, i, j) is a pullback of X and Y over S .

First, let f be surjective on 0-cells. For $y \in Y_0$, there exists $x \in X_0$ such that $f_0(x) = g_0(y)$. Then $(x, y) \in P_0$ and $j_0((x, y)) = y$. Therefore j is surjective on 0-cells.

Second, let f be full on k -cells. let $(x, y), (x', y') \in P_{k-1}$, $\phi \in \mathbf{Hom}(y, y')$, then $s_k g_k(\phi) = g_{k-1}(y) = f_{k-1} t_k g_k(\phi) = g_{k-1}(y') = f_{k-1}(x')$. Thus $g_k(\phi) \in \mathbf{Hom}(f_{k-1}(x), f_{k-1}(x'))$. From fullness, there exists $\psi \in \mathbf{Hom}(x, x')$ such that $f_k(\psi) = g_k(\phi)$. Then $(\psi, \phi) \in \mathbf{Hom}((x, y), (x', y'))$ and $j_k(\psi, \phi) = \phi$. Therefore j is full on k -cells.

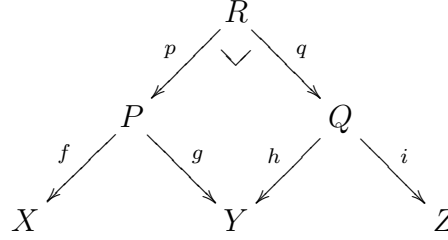
Third, let f be faithful on k -cells. let $(x, y), (x', y') \in P_{k-1}$, $\psi, \phi \in \mathbf{Hom}((x, y), (x', y'))$ such that $j_k(\psi) = j_k(\phi)$. Then $f_k i_k(\psi) = g_k j_k(\psi) = g_k j_k(\phi) = f_k i_k(\phi)$. From faithfulness, $i_k(\psi) = i_k(\phi)$, and $\psi = (i_k(\psi), j_k(\psi)) = (i_k(\phi), j_k(\phi)) = \phi$. Therefore j is faithful on k -cells.

2.5. REMARK. Let K be a monad on $n\text{-GSet}$. Pullbacks in $K\text{-Alg}$ are created by the forgetful functor $U : K\text{-Alg} \rightarrow n\text{-GSet}$.

2.6. PROPOSITION. *Let*



be span equivalences, then



is span equivalence.

PROOF. By proposition, p, q are surjective on 0-cells, full on k -cells for $1 \leq k \leq n$ and faithful on n -cells. Therefore $f \circ p, i \circ q$ are surjective on 0-cells, full on k -cells for $1 \leq k \leq n$ and faithful on n -cells. So the span is span equivalence.

2.7. THEOREM. *Span equivalence is equivalence relation on K -algebras.*

PROOF. It is straightforward from the definition and previous proposition that span equivalence is equivalence relation.

3. Characterizing equivalence of categories via spans

3.1. DEFINITION. *Let \mathcal{A} and \mathcal{B} be categories. We say that \mathcal{A} and \mathcal{B} are span equivalent if there exists a triple $\langle \mathcal{A}, u, v \rangle$ such that \mathcal{C} is a category, $u : \mathcal{C} \rightarrow \mathcal{A}$ and $v : \mathcal{C} \rightarrow \mathcal{B}$ are functors, surjective on objects, full and faithful.*

3.2. DEFINITION. *Let \mathcal{A} and \mathcal{B} be categories, let $\langle S : \mathcal{A} \rightarrow \mathcal{B}, T : \mathcal{B} \rightarrow \mathcal{A}, \eta : I_{\mathcal{A}} \rightarrow TS, \epsilon : ST \rightarrow I_{\mathcal{B}} \rangle$ be an adjoint equivalence between \mathcal{A} and \mathcal{B} . We define a category, equivalence fusion $\mathcal{A} \Downarrow \mathcal{B}$, as follows:*

- *object-set*

$$\mathbf{Ob}(\mathcal{A} \Downarrow \mathcal{B}) := \mathbf{Ob}(\mathcal{A}) \sqcup \mathbf{Ob}(\mathcal{B}) \quad (\text{disjoint})$$

- *hom-set*

$$\mathbf{Hom}(x, y) := \begin{cases} \{ \langle f, x, y \rangle \mid f \in \mathcal{A}(x, y) \} & (x, y \in \mathcal{A}) \\ \{ \langle f, x, y \rangle \mid f \in \mathcal{B}(x, y) \} & (x, y \in \mathcal{B}) \\ \{ \langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y) \} & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \{ \langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy) \} & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- *composition*

$$\begin{aligned} \tilde{\circ} : \mathbf{Hom}(y, z) \times \mathbf{Hom}(x, y) &\longrightarrow \mathbf{Hom}(x, z) \\ \langle \langle g, y, z \rangle, \langle f, x, y \rangle \rangle &\longmapsto \langle g, y, z \rangle \tilde{\circ} \langle f, x, y \rangle := \langle g \circ f, x, z \rangle \end{aligned}$$

$$g \circ f := \begin{cases} g \circ_{\mathcal{A}} f & (x, y, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x, y, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} Sf & (x, y \in \mathcal{A}, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{A}, y, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x, y \in \mathcal{B}, z \in \mathcal{A}) \\ Sg \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y, z \in \mathcal{A}) \\ \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}) \end{cases}$$

- *identities*

$$\text{id}_x := \begin{cases} \langle \text{id}_x, x, x \rangle & (x \in \mathcal{A}, \text{id}_x \in \mathcal{A}(x, x)) \\ \langle \text{id}_x, x, x \rangle & (x \in \mathcal{B}, \text{id}_x \in \mathcal{B}(x, x)) \end{cases}$$

3.3. PROPOSITION. *The equivalence fusion $\mathcal{A} \Downarrow \mathcal{B}$ forms a category.*

PROOF. It is easy to check that the composition $\tilde{\circ}$ is map from $\mathbf{Hom}(x, y) \times \mathbf{Hom}(y, z)$ to $\mathbf{Hom}(x, z)$. Now, we prove that the composition $\tilde{\circ}$ satisfies composition law and identity law by case analysis.

- *composition law*

$$\begin{aligned} &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}, \\ &\quad h \circ (g \circ f) = h \circ_{\mathcal{A}} (g \circ_{\mathcal{A}} f) \\ &\quad (h \circ g) \circ f = (h \circ_{\mathcal{A}} g) \circ_{\mathcal{A}} f \\ &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}, \\ &\quad h \circ (g \circ f) = h \circ (g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} S(g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} Sf) \\ &\quad (h \circ g) \circ f = (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} Sf \\ &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}, \\ &\quad h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y \circ_{\mathcal{A}} f \\ &\quad (h \circ g) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f \\ &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}, \\ &\quad h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} Sf) \\ &\quad (h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} Sf \\ &- x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}, \\ &\quad h \circ (g \circ f) = h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) = h \circ_{\mathcal{A}} \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &\quad (h \circ g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$

- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\
 &= h \circ_{\mathcal{B}} S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\
 &= h \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\
 &= h \circ_{\mathcal{B}} (\epsilon_{Sz} \circ_{\mathcal{B}} S\eta_z) \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\
 &= h \circ_{\mathcal{B}} \epsilon_{Sz} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\
 &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \circ_{\mathcal{B}} \epsilon_{Sx} \circ_{\mathcal{B}} S\eta_x \\
 &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \\
 (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f
 \end{aligned}$$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x \\
 (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(h \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x
 \end{aligned}$$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\
 (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f
 \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\
 (h \circ g) \circ f &= (h \circ_{\mathcal{A}} g) \circ f = S(h \circ_{\mathcal{A}} g) \circ_{\mathcal{B}} f = (Sh \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f
 \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\
 (h \circ g) \circ f &= (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f
 \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\
 (h \circ g) \circ f &= (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f \\
 &= S(\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{B}} f \\
 &= S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\
 &= (\epsilon_{Sw} \circ_{\mathcal{B}} S\eta_w) \circ_{\mathcal{B}} S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\
 &= \epsilon_{Sw} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\
 &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} \epsilon_{Sy} \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\
 &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f
 \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\
 (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f
 \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\
 (h \circ g) \circ f &= (Sh \circ_{\mathcal{B}} g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f
 \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,

$$\begin{aligned}
 h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\
 (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f
 \end{aligned}$$

- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A},$
 $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
 $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B},$
 $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
 $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$

- identity law

- $x \in \mathcal{A}, y \in \mathcal{A},$
 $f \circ \text{id}_x = f \circ_{\mathcal{A}} \text{id}_x = f$
 $\text{id}_y \circ f = \text{id}_y \circ_{\mathcal{A}} f = f$
- $x \in \mathcal{A}, y \in \mathcal{B},$
 $f \circ \text{id}_x = f \circ_{\mathcal{B}} S\text{id}_x = f \circ_{\mathcal{B}} \text{id}_{Sx} = f$
 $\text{id}_y \circ f = \text{id}_y \circ_{\mathcal{B}} f = f$
- $x \in \mathcal{B}, y \in \mathcal{A},$
 $f \circ \text{id}_x = f \circ_{\mathcal{B}} \text{id}_x = f$
 $\text{id}_y \circ f = S\text{id}_y \circ_{\mathcal{B}} f = \text{id}_{Sy} \circ_{\mathcal{B}} f = f$
- $x \in \mathcal{B}, y \in \mathcal{B},$
 $f \circ \text{id}_x = f \circ_{\mathcal{B}} \text{id}_x = f$
 $\text{id}_y \circ f = \text{id}_y \circ_{\mathcal{B}} f = f$

3.4. DEFINITION. Let $\langle S : \mathcal{A} \rightarrow \mathcal{B}, T : \mathcal{B} \rightarrow \mathcal{A}, \eta : I_{\mathcal{A}} \rightarrow TS, \epsilon : ST \rightarrow I_{\mathcal{B}} \rangle$ be an adjoint equivalence, let $\mathcal{A} \Downarrow \mathcal{B}$ be the equivalence fusion. We define the projections u, v as follows:

- $u : \mathcal{A} \Downarrow \mathcal{B} \longrightarrow \mathcal{A}$

object-function $u : \mathbf{Ob}(\mathcal{A} \Downarrow \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{A})$

$$x \longmapsto ux := \begin{cases} x & (x \in \mathcal{A}) \\ Tx & (x \in \mathcal{B}) \end{cases}$$

hom-functions $u : \mathbf{Hom}(x, y) \longrightarrow \mathcal{A}(ux, uy)$

$$\langle f, x, y \rangle \longmapsto uf := \begin{cases} f & (x, y \in \mathcal{A}) \\ Tf & (x, y \in \mathcal{B}) \\ Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \eta_y^{-1} \circ_{\mathcal{A}} Tf & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- $v : \mathcal{A} \Downarrow \mathcal{B} \longrightarrow \mathcal{B}$

object-function $v : \mathbf{Ob}(\mathcal{A} \Downarrow \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{B})$

$$x \longmapsto vx := \begin{cases} Sx & (x \in \mathcal{A}) \\ x & (x \in \mathcal{B}) \end{cases}$$

hom-functions $v : \mathbf{Hom}(x, y) \longrightarrow \mathcal{B}(vx, vy)$

$$\langle f, x, y \rangle \longmapsto vf := \begin{cases} Sf & (x, y \in \mathcal{A}) \\ f & (\text{others}) \end{cases}$$

3.5. PROPOSITION. *The projections u, v are functors.*

PROOF. We show that u, v preserve composition of morphisms and identity morphism by case analysis.

- u preserves composition of morphisms

$$\begin{aligned}
& - x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{A}} f) = g \circ_{\mathcal{A}} f \\
& \quad ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} f \\
& - x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} Sf) = T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\
& \quad ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\
& - x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\
& \quad ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x) \\
& - x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\
& \quad ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x) \\
& - x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{B}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{A}} Tf \\
& - x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = Tg \circ_{\mathcal{A}} Tf \\
& - x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} Tf \\
& - x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} Tf
\end{aligned}$$

- u preserves identity morphisms

$$\begin{aligned}
& - x \in \mathcal{A}, \\
& \quad u(\text{id}_x) = \text{id}_x = \text{id}_{ux} \\
& - x \in \mathcal{B}, \\
& \quad u(\text{id}_x) = T\text{id}_x = \text{id}_{Tx} = \text{id}_{ux}
\end{aligned}$$

- v preserves composition of morphisms

- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}$,
 $v(g \circ f) = v(g \circ_{\mathcal{A}} f) = S(g \circ_{\mathcal{A}} f) = Sg \circ_{\mathcal{B}} Sf$
 $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{A}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}$,
 $v(g \circ f) = v(g \circ_{\mathcal{B}} Sf) = g \circ_{\mathcal{B}} Sf$
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}$,
 $v(g \circ f) = v(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x)$
 $= S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x)$
 $= S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x$
 $= g \circ_{\mathcal{B}} f$
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}$,
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}$,
 $v(g \circ f) = v(Sg \circ_{\mathcal{B}} f) = Sg \circ_{\mathcal{B}} f$
 $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}$,
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}$,
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}$,
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$

• v preserves identity morphisms

- $x \in \mathcal{A}$
 $v(\text{id}_x) = S\text{id}_x = \text{id}_{Sx} = \text{id}_{vx}$
- $x \in \mathcal{B}$
 $v(\text{id}_x) = \text{id}_x \text{id}_{vx}$

3.6. PROPOSITION. *The projections u, v are surjective on objects, full and faithful.*

PROOF. It's trivial by definitions that u, v are surjective on objects. So we check fullness and faithfulness.

• u is full and faithful

- $x, y \in \mathcal{A}$,
 $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{A}(x, y)$ is bijective.
 - $x, y \in \mathcal{B}$,
 $T : \mathcal{B}(x, y) \rightarrow \mathcal{A}(Tx, Ty)$ is bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{A}(x, y) \ni \langle f, x, y \rangle \mapsto Tf \in \mathcal{A}(Tx, Ty) = \mathcal{A}(ux, uy)$ is bijective.
 - $x \in \mathcal{A}, y \in \mathcal{B}$,
 $\mathcal{B}(Sx, y) \ni f \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty)$ is the right adjunct of each f , and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y \rangle \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty) = \mathcal{A}(ux, uy)$ is bijective.
 - $x \in \mathcal{B}, y \in \mathcal{A}$,
 $\mathcal{B}(x, Sy) \ni f \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y)$ is the left adjunct of each f , and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y \rangle \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y) = \mathcal{A}(ux, uy)$
- v is full and faithful
- $x, y \in \mathcal{A}$,
 $S : \mathcal{A}(x, y) \rightarrow \mathcal{B}(Sx, Sy)$ is bijective. Therefore $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y \rangle \mapsto Sf \in \mathcal{B}(Sx, Sy) = \mathcal{B}(vx, vy)$ is bijective.
 - $x, y \in \mathcal{B}$,
 $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{B}(x, y) = \mathcal{B}(vx, vy)$ is bijective.
 - $x \in \mathcal{A}, y \in \mathcal{B}$,
 $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{B}(Sx, y) = \mathcal{B}(vx, vy)$ is bijective.
 - $x \in \mathcal{B}, y \in \mathcal{A}$,
 $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{B}(x, Sy) = \mathcal{B}(vx, vy)$ is bijective.

3.7. THEOREM. *Let \mathcal{A} and \mathcal{B} be categories. \mathcal{A} is ordinary equivalent to \mathcal{B} if and only if \mathcal{A} is span equivalent to \mathcal{B} .*

PROOF. Let \mathcal{A} be ordinary equivalent to \mathcal{B} , then \mathcal{A} is adjoint equivalent to \mathcal{B} . Thus there exists a adjoint equivalence between \mathcal{A} and \mathcal{B} . So we can construct the equivalence fusion and the projections. By Propositions, they are span equivalence. Therefore \mathcal{A} is span equivalent to \mathcal{B} .

On the other hand, let \mathcal{A} be span equivalent to \mathcal{B} . Then there exists a span equivalence $\langle \mathcal{C}, u, v \rangle$ between \mathcal{A} and \mathcal{B} , and \mathcal{C} is ordinary equivalent to both \mathcal{A} and \mathcal{B} . Therefore \mathcal{A} is ordinary equivalent to \mathcal{B} .

3.8. **REMARK.** Let \mathcal{A} be presheaf category. The forgetful functor

$$U : \mathcal{A}\text{-}\mathbf{Cat} \longrightarrow \mathcal{A}\text{-}\mathbf{Gph}$$

is monadic. (Proposition F 1.1 in [Leinster 2004])

Let $\mathcal{A} = \mathbf{Set}$, the induced monad T_1 is the free strict 1-category monad, $\mathbf{Set}\text{-}\mathbf{Grp}$ is the category of 1-globular sets. Hence

$$T_1\text{-}\mathbf{Alg} \cong \mathbf{Cat}$$

Moreover, the category of weak 1-category $\mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$ is isomorphic to $T_1\text{-}\mathbf{Alg}$. Therefore

$$\mathbf{Wk}\text{-}1\text{-}\mathbf{Cat} \cong T_1\text{-}\mathbf{Alg} \cong \mathbf{Cat}$$

3.9. **PROPOSITION.** *Let $F : \mathbf{Cat} \rightarrow \mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$ be the isomorphism above. let \mathcal{A} and \mathcal{B} be categories. \mathcal{A} is span equivalent to \mathcal{B} in \mathbf{Cat} if and only if $F(\mathcal{A})$ is span equivalent to $F(\mathcal{B})$ in $\mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$.*

PROOF.

3.10. **THEOREM.** *Let \mathcal{A} and \mathcal{B} be categories. \mathcal{A} is ordinary equivalent to \mathcal{B} in \mathbf{Cat} if and only if $F(\mathcal{A})$ is span equivalent to $F(\mathcal{B})$ in $\mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$.*

References

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