

# SPAN EQUIVALENCE BETWEEN WEAK $N$ -CATEGORIES

YUYA NISHIMURA

ABSTRACT.

## 1. Introduction

## 2. Span equivalence

2.1. DEFINITION. Let  $n \in \mathbb{N}$ . An  $n$ -globular set is a diagram

$$X = (X_n \xrightleftharpoons[t_n^X]{s_n^X} X_{n-1} \xrightleftharpoons[t_{n-1}^X]{s_{n-1}^X} \dots \xrightleftharpoons[t_1^X]{s_1^X} X_0)$$

of sets and maps such that

$$s_{k-1}^X s_k^X(x) = s_{k-1}^X t_k^X(x), \quad t_{k-1}^X s_k^X(x) = t_{k-1}^X t_k^X(x)$$

for all  $k \in \{2, \dots, n\}$  and  $x \in X_k$ .

Elements of  $X_k$  are called  $k$ -cells of  $X$ . We defined hom-sets of  $X$  as follows:

$$\mathbf{Hom}(x, y) := \{\alpha \in X_k \mid s_k^X(\alpha) = x, t_k^X(\alpha) = y\}$$

for all  $k \in \{1, \dots, n\}$  and  $x, y \in X_{k-1}$ .

Let  $X, Y$  be  $n$ -globular sets, A map of  $n$ -globular sets from  $X$  to  $Y$  is a collection  $f = \{f_k : X_k \rightarrow Y_k\}_{k \in \{1, \dots, n\}}$  of maps of sets such that

$$s_k^Y f_k(x) = f_{k-1} s_k^X(x), \quad t_k^Y f_k(x) = f_{k-1} t_k^X(x)$$

for all  $k \in \{1, \dots, n\}$  and  $x \in X_k$ .

The category of  $n$ -globular sets and their maps is denoted by  $n\text{-}\mathbf{GSet}$ .

2.2. DEFINITION. Let  $f : X \rightarrow Y$  be a map of  $n$ -globular sets.

- $f$  is surjective on  $k$ -cells  $:\Leftrightarrow f_k : X_k \rightarrow Y_k$  is surjective
- $f$  is injective on  $k$ -cells  $:\Leftrightarrow f_k : X_k \rightarrow Y_k$  is injective
- $f$  is full on  $k$ -cells  $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g \in \mathbf{Hom}_Y(f(x), f(x')), \\ \exists h \in \mathbf{Hom}_X(x, x') \text{ s.t. } f(h) = g \end{cases}$
- $f$  is faithful on  $k$ -cell  $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g, g' \in \mathbf{Hom}_X(f(x), f(x')), \\ g \neq g' \Rightarrow f(g) \neq f(g') \end{cases}$

---

2010 Mathematics Subject Classification: 00A00.

Key words and phrases: Weak  $n$ -category, span equivalence.

© Yuya Nishimura, 2016. Permission to copy for private use granted.

2.3. DEFINITION. Let  $K$  be an  $n$ -globular operad.  $K$ -algebras  $KX \rightarrow X$  and  $KY \rightarrow Y$  are span equivalent if there exists a triple  $\langle \psi, u, v \rangle$  such that  $\psi : KZ \rightarrow Z$  is a  $K$ -algebra,  $u : Z \rightarrow X$  and  $v : Z \rightarrow Y$  are maps of  $K$ -algebras, surjective on 0-cells, full on  $m$ -cells for all  $1 \leq m \leq n$ , and faithful on  $n$ -cells. The triple  $\langle \psi, u, v \rangle$  is referred to as a span equivalence of  $K$ -algebras.

2.4. PROPOSITION. In the pullback diagram in  $n\text{-GSet}$

$$\begin{array}{ccc} P & \xrightarrow{j} & Y \\ \downarrow i & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

- $f$  is surjective on 0-cells  $\Rightarrow j$  is surjective on 0-cells
- $f$  is full on  $k$ -cells  $\Rightarrow j$  is full on  $k$ -cells
- $f$  is faithful on  $k$ -cells  $\Rightarrow j$  is faithful on  $k$ -cells

PROOF. We define an  $n$ -globular set  $P$  as follows:

$$P_k := \{(x, y) \in X_k \times Y_k \mid f_k(x) = g_k(y)\}$$

$$s_l^P := (P_l \ni (x, y) \mapsto (s_l^X(x), s_l^Y(y)) \in P_{l-1})$$

$$t_l^P := (P_l \ni (x, y) \mapsto (t_l^X(x), t_l^Y(y)) \in P_{l-1})$$

for all  $k \in \{0, \dots, n\}$ ,  $l \in \{1, \dots, n\}$ , and maps of  $n$ -globular sets  $i, j$  as follows:

$$i_k := (P_k \ni (x, y) \mapsto x \in X_k), \quad j_k := (P_k \ni (x, y) \mapsto y \in Y_k)$$

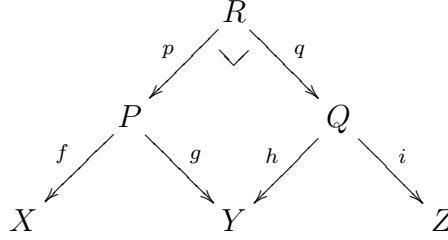
for all  $k \in \{0, \dots, n\}$ . Then  $(P, i, j)$  is a pullback of  $X$  and  $Y$  over  $S$ . It is enough to prove the proposition that we check the claims for  $(P, i, j)$ . Firstly, we prove surjectivity on 0-cells. For  $y \in Y_0$ , there exists  $x \in X_0$  such that  $f_0(x) = g_0(y)$ . So  $(x, y) \in P_0$  and  $j_0((x, y)) = y$ . which is the condition of surjectivity. To show fullness, we suppose  $(x, y), (x', y') \in P_{k-1}$ ,  $\phi \in \mathbf{Hom}(y, y')$ , we can see  $s_k g_k(\phi) = g_{k-1}(y) = f_{k-1}(x)$ ,  $t_k g_k(\phi) = g_{k-1}(y') = f_{k-1}(x')$ . Thus  $g_k(\phi) \in \mathbf{Hom}(f_{k-1}(x), f_{k-1}(x'))$ . For fullness, there exists  $\psi \in \mathbf{Hom}(x, x')$  such that  $f_k(\psi) = g_k(\phi)$ . Then  $(\psi, \phi) \in \mathbf{Hom}((x, y), (x', y'))$  and  $j_k(\psi, \phi) = \phi$ . Therefore  $j$  is full on  $k$ -cells. Lastly, let  $f$  be faithful on  $k$ -cells. let  $(x, y), (x', y') \in P_{k-1}$ ,  $\psi, \phi \in \mathbf{Hom}((x, y), (x', y'))$  such that  $j_k(\psi) = j_k(\phi)$ . Then  $f_k i_k(\psi) = g_k j_k(\psi) = g_k j_k(\phi) = f_k i_k(\phi)$ . From faithfulness,  $i_k(\psi) = i_k(\phi)$ , and  $\psi = (i_k(\psi), j_k(\psi)) = (i_k(\phi), j_k(\phi)) = \phi$ . Therefore  $j$  is faithful on  $k$ -cells.

2.5. REMARK. Let  $K$  be a monad on  $n\text{-GSet}$ . Pullbacks in  $K\text{-Alg}$  are created by the forgetful functor  $U : K\text{-Alg} \rightarrow n\text{-GSet}$ .

2.6. PROPOSITION. *Let*



*be span equivalences, then*



*is span equivalence.*

PROOF. By proposition,  $p, q$  are surjective on 0-cells, full on  $k$ -cells for  $1 \leq k \leq n$  and faithful on  $n$ -cells. Therefore  $f \circ p, i \circ q$  are surjective on 0-cells, full on  $k$ -cells for  $1 \leq k \leq n$  and faithful on  $n$ -cells. So the span is span equivalence.

2.7. THEOREM. *Span equivalence is equivalence relation on  $K$ -algebras.*

PROOF. It is straightforward from the definition and previous proposition that span equivalence is equivalence relation.

### 3. Characterizing equivalence of categories via spans

3.1. DEFINITION. *Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories. We say that  $\mathcal{A}$  and  $\mathcal{B}$  are span equivalent if there exists a triple  $\langle \mathcal{A}, u, v \rangle$  such that  $\mathcal{C}$  is a category,  $u : \mathcal{C} \rightarrow \mathcal{A}$  and  $v : \mathcal{C} \rightarrow \mathcal{B}$  are functors, surjective on objects, full and faithful.*

3.2. DEFINITION. *Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories, let  $\langle S : \mathcal{A} \rightarrow \mathcal{B}, T : \mathcal{B} \rightarrow \mathcal{A}, \eta : I_{\mathcal{A}} \rightarrow TS, \epsilon : ST \rightarrow I_{\mathcal{B}} \rangle$  be an adjoint equivalence between  $\mathcal{A}$  and  $\mathcal{B}$ . We define a category, equivalence fusion  $\mathcal{A} \Downarrow \mathcal{B}$ , as follows:*

- *object-set*

$$\mathbf{Ob}(\mathcal{A} \Downarrow \mathcal{B}) := \mathbf{Ob}(\mathcal{A}) \bigsqcup \mathbf{Ob}(\mathcal{B}) \quad (\text{disjoint})$$

- *hom-set*

$$\mathbf{Hom}(x, y) := \begin{cases} \{ \langle f, x, y \rangle \mid f \in \mathcal{A}(x, y) \} & (x, y \in \mathcal{A}) \\ \{ \langle f, x, y \rangle \mid f \in \mathcal{B}(x, y) \} & (x, y \in \mathcal{B}) \\ \{ \langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y) \} & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \{ \langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy) \} & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- *composition*

$$\begin{aligned} \tilde{\circ} : \mathbf{Hom}(y, z) \times \mathbf{Hom}(x, y) &\longrightarrow \mathbf{Hom}(x, z) \\ \langle \langle g, y, z \rangle, \langle f, x, y \rangle \rangle &\longmapsto \langle g, y, z \rangle \tilde{\circ} \langle f, x, y \rangle := \langle g \circ f, x, z \rangle \end{aligned}$$

$$g \circ f := \begin{cases} g \circ_{\mathcal{A}} f & (x, y, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x, y, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} Sf & (x, y \in \mathcal{A}, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{A}, y, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x, y \in \mathcal{B}, z \in \mathcal{A}) \\ Sg \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y, z \in \mathcal{A}) \\ \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}) \end{cases}$$

- *identities*

$$\text{id}_x := \begin{cases} \langle \text{id}_x, x, x \rangle & (x \in \mathcal{A}, \text{id}_x \in \mathcal{A}(x, x)) \\ \langle \text{id}_x, x, x \rangle & (x \in \mathcal{B}, \text{id}_x \in \mathcal{B}(x, x)) \end{cases}$$

3.3. PROPOSITION. *The equivalence fusion  $\mathcal{A} \Downarrow \mathcal{B}$  forms a category.*

PROOF. It is easy to check that the composition  $\tilde{\circ}$  is map from  $\mathbf{Hom}(x, y) \times \mathbf{Hom}(y, z)$  to  $\mathbf{Hom}(x, z)$ . Now, we prove that the composition  $\tilde{\circ}$  satisfies associative law and identity law by case analysis.

- *associative law*

$$\begin{aligned} &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}, \\ &\quad h \circ (g \circ f) = h \circ_{\mathcal{A}} (g \circ_{\mathcal{A}} f) \\ &\quad (h \circ g) \circ f = (h \circ_{\mathcal{A}} g) \circ_{\mathcal{A}} f \\ &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}, \\ &\quad h \circ (g \circ f) = h \circ (g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} S(g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} Sf) \\ &\quad (h \circ g) \circ f = (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} Sf \\ &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}, \\ &\quad h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y \circ_{\mathcal{A}} f \\ &\quad (h \circ g) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f \\ &- x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}, \\ &\quad h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} Sf) \\ &\quad (h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} Sf \\ &- x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}, \\ &\quad h \circ (g \circ f) = h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) = h \circ_{\mathcal{A}} \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &\quad (h \circ g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &\quad \quad \quad = \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$

- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} (\epsilon_{Sz} \circ_{\mathcal{B}} S\eta_z) \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} \epsilon_{Sz} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \circ_{\mathcal{B}} \epsilon_{Sx} \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(h \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{A}} g) \circ f = S(h \circ_{\mathcal{A}} g) \circ_{\mathcal{B}} f = (Sh \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f \\ &= S(\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{B}} f \\ &= S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= (\epsilon_{Sw} \circ_{\mathcal{B}} S\eta_w) \circ_{\mathcal{B}} S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= \epsilon_{Sw} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} \epsilon_{Sy} \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (Sh \circ_{\mathcal{B}} g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$ ,  

$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A},$   
 $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$   
 $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B},$   
 $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$   
 $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$

- identity law

- $x \in \mathcal{A}, y \in \mathcal{A},$   
 $f \circ \text{id}_x = f \circ_{\mathcal{A}} \text{id}_x = f$   
 $\text{id}_y \circ f = \text{id}_y \circ_{\mathcal{A}} f = f$
- $x \in \mathcal{A}, y \in \mathcal{B},$   
 $f \circ \text{id}_x = f \circ_{\mathcal{B}} S\text{id}_x = f \circ_{\mathcal{B}} \text{id}_{Sx} = f$   
 $\text{id}_y \circ f = \text{id}_y \circ_{\mathcal{B}} f = f$
- $x \in \mathcal{B}, y \in \mathcal{A},$   
 $f \circ \text{id}_x = f \circ_{\mathcal{B}} \text{id}_x = f$   
 $\text{id}_y \circ f = S\text{id}_y \circ_{\mathcal{B}} f = \text{id}_{Sy} \circ_{\mathcal{B}} f = f$
- $x \in \mathcal{B}, y \in \mathcal{B},$   
 $f \circ \text{id}_x = f \circ_{\mathcal{B}} \text{id}_x = f$   
 $\text{id}_y \circ f = \text{id}_y \circ_{\mathcal{B}} f = f$

3.4. DEFINITION. Let  $\langle S : \mathcal{A} \rightarrow \mathcal{B}, T : \mathcal{B} \rightarrow \mathcal{A}, \eta : I_{\mathcal{A}} \rightarrow TS, \epsilon : ST \rightarrow I_{\mathcal{B}} \rangle$  be an adjoint equivalence, let  $\mathcal{A} \Downarrow \mathcal{B}$  be the equivalence fusion. We define the projections  $u, v$  as follows:

- $u : \mathcal{A} \Downarrow \mathcal{B} \longrightarrow \mathcal{A}$

object-function  $u : \mathbf{Ob}(\mathcal{A} \Downarrow \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{A})$

$$x \longmapsto ux := \begin{cases} x & (x \in \mathcal{A}) \\ Tx & (x \in \mathcal{B}) \end{cases}$$

hom-functions  $u : \mathbf{Hom}(x, y) \longrightarrow \mathcal{A}(ux, uy)$

$$\langle f, x, y \rangle \longmapsto uf := \begin{cases} f & (x, y \in \mathcal{A}) \\ Tf & (x, y \in \mathcal{B}) \\ Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \eta_y^{-1} \circ_{\mathcal{A}} Tf & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- $v : \mathcal{A} \Downarrow \mathcal{B} \longrightarrow \mathcal{B}$

object-function  $v : \mathbf{Ob}(\mathcal{A} \Downarrow \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{B})$

$$x \longmapsto vx := \begin{cases} Sx & (x \in \mathcal{A}) \\ x & (x \in \mathcal{B}) \end{cases}$$

hom-functions  $v : \mathbf{Hom}(x, y) \longrightarrow \mathcal{B}(vx, vy)$

$$\langle f, x, y \rangle \longmapsto vf := \begin{cases} Sf & (x, y \in \mathcal{A}) \\ f & (\text{others}) \end{cases}$$

3.5. PROPOSITION. *The projections  $u, v$  are functors.*

PROOF. We show that  $u, v$  preserve composition of morphisms and identity morphism by case analysis.

- $u$  preserves composition of morphisms

$$\begin{aligned}
& - x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{A}} f) = g \circ_{\mathcal{A}} f \\
& \quad ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} f \\
& - x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} Sf) = T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\
& \quad ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\
& - x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\
& \quad ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x) \\
& - x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\
& \quad ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x) \\
& - x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{B}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{A}} Tf \\
& - x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = Tg \circ_{\mathcal{A}} Tf \\
& - x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} Tf \\
& - x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, \\
& \quad u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf \\
& \quad ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} Tf
\end{aligned}$$

- $u$  preserves identity morphisms

$$\begin{aligned}
& - x \in \mathcal{A}, \\
& \quad u(\text{id}_x) = \text{id}_x = \text{id}_{ux} \\
& - x \in \mathcal{B}, \\
& \quad u(\text{id}_x) = T\text{id}_x = \text{id}_{Tx} = \text{id}_{ux}
\end{aligned}$$

- $v$  preserves composition of morphisms

- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}$ ,  
 $v(g \circ f) = v(g \circ_{\mathcal{A}} f) = S(g \circ_{\mathcal{A}} f) = Sg \circ_{\mathcal{B}} Sf$   
 $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{A}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}$ ,  
 $v(g \circ f) = v(g \circ_{\mathcal{B}} Sf) = g \circ_{\mathcal{B}} Sf$   
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}$ ,  
 $v(g \circ f) = v(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x)$   
 $= S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x)$   
 $= S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x$   
 $= g \circ_{\mathcal{B}} f$   
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}$ ,  
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$   
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}$ ,  
 $v(g \circ f) = v(Sg \circ_{\mathcal{B}} f) = Sg \circ_{\mathcal{B}} f$   
 $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}$ ,  
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$   
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}$ ,  
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$   
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}$ ,  
 $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$   
 $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$

- $v$  preserves identity morphisms

- $x \in \mathcal{A}$   
 $v(\text{id}_x) = S\text{id}_x = \text{id}_{Sx} = \text{id}_{vx}$
- $x \in \mathcal{B}$   
 $v(\text{id}_x) = \text{id}_x \text{id}_{vx}$

3.6. PROPOSITION. *The projections  $u, v$  are surjective on objects, full and faithful.*

PROOF. It's trivial by definitions that  $u, v$  are surjective on objects. So we check fullness and faithfulness.

- $u$  is full and faithful



- $x, y \in \mathcal{A}$ ,  
 $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{A}(x, y)$  is bijective.
  - $x, y \in \mathcal{B}$ ,  
 $T : \mathcal{B}(x, y) \rightarrow \mathcal{A}(Tx, Ty)$  is bijective. Therefore  $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{A}(x, y) \ni \langle f, x, y \rangle \mapsto Tf \in \mathcal{A}(Tx, Ty) = \mathcal{A}(ux, uy)$  is bijective.
  - $x \in \mathcal{A}, y \in \mathcal{B}$ ,  
 $\mathcal{B}(Sx, y) \ni f \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty)$  is the right adjunct of each  $f$ , and bijective. Therefore  $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y \rangle \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty) = \mathcal{A}(ux, uy)$  is bijective.
  - $x \in \mathcal{B}, y \in \mathcal{A}$ ,  
 $\mathcal{B}(x, Sy) \ni f \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y)$  is the left adjunct of each  $f$ , and bijective. Therefore  $u : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y \rangle \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y) = \mathcal{A}(ux, uy)$
- $v$  is full and faithful
- $x, y \in \mathcal{A}$ ,  
 $S : \mathcal{A}(x, y) \rightarrow \mathcal{B}(Sx, Sy)$  is bijective. Therefore  $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y \rangle \mapsto Sf \in \mathcal{B}(Sx, Sy) = \mathcal{B}(vx, vy)$  is bijective.
  - $x, y \in \mathcal{B}$ ,  
 $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{B}(x, y) = \mathcal{B}(vx, vy)$  is bijective.
  - $x \in \mathcal{A}, y \in \mathcal{B}$ ,  
 $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{B}(Sx, y) = \mathcal{B}(vx, vy)$  is bijective.
  - $x \in \mathcal{B}, y \in \mathcal{A}$ ,  
 $v : \mathbf{Hom}(x, y) = \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y \rangle \mapsto f \in \mathcal{B}(x, Sy) = \mathcal{B}(vx, vy)$  is bijective.

**3.7. THEOREM.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be categories.  $\mathcal{A}$  is ordinary equivalent to  $\mathcal{B}$  if and only if  $\mathcal{A}$  is span equivalent to  $\mathcal{B}$ .*

**PROOF.** Let  $\mathcal{A}$  be ordinary equivalent to  $\mathcal{B}$ , then  $\mathcal{A}$  is adjoint equivalent to  $\mathcal{B}$ . Thus there exists a adjoint equivalence between  $\mathcal{A}$  and  $\mathcal{B}$ . So we can construct the equivalence fusion and the projections. By Propositions, they are span equivalence. Therefore  $\mathcal{A}$  is span equivalent to  $\mathcal{B}$ .

On the other hand, let  $\mathcal{A}$  be span equivalent to  $\mathcal{B}$ . Then there exists a span equivalence  $\langle \mathcal{C}, u, v \rangle$  between  $\mathcal{A}$  and  $\mathcal{B}$ , and  $\mathcal{C}$  is ordinary equivalent to both  $\mathcal{A}$  and  $\mathcal{B}$ . Therefore  $\mathcal{A}$  is ordinary equivalent to  $\mathcal{B}$ .

3.8. **REMARK.** Let  $\mathcal{A}$  be presheaf category. The forgetful functor

$$U : \mathcal{A}\text{-}\mathbf{Cat} \longrightarrow \mathcal{A}\text{-}\mathbf{Gph}$$

is monadic. (Proposition F 1.1 in [Leinster 2004])

Let  $\mathcal{A} = \mathbf{Set}$ , we can see  $\mathbf{Set}\text{-}\mathbf{Cat} = \mathbf{Cat}$ ,  $\mathbf{Set}\text{-}\mathbf{Grp} = \mathbf{1}\text{-}\mathbf{GSet}$ . and the induced monad  $T_1$  is the free strict 1-category monad on  $\mathbf{1}\text{-}\mathbf{GSet}$ . by the remark, the comparison functor

$$K : \mathbf{Cat} \longrightarrow (\mathbf{1}\text{-}\mathbf{GSet})^{T_1}$$

is isomorphic and arrow part of the functor is

$$K : f \longmapsto Uf.$$

Moreover, the category  $\mathbf{Wk}\text{-}\mathbf{1}\text{-}\mathbf{Cat}$  of Leinster's weak 1 category is the category  $T_1\text{-}\mathbf{Alg}$  of algebras for the monad. So the isomorphism  $K : \mathbf{Cat} \rightarrow \mathbf{Wk}\text{-}\mathbf{1}\text{-}\mathbf{Cat}$  preserve surjectivity, fullness and faithfulness. Hence,

3.9. **THEOREM.** *Let  $K : \mathbf{Cat} \rightarrow \mathbf{Wk}\text{-}\mathbf{1}\text{-}\mathbf{Cat}$  be the isomorphism above. let  $\mathcal{A}$  and  $\mathcal{B}$  be categories.  $\mathcal{A}$  is span equivalent to  $\mathcal{B}$  in  $\mathbf{Cat}$  if and only if  $K(\mathcal{A})$  is span equivalent to  $K(\mathcal{B})$  in  $\mathbf{Wk}\text{-}\mathbf{1}\text{-}\mathbf{Cat}$ .*

## References

L. Lamport, *Latex User's Guide & Reference Manual*. Addison-Wesley (fifth edition), 1986.

Email: