SPAN EQUIVALENCE BETWEEN WEAK N-CATEGORIES

YUYA NISHIMURA

Abstract.

1. Introduction

2. Span equivalence

2.1. Definition. *Let $f : X \to Y$ be a map of n-globular sets.*

- f *:surjective on k-cells* $:\Leftrightarrow$ $f_k : X_k \to Y_k$ *:surjective*
- f *:injective on k-cells* $:\Leftrightarrow$ $f_k : X_k \to Y_k$ *:injective*
- f *:full on k-cells* $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g \in \mathbf{Hom}_Y(f(x), f(x')), \\ \exists h \in \mathbf{Hom}_X(x, x') \text{ s.t. } f(h) = g \end{cases}$
- f *:faithful on k-cell* $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g, g' \in \mathbf{Hom}_X(f(x), f(x')), \\ g \neq g' \Rightarrow f(g) \neq f(g') \end{cases}$

2.2. Definition. *Let K be an n-globular operad. K-algebras $KX \to X$ and $KY \to Y$ are span equivalent if there exists a triple $\langle \psi, u, v \rangle$ such that $\psi : KZ \to Z$ is an K-algebra, $u : Z \to X$ and $v : Z \to Y$ are maps of K-algebras, surjective on 0-cells, full on m-cells for all $1 \leq m \leq n$, and faithful on n-cells.The triple $\langle \psi, u, v \rangle$ is referred to as an span equivalence of K-algebras.*

2.3. Proposition. *In the pullback diagram in $[\mathbf{G}_n^{op}, \mathbf{Set}]$*

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

- f *:surjective on 0-cells$\Rightarrow$ j :surjective on 0-cells*
- f *:full on k-cells $\Rightarrow$ j :full on k-cells*
- f *:faithful on k-cells $\Rightarrow$ j :faithful on k-cells*

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Proof.

2.4. Remark. Let K be a monad on $[\mathbf{G}_n^{op}, \mathbf{Set}]$. Pullbacks in K-$\mathbf{Alg}$ are created by the forgetful functor $U : K\text{-}\mathbf{Alg} \to [\mathbf{G}_n^{op}, \mathbf{Set}]$.

2.5. Proposition. *Let*

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & Q \arrow[dl] \arrow[dr] & \\ Y & & Z \end{tikzcd}$$

be span equivalences, then

$$\begin{tikzcd} & & R \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & P \arrow[dl] \arrow[dr] & & Q \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

is span equivalence.

Proof.

2.6. Theorem. *Span equivalence is equivalence relation on K-algebras.*

3. Characterizing equivalence of categories via spans

3.1. Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. We say that $\mathcal{A}$ and $\mathcal{B}$ are span equivalent if there exists a triple $\langle \mathcal{A}, u, v \rangle$ such that $\mathcal{C}$ is a category, $u : \mathcal{C} \to \mathcal{A}$ and $v : \mathcal{C} \to \mathcal{B}$ are functors, surjective on objects, full and faithful.*

3.2. Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, let $\langle S : \mathcal{A} \to \mathcal{B}, T : \mathcal{B} \to \mathcal{A}, \eta : I_{\mathcal{A}} \to TS, \epsilon : ST \to I_{\mathcal{B}} \rangle$ be an adjoint equivalence between $\mathcal{A}$ and $\mathcal{B}$. We define a category,* equivalence fusion $\mathcal{A} \uplus \mathcal{B}$ *, as follows:*

- *object-set*

$$\mathbf{Ob}(\mathcal{A} \uplus \mathcal{B}) := \mathbf{Ob}(\mathcal{A}) \bigsqcup \mathbf{Ob}(\mathcal{B}) \qquad \text{(disjoint)}$$

- *hom-set*

$$\mathbf{Hom}(x, y) := \begin{cases} \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x, y)\} & (x, y \in \mathcal{A}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, y)\} & (x, y \in \mathcal{B}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y)\} & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy)\} & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- *composition*

$$\begin{aligned} \tilde{\circ} : \mathbf{Hom}(y,z) \times \mathbf{Hom}(x,y) &\longrightarrow \mathbf{Hom}(x,z) \\ \langle \langle g,y,z\rangle, \langle f,x,y\rangle\rangle &\longmapsto \langle g,y,z\rangle \tilde{\circ} \langle f,x,y\rangle := \langle g\circ f, x, z\rangle \end{aligned}$$

$$g \circ f := \begin{cases} g \circ_{\mathcal{A}} f & (x,y,z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x,y,z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} Sf & (x,y \in \mathcal{A}, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{A}, y,z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} f & (x,y \in \mathcal{B}, z \in \mathcal{A}) \\ Sg \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y,z \in \mathcal{A}) \\ \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}) \end{cases}$$

- *identities*

$$\mathrm{id}_x := \begin{cases} \langle \mathrm{id}_x, x, x\rangle & (x \in \mathcal{A}, \mathrm{id}_x \in \mathcal{A}(x,x)) \\ \langle \mathrm{id}_x, x, x\rangle & (x \in \mathcal{B}, \mathrm{id}_x \in \mathcal{B}(x,x)) \end{cases}$$

3.3. Proposition. *The equivalence fusion $\mathcal{A} \uplus \mathcal{B}$ forms a category.*

Proof. It is easy to check that the composition $\tilde{\circ}$ is map from $\mathbf{Hom}(x,y) \times \mathbf{Hom}(y,z)$ to $\mathbf{Hom}(x,z)$. Now, we prove that the composition $\tilde{\circ}$ satisfies composition law and identity law by case analysis.

- composition law
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$,
    $h \circ (g \circ f) = h \circ_{\mathcal{A}} (g \circ_{\mathcal{A}} f)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{A}} g) \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$,
    $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} S(g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} Sf)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} Sf$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$,
    $$\begin{aligned} h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x \\ &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x \\ &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y \circ_{\mathcal{A}} f \end{aligned}$$
    $(h \circ g) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$,
    $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} Sf)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} Sf$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$,
    $$\begin{aligned} h \circ (g \circ f) = h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) &= h \circ_{\mathcal{A}} \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &= \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$
    $$\begin{aligned} (h \circ g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ f &= \eta_w^{-1} \circ_{\mathcal{A}} T(Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \\ &= \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$

– $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} (\epsilon_{Sz} \circ_{\mathcal{B}} S\eta_z) \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} \epsilon_{Sz} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \circ_{\mathcal{B}} \epsilon_{Sx} \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(h \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$

– $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{A}} g) \circ f = S(h \circ_{\mathcal{A}} g) \circ_{\mathcal{B}} f = (Sh \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f \\ &= S(\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{B}} f \\ &= S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= (\epsilon_{Sw} \circ_{\mathcal{B}} S\eta_w) \circ_{\mathcal{B}} S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= \epsilon_{Sw} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} \epsilon_{Sy} \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (Sh \circ_{\mathcal{B}} g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,
$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$,
    $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$,
    $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- identity law
  - $x \in \mathcal{A}, y \in \mathcal{A}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{A}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{A}} f = f$
  - $x \in \mathcal{A}, y \in \mathcal{B}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} S\mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_{Sx} = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{B}} f = f$
  - $x \in \mathcal{B}, y \in \mathcal{A}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = S\mathrm{id}_y \circ_{\mathcal{B}} f = \mathrm{id}_{Sy} \circ_{\mathcal{B}} f = f$
  - $x \in \mathcal{B}, y \in \mathcal{B}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{B}} f = f$

3.4. Definition. *Let $\langle S : \mathcal{A} \to \mathcal{B}, T : \mathcal{B} \to \mathcal{A}, \eta : I_{\mathcal{A}} \to TS, \epsilon : ST \to I_{\mathcal{B}} \rangle$ be an adjoint equivalence, let $\mathcal{A} \Cup \mathcal{B}$ be the equivalence fusion. We define the projections u, v as follows:*

- $u : \mathcal{A} \Cup \mathcal{B} \longrightarrow A$
  object-function $u : \mathbf{Ob}(\mathcal{A} \Cup \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{A})$
  $$x \longmapsto ux := \begin{cases} x & (x \in \mathcal{A}) \\ Tx & (x \in \mathcal{B}) \end{cases}$$
  hom-functions $u : \mathbf{Hom}(x, y) \longrightarrow \mathcal{A}(ux, uy)$
  $$\langle f, x, y \rangle \longmapsto uf := \begin{cases} f & (x, y \in \mathcal{A}) \\ Tf & (x, y \in \mathcal{B}) \\ Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \eta_y^{-1} \circ_{\mathcal{A}} Tf & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$
- $v : \mathcal{A} \Cup \mathcal{B} \longrightarrow B$
  object-function $v : \mathbf{Ob}(\mathcal{A} \Cup \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{B})$
  $$x \longmapsto vx := \begin{cases} Sx & (x \in \mathcal{A}) \\ x & (x \in \mathcal{B}) \end{cases}$$
  hom-functions $v : \mathbf{Hom}(x, y) \longrightarrow \mathcal{B}(ux, uy)$
  $$\langle f, x, y \rangle \longmapsto vf := \begin{cases} Sf & (x, y \in \mathcal{A}) \\ f & (\text{others}) \end{cases}$$

3.5. PROPOSITION. *The projections u, v are functors.*

PROOF. We show that u, v preserve composition of morphisms and identity morphism by case analysis.

- u preserves composition of morphisms
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}$,
    $u(g \circ f) = u(g \circ_{\mathcal{A}} f) = g \circ_{\mathcal{A}} f$
    $ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} Sf) = T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x$
    $ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}$,
    $u(g \circ f) = u(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
    $ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x)$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
    $ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x)$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}$,
    $u(g \circ f) = u(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{B}} Tf$
    $ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{A}} Tf$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf$
    $ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = Tg \circ_{\mathcal{A}} Tf$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf$
    $ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} Tf$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf$
    $ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} Tf$
- u preserves identity morphisms
  - $x \in \mathcal{A}$,
    $u(\mathrm{id}_x) = \mathrm{id}_x = \mathrm{id}_{ux}$
  - $x \in \mathcal{B}$,
    $u(\mathrm{id}_x) = T\mathrm{id}_x = \mathrm{id}_{Tx} = \mathrm{id}_{ux}$
- v preserves composition of morphisms

- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}$,
  $v(g \circ f) = v(g \circ_{\mathcal{A}} f) = S(g \circ_{\mathcal{A}} f) = Sg \circ_{\mathcal{B}} Sf$
  $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{A}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}$,
  $v(g \circ f) = v(g \circ_{\mathcal{B}} Sf) = g \circ_{\mathcal{B}} Sf$
  $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} Sf$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}$,
  $$\begin{aligned} v(g \circ f) &= v(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= g \circ_{\mathcal{B}} f \end{aligned}$$
  $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}$,
  $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
  $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}$,
  $v(g \circ f) = v(Sg \circ_{\mathcal{B}} f) = Sg \circ_{\mathcal{B}} f$
  $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}$,
  $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
  $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}$,
  $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
  $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}$,
  $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
  $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$

- v preserves identity morphisms
  - $x \in \mathcal{A}$
    $v(\mathrm{id}_x) = S\mathrm{id}_x = \mathrm{id}_{Sx} = \mathrm{id}_{vx}$
  - $x \in \mathcal{B}$
    $v(\mathrm{id}_x) = \mathrm{id}_x \ \mathrm{id}_{vx}$

3.6. Proposition. *The projections u, v are surjective on objects, full and faithful.*

Proof. It's trivial by definitions that u, v are surjective on objects. So we check fullness and faithfulness.

- u is full and faithful

  - $x, y \in \mathcal{A}$,
    $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{A}(x, y)$ is bijective.
  - $x, y \in \mathcal{B}$,
    $T : \mathcal{B}(x, y) \to \mathcal{A}(Tx, Ty)$ is bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{A}(x, y) \ni \langle f, x, y\rangle \mapsto Tf \in \mathcal{A}(Tx, Ty) = \mathcal{A}(ux, uy)$ is bijective.
  - $x \in \mathcal{A}, y \in \mathcal{B}$,
    $\mathcal{B}(Sx, y) \ni f \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty)$ is the right adjunct of each f, and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y\rangle \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty) = \mathcal{A}(ux, uy)$ is bijective.
  - $x \in \mathcal{B}, y \in \mathcal{A}$,
    $\mathcal{B}(x, Sy) \ni f \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y)$ is the left adjunct of each f, and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y\rangle \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y) = \mathcal{A}(ux, uy)$

- v is full and faithful

  - $x, y \in \mathcal{A}$,
    $S : \mathcal{A}(x, y) \to \mathcal{B}(Sx, Sy)$ is bijective. Therefore $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y\rangle \mapsto Sf \in \mathcal{B}(Sx, Sy) = \mathcal{B}(vx, vy)$ is bijective.
  - $x, y \in \mathcal{B}$,
    $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(x, y) = \mathcal{B}(vx, vy)$ is bijective.
  - $x \in \mathcal{A}, y \in \mathcal{B}$,
    $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(Sx, y) = \mathcal{B}(vx, vy)$ is bijective.
  - $x \in \mathcal{B}, y \in \mathcal{A}$,
    $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(x, Sy) = \mathcal{B}(vx, vy)$ is bijective.

3.7. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. $\mathcal{A}$ is ordinary equivalent to $\mathcal{B}$ if and only if $\mathcal{A}$ is span equivalent to $\mathcal{B}$.*

Proof. Let $\mathcal{A}$ be ordinary equivalent to $\mathcal{B}$, then $\mathcal{A}$ is adjoint equivalent to $\mathcal{B}$. Thus there exists a adjoint equivalence between $\mathcal{A}$ and $\mathcal{B}$. So we can construct the equivalence fusion and the projections. By Propositions, they are span equivalence. Therefore $\mathcal{A}$ is span equivalent to $\mathcal{B}$.
On the other hand, let $\mathcal{A}$ be span equivalent to $\mathcal{B}$. Then there exists a span equivalence $\langle \mathcal{C}, u, v\rangle$ between $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{C}$ is ordinary equivalent to both $\mathcal{A}$ and $\mathcal{B}$. Therefore $\mathcal{A}$ is ordinary equivalent to $\mathcal{B}$.

3.8. Remark. Let $\mathcal{A}$ be presheaf category. The forgetful functor

$$U : \mathcal{A}\text{-}\mathbf{Cat} \longrightarrow \mathcal{A}\text{-}\mathbf{Gph}$$

is monadic.

3.9. Proposition. *Let $F : \mathbf{Cat} \to \textbf{Wk-1-Cat}$ be the isomorphism above. let $\mathcal{A}$ and $\mathcal{B}$ be categories. $\mathcal{A}$ is span equivalent to $\mathcal{B}$ in* **Cat** *if and only if $F(\mathcal{A})$ is span equivalent to $F(\mathcal{B})$ in* **Wk-1-Cat**.

Proof.

3.10. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. $\mathcal{A}$ is ordinary equivalent to $\mathcal{B}$ in* **Cat** *if and only if $F(\mathcal{A})$ is span equivalent to $F(\mathcal{B})$ in* **Wk-1-Cat**.

References

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