SPAN EQUIVALENCE BETWEEN WEAK N-CATEGORIES

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ABSTRACT.

1. Introduction

2. Preliminary

2.1. DEFINITION. *Let $n \in \mathbb{N}$. An n-*globular set *is a diagram*

$$X = (X_n \underset{t_n^X}{\overset{s_n^X}{\rightrightarrows}} X_{n-1} \underset{t_{n-1}^X}{\overset{s_{n-1}^X}{\rightrightarrows}} \cdots \underset{t_1^X}{\overset{s_1^X}{\rightrightarrows}} X_0)$$

of sets and maps such that

$$s_{k-1}^X s_k^X(x) = s_{k-1}^X t_k^X(x), \quad t_{k-1}^X s_k^X(x) = t_{k-1}^X t_k^X(x)$$

for all $k \in \{2, ..., n\}$ and $x \in X_k$.
*Elements of X_k are called k-*cells *of X. We defined* hom-sets *of X as follows:*

$$\mathbf{Hom}(x, y) := \{\alpha \in X_k \mid s_k^X(\alpha) = x, t_k^X(\alpha) = y\}$$

for all $k \in \{1, ...n\}$ and $x, y \in X_{k-1}$.
Let X, Y be n-globular sets, A map *of n-globular sets from X to Y is a collection $f = \{f_k : X_k \to Y_k\}_{k \in \{1,...,n\}}$ of maps of sets such that*

$$s_k^Y f_k(x) = f_{k-1} s_k^X(x), \quad t_k^Y f_k(x) = f_{k-1} t_k^X(x)$$

for all $k \in \{1, ..., n\}$ and $x \in X_k$.
*The category of n-globular sets and their maps is denoted by n-**GSet**.*

2.2. DEFINITION. *A category is* cartesian *if it has all pullbacks. A functor is* cartesian *if it preserves pullbacks. A natural transformation is* cartesian *if it all of its naturality squares are pullbacks squares. A monad is* cartesian *if its functor part, unit and counit are cartesian. A map of monad is* cartesian *if its underlying natural transformation is cartesian.*

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2.3. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1. and T be a cartesian monad on $\mathcal{C}$. The category of T-*collections *is the slice category $\mathcal{C}/T1$. The category has a monoidal structure: let $k : K \to T1, k' : K' \to T1$ be collections; then their tensor product is defined to be the composite along the top of the diagram*

$$\begin{tikzcd}
K \otimes K' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TK' \arrow[r, "Tk'"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu_1^T"] & T1 \\
K \arrow[r, "k"'] & T1
\end{tikzcd}$$

where ! is the unique map $K' \to 1$. the unit for this tensor product is the collection

$$\begin{tikzcd}
1 \arrow[d, "\mu_1^T"] \\
T1
\end{tikzcd}$$

*The monoidal category is denoted by T-**Coll**.*

2.4. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1, and T be a cartesian monad on $\mathcal{C}$. A T-*operad *is a monoid in the monoidal category T-**Coll**. In the case in which T is the free strict n-category monad on n-**GSet**, a T-operad is called an* n-globular operad.

2.5. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1, T be a cartesian monad on $\mathcal{C}$ and K be a T-operad. Then there is an* induced monad *on $\mathcal{C}$, which by abuse of notation we denote (K, η^K, μ^K): The endfunctor*

$$K : \mathcal{C} \to \mathcal{C}$$

is defined as follows; The object part of the functor, for $X \in \mathcal{C}$, KX is defined by the pullback:

$$\begin{tikzcd}
KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\
TX \arrow[r, "T!"'] & T1
\end{tikzcd}$$

The arrow part of the functor, for $Y \in \mathcal{C}, u : X \to Y$, Ku is defined by the unique property of the pullback:

$$\begin{tikzcd}
KX \arrow[r, dotted, "Ku"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "K!"] & KY \arrow[r, "K!"] \arrow[d, "K_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\
TX \arrow[r, "Tu"'] \arrow[rr, bend right, "T!"'] & TY \arrow[r, "T!"'] & T1
\end{tikzcd}$$

Components η^K_X, μ^K_X of the unit map $\eta^K : 1 \Rightarrow K$ and $\mu^K : K^2 \Rightarrow K$ are defined by the following diagrams:

$$
\begin{tikzcd}
X \arrow[rr, "!"] \arrow[dr, dotted, "\eta^K_X"] \arrow[ddr, bend right, "\eta^T_X"'] & & 1 \arrow[d, "\epsilon"] \arrow[dd, bend left, "\eta^T_1"] \\
& KX \arrow[r, "K!"] \arrow[d, "k_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "k"] \\
& TX \arrow[r, "T!"'] & T1
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& & & K^2X \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[dddd, dotted, "\mu^K_X"] & & & \\
& & K \otimes K \arrow[dl] \arrow[dr] \arrow[dddd, bend right=60, "\mu^K"'] & & TKX \arrow[dl] \arrow[dr] & & \\
& K \arrow[dr, "k"'] & & TK \arrow[dl, "T!"] \arrow[dr, "Tk"'] & & T^2X \arrow[dl, "T^2!"] \arrow[dddl, bend left, "\mu^T_X"] & \\
& & T1 & & T^21 & & \\
& & & KX \arrow[dl] \arrow[dr] & & & \\
& & K \arrow[dr, "k"'] & & TX \arrow[dl, "T!"] & & \\
& & & T1 & & &
\end{tikzcd}
$$

2.6. Definition. *Let $\mathcal{C}$ be a cartesian category with a terminal object 1, T be a cartesian monad on $\mathcal{C}$ and K be a T-operad. We define a K-*algebra *as an algebra for the induced monad (K, η^K, μ^K). Similarly, a* map *of algebras for T-operad K is a map of algebras for the induced monad. The category of K-algebras and thier maps is denoted by K-**Alg**.*

3. Span equivalence

3.1. Definition. *Let $f : X \to Y$ be a map of n-globular sets.*

- f *is* surjective on k-cells $:\Leftrightarrow$ $f_k : X_k \to Y_k$ *is surjective*
- f *is* injective on k-cells $:\Leftrightarrow$ $f_k : X_k \to Y_k$ *is injective*
- f *is* full on k-cells $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g \in \mathbf{Hom}_Y(f(x), f(x')), \\ \exists h \in \mathbf{Hom}_X(x, x') \text{ s.t. } f(h) = g \end{cases}$
- f *is* faithful on k-cell $:\Leftrightarrow \begin{cases} \forall x, x' \in X_{k-1}, g, g' \in \mathbf{Hom}_X(f(x), f(x')), \\ g \neq g' \Rightarrow f(g) \neq f(g') \end{cases}$

3.2. Definition. *Let K be an n-globular operad. K-algebras $KX \to X$ and $KY \to Y$ are* span equivalent *if there exists a triple $\langle \psi, u, v \rangle$ such that $\psi : KZ \to Z$ is an K-algebra, $u : Z \to X$ and $v : Z \to Y$ are maps of K-algebras, surjective on 0-cells, full on m-cells for all $1 \leq m \leq n$, and faithful on n-cells. The triple $\langle \psi, u, v \rangle$ is referred to as an span equivalence of K-algebras.*

3.3. Proposition. *In the pullback diagram in n-**GSet***

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

- *f is surjective on 0-cells$\Rightarrow$ j is surjective on 0-cells*
- *f is full on k-cells $\Rightarrow$ j is full on k-cells*
- *f is faithful on k-cells $\Rightarrow$ j is faithful on k-cells*

Proof. We define an n-globular set P as follows:

$$P_k := \{(x, y) \in X_k \times Y_k \mid f_k(x) = g_k(y)\}$$

$$s_l^P := (P_l \ni (x, y) \mapsto (s_l^X(x), s_l^Y(y)) \in P_{l-1})$$
$$t_l^P := (P_l \ni (x, y) \mapsto (t_l^X(x), t_l^Y(y)) \in P_{l-1})$$

for all $k \in \{0, ..., n\}, l \in \{1, ..., n\}$, and maps of n-globular sets i, j as follows:

$$i_k := (P_k \ni (x, y) \mapsto x \in X_k), \quad j_k := (P_k \ni (x, y) \mapsto y \in Y_k)$$

for all $k \in \{0, ..., n\}$. Then (P, i, j) is a pullback of X and Y over S. It is enough to prove the proposition that we check the claims for (P, i, j). Firstly, we prove surjectivity on 0-cells. For $y \in Y_0$, there exists $x \in X_0$ such that $f_0(x) = g_0(y)$, So $(x, y) \in P_0$ and $j_0((x, y)) = y$. which is the condition of surjectivity. To show fullness, we suppose $(x, y), (x', y') \in P_{k-1}, \phi \in \mathbf{Hom}(y, y')$, we can see $s_k g_k(\phi) = g_{k-1}(y) = f_{k-1}, t_k g_k(\phi) = g_{k-1}(y') = f_{k-1}(x')$. Thus $g_k(\phi) \in \mathbf{Hom}(f_{k-1}(x), f_{k-1}(x'))$. For fullness, there exists $\psi \in \mathbf{Hom}(x, x')$ such that $f_k(\psi) = g_k(\phi)$. Then $(\psi, \phi) \in \mathbf{Hom}((x, y), (x', y'))$ and $j_k(\psi, \phi) = \phi$. Therefore j is full on k-cells. Lastly, let f be faithful on k-cells. let $(x, y), (x', y') \in P_{k-1}, \psi, \phi \in \mathbf{Hom}((x, y), (x', y'))$ such that $j_k(\psi) = j_k(\phi)$. Then $f_k i_k(\psi) = g_k j_k(\psi) = g_k j_k(\phi) = f_k i_k(\phi)$. From faithfulness, $i_k(\psi) = i_k(\phi)$, and $\psi = (i_k(\psi), j_k(\psi)) = (i_k(\phi), j_k(\phi)) = \phi$. Therefore j is faithful on k-cells.

3.4. Remark. Let K be a monad on n-**GSet**. Pullbacks in K-**Alg** are created by the forgetful functor $U : K\text{-}\mathbf{Alg} \to n\text{-}\mathbf{GSet}$.

3.5. Proposition. *Let*

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & Q \arrow[dl, "h"'] \arrow[dr, "i"] & \\ Y & & Z \end{tikzcd}$$

be span equivalences, then

$$\begin{tikzcd} & & R \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & P \arrow[dl, "f"'] \arrow[dr, "g"] & & Q \arrow[dl, "h"'] \arrow[dr, "i"] & \\ X & & Y & & Z \end{tikzcd}$$

is span equivalence.

Proof. By proposition, p, q are are surjective on 0-cells, full on k-cells for $1 \leq k \leq n$ and faithful on n-cells. Therefore $f \circ p, i \circ q$ are surjective on 0-cells, full on k-cells for $1 \leq k \leq n$ and faithful on n-cells. So the span is span equivalence.

3.6. Theorem. *Span equivalence is equivalence relation on K-algebras.*

Proof. It is straightforward from the definition and previous proposition that span equivalence is equivalence relation.

4. Characterizing equivalence of categories via spans

4.1. Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. We say that $\mathcal{A}$ and $\mathcal{B}$ are* span equivalent *if there exists a triple $\langle \mathcal{A}, u, v \rangle$ such that $\mathcal{C}$ is a category, $u : \mathcal{C} \to \mathcal{A}$ and $v : \mathcal{C} \to \mathcal{B}$ are functors, surjective on objects, full and faithful.*

4.2. Definition. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, let $\langle S : \mathcal{A} \to \mathcal{B}, T : \mathcal{B} \to \mathcal{A}, \eta : I_{\mathcal{A}} \to TS, \epsilon : ST \to I_{\mathcal{B}} \rangle$ be an adjoint equivalence between $\mathcal{A}$ and $\mathcal{B}$. We define a category,* equivalence fusion $\mathcal{A} \Cup \mathcal{B}$ *, as follows:*

- *object-set*

$$\mathbf{Ob}(\mathcal{A} \Cup \mathcal{B}) := \mathbf{Ob}(\mathcal{A}) \bigsqcup \mathbf{Ob}(\mathcal{B}) \qquad \text{(disjoint)}$$

- *hom-set*

$$\mathbf{Hom}(x, y) := \begin{cases} \{\langle f, x, y \rangle \mid f \in \mathcal{A}(x, y)\} & (x, y \in \mathcal{A}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, y)\} & (x, y \in \mathcal{B}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(Sx, y)\} & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \{\langle f, x, y \rangle \mid f \in \mathcal{B}(x, Sy)\} & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- *composition*

$$\begin{aligned}\tilde{\circ} : \mathbf{Hom}(y,z) \times \mathbf{Hom}(x,y) &\longrightarrow \mathbf{Hom}(x,z)\\ \langle\langle g,y,z\rangle, \langle f,x,y\rangle\rangle &\longmapsto \langle g,y,z\rangle \tilde{\circ} \langle f,x,y\rangle := \langle g\circ f, x, z\rangle\end{aligned}$$

$$g \circ f := \begin{cases} g \circ_{\mathcal{A}} f & (x,y,z \in \mathcal{A})\\ g \circ_{\mathcal{B}} f & (x,y,z \in \mathcal{B})\\ g \circ_{\mathcal{B}} Sf & (x,y \in \mathcal{A}, z \in \mathcal{B})\\ g \circ_{\mathcal{B}} f & (x \in \mathcal{A}, y,z \in \mathcal{B})\\ g \circ_{\mathcal{B}} f & (x,y \in \mathcal{B}, z \in \mathcal{A})\\ Sg \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y,z \in \mathcal{A})\\ \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A})\\ g \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}) \end{cases}$$

- *identities*

$$\mathrm{id}_x := \begin{cases} \langle \mathrm{id}_x, x, x\rangle & (x \in \mathcal{A}, \mathrm{id}_x \in \mathcal{A}(x,x))\\ \langle \mathrm{id}_x, x, x\rangle & (x \in \mathcal{B}, \mathrm{id}_x \in \mathcal{B}(x,x)) \end{cases}$$

4.3. Proposition. *The equivalence fusion $\mathcal{A} \uplus \mathcal{B}$ forms a category.*

Proof. It is easy to check that the composition $\tilde{\circ}$ is map from $\mathbf{Hom}(x,y) \times \mathbf{Hom}(y,z)$ to $\mathbf{Hom}(x,z)$. Now, we prove that the composition $\tilde{\circ}$ satisfies associative law and identity law by case analysis.

- associative law
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$,
    $h \circ (g \circ f) = h \circ_{\mathcal{A}} (g \circ_{\mathcal{A}} f)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{A}} g) \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$,
    $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} S(g \circ_{\mathcal{A}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} Sf)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} Sf$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$,
    $$\begin{aligned} h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x\\ &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x\\ &= \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y \circ_{\mathcal{A}} f \end{aligned}$$
    $(h \circ g) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f = (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$,
    $h \circ (g \circ f) = h \circ (g \circ_{\mathcal{B}} Sf) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} Sf)$
    $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} Sf$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$,
    $$\begin{aligned} h \circ (g \circ f) = h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) &= h \circ_{\mathcal{A}} \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x\\ &= \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$
    $$\begin{aligned} (h \circ g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ f &= \eta_w^{-1} \circ_{\mathcal{A}} T(Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x\\ &= \eta_w^{-1} \circ_{\mathcal{A}} TSh \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$

– $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ (\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= h \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} (\epsilon_{Sz} \circ_{\mathcal{B}} S\eta_z) \circ_{\mathcal{B}} S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} \epsilon_{Sz} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \circ_{\mathcal{B}} \epsilon_{Sx} \circ_{\mathcal{B}} S\eta_x \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = \eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ f = \eta_w^{-1} \circ_{\mathcal{A}} T(h \circ_{\mathcal{B}} g) \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x \end{aligned}$$

– $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{A}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{A}} g) \circ f = S(h \circ_{\mathcal{A}} g) \circ_{\mathcal{B}} f = (Sh \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}, w \in \mathcal{B}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ (Sg \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (Sg \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} Sg) \circ f = (h \circ_{\mathcal{B}} Sg) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{A}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ f \\ &= S(\eta_w^{-1} \circ_{\mathcal{A}} Th \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{B}} f \\ &= S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= (\epsilon_{Sw} \circ_{\mathcal{B}} S\eta_w) \circ_{\mathcal{B}} S\eta_w^{-1} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= \epsilon_{Sw} \circ_{\mathcal{B}} ST(h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} \epsilon_{Sy} \circ_{\mathcal{B}} S\eta_y \circ_{\mathcal{B}} f \\ &= h \circ_{\mathcal{B}} g \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}, w \in \mathcal{B}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{A}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ (g \circ_{\mathcal{B}} f) = Sh \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (Sh \circ_{\mathcal{B}} g) \circ f = (Sh \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

– $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}, w \in \mathcal{B}$,

$$\begin{aligned} h \circ (g \circ f) &= h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f) \\ (h \circ g) \circ f &= (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f \end{aligned}$$

- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{A}$,
  $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$
- $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}, w \in \mathcal{B}$,
  $h \circ (g \circ f) = h \circ_{\mathcal{B}} (g \circ_{\mathcal{B}} f)$
  $(h \circ g) \circ f = (h \circ_{\mathcal{B}} g) \circ_{\mathcal{B}} f$

- identity law
  - $x \in \mathcal{A}, y \in \mathcal{A}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{A}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{A}} f = f$
  - $x \in \mathcal{A}, y \in \mathcal{B}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} S\mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_{Sx} = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{B}} f = f$
  - $x \in \mathcal{B}, y \in \mathcal{A}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = S\mathrm{id}_y \circ_{\mathcal{B}} f = \mathrm{id}_{Sy} \circ_{\mathcal{B}} f = f$
  - $x \in \mathcal{B}, y \in \mathcal{B}$,
    $f \circ \mathrm{id}_x = f \circ_{\mathcal{B}} \mathrm{id}_x = f$
    $\mathrm{id}_y \circ f = \mathrm{id}_y \circ_{\mathcal{B}} f = f$

4.4. Definition. *Let $\langle S : \mathcal{A} \to \mathcal{B}, T : \mathcal{B} \to \mathcal{A}, \eta : I_{\mathcal{A}} \to TS, \epsilon : ST \to I_{\mathcal{B}} \rangle$ be an adjoint equivalence, let $\mathcal{A} \Cup \mathcal{B}$ be the equivalence fusion. We define the* projections u, v *as follows:*

- $u : \mathcal{A} \Cup \mathcal{B} \longrightarrow A$
  object-function $u : \mathbf{Ob}(\mathcal{A} \Cup \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{A})$
  $$x \longmapsto ux := \begin{cases} x & (x \in \mathcal{A}) \\ Tx & (x \in \mathcal{B}) \end{cases}$$
  hom-functions $u : \mathbf{Hom}(x, y) \longrightarrow \mathcal{A}(ux, uy)$
  $$\langle f, x, y \rangle \longmapsto uf := \begin{cases} f & (x, y \in \mathcal{A}) \\ Tf & (x, y \in \mathcal{B}) \\ Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B}) \\ \eta_y^{-1} \circ_{\mathcal{A}} Tf & (x \in \mathcal{B}, y \in \mathcal{A}) \end{cases}$$

- $v : \mathcal{A} \Cup \mathcal{B} \longrightarrow B$
  object-function $v : \mathbf{Ob}(\mathcal{A} \Cup \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{B})$
  $$x \longmapsto vx := \begin{cases} Sx & (x \in \mathcal{A}) \\ x & (x \in \mathcal{B}) \end{cases}$$
  hom-functions $v : \mathbf{Hom}(x, y) \longrightarrow \mathcal{B}(ux, uy)$
  $$\langle f, x, y \rangle \longmapsto vf := \begin{cases} Sf & (x, y \in \mathcal{A}) \\ f & (\text{others}) \end{cases}$$

4.5. Proposition. *The projections u, v are functors.*

Proof. We show that u, v preserve composition of morphisms and identity morphism by case analysis.

- u preserves composition of morphisms
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A}$,
    $u(g \circ f) = u(g \circ_{\mathcal{A}} f) = g \circ_{\mathcal{A}} f$
    $ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} f$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} Sf) = T(g \circ_{\mathcal{B}} Sf) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x$
    $ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} f = Tg \circ_{\mathcal{A}} TSf \circ_{\mathcal{A}} \eta_x$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A}$,
    $u(g \circ f) = u(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
    $ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x)$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) \circ_{\mathcal{A}} \eta_x = Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x$
    $ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} (Tf \circ_{\mathcal{A}} \eta_x)$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A}$,
    $u(g \circ f) = u(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(Sg \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{B}} Tf$
    $ug \circ_{\mathcal{A}} uf = g \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = \eta_z^{-1} \circ_{\mathcal{A}} TSg \circ_{\mathcal{A}} Tf$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf$
    $ug \circ_{\mathcal{A}} uf = (Tg \circ_{\mathcal{A}} \eta_y) \circ_{\mathcal{A}} (\eta_y^{-1} \circ_{\mathcal{A}} Tf) = Tg \circ_{\mathcal{A}} Tf$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} T(g \circ_{\mathcal{B}} f) = \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf$
    $ug \circ_{\mathcal{A}} uf = (\eta_z^{-1} \circ_{\mathcal{A}} Tg) \circ_{\mathcal{A}} Tf$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B}$,
    $u(g \circ f) = u(g \circ_{\mathcal{B}} f) = T(g \circ_{\mathcal{B}} f) = Tg \circ_{\mathcal{A}} Tf$
    $ug \circ_{\mathcal{A}} uf = Tg \circ_{\mathcal{A}} Tf$
- u preserves identity morphisms
  - $x \in \mathcal{A}$,
    $u(\mathrm{id}_x) = \mathrm{id}_x = \mathrm{id}_{ux}$
  - $x \in \mathcal{B}$,
    $u(\mathrm{id}_x) = T\mathrm{id}_x = \mathrm{id}_{Tx} = \mathrm{id}_{ux}$
- v preserves composition of morphisms

  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{A},$
    $v(g \circ f) = v(g \circ_{\mathcal{A}} f) = S(g \circ_{\mathcal{A}} f) = Sg \circ_{\mathcal{B}} Sf$
    $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{A}} Sf$
  - $x \in \mathcal{A}, y \in \mathcal{A}, z \in \mathcal{B},$
    $v(g \circ f) = v(g \circ_{\mathcal{B}} Sf) = g \circ_{\mathcal{B}} Sf$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} Sf$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{A},$
    $$\begin{aligned} v(g \circ f) &= v(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= S(\eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x) \\ &= S\eta_z^{-1} \circ_{\mathcal{B}} ST(g \circ_{\mathcal{B}} f) \circ_{\mathcal{B}} S\eta_x \\ &= g \circ_{\mathcal{B}} f \end{aligned}$$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
  - $x \in \mathcal{A}, y \in \mathcal{B}, z \in \mathcal{B},$
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{A},$
    $v(g \circ f) = v(Sg \circ_{\mathcal{B}} f) = Sg \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = Sg \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{A}, z \in \mathcal{B},$
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{A},$
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$
  - $x \in \mathcal{B}, y \in \mathcal{B}, z \in \mathcal{B},$
    $v(g \circ f) = v(g \circ_{\mathcal{B}} f) = g \circ_{\mathcal{B}} f$
    $vg \circ_{\mathcal{B}} vf = g \circ_{\mathcal{B}} f$

- v preserves identity morphisms
  - $x \in \mathcal{A}$
    $v(\mathrm{id}_x) = S\mathrm{id}_x = \mathrm{id}_{Sx} = \mathrm{id}_{vx}$
  - $x \in \mathcal{B}$
    $v(\mathrm{id}_x) = \mathrm{id}_x \ \mathrm{id}_{vx}$

4.6. Proposition. *The projections u, v are surjective on objects, full and faithful.*

Proof. It's trivial by definitions that u, v are surjective on objects. So we check fullness and faithfulness.

- u is full and faithful

- $x, y \in \mathcal{A}$,
  $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{A}(x, y)$ is bijective.
- $x, y \in \mathcal{B}$,
  $T : \mathcal{B}(x, y) \to \mathcal{A}(Tx, Ty)$ is bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{A}(x, y) \ni \langle f, x, y\rangle \mapsto Tf \in \mathcal{A}(Tx, Ty) = \mathcal{A}(ux, uy)$ is bijective.
- $x \in \mathcal{A}, y \in \mathcal{B}$,
  $\mathcal{B}(Sx, y) \ni f \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty)$ is the right adjunct of each f, and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y\rangle \mapsto Tf \circ_{\mathcal{A}} \eta_x \in \mathcal{A}(x, Ty) = \mathcal{A}(ux, uy)$ is bijective.
- $x \in \mathcal{B}, y \in \mathcal{A}$,
  $\mathcal{B}(x, Sy) \ni f \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y)$ is the left adjunct of each f, and bijective. Therefore $u : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y\rangle \mapsto \eta_y^{-1} \circ_{\mathcal{A}} Tf \in \mathcal{A}(Tx, y) = \mathcal{A}(ux, uy)$

• v is full and faithful

- $x, y \in \mathcal{A}$,
  $S : \mathcal{A}(x, y) \to \mathcal{B}(Sx, Sy)$ is bijective. Therefore $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{A}(x, y)\} \ni \langle f, x, y\rangle \mapsto Sf \in \mathcal{B}(Sx, Sy) = \mathcal{B}(vx, vy)$ is bijective.
- $x, y \in \mathcal{B}$,
  $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(x, y) = \mathcal{B}(vx, vy)$ is bijective.
- $x \in \mathcal{A}, y \in \mathcal{B}$,
  $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(Sx, y)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(Sx, y) = \mathcal{B}(vx, vy)$ is bijective.
- $x \in \mathcal{B}, y \in \mathcal{A}$,
  $v : \mathbf{Hom}(x, y) = \{\langle f, x, y\rangle \mid f \in \mathcal{B}(x, Sy)\} \ni \langle f, x, y\rangle \mapsto f \in \mathcal{B}(x, Sy) = \mathcal{B}(vx, vy)$ is bijective.

4.7. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. $\mathcal{A}$ is ordinary equivalent to $\mathcal{B}$ if and only if $\mathcal{A}$ is span equivalent to $\mathcal{B}$.*

Proof. Let $\mathcal{A}$ be ordinary equivalent to $\mathcal{B}$, then $\mathcal{A}$ is adjoint equivalent to $\mathcal{B}$. Thus there exists a adjoint equivalence between $\mathcal{A}$ and $\mathcal{B}$. So we can construct the equivalence fusion and the projections. By Propositions, they are span equivalence. Therefore $\mathcal{A}$ is span equivalent to $\mathcal{B}$.
On the other hand, let $\mathcal{A}$ be span equivalent to $\mathcal{B}$. Then there exists a span equivalence $\langle \mathcal{C}, u, v\rangle$ between $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{C}$ is ordinary equivalent to both $\mathcal{A}$ and $\mathcal{B}$. Therefore $\mathcal{A}$ is ordinary equivalent to $\mathcal{B}$.

4.8. Remark. Let $\mathcal{A}$ be presheaf category. The forgetful functor

$$U : \mathcal{A}\text{-}\mathbf{Cat} \longrightarrow \mathcal{A}\text{-}\mathbf{Gph}$$

is monadic. (Proposition F 1.1 in [Leinster 2004])

Let $\mathcal{A} = \mathbf{Set}$, we can see $\mathbf{Set}\text{-}\mathbf{Cat} = \mathbf{Cat}$, $\mathbf{Set}\text{-}\mathbf{Grp} = 1\text{-}\mathbf{GSet}$. and the induced monad T_1 is the free strict 1-category monad on $1\text{-}\mathbf{GSet}$. by the remark, the comparison functor

$$K : \mathbf{Cat} \longrightarrow T_1\text{-}\mathbf{Alg}$$

is isomorphic and arrow part of the functor is

$$K : f \longmapsto Uf.$$

Moreover, the category $\mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$ of Leinster's weak 1 category is the category $T_1\text{-}\mathbf{Alg}$ of algebras for the monad. (As for details, refer to the proof of Theorem 9.1.4 in [Leinster 2004].) So the isomorphism $K : \mathbf{Cat} \to \mathbf{Wk}\text{-}1\text{-}\mathbf{Cat}$ preserve surjectivity, fullness and faithfullness. Hence,

4.9. Theorem. *Let $K : \mathbf{Cat} \to \mathbf{Wk\text{-}1\text{-}Cat}$ be the isomorphism above. let $\mathcal{A}$ and $\mathcal{B}$ be categories. $\mathcal{A}$ is span equivalent to $\mathcal{B}$ in $\mathbf{Cat}$ if and only if $K(\mathcal{A})$ is span equivalent to $K(\mathcal{B})$ in $\mathbf{Wk\text{-}1\text{-}Cat}$.*

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