Span equivalence between weak n-categories

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1 Background

- In this talk, we suppose that a weak n-category is defined as an algebra for "special" n-globular operad.
- An n-globular operad is an n-globular set whose elements are operations that has a (globular) pasting diagram as its arity.
- An n-globular operad induces a monad. the algebra for n-globular operad is the algebra for the monad.

2 Motivation

- It is interesting question what equivalence is "good" for weak n-categories.

Definition. "Cottrell's equivalence"
Let K be an n-globular operad. K-algebras $KX \to X$ and $KY \to Y$ are equivalent if there exists a map of K-algebras $u : X \to Y$ or $u : Y \to X$ such that u is surjective on 0-cells, full on m-cells for all $1 \leq m \leq n$, and faithful on n-cells.

- Cottrell's equivalence is defined for his first coherence theorem (every free K-algebra is equivalent to a free strict n-caregory).
- and he says

"this definition of equivalence is much more strict (and thus much less general) than it ought to be. ... If we required a more general definition of equivalence of K-algebra, ... Another option is to replace the map u with a span of maps of K-algebra, ..." [Cottrell 2015]

DEFINITION. "Span equivalence"
Let$K$ be an n-globular operad. K-algebras $KX \to X$ and $KY \to Y$ are *span equivalent* if there exists a triple $\langle \psi, u, v \rangle$ such that $\psi : KZ \to Z$ is K-algebra,$u : Z \to X$ and $v : Z \to Y$ are maps of K-algebras, surjective on 0-cells, full on m-cells for all $1 \leq m \leq n$, and faithful on n-cells.

- What properties span equivalence satisfies.

3 Main Theorems

THEOREM.1

Span equivalence is equivalence relation on K-algebras

THEOREM.2

ordinary equivalent $\Longleftrightarrow$ span equivalent in $\mathbf{Cat}$

4 Another form for Theorem.2

THEOREM.2
ordinary equivalent $\Longleftrightarrow$ span equivalent in $\mathbf{Cat}$

DEFINITION Let $\mathcal{A}$ and $\mathcal{B}$ be categories. We say that $\mathcal{A}$ and $\mathcal{B}$ are *span equivalent* if there exists a triple $\langle \mathcal{C}, u, v \rangle$ such that $\mathcal{C}$ is a category, $u : \mathcal{C} \to \mathcal{A}$ and $v : \mathcal{C} \to \mathcal{B}$ are functors, surjective on objects, full and faithful.

PROPOSITION Let A be presheaf category. The forgetful functor

$$U : \mathcal{A}\text{-}\mathbf{Cat} \longrightarrow \mathcal{A}\text{-}\mathbf{Gph}$$

is monadic. (prop F.1.1 in [Leinster 2004a])

Let $\mathcal{A} = \mathbf{Set}$, the induced monad T_1 is the free strict 1-category monad, $\mathbf{Set}\text{-}\mathbf{Gph}$ is the category of 1-globular sets. Hence

$$T_1\text{-}\mathbf{Alg} \cong \mathbf{Cat}.$$

Morever, the category of weak 1-category $\mathbf{Wk}1\text{-}\mathbf{Cat}$ is isomorphic to $T_1\text{-}\mathbf{Alg}$. Thus

$$\mathbf{Wk}1\text{-}\mathbf{Cat} \cong T_1\text{-}\mathbf{Alg} \cong \mathbf{Cat}.$$

Let $K : \mathbf{Cat} \to \mathbf{Wk}1\text{-}\mathbf{Cat}$ be the isomorphism, C and D be categories. C is span equivalent to D if and only if $K(C)$ is span equivalent to $K(D)$.

Theorem.2 can be stated in another way

THEOREM.2' Let C and D be categories. C is ordinary equivalent to D if and only if $K(C)$ is span equivalent to $K(D)$.

5 Preparation of Theorem.1

THEOREM.1

Span equivalence is equivalence relation on K-algebras

- It is trivial that span equivalence is reflexive and symmetric relation.
- In order to show that span equivalence is transitive relation, We show that some properties of map of n-globular sets are stable under pullback.

Let $f : X \to Y$ be a map of n-globular sets.

- f:surjective (resp. injective) on k-cells
  :$\Longleftrightarrow f_k : X_k \to Y_k$:surjective (resp. injective)
- f:full on k-cells
  :$\Longleftrightarrow \forall x, x' \in X_{k-1}, g \in \mathbf{Hom}_Y(f(x), f(x')),$
  $$\exists h \in \mathbf{Hom}_X(x, x')\text{s.t.} f(h) = g$$
- f:faithful on k-cells
  :$\Longleftrightarrow \forall x, x' \in X_{k-1}, g, g' \in \mathbf{Hom}_X(x, x'),$
  $$\text{“}g \neq g' \Rightarrow f(g) \neq f(g')\text{”}$$

PROPOSITION. the pullback diagram in $[\mathbf{G}_n^{\mathrm{op}}, \mathbf{Set}]$

$$\begin{tikzcd} P \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

- f:surjective on 0-cell, full on k-cell for $1 \leq m \leq n \Rightarrow j$:surjective on 0-cell, full on m-cell for $1 \leq m \leq n$.
- f:faithful on k-cell $\Rightarrow j$:faithful on k-cell.

Corollary. let

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & & & Q \arrow[dl] \arrow[dr] & \\ X & & Y, & Y & & Z \end{tikzcd}$$

be span equivalences, then

$$\begin{tikzcd} & & R \arrow[dl] \arrow[dr] & & \\ & P \arrow[dl] \arrow[dr] & & Q \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

is span equivalence.

6 Proof of Theorem.1

THEOREM.1
Span equivalence is equivalence relation on K-algebras

Proof. By definition, it is trivial that span equivalence is reflexive and symmetric relation. By corollary, span equivalence is transitive relation.

7 Preparation of Theorem.2

THEOREM.2

ordinary equivalent $\Longleftrightarrow$ span equivalent in $\mathbf{Cat}$

- In order to prove theorem.2. We construct a triple $\langle C, u, v\rangle$ of adjoint equivalence. (C is span equivalence, u and v are its projections.)
- We show that the triple $\langle C, u, v\rangle$ is span equivalence in **Cat**.

8 Equivalence Fusion

DEFINITION. Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and $\langle S, T, \eta, \epsilon \rangle$ be adjoint equivalence between $\mathcal{A}$ and $\mathcal{B}$. We define a category $\mathcal{A} \Perp \mathcal{B}$, *equivalence fusion*,as follows:

- object-set

$$\mathbf{Ob}(\mathcal{A} \Perp \mathcal{B}) := \mathbf{Ob}(\mathcal{A}) \bigsqcup \mathbf{Ob}(\mathcal{B}) \ \ \text{(disjoint)}$$

- hom-set

$$\mathbf{Hom}(x,y) := \begin{cases} \{\langle f,x,y\rangle \mid f \in \mathcal{A}(x,y)\} & (x, y \\ \{\langle f,x,y\rangle \mid f \in \mathcal{B}(x,y)\} & (x, y \\ \{\langle f,x,y\rangle \mid f \in \mathcal{B}(Sx,y)\} & (x \in \\ \{\langle f,x,y\rangle \mid f \in \mathcal{B}(x,Sy)\} & (x \in \end{cases}$$

- identities

$$\mathrm{id}_x := \begin{cases} \langle \mathrm{id}_x, x, x\rangle & (x \in \mathcal{A}, \mathrm{id}_x \in \mathcal{A}(x,x)) \\ \langle \mathrm{id}_x, x, x\rangle & (x \in \mathcal{B}, \mathrm{id}_x \in \mathcal{B}(x,x)) \end{cases}$$

- composition

$$\langle\langle g, y, z\rangle, \langle f, x, y\rangle\rangle \mapsto \langle g\tilde{\circ} f, x, z\rangle$$

$$g\tilde{\circ} f := \begin{cases} g \circ_{\mathcal{A}} f & (x, y, z \in \mathcal{A}) \\ g \circ_{\mathcal{B}} f & (x, y, z \in \mathcal{B}) \\ g \circ_{\mathcal{B}} Sf & (x, y \in \mathcal{A}, z \in \\ g \circ_{\mathcal{B}} f & (x \in \mathcal{A}, y, z \in \\ g \circ_{\mathcal{B}} f & (x, y \in \mathcal{B}, z \in \\ Sg \circ_{\mathcal{B}} f & (x \in \mathcal{B}, y, z \in \\ \eta_z^{-1} \circ_{\mathcal{A}} Tg \circ_{\mathcal{A}} Tf \circ_{\mathcal{A}} \eta_x & (x, z \in \mathcal{A}, y \in \\ g \circ_{\mathcal{B}} f & (x, z \in \mathcal{B}, y \in \end{cases}$$

DEFINITION. Let $\mathcal{A} \uplus \mathcal{B}$ be equivalence fusion, the projection u, v are defined by followings.

- $u : \mathcal{A} \uplus \mathcal{B} \longrightarrow \mathcal{A}$
  object-function $u : \mathbf{Ob}(\mathcal{A} \uplus \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{A})$

$$x \longmapsto ux := \begin{cases} x & (x \in \mathcal{A}) \\ Tx & (x \in \mathcal{B}) \end{cases}$$

hom-functions $u : \mathbf{Hom}(x, y) \longrightarrow \mathcal{A}(ux, uy)$

$$\langle f, x, y \rangle \longmapsto uf := \begin{cases} f & (x, y \in \mathcal{A}) \\ Tf & (x, y \in \mathcal{B}) \\ Tf \circ_{\mathcal{A}} \eta_x & (x \in \mathcal{A}, y \in \mathcal{B} \\ \eta_y^{-1} \circ_{\mathcal{A}} Tf & (x \in \mathcal{B}, y \in \mathcal{A} \end{cases}$$

- $u : \mathcal{A} \uplus \mathcal{B} \longrightarrow \mathcal{B}$

  object-function $v : \mathbf{Ob}(\mathcal{A} \uplus \mathcal{B}) \longrightarrow \mathbf{Ob}(\mathcal{B})$

$$x \longmapsto vx := \begin{cases} Sx & (x \in \mathcal{A}) \\ x & (x \in \mathcal{B}) \end{cases}$$

  hom-functions $v : \mathbf{Hom}(x, y) \longrightarrow \mathcal{B}(ux, uy)$

$$\langle f, x, y \rangle \longmapsto vf := \begin{cases} Sf & (x, y \in \mathcal{A}) \\ f & (\text{others}) \end{cases}$$

Proposition.
An equivalence fusion forms a category.

Proposition.
u, v are functors, surjective on objects, full and faithful.

9 Proof of Theorem.2

THEOREM.2

ordinary equivalent $\Longleftrightarrow$ span equivalent in $\mathbf{Cat}$

Proof. $(\Leftarrow)$ Let $\mathcal{A}$ and $\mathcal{B}$ are span equivalent, $\langle \mathcal{C}, u, v \rangle$ be the span equivalence. Then u, v are equivalence. So the categories are equivalent.
$(\Rightarrow)$ Let $\mathcal{A}$ and $\mathcal{B}$ are equivalent, thus are adjoint equivalent. Then we can construct equivalence fusion. By propositions, $\mathcal{A}$ and $\mathcal{B}$ are span equivalent.

10 Reference

[Cottrell 2015] Thomas Cottrell. OPERADIC DEFINITIONS OF WEAK N-CATEGORY: COHERENCE AND COMPARISONS. Theory and Applications of Categories, Vol.30, No13, 2015, pp. 433-488.

[Leinster 2004a] Tom Leinster. *Higher operads, higher categories.* Volume 298 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2004.