当 (4.20) 中模糊集取 $\mathcal{F} \triangleq \mathcal{F}_{MV}$ 且其中的 $\boldsymbol{\mu} \triangleq \mathbf{0}$ 时，Ghaoui et al. (2003) 的研究表明，(4.20) 等价于

$$y^0(\boldsymbol{x}) + \sqrt{\frac{1-\epsilon}{\epsilon}}\sqrt{\boldsymbol{y}(\boldsymbol{x})^\top \boldsymbol{\Sigma} \boldsymbol{y}(\boldsymbol{x})} \leq 0. \tag{4.21}$$

这里，令 $\boldsymbol{\mu} \triangleq \mathbf{0}$ 并不失一般性，因为如果 $\boldsymbol{\mu} \neq \mathbf{0}$ 则可通过变量替换的方法，令 $\tilde{\boldsymbol{\xi}} \triangleq \tilde{\boldsymbol{\varepsilon}} - \boldsymbol{\mu}$ 并将 $\tilde{\boldsymbol{\xi}}$ 视为 (4.20) 中的 $\tilde{\boldsymbol{\varepsilon}}$。有趣的是，(4.21) 恰好等价于传统鲁棒优化约束式

$$y^0(\boldsymbol{x}) + \boldsymbol{y}(\boldsymbol{x})^\top \boldsymbol{\varepsilon} \leq 0 \quad \forall \boldsymbol{\varepsilon} \in \Xi(\epsilon), \tag{4.22}$$

其中，不确定集为椭球形，给定为

$$\Xi(\epsilon) \triangleq \left\{ \boldsymbol{\varepsilon} \in \mathbb{R}^J \,\middle|\, \|\boldsymbol{\Sigma}^{1/2}\boldsymbol{\varepsilon}\|_2 \leq \sqrt{\frac{1-\epsilon}{\epsilon}} \right\}.$$

关于 (4.21) 与 (4.22) 的关系，可参见 Natarajan et al. (2009)。

当 (4.20) 中模糊集取 $\mathcal{F} \triangleq \mathcal{F}_{MVS}$ 时，分布鲁棒机会约束规划一般不可处理。研究者们提出了基于**条件风险值** (Conditional Value-at-Risk) 的近似方法进行处理，感兴趣的读者可参见 Chen et al. (2010); Zymler et al. (2013) 等。

当 $\mathcal{F} \triangleq \mathcal{F}_{WKS}$ 且其中 $K = 1, \underline{p}_1 = \overline{p}_1 = 1$ 时，Hanasusanto et al. (2015) 证明了 (4.20) 等价于如下一系列锥优化约束：

$$\begin{aligned}
&\beta + \boldsymbol{b}^\top \boldsymbol{\gamma} \geq (1-\epsilon)\tau, && \beta + \boldsymbol{c}_1^\top \boldsymbol{\phi} \leq \tau, \quad \beta + \boldsymbol{c}_1^\top \boldsymbol{\psi} \leq -y^0(\boldsymbol{x}),\\
&\boldsymbol{A}^\top \boldsymbol{\gamma} = \boldsymbol{C}_1^\top \boldsymbol{\phi}, && \boldsymbol{B}^\top \boldsymbol{\gamma} = \boldsymbol{D}_1^\top \boldsymbol{\phi},\\
&\boldsymbol{A}^\top \boldsymbol{\gamma} + \boldsymbol{y}(\boldsymbol{x}) = \boldsymbol{C}_1^\top \boldsymbol{\psi}, && \boldsymbol{B}^\top \boldsymbol{\gamma} = \boldsymbol{D}_1^\top \boldsymbol{\psi},\\
&\beta \in \mathbb{R}, \boldsymbol{\gamma} \in \mathbb{R}^L, \tau \in \mathbb{R}_+, && \boldsymbol{\phi}, \boldsymbol{\psi} \in \mathcal{K}_1^*
\end{aligned}$$

其中 $\mathcal{K}_1^*$ 表示 $\mathcal{K}_1$ 的对偶锥 (Ben-Tal and Nemirovski, 2001)。考虑到 $\mathcal{F}_M$ 和 $\mathcal{F}_{DY}$ 均为 $\mathcal{F}_{WKS}$ 的特例，当 $\mathcal{F} \triangleq \mathcal{F}_M$ (其中支撑集为 (4.10) 的形式) 或 $\mathcal{F} \triangleq \mathcal{F}_{DY}$ 时，(4.4) 可等价转化成锥优化约束形式。

作为 Hanasusanto et al. (2015) 的扩展，Xie and Ahmed (2018) 考虑了更一般的模糊集，研究了分布鲁棒独立机会约束规划和联合机会约束规划的等价凸优化形式。对于考虑均值、散度上界、支撑集的一类模糊集，Hanasusanto et al. (2017) 研究了其分布鲁棒联合机会约束规划的计算复杂度及求解方法。

当 (4.20) 中模糊集取 $\mathcal{F} \triangleq \mathcal{F}_{KL}$ 时，Jiang and Guan (2016) 的研究表明，(4.20) 等价于

$$\hat{\mathbb{P}}[y^0(\boldsymbol{x}) + \boldsymbol{y}(\boldsymbol{x})^\top \tilde{\boldsymbol{\varepsilon}} \leq 0] \geq 1 - \bar{\epsilon}. \tag{4.23}$$

其中，

$$\bar{\epsilon} \triangleq 1 - \inf_{t \in (0,1)} \frac{e^{-\theta} t^{1-\epsilon} - 1}{t - 1}.$$

由此可见，它与随机规划中基于采样平均近似 (sample average approximation) 的机会约束式 (4.24) 相比，仅仅是具有不同的概率界 $\bar{\epsilon}$ 而已。此外，对于一般的 ϕ 散度形式

下分布鲁棒联合机会约束规划的处理方法，可详见 Jiang and Guan (2016)。

$$\hat{\mathbb{P}}[y^0(\boldsymbol{x}) + \boldsymbol{y}(\boldsymbol{x})^\top \tilde{\boldsymbol{\varepsilon}} \le 0] \ge 1 - \epsilon. \tag{4.24}$$

作为常用技巧，(4.24) 可通过引入 0-1 辅助决策变量的方法等价转换为

$$\begin{aligned}
& y^0(\boldsymbol{x}) + \boldsymbol{y}(\boldsymbol{x})^\top \hat{\boldsymbol{\varepsilon}}_\omega \le M_0(1 - z_\omega) \quad \forall \omega \in [N], \\
& \frac{1}{N} \sum_{\omega \in [N]} z_\omega \ge 1 - \epsilon, \\
& \boldsymbol{z} \in \{0, 1\}^N.
\end{aligned} \tag{4.25}$$

其中，M_0 为一个足够大的实数。观察可知，当 $y^0(\boldsymbol{x}) + \boldsymbol{y}(\boldsymbol{x})^\top \hat{\boldsymbol{\varepsilon}}_\omega \le 0$ 时，z_ω 可取值 1，代表该约束在第 ω 个场景中成立，否则不得不取值 0。

当 $\mathcal{F} \triangleq \mathcal{F}_W$ 时，Chen et al. (2018); Xie (2019) 讨论了如何处理分布鲁棒（独立和联合）机会约束规划问题，他们用不同的方法得到了相同的结论。此时，独立机会约束 (4.20) 等价于如下混合 0-1 锥优化约束

$$\begin{aligned}
& \epsilon N t - \boldsymbol{e}^\top \boldsymbol{s} \ge \theta N \|\boldsymbol{y}(\boldsymbol{x})\|_*, \\
& -\boldsymbol{y}(\boldsymbol{x})^\top \hat{\boldsymbol{\varepsilon}}_\omega - y^0(\boldsymbol{x}) + M_0 z_\omega \ge t - s_\omega \quad && \forall \omega \in [N], \\
& M_0(1 - z_\omega) \ge t - s_\omega && \forall \omega \in [N], \\
& t \in \mathbb{R}, \boldsymbol{z} \in \{0, 1\}^N, \boldsymbol{s} \in \mathbb{R}^N.
\end{aligned}$$

其中 $\boldsymbol{e}$ 代表长度为 N、元素全为 1 的向量，$\|\cdot\|_*$ 为 (4.17) 中 $\|\cdot\|$ 对应的对偶范数 (Boyd et al., 2004)。

4.3 分布鲁棒线性优化（Distributionally robust linear optimization）

现实世界中很多优化问题可建模或近似为线性规划问题。线性约束不仅本身可描述许多现实问题的资源约束，而且可建模或近似更复杂的资源约束。作为抛砖引玉，本节讨论如何处理分布鲁棒线性优化约束式 (4.5)，读者可进而自行研究更一般也更难处理的非线性约束，例如：

$$\mathbb{E}_{\mathbb{P}}\Big[\max_{k \in [K]} \{y_k^0(\boldsymbol{x}) + \boldsymbol{y}_k(\boldsymbol{x})^\top \tilde{\boldsymbol{\varepsilon}}\}\Big] \le 0 \quad \forall \mathbb{P} \in \mathcal{F}, \tag{4.26}$$

其左端项为关于 $\boldsymbol{x}$ 和 $\tilde{\boldsymbol{\varepsilon}}$ （各自）的分段线性凸函数；此形式出现在许多管理科学问题中，如库存管理 (See and Sim, 2010; Mamani et al., 2017)、预约调度 (Mak et al., 2015; Kong et al., 2013; Qi, 2017)、带时间窗的车辆路径问题 (Zhang et al., 2019)，等等。

接下来探讨如何处理分布鲁棒线性优化约束式 (4.5)。易知，它等价于

$$\sup_{\mathbb{P} \in \mathcal{F}} \mathbb{E}_{\mathbb{P}}[y^0(\boldsymbol{x}) + \boldsymbol{y}(\boldsymbol{x})^\top \tilde{\boldsymbol{\varepsilon}}] \le 0. \tag{4.27}$$

处理该约束的关键是考察左端项中优化问题

$$Z_P(\boldsymbol{x}) = \sup_{\mathbb{P}\in\mathcal{F}} \mathbb{E}_{\mathbb{P}}[\boldsymbol{y}(\boldsymbol{x})^\top \tilde{\boldsymbol{\varepsilon}}] \tag{4.28}$$

的对偶问题。注意，该优化问题中 $\boldsymbol{x}$ 被视为给定参数，而概率分布 $\mathbb{P}$ 才是决策变量。抽象地，(4.28) 的对偶问题形式为

$$Z_D(\boldsymbol{x}) = \inf_{\boldsymbol{p}\in\mathcal{P}(\boldsymbol{x})} f(\boldsymbol{p}). \tag{4.29}$$

其中 $\boldsymbol{p}$ 为对偶决策变量，$\mathcal{P}(\boldsymbol{x})$ 为其可行域，$f(\boldsymbol{p})$ 为目标函数，$\boldsymbol{y}(\boldsymbol{x})$ 作为参数被包含于 $\mathcal{P}(\boldsymbol{x})$ 中。在某些条件下，强对偶定理对此成立，则 $Z_P = Z_D$。于是，(4.27) 等价于

$$\begin{aligned} & y^0(\boldsymbol{x}) + f(\boldsymbol{p}) \leq 0, \\ & \boldsymbol{p} \in \mathcal{P}(\boldsymbol{x}). \end{aligned} \tag{4.30}$$

因此，技术上主要关注如何在取不同模糊集 $\mathcal{F}$ 的情况下求解 (4.28) 的对偶问题并证明强对偶定理成立。

当 $\mathcal{F} \triangleq \mathcal{F}_{MV}$ 或 $\mathcal{F} \triangleq \mathcal{F}_{MVS}$ 时，由于已知 $\tilde{\boldsymbol{\varepsilon}}$ 的均值 $\boldsymbol{\mu}$，故 (4.28) 等价于 $Z_P(\boldsymbol{x}) = \boldsymbol{y}(\boldsymbol{x})^\top \boldsymbol{\mu}$。Popescu (2007) 针对 $\mathcal{F} \triangleq \mathcal{F}_{MV}$ 且 (4.28) 目标函数变为某一类非线性函数的情形，研究了其等价模型与求解方法。

当 (4.28) 中 $\mathcal{F} \triangleq \mathcal{F}_{DY}$ 时，则在某些技术性条件下，Delage and Ye (2010) 推导出其对偶问题 (4.29) 的具体形式：

$$\begin{aligned} Z_D(\boldsymbol{x}) = \min_{\boldsymbol{Q},\boldsymbol{q},r,t} \quad & r + t \\ \text{s.t.} \quad & r \geq \boldsymbol{y}(\boldsymbol{x})^\top \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^\top \boldsymbol{Q} \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^\top \boldsymbol{q} \quad \forall \boldsymbol{\varepsilon} \in \Xi, \\ & t \geq (\gamma_2 \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^\top) \bullet \boldsymbol{Q} + \boldsymbol{\mu}^\top \boldsymbol{q} + \sqrt{\gamma_1} \|\boldsymbol{\Sigma}^{1/2}(\boldsymbol{q} + 2\boldsymbol{Q}\boldsymbol{\mu})\|, \\ & \boldsymbol{Q} \succeq \boldsymbol{0}, \end{aligned} \tag{4.31}$$

其中 “$\bullet$” 表示矩阵间的弗罗贝尼乌斯内积。注意到 (4.31) 中的第一个约束实则为（传统）鲁棒优化约束，因此求解 $\mathcal{F}_{DY}$ 模糊集下的分布鲁棒优化问题 (4.28) 等价于求解鲁棒优化问题 (4.31)，而上一章已讲述如何求解鲁棒优化问题。

当 (4.28) 中 $\mathcal{F} \triangleq \mathcal{F}_{WKS}$ 时，则在某些技术性条件下，Wiesemann et al. (2014) 推导出其对偶问题 (4.29) 的具体形式：

$$\begin{aligned} Z_D(\boldsymbol{x}) = \min_{\boldsymbol{\beta},\boldsymbol{\eta},\boldsymbol{\lambda},\boldsymbol{\phi}} \quad & \boldsymbol{b}^\top \boldsymbol{\beta} + \sum_{k\in[K]} \overline{p}_k \eta_k - \underline{p}_k \lambda_k \\ \text{s.t.} \quad & \boldsymbol{c}_k^\top \boldsymbol{\phi}_k \leq \sum_{k'\in\mathcal{A}(k)} (\eta_{k'} - \lambda_{k'}) && \forall k \in [K], \\ & \boldsymbol{C}_k^\top \boldsymbol{\phi}_k + \boldsymbol{A}^\top \boldsymbol{\beta} = \boldsymbol{y}(\boldsymbol{x}) && \forall k \in [K], \\ & \boldsymbol{D}_k^\top \boldsymbol{\phi}_k + \boldsymbol{B}^\top \boldsymbol{\beta} = \boldsymbol{0} && \forall k \in [K], \\ & \boldsymbol{\phi}_k \in \mathcal{K}_k^* && \forall k \in [K], \end{aligned} \tag{4.32}$$

其中，$\mathcal{A}(k) \triangleq \{k' \in [K] \mid \Xi_{k'}$严格包含于$\Xi_k\}$，$\mathcal{K}_k^*$ 表示 $\mathcal{K}$ 的对偶锥。

当 (4.28) 中 $\mathcal{F} \triangleq \mathcal{F}_{KL}$ 时，则在某些技术性条件下，Hu and Hong (2013) 推导出其对偶问题 (4.29) 的具体形式：

$$Z_D(\boldsymbol{x}) = \min_{\alpha \geq 0} \quad \alpha \log \mathbb{E}_{\hat{\mathbb{P}}}[e^{\boldsymbol{y}(\boldsymbol{x})^\top \tilde{\boldsymbol{\varepsilon}}^\dagger / \alpha}] + \alpha\theta. \tag{4.33}$$

其中的目标函数为凸函数，因此可用内点法 (Ben-Tal et al., 2013) 或分段线性函数逼近 (Long and Qi, 2014) 等方法进行处理。

当 (4.28) 中 $\mathcal{F} \triangleq \mathcal{F}_W$ 且其中的支撑集为 $\Xi \triangleq \{\boldsymbol{\varepsilon} \in \mathbb{R}^I \mid \boldsymbol{C}\boldsymbol{\varepsilon} \leq \boldsymbol{d}\}$ 时，则在某些技术性条件下，Esfahani and Kuhn (2018) 推导出其对偶问题 (4.29) 的具体形式：

$$\begin{aligned} Z_D(\boldsymbol{x}) = \inf_{\lambda, s, \boldsymbol{\gamma}} \quad & \lambda\theta + \frac{1}{N} \sum_{\omega \in [N]} s_\omega \\ \text{s.t.} \quad & \boldsymbol{y}(\boldsymbol{x})^\top \hat{\boldsymbol{\varepsilon}}_\omega + \boldsymbol{\gamma}_\omega^\top (\boldsymbol{d} - \boldsymbol{C}\hat{\boldsymbol{\varepsilon}}_\omega) \leq s_\omega \quad && \forall \omega \in [N], \\ & \|\boldsymbol{C}^\top \boldsymbol{\gamma}_\omega - \boldsymbol{y}(\boldsymbol{x})\|_* \leq \lambda && \forall \omega \in [N], \\ & \boldsymbol{\gamma}_\omega \geq \boldsymbol{0} && \forall \omega \in [N]. \end{aligned} \tag{4.34}$$

将以上各结果嵌入到 (4.30) 即可得到 (4.27) 的等价形式，进而求解鲁棒线性规划问题。

事实上，Popescu (2007); Delage and Ye (2010); Wiesemann et al. (2014); Esfahani and Kuhn (2018) 的研究均解决了 (4.26) 的等价转化问题，而上述结论针对 (4.27)，只是 (4.26) 中 $K = 1$ 时的特例，感兴趣的读者可细读他们的论文。

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