Costly word-of-mouth learning in networks☆

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Abstract

This paper develops a framework for word-of-mouth learning in networks where agents strategically decide when to take an irreversible action. Agents face a trade-off between taking an irreversible action early enough to avoid the cost of waiting and waiting to receive more information to increase confidence in their choice. We characterize equilibrium exit times and establish conditions for correct learning in large societies. We show that exit times are shorter for highly connected networks and agents. The necessary conditions for correct learning are: (*i*) no single or small group of agents should have unbounded influence as measured by *conditional in-degree*; (*ii*) The underlying network must have a bounded diameter. Finally, we show that the presence of noise in signals prolongs exit times, and hence increases the likelihood of asymptotic learning.

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1. Introduction

Many economic and social decisions are made without knowledge of the associated rewards. Individuals rely on their beliefs about the potential rewards when weighing available options. In situations where individuals can communicate their opinions and experiences about the rewards, their beliefs will in turn evolve. If communications are restricted to friends, colleagues, acquaintances, and relatives, then the social network governing such relations imposes a restriction on how fast information travels across the population. If actions are irreversible, then individuals must wait to receive information until their belief confidence is sufficiently high. In many situations however, waiting is costly, resulting from either material losses or the urgency of the decision at hand. This implies that individuals face a decision on how long to wait to receive more information before taking an irreversible action.

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As a motivating example, consider a society of individuals to whom a new vaccination program has been introduced. The willingness to vaccinate depends on individual perceptions or beliefs about the vaccine safety and its consequences for disease control. Individuals start with prior beliefs about the benefits of the vaccination program. If the level of confidence in their belief is low, they will wait to receive more information from those in their social network, which may include media sources. Once their confidence reaches a sufficient level, they then decide on whether to participate in the vaccination program. There exists empirical evidence in support of social learning in such settings (Henrich and Holmes, 2011; Miguel et al., 2003).[1]

This paper develops a framework for studying word-of-mouth learning in social networks where agents strategically decide when to take an irreversible action. The underlying network consists of directed relations capturing the direction of flow of information. Agents face payoff uncertainty regarding available choices and an underlying state of nature determines the action with the highest payoff. Each agent is endowed with a prior belief, and receives a signal that is correlated with the true state of nature. Over time, agents truthfully communicate their posterior belief to their neighbours. At each period, they decide whether to wait to receive more information or irreversibly choose an action. Waiting for more information increases one's confidence in their belief but it has an associated cost in the form of discounted payoffs. Agents' objective is then to choose the optimal time to stop waiting. We refer to the stopping time as *exit time* in the sense that an agent exits the game. After exit, an agent stops actively waiting for new information but continues to communicate her posterior belief thus far. This assumption captures the notion that an agent's decision to make an irreversible choice implies that their level of confidence is sufficiently high. Hence there is no incentive to keep asking for neighbours' opinions; but if others who are still active ask for her opinion then she communicates it to them. Moreover, after exit, any new information that arrives does not dramatically change once posterior belief.

Given the model parameters, we study (i) the relationship between network structure and equilibrium exit times; (ii) conditions under which agents learn the true state of nature. We find the network measure that directly influence exit times to be *neighbourhood growth*: the distribution of sizes of r-order neighbourhoods. Neighbourhood growth determines the number of signals an agent receives at each period and hence the rate of growth of belief confidence. For each agent, neighbourhood distributions are directly computable and assume various functional forms: e.g.

[1] Henrich and Holmes (2011) conduct an empirical study of online discussions regarding the vaccination program for the 2009 H1N1 pandemic in Canada and find that they reflected actual public opinion and hence decision making process about vaccination. Miguel et al. (2003) examine social learning using data from a program that promoted use of deworming medicine in Kenyan schools. They find effects of information exchange through social networks, which in turn influenced individual beliefs and decisions.

linear, exponential, Gaussian, Poisson, and power-law. Each network however generally exhibits well-defined neighbourhood growths.

We show that equilibrium exit times are generally shorter for highly connected networks and/or agents. For example, exit times in networks with exponentially growing neighbourhood are shorter than those in networks with linear neighbourhood growth. If the discount rate is sufficiently high, it is possible that agents in highly connected networks exit after receiving information from only their first-order neighbours. This result has direct implications on equilibrium beliefs and actions.

Consider for example a network that is homophilous; that is, made up of densely connected subgroups whereby agents within subgroups receive strongly correlated signals. This can occur in situations where agents within the subgroup obtain information from the same sources. The implications are: (i) dense connections imply that agents exit after receiving information from say only first and second order neighbours. (ii) Strong correlation among signals also implies strong correlation of equilibrium posterior beliefs and actions. The result is a society with strong correlation in intra-subgroup beliefs and actions but not across subgroups. Hong et al. (2005) observe this phenomenon in portfolio investment. They observe that a mutual-fund manager is more likely to hold a particular stock in any quarter if other managers in the same city are holding that same stock. They explain their findings as a result of information sharing through word-of-mouth.

We also investigate the structure of equilibrium beliefs and actions when the society becomes large. Specifically, we say that asymptotic learning occurs if for a sequence of games indexed by the population size, agents learn the true state of nature and hence correct action. We identify two conditions for asymptotic learning. First, no single agent or small group of agents should have unbounded influence as captured by *conditional in-degree.* Consider a network G where a link from i to k implies that i listens to k's posterior belief. For any pair i and j, let G_{ij} be a sub-network derived from G by considering all paths starting from i and terminating at j. The in-degree ind_{ij}, of j in G_{ij} is the conditional in-degree of i and j. The paths in G_{ij} essentially represent the routes that information follows through from j to i.

Conditional in-degree is a network measure that we find to be appropriate to capture individual influence in our model of word-of-mouth learning. This is opposed to the well-known measures of centrality that are shown to be accurate measures of influence in other forms of learning. For example, eigenvector centrality measures influence in the models of naïve learning (Demarzo et al., 2003; Golub and Jackson, 2010); and Bonacich centrality determines equilibrium behaviour in models of strategic interactions in networks (Ballester et al., 2006).

The second condition for asymptotic learning is that the network must have a bounded diameter. This condition rules out most networks in which diameter grows with population size. Examples include networks with linear neighbourhood growth and scale-free networks which are representative of many real-world networks. A sufficient condition for asymptotic learning is existence of central agents that collect and in turn disseminate information. Such networks include media broadcast-audience relations and some social networks in which fewer individual have unbounded neighbourhoods. For learning to occur, it is however necessary that such central figures are involved in collecting information from others or in research that they in turn disseminate to others. Asymptotic learning can also occur in Erdös-Rényi random networks in which average degree grows with the population size. Erdös-Rényi random networks are particularly interesting to study because they share neighbourhood growth properties with some online social networks. We demonstrate this with three examples of empirical social networks from the Stanford Large Network Data Set Collection (Leskovec and Krevl, 2014). Moreover, most online social networks also exhibit bounded diameters as supported by the phenomenon of *six degrees of separation* (Ugander et al., 2011; Backstrom et al., 2012).

Our paper is related to strands of literature on word-of-mouth communication and social learning on networks. The first strand of literature studies the process of information diffusion through word-of-mouth. This literature focuses on optimal strategies that firms can employ to diffuse their product through word-of-mouth. It characterizes the interaction between social network and firm pricing strategies. In Candogan et al. (2010), Campbell (2012) and Crapis et al. (2016), consumers pass on information about the product only after they have purchased it. In Galeotti and Goyal (2007), firms initially advertise to consumers, and then through word-of-mouth, information travels a distance of one in the social network. A recurring theme in these models is that firm strategies which increases the propensity of any consumer to hold information will facilitate information diffusion. Our paper instead focuses on costly word-of-mouth learning, and the interaction between the social network and exit times.[2]

The second strand of literature is the line of research on social learning through observation. In these models, agents repeatedly interact and learn an unobserved state of nature by observing each other's actions and payoffs. They then choose actions that maximize average payoffs (Ellison and Fudenberg, 1995; Bala and Goyal, 1998; Banerjee and Fudenberg, 2004). In other cases, agents observe actions of their neighbours, which are then used to deduce signals about the state of nature (Gale and Kariv, 2003; Rosenberg et al., 2009; Mueller-Frank, 2013). These papers find that learning converges to a state in which payoffs and hence actions are identical across players. Here, we focus on information exchange through communication rather than observational learning.[3]

[2] A related literature studies incentives for individuals to engage in word-of-mouth, e.g. Sundaram et al. (1998), Hennig-Thurau et al. (2004) and Campbell et al. (2017).

[3] A related strand of literature studies sequential Bayesian learning, where agents make a decision once in a lifetime in an exogenously predefined order. When it is an agent's turn to act, she observes the history of actions of all agents that acted before her. The primary concern of this literature is establishing conditions under which informational

Our paper is also related to the literature on information percolation in which rational agents exchange private signals (Duffie et al., 2009; Acemoglu et al., 2014). In Duffie et al. (2009), a continuum of agents are randomly matched according to search intensities determined by individual specific effort to gather information. At each period, the set of agents that happen to meet share their signals. They characterize equilibrium search intensities as a function of information gathered. Acemoglu et al. (2014) consider a similar approach but agents interact through a fixed network. They show that learning in large societies occurs in the presence of "information hubs", which receive and distribute a large amount of information. In our model, agents communicate their posterior beliefs and not signals directly. As a result, we find that for learning in large societies to occur, no finite subgroup of agents should have unlimited influence. We also show that although the presence of information hubs is a sufficient condition, asymptotic learning can still occur in some random networks with bounded diameters.

Finally, our paper is related to the empirical literature on word-of-mouth learning partly discussed in the motivational example above. Cai et al. (2015) study the influence of social networks on weather insurance adoption and the mechanisms through which social networks operate in rural China. They find positive evidence of social network effects, which is not driven by the diffusion of information on purchase decisions, but instead by the diffusion of knowledge about insurance. Their results thus highlight evidence of word-of-mouth communication rather than observation of others' actions. Duflo and Saez (2002, 2003) provide evidence of social networks effects in retirement decisions by employees in a university. A similar study and result is established by Sorensen (2006). There is also strong evidence of word-of-mouth communication and social network effects in adoption of new technologies (Bandiera and Rasul, 2006; Conley and Udry, 2010) and investment decisions (Hong et al., 2004).

The remainder of the paper is organized as follows. Section 2 outlines a framework of costly word-of-mouth learning. Section 3 presents a discussion and results on equilibrium beliefs and exit times. Section 4 characterizes learning in large societies, where we establish conditions under which it occurs. Section 5 characterizes asymptotic learning when prior beliefs of opponents are unobservable. In Section 6, we discusses implications of the assumptions made in the model. A conclusion is offered in Section 7, and lengthy proofs are relegated to the Appendix.

2. The model

The model we develop aims to capture decision processes under payoff uncertainty in which individual level of confidence in their belief plays a role in deciding when to act. Agents are faced

cascades and herding behaviour occurs. The main contributions are Banerjee (1992), Bikhchandani et al. (1992), Smith and Sorensen (2000) and Acemoglu et al. (2011).

with a trade-off between taking an irreversible action early enough to avoid the cost of waiting and waiting to receive more information from their neighbours to increase confidence in their choice. We start with a stylized model that combines a simple quadratic payoff structure and Gaussian beliefs and signals. We then discuss the implications of the assumptions of the stylized model in Section 6.

2.1. Actions, payoffs, information and communication structures

We consider an information externality game played by a set $N^n = \{1, 2, \cdots, n\}$ of agents, each of whom chooses an irreversible action $x \in \mathbb{R}$. For the sake of concreteness, we start with a simple payoff structure in which the payoff $U(x, \theta)$ to action x is a quadratic function of x and a *state of nature* θ; that is $U(x, \theta) = h - (x - \theta)^2$, where h is a constant. As will become clear in the following sections, what matters is for the optimal value of $U(x, \theta)$ to reflect individual confidence in their opinion; so $U(x, \theta)$ could take any form (as we discuss in Section 6) provided this property is preserved.

The true state of nature $\bar{\theta}$ is unknown to all agents and is drawn from a normal distribution with mean θ and variance σ_θ^2. Each agent observes a noisy signal s_i that is informative about $\bar{\theta}$ and is of the form $s_i = \bar{\theta} + \varepsilon_i$, where across all i, $\bar{\theta}$ and $\varepsilon_1, \cdots, \varepsilon_n$ are independently distributed, and it is common knowledge that $\varepsilon_i \sim \mathcal{N}(0, \sigma_\varepsilon^2)$.

Agents exchange information through word-of-mouth communication and their interactions are governed by a network $\mathcal{G}^n = (N^n, \mathcal{E}^n)$, where $\mathcal{E}^n$ is the set of links connecting agents. We assume that agents truthfully communicate their posterior beliefs. That is, they do not have incentives to hide information from or send deceptive messages to their neighbours.[4] The network imposes constraints on information exchange in that agents can only communicate to those in their neighbourhood. We write G^n, with elements g_{ij}, for the adjacency matrix of $\mathcal{G}^n$. If $g_{ij} = 1$, then j communicates (sends a message) to i, or that i listens to j, and $g_{ij} = 0$ implies otherwise. Communication is simultaneous, deterministic and occurs at discrete time intervals $t = 1, 2, \cdots$.

The following definitions and notations regarding networks will be used throughout the paper. The set of agents at a radius d from i is denoted by $N_{i,d}^n$, such that $N_{i,1}^n$ is the set of agents that i directly listens to; that is, the immediate neighbours of i. A network is said to be *strongly connected* if for every pair of agents i and j, there exists a directed path from i to j and vice versa. The *geodesic* $d_{ij}(\mathcal{G}^n)$ is the shortest path from i to j. The *diameter* $d_i(\mathcal{G}^n) = \max_j d_{ij}(\mathcal{G}^n)$ of i is maximum geodesic of i. The diameter $d(\mathcal{G}^n)$ of network $\mathcal{G}^n$ is the longest geodesic, that is, $d(\mathcal{G}^n) = \max_i d_i(\mathcal{G}^n)$. Finally, we define the notion of neighbourhood degrees. Let $\mathcal{N}_{i,d}^n$ be the union of the immediate neighbours of all agents at a radius $d - 1$ from i but excluding those in $d - 2$.

[4] See for example Acemoglu et al. (2014) for a model that is closely related to ours in which they study strategic communication. Here, we focus on truthful communication to provide an in-depth analysis of equilibrium behaviour.

That is, for all $j \in N^n_{i,d-1}$, $\mathcal{N}^n_{i,d} = \bigcup_{j \in N^n_{i,d-1}} N^n_{j,1} - N^n_{i,d-2}$. We refer to $\mathcal{N}^n_{i,d}$ as the d-order degree neighbourhood of i, and write $k^n_{i,d}$ for its cardinality.

2.2. Dynamics and equilibrium of information externality game

In addition to the constraints imposed by the communication network on information exchange, there is a cost associated with delaying an irreversible action. At every period, each agent i chooses between taking an irreversible action and "exit the game" or "wait" to receive more information. By "exit the game" we mean an agent stops incorporating new information but continues to communicate information collected thus far. This assumption captures the notion that after taking an action irreversibly, i may still receive new information but she sees no reason to pay attention to it since it will not change the action taken. Alternatively, at the time of exit, i is highly confident in her belief that the information that arrives after is not taken seriously for it to radically change her posterior belief.

Waiting is costly in that the payoffs are discounted over time with discount rate of r. That is at time t, i's payoff from action x_i is $(h - (x_i - \theta)^2)e^{-rt}$. Let $\mu^n_{i,t}$ and $\rho^n_{i,t}$ denote the mean and precision (that is the reciprocal of the variance $\text{var}^n_{i,t}$) of i's posterior belief at t. At each period t, each i listens to the posterior beliefs of her neighbours, from which she deduces new information by subtracting their posterior beliefs at t from those at $t-1$ in a Bayesian manner.

The following additional notation is useful in characterizing the evolution of posterior beliefs. Let $\bar{\mathcal{N}}^n_{i,d}$ be a set of all agents in $\mathcal{N}^n_{i,d}$ that also belong to $N^n_{i,d-1}$. This set captures triadic interactions. For example, let $k \in N^n_{i,d-2}$ and $\{j,l\} \in N^n_{i,d-1}$, that is, k is a distance $d-2$ from i, and j and l are a distance $d-1$ away from i. Let j and l also be immediate neighbours of k, that is $j \in N^n_{k,1}$ and $l \in N^n_{k,1}$. From above definitions $\mathcal{N}^n_{i,d}$ contains all immediate neighbours of j and l but excluding k. If in addition j is also an immediate neighbour of l, then j belongs to $\bar{\mathcal{N}}^n_{i,d}$. The reason for this categorization becomes clear below but in brief, it has implications on posterior belief updating. For the above example, in the second period k learns the signals of j and l. In the third period j and l reveal the total signal of their immediate neighbours to k; The total signal l reveals to k in the third period includes j's signal. But since k already knows j's signal from period two, she subtracts it from l's total signal. Through an iterative process, the total signal that $i's$ immediate neighbours reveal to her in the tth period includes signals of l's immediate neighbours but excluding j. We write $\bar{k}^n_{i,d}$ for the cardinality of $\bar{\mathcal{N}}^n_{i,d}$. Note that the index d on $\bar{\mathcal{N}}^n_{i,d}$ corresponds to the communication round t, so for the remainder of the paper we write $\bar{\mathcal{N}}^n_{i,t}$ for the respective set when $t = d$. The same notion applies to $\mathcal{N}^n_{i,t}$, $N^n_{i,t}$, $\bar{k}^n_{i,t}$ and $k^n_{i,t}$. The posterior beliefs then evolve as described by Lemma 1 below.

Lemma 1. *Let $r = 0$; if the network is common knowledge and prior beliefs are observable, then* *the posterior belief of each i at t is normally distributed with mean and precision*

$$\mu_{i,t}^n = \frac{\sigma_\varepsilon^2}{b_{i,t}^n\sigma_\theta^2 + \sigma_\varepsilon^2}\mu_{j,0} + \frac{\sigma_\theta^2}{b_{i,t}^n\sigma_\theta^2 + \sigma_\varepsilon^2}\sum_{\tau=1}^{t} S_{i,\tau} \quad \text{and } \rho_{i,t}^n = \rho_\theta + b_{i,t}^n \rho_\varepsilon. \tag{1}$$

where $S_{i,\tau} = \sum_{j\in\mathcal{N}_{i,\tau}} s_j - \sum_{l\in\bar{\mathcal{N}}_{i,\tau}} s_l$ *and* $b_{i,t}^n = \sum_{\tau=1}^{t}\left[k_{i,\tau}^n - \bar{k}_{i,\tau}^n\right]$ *is the size of the corrected t-order neighbourhood* $S_{i,\tau}$.

Proof. See Appendix Appendix A.1 □

The dynamic process described above entails an information externality game. Individual actions indirectly depend on others' strategies through information sharing. For each i and t, let $\mathcal{I}_{i,t}^n$ denote the set of all possible information sets, whose elements are denoted by $I_{i,t}^n$. Agent i's strategy at t is then a mapping $a_{i,t}^n : \mathcal{I}_{i,t}^n \to \mathbb{R} \cup \{\text{"}wait''\}$, from an information set to an action set. We write $\mathbf{a}_t^n$ for the strategy profile at time t, and $\mathbf{a}_{-i,t}^n$ for a profile that excludes i. Let $b_{i,t}^{n,\mathbf{a}}$ be the corresponding $b_{i,t}^n$ when the strategy profile is $\mathbf{a}^n$. Let also $\mu_{i,t}^{n,\mathbf{a}}$ and $\mathrm{var}_{i,t}^{n,\mathbf{a}}$ be the respective mean and variance of i posterior belief given strategy profile $\mathbf{a}^n$. The optimal irreversible action if i decides to exit at t is given $\mathbf{a}^n$ is

$$x_{i,t}^{n,\mathbf{a}} = \arg\min_{x\in\mathbb{R}} \mathbb{E}\left[h - (x-\theta)^2 | I_{i,t}^n\right] = \mathbb{E}\left[\theta | I_{i,t}^n\right] = \mu_{i,t}^{n,\mathbf{a}}$$

and the corresponding optimal payoff is

$$U_{it}(I_{i,t}^n) = \max_x \mathbb{E}\left[(h-(x-\theta)^2)e^{-rt} | I_{i,t}^n\right] = (h - \mathrm{var}_{i,t}^{n,\mathbf{a}})e^{-rt} = \left(h - \frac{1}{\rho_\theta + b_{i,t}^{n,\mathbf{a}}\rho_\varepsilon}\right)e^{-rt}$$

Let $\mathbb{E}_\mathbf{a}$ denote the expectation conditional on the action profile $\mathbf{a}$. Then the respective value function for i at t is

$$V_{i,t}^n(I_{i,t}^n) = \max \begin{cases} h - \frac{1}{\rho_\theta + \rho_\varepsilon b_{i,t}^{n,\mathbf{a}}} & \text{when } a_{i,t}^n = x_{i,t}^{n,\mathbf{a}} \\ \mathbb{E}\left[\mathbb{E}_\mathbf{a}(V_{i,t+1}^n | I_{i,t+1}^n) | I_{i,t}^n\right] e^{-r} & \text{when } a_{i,t}^n = wait \end{cases}$$

Given the value function above, equilibrium of information externality game is then defined as follows.

Definition 1. *An action profile $\mathbf{a}^{n,*}$ is a pure-strategy perfect Bayesian equilibrium of information externality game if for every $i \in N$, t and action profile $\mathbf{a}_{-i}^{n,*}$, action $a_{i,t}^{n,*}$ yields to i an expected payoff equal to i's value function at t, $V_{i,t}^n(I_{i,t}^n)$. We denote the set of equilibria of the game by $A^*(\mathcal{G}^n)$*

An equilibrium strategy profile induces an equilibrium *exit time* profile $\mathbf{t}_e^{n,\mathbf{a}}$. Each $t_{i,e}^{n,\mathbf{a}}$ is the time at which i takes an irreversible action and exits the game. After taking an irreversible action i stops incorporating new information and can only transmit information acquired until $t = t_{i,e}^{n,\mathbf{a}}$.

2.3. Learning in large societies

Given the learning mechanism, equilibrium strategies and network structure, we are interested in determining whether correct learning occurs at equilibrium; equivalently, the conditions under which equilibrium behaviour leads to complete aggregation of decentralized information. We particularly focus on the network topologies that lead to correct learning. We define correct learning or more accurately *asymptotic learning* as the convergence in probability to the correct equilibrium action as prescribed by the true state of nature.

Definition 2. *Given a sequence of networks* $\{\mathcal{G}^n\}_{n\geq 2}$ *and some* $\epsilon > 0$*, asymptotic learning is said to occur if* $\lim_{n\to\infty}\mathbb{P}\left(|x_{i,e}^{n,\mathbf{a}} - \bar{\theta}| > \epsilon\right) = 0$ *for all* $i \in N$*, where* $x_{i,e}^{n,\mathbf{a}}$ *is the choice made by* i *after exiting play.*

Note that the correct action for the model considered here is equivalent to the true state of nature $\bar{\theta}$. Asymptotic learning condition of Definition 2 requires point-wise convergence of beliefs. Agents must be at most ϵ-away from learning the truth state of nature and hence taking an action that is ϵ-close to the optimal action. An alternative definition, commonly employed in convergence analysis, is to get all but a negligible proportion of agents arbitrarily close to the truth. That is, for some $\epsilon > 0$, let $A_i^{n,\mathbf{a}} = 1$ if and only if i chooses an irreversible action that is at most ϵ-away from the optimal action: $|x_{i,e}^{n,\mathbf{a}} - \bar{\theta}| \leq \epsilon$, and $A_i^{n,\mathbf{a}} = 0$ otherwise. Then asymptotic learning occurs if

$$\lim_{n\to\infty}\mathbb{P}\left(\left[\frac{1}{n}\sum_{i=1}^{n}(1 - A_i^{n,\epsilon})\right] > \epsilon\right) = 0. \tag{2}$$

Point-wise convergence in definition 2 is stronger than condition (2) and we discuss the difference and implication between the two in Section 4.

3. Equilibrium beliefs and exit times

This section presents a detailed discussion of the nature of equilibrium posterior beliefs and how to compute equilibrium exit times.

3.1. Equilibrium posterior beliefs

It is well known from a theoretical point of view (e.g. Bala and Goyal (1998), Acemoglu et al. (2011) and Jiménez-Martínez (2015)) that the network structure affects the nature of posterior beliefs and hence whether correct learning occurs. Of importance in this paper is how agents' network measure of influence affects posterior beliefs. This section considers the case for $r = 0$; we then study the case for $r > 0$ in Section 3.2. By doing so, we establish a benchmark for the analysis of asymptotic learning in Section 4.

Example 1: The following example helps fix ideas regarding the structure of beliefs in equilibrium. Consider a network structure in Figure 1 and that the assumptions considered in Lemma

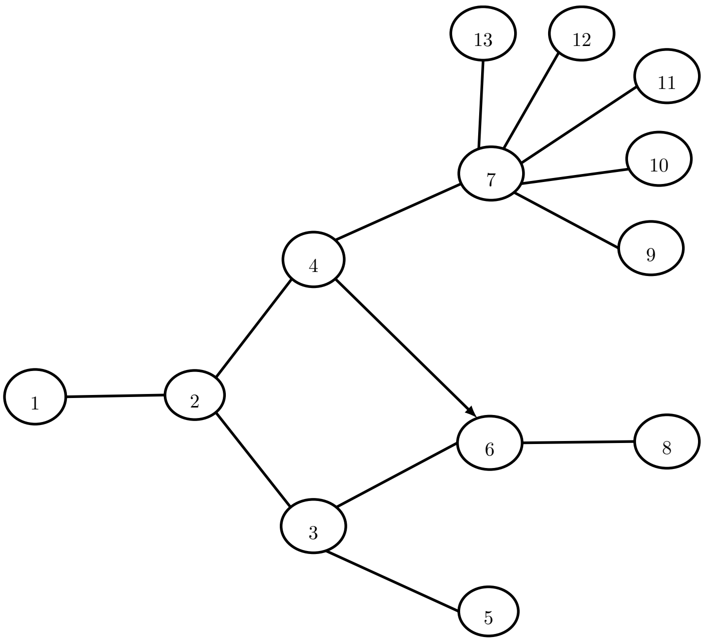


Figure 1: All direct relations, except that between agents 4 and 6, are reciprocal.

1 hold. The posterior beliefs of agents 1, 2 and 6 are as follows. For agent 2, the second period messages reveal the signals of her immediate neighbours, s_1, s_3 and s_4, and hence a new total signal $\frac{s_1+s_3+s_4}{3}$ with corresponding variance of $\frac{\sigma_\varepsilon^2}{3}$. From Lemma 1 it follows that agent 2's posterior belief at the end of period 1 is normally distributed with mean and precision of

$$\mu_{2,1} = \frac{\sigma_\varepsilon^2}{4\sigma_\theta^2 + \sigma_\varepsilon^2}\mu_{j,0} + \frac{\sigma_\theta^2}{4\sigma_\theta^2 + \sigma_\varepsilon^2}(s_1 + s_2 + s_3 + s_4) \quad \text{and} \quad \rho_{2,1} = \rho_\theta + 4\rho_\varepsilon. \tag{3}$$

Period 2's messages of agent 2's first-order neighbours reveal signal statistic $s_2 + s_6 + s_7$ and $s_2 + s_5 + s_6$ respectively. But since agent 2 know her own realized signal, she subtracts it and deduces the signal statistic $s_5 + 2s_6 + s_7$. Hence, the new signal for agent 2 in period 2 is $\frac{s_5+2s_6+s_7}{4}$ with corresponding variance of $\frac{\sigma_\varepsilon^2}{4}$. From Lemma 1, it follows that agent 2's posterior belief has the mean and precision of

$$\mu_{2,2} = \frac{\sigma_\varepsilon^2}{8\sigma_\theta^2 + \sigma_\varepsilon^2}\mu_{j,0} + \frac{\sigma_\theta^2}{8\sigma_\theta^2 + \sigma_\varepsilon^2}(s_1 + s_2 + s_3 + s_4 + s_5 + 2s_6 + s_7) \tag{4}$$

$$\rho_{2,2} = \rho_\theta + 8\rho_\varepsilon$$

In period 3, the messages of agent 2's first-order neighbours reveal signal statistic $s_8 + \cdots + s_{13}$, which results into posterior belief with mean and precision of

$$\mu_{2,3} = \frac{\sigma_\varepsilon^2}{14\sigma_\theta^2 + \sigma_\varepsilon^2}\mu_{j,0} + \frac{\sigma_\theta^2}{14\sigma_\theta^2 + \sigma_\varepsilon^2}(s_1 + s_2 + \cdots + 2s_6 + s_7 + \cdots + s_{13}) \tag{5}$$

$$\rho_{2,3} = \rho_\theta + 14\rho_\varepsilon$$

Agent 2 then stops learning after three steps. Following similar steps, agents 1 and 6 stop learning after four and five steps respectively, and the corresponding posterior beliefs are normally distributed

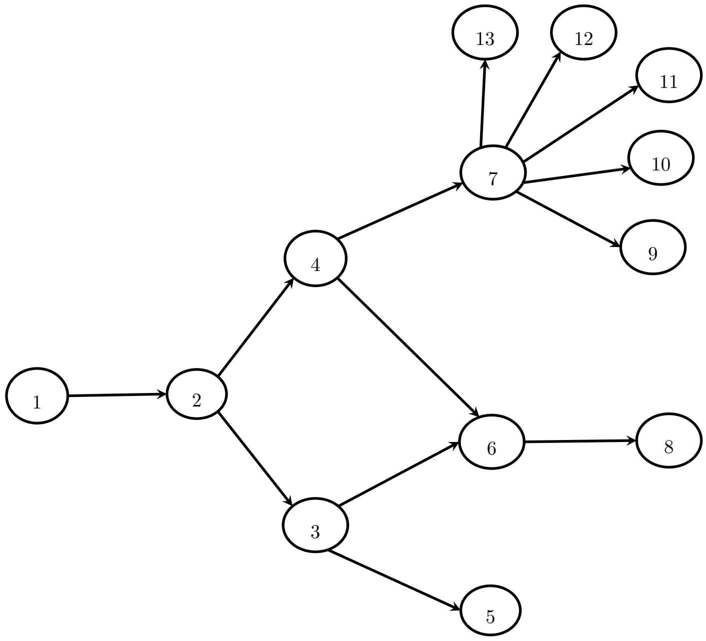


Figure 2: Distribution of in-degrees conditional on starting from agent 1.

with the following respective means and precisions.

$$\mu_{1,4} = \frac{\sigma_\varepsilon^2}{14\sigma_\theta^2 + \sigma_\varepsilon^2}\mu_{j,0} + \frac{\sigma_\theta^2}{14\sigma_\theta^2 + \sigma_\varepsilon^2}(s_1 + s_2 + \cdots + 2s_6 + s_7 + \cdots + s_{13}) \tag{6}$$

$$\rho_{1,4} = \rho_\theta + 14\rho_\varepsilon$$

$$\mu_{6,5} = \frac{\sigma_\varepsilon^2}{13\sigma_\theta^2 + \sigma_\varepsilon^2}\mu_{j,0} + \frac{\sigma_\theta^2}{13\sigma_\theta^2 + \sigma_\varepsilon^2}(s_1 + s_2 + \cdots + s_{13}) \tag{7}$$

$$\rho_{6,5} = \rho_\theta + 13\rho_\varepsilon$$

Example 1 illustrates two aspects regarding word-of mouth learning. First, the time it takes each agent to fully aggregate private information of all other agents depends on her diameter $d_i(\mathcal{G}^n)$. For the network in Figure 1, agent 2 has the shortest diameter of only three steps. Agents $8, \cdots, 13$ have the longest diameters, of six-steps each. Since we consider learning environments in which waiting to be more informed is costly, the lengths of diameters thus strongly influence information aggregation and hence whether asymptotic learning occurs.

The second aspect is agents' influence on each other's' posterior beliefs. The example illustrates that it is neither the first-order degree nor eigenvector centrality, often identified in the case of naïve learning, that determines how influential an agent is. Rather, it is a measure we refer to as *conditional in-degree* that matters.

Definition 3. *Given network $\mathcal{G}^n$ and a pair $i, j \in N^n$, define a directed subgraph of $\mathcal{G}^n$ that is made up of directed paths starting from i and terminating in j. Then the conditional in-degree $ind^n_{i,j}(\mathcal{G}^n)$ of j relative to i is the in-degree of j in this subgraph, excluding all g_{lj} for which $j \in N_{k,1}$ and $l \in N_{k,1}$.*

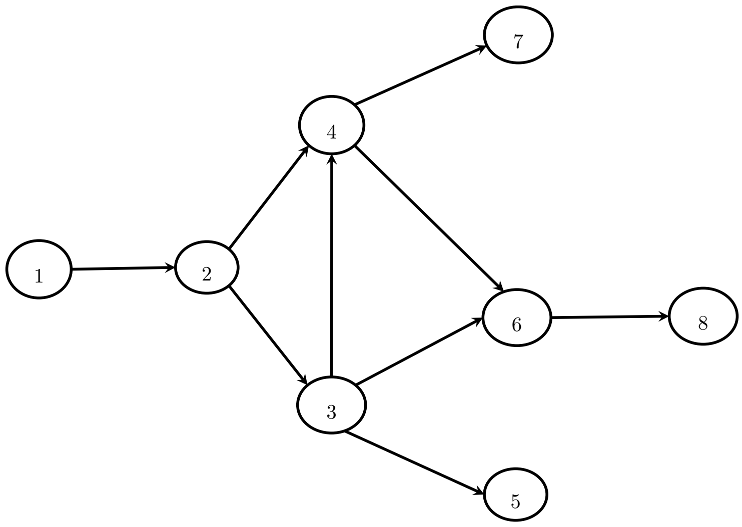


Figure 3: Distribution of in-degrees conditional on starting from agent 1.

Example 2: Consider the network in Figure 2 derived from that Figure 1 and showing the in-degrees of all agents conditional on agent 1. Agent 6 has the largest conditional in-degree of $\text{ind}^n_{1,6} = 2$ and all other agents have $\text{ind}^n_{1,j} = 1$. This makes agent 6 the most influential on the posterior belief of agent 1. In other cases, agent 3 is the most influential to the posterior beliefs of agents 4, 7, 9, 10, 11, 12 and 13, with conditional in-degrees of also two. On the contrary, agent 7 has the largest overall in- and out-degree of six but has lesser overall influence on other agents' beliefs.

To illustrate the second condition of Definition 3, which excludes all g_{ij} for which $j \in N_{k,1}$ and $l \in N_{k,1}$, consider the network in Figure 3 pivoted at agent 1. In this network, agent 6, has conditional in-degree $\text{ind}^n_{1,6} = 2$ and agent 4 appears to have $\text{ind}^n_{1,4} = 2$ but its net conditional in-degree is in fact one. The link g_{34} belongs to the relations in $\bar{\mathcal{N}}_{1,3}$ as defined above. That is, in period 2, the messages of agents 3 and 4 reveal the signal statistic $s_4+s_5+s_6$ and s_6+s_7 respectively to agent 2. But agent 2 can distinguish the signals of her first-order neighbours, agents 3 and 4, from period 1 and so subtracts s_4 from agent 3's message. Hence the link $3 \to 4$ is redundant in conveying new information.

These examples highlight how the concept of conditional in-degree can be employed to capture influence of agents on each other's posterior beliefs and hence whether asymptotic learning obtains. From (1) of Lemma 1, we can then rewrite equilibrium beliefs $\mu^n_{i,\infty}$ when $r = 0$ as

$$\mu^n_{i,\infty} = \frac{\sigma^2_\varepsilon}{b^n_{i,\infty}\sigma^2_\theta + \sigma^2_\varepsilon}\mu_{j,0} + \frac{\sigma^2_\theta}{b^n_{i,\infty}\sigma^2_\theta + \sigma^2_\varepsilon}\sum_{j\in N}\text{ind}^n_{i,j}s_j \quad \text{and } \rho^n_{i,\infty} = \rho_\theta + b^n_{i,\infty}\rho_\varepsilon. \tag{8}$$

where $b^n_{i,\infty} = \sum_{j\in N}\text{ind}^n_{i,j}$ and $\text{ind}^n_{i,j} = 0$ if no path exists from i to j.

3.2. Equilibrium exit times

Equilibrium exit times depend on the discount rate and precision of posterior beliefs, which itself is a function of the network measure $b_{i,t}^{n,\mathbf{a}}$. The distributional form of $b_{i,t}^{n,\mathbf{a}}$ depends on the family of networks in consideration. Each network type has a unique distributional form for $b_{i,t}^{n,\mathbf{a}}$. The main functional forms we discuss in the is paper are: linear form that is characteristic of some families of deterministic networks, exponential form that is characteristic of hierarchical and tree-like networks, Poisson distributional form that is characteristic of Erdös-Rényi random networks and some empirical networks, and power-law distributional forms that are characteristic of scale-free networks. We then examine in the next section which of these distributional forms support asymptotic learning. The following lemma provides a method for computing equilibrium exit times.

Lemma 2. *Suppose that the precision of agents' beliefs evolve as in* (1)*. Then equilibrium exit times are a solution to the simultaneous set of equations*

$$\Delta b_{i,t}^{n,\mathbf{a}} = \frac{\beta\left(\left(\rho_\theta + b_{i,t}^{n,\mathbf{a}}\rho_\varepsilon\right)h - 1\right)\left(\rho_\theta + b_{i,t}^{n,\mathbf{a}}\rho_\varepsilon\right)}{\rho_\varepsilon\left(1 - h\beta(\rho_\theta + b_{i,t}^{n,\mathbf{a}}\rho_\varepsilon)\right)} \quad \text{for } i = 1, \cdots, n \tag{9}$$

where $\beta = (1 - e^{-r})$.

Proof. See Appendix Appendix A.2 □

The size of an agent's corrected t-order neighbourhood $b_{i,t}^{n,\mathbf{a}}$, is non-decreasing function of t. For each i, the variation of $b_{i,t}^{n,\mathbf{a}}$ with t depends on network topology and i's position in the network. The equilibrium exit times $t_{i,e}^{n,\mathbf{a}}$ then depend on the distributional forms that $b_{i,t}^{n,\mathbf{a}}$ assumes. The following Proposition provides insights into the dependence of $t_{i,e}^{n,\mathbf{a}}$ on $b_{i,t}^{n,\mathbf{a}}$.

Proposition 1. *Consider two networks $\mathcal{G}^n$ and $\mathcal{G}^{n'}$ with respective neighbourhood growth functions $b_{i,t}^{n,\mathbf{a}}$ and $b_{i,t}^{n,\mathbf{a}'}$, identical for all i in a given network. If $b_{i,t}^{n,\mathbf{a}}$ first order stochastically dominates $b_{i,t}^{n,\mathbf{a}'}$, then the respective exit times $t_{i,e}^{n,\mathbf{a}}$ and $t_{i,e}^{n,\mathbf{a}'}$ are such that $t_{i,e}^{n,\mathbf{a}} \leq t_{i,e}^{n,\mathbf{a}'}$, for r sufficiently small.*

Proof. See Appendix Appendix A.3 □

Proposition 1 shows that higher connectivity reduces exit times. For any two networks in which one is at least as connected as the other, the exit times for the latter will be at least greater than for the former. The underlying reason, as observable from (1), is that each agent's confidence grows at the rate of $b_{i,t}^{n,\mathbf{a}}$. Since highly connected networks have higher neighbourhood growth rates $b_{i,t}^{n,\mathbf{a}}$, the corresponding exit times are thus shorter, caeteris paribus. We demonstrate this relationship further with the following examples presented as corollaries.

Figure 4: A 3-regular infinite grid network.

Corollary 1. *Consider a network in which $b_{i,t}^{n,\mathbf{a}} = at + c$ for all i, where a and c are some constants. Let $h = \rho_\theta = \rho_\varepsilon = 1$. Then the exit times assume the form*

$$t_{i,e}^{n,\mathbf{a}} = \frac{1}{\phi_1}\left(\left[\phi_2 + \frac{\phi_4}{\beta}\right]^{\frac{1}{2}} - \phi_3\right) \quad \textit{for } i = 1, \cdots, n \tag{10}$$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are increasing functions of a and c.[5]

Proof. See Appendix Appendix A.4 □

Corollary 2. *Consider a network in which $b_{i,t}^{n,\mathbf{a}} = a^t + c$ for all i, where a and c are some constants. Let $h = \rho_\theta = \rho_\varepsilon = 1$. Then the exit times assume the form*

$$t_{i,e}^{n,\mathbf{a}} = \frac{1}{\ln(a)} \ln\left[\frac{1}{2a}\left(\psi_\beta - \psi_1 + \left[\psi_\beta^2 - 2\psi_1\psi_\beta + \psi_2\right]^{\frac{1}{2}}\right)\right] \quad \textit{for } i = 1, \cdots, n \tag{11}$$

where $\psi_\beta = \left(1 + \frac{a_1}{\beta}\right)$, and ϕ_1 and ϕ_2 are increasing functions of a and c.[6]

Proof. See Appendix Appendix A.5 □

The neighbourhood growth function $b_{i,t}^{n,\mathbf{a}}$ for the network in Corollary 2 (exponential neighbourhood growth) stochastically dominates that for the network in Corollary 1 (linear neighbourhood growth). We also observe that the exit times for the latter are greater than those for the former, which are logarithmic in nature.

An example of networks with linear neighbourhood growth is depicted in Figure 4. An infinite regular network with $b_{i,t}^{n,\mathbf{a}} = 8t - 4$ for each i. That is, at $t = 1$ each i receives three signals from the first order neighbours plus their own, and from $t \geq 2$ onward receives eight signals periodically. Substituting for $a = 8$ and $c = -4$ into (10), and for $\beta = 0.001$, we find that each agent exits after 11 periods.

An example of networks with exponential neighbourhood growth is depicted in Figure 5. An infinite regular network with $b_{i,t}^{n,\mathbf{a}} = 3\left(2^t - 1\right)$ for each i. Substituting for $a = 2$ and $c = -3$ into (11) while accounting for the factor 3, then for $\beta = 0.001$ we find that each agent exits after approximately 9 periods.

[5] More specifically, $\phi_1 = 2a^2$, $\phi_2 = 4a^4 - 2a^3 + a^2$, $\phi_3 = a^2 + 2ac + a$ and $\phi_4 = 4a^3$.

[6] More specifically, $\psi_1 = (1+a)(1+c)$ and $\psi_2 = (1+a)^2(1+c)^2 - 4ac(1+c)$.

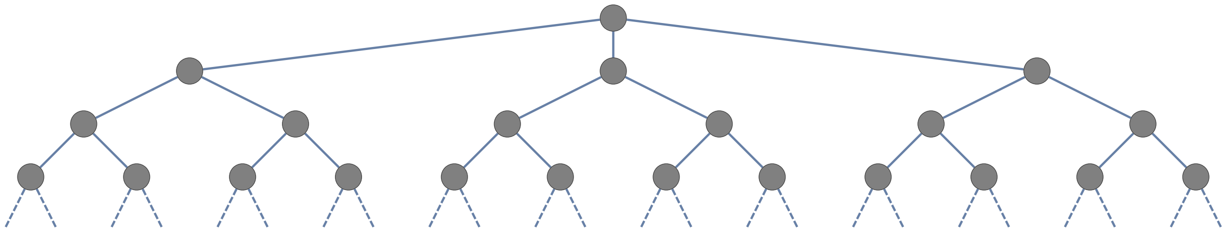


Figure 5: A 3-regular infinite tree network.

In closing this section, there are two main highlights from above results. The first being that highly connected networks tend to have shorter equilibrium exit times in general. The second, in a network with heterogeneous connections, Proposition 1 also implies that the most connected agents will exit first. The direct implication is that if the underlying network is densely connected, then it is possible that agents exit after receiving information from only say their first and second order neighbours. This will intern lead to localization of information in equilibrium. If for example agents within different regions of the network observe correlated signals, then their equilibrium posterior beliefs and action will also be strongly correlated since they exit the game without receiving information from other regions of the network. The direct result is a society in which each subregion or subgroup has correlated beliefs and actions but not across subregions/subgroups.

4. Asymptotic learning

This section establishes conditions for asymptotic learning by rational agents. As highlighted in section 2.2, the interdependence between exit times (induced by Bayesian perfect equilibrium) of agents through informational externalities influences information aggregation and hence asymptotic learning.

For asymptotic learning to obtain, no agents should be forced to exit the game "too early" and that no single agent or small group of agents has unlimited influence on others' posterior beliefs. By too early we mean before fully aggregating information. The analysis for asymptotic learning thus anchors on comparing for each agent their exit time to the time it would take to fully aggregate information if waiting were not costly. Denote such exit time by $t^n_{i,e}$ and call it the *perfect observational radius* (see Acemoglu et al. (2014) for a similar characterization). We saw from Section 3.1 that when $r = 0$, i stops learning after a period that is equal to her diameter in the network. At this period, i receives information from all other agents. The perfect observation radius $t^n_{i,e}$ is thus equivalent to i's diameter.

Theorem 3. *Given a sequence $\{\mathcal{G}^n\}_{n\geq 2}$ of strongly connected communication networks, let $\{d_i(\mathcal{G}^n)\}_{i\in N,n\geq 2}$ be the corresponding sequence of agents' maximum geodesics and $\{ind^n_{i,j}\}_{i\in N,n\geq 2}$ the respective conditional in-degrees. Then asymptotic learning with observable priors obtains if and only if* *(i)* $\frac{1}{n}\sum_{i=1}^{n}\left(t_{i,e}^{n,\mathbf{a}} - d_i(\mathcal{G}^n)\right) \to 0$ *as* $n \to \infty$*, in probability*

(ii) $\lim_{n\to\infty} \frac{1}{n} ind_{i,j}^{n} = 0$ *for all pairs of* i *and* j*.*

Proof. See Appendix Appendix A.6 □

Theorem 3 establishes necessary and sufficient conditions for asymptotic learning. The necessary condition is for the network to be strongly connected. If the network were not strongly connected then messages/information that is localized within one region of the network does not get passed on to some other regions of the network. Being strongly connected is however not sufficient for asymptotic learning for two reasons. First, if information takes too long to diffuse from one region of the network to another, then most agents could act before fully aggregating information. Condition (i) of Theorem 3 then requires equilibrium exit times to be equal or close to the perfect observational radii.

Secondly, if the posterior beliefs of agents are influenced by a single agent or small subgroup of agents, then opinions of the respective subgroup influence their final choices. Consequently, the final choices need not be asymptotically correct. To prevent such cases, no small subset of agents must possess unlimited influence on others' beliefs. This is captured by the second condition of Theorem 3, which states that the conditional in-degrees must be asymptotically negligible relative to the population size.

The derivation of Theorem 3 is based on the point-wise convergence of beliefs as stated in definition 2. It however still holds if asymptotic learning were defined as convergence in sum of individual actions; the difference lies in convergence rates to correct choices. To see why, note that the convergence rate to correct learning under definition 2 is given by the variance of individual choices made in equilibrium $\text{var}[x_{i,e}^{n,\mathbf{a}}]$ while under definition (2) it is the variance of $\frac{1}{n}\sum_{i=1}^{n} A_i^{n,\epsilon}$.[7] In the proof of Theorem 3 in Appendix Appendix A.6, we showed that the variance $\text{var}[x_{i,e}^{n,\mathbf{a}}]$ decays at the rate $1/b_{i,e}^{n,\mathbf{a}}$.[8] The variance of $\frac{1}{n}\sum_{i=1}^{n} A_i^{n,\epsilon}$ on the other hand decays at the rate of $\frac{1}{n}e^{\left(-\epsilon\mathcal{O}\left(\sqrt{1/b_e^{n,\mathbf{a}}}\right)\right)}$. More specifically, since $\mu_{i,t}^{n,\mathbf{a}}$ and hence $x_{i,t}^{n,\mathbf{a}}$ are normally distributed with variance $\text{var}[x_{i,e}^{n,\mathbf{a}}]$, the probability that $A_i^{n,\epsilon} = 0$ is equal to the probability that the error does not belong to the closed interval $[-\epsilon, \epsilon]$. That is

$$\mathbb{P}(A_i^{n,\epsilon} = 0) = 1 - \text{erf}\left(\epsilon\sqrt{\text{var}[x_{i,e}^{n,\mathbf{a}}]/2}\right) \tag{12}$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\tau^2}d\tau$ is the error function with $\text{erf}(0) = 0$ and $\text{erf}(\infty) = 1$. The error function can be numerically approximated by (see Tsay et al. (2009)) $\text{erf}(x) \approx 1 - e^{c_1 x + c_2 x^2}$ (for

[7] This follows from Chebyshev's inequality whereby if $\bar{A}$ is the mean of variable A, then $\mathbb{P}\left(\left[\bar{A} - A\right] > \epsilon\right) < \frac{\text{var}[A]}{\epsilon^2}$.

[8] More specifically, as $\text{var}[\mu_{i,e}^{n,\mathbf{a}}] \to 0$ at the limit of n, the slowest term is $\frac{\sigma_\theta^4}{\left(b_{i,e}^{n,\mathbf{a}}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\left(\text{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}} S_{i,\tau}\right]\right)$. The variances $\text{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}} S_{i,\tau}\right] = b_{i,e}^{n,\mathbf{a}}\sigma_\varepsilon^2$. Hence $\text{var}[\mu_{i,e}^{n,\mathbf{a}}] \to 0$ at the rate of $\frac{b_{i,e}^{n,\mathbf{a}}\sigma_\varepsilon^2\sigma_\theta^4}{\left(b_{i,e}^{n,\mathbf{a}}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}$.

$c_1 = -1.0950081470333$ and $c_2 = -0.75651138383854$). Hence the variance of $\frac{1}{n}\sum_{i=1}^{n} A_i^{n,\epsilon}$ can be expressed as

$$\operatorname{var}\left[\frac{1}{n}\sum_{i=1}^{n} A_i^{n,\epsilon}\right] = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{var}[A_i^{n,\epsilon}] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{P}(A_i^{n,\epsilon}=1)\mathbb{P}(A_i^{n,\epsilon}=0) \tag{13}$$

where the second equality follows from the variance of binary variables, that is, $\operatorname{var}[A_i^{n,\epsilon}] = \mathbb{P}(A_i^{n,\epsilon} = 1)(1-\mathbb{P}(A_i^{n,\epsilon}=1))$. From (12), and the approximation for the error function,

$$\operatorname{var}\left[\frac{1}{n}\sum_{i=1}^{n} A_i^{n,\epsilon}\right] \leq \frac{1}{n^2}\sum_{i=1}^{n}\left(e^{c_1\left(\epsilon\sqrt{\operatorname{var}[x_{i,e}^{n,\mathbf{a}}]/2}\right)+c_2\left(\epsilon\sqrt{\operatorname{var}[x_{i,e}^{n,\mathbf{a}}]/2}\right)^2}\right) \tag{14}$$

Let $\operatorname{var}[x_e^{n,\mathbf{a}}] = \min_{i\in N}\operatorname{var}[x_{i,e}^{n,\mathbf{a}}]$, and $b_e^{n,\mathbf{a}} = \arg\max_{a_j\in A}\operatorname{var}[x_{i,e}^{n,\mathbf{a}}]$. Then $\operatorname{var}\left[\frac{1}{n}\sum_{i=1}^{n} A_i^{n,\epsilon}\right] \leq \frac{1}{n}e^{c_1\left(\epsilon\sqrt{\operatorname{var}[x_e^{n,\mathbf{a}}]/2}\right)}$. Since the $\operatorname{var}[x_e^{n,\mathbf{a}}]$ decays at the rate $1/b_e^{n,\mathbf{a}}$, convergence to correct choices thus occurs at the rate $\frac{1}{n}e^{\left(-\epsilon\mathcal{O}\left(\sqrt{1/b_e^{n,\mathbf{a}}}\right)\right)}$, which is faster than for point-wise convergence in definition 2. In both cases however, Theorem 3 holds since condition (i) of Theorem 3 must hold for $\lim_{n\to 0}\operatorname{var}\left[\frac{1}{n}\sum_{i=1}^{n} A_i^{n,\epsilon}\right] = 0$.

We now elaborate on the implications of Theorem 3. First note that condition (i) can equivalently be stated as $\lim_{n\to\infty}\frac{t_{i,e}^{n,\mathbf{a}}}{d_i(\mathcal{G}^n)} = 1$ for all i. The strength of Theorem 3 lies in the notion that for every sequence of networks, one can derive a range of parameter values under which asymptotic learning occurs. For a given network structure, exit times $t_{i,e}^{n,\mathbf{a}}$ can be derived from (9) and geodesics $d_i(\mathcal{G}^n)$ together with conditional in-degrees directly from the network. Once expressions for these measures are known, the range of parameter values of r, ρ_ε and ρ_θ for which asymptotic learning obtains are those that satisfy the conditions in Theorem 3.

Given condition (ii) of Theorem 3, and that $r > 0$ the following Corollary establishes the necessary condition on the network for condition (i) of Theorem 3 to hold.

Corollary 3. *Given condition (ii) of Theorem 3, the necessary condition for asymptotic learning is for $\lim_{n\to\infty}\max_{i\in N} d_i(\mathcal{G}^n) < \infty$.*

Proof. See Appendix Appendix A.7 □

Corollary 3 follows from the fact that exit times are always finite whenever $r > 0$. Hence, any set of agents with unbounded perfect observational radius will exit play before fully aggregating information.

4.1. Deterministic networks

The requirement for the sequence of networks to have bounded diameter already rules out most families of deterministic networks. Consider the network families discussed in Section 3.2, with linear and exponential neighbourhood growth. For the former, the diameter $d(\mathcal{G}^n) = \max_{i\in N} d_i(\mathcal{G}^n) =$

$\mathcal{O}(n)$, that is, increases monotonically with n. For example, for the network in Figure 4, $d(\mathcal{G}^n) = \frac{n}{4}$; and hence $\lim_{n\to\infty} d(\mathcal{G}^n) = \infty$. For tree-networks depicted in Figure 5, but with degree k, the diameter when the network is finite $d(\mathcal{G}^n) = \frac{(k-1)}{\ln(k-1)} \ln\left(\frac{n+(k-1)}{k}\right)$, which still implies that $\lim_{n\to\infty} d(\mathcal{G}^n) = \infty$.

A sufficient condition for asymptotic learning in deterministic networks is existence of *central agents* that help aggregate information. These agents must have unbounded in-degree and bounded conditional in-degree. Formally, denote the central set of agents by C^n, that is, $C^n = \{i : \lim_{n\to\infty} \#N^n_{i,1} = \infty\}$. The set C^n is finite otherwise $\mathcal{G}^n$ would be a complete network.[9]

Corollary 4. *Let $d^{c,n}_i$ denote the geodesic between i and some $j \in C^n$, and let $S^n = \{i : d^{c,n}_i \leq t^n_{i,e}\}$. Then asymptotic learning occurs whenever $\lim_{n\to\infty} \frac{1}{n}\#S^n = 1$, provided condition (ii) of Theorem 3 holds.*

Corollary 4 highlights the role of central agents in aggregating information and states that if all other agents are within the perfect observational radius to the central agents, then asymptotic learning is guaranteed to occur. Networks with central agents also necessarily satisfy condition (*ii*) of Theorem 3 since unbounded in-degree need not imply unbounded conditional in-degree.

Example of real-world networks that exhibit a structure with central agents is media-audience networks, where the broadcasters are central and the audience are peripheral. For asymptotic learning to occur in such networks however, first the diameter should be bounded; second the central figures must be involved in some form of information aggregation process. In the case of media-audience networks, this may involve collecting opinion polls about a given topic and then in turn broadcasting it to the wider audience. This thus rules out media-audience networks in which the broadcaster simply disseminates his/her individual opinion.

4.2. Random and empirical networks

The result in Corollary 3 also rules out most random networks. A family of networks ruled out is the power-law networks formed from the process of preferential attachment. Some real-world networks have been shown to assume this structure. Networks that form through preferential attachment possess unbounded diameter in the order of $\ln n$ (Newman et al., 2001).

A family of random networks with bounded diameter is Erdös-Rényi random networks in which average degree np grows with n, that is $np \to \infty$. Erdös-Rényi random network $\mathcal{G}^n = (N, p)$ is constructed by connecting every pair of nodes randomly and independently with probability p. Chung and Lu (2001) show that if $np \to \infty$, then the diameter is approximately equal to $\ln n/\ln(np)$. Moreover, if $np/\ln n > 8$ then the diameter is concentrated around two values.[10]

[9] We write $\#N$ for the cardinality of N.

[10] If however $np \to 1$, Chung and Lu (2001) show that the diameter is approximately $\ln n$.

The Erdös-Rényi family of networks with bounded diameters are particularly interesting to study because some online social networks share functional properties of neighbourhood growth with the former. This can be seen from the quantile-quantile plots of Figure 6 for the log values of $\Delta b^n_{i,t}$ for a randomly chosen i.

The networks plotted in Figure 6 are: (a) Erdös-Rényi random network with network parameter values $n = 5000$ and $p = 1 \times 10^{-3}$. (b) The "Facebook friendship" network consisting of ego networks for 4,039 nodes. (c) Epinions social network is a network of who-trust-whom in a general consumer review site Epinions.com. Members of the site can decide whether to "trust" each other. All the trust relationships interact and form the Web of Trust which is then combined with review ratings to determine which reviews are shown to the user. (d) Slashdot, is a network of friendship in the technology-related news website *slashdot.org*. The website features user-submitted and editor-evaluated current primarily technology oriented news. The network was obtained in February 2009 (Leskovec and Krevl, 2014).

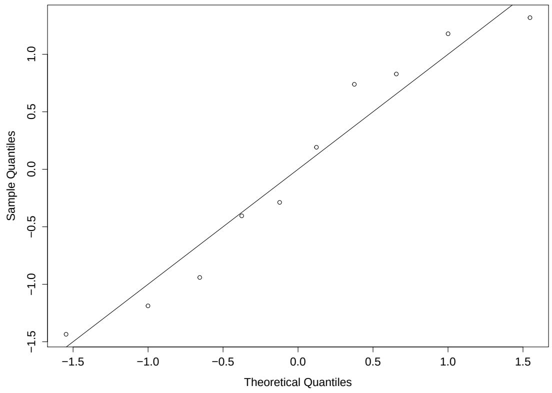
Most of these online social networks also exhibit short geodesics between agents. Ugander et al. (2011) study a social Facebook network of 721 million active individuals and show that it is nearly fully connected, with 99.91% of individuals belonging to a single large connected component. They also confirm the 'six degrees of separation' phenomenon on a global scale. Six degrees of separation is the idea that any two agents are a maximum of five friends away from each other.[11] In a follow up paper, Backstrom et al. (2012) find that for a same size Facebook network above, the average distance is 3.74 degrees of separation, showing that the world is even smaller than six degrees of separation.

The functional form for the neighbourhood growth of most online social networks closely follows a Poisson distribution (hence the reason for the Q-Q normal plots in Figure 6). Figure 7 shows the distribution of $\Delta b^n_{i,t}$ for Facebook friendship network also obtained from the Stanford Large Network Data set Collection (Leskovec and Krevl, 2014). Given a vector b^n_i of i's values of $b^n_{i,t}$, let $\|b^n_i\|$ be its norm. Then for some parameter λ, which is the mean of $\frac{1}{\|b^n_i\|} b^n_i$

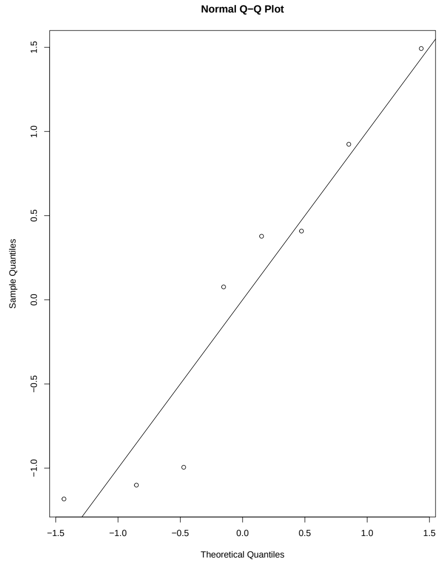
$$\Delta b^n_{i,t} = \|b^n_i\| e^{-\lambda} \frac{\lambda^t}{t!} \tag{15}$$

Corollary 5. *For a sequence* $\{\mathcal{G}\}_{n \geq 2}$ *of strongly connected networks whose neighbourhood sizes follows a Poisson distribution, let the network diameter be bounded. For such networks, and the Erdös-Rényi networks with* $np \to \infty$, *there exists a real number* $\bar{r} > 0$ *such that asymptotic learning occurs whenever* $r < \bar{r}$.

[11] The six degrees of separation phenomenon follows from Stanley Milgram famous experiment in which he challenged people to route postcards to a fixed recipient by passing them only through direct acquaintances. The average number of intermediaries on the path of the postcards lay between 4.4 and 5.7, depending on the sample of people chosen



(a)



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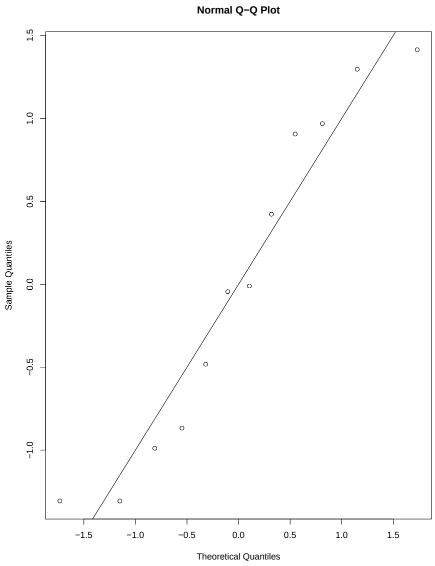

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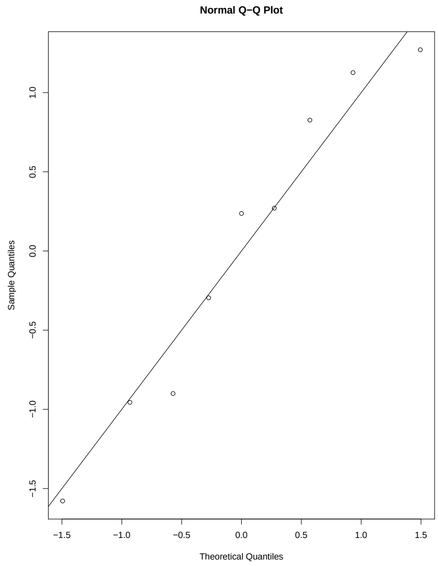
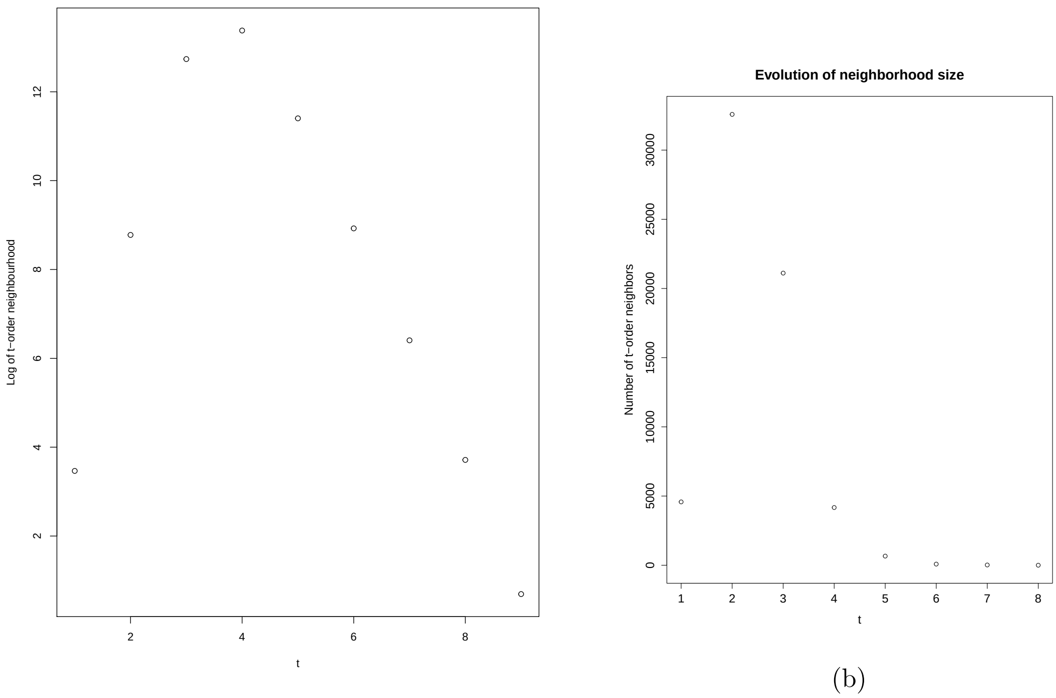


(d)

Figure 6: The quantile-quantile plots of the log values of $\Delta b_{i,t}^n$ for a randomly selected player. (a) is Q-Q plot for Erdös-Rényi random network; (b), (c) and (d) are Q-Q plots for the Facebook friendship network, Epinions social network and Slashdot network respectively. All the three empirical networks were obtained from Stanford Large Network Data Set Collection (Leskovec and Krevl, 2014). See the text for descriptions.



(a)

(b)

Figure 7: Figure 7a is a plot of the log values of $\Delta b_{i,t}^{n}$ for a randomly selected player in the Slashdot network. Figure 7b is a distribution of values of $\Delta b_{i,t}^{n}$ for a randomly selected player in Facebook friendship network.

Proof. See Appendix Appendix A.8 □

We conclude this section by highlighting the novelty of the above analysis in relation to existing literature. As already mentioned above, a closely related study is Acemoglu et al. (2014). In addition to the fundamental differences discussed in Section 1, Acemoglu et al. (2014) find that asymptotic learning obtains in the presence of "information hubs" which receive and distribute a large amount of information. Here, we have been able to show that the necessary condition for asymptotic learning is for the diameter of the network, hence the perfect observational radii, to be asymptotically bounded. This holds provided that no small group of agents has unlimited influence on others' posterior beliefs. Asymptotic learning in our model thus occurs even in some random and empirical networks provided that above conditions are satisfied.

5. Bayesian learning with unobservable prior beliefs

In this section, we consider the case of noisy learning, whereby prior beliefs of neighbours are not observable. Such uncertainty in prior beliefs leads to noisy signals as agents can only deduce expected signals based on their prior beliefs. We examine the effect of such noise on asymptotic learning.

To capture prior belief uncertainty, we assume that prior beliefs are independently and normally distributed across agents. That is, $\boldsymbol{\mu}_0 \sim \mathcal{N}(\boldsymbol{\nu}, \sigma_p^2 \mathbf{I})$, where $\boldsymbol{\mu}_0$ is a vector of prior beliefs, $\boldsymbol{\nu}$ is the vector of the means of prior beliefs and $\mathbf{I}$ is an identity matrix. Under this set up, the precision of i's posterior belief at time t is given by (see Lemma 4 in the proof of Proposition 2)

$$\rho_{i,t}^{n,\mathbf{a}} = \rho_\theta + \rho_\varepsilon + \gamma b_{i,t}^{n,\mathbf{a}} \tag{16}$$

where $\gamma = \frac{\rho_p \rho_\varepsilon^2}{\rho_\theta^2(1+\rho_p\rho_\varepsilon)}$ and $\rho_p = \frac{1}{\sigma_p^2}$ is the precision of prior beliefs. Consequently, as in Lemma 2, the exit times are the solutions to the difference equations

$$\Delta b_{i,t}^{n,\mathbf{a}} = \frac{\beta\left(\left(\rho_\theta + \rho_\varepsilon + \gamma b_{i,t}^{n,\mathbf{a}}\right)h - 1\right)\left(\rho_\theta + \rho_\varepsilon + \gamma b_{i,t}^{n,\mathbf{a}}\right)}{\rho_\varepsilon\left(1 - h\beta(\rho_\theta + \rho_\varepsilon + \gamma b_{i,t}^{n,\mathbf{a}})\right)} \quad \text{for } i = 1, \cdots, n \tag{17}$$

The presence of noise in signals due to unobservable priors generally prolongs exit times. This is observable from (16) where the rate at which the neighbourhood growth contributes to growth in confidence is a factor $\gamma < 1$. Note also that $\gamma < \rho_\varepsilon$. Hence for any values of r, ρ_θ, ρ_ε and ρ_p, and functional form of $b_{i,t}^{n,\mathbf{a}}$, $\rho_{i,t}^{n,\mathbf{a}}$ at any t under unobservable priors is less than $\rho_{i,t}^{n,\mathbf{a}}$ under observable priors. The respective exit times for the former are also generally less than for the later. As an example, consider the case of liner neighbourhood growth discussed in Section 3.2, where $b_{i,t}^{n,\mathbf{a}} = at + c$. Corollary 1 provides equilibrium exit times for the case of observable priors, where we find that for $a = 2$, $c = -3$, $h = \rho_\theta = \rho_\varepsilon = 1$ and $\beta = 0.001$, the exit time is 11 rounds of updating. Following similar steps and for $\rho_p = 1$, equilibrium exit times for unobservable priors is

$$t_{i,e}^{n,\mathbf{a}} = \frac{1}{\phi_1}\left(\left[\phi_2 + \frac{\phi_4}{\beta}\right]^{\frac{1}{2}} - \phi_3\right) \quad \text{for } i = 1, \cdots, n \tag{18}$$

where $\phi_1 = \frac{a^2}{2}$, $\phi_2 = \frac{1}{14}a^4 + \frac{1}{4}a^3 + \frac{1}{4}a^2$, $\phi_3 = \frac{3}{2}a + \frac{1}{4}a^2 + \frac{1}{2}ac$ and $\phi_4 = \frac{a^3}{2}\frac{1-\beta}{\beta}$

Substituting for $a = 2$, $c = -3$ and $\beta = 0.001$ results to exits times of 31 updating rounds, which is almost three times higher than the case for observable priors.

Given (17), the following proposition shows that asymptotic learning obtains when prior beliefs are unobservable.

Proposition 2. *Given a sequence $\{\mathcal{G}^n\}_{n\geq 2}$ of strongly connected networks, let $\{d_i(\mathcal{G}^n)\}_{i\in N, n\geq 2}$ be the corresponding sequence of agents' maximum geodesics and $\{ind_{i,j}^n\}_{i\in N, n\geq 2}$ the respective conditional in-degrees. Then asymptotic learning under unobservable priors obtains if*

(i) $\frac{1}{n}\sum_{i=1}^n \left(t_{i,e}^{n,\mathbf{a}} - d_i(\mathcal{G}^n)\right) \to 0$ *as $n \to \infty$, in probability*

(ii) $\lim_{n\to\infty} \frac{1}{n} ind_{i,j}^n = 0$ *for all pairs of i and j.*

Proof. See Appendix Appendix A.9 □

Proposition 2 shows that asymptotic learning under prior beliefs uncertainty obtains under similar conditions as for observable priors. The convergence rates to the true state of nature is however generally slower for the case of unobservable priors. For example if $\sigma_\varepsilon = \sigma_\theta = \sigma_p = \sigma$, then the convergence rate under unobservable priors is at least twice as slow compared to observable priors.[12] Similarly, as mentioned above, the actual values of equilibrium exit times are generally longer under unobservable priors by a factor of γ. The tendency for learning to converge relatively slower in the presence of noise was also observed by Vives (1995) in the model of learning in financial markets. Such delays in convergence in turn increase the likelihood of asymptotic learning.

The finding that asymptotic learning occurs under prior belief uncertainty implies that the noisiness of signals does not hinder correct aggregation of decentralized information, but only slows it. That is, the network does not act to amplify such noise in the signals at the aggregate level. This is contrary to other forms of learning where the network can amplify individual noise at aggregate level. For example, Acemoglu et al. (2012) study a model of inter-sectoral input–output linkages and show how the network capturing such linkages may lead to amplification of idiosyncratic shocks at the aggregate level.

6. Functional dependence between payoffs and signals

The analysis in above sections assumed functional forms in which payoffs are quadratic and beliefs/signals combination to be normally distributed. The question that arises is to what extent the above results can be generalized to other functional form relations? As noted in Sections 1 and 2, the main behavioural property our model captures is decision processes in which individual level of confidence plays a role. More specifically, the optimal payoff at any given period and hence the optimal time to act are functions of individual level of confidence in their belief. The quadratic payoff structure clearly captures this property. Quadratic payoffs are widely employed in economic models of individual behaviour both in non-strategic (as in the case considered above) and strategic interactions. For non-strategic settings, a more general payoff structure to that in Section 2.1 is $U_i(x_i, \theta) = h_i + \theta x_i - \frac{1}{2}x_i^2$, where h_i is i's intrinsic payoff, θx_i is the benefit to taking action x_i (e.g. the productivity of an investment) with θ representing some form of payoff uncertainty, x_i^2 is the cost of action x_i. For strategic interactions (see Jackson et al. (2015) for a survey of strategic interactions on networks), the payoffs are of the form $U_i(x_i, \theta, \mathbf{x}) = h_i + \theta x_i - \frac{1}{2}x_i^2 + \sum_{j=1}^{n} g_{ij} x_i x_j$, where the last term represents strategic interactions with neighbours.

In both cases, strategic and non-strategic interactions, the quadratic nature of the payoff function

[12] This follows from the fact that as $\text{var}[\mu_{i,e}^{n,\mathbf{a}}] \to 0$ at the limit of n, the slowest term is $\left(\frac{\gamma\alpha}{1+\gamma\alpha b_{i,e}^{n,\mathbf{a}}}\right)^2 \text{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}} B_{i,\tau}^n\right]$ for the case of unobservable priors, and $\frac{\sigma_\theta^4}{\left(b_{i,e}^{n,\mathbf{a}}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\left(\text{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}} S_{i,\tau}\right]\right)$ for the case of observable priors. The respective variances are: $\text{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}} S_{i,\tau}\right] = b_{i,e}^{n,\mathbf{a}}\sigma_\varepsilon^2$ and $\text{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}} B_{i,\tau}^n\right] = b_{i,e}^{n,\mathbf{a}}\left(\frac{\sigma_\varepsilon^4\sigma_p^2}{\sigma_\theta^4} + \sigma_\varepsilon^2\right)$. If $\sigma_\varepsilon = \sigma_\theta = \sigma_p = \sigma$, then $\text{var}[\mu_{i,e}^{n,\mathbf{a}}] \to 0$ at the rate of $\frac{2\sigma^{10} b_{i,e}^{n,\mathbf{a}}}{\left(1+\sigma^4 b_{i,e}^{n,\mathbf{a}}\right)^2}$ under unobservable priors, and $\frac{\sigma^2 b_{i,e}^{n,\mathbf{a}}}{\left(1+b_{i,e}^{n,\mathbf{a}}\right)^2}$ under observable priors.

ensures that the optimal payoff at any given period is a function of individual level of confidence in their belief. Therefore, agents face a trade-off between taking an action early enough to avoid costs of waiting versus waiting to receive more information to increase their confidence.

Unlike the necessary condition that payoffs be quadratic in nature, beliefs/signal combination need not be Gaussian. Gaussian beliefs/signal combination has a special property that individual confidence increases linearly with the number of signals, and the expected changes in confidence from the next signals received is not a function of current beliefs; such linearity and monotonicity simplify analysis. These properties are however not necessary for Theorem 3 to hold. They only influence the type of networks that satisfy condition (i) of Theorem 3. We demonstrate this assertion with the case of discrete beliefs/signals combination.

Let the payoffs be quadratic as in Section 2 above. Consider a binary choice set $\{0,1\}$, signal set $S=\{0,1\}$ and two possible states of nature $\Theta=\{\theta_0,\theta_1\}$. Assume that the conditional probabilities of signals are symmetric, that is $\mathbb{P}(s=0|\theta_0)=\mathbb{P}(s=1|\theta_1)=q$, and that $2q>1$, in which case signal $s=1$ increases the probability of θ_1; this follows from the likelihood ratio,

$$\frac{\mathbb{P}(\theta_1|s=1)}{\mathbb{P}(\theta_0|s=1)}=\frac{\mathbb{P}(s=1|\theta_1)}{\mathbb{P}(s=1|\theta_0)}\frac{\mathbb{P}(\theta_1)}{\mathbb{P}(\theta_0)}=\left(\frac{q}{1-q}\right)\frac{\mathbb{P}(\theta_1)}{\mathbb{P}(\theta_0)},$$

which implies that $\frac{\mathbb{P}(\theta_1|s=1)}{\mathbb{P}(\theta_0|s=1)}>\frac{\mathbb{P}(\theta_1)}{\mathbb{P}(\theta_0)}$ whenever $2q>1$. Assume also uniform prior beliefs such that the log-likelihood $\ln\left(\frac{\mathbb{P}(\theta_1)}{\mathbb{P}(\theta_0)}\right)=0$. Denote the log-likelihood ratio for i at time t after observing $b_{it}^{n,\mathbf{a}}$ signals, $\ln\left(\frac{\mathbb{P}(\theta_1|b_{it}^{n,\mathbf{a}})}{\mathbb{P}(\theta_0|b_{it}^{n,\mathbf{a}})}\right)$ by $\psi_{it}(b_{it}^{n,\mathbf{a}})$. If n_{it}^1 is the number of $s=1$ signals received by i at t, then[13]

$$\psi_{it}(b_{it}^{n,\mathbf{a}})=n_{it}^1\ln\left(\frac{q}{1-q}\right)+\left(b_{it}^{n,\mathbf{a}}-n_{it}^1\right)\ln\left(\frac{1-q}{q}\right)\tag{19}$$

The posterior belief $\mathbb{P}(\theta_1|b_{it}^{n,\mathbf{a}})$ of i after receiving $b_{it}^{n,\mathbf{a}}$ signal is then given by[14]

$$\mu_{it}^{n,\mathbf{a}}=\mathbb{P}(\theta_1|b_{it}^{n,\mathbf{a}})=\frac{\exp(\psi_{it}(b_{it}^{n,\mathbf{a}}))}{1+\exp(\psi_{it}(b_{it}^{n,\mathbf{a}}))}\tag{20}$$

The corresponding variance is

$$\mathrm{var}[\theta]=\mu_{it}^{n,\mathbf{a}}(1-\mu_{it}^{n,\mathbf{a}})=\frac{\exp(\psi_{it}(b_{it}^{n,\mathbf{a}}))}{\left[1+\exp(\psi_{it}(b_{it}^{n,\mathbf{a}}))\right]^2}\tag{21}$$

Compared to the case of Gaussian belief/signal combination, two properties stand out. First, the expected change in variance and hence confidence is non-monotone and is not independent of

[13]Relation (19) follows from the expression of the log-likelihood ratio and the assumption $\ln\left(\frac{\mathbb{P}(\theta_1)}{\mathbb{P}(\theta_0)}\right)=0$. That is,

$$\ln\left(\frac{\mathbb{P}(\theta_1|b_{it}^{n,\mathbf{a}})}{\mathbb{P}(\theta_0|b_{it}^{n,\mathbf{a}})}\right)=\ln\left(\frac{\mathbb{P}(s=1|\theta_1)}{\mathbb{P}(s=1|\theta_0)}\right)^{n_{it}^1}\left(\frac{\mathbb{P}(s=0|\theta_1)}{\mathbb{P}(s=0|\theta_0)}\right)^{b_{it}^{n,\mathbf{a}}-n_{it}^1}$$

[14]Relation (20) follows from the fact that $\mathbb{P}(\theta_1|b_{it}^{n,\mathbf{a}})=\mathbb{P}(\theta_0|b_{it}^{n,\mathbf{a}})\exp(\psi_{it}(b_{it}^{n,\mathbf{a}}))$. Substituting for $\mathbb{P}(\theta_0|b_{it}^{n,\mathbf{a}})=1-\mathbb{P}(\theta_1|b_{it}^{n,\mathbf{a}})$ yields the desired result.

current beliefs; it can increase or reduce depending on the signals received. Second, both posterior probabilities and variance evolve exponentially.

Consider the case in which each agent's exit time is equal or close to their perfect observational radius; as seen from Theorem 3, this corresponds to the situation in which each agent fully aggregates information. Let $b_{ie}^{n,\mathbf{a}}$ be the respective total number of signals i receives at equilibrium. For $b_{ie}^{n,\mathbf{a}}$ sufficiently large, $n_{it}^1 \approx b_{it}^{n,\mathbf{a}} q_1$, where q_1 is the probability of signal $s = 1$ being realized. The corresponding $\psi_{ie}(b_{ie}^{n,\mathbf{a}})$ is

$$\psi_{ie}(b_{ie}^{n,\mathbf{a}}) = b_{ie}^{n,\mathbf{a}} \left(2q_1 \ln \left(\frac{q}{1-q} \right) - \ln \left(\frac{q}{1-q} \right) \right)$$

If $2q_1 > 1$ and action 1 is optimal in state θ_1, then at the limit of n, $\mu_{ie}^{n,\mathbf{a}} \to 1$ and $\mathrm{var}[\theta] \to 0$ (see equations (20) and (21)), and hence action 1 is played in equilibrium; otherwise, action 0 is played in equilibrium. If action θ_1 is the true state of nature, and that $s = 1$ is informative of θ_1, that is $2q_1 > 1$, then agents correctly learn the true state and hence play the optimal action.

To arrive at this conclusion, we assumed that exit times are close or equal to the perfect observation radius for each agent. This holds under conditions of Theorem 3. Hence, the main conditions for asymptotic learning to obtain are the same for both discrete and Gaussian beliefs/signals combination. The difference however arises in convergence rates. We saw earlier that the convergence rate for Gaussian beliefs/signals combination is $1/b_{ie}^{n,\mathbf{a}}$. For discrete beliefs/signals combination however, we see from (21) that the variance decays at an exponential rate of $1/b_{ie}^{n,\mathbf{a}}$. Over all, equilibrium exit times given evolution of variance in (21) is a solution to the set of equations (where we write $\psi_{it}^{n,\mathbf{a}}$ as the short form for $\psi_{it}(b_{it}^{n,\mathbf{a}})$)[15]

$$\begin{aligned}
&h\beta(1 + \exp(\psi_{it}^{n,\mathbf{a}}))^2 (1 + \exp(\psi_{it}^{n,\mathbf{a}} + \Delta\psi_{it}^{n,\mathbf{a}}))^2 \\
&= \exp(\psi_{it}^{n,\mathbf{a}}) + \exp(3\psi_{it}^{n,\mathbf{a}} + 2\Delta\psi_{it}^{n,\mathbf{a}}) - \exp(\psi_{it}^{n,\mathbf{a}} + \Delta\psi_{it}^{n,\mathbf{a}} - r) - \exp(4\psi_{it}^{n,\mathbf{a}} - r) \\
&\quad \text{for } i = 1, \cdots, n.
\end{aligned}$$

7. Conclusion

We have developed a framework of word-of-mouth learning in networks whereby agents are faced with two constraints on information acquisition; the localization of information imposed by a network and the costs associated with waiting to become more informed. The benefit of waiting

[15] The proof follows that for Lemma 2. That is, by equating to zero the time difference of the payoff

$$U_{it}(I_{it}^n) = (h - \mu_{it}^{n,\mathbf{a}}(1 - \mu_{it}^{n,\mathbf{a}}))e^{-rt} = he^{-rt} - \frac{\exp(\psi_{it}(b_{it}^{n,\mathbf{a}}) - rt)}{[1 + \exp(\psi_{it}(b_{it}^{n,\mathbf{a}}))]^2}$$

to become more informed is that it increases ones confidence in their belief. The framework is that of an information externality game in which agents choose optimal times to act. Several social decisions fit into this framework and they range from consumer choices on products, technologies and services to political and investment decisions.

We have provided an explicit characterization of the relationship between equilibrium exit times and the network structure. We showed that highly connected networks generally have shorter exit times than less connected networks. We also study asymptotic learning, that is whether agents learn the true state of nature and hence the correct action. Asymptotic learning occurs under two conditions: (i) no small group of agents should have unlimited influence as measured by conditional in-degree. (ii) The underlying network must have a bounded diameter. Our results in part help explain patterns of social interactions in which there is localization of beliefs and choices, for example in portfolio investment (Hong et al., 2005). In a broader sense, our study shows that the role of network structure in the process of word-of-mouth learning is relatively different when compared to other learning mechanism in networks such as naïve learning, information percolation and strategic interactions.

Appendix A. Appendix

Appendix A.1. Proof of Lemma 1

The proof follows by iteration over $t = 1, 2, 3, \cdots$ At the beginning of $t = 1$, each i incorporates the observed signal s_i through Bayesian updating, and the resulting posterior belief is Normally distributed with mean and precision,

$$\mu_{i,1} = \frac{\sigma_\varepsilon^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}\mu_{i,0} + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}s_i \quad \text{and } \rho_{i,1}^n = \rho_\theta + \rho_\varepsilon. \tag{A.1}$$

Relation (A.1) follows from the fact that given $\theta \sim \mathcal{N}(\mu, \sigma_\theta^2)$ and $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, if $s = \theta + \varepsilon$ then Bayes Theorem implies

$$\mathbb{E}\left[\theta|s\right] = \frac{\sigma_\varepsilon^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}\mu + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2}s.$$

with a conditional variance of $\text{var}\left[\theta|s\right] = \frac{\sigma_\varepsilon^2\sigma_\theta^2}{\sigma_\theta^2+\sigma_\varepsilon^2}$. The precision is then $\rho_{i,1}^n = \frac{1}{\text{var}[\theta|s]} = \rho_\theta + \rho_\varepsilon$.

From (A.1), agent i deduces the signals of her first-order neighbours, all $j \in N_{i,1}$, by observing that

$$s_j = \left(\sigma_\theta^2 + \sigma_\varepsilon^2\right)\left(\frac{1}{\sigma_\theta^2}\mu_{j,1} - \frac{1}{\sigma_\varepsilon^2}\mu_{j,0}\right) \tag{A.2}$$

At the end of period $t = 1$ and beginning of period 2, agent i's new signal is then $\frac{1}{k_{i,1}^n}\sum_{j\in N_{i,1}} s_j$, which has a corresponding variance of $\frac{\sigma_\varepsilon^2}{k_{i,1}^n}$. The posterior belief for each $i \in N$ at the beginning of period 2 is normally distributed with mean and precision of

$$\begin{aligned}
\mu_{i,2}^{n} &= \frac{\sigma_\varepsilon^2/k_{i,1}^n}{\sigma_\theta^2\sigma_\varepsilon^2/(\sigma_\theta^2+\sigma_\varepsilon^2)+\sigma_\varepsilon^2/k_{i,1}^n}\mu_{i,1} + \frac{\sigma_\theta^2\sigma_\varepsilon^2/(\sigma_\theta^2+\sigma_\varepsilon^2)}{\sigma_\theta^2\sigma_\varepsilon^2/(\sigma_\theta^2+\sigma_\varepsilon^2)+\sigma_\varepsilon^2/k_{i,1}^n}\left(\frac{1}{k_{i,1}^n}\sum_{j\in N_{i,1}} s_j\right)\\
&= \frac{(\sigma_\theta^2+\sigma_\varepsilon^2)}{\sigma_\theta^2 k_{i,1}^n+(\sigma_\theta^2+\sigma_\varepsilon^2)}\mu_{i,1} + \frac{\sigma_\theta^2}{\sigma_\theta^2 k_{i,1}^n+(\sigma_\theta^2+\sigma_\varepsilon^2)}\sum_{j\in N_{i,1}} s_j\\
&= \frac{\sigma_\varepsilon^2}{(1+k_{i,1}^n)\sigma_\theta^2+\sigma_\varepsilon^2}\mu_{i,0} + \frac{\sigma_\theta^2}{(1+k_{i,1}^n)\sigma_\theta^2+\sigma_\varepsilon^2}\left(s_i+\sum_{j\in N_{i,1}} s_j\right) \qquad \text{(A.3)}
\end{aligned}$$

$$\begin{aligned}
\rho_{i,2}^{n} &= \left(\frac{\sigma_\theta^2\sigma_\varepsilon^2/(\sigma_\theta^2+\sigma_\varepsilon^2)\sigma_\varepsilon^2/k_{i,1}^n}{\sigma_\theta^2\sigma_\varepsilon^2/(\sigma_\theta^2+\sigma_\varepsilon^2)+\sigma_\varepsilon^2/k_{i,1}^n}\right)^{-1} = \left(\frac{\sigma_\theta^2\sigma_\varepsilon^4}{\sigma_\theta^2\sigma_\varepsilon^2 k_{i,1}^n+\sigma_\varepsilon^2(\sigma_\theta^2+\sigma_\varepsilon^2)}\right)^{-1}\\
&= \rho_\theta + (1+k_{i,1}^n)\rho_\varepsilon \qquad \text{(A.4)}
\end{aligned}$$

From period 2's messages, i can deduce the signals of her first-order neighbours' neighbours and subtracts her own and first-order neighbours' signals since they are known from period 1's messages. That is, for each $j \in N_{i,1}$, i deduces the signal statistic $s_i + \sum_{l\in N_{j,1}} s_l$ from j's message at the end of period 2 and subtracts signals s_i and that of any $l \in N_{j,1}$ who is also in $N_{i,1}$. Put together for all $j \in N_{i,1}$, the new signal at the beginning of period 3 is

$$S_{i,3} = \frac{1}{k_{i,2}^n - \bar{k}_{i,2}^n}\left(\sum_{j\in\mathcal{N}_{i,2}} s_j - \sum_{l\in\bar{\mathcal{N}}_{i,2}} s_l\right),$$

with the corresponding variance of $\sigma_\varepsilon^2/(k_{i,2}^n - \bar{k}_{i,2}^n)$. Following similar steps as in (A.3) and (A.4) above, the respective mean and precision for the posterior belief in period 3 are,

$$\begin{aligned}
\mu_{i,3}^{n} &= \frac{\sigma_\varepsilon^2}{(1+k_{i,1}^n+[k_{i,2}^n-\bar{k}_{i,2}^n])\sigma_\theta^2+\sigma_\varepsilon^2}\mu_{i,0}\\
&\quad + \frac{\sigma_\theta^2}{(1+k_{i,1}^n+[k_{i,2}^n-\bar{k}_{i,2}^n])\sigma_\theta^2+\sigma_\varepsilon^2}\left(s_i+\sum_{j\in N_{i,1}} s_j + \left[\sum_{j\in\mathcal{N}_{i,2}} s_j - \sum_{l\in\bar{\mathcal{N}}_{i,2}} s_l\right]\right)\\
&= \frac{\sigma_\varepsilon^2}{b_{i,2}^n\sigma_\theta^2+\sigma_\varepsilon^2}\mu_{j,0} + \frac{\sigma_\theta^2}{b_{i,2}^n\sigma_\theta^2+\sigma_\varepsilon^2}\sum_{\tau=1}^{3} S_{i,\tau} \qquad \text{(A.5)}
\end{aligned}$$

$$\rho_{i,3}^{n} = \rho_\theta + \left(1+k_{i,1}^n+[k_{i,2}^n-\bar{k}_{i,2}^n]\right)\rho_\varepsilon = \rho_\theta + b_{i,2}^n\rho_\varepsilon \qquad \text{(A.6)}$$

In period 4, each $j \in N_{i,1}$ announces to i, posterior beliefs of the form in (A.5) and (A.6). Agent i then subtracts the total signal received until period 3. The total signal deduced in period 4 then becomes

$$S_{i,4} = \frac{1}{k_{i,3}^n - \bar{k}_{i,3}^n}\left(\sum_{j\in\mathcal{N}_{i,3}} s_j - \sum_{l\in\bar{\mathcal{N}}_{i,3}} s_l\right),$$

The respective posterior belief of i at end of period 4 is normally distributed with mean and precision of

$$\mu^n_{i,4} = \frac{\sigma^2_\varepsilon}{b^n_{i,3}\sigma^2_\theta + \sigma^2_\varepsilon}\mu_{j,0} + \frac{\sigma^2_\theta}{b^n_{i,3}\sigma^2_\theta + \sigma^2_\varepsilon}\sum_{\tau=1}^{4} S_{i,\tau}$$

$$\rho^n_{i,4} = \rho_\theta + \left(1 + k^n_{i,1} + [k^n_{i,2} - \bar{k}^n_{i,2}] + [k^n_{i,3} - \bar{k}^n_{i,3}]\right)\rho_\varepsilon = \rho_\theta + b^n_{i,3}\rho_\varepsilon \tag{A.7}$$

It then follows by an iteration process that i's posterior belief at $\tau = t$ is as in (1).

Appendix A.2. Proof of Lemma 2

Given the value function, an agent's problem is to choose the optimal time to take an irreversible action and exit the game. This can be achieved by optimizing with respect to t the objective function $U^n_{it}(I^n_{i,t}) = \left(h - \frac{1}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon}\right)e^{-rt}$.

The first order condition yields

$$\begin{aligned}\frac{\Delta U_{i,t}(I^n_{i,t})}{\Delta t} &= U_{i,t+1t}(I^n_{i,t+\Delta t}) - U_{i,t}(I^n_{i,t}) \\ &= \left(h - \frac{1}{\rho_\theta + b^{n,\mathbf{a}}_{i,t+1}\rho_\varepsilon}\right)e^{-r(t+1)} - \left(h - \frac{1}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon}\right)e^{-rt} = 0.\end{aligned} \tag{A.8}$$

where we substitute for $\Delta t = 1$ in the first equality of (A.8), which can be simplified as follows.

$$e^{-rt}\left\{-h(1-e^{-r}) + \frac{1}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon}\left\{\frac{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t} - e^{-r}(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon)}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t}}\right\}\right\} = 0$$

$$-h(1-e^{-r}) + \frac{1}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon}\left\{\frac{\left(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon\right)(1-e^{-r}) + \rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t}}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t}}\right\} = 0 \tag{A.9}$$

Let $\beta = (1 - e^{-r})$. Then

$$-h\beta\left\{\left(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon\right)^2 + \left(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon\right)\rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t}\right\} + \left(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon\right)\beta + \rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t} = 0 \tag{A.10}$$

Simplifying (A.10) yields

$$\Delta b^{n,\mathbf{a}}_{i,t} = \frac{\beta\left(\left(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon\right)h - 1\right)\left(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon\right)}{\rho_\varepsilon\left(1 - h\beta(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon)\right)} \tag{A.11}$$

Appendix A.3. Proof of Proposition 1

From (A.9) in the proof of Lemma 2, we see that for a given $b^{n,\mathbf{a}}_{i,t}$ an equilibrium exit time is the t at the intersection of the two functions $f(\beta) = h\beta$ and

$$g(t) = \frac{\beta}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t}} + \frac{\rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t}}{(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon)(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon\Delta b^{n,\mathbf{a}}_{i,t})}$$

Note that $b^{n,\mathbf{a}}_{i,t}$ is a monotonically increasing function of t. Hence if $b^{n,\mathbf{a}}_{i,t}$ first order stochastically dominates $b^{n,\mathbf{a}'}_{i,t}$, then the respective functions $g(t)$ and $g'(t)$ are such that $g(t) \le g'(t)$. Consequently, the values of t for which $g(t) = f(\beta)$ and $g'(t) = f(\beta)$ are such that $t^{n,\mathbf{a}}_{i,e} \le t^{n,\mathbf{a}'}_{i,e}$.

Appendix A.4. Proof of Corollary 1

Given $b_{i,t}^{n,\mathbf{a}} = at + c$, $\Delta b_{i,t}^{n,\mathbf{a}} = a$. Substituting for $h = 1$, $\rho_\theta = \rho_\varepsilon = 1$, and $\Delta b_{i,t}^{n,\mathbf{a}} = a$ into (2) yields,

$$a(1 - \beta(at + (1 + c))) = \beta(at + (1 + c))^2 - \beta(at + (1 + c))$$

$$a^2\beta t^2 + a\beta(a + 2c + 1)t + \beta(1 + c)(a + c) - a = 0 \tag{A.12}$$

A solution to (A.12) is

$$t_e = \frac{1}{2a^2}\left(\left[4a^4 - 2a^3 + a^2 + \frac{4a^3}{\beta}\right]^{\frac{1}{2}} - (a^2 + 2ac + a)\right)$$

By letting $\phi_1 = 2a^2$, $\phi_2 = 4a^4 - 2a^3 + a^2$, $\phi_3 = a^2 + 2ac + a$ and $\phi_4 = 4a^3$, we obtain equation (10).

Appendix A.5. Proof of Corollary 2

Given $b_{i,t}^{n,\mathbf{a}} = a^t + c$, $\Delta b_{i,t}^{n,\mathbf{a}} = a^t a_1$, where $a_1 = a - 1$. Substituting for $h = \rho_\theta = \rho_\varepsilon = 1$, and $\Delta b_{i,t}^{n,\mathbf{a}} = a^t a_1$ into (2) yields,

$$a^t a_1(1 - \beta(a^t + (1 + c))) = \beta(a^t + (1 + c))^2 - \beta(a^t + (1 + c))$$

$$-a\beta a^{2t} + [\beta(1 - (1 + c)(a + 1)) + a_1]a^t - \beta c(1 + c) = 0 \tag{A.13}$$

Let $z = a^t$; substituting in (A.13) yields,

$$az^2 - \left[(1 - (1 + c)(a + 1)) + \frac{a_1}{\beta}\right] z + c(1 + c) = 0 \tag{A.14}$$

Let $\psi_\beta = \left(1 + \frac{a_1}{\beta}\right)$, $\psi_1 = (1 + a)(1 + c)$. A solution to (A.14) is

$$z = \frac{1}{2a}\left(\psi_\beta - \psi_1 + \left[\psi_\beta^2 - 2\psi_1\psi_\beta + \psi_2\right]^{\frac{1}{2}}\right)$$

where $\psi_2 = (1 + a)^2(1 + c)^2 - 4ac(1 + c)$. The respective equilibrium exit times then assume the form

$$t_e = \frac{1}{\ln(a)} \ln\left[\frac{1}{2a}\left(\psi_\beta - \psi_1 + \left[\psi_\beta^2 - 2\psi_1\psi_\beta + \psi_2\right]^{\frac{1}{2}}\right)\right]$$

Appendix A.6. Proof of Theorem 3

The first step to the proof is realizing that in the absence of costs associated with waiting, an agent stops learning (acquiring new information) at the time equal to their maximum geodesic. This also implies that learning at the society level stops at the period equal to the maximum geodesic. This relation follows directly from the fact that under rational learning with observable priors, at every t, agent i receives signals from the t-order neighbours. So at $t = d_i(\mathcal{G}^n)$ agent i will have received signals from all other agents in the population.

Now, consider the contrary case in which the condition

$$\frac{1}{n}\sum_{i-1}^{n}\left(t_{i,e}^{n,\mathbf{a}}-d_i(\mathcal{G}^n)\right)\to 0 \qquad \text{as } n\to\infty \text{, in probability}$$

is not fulfilled. That is,

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{1}{n}\sum_{i-1}^{n}|t_{i,e}^{n,\mathbf{a}}-d_i(\mathcal{G}^n)|>\epsilon\right)>0 \tag{A.15}$$

Let $Y^n=\frac{1}{n}\sum_{i-1}^{n}\left(t_{i,e}^{n,\mathbf{a}}-d_i(\mathcal{G}^n)\right)$, then from Chebyshev's inequality

$$\mathbb{P}\left(\frac{1}{n}\sum_{i-1}^{n}|t_{i,e}^{n,\mathbf{a}}-d_i(\mathcal{G}^n)|>\epsilon\right)\leq\frac{\mathrm{var}(Y^n)}{\epsilon^2}$$

where $\mathrm{var}(Y^n)$ is the variance of Y^n. Equation (A.15) then implies that the limit of $\mathrm{var}(Y^n)$ as $n\to\infty$ is positive. This in turn implies that the number of agents for whom $t_{i,e}^{n,\mathbf{a}}<d_i(\mathcal{G}^n)$ grows with n. Denote this set of agents by Q_e^n. Denote also by $x_{i,e}^{n,\mathbf{a}}$ for the corresponding action taken by an agent $i\in Q_e^n$ at the end of learning process, and by $n_{i,e}$ as the number of agents whose signals i received before exiting the game. Let $n_e=\max_{i\in Q_e^n}\{n_{i,e}\}$.

For the model considered in this paper $x_{i,e}^{n,\mathbf{a}}=\mathbb{E}[\theta|I_{i,e}^{n,\mathbf{a}}]=\mu_{i,e}^{n,\mathbf{a}}$, where $I_{i,e}^{n,\mathbf{a}}$ and $\mu_{i,e}^{n,\mathbf{a}}$ are the information set and posterior mean after i takes an irreversible action and exits play. Let $b_{i,e}^{n,\mathbf{a}}$ be the $b_{i,t}^{n,\mathbf{a}}$ at the end of learning process.

Recall that for a finite network $\mathcal{G}^n$, if waiting is not costly then agent i's belief at the end of learning process will be normally distributed with mean

$$\mu_{i,\infty}^{n,\mathbf{a}}=\frac{\sigma_\varepsilon^2}{b_{i,e}^{n,\mathbf{a}}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2}\mu_{i,0}+\frac{1}{b_{i,e}^{n,\mathbf{a}}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2}\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}}S_{i,\tau}$$

Given the sequence of networks $\{\mathcal{G}^n\}_{n\geq 2}$ and an agent $i\in Q_e^n$ for whom $n_{i,e}=n_e$ such that $b_{i,e}^{n,\mathbf{a}}:=b_{i,e}^{n,\mathbf{a}}(n_e)$ is a linear function of n_e, the following relation holds

$$\begin{aligned}\mathrm{var}[x_{i,e}^{n,\mathbf{a}}]=\mathrm{var}[\mu_{i,e}^{n,\mathbf{a}}]&=\mathrm{var}\left[\frac{\sigma_\varepsilon^2}{b_{i,e}^{n,\mathbf{a}}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2}\mu_{i,0}+\frac{\sigma_\theta^2}{b_{i,e}^{n,\mathbf{a}}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2}\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}}S_{i,\tau}\right]\\&=\frac{\sigma_\varepsilon^4\sigma_\theta^2}{\left(b_{i,e}^{n,\mathbf{a}}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}+\frac{\sigma_\theta^4}{\left(b_{i,e}^{n,\mathbf{a}}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\mathrm{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}}S_{i,\tau}\right]\end{aligned}$$

The correct action x^* obtains when private information is fully aggregated such that $x^*=$

$\lim_{n\to\infty}\mathbb{E}[\theta|I^n_{i,\infty}]=\bar{\theta}$, the true state of the world. From Chebyshev's inequality, it follows that

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{P}\left(|x^{n,\mathbf{a}}_{i,e}-\bar{\theta}|>\epsilon\right) &> \lim_{n\to\infty}\left(1-\frac{\text{var}[x^{n,\mathbf{a}}_{i,e}]}{\epsilon^2}\right)\\
&= \lim_{n\to\infty}\left(1-\frac{\sigma_\varepsilon^4\sigma_\theta^2}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}+\frac{\sigma_\theta^4}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\text{var}\left[\sum_{\tau=1}^{t^{n,\mathbf{a}}_{i,e}}S_{i,\tau}\right]\right)\\
&= 1-\frac{\sigma_\varepsilon^4\sigma_\theta^2}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}+\frac{\sigma_\theta^4}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}(n_e)\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\lim_{n\to\infty}\left(\text{var}\left[\sum_{\tau=1}^{t^{n,\mathbf{a}}_{i,e}}S_{i,\tau}\right]\right)\\
&> 0
\end{aligned}
\tag{A.16}
$$

in which case $\lim_{n\to\infty}x^{n,\mathbf{a}}_{i,e}\neq\bar{\theta}$. If this is true for an agent $i\in Q^n_e$ for whom $n_{i,e}=n_e$, and from the fact that for all $i\in Q^n_e$ $n_{i,e}\leq n_e$, then it must be true for all $i\in Q^n_e$.

The second condition of Theorem 3 follows from examining the term var $\left[\sum_{\tau=1}^{t^{n,\mathbf{a}}_{i,e}}S_{i,\tau}\right]$ in (A.16). Consider the case in which a small subgroup N_s of agents of size n_s possess unlimited influence on others'posterior beliefs such that their in-degrees are increasing functions of the population size. That is, for all $i\neq j$ where $j\in N_s$, $\text{ind}^n_{i,j}:=f(n)$, where $f(n)$ is an increasing function of n. Assume that Condition (i) of Theorem 3 holds such that each $i\in N$ fully aggregate information in equilibrium. Then

$$\text{var}\left[\sum_{\tau=1}^{t^{n,\mathbf{a}}_{i,e}}S_{i,\tau}\right]=(b^{n,\mathbf{a}}_{i,e}-n_sf(n))\sigma_\varepsilon^2+n_sf(n)^2\sigma_\varepsilon^2$$

where $g(n)=b^{n,\mathbf{a}}_{i,e}-n_sf(n)$ also increases linearly with n. From Chebyshev's inequality, the limit of $\mathbb{P}\left(|x^{n,\mathbf{a}}_{i,e}-\bar{\theta}|>\epsilon\right)$ becomes

$$
\begin{aligned}
\lim_{n\to\infty}\mathbb{P}\left(|x^{n,\mathbf{a}}_{i,e}-\bar{\theta}|>\epsilon\right) &= \lim_{n\to\infty}\left(1-\frac{\sigma_\varepsilon^4\sigma_\theta^2}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}+\frac{\sigma_\theta^4}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\text{var}\left[\sum_{\tau=1}^{t^{n,\mathbf{a}}_{i,e}}S_{i,\tau}\right]\right)\\
&= 1-\lim_{n\to\infty}\left(\frac{\sigma_\theta^4}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\text{var}\left[\sum_{\tau=1}^{t^{n,\mathbf{a}}_{i,e}}S_{i,\tau}\right]\right)\\
&= 1-\lim_{n\to\infty}\left(\frac{\sigma_\theta^4((b^{n,\mathbf{a}}_{i,e}-n_sf(n))\sigma_\varepsilon^2+n_sf(n)^2\sigma_\varepsilon^2)}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\right)\\
&= 1-\lim_{n\to\infty}\left(\frac{\sigma_\theta^4\sigma_\varepsilon^2n_sf(n)^2}{\epsilon^2\left(b^{n,\mathbf{a}}_{i,e}\sigma_\theta^2+\sigma_\varepsilon^2\right)^2}\right)
\end{aligned}
$$

From the definition of $g(n)$ above, we can write $b^{n,\mathbf{a}}_{i,e}=n_sf(n)+g(n)$. Since $g(n)$ and $f(n)$ are both linear in n, then $b^{n,\mathbf{a}}_{i,e}$ must also be linear in n. Hence,

$$\lim_{n\to\infty}\mathbb{P}\left(|x^{n,\mathbf{a}}_{i,e}-\bar{\theta}|>\epsilon\right)=1-\frac{\sigma_\theta^4\sigma_\varepsilon^2n_sC}{\epsilon^2}>0 \tag{A.17}$$

where $0 < C < 1$ is some real number. For the variance $\text{var}[x^{n,\mathbf{a}}_{i,e}]$ to be asymptotically equal zero, and hence asymptotic learning to occur, both conditions (i) and (ii) of Theorem 3 must thus be fulfilled.

Appendix A.7. Proof of Corollary 3

From the proof of 1, $t^{n,\mathbf{a}}_{i,e}$ is the t at the intersection of $f(\beta) = h\beta$ and

$$g(t) = \frac{\beta}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon \Delta b^{n,\mathbf{a}}_{i,t}} + \frac{\rho_\varepsilon \Delta b^{n,\mathbf{a}}_{i,t}}{(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon)(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon \Delta b^{n,\mathbf{a}}_{i,t})}$$

Provided $r > 0$, and hence $\beta > 0$, such a t is non-negative and finite for any non-decreasing functional form of $b^{n,\mathbf{a}}_{i,t}$. Since the condition for asymptotic learning is for $\lim_{n\to\infty} \frac{t^{n,\mathbf{a}}_{i,e}}{d_i(\mathcal{G}^n)} = 1$, it follows that $d_i(\mathcal{G}^n)$ must be bounded for each i.

Appendix A.8. Proof of Corollary 5

Recall again that $t^{n,\mathbf{a}}_{i,e}$ is the t at which $f(\beta)$ intersects $g(t)$, where

$$g(t) = \frac{\beta}{\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon \Delta b^{n,\mathbf{a}}_{i,t}} + \frac{\rho_\varepsilon \Delta b^{n,\mathbf{a}}_{i,t}}{(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon)(\rho_\theta + b^{n,\mathbf{a}}_{i,t}\rho_\varepsilon + \rho_\varepsilon \Delta b^{n,\mathbf{a}}_{i,t})}$$

Given the perfect observational radius $t^n_{i,e}$ for each i, let $b^n_{i,e}$ be the corresponding value of $t^n_{i,e}$-order neighbourhood size. Denote by $g(t^n_{i,e})$ the corresponding value of $g(t)$.

For any non-decreasing functional form of $b^n_{i,t}$, including that in (15), $b^n_{i,e}$ is positive and finite provided $t^n_{i,e}$ is asymptotically bounded. For each $b^n_{i,e} > 0$, there exists a value of $r > 0$ and hence $\beta > 0$ at which $t^{n,\mathbf{a}}_{i,e} = t^n_{i,e}$ for each i. Under this condition, each i fully aggregate information. For a sequence $\{\mathcal{G}\}_{n\geq 2}$ of networks with bounded diameter and hence perfect observational radius $t^n_{i,e}$ for each i, even if $g(t^n_{i,e}) \to 0$ at the limit of n, $t^\infty_{i,e}$ is finite; where $t^\infty_{i,e} = \lim_{n\to\infty} t^n_{i,e}$. It follows that there exists a value of r, $\bar{r} > 0$, at which $g(t^\infty_{i,e} - 1) > f(\beta)$. For any $r \leq \bar{r}$, $t^{n,\mathbf{a}}_{i,e} = t^n_{i,e}$, in which case asymptotic learning obtains.

Appendix A.9. Proof of Proposition 2

The first step to the proof is to derive the structure of posterior beliefs at the end of the learning process.

Lemma 4. *Given a strongly connected network $\mathcal{G}^n$ and that $\boldsymbol{\mu}_0 \sim \mathcal{N}(\boldsymbol{\nu}, \sigma_p^2\mathbf{I})$, if information is fully aggregated at the end of the learning process (or equivalently if $r = 0$) then the posterior belief of each i after t periods is normally distributed with mean and precision*

$$\mu^{n,\mathbf{a}}_{i,t} = \frac{1}{1+\gamma\alpha b^{n,\mathbf{a}}_{i,t}}\mu_{i,1} + \frac{\gamma\alpha}{1+\gamma\alpha b^{n,\mathbf{a}}_{i,t}}\sum_{\tau=1}^{t} B^n_{i,\tau}$$

$$\rho^{n,\mathbf{a}}_{i,t} = \rho_\theta + \rho_\varepsilon + \gamma b^{n,\mathbf{a}}_{i,t}$$

where $\mathbb{E}_j[s_l|\nu_l] = \frac{\rho_\theta}{\rho_\varepsilon}(\mu_{l,0} - \nu_l) + s_l$ *is the expected signal of* l *according to* j *given* ν_l*;* $\alpha = \frac{\sigma_\theta^2\sigma_\varepsilon^2}{\sigma_\theta^2+\sigma_\varepsilon^2}$ *is the variance of* $\mu_{i,1}$*; and*

$$B^n_{i,\tau} = \sum_{j\in N_{i,t-1}} \left(\sum_{l\in N_{j,1}} \mathbb{E}_j[s_l|\nu_l] - \sum_{m\in \bar{N}_{j,1}} \mathbb{E}_j[s_m|\nu_m] \right),$$

Proof. Let $\mathbb{E}_j[s_l|\nu_l]$ for each $l \in N_{j,1}$ denote the expected signal of l according to j given the distribution of l's prior belief. After observing his neighbours first period announcements, i deduces the expected signals of each $j \in N_{i,1}$ to be

$$\mathbb{E}_i[s_j|\nu_j] = \frac{\sigma_\theta^2+\sigma_\varepsilon^2}{\sigma_\theta^2}\mu_{j,1} - \frac{\sigma_\varepsilon^2}{\sigma_\theta^2}\nu_j = \frac{\rho_\theta}{\rho_\varepsilon}(\mu_{j,0} - \nu_j) + s_j$$

where the second equality results from substituting for $\mu_{j,1}$. The corresponding variance of the expected signal according to i is $\text{var}[\mathbb{E}_i[s_j|\nu_j]] = \frac{\sigma_\varepsilon^4\sigma_p^2}{\sigma_\theta^4} + \sigma_\varepsilon^2 = \frac{\rho_\theta^2(1+\rho_p\rho_\varepsilon)}{\rho_p\rho_\varepsilon^2} \equiv \frac{1}{\gamma}$.

After incorporating the expected signals from his neighbours, and following similar steps as in the proof of Lemma 1, each $i \in N$ updates his beliefs to a normal distribution with mean and precision

$$\begin{aligned}\mu^{n,\mathbf{a}}_{i,2} &= \frac{\gamma^{-1}/k_i}{\gamma^{-1}/k_i+\alpha}\mu_{i,1} + \frac{\alpha}{\gamma^{-1}/k_i+\alpha}\frac{1}{k_i}\sum_{j\in N_{i,1}}\mathbb{E}_i[s_j|\nu_j] \\ &= \frac{1}{1+\gamma\alpha k_i}\mu_{i,1} + \frac{\gamma\alpha}{1+\gamma\alpha k_i}\sum_{j\in N_{i,1}}\mathbb{E}_i[s_j|\nu_j] \end{aligned} \tag{A.18}$$

and noting that $\text{var}[\mu_{i,1}] = \frac{\sigma_\theta^2\sigma_\varepsilon^2}{\sigma_\theta^2+\sigma_\varepsilon^2} = \frac{1}{\rho_\theta+\rho_\varepsilon}$

$$\rho^{n,\mathbf{a}}_{i,t} = \frac{1}{\text{var}^{n,\mathbf{a}}_{i,2}} = \frac{\frac{1}{\rho_\theta+\rho_\varepsilon} + \gamma^{-1}/k_i}{\frac{1}{\rho_\theta+\rho_\varepsilon}\gamma^{-1}/k_i} = \rho_\theta + \rho_\varepsilon + \gamma k_i$$

In period 3, there will be $k^n_{i,2} - \bar{k}^n_{i,2}$ new signals, that is $\sum_{j\in N_{i,1}}\sum_{l\in N_{j,1}}\mathbb{E}_j[s_l|\nu_l] - \sum_{j\in N_{i,1}}\sum_{m\in\bar{N}_{j,1}}\mathbb{E}_j[s_m|\nu_m]$, each with variance γ^{-1}. Through an iterative process as in the proof of Lemma 1, and letting

$$B^n_{i,\tau} = \sum_{j\in N_{i,t-1}} \left(\sum_{l\in N_{j,1}} \mathbb{E}_j[s_l|\nu_l] - \sum_{m\in \bar{N}_{j,1}} \mathbb{E}_j[s_m|\nu_m] \right),$$

the posterior belief of i at period t is normally distributed with mean and precisions of

$$\mu^{n,\mathbf{a}}_{i,t} = \frac{1}{1+\gamma\alpha b^{n,\mathbf{a}}_{i,t}}\mu_{i,1} + \frac{\gamma\alpha}{1+\gamma\alpha b^{n,\mathbf{a}}_{i,t}}\sum_{\tau=1}^{t} B^n_{i,\tau}$$

$$\rho^{n,\mathbf{a}}_{i,t} = \rho_\theta + \rho_\varepsilon + \gamma b^{n,\mathbf{a}}_{i,t}$$

□

The variance of $\mu^{n,\mathbf{a}}_{i,e}$ at the end of the learning process is

$$\text{var}[\mu^{n,\mathbf{a}}_{i,e}] = \left(\frac{1}{1+\gamma\alpha b^{n,\mathbf{a}}_{i,e}}\right)^2 \alpha + \left(\frac{\gamma\alpha}{1+\gamma\alpha b^{n,\mathbf{a}}_{i,e}}\right)^2 \text{var}\left[\sum_{\tau=1}^{t^{n,\mathbf{a}}_{i,e}} B^n_{i,\tau}\right]$$

From Chebyshev's inequality, it follows that

$$\lim_{n\to\infty} \mathbb{P}\left(|x_{i,e}^{n,\mathbf{a}} - \bar{\theta}| > \epsilon\right) > \lim_{n\to\infty}\left(1 - \frac{\mathrm{var}[x_{i,e}^{n,\mathbf{a}}]}{\epsilon^2}\right)$$
$$= \lim_{n\to\infty}\left(1 - \frac{1}{\epsilon^2}\left(\frac{1}{1+\gamma\alpha b_{i,e}^{n,\mathbf{a}}}\right)^2 \alpha + \frac{1}{\epsilon^2}\left(\frac{\gamma\alpha}{1+\gamma\alpha b_{i,e}^{n,\mathbf{a}}}\right)^2 \mathrm{var}\left[\sum_{\tau=1}^{t_{i,e}^{n,\mathbf{a}}} B_{i,\tau}^{n}\right]\right)$$

Following similar steps as in the proof of Theorem 3, it follows that the variance $\mathrm{var}[\mu_{i,e}^{n,\mathbf{a}}]$ is asymptotically zero under conditions (i) and (ii) of Proposition 2.

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