LECTURE 3

PREVIOUS LECTURE RECAP

- The number of dimensions that the quantum state is living in will be clear from the context.

- We need to define inner product.

If $|\phi\rangle = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$ then $\langle\phi| = \begin{bmatrix} \alpha_0^* & \cdots & \alpha_{N-1}^* \end{bmatrix}$

read "ket ϕ"

read "bra ϕ"

conjugate transpose of $|\phi\rangle$

We have inner product of $|\phi\rangle$ and $|\psi\rangle$ defined as

$\langle\phi| . |\psi\rangle$ denoted as $\langle\phi|\psi\rangle$

Notice: For $|\phi\rangle = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$,

$$\langle\phi|\phi\rangle = \begin{bmatrix} \alpha_0^* & \cdots & \alpha_{N-1}^* \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = \sum_{i=0}^{N-1} \alpha_i^* \alpha_i$$

$$= \sum_{i=0}^{N-1} |\alpha_i|^2$$

Measuring a quantum state:

Q.M. Law:

- We can measure a quantum state $|\phi\rangle = \begin{bmatrix} \alpha_0 \\ \vdots \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$

- The state collapses to $|j\rangle$ w.p. $|\alpha_j|^2$.

- The original superposition is lost.

We assume: $\sum_{i=0}^{N-1} |\alpha_i|^2 = 1.$

Quantum State in another basis

Suppose $|\phi\rangle = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$

We can write $|\phi\rangle$ in any orthonormal basis

$v_0, \ldots, v_{N-1}$

$\langle v_i | v_j \rangle = 0$ if $i \neq j$
$= 1$ if $i = j$

$$|\phi\rangle = \beta_0 |v_0\rangle + \ldots + \beta_{N-1} |v_{N-1}\rangle$$

β_i : "Measure of projection" of $|\phi\rangle$ on $|v_i\rangle$

Mathematically $\beta_i = \langle v_i | \phi \rangle$

$\langle v_i | \left(\beta_0 |v_0\rangle + \ldots\ldots + \beta_i | v_{N-1} \rangle \right)$

Unitary transformation :

We start with $|\phi\rangle = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = \alpha_0|0\rangle + \alpha_1|1\rangle + \ldots + \alpha_{N-1}|N-1\rangle$

Q.M. Law : We can transform the state to

$\alpha_0|u_0\rangle + \ldots + \alpha_{N-1}|u_{N-1}\rangle$ where $|u_0\rangle, \ldots, |u_{N-1}\rangle$

is an orthonormal basis.

The transformation that does this is

$$U = \begin{bmatrix} | & | & & | \\ u_0 & u_1 & \cdots & u_{N-1} \\ | & | & & | \end{bmatrix}$$

Transform basis vectors correctly.
Linearity does the rest.

$$U|\phi\rangle = \alpha_0 |u_0\rangle + \cdots + \alpha_{N-1} |u_{N-1}\rangle \quad \text{(Check!)}$$

Also $U^\dagger := (U^*)^T = U^{-1}$ (From orthonormality)

Thus, given

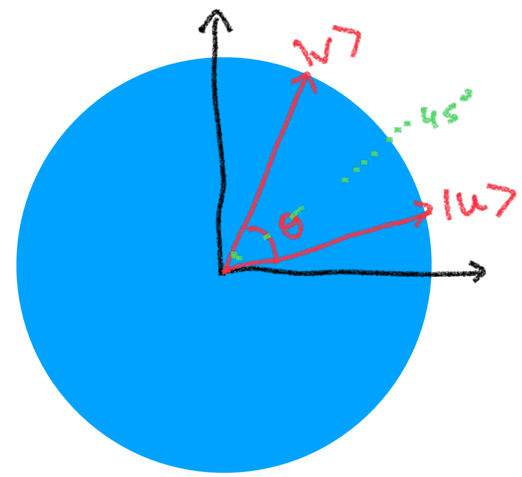
$$|\psi\rangle = \beta_0 |u_0\rangle + \ldots \ldots + \beta_{N-1} |u_{N-1}\rangle$$

$$U^{-1}|\psi\rangle = U^{\dagger}|\psi\rangle = \beta_0 |0\rangle + \ldots \ldots + \beta_{N-1} |u_{N-1}\rangle$$

Question: Distinguish $|u\rangle$ from $|v\rangle$

$|v\rangle$: good

$|u\rangle$: bad



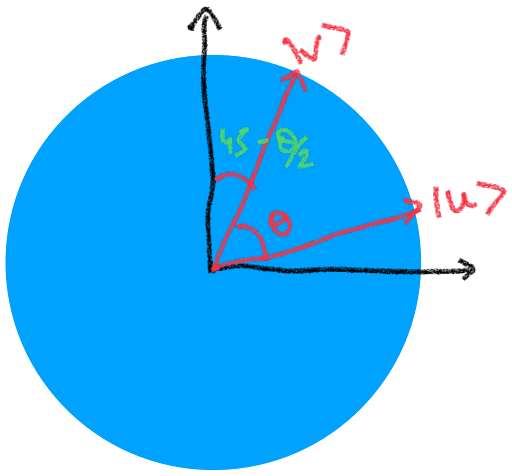
w.l.o.g. assume $|u\rangle = \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)|1\rangle$

$|v\rangle = \cos\left(\pi/4 + \frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)|1\rangle$

Measure in standard basis

Measure $|1\rangle$: Output $\underline{|v\rangle}$

Measure $|0\rangle$: Output $\underline{|u\rangle}$



Probability[error] =

Given $|v\rangle$, $P_r[\text{output } |v\rangle] = \cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

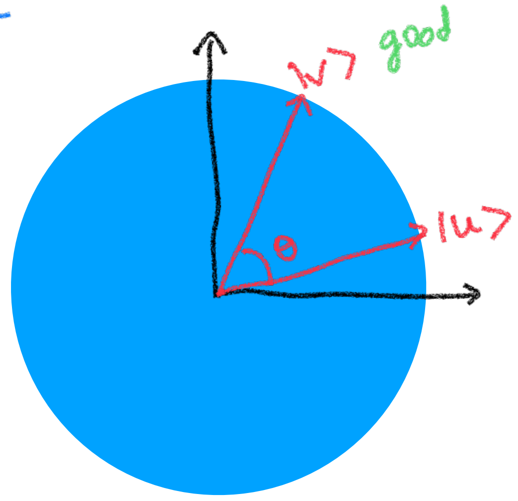
$P_r[\text{output } |u\rangle] = \sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

$P_r[\text{error}] = \sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$

Measure in $|v\rangle$, $|v^\perp\rangle$

$\Pr[\text{False negative}] = 0$

$\Pr[\text{False positive}] = \cos^2\theta$

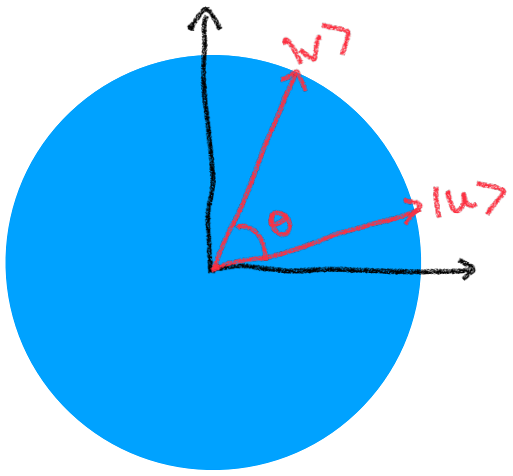


$\text{Prob}[\text{error}] = \dfrac{\cos^2\theta}{2}$

Measure in $|u\rangle$, $|u^{\perp}\rangle$

$\Pr[\text{False positive}] = 0$

$\Pr[\text{False negative}] = \cos^2\theta$



$$\Pr[\text{error}] = \frac{\cos^2\theta}{2}$$

How can we make $\Pr[\text{Wrong answer}] = 0$

Q : Can we distinguish $|u\rangle$ and $-|u\rangle$?

$|u\rangle$ and $i|u\rangle$?

Today: All remaining laws of quantum mechanics

The intuition from prob. theory will guide you.

4-dimensional quantum state.

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \qquad \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle + \alpha_3|3\rangle$$

- just names given to basic states
- Could use other names

e.g. $\alpha_0|red\rangle + \alpha_1|blue\rangle + \alpha_2|green\rangle$
$+ \alpha_3|orange\rangle$

The most common 4 dim state is 2 qubits

$$|\phi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$|00\rangle$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $|01\rangle$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $|10\rangle$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $|11\rangle$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$|\phi\rangle = \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

Note : we write $|00\rangle$ instead of $|0\rangle|0\rangle$

Suppose Alice has $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

Bob has $|\phi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$

Q1. What is the joint 4 dim state?

Q 2 If Alice applies U to $|\psi\rangle$, how does the joint state change?

Q3 If Bob measures his qubit, what is the joint state?

Your intuition from prob. theory will tell you the answers.

What is the joint state $\begin{bmatrix} \gamma_{00} \\ \gamma_{01} \\ \gamma_{10} \\ \gamma_{11} \end{bmatrix}$

$\gamma_{00} = \alpha_0 \beta_0 \quad \gamma_{01} = \alpha_0 \beta_1 \quad \gamma_{10} = \alpha_1 \beta_0 \quad \gamma_{11} = \alpha_1 \beta_1$

Check: $|\gamma_{00}|^2 + |\gamma_{01}|^2 + |\gamma_{10}|^2 + |\gamma_{11}|^2 = |\alpha_0 \beta_0|^2 + |\alpha_0 \beta_1|^2 + |\alpha_1 \beta_0|^2 + |\alpha_1 \beta_1|^2 = (|\alpha_0|^2 + |\alpha_1|^2)(|\beta_0|^2 + |\beta_1|^2)$

More generally:

Joint state of

$$
\begin{array}{c}
|0\rangle \\ |1\rangle \\ \vdots \\ |M-1\rangle
\end{array}
\begin{bmatrix} \alpha_0 \\ \vdots \\ \vdots \\ \alpha_{M-1} \end{bmatrix}
\quad \text{and} \quad
\begin{array}{c}
|0\rangle \\ \vdots \\ \vdots \\ |N-1\rangle
\end{array}
\begin{bmatrix} \beta_0 \\ \vdots \\ \vdots \\ \beta_{N-1} \end{bmatrix}
\quad \text{is} \quad
\begin{array}{c}
|00\rangle \\ |01\rangle \\ \\ \\ |M\text{-}1\ N\text{-}1\rangle
\end{array}
\begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \vdots \\ \vdots \\ \alpha_{M-1}\beta_{N-1} \end{bmatrix}
$$

What is the operation?

Joint state of

$$|\psi\rangle = \begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \vdots\end{bmatrix} \text{ and } |\phi\rangle = \begin{bmatrix}\beta_0\\ \beta_1\\ \vdots\\ \vdots\end{bmatrix} \text{ is } |\psi\rangle \otimes |\phi\rangle = \begin{bmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \vdots\\ \vdots\\ \vdots\end{bmatrix}$$

tensor product

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & & \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}$$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & \\ a_{m1}B & \cdots & & a_{mn}B \end{bmatrix} \quad mp \times nq$$

Examples

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |+\rangle = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \quad |0+\rangle = |0\rangle \otimes |+\rangle$$

$$= \left[\begin{array}{c} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{array}\right] \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

$$|+\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad |-\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$|+-\rangle = |+\rangle \otimes |-\rangle = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$$

$$= \frac{1}{2}|00\rangle + \left(-\frac{1}{2}\right)|01\rangle + \frac{1}{2}|10\rangle + \left(-\frac{1}{2}\right)|11\rangle$$

Properties

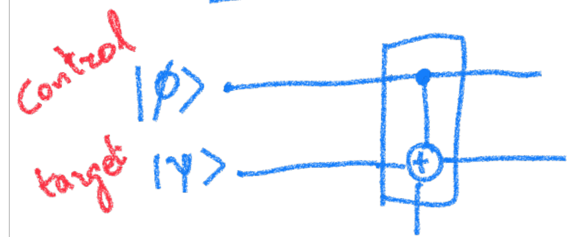
$$A \otimes (B+C) = (A \otimes B) + (A \otimes C)$$

$$(A+B) \otimes C = (A \otimes C) + (B \otimes C)$$

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C = A \otimes B \otimes C$$

Can generalize what we talked about to multiple qubits.

Controlled NOT gate



If control qubit is $|0\rangle$, do nothing

If control qubit is $|1\rangle$,
flip the target qubit.

$|00\rangle \to |00\rangle$

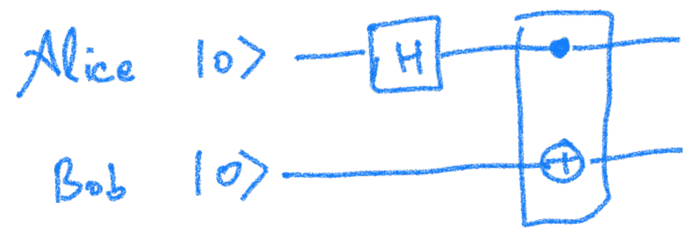
$|01\rangle \to |01\rangle$

$|10\rangle \to |11\rangle$

$|11\rangle \to |10\rangle$

$$\mathrm{CNOT} = \begin{array}{c} \begin{matrix} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{11} \\ \alpha_{10} \end{bmatrix}$$

EPR Pair



time 1 2 3

State at time

1

$|00\rangle$

2

$|{+}0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$

3

$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

Bell state / EPR Pair

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Einstein-Podolsky-Rosen used this state for important experiments.

$$= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Can this state be written as $|\phi\rangle \otimes |\psi\rangle$

$$= \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_0 \beta_0 \\ \alpha_0 \beta_1 \\ \alpha_1 \beta_0 \\ \alpha_1 \beta_1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

α_0 or $\beta_1 = 0$

Entangled state : Not unentangled

: Not of the form $|\phi\rangle \otimes |\psi\rangle$

Contrast with dependent random variables.

Consider the 2 qubit state

$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

Suppose we apply U to 2nd qubit

$$U = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$$

$|0\rangle \otimes (U|0\rangle)$

$$|00\rangle \rightarrow \begin{bmatrix} p \\ q \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle \rightarrow \begin{bmatrix} r \\ s \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle \rightarrow \begin{bmatrix} 0 \\ 0 \\ p \\ q \end{bmatrix}, \quad |11\rangle \rightarrow \begin{bmatrix} 0 \\ 0 \\ r \\ s \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} = I \otimes U \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix}$$

$$= \begin{bmatrix} p & r & 0 & 0 \\ q & s & 0 & 0 \\ 0 & 0 & p & r \\ 0 & 0 & q & s \end{bmatrix} . = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} p & r \\ q & s \end{bmatrix}$$

Thm More generally if we apply U to $|\phi\rangle$

V to $|\psi\rangle$,

the joint state $|\phi\,\psi\rangle$ gets transformed

to $U \otimes V\, |\phi\,\psi\rangle$

Proof: Exercise

Theorem (stated without proof)

Every unitary on n qubits can be realized by a circuit of

- 1 qubit unitaries
- Controlled NOT.

Measurement of Joint State

Alice has qubit $|\phi\rangle$

Bob has qubit $|\psi\rangle$

State $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

Alice measures her qubit.

What do you expect?

Use intuition from prob theory

Use current understanding of Q.C.

- Alice's state should be one of $|0\rangle$ or $|1\rangle$.

  with what probabilities?

- Bob's state should still be in superposition.

  with what amplitudes?

$\Pr[\text{Alice reads } |0\rangle] = |\alpha_{00}|^2 + |\alpha_{01}|^2$

State collapses to $\underline{\gamma_0 |00\rangle + \gamma_1 |01\rangle}$ with this probability

$$\gamma_0 = \frac{\alpha_{00}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \qquad \gamma_1 = \frac{\alpha_{01}}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

$\Pr[\text{Alice reads } |1\rangle] = |\alpha_{10}|^2 + |\alpha_{11}|^2$

State collapses to $\underline{\gamma_0' |10\rangle + \gamma_1' |11\rangle}$ with this probability

$$\gamma_0' = \alpha_{10} / \sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}, \quad \gamma_1' = \alpha_{11} / \sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}$$

Analogous of classical prob theory

p_{00}, p_{01}, p_{10}, p_{11} for X, Y

$X = 0$ w.p. $p_{00} + p_{01}$

Joint distribution conditioned on $X=0$ is

00 w.p. $\dfrac{p_{00}}{p_{00}+p_{01}}$, 01 w.p. $\dfrac{p_{01}}{p_{00}+p_{01}}$

Mixed State

- After Alice measures, the joint state is a classical probability distribution over pure quantum states.

The generalization to qudits is straightforward.

$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \vdots \\ \vdots \\ \alpha_{M-1\,N-1} \end{bmatrix} = |\phi\,\psi\rangle$$

ϕ is M dim

ψ is N dim

what happens when we measure $|\phi\rangle$

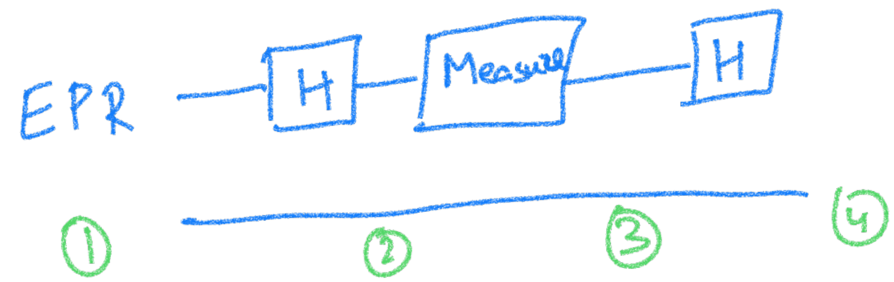
Prob that we measure $|0\rangle$

$$= |\alpha_{00}|^2 + |\alpha_{01}|^2 + \dots + |\alpha_{0\,N-1}|^2$$

with this prob the state collapses to

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \dots + \alpha_{0\,N-1}|0\,N-1\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2 + \dots + |\alpha_{0\,N-1}|^2}}$$

Example



① $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

② $\frac{1}{\sqrt{2}}|{+}0\rangle + \frac{1}{\sqrt{2}}|{-}1\rangle = \underline{\frac{1}{2}|00\rangle} + \frac{1}{2}|10\rangle + \underline{\frac{1}{2}|01\rangle}$
$- \frac{1}{2}|11\rangle$

(3) w.p. $\frac{1}{2}$ $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$

w.p. $\frac{1}{2}$ $\frac{1}{\sqrt{2}}|10\rangle - \frac{1}{\sqrt{2}}|11\rangle$

(4) w.p. $\frac{1}{2}$ $\frac{1}{\sqrt{2}}|+0\rangle + \frac{1}{\sqrt{2}}|+1\rangle$

$\frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle$

w.f. $\frac{1}{2}$

$$\frac{1}{\sqrt{2}}|-0\rangle - \frac{1}{\sqrt{2}}|-1\rangle$$

$$\frac{1}{2}|00\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle$$