CS5275 The Algorithm Designer's Toolkit
S2 AY2025/26
Problem Set 2

Submission is due at 23:59, February 26 (Thu).

Spectral preliminaries. Let $G = (V, E)$ be a connected graph, not necessarily regular. Let $\mathbf{D}$ be the diagonal degree matrix and $\mathbf{A}$ the adjacency matrix. The *normalized Laplacian* of G is defined as

$$\mathbf{N} := \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}.$$

Let

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$$

denote the eigenvalues of $\mathbf{N}$.

A useful variational characterization of the second smallest eigenvalue is

$$\lambda_2 = \min_{\substack{f:V\to\mathbb{R}\\ f\not\equiv 0}} \frac{\sum_{\{u,v\}\in E}\big(f(u)-f(v)\big)^2}{\sum_{v\in V}\deg(v)\, f(v)^2} \quad \text{subject to} \ \sum_{v\in V}\deg(v)\, f(v) = 0.$$

(You may use this without proof. For the proof, you can see our main references.)

Lazy random walk and mixing time. Consider the *lazy random walk* on G. Its transition matrix is defined as

$$\tilde{\mathbf{W}} \ := \ \frac{1}{2}\mathbf{I} \ + \ \frac{1}{2}\mathbf{A}\mathbf{D}^{-1}.$$

Let $\boldsymbol{\pi}$ denote the stationary distribution of this walk, viewed as a column vector.

For $\varepsilon \in (0, 1)$, the *mixing time* is defined as

$$T_{\mathrm{mix}}(G, \varepsilon) := \max_{\mathbf{x}} \min\left\{t : d_{\mathrm{TV}}\left(\tilde{\mathbf{W}}^{\top t}\mathbf{x}, \boldsymbol{\pi}\right) \leq \varepsilon\right\},$$

where the maximum is over all probability distributions $\mathbf{x} \in \mathbb{R}^V$.

Important: In all problems below, the graph G is **not necessarily regular**.

Problem 1 (Expander decomposition, 4 marks)**.** Consider the following procedure. Starting from an n-vertex graph $G = (V, E)$, iteratively do the following: for any connected component C, if there exists a subset $S \subseteq V(C)$ with

$$\Phi_C(S) \leq \phi,$$

remove the edges in the cut $(S, V(C) \setminus S)$, thereby splitting C into two components. Repeat until no such cut exists in any component.

Prove that the total number of edges removed during this process is at most

$$O(\phi \cdot |E| \cdot \log n).$$

Here $|E|$ refers to the original number of edges.

Problem 2 (Eigenvalue lower bound)**.** In this problem, the goal is to show that $\Theta\left(\frac{1}{n^3}\right)$ is the smallest-possible value of $\lambda_2(\mathbf{N})$ for connected graphs.

2.a. (2 marks) For every n, construct a connected graph $G = (V, E)$ with n vertices, such that the second-smallest eigenvalue of its normalized Laplacian $\mathbf{N}$ satisfies

$$\lambda_2(\mathbf{N}) \in O\left(\frac{1}{n^3}\right).$$

2.b. (2 marks) Prove that for every connected graph G on n vertices, the second-smallest eigenvalue of the normalized Laplacian $\mathbf{N}$ satisfies

$$\lambda_2(\mathbf{N}) \in \Omega\left(\frac{1}{n^3}\right).$$

Problem 3 (Starting from a single vertex, 4 marks)**.** Show that the definition of mixing time remains unchanged if the maximum in the definition of $T_{\text{mix}}(G, \varepsilon)$ is taken only over initial distributions $\mathbf{x}$ that are supported on a single vertex, i.e., $x_v = 1$ for some vertex v and $x_u = 0$ for all $u \neq v$.

Problem 4 (Spectral bound via dense subgraphs, 4 marks)**.** Let $G = (V, E)$ be any graph with adjacency matrix $\mathbf{A}$. Prove that the largest eigenvalue of $\mathbf{A}$ satisfies

$$\lambda_{\max}(\mathbf{A}) \;\geq\; 2 \cdot \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|},$$

where the maximum is over all (not necessarily induced) non-null subgraphs H of G.