CS5275 The Algorithm Designer's Toolkit
S2 AY2025/26
Problem set 1

Submission is due at 23:59, February 5 (Thu).

Problem 1 (Vertex expansion in trees, 4 marks)**.** Let $T = (V, E)$ be an n-vertex tree.

1.a. (2 marks) Prove that $h_{\text{out}}(T) \in O(1/n)$.

1.b. (2 marks) Prove that $h_{\text{out}}(T) \in \Omega(1/n)$.

Problem 2 (Trimming sparse cuts, 4 marks)**.** Let $G = (V, E)$ be a graph. Suppose there exists a subgraph $G' = (V', E')$ of G such that

$$|E'| \geq (1 - \varepsilon)|E| \qquad \text{and} \qquad \Phi(G') \geq \phi,$$

Consider the following procedure, where we write $G[W]$ to denote the induced subgraph of G on W.

1. Initialize $W \leftarrow V$.
2. While there exists a subset $S \subseteq W$ with

$$\mathrm{vol}_{G[W]}(S) \leq \frac{\mathrm{vol}_{G[W]}(W)}{2} \qquad \text{and} \qquad \Phi_{G[W]}(S) \leq \frac{\phi}{2},$$

   set $W \leftarrow W \setminus S$.
3. Return W.

Prove that the output W_{final} satisfies

$$\mathrm{vol}_G(W_{\text{final}}) \in (1 - O(\varepsilon)) \cdot \mathrm{vol}_G(V),$$

where the hidden constant in $O(\varepsilon)$ is independent of G, n, ϕ.

(Hint: Intuitively, the union of all the subsets trimmed should yield a low-conductance cut in G; consider restricting this cut to G'.)

Problem 3 (Integrality gap of Leighton–Rao LP relaxation, 4 marks)**.** Let $G = (V, E)$ be any r-regular graph on n vertices with $r \in O(1)$ and $\Phi(G) \in \Omega(1)$.

3.a. (2 marks) Prove that there exist $\Omega(n^2)$ pairs $(u, v) \in V \times V$ satisfying

$$\mathsf{dist}_G(u, v) \in \Omega(\log n).$$

3.b. (2 marks) Let H be the clique K_n on the same vertex set V. Let $\mathrm{LR}(G, H)$ denote the optimum value of the Leighton–Rao LP relaxation for this instance. Prove that

$$\mathrm{LR}(G, H) \in O\left(\frac{\Phi(G, H)}{\log n}\right).$$

(Hint: Consider the shortest-path metric on G. You may use the result from (3.a).)

Problem 4 (Bourgain's embedding with a single sampling probability, 4 marks)**.** Let $G = (V, E)$ be a connected unweighted graph with shortest-path metric dist_G. Sample a random subset $A \subseteq V$ by including each vertex independently with probability p, and define the embedding into the line:

$$f_A(x) := \mathsf{dist}_G(x, A) := \min_{a \in A} \mathsf{dist}_G(x, a),$$

which yields the following line metric:

$$d_A(x, y) = \begin{cases} |f_A(x) - f_A(y)|, & \text{if } A \neq \emptyset; \\ 0, & \text{if } A = \emptyset. \end{cases}$$

4.a. (2 marks) For every n, construct a connected unweighted graph $G = (V, E)$ on n vertices, with a pair of distinguished vertices $x, y \in V$, such that the embedding with $p = \frac{1}{2}$ has large distortion in expectation, in the sense that

$$\mathbb{E}[d_A(x, y)] \in O\big(n^{-0.01}\big) \cdot \mathsf{dist}_G(x, y).$$

4.b. (2 marks) For every n, construct a connected unweighted graph $G = (V, E)$ on n vertices, with a pair of distinguished vertices $x, y \in V$, such that the embedding with $p = \frac{1}{n}$ has large distortion in expectation, in the sense that

$$\mathbb{E}[d_A(x, y)] \in O\big(n^{-0.01}\big) \cdot \mathsf{dist}_G(x, y).$$

In both cases, prove your answer, including showing the calculation for $\mathbb{E}[d_A(x, y)] \in O(n^{-0.01}) \cdot \mathsf{dist}_G(x, y)$.