CS5275 – The Algorithm Designer’s Toolkit (S2 AY2025/26)

Lecture 1:

Expanders – robustness against failures

Measuring connectivity

- **Edge Connectivity**
  - Minimum number of edges whose removal disconnects the graph.
  - Captures global robustness against edge failures.

- **Vertex Connectivity**
  - Minimum number of vertices whose removal disconnects the graph.
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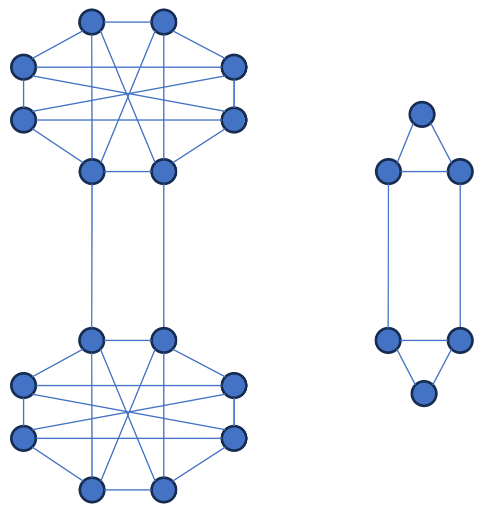

Key limitations of Edge & Vertex Connectivity:
- Ignore the size of the affected region.
- Larger cities → more roads

Note: We focus on **simple unweighted graphs**, but the definitions also extend to weighted graphs with multi-edges and self-loops.

Conductance

Consider a graph $G = (V, E)$.

Volume of a vertex set S:

- $\mathrm{vol}(S) = \sum_{v \in S} \deg(v)$.

It measures the size of a region.

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Conductance of a cut $(S, V \setminus S)$:

- $\Phi(S) = \frac{|E(S, V \setminus S)|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}}$, where $E(A, B) = \{\, \{u, v\} \in E \mid u \in A \text{ and } v \in B \,\}$.

It measures the ratio between the size of a cut and the size of the region it separates.

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It measures the ratio between the size of a cut and the size of the region it separates.

Some facts:

- If $A \cap B = \emptyset$, then $\mathrm{vol}(A \cup B) = \mathrm{vol}(A) + \mathrm{vol}(B)$.
- $0 \le \Phi(S) \le 1$.

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Conductance of a graph G:

- $\Phi(G) = \min_{S \subseteq V \,:\, S \neq V \text{ and } S \neq \emptyset} \Phi(S)$.

The minimum conductance over all cuts in the graph.

Intuition:

- What is the weakest cut in the network?
- Which region is easiest to separate relative to its size?

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Conductance

Consider a graph $G = (V, E)$.

Some facts:

- $0 \le \Phi(G) \le 1$.
- If $G = (V, E)$ is connected, then $\Phi(G) \ge 1/|E|$.

Volume of a vertex set S:

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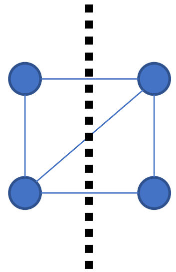

Conductance $= \frac{3}{\min\{2+3,\, 2+3\}} = \frac{3}{5} = 0.6$

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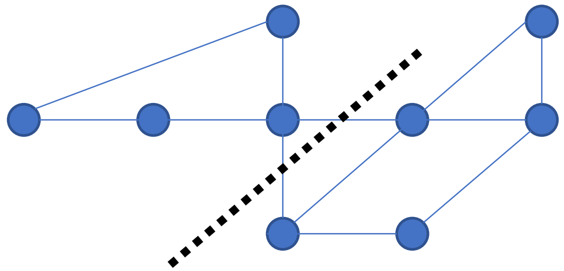

Conductance $= \frac{2}{\min\{4+2+2+2,\, 4+3+3+2+2\}} = \frac{2}{10} = 0.2$

Expanders

- A graph G is a **$\boldsymbol{\phi}$-expander** $\leftrightarrow \Phi(G) \geq \phi$.

Key takeaways:

- Conductance provides a size-sensitive notion of expansion.
- It better captures robustness and connectivity in real networks.

Comparing with vertex & edge connectivity

Expanders

Two $(n/2)$-cliques connected by an edge

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Graph	Conductance
Dumbbells	$\Theta(1/n^2)$
Paths	$\Theta(1/n)$
Cycles	$\Theta(1/n)$
$(\sqrt{n} \times \sqrt{n})$-grids	$\Theta(1/\sqrt{n})$
Hypercubes	$\Theta(1/\log n)$
Stars	$\Theta(1)$
Complete graphs / cliques	$\Theta(1)$

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Exercises!

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Comparing with vertex & edge connectivity

Robustness against edge deletions

- Suppose we remove a subset of edges $D \subseteq E$ from a ϕ-expander $G = (V, E)$.
  - The resulting graph is $G' = (V, E \setminus D)$.
  - We say that a connected component S of G' is **small** if $\mathrm{vol}_G(S) \leq \frac{\mathrm{vol}_G(V)}{2}$.

Note: There can be at most one component that is not small.

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- **Claim:** The total volume of small connected components of G' is $O\left(\frac{|D|}{\phi}\right)$.

In other words, deleting the edges $D \subseteq E$ can only disconnect a small subset of volume $O\left(\frac{|D|}{\phi}\right)$.

- Therefore, expanders are **robust against edge deletions**.

Useful in fault-tolerant distributed computing

Robustness against edge deletions

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- **Claim:** The total volume of small connected components of G' is $O\left(\frac{|D|}{\phi}\right)$.

Proof of the claim – Case 1: There is a connected component that is not small.

- Let X be the union of all small connected components.

$$\boxed{\mathrm{vol}_G(X) \leq \frac{\mathrm{vol}_G(V)}{2}} \longrightarrow \boxed{\phi \leq \frac{|E(X,\ V \setminus X)|}{\mathrm{vol}_G(X)} \leq \frac{|D|}{\mathrm{vol}_G(X)}} \longrightarrow \boxed{\mathrm{vol}_G(X) \leq \frac{|D|}{\phi}}$$

Robustness against edge deletions

Proof of the claim – Case 2: All connected components are small.

- We show that there is $X =$ union of <u>some</u> small connected components such that:
  - $\frac{\mathrm{vol}_G(V)}{3} \leq \mathrm{vol}_G(X) \leq \frac{2\mathrm{vol}_G(V)}{3}$

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$$\phi \leq \frac{|E(X,\ V \setminus X)|}{\min\{\mathrm{vol}_G(X), \mathrm{vol}_G(V \setminus X)\}} \leq \frac{|D|}{\mathrm{vol}_G(V)/3}$$

$$\min\{\mathrm{vol}_G(X), \mathrm{vol}_G(V \setminus X)\} \geq \frac{\mathrm{vol}_G(V)}{3}$$

The total volume of small connected components $= \mathrm{vol}_G(V) \leq \frac{3|D|}{\phi}$

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Selection of $\boldsymbol{X}$:

- If there is a small connected component S with $\mathrm{vol}_G(S) \geq \frac{\mathrm{vol}_G(V)}{3}$:
  - We may select $X \leftarrow S$.
- Otherwise:
  - Initialize $X \leftarrow \emptyset$.
  - While ($\mathrm{vol}_G(X) < \frac{\mathrm{vol}_G(V)}{3}$)
    - $X \leftarrow$ union of X and some small connected component.

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Next: We will demonstrate an application of this result to sublinear–time algorithms.

Fast connectivity check under edge failures

- **Input:**
  - A ϕ-expander $G = (V, E)$.
  - A subset of edges $D \subseteq E$.
  - Two vertices s and t.
- **Goal:**
  - Decide whether s and t are connected in $G' = (V, E \setminus D)$.

- Of course, the problem can be solved in linear time using BFS.
- Here, however, our goal is to solve it in **sublinear time**.

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It suffices to do BFS to explore up to $O\left(\frac{|D|}{\phi}\right)$ volume from both s and t.

The volume is sufficient for us to decide whether s and t belong to small connected components.

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It suffices to do BFS to explore up to $O\left(\frac{|D|}{\phi}\right)$ volume from both s and t.

The volume is sufficient for us to decide whether s and t belong to small connected components.

If both s and t are not in small connected components, then the algorithm returns **YES**.

Otherwise, at least one of s and t belongs to a small connected component, and the volume is sufficient for us to search the entire component to decide whether s and t are connected.

Expanders have small diameter

- **Claim:** The diameter of any n-vertex ϕ-expander is $O\left(\frac{\log n}{\phi}\right)$.

Maximum shortest-path distance between any two vertices

Intuition:

- In a well-connected network:
    - Everything is only a few steps away.

Algorithmic Advantages:

- Quick broadcast and aggregation:
    - Messages reach the whole network rapidly.

Expanders have small diameter

- **Claim:** The <u>diameter</u> of any n-vertex ϕ-expander is $O\left(\frac{\log n}{\phi}\right)$.

We prove the claim using a **ball-growing** argument:

- $B(v, r) =$ the ball of radius r around v.

- It suffices to show that:
  - There exists $r^* \in O\left(\frac{\log n}{\phi}\right)$ such that $\mathrm{vol}\big(B(v, r^*)\big) > \frac{\mathrm{vol}(V)}{2}$ for every vertex v.

$\downarrow$

$B(u, r^*) \cap B(v, r^*) \neq \emptyset$ for any u and v.

$\downarrow$

Graph diameter $\leq 2r^* \in O\left(\frac{\log n}{\phi}\right)$.

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If $\mathrm{vol}(B(v,r)) \leq \frac{\mathrm{vol}(V)}{2}$, then $\mathrm{vol}(B(v,r+1)) \geq (1+\phi)\mathrm{vol}(B(v,r))$.

$$
\begin{aligned}
&\mathrm{vol}(B(v,r+1)) \\
&\geq \mathrm{vol}(B(v,r)) + E(B(v,r), V \setminus B(v,r)) \\
&\geq (1+\phi)\mathrm{vol}(B(v,r))
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If $\mathrm{vol}(B(v,r)) \leq \frac{\mathrm{vol}(V)}{2}$, then $\mathrm{vol}(B(v,r)) > e^{\frac{r\phi}{2}}$.

$$\mathrm{vol}(B(v,r)) \geq (1+\phi)^r \mathrm{vol}(B(v,0)) \geq (1+\phi)^r > e^{\frac{r^2\phi}{r\phi+r}} \geq e^{\frac{r^2\phi}{2r}} = e^{\frac{r\phi}{2}}$$

$$\left(1+\frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}$$

https://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf

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$\downarrow$

If $\mathrm{vol}\big(B(v,r)\big) \leq \frac{\mathrm{vol}(V)}{2}$, then $\mathrm{vol}\big(B(v,r)\big) > e^{\frac{r\phi}{2}}$. $\rightarrow \quad e^{\frac{r\phi}{2}} < \frac{\mathrm{vol}(V)}{2} \quad \rightarrow \quad r < \frac{2\ln\frac{\mathrm{vol}(V)}{2}}{\phi}$

$\downarrow$

There exists $r^* \in O\left(\frac{\log n}{\phi}\right)$ such that $\mathrm{vol}\big(B(v,r^*)\big) > \frac{\mathrm{vol}(V)}{2}$.

Construction of expanders

- **Stars** and **cliques** already have very good conductance:
  - $\Phi(G) \in \Omega(1)$
- However, they are undesirable in that they have high-degree vertices.

Construction of expanders

- **Stars** and **cliques** already have very good conductance:
  - $\Phi(G) \in \Omega(1)$
- However, they are undesirable in that they have <u>high-degree vertices</u>.
- Can we simultaneously achieve both of the following?
  - $\Phi(G) \in \Omega(1)$
  - Maximum degree $\in O(1)$
- **Hypercubes** nearly achieve this goal, up to a factor of $O(\log n)$.

Randomized construction 1

- Erdős-Renyi random graph $\mathcal{G}(n, p)$:
  - An n-vertex graph $G = (V, E)$ such that:
    - for each pair $\{u, v\}$ of vertices in V, $\{u, v\} \in E$ with probability p independently.

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- There is a choice of sampling probability $p \in \Theta\left(\frac{\log n}{n}\right)$ such that:
  - With probability $1 - 1/\mathrm{poly}(n)$,
    - The maximum degree of G is $O(\log n)$.
    - $\Phi(G) \in \Omega(1)$.

Intuitively, this means that <u>almost all</u> graphs are good expanders!

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This is an **exercise**:

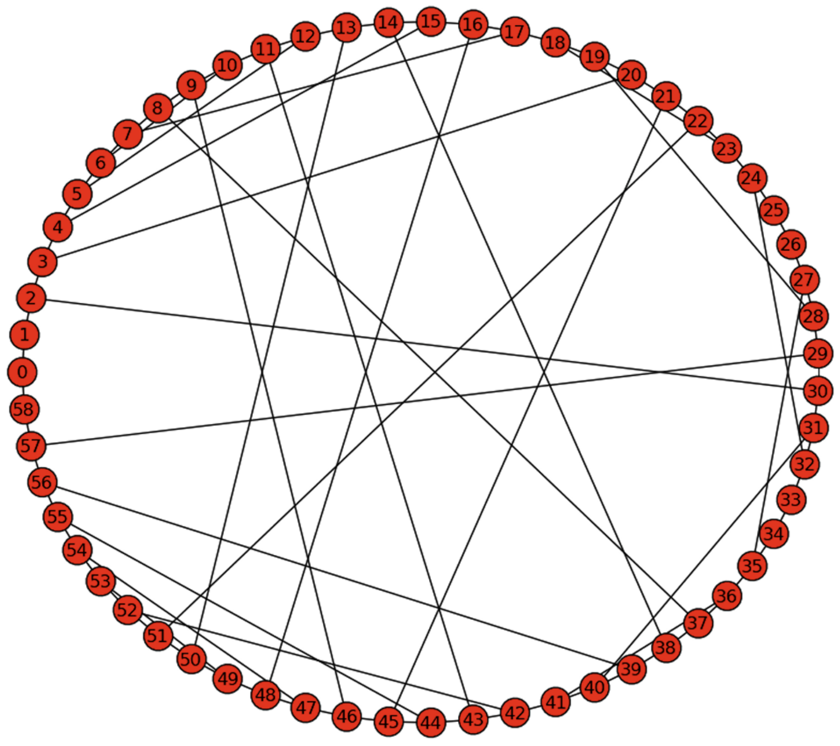
- Apply a Chernoff bound for every cut.
- Sum up the error probability.

Randomized construction 2

- Generate d random perfect matching over n vertices.
- Take the union of all edges in the matchings.
- There exists a constant d such that the resulting graph G satisfies $\Phi(G) \in \Omega(1)$ with probability $1 - 1/\mathrm{poly}(n)$.

https://lucatrevisan.github.io/teaching/expanders2016/lecture19.pdf

Deterministic construction



- Let p be a prime.
- Define the graph $G_p = \left(V_p, E_p\right)$ as follows.
  - $V_p = \{0, 1, \dots, p-1\}$
  - Each vertex $a \in V_p \setminus \{0\}$ is connected to:
    - $a - 1 \bmod p$
    - $a + 1 \bmod p$
    - a^{-1} (multiplicative inverse: the unique element $a^{-1} \in V_p$ such that $a \cdot a^{-1} \bmod p = 1$)
  - The vertex 0 is connected to 1, to $p - 1$, and has a self-loop.
- $\Phi\left(G_p\right) \in \Omega(1)$
- The degree of every vertex in G_p equals 3.

https://lucatrevisan.github.io/teaching/expanders2016/lecture16.pdf

Variations

- Given a vertex set $S \subseteq V$, define:
  - $\partial_{\text{in}}(S)$ = the set of vertices in S adjacent to $V \setminus S$.
  - $\partial_{\text{out}}(S)$ = the set of vertices in $V \setminus S$ adjacent to S.

- **Edge expansion:**
  - $h(G) = \min_{0<|S|\leq\frac{n}{2}} \frac{|E(V, V\setminus S)|}{|S|}$
- **Vertex expansion:**
  - $h_{\text{in}}(G) = \min_{0<|S|\leq\frac{n}{2}} \frac{|\partial_{\text{in}}(S)|}{|S|}$
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Claim: If the maximum degree Δ of G is $O(1)$, then $\Phi(G)$, $h(G)$, $h_{\text{in}}(G)$, $h_{\text{out}}(G)$ are within a constant factor of each other.

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What about $\Phi(G)$?

$h(G)$, $h_{\text{in}}(G)$, $h_{\text{out}}(G)$ are within a constant factor of each other.

- $|\partial_{\text{in}}(S)| \le |E(V, V \setminus S)| \le \Delta|\partial_{\text{in}}(S)|$
- $|\partial_{\text{out}}(S)| \le |E(V, V \setminus S)| \le \Delta|\partial_{\text{out}}(S)|$

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- It remains to show that the two parameters are within an $O(1)$-factor:
  - $h(G) = \min_{0<|S|\leq\frac{n}{2}} \frac{|E(V, V\backslash S)|}{|S|}$
  - $\Phi(G) = \min_{S\subseteq V\,:\,S\neq V \text{ and } S\neq\emptyset} \frac{|E(S, V\backslash S)|}{\min\{\text{vol}(S), \text{vol}(V\backslash S)\}} = \min_{0<|S|\leq\frac{n}{2}} \frac{|E(S, V\backslash S)|}{\min\{\text{vol}(S), \text{vol}(V\backslash S)\}}$

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- To do so, it suffices to show that for any $0 < |S| \leq \frac{n}{2}$, the two parameters are within an $O(1)$-factor:
  - $|S|$
  - $\min\{\text{vol}(S), \text{vol}(V \setminus S)\}$

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Claim: If the maximum degree Δ of G is $O(1)$, then $\Phi(G)$, $h(G)$, $h_{\text{in}}(G)$, $h_{\text{out}}(G)$ are within a constant factor of each other.

- It remains to show that the two parameters are within an $O(1)$-factor:
  - $h(G) = \min_{0<|S|\le\frac{n}{2}} \frac{|E(V, V\setminus S)|}{|S|}$
  - $\Phi(G) = \min_{S\subseteq V\,:\,S\neq V \text{ and } S\neq\emptyset} \frac{|E(S, V\setminus S)|}{\min\{\text{vol}(S), \text{vol}(V\setminus S)\}} = \min_{0<|S|\le\frac{n}{2}} \frac{|E(S, V\setminus S)|}{\min\{\text{vol}(S), \text{vol}(V\setminus S)\}}$
- To do so, it suffices to show that for any $0 < |S| \le \frac{n}{2}$, the two parameters are within an $O(1)$-factor:
  - $|S|$
  - $\min\{\text{vol}(S), \text{vol}(V \setminus S)\}$
- Indeed, $|S| \le \min\{|S|, |V \setminus S|\} \le \min\{\text{vol}(S), \text{vol}(V \setminus S)\} \le \text{vol}(S) \le \Delta|S|$.

  - This step requires that there is no isolated vertex.
  - If there is an isolated vertex, then all the parameters $\Phi(G)$, $h(G)$, $h_{\text{in}}(G)$, $h_{\text{out}}(G)$ are zero.

Vertex expansion vs. conductance

- We have shown the following results for ϕ-expanders:
  - Robustness against <u>edge</u> deletions.
  - Small diameter.

- **Exercise:** Extend the above results to <u>graphs with high vertex expansion</u>:
  - Robustness against <u>vertex</u> deletions.
  - Small diameter.

Vertex expansion vs. conductance

- Vertex expansion and conductance can be very different!

There is an n-vertex graph G with

- $\Phi(G) \in \Omega(1)$
- $h_{\text{in}}(G) \in O\left(\frac{1}{n}\right)$
- $h_{\text{out}}(G) \in O\left(\frac{1}{n}\right)$

There is an n-vertex graph G with

- $\Phi(G) \in O\left(\frac{1}{n}\right)$
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Star graphs

There is an n-vertex graph G with
- $\Phi(G) \in O\left(\frac{1}{n}\right)$
- $h_{\text{in}}(G) \in \Omega(1)$
- $h_{\text{out}}(G) \in \Omega(1)$

Two $(n/2)$-cliques with a perfect matching between them

Outlook

- So far, we mostly talk about nice properties of expanders, focusing on the robustness against failures.

- **Next:**
  - How to compute the conductance of a graph efficiently? ← We will do this first.
  - What are the applications of expanders in algorithm design?

References

- **Main reference:**
  - Lecture 1.2 of https://sites.google.com/site/thsaranurak/teaching/Expander

- **Additional/optional reading:**
  - More about expander graphs:
    - https://en.wikipedia.org/wiki/Expander_graph
  - An application in distributed computing:
    - Ghosh, Bhaskar, et al. “Tight analyses of two local load balancing algorithms.” SIAM Journal on Computing 29.1 (1999): 29-64.
    - https://epubs.siam.org/doi/10.1137/S0097539795292208