CS5275 The Algorithm Designer's Toolkit
S2 AY2025/26
Tutorial problem set 2

(There is no need to submit your solution for tutorial problem sets)

Problem 1 (Alternative definition of total variation distance)**.** Let μ and ν be two probability distributions on a finite set Ω. The total variation distance is defined as

$$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|.$$

Prove that this is equivalent to the following characterization

$$d_{\mathrm{TV}}(\mu, \nu) = \max_{A \subseteq \Omega} \big(\mu(A) - \nu(A)\big).$$

Problem 2 (Mixing time lower bound)**.** Let G be any connected graph. Prove that

$$T_{\mathrm{mix}}(G, 0.01) \in \Omega\left(\frac{1}{\Phi(G)}\right).$$

Problem 3 (Convergence of random walks in non-bipartite graphs)**.** Let G be a connected, non-bipartite d-regular graph. Let $\mathbf{W} = \frac{1}{2}\mathbf{A}$ denote the transition matrix of the standard (non-lazy) random walk.

Prove that for every initial distribution $\mathbf{x}$,

$$\lim_{t \to \infty} \left(\mathbf{W}^{\top}\right)^{t} \mathbf{x} = \boldsymbol{\pi},$$

where $\boldsymbol{\pi}$ is the stationary (uniform) distribution.

Problem 4 (Large bipartition without crossing edges)**.** For any $d \geq 1$, prove that there exists $n_0 = n_0(d)$ such that for every d-regular graph $G = (V, E)$ on $n > n_0$ vertices, there exist vertex sets $S, T \subseteq V$ satisfying

$$|S|\,|T|\,d \geq 0.01\,n^2$$

such that there is *no edge* between S and T.
(Remark: This is an exercise on page 23 of Lecture 8)