

# CS5275 – The Algorithm Designer’s Toolkit (S2 AY2025/26)

Lecture 4:  
Expanders – spectral graph theory

# Matrix representation of graphs

- Let  $G = (V, E)$  be a graph, with  $V = \{1, 2, \dots, n\}$ .

The adjacency matrix  $\mathbf{A}$  is an  $(n \times n)$ -matrix:

$$\mathbf{A}[i, j] = \begin{cases} 0, & \{i, j\} \notin E \\ 1, & \{i, j\} \in E \end{cases}$$

The degree matrix  $\mathbf{D}$  is an  $(n \times n)$ -matrix:

$$\mathbf{D}[i, j] = \begin{cases} \deg(i), & i = j \\ 0, & i \neq j \end{cases}$$

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The degree matrix  $D$  is an  $(n \times n)$ -matrix:

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The Laplacian matrix:  $L = D - A$

The normalized Laplacian matrix:  $N = D^{-1/2} L D^{-1/2}$

# Connection to cuts

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The Laplacian matrix:  $\mathbf{L} = \mathbf{D} - \mathbf{A}$

The normalized Laplacian matrix:  $\mathbf{N} = \mathbf{D}^{-1/2} \mathbf{L} \mathbf{D}^{-1/2}$

Let  $\mathbf{x} \in \mathbb{R}^n$  be a column vector.

$$\mathbf{x}^\top \mathbf{L} \mathbf{x} = \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

If  $\mathbf{x} = \mathbf{1}_S$  is the indicator vector for a cut  $S \subseteq V$ , then  $\mathbf{x}^\top \mathbf{L} \mathbf{x} = |E(S, V \setminus S)|$  is the size of the cut.

# Regular graphs

- Let  $G = (V, E)$  be a  **$d$ -regular** graph, with  $V = \{1, 2, \dots, n\}$ .

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The degree matrix  $\mathbf{D}$  is an  $(n \times n)$ -matrix:

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The Laplacian matrix:

$$\mathbf{L} = \mathbf{D} - \mathbf{A} = \mathbf{dI} - \mathbf{A}$$

The normalized Laplacian matrix:

$$\mathbf{N} = \mathbf{D}^{-1/2} \mathbf{L} \mathbf{D}^{-1/2} = \mathbf{I} - \frac{1}{d} \mathbf{A}$$

For simplicity, we restrict our attention to regular graphs, noting that all results extend to general graphs.

$$\mathbf{x}^\top \mathbf{L} \mathbf{x} = \sum_{\{u, v\} \in E} (x_u - x_v)^2$$

$$\mathbf{x}^\top \mathbf{N} \mathbf{x} = \frac{1}{d} \sum_{\{u, v\} \in E} (x_u - x_v)^2$$

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All four types of matrices  $\mathbf{A}, \mathbf{D}, \mathbf{L}, \mathbf{N}$  are symmetric.

We start by reviewing some basic facts about such matrices.

# Symmetric real matrices

- Let  $M$  be a real symmetric  $(n \times n)$ -matrix.

$$M[i,j] \in \mathbb{R} \quad M[i,j] = M[j,i]$$


# Symmetric real matrices

- Let  $M$  be a real symmetric  $(n \times n)$ -matrix.

**Theorem:** There exist

- $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$  (they are called eigenvalues)
  - Orthonormal vectors  $v_1, v_2, \dots, v_n \in \mathbb{R}^n$  (they are called eigenvectors)
- such that:

$$M = \sum_{i=1}^n \lambda_i v_i v_i^\top$$

Which implies: (for all i)

$$Mv_i = \lambda_i v_i$$

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  - Orthonormal vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$  (they are called eigenvectors)
- such that:

$$\mathbf{M} = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{v}_i^\top$$

Which implies: (for all i)

$$\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

If we change the axes of the Euclidean space  $\mathbb{R}^n$  to the eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ , then the matrix  $\mathbf{M}$  becomes diagonal, with  $\lambda_1, \dots, \lambda_n$  on the diagonal.

In other words, symmetric matrices stretch space independently along orthogonal directions.

# Variational characterizations

- Let  $M$  be a real symmetric  $(n \times n)$ -matrix.
  - Eigenvalues:  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$
  - Eigenvectors  $v_1, v_2, \dots, v_n$

Rayleigh quotient of  $x$  with respect to  $M$ :

$$R_M(x) = \frac{x^\top M x}{x^\top x}$$

Related to cuts

$$x^\top L x = \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

$$x^\top N x = \frac{1}{d} \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

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**Theorem:**

$$\lambda_k = \min_{k\text{-dim space } \mathcal{V}} \max_{x \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(x)$$

Rayleigh quotient of  $x$  with respect to  $\mathbf{M}$ :

$$R_{\mathbf{M}}(x) = \frac{x^T \mathbf{M} x}{x^T x}$$

$$x^T L x = \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

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Rayleigh quotient of  $\mathbf{x}$  with respect to  $\mathbf{M}$ :

$$R_{\mathbf{M}}(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

**Theorem:**

$$\lambda_k = \min_{k-\text{dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

**Proof ( $\geq$ ):**

Consider  $\mathcal{V} = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$

- Write  $\mathbf{x} \in \mathcal{V}$  as  $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i$ .
- $R_{\mathbf{M}}(\mathbf{x}) = \frac{\sum_{i=1}^k \alpha_i^2 \lambda_i}{\sum_{i=1}^k \alpha_i^2}$ .
- $\max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x}) = \lambda_k$ .
- $\lambda_k \geq \min_{k-\text{dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$ .

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**Proof ( $\leq$ ):**

Consider any  $k$ -dimensional space  $\mathcal{V}$ .

- $\exists \mathbf{x} \in \mathcal{V} \cap \text{span}\{\mathbf{v}_k, \dots, \mathbf{v}_n\} \setminus \{\mathbf{0}\}$ .
  - $\mathbf{x} = \sum_{i=k}^n \alpha_i \mathbf{v}_i$ .
  - $R_{\mathbf{M}}(\mathbf{x}) = \frac{\sum_{i=k}^n \alpha_i^2 \lambda_i}{\sum_{i=k}^n \alpha_i^2} \geq \lambda_k$ .
- $\lambda_k \leq \min_{k-\text{dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$ .

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$$\lambda_1 = \min_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(x)$$

Moreover, any minimizer  $x$  is a corresponding eigenvector.

Consider  $-\mathbf{M}$ .

$$\lambda_n = \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(x)$$

Moreover, any maximizer  $x$  is a corresponding eigenvector.

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$$\lambda_2 = \min_{\mathbf{x} \neq \mathbf{0}, \mathbf{x} \perp \mathbf{v}_1} R_{\mathbf{M}}(\mathbf{x})$$

A natural extension

Consider  $-\mathbf{M}$ .

$$\lambda_n = \max_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

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# Spectral graph theory

- The eigenvalues of  $A$  (adjacency),  $L$  (Laplacian), and  $N$  (normalized Laplacian) reveal information about:
  - Average density of cuts.
  - Bipartiteness.
  - Chromatic number.
  - Conductance.
  - Hamiltonicity.
  - Size of a maximum independent set.
  - Size of a maximum matching.
  - Toughness of a graph.
  - Number of connected components.

<https://adga-workshop.org/2025/tijn.pdf>

# The smallest eigenvalue $\lambda_1$

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**Theorem:**

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Rayleigh quotient of  $x$  with respect to  $N$ :

$$R_N(x) = \frac{\mathbf{x}^\top N \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2} \geq 0$$

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$$\lambda_1 = \min_{x \in \mathbb{R}^n \setminus \{0\}} R_N(x) \leq R_N(\mathbf{1}) = \frac{\sum_{\{u,v\} \in E} (1 - 1)^2}{d \sum_{v \in V} 1^2} = 0$$

$$\lambda_1 \geq 0$$

$$\lambda_1 \leq 0$$

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For every connected component  $S$ ,  
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$G$  has at least  $k$  connected components  $S_1, \dots, S_k$ .

Set  $\mathcal{V} = \text{span}\{\mathbf{1}_{S_1}, \dots, \mathbf{1}_{S_k}\}$ .

- $\forall x \in \mathcal{V} \setminus \{\mathbf{0}\}, R_N(x) = 0$

$$\lambda_k = \min_{k-\text{dim space } \mathcal{V}} \max_{x \in \mathcal{V} \setminus \{\mathbf{0}\}} R_N(x)$$

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if and only if

For every connected component  $S$ ,  
 $x_u = x_v \quad \forall u, v \in S$

$$\lambda_k = 0$$

There is a  $k$ -dimensional subspace  $\mathcal{V}$  of  $\mathbb{R}^n$  such that:

- $\forall x \in \mathcal{V} \setminus \{\mathbf{0}\}, R_N(x) = \mathbf{0}$

The value of  $x$  is constant within each connected component.

$k = \text{dimension of } \mathcal{V} \leq \text{the number of connected components.}$

# The largest eigenvalue $\lambda_n$

- Consider the normalized Laplacian  $N$  of a graph  $G = (V, E)$ .
  - Eigenvalues:  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

**Theorem:**

- $\lambda_n \leq 2$ .
- $\lambda_n = 2$  if and only if  $G$  has a bipartite connected component.

$$\lambda_n = \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_N(x)$$

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$$\lambda_n = \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_N(x) = 2 - \min_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} \leq 2$$

$$\begin{aligned} R_N(x) &= \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2} \\ &= \frac{2 \sum_{\{u,v\} \in E} (x_u^2 + x_v^2) - \sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} \\ &= 2 - \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} \end{aligned}$$

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## Theorem:

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- $\lambda_n = 2$  if and only if  $G$  has a bipartite connected component.

$$\lambda_n = 2 - \min_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2}$$

- Suppose  $G$  has a bipartite connected component  $S$  with bipartition  $S = A \cup B$ .
- Setting  $x = \mathbf{1}_A - \mathbf{1}_B$  makes  $\frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} = 0$ .
- Therefore,  $\lambda_n = 2$ .

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- Suppose  $\lambda_n = 2$ .
- There exists  $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  such that  $\frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} = 0$ .
- Define:
  - $A = \{v \in V \mid x_v > 0\} \neq \emptyset$
  - $B = \{v \in V \mid x_v < 0\} \neq \emptyset$
- $A \cup B$  is a union of bipartite connected components.

# Outlook

- **Next:** We will show that:

$$\frac{\lambda_2}{2} \leq \Phi(G) \leq \sqrt{2\lambda_2} \quad \text{Cheeger's inequality.}$$

where  $\lambda_2$  is the second eigenvalue of the normalized Laplacian  $N$  of  $G$ .

Moreover, given the eigenvector  $v_2$ , we can obtain a cut of conductance at most  $\sqrt{2\lambda_2}$  in polynomial time.

# References

- **Main reference:**
  - Lecture 4.1 of <https://sites.google.com/site/thsaranurak/teaching/Expander>
  - Chapters 1, 2, and 3 of <https://lucatrevisan.github.io/books-expanders-2016.pdf>