CS5275 Tutorial Problem Set 1

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Problem 1

1. Take the graph $H = (V, E_H)$ where $E_H = \{\{s,t\}\}$.

   Recall that the generalised conductance is defined by
   $$\Phi(G,H) = \min_{S : E_H(S, V\setminus S) \neq \varnothing} \frac{|E_G(S, V\setminus S)|}{|E_H(S, V\setminus S)|}.$$
   Note that when $E_H(S, V\setminus S) \neq \varnothing$, we must have $\{s,t\} \in E_H(S, V\setminus S)$. WLOG say $s \in S$ and $t \in V \setminus S$.

   Noticing that $|E_H(S, V\setminus S)| = 1$, we have
   $$\Phi(G,H) = \min_{S : s\in S \wedge t \in V\setminus S} |E_G(S, V\setminus S)| = \min_{S : s\in S \wedge t \in V\setminus S} |\{\{s,t\} \in E\}|,$$
   which is precisely the size of a minimum s–t cut in G.

Problem 2 First of all, recall the variational characterisation of λ_2 as
$$\lambda_2(\mathbf{N}) = \min_{\mathbf{x}\neq 0, \mathbf{x}\perp \mathbf{v}_1} R_{\mathbf{N}}(\mathbf{x}),$$
where $\mathbf{v}_1 = \mathbf{1}$, and the Rayleigh quotient can be expressed as
$$R_{\mathbf{N}}(\mathbf{x}) = \frac{\mathbf{x}^\top \mathbf{N}\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{\sum_{\{u,v\}\in E}(x_u - x_v)^2}{d\sum_{v\in V} x_v^2}.$$

1. Take the vector $\mathbf{x} = (x_1, \dots, x_n)$ such that $x_i = \cos\left(\frac{2\pi i}{n}\right)$. Then $\sum_{i=1}^{n} x_i = 0$, so $\mathbf{x} \perp \mathbf{1}$ and thus $\lambda_2(\mathbf{N}) \le R_{\mathbf{N}}(\mathbf{x})$.

   For an n-vertex cycle, we have $V = \{1, 2, \dots, n\}$, $E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\}$ and $d = 2$. Thus we can compute
   $$\sum_{v\in V} x_v^2 = \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} \cos^2\left(\frac{2\pi i}{n}\right) = \frac{n}{2},$$
   $$\sum_{\{u,v\}\in E} (x_u - x_v)^2 = \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 + (x_n - x_1)^2 = 2n \sin^2\left(\frac{\pi}{n}\right),$$
   and therefore, by $\sin x \le x$, we conclude that
   $$\lambda_2(\mathbf{N}) \le R_{\mathbf{N}}(\mathbf{x}) = \frac{2n\sin^2\left(\frac{\pi}{n}\right)}{2 \cdot \frac{n}{2}} = 2\sin^2\left(\frac{\pi}{n}\right) \le 2\left(\frac{\pi}{n}\right)^2 \in O\left(\frac{1}{n^2}\right).$$

2. Recall that part of the Cheeger inequality states $\lambda_2 \leq 2\Phi(G)$.

   To compute $\Phi(G)$, intuitively a cut across the two edges $\{u, w\}$ and $\{v, x\}$ yields the lowest conductance. Note that the resulting graph is still d-regular with $d = n/2-1$.

   For this cut, $S = A$ and $V \setminus S = B$. Thus $\min(\mathrm{vol}(S), \mathrm{vol}(V \setminus S)) = d(n/2) = n(n-2)/4$ and $|E(S, V \setminus S)| = 2$, and

$$\Phi(S) = \frac{|E(S, V \setminus S)|}{\min(\mathrm{vol}(S), \mathrm{vol}(V \setminus S))} = \frac{2}{n(n-2)/4} \in O\left(\frac{1}{n^2}\right).$$

   Thus we have

$$\lambda_2 \leq 2\Phi(G) \leq 2\Phi(S) \in O\left(\frac{1}{n^2}\right).$$

3. As the diameter is at most D, for any two vertices, the shortest path connecting them has length $D' \leq D$. Denote this path as $w_1, \ldots, w_{D'}$. Let $M := \max\limits_{v \in V} x_v$, $m := \min\limits_{v \in V} x_v$. We have, via telescoping sum, that

$$\sum_{i=1}^{D'-1} (x_{w_{i+1}} - x_{w_i}) = x_{w_{D'}} - x_{w_1} \geq M - m,$$

   so by Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{D'-1} (x_{w_{i+1}} - x_{w_i})^2 \geq \frac{\left(\sum_{i=1}^{D'-1} (x_{w_{i+1}} - x_{w_i})\right)^2}{D'-1} \geq \frac{(M-m)^2}{D'-1}.$$

   Note that the sum of squared differences over all edges is greater than or equal to the sum of squared differences along the path, i.e.,

$$\sum_{\{u,v\} \in E} (x_u - x_v)^2 \geq \sum_{i=1}^{D'-1} (x_{w_{i+1}} - x_{w_i})^2 \geq \frac{(M-m)^2}{D'-1}.$$

   As $\mathbf{x} \neq 0, \mathbf{x} \perp \mathbf{1}$, we have $M > 0 > m$, so that $(M-m)^2 > M^2$, and

$$M^2 = (\max_v x_v)^2 = \max_v x_v^2 \geq \frac{\sum_{v \in V} x_v^2}{n}.$$

   Therefore, we can now bound $R_{\mathbf{N}}(\mathbf{x})$ as

$$R_{\mathbf{N}}(\mathbf{x}) = \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2} \geq \frac{(M-m)^2/(D'-1)}{d(nM^2)} > \frac{1}{ndD},$$

   giving the bound $\lambda_2(\mathbf{N}) \in \Omega(1/(ndD))$ as required.

Problem 3

1. We show that the function $p = \dfrac{C \log n}{n} \in O\left(\dfrac{\log n}{n}\right)$ satisfies the requirement for some large constant C.

Recall that for $X \sim \mathrm{Bin}(n,p)$ with $\mu = \mathbb{E}[X] = np$, the Chernoff bounds state that, for $0 \le \delta \le 1$,

$$\Pr[X \ge (1+\delta)\mu] \le \mathrm{e}^{-\frac{\delta^2\mu}{3}}, \quad \Pr[X \le (1-\delta)\mu] \le \mathrm{e}^{-\frac{\delta^2\mu}{2}}.$$

In the graph $G \sim \mathcal{G}(n,p)$, the degree for each vertex v follows $\deg(v) \sim \mathrm{Bin}(n-1,p)$, so $\mu = (n-1)p = C\log n(1-1/n) \in \Theta(\log n)$. Taking $\delta = 1/2$ gives

$$\Pr\left[\deg(v) \ge \frac{3}{2}\mu\right] \le \mathrm{e}^{-\frac{\mu}{12}} = n^{-\frac{C}{12}\left(1-\frac{1}{n}\right)}, \quad \Pr\left[\deg(v) \le \frac{1}{2}\mu\right] \le \mathrm{e}^{-\frac{\mu}{8}} = n^{-\frac{C}{8}\left(1-\frac{1}{n}\right)}.$$

For large n and C, we have

$$\Pr\left[\deg(v) \notin \left[\frac{1}{2}\mu, \frac{3}{2}\mu\right]\right] < n^{-C/12} + n^{-C/8} < \frac{1}{2n},$$

i.e., $\deg(v) \in \Theta(\log n)$ for every $v \in V$ with probability at least $1 - 1/(2n)$. $(*)$

Now consider any $S \subseteq V$ with $S \ne V$ and $S \ne \varnothing$. WLOG assume that $s = |S| \le n/2$. Then we have $|E(S, V\setminus S)| \sim \mathrm{Bin}(s(n-s),p)$. Then $\mu = s(n-s)p \ge s(n/2)(C\log n/n) = Cs\log n/2$. Therefore taking $\delta = 1/2$ gives

$$\Pr\left[|E(S,V\setminus S)| \le \frac{1}{2}\mu\right] \le \mathrm{e}^{-\frac{\mu}{8}} = n^{-\frac{Cs}{16}}.$$

The total number of such sets S is, via Stirling's approximation,

$$\binom{n}{s} = \frac{n(n-1)\cdots(n-s+1)}{s!} \le \frac{n^s}{s!} \le \left(\frac{n\mathrm{e}}{s}\right)^s,$$

so the probability

$$\Pr\left[\exists S : |S| = k, |E(S,V\setminus S)| \le \frac{1}{2}\mu\right] \le \left(\frac{n\mathrm{e}}{s}\right)^s n^{-\frac{Cs}{16}} = \left(\frac{\mathrm{e}}{s}n^{1-C/16}\right)^s.$$

For large n and C, summing over all $s = 1, \ldots, \lfloor n/2 \rfloor$ gives

$$\begin{aligned}\Pr\left[\forall S : 0 < |S| \le \frac{n}{2}, |E(S,V\setminus S)| \le \frac{1}{2}\mu\right] &\le \sum_{s=1}^{\lfloor n/2 \rfloor} \left(\frac{\mathrm{e}}{s}n^{1-C/16}\right)^s \\ &< \sum_{s=1}^{\infty} \left(\mathrm{e}n^{1-C/16}\right)^s \\ &= \frac{\mathrm{e}n^{1-C/16}}{1-\mathrm{e}n^{1-C/16}} < \frac{1}{2n},\end{aligned}$$

i.e., $|E(S,V\setminus S)| \in O(s\log n)$ with probability at least $1 - 1/(2n)$.

We have, via $(*)$, that $\mathrm{vol}(S) = \sum_{v\in S} \deg(v) \in \Theta(s\log n)$, and the conductance

$$\Phi(S) = \frac{|E(S,V\setminus S)|}{\mathrm{vol}(S)} \in \frac{O(s\log n)}{\Theta(s\log n)} = O(1),$$

i.e., $\Phi(G) \in \Omega(1)$ with probability at least $1 - 1/(2n)$. $(**)$

From $(*)$ and $(**)$, the probability that both properties hold is at least $1 - 1/(2n) - 1/(2n) = 1 - 1/n$, as required.