

CS5275 – The Algorithm Designer's Toolkit
(S2 AY2025/26)

Lecture 4:

Expanders – spectral graph theory

Matrix representation of graphs

- Let $G = (V, E)$ be a graph, with $V = \{1, 2, \dots, n\}$.

The adjacency matrix \mathbf{A} is an $(n \times n)$ -matrix:

$$\mathbf{A}[i, j] = \begin{cases} 0, & \{i, j\} \notin E \\ 1, & \{i, j\} \in E \end{cases}$$

The degree matrix \mathbf{D} is an $(n \times n)$ -matrix:

$$\mathbf{D}[i, j] = \begin{cases} \deg(i), & i = j \\ 0, & i \neq j \end{cases}$$

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The Laplacian matrix: $L = D - A$

The normalized Laplacian matrix: $N = D^{-1/2} L D^{-1/2}$

Connection to cuts

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Let $\mathbf{x} \in \mathbb{R}^n$ be a column vector.

$$\mathbf{x}^\top L \mathbf{x} = \sum_{\{u, v\} \in E} (x_u - x_v)^2$$

If $\mathbf{x} = \mathbf{1}_S$ is the indicator vector for a cut $S \subseteq V$, then $\mathbf{x}^\top L \mathbf{x} = |E(S, V \setminus S)|$ is the size of the cut.

Regular graphs

- Let $G = (V, E)$ be a **d -regular** graph, with $V = \{1, 2, \dots, n\}$.

The adjacency matrix A is an $(n \times n)$ -matrix:

$$A[i, j] = \begin{cases} 0, & \{i, j\} \notin E \\ 1, & \{i, j\} \in E \end{cases}$$

The degree matrix D is an $(n \times n)$ -matrix:

$$D[i, j] = \mathbf{dI} = \begin{cases} d, & i = j \\ 0, & i \neq j \end{cases}$$

The Laplacian matrix:

$$L = D - A = \mathbf{dI} - A$$

The normalized Laplacian matrix:

$$N = D^{-1/2} L D^{-1/2} = \mathbf{I} - \frac{1}{d} A$$

For simplicity, we restrict our attention to regular graphs, noting that all results extend to general graphs.

$$\mathbf{x}^\top L \mathbf{x} = \sum_{\{u, v\} \in E} (x_u - x_v)^2$$

$$\mathbf{x}^\top N \mathbf{x} = \frac{1}{d} \sum_{\{u, v\} \in E} (x_u - x_v)^2$$

Regular graphs

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$$D[i, j] = dI = \begin{cases} d, & i = j \\ 0, & i \neq j \end{cases}$$

The Laplacian matrix:

$$L = D - A = dI - A$$

The normalized Laplacian matrix:

$$N = D^{-1/2} L D^{-1/2} = I - \frac{1}{d} A$$

For simplicity, we restrict our attention to regular graphs, noting that all results extend to general graphs.

All four types of matrices A, D, L, N are symmetric.

We start by reviewing some basic facts about such matrices.

Symmetric real matrices

- Let \mathbf{M} be a real symmetric $(n \times n)$ -matrix.

$$\mathbf{M}[i, j] \in \mathbb{R}$$

$$\mathbf{M}[i, j] = \mathbf{M}[j, i]$$

Symmetric real matrices

- Let \mathbf{M} be a real symmetric $(n \times n)$ -matrix.

Theorem: There exist

- $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ (they are called eigenvalues)
 - Orthonormal vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$ (they are called eigenvectors)
- such that:

$$\mathbf{M} = \sum_{i=1}^n \lambda_i \mathbf{v}_i \mathbf{v}_i^{\top}$$

Which implies: (for all i)

$$\mathbf{M} \mathbf{v}_i = \lambda_i \mathbf{v}_i$$

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If we change the axes of the Euclidean space \mathbb{R}^n to the eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, then the matrix \mathbf{M} becomes diagonal, with $\lambda_1, \dots, \lambda_n$ on the diagonal.

In other words, symmetric matrices stretch space independently along orthogonal directions.

Variational characterizations

- Let \mathbf{M} be a real symmetric $(n \times n)$ -matrix.
 - Eigenvalues: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$
 - Eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$

Rayleigh quotient of \mathbf{x}
with respect to \mathbf{M} :

$$R_{\mathbf{M}}(\mathbf{x}) = \frac{\mathbf{x}^\top \mathbf{M} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}}$$

Related to cuts

$$\mathbf{x}^\top \mathbf{L} \mathbf{x} = \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

$$\mathbf{x}^\top \mathbf{N} \mathbf{x} = \frac{1}{d} \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

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Theorem:

$$\lambda_k = \min_{k\text{-dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

Rayleigh quotient of \mathbf{x}
with respect to \mathbf{M} :

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Theorem:

$$\lambda_k = \min_{k\text{-dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

Proof (\geq):

Consider $\mathcal{V} = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$

- Write $\mathbf{x} \in \mathcal{V}$ as $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i$.
- $R_{\mathbf{M}}(\mathbf{x}) = \frac{\sum_{i=1}^k \alpha_i^2 \lambda_i}{\sum_{i=1}^k \alpha_i^2}$.
- $\max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x}) = \lambda_k$.
- $\lambda_k \geq \min_{k\text{-dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$.

Variational characterizations

Rayleigh quotient of \mathbf{x} with respect to \mathbf{M} :

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$$\lambda_k = \min_{k\text{-dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

Proof (\leq):

Consider any k -dimensional space \mathcal{V} .

- $\exists \mathbf{x} \in \mathcal{V} \cap \text{span}\{\mathbf{v}_k, \dots, \mathbf{v}_n\} \setminus \{\mathbf{0}\}$.
 - $\mathbf{x} = \sum_{i=k}^n \alpha_i \mathbf{v}_i$.
 - $R_{\mathbf{M}}(\mathbf{x}) = \frac{\sum_{i=k}^n \alpha_i^2 \lambda_i}{\sum_{i=k}^n \alpha_i^2} \geq \lambda_k$.
- $\lambda_k \leq \min_{k\text{-dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$.

Variational characterizations

Rayleigh quotient of \mathbf{x}
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$$\lambda_k = \min_{k\text{-dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

$$\lambda_1 = \min_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

Moreover, any minimizer \mathbf{x} is a corresponding eigenvector.

Consider $-\mathbf{M}$.

$$\lambda_n = \max_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

Moreover, any maximizer \mathbf{x} is a corresponding eigenvector.

Variational characterizations

- Let \mathbf{M} be a real symmetric $(n \times n)$ -matrix.
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Consider $-\mathbf{M}$.

$$\lambda_n = \max_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_{\mathbf{M}}(\mathbf{x})$$

A natural extension

$$\lambda_2 = \min_{\mathbf{x} \neq \mathbf{0}, \mathbf{x} \perp \mathbf{v}_1} R_{\mathbf{M}}(\mathbf{x})$$

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Theorem: We omit the proof.

$$\lambda_k = \min_{\mathbf{x} \neq \mathbf{0}, \mathbf{x} \perp \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}\}} R_{\mathbf{M}}(\mathbf{x})$$

$$\lambda_2 = \min_{\mathbf{x} \neq \mathbf{0}, \mathbf{x} \perp \mathbf{v}_1} R_{\mathbf{M}}(\mathbf{x})$$

A natural extension

Spectral graph theory

- The eigenvalues of A (adjacency), L (Laplacian), and N (normalized Laplacian) reveal information about:
 - Average density of cuts.
 - Bipartiteness.
 - Chromatic number.
 - Conductance.
 - Hamiltonicity.
 - Size of a maximum independent set.
 - Size of a maximum matching.
 - Toughness of a graph.
 - Number of connected components.

<https://adga-workshop.org/2025/tijn.pdf>

The smallest eigenvalue λ_1

- Consider the normalized Laplacian N of a graph $G = (V, E)$.
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Theorem:

- $\lambda_1 = 0$, with $\mathbf{1}$ being a corresponding eigenvector.

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Rayleigh quotient of \mathbf{x} with respect to N :

$$R_N(\mathbf{x}) = \frac{\mathbf{x}^\top N \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2} \geq 0$$

$$\lambda_1 \geq 0$$


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$$\lambda_1 \geq 0$$

$$\lambda_1 = \min_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} R_N(\mathbf{x}) \leq R_N(\mathbf{1}) = \frac{\sum_{\{u,v\} \in E} (1 - 1)^2}{d \sum_{v \in V} 1^2} = 0$$

$$\lambda_1 \leq 0$$

The eigenvalue λ_k

- Consider the normalized Laplacian N of a graph $G = (V, E)$.
 - Eigenvalues: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

Theorem:

- $\lambda_k = 0$ if and only if G has at least k connected components.

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if and only if

For every connected component S ,
 $x_u = x_v \quad \forall u, v \in S$

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if and only if

For every connected component S ,
 $x_u = x_v \quad \forall u, v \in S$

G has at least k connected components S_1, \dots, S_k .

Set $\mathcal{V} = \text{span}\{\mathbf{1}_{S_1}, \dots, \mathbf{1}_{S_k}\}$.

- $\forall \mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}, R_N(\mathbf{x}) = 0$

$$\lambda_k = \min_{k\text{-dim space } \mathcal{V}} \max_{\mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}} R_N(\mathbf{x})$$

$$\lambda_k = 0$$

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if and only if

For every connected component S ,
 $x_u = x_v \quad \forall u, v \in S$

$k = \text{dimension of } \mathcal{V} \leq \text{the number of connected components.}$

$$\lambda_k = 0$$

There is a k -dimensional subspace \mathcal{V} of \mathbb{R}^n such that:

- $\forall \mathbf{x} \in \mathcal{V} \setminus \{\mathbf{0}\}, R_N(\mathbf{x}) = 0$

The value of \mathbf{x} is constant within each connected component.

The largest eigenvalue λ_n

- Consider the normalized Laplacian N of a graph $G = (V, E)$.
 - Eigenvalues: $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$

Theorem:

- $\lambda_n \leq 2$.
- $\lambda_n = 2$ if and only if G has a bipartite connected component.

$$\lambda_n = \max_{x \in \mathbb{R}^n \setminus \{0\}} R_N(x)$$

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$$\lambda_n = \max_{x \in \mathbb{R}^n \setminus \{0\}} R_N(x) = 2 - \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} \leq 2$$

$$\begin{aligned} R_N(x) &= \frac{\sum_{\{u,v\} \in E} (x_u - x_v)^2}{d \sum_{v \in V} x_v^2} \\ &= \frac{2 \sum_{\{u,v\} \in E} (x_u^2 + x_v^2) - \sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} \\ &= 2 - \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} \end{aligned}$$

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$$\lambda_n = 2 - \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2}$$

- Suppose G has a bipartite connected component S with bipartition $S = A \cup B$.
- Setting $x = \mathbf{1}_A - \mathbf{1}_B$ makes $\frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} = 0$.
- Therefore, $\lambda_n = 2$.

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Theorem:

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- Suppose $\lambda_n = 2$.
- There exists $x \in \mathbb{R}^n \setminus \{0\}$ such that $\frac{\sum_{\{u,v\} \in E} (x_u + x_v)^2}{d \sum_{v \in V} x_v^2} = 0$.
- Define:
 - $A = \{v \in V \mid x_v > 0\} \neq \emptyset$
 - $B = \{v \in V \mid x_v < 0\} \neq \emptyset$
- $A \cup B$ is a union of bipartite connected components.

Outlook

- **Next:** We will show that:

$$\frac{\lambda_2}{2} \leq \Phi(G) \leq \sqrt{2\lambda_2} \quad \text{Cheeger's inequality.}$$

where λ_2 is the second eigenvalue of the normalized Laplacian N of G .

Moreover, given the eigenvector \mathbf{v}_2 , we can obtain a cut of conductance at most $\sqrt{2\lambda_2}$ in polynomial time.

References

- **Main reference:**
 - Lecture 4.1 of <https://sites.google.com/site/th saranurak/teaching/Expander>
 - Chapters 1, 2, and 3 of <https://lucatrevisan.github.io/books/expanders-2016.pdf>