CS5275 – The Algorithm Designer's Toolkit (S2 AY2025/26)

Lecture 8:

Expanders – Ramanujan graphs

Quasirandomness

- **Observation:**
  - For n-vertex d-regular graphs, the following two statements are equivalent.

For any cut $(S, V \setminus S)$, we have:

$$|E(S, V \setminus S)| \in \Theta\left(\frac{d}{n} \cdot |S||V \setminus S|\right)$$

$G = (V, E)$ is an $\Omega(1)$-expander.

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This is the expected number of edges in $E(S, V \setminus S)$ if G is a random n-vertex d-regular graph.

$G = (V, E)$ is an $\Omega(1)$-expander.

This can be seen as a **quasirandomness** property.

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Question: Can we extend this to $E(S, T)$ for any two subsets $S \subseteq V$ and $T \subseteq V$?

Notation

- For simplicity, we restrict ourselves to n-vertex d-regular graphs.

$e(A,B) = |(a,b) \in A \times B : \{a,b\} \in E|$

- This is the number of edges between A and B.
- If $\{u,v\} \in E$ satisfies $\{u,v\} \subseteq A \cap B$, then $\{u,v\}$ is counted twice.

$\sigma_2 = \max\{|\lambda_2|, |\lambda_3|, \dots, |\lambda_n|\}$,

- where $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -d$ are the eigenvalues of the adjacency matrix $\boldsymbol{A}$.

Expander mixing lemma

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Expander mixing lemma:

$$\left| e(S,T) - \frac{d}{n} \cdot |S||T| \right| \leq \sigma_2 \cdot \sqrt{|S||T|}$$

The number of edges between S and T is close to the expected number of edges between them in a random d-regular graph.

Proof of the lemma

- $e(S,T) = \mathbf{1}_S^\top \boldsymbol{A} \mathbf{1}_T$
- $|S||T| = \mathbf{1}_S^\top \boldsymbol{J} \mathbf{1}_T$, where $\boldsymbol{J}$ is the all-1 $(n \times n)$ matrix.

$$\left| e(S,T) - \frac{d}{n} \cdot |S||T| \right| = \left| \mathbf{1}_S^\top \boldsymbol{A} \mathbf{1}_T - \frac{d}{n} \cdot \mathbf{1}_S^\top \boldsymbol{J} \mathbf{1}_T \right| = \left| \mathbf{1}_S^\top \left(\boldsymbol{A} - \frac{d}{n} \boldsymbol{J} \right) \mathbf{1}_T \right|$$

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$$\leq \|\mathbf{1}_S\| \cdot \left\| \left(\boldsymbol{A} - \frac{d}{n} \boldsymbol{J} \right) \mathbf{1}_T \right\|$$

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Eigenvalues

Eigenvectors	$\boldsymbol{A}$	$\frac{d}{n}\boldsymbol{J}$	$\boldsymbol{A} - \frac{d}{n}\boldsymbol{J}$
$\boldsymbol{v}_1 = \frac{1}{\sqrt{n}}\mathbf{1}$	$\lambda_1 = d$	d	0
$\boldsymbol{v}_2$	λ_2	0	λ_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{v}_n$	λ_n	0	λ_n

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$$\left\| \left(\boldsymbol{A} - \frac{d}{n} \boldsymbol{J} \right) \mathbf{1}_T \right\| = \left\| \sum_{i=2}^{n} \lambda_i \langle \mathbf{1}_T, \boldsymbol{v}_i \rangle \boldsymbol{v}_i \right\|$$

$$= \sqrt{\sum_{i=2}^{n} \lambda_i^2 \langle \mathbf{1}_T, \boldsymbol{v}_i \rangle^2} \leq \sigma_2 \cdot \sqrt{\sum_{i=2}^{n} \langle \mathbf{1}_T, \boldsymbol{v}_i \rangle^2} \leq \sigma_2 \cdot \|\mathbf{1}_T\|$$

$$\sigma_2 = \max\{|\lambda_2|, |\lambda_3|, \dots, |\lambda_n|\}$$

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$$\leq \|\mathbf{1}_S\| \cdot \left\| \left(\boldsymbol{A} - \frac{d}{n} \boldsymbol{J} \right) \mathbf{1}_T \right\| \leq \|\mathbf{1}_S\| \cdot \sigma_2 \cdot \|\mathbf{1}_T\| = \sigma_2 \cdot \sqrt{|S||T|}$$

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Ramanujan graphs

- How small σ_2 can be?
  - **Answer:** $2\sqrt{d-1}$

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Alon–Boppana bound: For every d and $\epsilon > 0$, there exists n_0 such that all graphs with $\geq n_0$ vertices have $\sigma_2 > 2\sqrt{d-1} - \epsilon$.

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Proof of a weaker bound:

$\sigma_2 \in \Omega\left(\sqrt{d}\right)$ when $d \leq 0.99 \cdot n$

$$\mathrm{tr}(\boldsymbol{A}^2) = nd$$

The trace of a matrix is the sum of its diagonal entries and equals the sum of its eigenvalues.

$$\mathrm{tr}(\boldsymbol{A}^2) = \sum_{i=1}^{n} \lambda_i^2 \leq d^2 + (n-1)\sigma_2^2$$

$$\sigma_2 \geq \sqrt{\frac{nd - d^2}{n-1}} \in \Omega\left(\sqrt{d}\right)$$

Ramanujan graphs

The name comes from the Ramanujan–Petersson conjecture, which was used in a construction of some of these graphs.

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The graphs with $\sigma_2 \leq 2\sqrt{d-1}$ are called **Ramanujan graphs**.

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Eigenvalues of $\boldsymbol{N}$:

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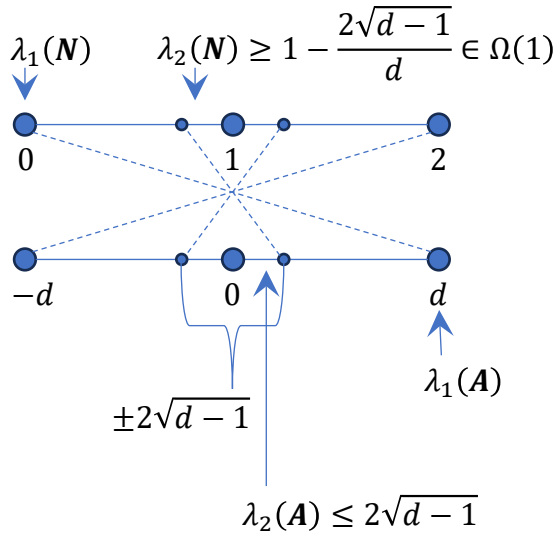

$$\boldsymbol{N} = \boldsymbol{I} - \frac{1}{d}\boldsymbol{A}$$

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Some $\Omega(1)$-expanders are not Ramanujan.

$\sigma_2 = d$ for any bipartite graph.

Random graphs are nearly Ramanujan

- For every d and $\epsilon > 0$, the probability that a random d-regular graph satisfies $\sigma_2 < 2\sqrt{d-1} + \epsilon$ tends to 1 as $n \to \infty$.

  - Joel Friedman (Duke Mathematical Journal, 2003)

Algebraic constructions

- There is an infinite family of d-regular Ramanujan graphs, whenever $d-1$ is a prime power.

  - Alexander Lubotzky, Ralph Phillips, and Peter Sarnak (Combinatorica, 1988)
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Open problem: Show this for all $d \geq 3$.

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Open problem: Show this for all $d \geq 3$.

Solved for <u>bipartite</u> Ramanujan graphs:

$$\max\{|\lambda_2|, |\lambda_3|, \dots, |\lambda_{n-1}|\} \leq 2\sqrt{d-1}$$

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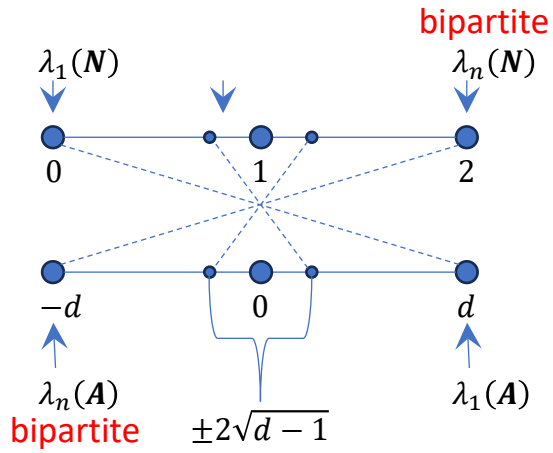

- Adam Marcus, Daniel Spielman and Nikhil Srivastava (Annals of Mathematics, 2015)

Properties of Ramanujan graphs

- **Claim:** For any $S, T \subseteq V$, if $|S| \cdot |T| \cdot d \geq 4n^2$, then $e(S,T) > 0$.

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$$\sqrt{d|S||T|} \geq 2n$$

$$\downarrow$$

$$\frac{d}{n} \cdot |S||T| \geq 2\sqrt{d|S||T|} > \sigma_2 \cdot \sqrt{|S||T|}$$

$$\downarrow$$

$$e(S,T) \geq \frac{d}{n} \cdot |S||T| - \sigma_2 \cdot \sqrt{|S||T|} > 0$$

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Exercise: The result is asymptotically the best possible.
- If you change $4n^2$ to ϵn^2 for some sufficiently small constant $\epsilon > 0$,
  - then this claim is false for every large enough d-regular graph.

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Corollary 1: Any independent set S has size $|S| < \frac{2n}{\sqrt{d}}$.

If $|S| \geq \frac{2n}{\sqrt{d}}$, then $|S|^2 d \geq 4n^2$, so $e(S,S) > 0$.

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Corollary 2: Chromatic number $> \frac{\sqrt{d}}{2}$.

If a proper coloring with $\leq \frac{\sqrt{d}}{2}$ colors exists, then $n < \frac{\sqrt{d}}{2} \cdot \frac{2n}{\sqrt{d}} = n$, which is impossible.

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These results have been used to give lower bounds on the number of communication rounds needed to compute certain colorings and independent sets in a distributed network.

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- Marijke H.L. Bodlaender, Magnús M. Halldórsson, Christian Konrad, and Fabian Kuhn. "Brief announcement: Local independent set approximation." *Proceedings of the ACM Symposium on Principles of Distributed Computing* (PODC 2016).
- Nathan Linial. "Locality in distributed graph algorithms." *SIAM Journal on computing* (1992).

Success probability amplification

It uses r random bits and takes t time.

- Consider a randomized algorithm that succeeds with probability $\frac{1}{2}$.
- **Goal:** Amplify the success probability to $1 - f$.

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Probability of no success is $\frac{1}{2^x} \leq f$.

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Probability of no success is $\frac{1}{2^x} \leq f$.

Question: Is it possible to reduce the failure probability without using more than r random bits?

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- Consider a randomized algorithm that succeeds with probability $\frac{1}{2}$.
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- Set $d = \left\lceil \frac{8}{f} \right\rceil$.
- Take a d-regular Ramanujan graph with $n = 2^r$ vertices.
  - Each vertex corresponds to an r-bit string.

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  - Each vertex corresponds to an r-bit string.
  - $S =$ the set of vertices that lead to a successful execution of the algorithm.
    - $|S| = \frac{n}{2}$.
  - $T =$ the set of vertices that do not have a neighbor in S.
    - $|S| \cdot |T| \cdot d < 4n^2 \rightarrow |T| < fn$.

Recall: For any $S, T \subseteq V$, if $|S| \cdot |T| \cdot d \geq 4n^2$, then $e(S,T) > 0$.

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- Sample a vertex v uniformly at random, for each neighbor u of v, run the algorithm using the r-bit string of u.
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Standard	$O\left(t \log \frac{1}{f}\right)$	$O\left(r \log \frac{1}{f}\right)$
Ramanujan	$O\left(\frac{t}{f}\right)$	r

Here we omit the cost for simulating a Ramanujan graph.

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Ramanujan	$O\left(\frac{t}{f}\right)$	r
Next	$O\left(t \log \frac{1}{f}\right)$	$O\left(r + \log \frac{1}{f}\right)$

A Chernoff Bound for random walks

- Let $G = (V, E)$ be an n-vertex d-regular regular graph.
- Let $(v_1, v_2, \ldots, v_\ell)$ be an $(\ell - 1)$-step random walk starting from a uniformly random vertex.
- Let $f : V \to [0, 1]$ be a function.
- Let $\mu = \frac{1}{n}\sum_{v \in V} f(v)$.
- Let $\epsilon > 0$.

$$\Pr\left[\frac{1}{\ell}\sum_{i=1}^{\ell} f(v_i) \geq \mu + \epsilon + \frac{\sigma_2}{d}\right] \in e^{-\Omega(\epsilon^2 \ell)}$$

The proof is omitted.

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New approach:

- Take $d \in O(1)$ and $n = 2^r$.
- Construct a random walk $(v_1, v_2, \dots, v_\ell)$ with $\ell \in O\left(\log\frac{1}{f}\right)$.
- Run the algorithm with these bit strings.

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It uses r random bits and takes t time.

- Consider a randomized algorithm that succeeds with probability $\frac{1}{2}$.
- **Goal:** Amplify the success probability to $1 - f$.

$n = 2^r$ $\quad$ $d \in O(1)$ is large enough so that $\frac{\sigma_2}{d} \leq \frac{1}{8}$.

- Let $G = (V, E)$ be an n-vertex d-regular regular graph.
- Let $(v_1, v_2, \ldots, v_\ell)$ be an $(\ell - 1)$-step random walk starting from a uniformly random vertex.
- Let $f : V \to [0, 1]$ be a function. (0 : the algorithm succeeds. 1 : the algorithm fails.)
- Let $\mu = \frac{1}{n}\sum_{v \in V} f(v)$.
- Let $\epsilon > 0$.

$$\Pr\left[\frac{1}{\ell}\sum_{i=1}^{\ell} f(v_i) \geq \underbrace{\mu + \epsilon + \frac{\sigma_2}{d}}_{\leq \frac{3}{4}}\right] \in e^{-\Omega(\epsilon^2 \ell)}$$

($\mu \leq \frac{1}{2}$, $\epsilon \leq \frac{1}{8}$, $\frac{\sigma_2}{d} \leq \frac{1}{8}$)

$\ell \in O\left(\log \frac{1}{f}\right)$ is large enough so that the error probability $e^{-\Omega(\epsilon^2 \ell)}$ is at most f.

$\frac{1}{\ell}\sum_{i=1}^{\ell} f(v_i) < 1 \rightarrow$ some of $v_1, v_2, \ldots, v_\ell$ make the algorithm successful.

Further applications

- Ramanujan graphs can be used to construct error correcting codes.

  Michael Sipser and Daniel A. Spielman.
  “Expander codes.”
  IEEE transactions on Information Theory, 2002.

- Ramanujan graphs can be used to construct cryptographic hash functions.

  Denis X. Charles, Eyal Z. Goren, and Kristin E. Lauter.
  “Cryptographic hash functions from expander graphs.”
  Journal of Cryptology, 2009.

Outlook

- **Next:**
  - An interesting application of Ramanujan graphs for graph algorithm design.

for general graphs

References

- **Main reference:**
  - Lecture 5.1 of https://sites.google.com/site/thsaranurak/teaching/Expander
  - Chapter 21 of https://lucatrevisan.github.io/books/expanders-2016.pdf

- **Additional/optional reading:**
  - https://en.wikipedia.org/wiki/Expander_mixing_lemma
  - https://en.wikipedia.org/wiki/Ramanujan_graph
  - Shlomo Hoory, Nathan Linial, and Avi Wigderson. “Expander graphs and their applications.” *Bulletin of the American Mathematical Society* 43.4 (2006): 439-561.