CS5275 – The Algorithm Designer's Toolkit (S2 AY2025/26)

Lecture 2:

Expanders – approximating conductance

Computing conductance

- **Racall:**
  - Conductance of a graph measures how well-connected it is.
  - Graphs with high conductance are robust against failures.

- It is **NP-hard** to compute the conductance <u>exactly</u>.

Computing conductance

- **Racall:**
  - Conductance of a graph measures how well-connected it is.
  - Graphs with high conductance are robust against failures.

- It is **NP-hard** to compute the conductance exactly.

- **Next:** $O(\log n)$-approximation in polynomial time. The Leighton–Rao algorithm

Computing an estimate ϕ such that $\Phi(G) \le \phi \le \alpha \cdot \Phi(G)$ for some $\alpha \in O(\log n)$.

Tool 1: LP relaxation

- A **general strategy** for **approximation algorithms** for an NP-hard problem:
  - Relax it to a problem that can be solved in polynomial time.
  - Apply an efficient algorithm to solve the relaxed problem.
  - Transform the solution back to a valid solution of the original problem.

Tool 1: LP relaxation

- A **general strategy** for **approximation algorithms** for an NP-hard problem:
  - Relax it to a <u>problem</u> that can be solved in polynomial time.

    In our case, linear programming (LP).
  - Apply an efficient algorithm to solve the relaxed problem.
  - <u>Transform</u> the solution back to a valid solution of the original problem.

    This step incurs some approximation error.

- Maximizing/minimizing a linear objective function.
  - Subject to linear constraints (equalities and inequalities).
- Variables take real values.

Metric space

- A **metric space** is an ordered pair (X, d), where
  - X is a set.
  - $d: X \times X \to \mathbb{R}_{\geq 0}$ is a **distance function (metric)** on X meeting the conditions:
    - $d(x, x) = 0$
    - $d(x, y) = d(y, x)$ (symmetry)
    - $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

Note: In mathematics, the definition of a metric space includes an additional requirement: if $x \neq y$, then $d(x, y) \neq 0$. However, in many applications in computer science, this condition is not satisfied.

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Connections to LP:
- They are linear constraints.

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Closure under addition:

- If d_1 and d_2 are metrics on the same set X, then:
  - $d_1 + d_2$ is also a metric on X.

Closure under scalar multiplication:

- If d is a metric on X and α is a scalar, then
  - $\alpha \cdot d$ is also a metric on X.

Line metric

- **Line metric:**
  - Given a function $f\colon X \to \mathbb{R}$, define
    $$d(x, y) = |f(x) - f(y)|$$

ℓ_1 metric

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- **ℓ_1 metric:**
  - Given a function $f\colon X \to \mathbb{R}^k$, define
$$d(x, y) = \|f(x) - f(y)\|_1 = \sum_{i=1}^{k} |f_i(x) - f_i(y)|$$

ℓ_1 metric

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Observation: A k-dimensional ℓ_1 metric is the sum of k line metrics.

Cut metric

- **Line metric:**
  - Given a function $f\colon X \to \mathbb{R}$, define
$$d(x,y) = |f(x) - f(y)|$$

- **Cut metric:**
  - Given a subset $S \subseteq X$, define
$$d(x,y) = \begin{cases} 1 & \text{if } \big((x \in S) \wedge (y \in X \setminus S)\big) \vee \big((x \in X \setminus S) \wedge (y \in S)\big) \\ 0 & \text{otherwise} \end{cases}$$

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Observation: Cut metric is a special case of line metric, by setting $d(x,y) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$

Cut metric

Observation: A line metric is a linear combination of $|X| - 1$ cut metrics.

Order $X = \{x_1, \dots, x_n\}$ so that $f(x_1) \le \cdots \le f(x_n)$.

- Set $\alpha_i = f(x_{i+1}) - f(x_i)$
- Set d_i to be a cut metric with $S = \{x_1, \dots, x_i\}$.
- Then $d = \sum_{i=1}^{n-1} \alpha_i d_i$.

- **Line metric:**
  - Given a function $f\colon X \to \mathbb{R}$, define

$$d(x, y) = |f(x) - f(y)|$$

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$$d(x, y) = \begin{cases} 1 & \text{if } \big((x \in S) \wedge (y \in X \setminus S)\big) \vee \big((x \in X \setminus S) \wedge (y \in S)\big) \\ 0 & \text{otherwise} \end{cases}$$

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$+$

Observation: A k-dimensional ℓ_1 metric is the sum of k line metrics.

$\downarrow$

Observation: A k-dimensional ℓ_1 metric is a linear combination of $k(|X| - 1)$ cut metrics.

Shortest path metric

- **Shortest path metric:**
  - Given a graph $G = (X, E)$, define
    $$d(x, y) = \text{shortest path distance between } x \text{ and } y$$

Tool 2: metric embedding

- **Motivation:** Computing the distance in some metrics is more expensive than in other metrics.

Compare cut metric with shortest path metric.

Tool 2: metric embedding

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Embed a complicated metric into a simpler one with **small distortion**.

↓

Greatly reduce the cost of distance computation, at the expense of only a **small approximation error**.

Tool 2: metric embedding

- **Motivation:** Computing the distance in some metrics is more expensive than in other metrics.

Embed a complicated metric into a simpler one with **small distortion**.

Greatly reduce the cost of distance computation, at the expense of only a **small approximation error**.

Next: Bourgain's theorem allows us to embed any n-point metric into an $\underbrace{O(\log^2 n)\text{-dimensional } \ell_1 \text{ metric space}}_{\text{A linear combination of } O(n \log^2 n) \textbf{ cut metrics}!}$ with $O(\log n)$ distortion.

Tool 2: metric embedding

- **Bourgain's theorem for metric embeddings:**
  - Given any n-point metric space (X, d),
    - there is an embedding $f\colon X \to \mathbb{R}^{O(\log^2 n)}$ such that for any $x, y \in X$,
      - $\|f(x) - f(y)\|_1 \leq d(x, y) \in O(\log n) \cdot \|f(x) - f(y)\|_1$

If $d(x, y)$ is given for all $x, y \in X$, then the mapping f can be computed in (randomized) **polynomial time**.

A quick application

- **Bourgain's theorem for metric embeddings:**
  - Given any n-point metric space (X, d),
    - there is an embedding $f: X \to \mathbb{R}^{O(\log^2 n)}$ such that for any $x, y \in X$,
      - $\|f(x) - f(y)\|_1 \le d(x, y) \in O(\log n) \cdot \|f(x) - f(y)\|_1$

Apply Bourgain's theorem to the shortest path metric.

We can label each vertex with a vector of $O(\log^2 n)$ values so that:

- Given any two vertices u and v:
  - We can obtain an $O(\log n)$-approximation of $\mathrm{dist}(u, v)$ in **only** $O(\log^2 n)$ time by reading the labels of u and v.

Generalized conductance

with positive edge weights

- Let $G = (V, E_G)$ and $H = (V, E_H)$ be weighted graphs on the same vertex set V.

$$\Phi(G) = \min_{S\,:\,S \neq V \text{ and } S \neq \emptyset} \frac{|E_G(S, V \setminus S)|}{\min\{\mathrm{vol}_G(S), \mathrm{vol}_G(V \setminus S)\}}$$

$$\Phi(G, H) = \min_{S\,:\,E_H(S, V \setminus S) \neq \emptyset} \frac{|E_G(S, V \setminus S)|}{|E_H(S, V \setminus S)|}$$

Here we slightly abuse the notation to write:

- $|E_G(S, V \setminus S)| = \sum_{e \in E_G(S, V \setminus S)} w_G(e)$
- $|E_H(S, V \setminus S)| = \sum_{e \in E_H(S, V \setminus S)} w_H(e)$
- $\mathrm{vol}_G(S) = \sum_{v \in S} \sum_{e\,:\,v \in e} w_G(e)$

Generalized conductance

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Claim: Given G, there is a choice of H such that $\Phi(G)$ and $\Phi(G, H)$ are within a constant factor.

$\downarrow$

$O(\log n)$-approximation of $\Phi(G, H)$ $\longrightarrow$ $O(\log n)$-approximation of $\Phi(G)$

Generalized conductance

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$$\frac{|E_G(S, V \setminus S)|}{\min\{\mathrm{vol}_G(S), \mathrm{vol}_G(V \setminus S)\}} \xleftrightarrow{\text{within a constant factor}} \frac{|E_G(S, V \setminus S)|}{\left(\frac{\mathrm{vol}_G(S) \cdot \mathrm{vol}_G(V \setminus S)}{\mathrm{vol}_G(V)}\right)}$$

$$\frac{\mathrm{vol}_G(S) \cdot \mathrm{vol}_G(V \setminus S)}{\mathrm{vol}_G(V)} = \min\{\mathrm{vol}_G(S), \mathrm{vol}_G(V \setminus S)\} \cdot \overbrace{\frac{\max\{\mathrm{vol}_G(S), \mathrm{vol}_G(V \setminus S)\}}{\mathrm{vol}_G(V)}}^{\in [0.5, 1]}$$

Generalized conductance

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Connection to cut metric

- Let $G = (V, E_G)$ and $H = (V, E_H)$ be weighted graphs on the same vertex set V.

- Let d_S be a cut metric for the cut S of V.

$$\Phi(G,H) = \min_{S \,:\, E_H(S, V\setminus S) \neq \emptyset} \frac{|E_G(S, V \setminus S)|}{|E_H(S, V \setminus S)|}$$

equivalent

$$\Phi(G,H) = \min_{S \,:\, \sum_{u,v} w_H(u,v) \cdot d_S(u,v) \neq 0} \frac{\sum_{u,v} w_G(u,v) \cdot d_S(u,v)}{\sum_{u,v} w_H(u,v) \cdot d_S(u,v)}$$

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equivalent

$\Phi(G,H)$

Minimize:
- $\sum_{u,v} w_G(u,v) \cdot d(u,v)$

Subject to:
- $\sum_{u,v} w_H(u,v) \cdot d(u,v) = 1$
- d is a cut metric scaled by a real value.

LP relaxation

$\Phi(G, H)$

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LP relaxation $\longrightarrow$

Leighton–**R**ao

$\mathrm{LR}(G, H)$

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- d is a **metric**, i.e.,
  - $d(x, x) = 0\ \forall x$
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  - $d(x, z) \leq d(x, y) + d(y, z)\ \forall x, y, z$

LP relaxation

This is a relaxation in the sense that:

Any cut metric scaled by a real value is a metric. → $\mathrm{LR}(G,H)$ considers a wider range of search space than $\mathrm{LR}(G,H)$. → $\mathrm{LR}(G,H) \leq \Phi(G,H)$

$\Phi(G,H)$

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LP relaxation →

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Approximation ratio

An $O(\log n)$-approximation of conductance can be computed in polynomial time by computing $\mathrm{LR}(G, H)$ using a linear programming solver.

Claim: $\mathrm{LR}(G, H) \in \Omega\left(\frac{\Phi(G,H)}{\log n}\right)$

$\mathrm{LR}(G, H) \leq \Phi(G, H)$

$\Phi(G, H)$

Minimize:
- $\sum_{u,v} w_G(u, v) \cdot d(u, v)$

Subject to:
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LP relaxation $\rightarrow$

$\mathrm{LR}(G, H)$

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Proof of the claim

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$$\mathrm{LR}(G,H) = \frac{\sum_{u,v} w_G(u,v) \cdot d_{\mathrm{OPT}}(u,v)}{\sum_{u,v} w_H(u,v) \cdot d_{\mathrm{OPT}}(u,v)}$$

d_{OPT} = an optimal metric of

Proof of the claim

Claim: $\mathrm{LR}(G,H) \in \Omega\left(\frac{\Phi(G,H)}{\log n}\right)$

Apply **Bourgain's theorem** to d_{OPT}.

There is an $O(\log^2 n)$-dimensional ℓ_1 metric d^* such that:
$d^*(u,v) \leq d_{\mathrm{OPT}}(u,v) \in O(\log n) \cdot d^*(u,v)$

$$\mathrm{LR}(G,H) = \frac{\sum_{u,v} w_G(u,v) \cdot d_{\mathrm{OPT}}(u,v)}{\sum_{u,v} w_H(u,v) \cdot d_{\mathrm{OPT}}(u,v)}$$

Proof of the claim

Recall: d^* is a linear combination of $O(n \log^2 n)$ cut metrics.

There is a collection C of $O(n \log^2 n)$ cuts and coefficients $\{\alpha_S \mid S \in C\}$ such that:

$$d^* = \sum_{S \in C} \alpha_S \, d_S,$$

where d_S is the cut metric of the cut S.

Apply **Bourgain's theorem** to d_{OPT}.

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$$\sum_{S \in C} \alpha_S \, d_S(u,v) \leq d_{\mathrm{OPT}}(u,v) \in O(\log n) \cdot \sum_{S \in C} \alpha_S \, d_S(u,v)$$

$$\mathrm{LR}(G,H) = \frac{\sum_{u,v} w_G(u,v) \cdot d_{\mathrm{OPT}}(u,v)}{\sum_{u,v} w_H(u,v) \cdot d_{\mathrm{OPT}}(u,v)} \in \Omega\left(\frac{1}{\log n}\right) \cdot \frac{\sum_{u,v} w_G(u,v) \cdot \sum_{S \in C} \alpha_S \, d_S(u,v)}{\sum_{u,v} w_H(u,v) \cdot \sum_{S \in C} \alpha_S \, d_S(u,v)}$$

Proof of the claim

Claim: $\mathrm{LR}(G,H) \in \Omega\left(\frac{\Phi(G,H)}{\log n}\right)$

Proof idea:

- Among all $O(n \log^2 n)$ cuts $S \in C$:
  - Choose the sparsest one.
  - $\frac{\sum_{u,v} w_G(u,v) \cdot \sum_{S\in C} \alpha_S d_S(u,v)}{\sum_{u,v} w_H(u,v) \cdot \sum_{S\in C} \alpha_S d_S(u,v)} \geq$ sparsity of the cut $\geq \Phi(G,H)$

$$\sum_{S\in C} \alpha_S \, d_S(u,v) \leq d_{\mathrm{OPT}}(u,v) \in O(\log n) \cdot \sum_{S\in C} \alpha_S \, d_S(u,v)$$

$$\mathrm{LR}(G,H) = \frac{\sum_{u,v} w_G(u,v) \cdot d_{\mathrm{OPT}}(u,v)}{\sum_{u,v} w_H(u,v) \cdot d_{\mathrm{OPT}}(u,v)} \in \Omega\left(\frac{1}{\log n}\right) \cdot \frac{\sum_{u,v} w_G(u,v) \cdot \sum_{S\in C} \alpha_S \, d_S(u,v)}{\sum_{u,v} w_H(u,v) \cdot \sum_{S\in C} \alpha_S \, d_S(u,v)} \in \Omega\left(\frac{\Phi(G,H)}{\log n}\right)$$

Proof of the claim

Claim: $\mathrm{LR}(G,H) \in \Omega\left(\frac{\Phi(G,H)}{\log n}\right)$

$$\frac{\sum_{u,v} w_G(u,v) \cdot \sum_{S\in C} \alpha_S\, d_S(u,v)}{\sum_{u,v} w_H(u,v) \cdot \sum_{S\in C} \alpha_S\, d_S(u,v)} = \frac{\sum_{S\in C} \alpha_S \cdot \sum_{u,v} w_G(u,v) d_S(u,v)}{\sum_{S\in C} \alpha_S \cdot \sum_{u,v} w_H(u,v) d_S(u,v)} \geq \min_{S\in C} \frac{\sum_{u,v} w_G(u,v) \cdot d_S(u,v)}{\sum_{u,v} w_H(u,v) \cdot d_S(u,v)} \geq \Phi(G,H)$$

$$\frac{\sum_i a_i}{\sum_i b_i} \geq \min_i \frac{a_i}{b_i}$$

$$\mathrm{LR}(G,H) = \frac{\sum_{u,v} w_G(u,v) \cdot d_{\mathrm{OPT}}(u,v)}{\sum_{u,v} w_H(u,v) \cdot d_{\mathrm{OPT}}(u,v)} \in \Omega\left(\frac{1}{\log n}\right) \cdot \frac{\sum_{u,v} w_G(u,v) \cdot \sum_{S\in C} \alpha_S\, d_S(u,v)}{\sum_{u,v} w_H(u,v) \cdot \sum_{S\in C} \alpha_S\, d_S(u,v)} \in \Omega\left(\frac{\Phi(G,H)}{\log n}\right)$$

LP rounding

- Using a linear programming solver, we can obtain:
  - $\mathrm{LR}(G, H)$ → $O(\log n)$-approximation to the conductance $\Phi(G)$
  - d_{OPT}

Exercise:

Can we use it to obtain a cut $S \subseteq V$ whose conductance is $\Omega\left(\frac{\Phi(G)}{\log n}\right)$?

Summary

- Computing conductance can be viewed as minimizing a **linear objective function** subject to **linear constraints**, where the search space consists of all **cut metrics** scaled by real values.

- **Tool 1:**
  - By relaxing the search space from cut metrics to **all metrics**, we obtain a **linear programming** formulation, whose optimal solution can be computed in polynomial time.

- **Tool 2:**
  - By **Bourgain's theorem**, any metric can be approximated within an $O(\log n)$ factor by a linear combination of cut metrics. Consequently, the LP solution yields an $O(\log n)$-approximation to the conductance.

Outlook

- **Next:** Proof of Bourgain's theorem.
- **Natural question:** Is it possible to do better than $O(\log n)$-approximation?

Outlook

- **Next:** Proof of Bourgain's theorem.

- **Natural question:** Is it possible to do better than $O(\log n)$-approximation?

- The bound $\mathrm{LR}(G,H) \in \Omega\left(\frac{\Phi(G,H)}{\log n}\right)$ is already tight, so we will need a different approach.

- By considering some special class of metrics that admits better embedding to ℓ_1 metrics, the approximation ratio can be improved to $O\left(\sqrt{\log n} \log \log n\right)$.
  - This requires the use of semidefinite programming (SDP).

https://dl.acm.org/doi/pdf/10.1145/1060590.1060673 Not covered in this course

References

- **Main reference:**
  - Lecture 2.1 of https://sites.google.com/site/thsaranurak/teaching/Expander
  - Chapter 10 of https://lucatrevisan.github.io/books/expanders-2016.pdf

- **Additional/optional reading:**
  - Polylogarithmic approximations of conductance can also be obtained via cut-matching games.
    - Rohit Khandekar, Satish Rao, and Umesh Vazirani. "Graph partitioning using single commodity flows." J. ACM, 56(4)
      - https://dl.acm.org/doi/abs/10.1145/1538902.1538903
    - Rohit Khandekar, Subhash Khot, Lorenzo Orecchia, and Nisheeth K Vishnoi. "On a cut-matching game for the sparsest cut problem." Univ. California, Berkeley, CA, USA, Tech. Rep. UCB/EECS-2007-177, 6(7):12, 2007.
      - https://www2.eecs.berkeley.edu/Pubs/TechRpts/2007/EECS-2007-177.pdf