CS5275 Tutorial Problem Set 2

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Problem 1 For any two probability distributions μ and ν on a finite set Ω, define $\Delta(x) := \mu(x) - \nu(x)$ for $x \in \Omega$. Partition Ω into $\Omega^+ := \{x \in \Omega : \Delta(x) \geq 0\}$ and $\Omega^- := \{x \in \Omega : \Delta(x) < 0\}$.

By definition we have

$$\sum_{x\in\Omega} \Delta(x) = \sum_{x\in\Omega} \mu(x) - \sum_{x\in\Omega} \nu(x) = 1 - 1 = 0,$$

so that

$$\sum_{x\in\Omega} \Delta(x) = \sum_{x\in\Omega^+} \Delta(x) + \sum_{x\in\Omega^-} \Delta(x) = 0 \implies \sum_{x\in\Omega^+} \Delta(x) = -\sum_{x\in\Omega^-} \Delta(x).$$

Thus

$$\begin{aligned} d_{\mathrm{TV}}(\mu,\nu) &= \frac{1}{2}\sum_{x\in\Omega} |\Delta(x)| \\ &= \frac{1}{2}\left(\sum_{x\in\Omega^+} \Delta(x) + \sum_{x\in\Omega^-} (-\Delta(x))\right) \\ &= \sum_{x\in\Omega^+} \Delta(x). \end{aligned}$$

Now for any $A \subseteq \Omega$, we have

$$\begin{aligned} \mu(A) - \nu(A) &= \sum_{x\in A} \Delta(x) \\ &= \underbrace{\sum_{x\in A\cap\Omega^+} \Delta(x)}_{\leq \sum_{x\in\Omega^+} \Delta(x)} + \underbrace{\sum_{x\in A\cap\Omega^-} \Delta(x)}_{\leq 0} \\ &\leq \sum_{x\in\Omega^+} \Delta(x), \end{aligned}$$

and note that equality is attained when $A = \Omega^+$. Thus $d_{\mathrm{TV}}(\mu,\nu) = \max\limits_{A\subseteq\Omega} (\mu(A) - \nu(A))$ as required.

Problem 2

Problem 3 Consider the normalized Laplacian matrix $\mathbf{N} = \mathbf{I} - \frac{1}{d}\mathbf{A}$. Recall from Lecture 4 that

- its smallest eigenvalue is 0 with eigenvector $\mathbf{1}$;
- its second smallest eigenvalue is 0 if and only if the graph has at least 2 connected components;
- its largest eigenvalue is at most 2 and equals 2 if and only if the graph has a bipartite connected component.

As G is connected and non-bipartite, for the transition matrix $\mathbf{W} = \frac{1}{d}\mathbf{A} = \mathbf{I} - \mathbf{N}$, the largest eigenvalue is 1, the second largest eigenvalue is smaller than 1 and the smallest eigenvalue is larger than -1. Thus $\mathbf{W}$ has an orthonormal eigenbasis $\{\mathbf{w}_1 = \frac{1}{\sqrt{n}}\mathbf{1}, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ with eigenvalues $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n > -1$, and $\mathbf{W}^\top = \mathbf{W}$.

For any initial distribution $\mathbf{x}$, we can decompose it into the eigenbasis as $\sum_{i=1}^n c_i \mathbf{w}_i$. Hence we have

$$\left(\mathbf{W}^\top\right)^t \mathbf{x} = \mathbf{W}^t \mathbf{x} = \mathbf{W}^t \left(c_1 \mathbf{w}_1 + \sum_{i=2}^n c_i \mathbf{w}_i\right)$$
$$= c_1 \left(\frac{1}{\sqrt{n}}\mathbf{1}\right) + \sum_{i=2}^n c_i \lambda_i^t \mathbf{w}_i.$$

As $|\lambda_i| < 1$ for each $i = 2, \ldots, n$, the latter term goes to 0 as $t \to \infty$. Also notice that c_1 is the dot product between $\mathbf{x}$ and $\mathbf{w}_1$. Therefore,

$$\lim_{t\to\infty} \left(\mathbf{W}^\top\right)^t \mathbf{x} = \left(\sum_{i=1}^n x_i \frac{1}{\sqrt{n}}\right)\left(\frac{1}{\sqrt{n}}\mathbf{1}\right) + 0 = \frac{1}{n}\mathbf{1} = \pi.$$

Problem 4 Construct $S \subseteq V$ by independently choosing to include each vertex with probability $p = 1/(d+1)$. Define T as the set of vertices that are neither in S nor have any neighbors in S: $T = \{v \in V \setminus S \mid N(v) \cap S = \varnothing\}$. We show that $|S||T|d \geq 0.01n^2$.

Define indicator random variables for each vertex $i, j \in V$: $I_i = 1$ if $i \in S$ and 0 otherwise, $J_j = 1$ if $j \in T$ and 0 otherwise. Then $|S| = \sum_{i=1}^n I_i$ and $|T| = \sum_{j=1}^n J_j$.

Consider the three possible cases for $I_i J_j$:

- $i = j$: By definition, $I_i J_j = 0$ as a vertex cannot be both in S and in T.
- $i \in N(j)$: If $j \in T$, none of its neighbors can be in S. Thus $I_i = 0$, so $I_i J_j = 0$.
- $i \neq j$ and $i \notin N(j)$: The event that $i \in S$ depends only on vertex i. The event that $j \in T$ depends only on vertex j and its neighborhood $N(j)$. Since $i \notin N(j) \cup \{j\}$, I_i and J_j are independent.

Since G is d-regular, the set $\{j\} \cup N(j)$ contains exactly $d+1$ vertices. The probability that none of these $d+1$ vertices are in S is $(1-p)^{d+1}$. Therefore

$$\mathbb{E}[I_i J_j] = \mathbb{E}[I_i]\,\mathbb{E}[J_j] = p(1-p)^{d+1}.$$

For each of the n choices of j, we exclude j itself and the neighbors of j. This leaves exactly $n-d-1$ choices for i. Therefore

$$\mathbb{E}[|S||T|] = \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} I_j J_j\right] = n(n-d-1)p(1-p)^{d+1}.$$

By substituting $p = \frac{1}{d+1}$, we have

$$\mathbb{E}[|S||T|] = n(n-d-1)\left(\frac{1}{d+1}\right)\left(1-\frac{1}{d+1}\right)^{d+1} = n(n-d-1)\left(\frac{1}{d+1}\right)\left(\frac{d}{d+1}\right)^{d+1},$$

So that

$$\mathbb{E}[|S||T|d] = n(n-d-1)\left(\frac{d}{d+1}\right)^{d+2}.$$

As the function $\left(\frac{d}{d+1}\right)^{d+2}$ strictly increases with d, for $d \geq 1$ its minimum is $1/8$, attained at $d=1$. Therefore, for any $d \geq 1$, we have

$$\mathbb{E}[|S||T|d] \geq \frac{1}{8}n(n-d-1) = \frac{1}{8}n^2\left(1-\frac{d+1}{n}\right).$$

Finally, we take $n_0 = n_0(d) = 25(d+1)/23$, so that

$$\mathbb{E}[|S||T|d] \geq \frac{1}{8}n^2\left(1-\frac{23}{25}\right) = 0.01n^2,$$

which implies that there exists at least one outcome of the random choice of S and T so that $|S||T|d \geq 0.01n^2$ as required.