
University of Michigan–Ann Arbor

Department of Electrical Engineering and Computer Science

EECS 498 004 **Advanced Graph Algorithms**, Fall 2021

Lecture 6: Cut/Flow Equivalences Among Expanders

September 16, 2021

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1 Overview and Preliminaries

As is discussed in the previous lectures, an expander behaves like complete graph (the most well-connected graph) such that it is **robust against deletions**. This lecture is going to show expanders are the most well-connected graphs in the senses of **cut and flow**.

We first introduce some definitions.

Definition 1.1. Given a graph $G = \{V, E\}$ and a function $\mathbf{d} : V \rightarrow \mathbb{R}$, we say G has **degree profile** \mathbf{d} if $d(u) = \deg_G(u), \forall u \in V$. And graph is **\mathbf{d} -restricted** if $\deg_G(u) \leq \mathbf{d}(u), \forall u \in V$.

Definition 1.2. The **\mathbf{d} -product graph** $K = \{E, V\}$ is a complete graph such that the capacity/weight of each edge (u, v) is $\kappa_K(u, v) = \frac{\mathbf{d}(u)\mathbf{d}(v)}{\mathbf{d}(V)}$.

- Note that function \mathbf{d} is exactly the degree profile of K , since $\forall u \in V, \deg_K(u) = \sum_{v \in V} \mathbf{d}(u) \frac{\mathbf{d}(v)}{\mathbf{d}(V)} = \mathbf{d}(u)$.
- when $\forall u \in V, \mathbf{d}(u) = 1$, the graph is a normalized complete graph where each edge has weight $\frac{1}{n}$.

Definition 1.3. Given graphs $G = \{V, E\}$ and $H = \{V, E'\}$, $H \preceq^{\text{cut}} G$ if $\kappa_H(\partial_H S) \leq \kappa_G(\partial_G S), \forall S \subseteq V$. $H \preceq^{\text{flow}} G$ if H is embeddable into G with no congestion. And $H \preceq^{\text{deg}} G$ if $\deg_H(u) \leq \deg_G(u)$ for all u .

2 Product Graphs and Their Cut/Flow Connectivity

Our goal of this section is to show that among all \mathbf{d} -restricted graph, the \mathbf{d} -product graph K is the most well-connected graph w.r.t. both \preceq^{cut} and \preceq^{flow} .

2.1 Cut Connectivity of Product Graphs

Every cut of product graph is almost maximally big. Formally, we have the following lemma.

Lemma 2.1. *For all d -restricted graphs H and the d -product graph K , we have $H \leq^{\text{cut}} 2K$.*

Proof. For any cut S where $d(S) \leq d(V \setminus S)$, we have $\kappa_H(\partial_H S) \leq \text{vol}_H(S) = \sum_{u \in S} \deg_H(u) \leq \sum_{u \in S} \deg_K(u) = d(S)$ since H is d -restricted.

For each $u \in S$, we have $\kappa_K(E_K(u, V \setminus S)) = \sum_{v \notin S} \frac{d(u)d(v)}{d(V)} = d(u) \sum_{v \notin S} \frac{d(v)}{d(V)} = d(u) \frac{d(V \setminus S)}{d(V)} \geq d(u)/2$. So $\kappa_K(\partial_K S) \geq d(S)/2$. \square

- Note: this is the same as proving that $\Phi(K, H) \geq 1/2$.

2.2 Flow Connectivity of Product Graphs

All d -restricted demands are routable on the d -product graph. To see that, let us first get some intuition and discuss about routing demands with small congestion on arbitrary graphs.

Given any d -restricted demand H and an arbitrary graph G ,

- If there is a node u in G where $\deg_G(u) \ll d(u)$, then clearly routing some H in G must cause large congestion.
- What if $\deg_G(u) \geq d(u)$ for all u ? Is this enough to route H in G in general? No, for example...



However, if G is a d -product graph, condition " $\deg_G(u) \geq d(u), \forall u$ " is enough to make demand H routable in G .

Lemma 2.2. *All d -restricted demands H are routable in the d -product graph K with congestion 2, i.e. $H \leq^{\text{flow}} 2K$.*

Proof. To embed H into K , we construct the following flow $\mathcal{F} = \{f_{(s,t)}\}_{(s,t) \in E(H)}$: For each $(s,t) \in E(H)$, the flow $f_{(s,t)}$ send flow through the two-hop paths $P_{svt} = (s, v, t)$ with value $f(P_{svt}) = \kappa_H(s, t) \cdot \frac{d(v)}{d(V)}$. Note that $\|f_{(s,t)}\| = \sum_v \kappa_H(s, t) \cdot \frac{d(v)}{d(V)} = \kappa_H(s, t)$. So, \mathcal{F} indeed embeds H into K .

Now, we bound the congestion when we implement flow \mathcal{F} in K . For each edge $(u, v) \in E(K)$, it is contained in flow paths $\{P_{suv}\}_{s \in V}$ and $\{P_{uvt}\}_{t \in V}$ with total flow value

$$\sum_{s \in V} \kappa_H(s, v) \cdot \frac{d(u)}{d(V)} + \sum_{t \in V} \kappa_H(u, t) \cdot \frac{d(v)}{d(V)} = \frac{\deg_H(v)d(u)}{d(V)} + \frac{\deg_H(u)d(v)}{d(V)} \leq \frac{d(v)d(u)}{d(V)} + \frac{d(u)d(v)}{d(V)} \leq 2\kappa_K(u, v)$$

\square

Therefore, \mathcal{F} embeds H into K with congestion 2.

3 Cut/Flow Connectivity of Expanders

Given an expander G , if you want to prove that $H \preceq^{\text{cut}} G$ or $H \preceq^{\text{flow}} G$ (within some constant factor), you can just show that **degree of G is big enough!** Formally, we have the following lemma.

Lemma 3.1. *Let G be ϕ -expander. Let H be any graph where $H \preceq^{\text{deg}} G$. Then, $H \preceq^{\text{cut}} \frac{1}{\phi} G$.*

Proof. For any $S \subset V$ where $\text{vol}_G(S) \leq \text{vol}_G(V \setminus S)$, we have that

$$\kappa_H(\partial_H S) \leq \text{vol}_H(S) \leq \text{vol}_G(S)$$

because $H \preceq^{\text{deg}} G$. At the same time we have

$$\kappa_G(\partial_G S) \geq \phi \text{vol}_G(S).$$

So $\kappa_H(\partial_H S) \leq \frac{1}{\phi} \kappa_G(\partial_G S)$ and so $H \preceq^{\text{cut}} \frac{1}{\phi} G$. □

By the approximate max-flow min-cut theorem, we know that $H \preceq^{\text{cut}} \frac{1}{\phi} G$ implies $H \preceq^{\text{flow}} \frac{O(\log n)}{\phi} G$. So we have

Corollary 3.2. *Let G be ϕ -expander. Let H be any graph where $H \preceq^{\text{deg}} G$. Then, $H \preceq^{\text{flow}} \frac{O(\log n)}{\phi} G$.*

4 Cut/Flow Connectivity Equivalence of Expanders

We are going to show that expanders with same degree profile are approximately equivalent in some sense. We first formalize notations of equivalence.

4.1 Notations of Equivalence

Definition 4.1. G_1 and G_2 are α -**cut-equivalent** if there are $\alpha_1, \alpha_2 > 0$ where $\alpha_1 \cdot \alpha_2 \leq \alpha$ such that $G_2 \preceq^{\text{cut}} \alpha_1 G_1$ and $G_1 \preceq^{\text{cut}} \alpha_2 G_2$. Denote this by $G_1 \approx_\alpha^{\text{cut}} G_2$.

- Our definition is designed such that it measures how much the cut size of G_1 and G_2 are similar *modulo some scaling*. See the example below.

Example 4.2. G_1 and $G_2 = 100 \cdot G_1$ are 1-cut-equivalent, because $G_2 \preceq^{\text{cut}} 100 G_1$ but $G_1 \preceq^{\text{cut}} \frac{1}{100} G_2$.

Definition 4.3. G_1 and G_2 are α -**flow-equivalent** if there are $\alpha_1, \alpha_2 > 0$ where $\alpha_1 \cdot \alpha_2 \leq \alpha$ such that $G_2 \preceq^{\text{flow}} \alpha_1 G_1$ and $G_1 \preceq^{\text{flow}} \alpha_2 G_2$. Denote this by $G_1 \approx_\alpha^{\text{flow}} G_2$.

- Our definition is designed such that it measures the congestion when we try to embed G_1 and G_2 into each other.

Exercise 4.4. Show that:

- If $G_1 \approx_\alpha^{\text{flow}} G_2$, then $G_1 \approx_\alpha^{\text{cut}} G_2$.
- If $G_1 \approx_\alpha^{\text{cut}} G_2$, then $G_1 \approx_{O(\alpha \log^2 n)}^{\text{flow}} G_2$.

4.2 Equivalence and Well-Connectivity of Expanders

Based on previous conclusions given by simple proofs, We can obtain statements of strong conceptual messages.

In a sentence, **Expanders with Degree Profile d are all equivalent in the sense of cut/flow connectivity and they are the most well-connected d -restricted graphs.**

First, we have that $\tilde{\Omega}(1)$ -expanders with the same degree profile are approximately equivalent to each other w.r.t. both \leq^{cut} and \leq^{flow} .

Corollary 4.5. *Let G and G' be ϕ -expanders with degree profile d . We have*

- $G \leq^{\text{cut}} \frac{1}{\phi} G'$ and $G' \leq^{\text{cut}} \frac{1}{\phi} G$. So G and G' are $O(\frac{1}{\phi^2})$ -cut-equivalent.
- $G \leq^{\text{flow}} O(\frac{\log n}{\phi}) G'$ and $G' \leq^{\text{flow}} O(\frac{\log n}{\phi}) G$. So G and G' are $O(\frac{\log^2 n}{\phi^2})$ -flow-equivalent.

Notice first that this statement is robust against perturbation of d — even if G and G' have different degree profile within a factor of some constant, which is going to add some extra factor on the equivalence factor, they are still $O(\frac{1}{\phi^2})$ -cut- and $O(\frac{\log^2 n}{\phi^2})$ -flow-equivalent.

This Corollary shows that, in other words, $\tilde{\Omega}(1)$ -expanders with the same degree profile form a $\tilde{O}(1)$ -cut- and -flow-equivalence class, where everything within the class have a similar cut and flow structure. Moreover, the most well-connected d -restricted graph, i.e. the d -product graph K , is also in this equivalence class, since it is an $\Omega(1)$ -expander with degree profile d . The approximation factor is a bit better too.

Corollary 4.6. *Let G be ϕ -expander with degree profile d . Let K be the d -product graph. We have*

- $G \leq^{\text{cut}} 2K$ and $K \leq^{\text{cut}} \frac{1}{\phi} G$. So G and K are $O(\frac{1}{\phi})$ -cut-equivalent.
- $G \leq^{\text{flow}} 2K$ and $K \leq^{\text{flow}} O(\frac{\log n}{\phi}) G$. So G and K are $O(\frac{\log n}{\phi})$ -flow-equivalent.

This is nice because G can have much fewer edges than K , yet G is as “good” as K .

Because of this, any expander with degree profile d is approximately the most well-connected d -restricted graph w.r.t. both \leq^{cut} and \leq^{flow} .

Corollary 4.7. *Let G be a ϕ -expander with degree profile d .*

- *For all d -restricted graphs H , we have $H \leq^{\text{cut}} O(\frac{1}{\phi})G$.*
- *For all d -restricted graphs H , we have $H \leq^{\text{flow}} O(\frac{\log n}{\phi})G$.*

4.3 Certificate For Well-Connectivity via Expanders

- Lastly, suppose you want to show that some arbitrary d -restricted graph G' is the most well-connected among all d -restricted graphs.
- What is the *certificate* for this?

Corollary 4.8. *Let G be a ϕ -expander with degree profile d . Let G' be any d -restricted graph. If $G \leq^{\text{flow}} G'$, then $H \leq^{\text{flow}} O(\log(n)/\phi) \cdot G'$ for all d -restricted demand H .*

- That is, a feasible flow embedding that embeds any expander G into G' is the certificate.

5 Summary

Let's fix the degree profile d . We want to compare "well-connectivity" of graphs with this degree profile d .

- First, we prove that the d -product graph K is basically the most well-connected graph. In both cut and flow perspective, we have

$$H \preceq^{\text{cut}} 2K \text{ and } H \preceq^{\text{flow}} 2K$$

for any graph H where $\deg_H \leq d$ (i.e. d -restricted graph).

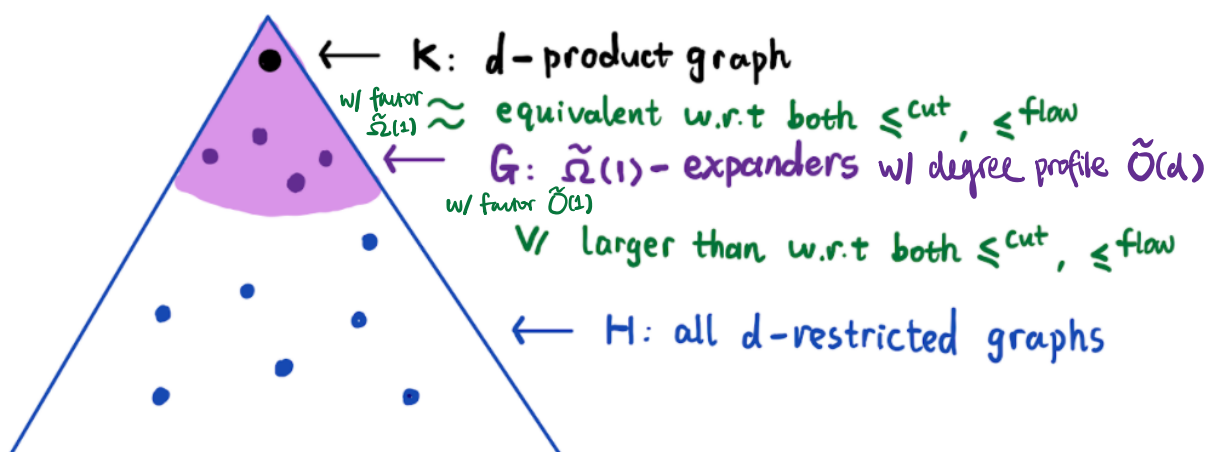
- Second, we show that if $H \preceq^{\text{deg}} G$ and G is a ϕ -expander, then $H \preceq^{\text{cut}} \frac{1}{\phi} G$.
 - Therefore, for any two $\tilde{\Omega}(1)$ -expanders G_1 and G_2 with the same degree profile d (i.e. $G_1 \preceq^{\text{deg}} G_2$), we have

$$G_1 \approx^{\text{cut}} G_2$$
 - Combining with the approximate max-flow min-cut theorem (i.e. $H \preceq^{\text{cut}} G \iff H \preceq^{\text{flow}} G$), we have

$$G_1 \approx^{\text{flow}} G_2$$
 - That is, expanders with the same degree profile are all approximately equivalent from both cut and flow perspectives.
 - Note that K is in the equivalent class too (as K is a $\Omega(1)$ -expander).
- Now, as K is the most well-connected graph w.r.t \preceq^{cut} and \preceq^{flow} , the whole class of expanders are basically most well-connected as well.
 - That is, for any $\tilde{\Omega}(1)$ -expander G and a d -restricted graph H , we have

$$H \preceq^{\text{cut}} G \text{ and } H \preceq^{\text{flow}} G$$

- This is the picture you should have in mind



- To summarize, if you know $H \leq^{\deg} G$ and G is an expander, then $H \lesssim^{\text{cut}} G$ and $H \lesssim^{\text{flow}} G$.
 - This is important.
 - For both flow and cut, all that “matters” for expanders is to compare degree cuts (when we allow some approximate)
- Lastly, suppose you want to show that some graph G' is the most well-connected among graphs with the same degree profile d .
 - It is enough to show $K \lesssim^{\text{cut}} G'$ or $K \lesssim^{\text{flow}} G'$.
 - But it is enough to show $G \lesssim^{\text{cut}} G'$ or $G \lesssim^{\text{flow}} G'$ for **any** expander G too.
 - That is, if you can embed an expander G into G' , then you are done.

6 The “Right” Way to Think of Expanders

- Let $G = (V, E)$ be a ϕ -expander.
 - By the previous discussion, if G has uniform degree, then G is cut/flow-equivalent to the corresponding d -product graph, which is essentially the complete graph on V with unit weight on each edge (after scaling by some constant.) Hence we can think of G as an unweighted complete graph on V , and this is a nice picture to have in mind.
 - But, otherwise, it is equivalent to the \deg_G -product graph, which is quite hard to think about since it has different weights on different nodes.
- I find it much more convenient to think of an expander in the following way.
- Suppose that $G = (V, E)$ is unweighted for now (otherwise make parallel edges).
- Let G' be obtained from G by subdividing each edge $e = (u, v)$ into (u, x_e) and (x_e, v) .
 - We call x_e the *split node* of e .
 - Let $X_E = \{x_e \mid e \in E\}$.
 - So $V(G') = V(G) \cup X_E$.
- Here is the “better” way to think about an expander. Let K_E be a **unweighted clique** on X_E (no edges on V).
 - Note: every node in $\frac{1}{m-1}K_E$ has degree 1.
- Essentially, G is an expander iff an unweighted clique on split-nodes X_E (after scaling by $\frac{1}{m}$) is embeddable into G' . Namely, we can think of G as an unweighted complete graph on X_E .

Intuitively this is saying that for a graph to be an expander, all of its edges should be pairwise “well connected”, in the sense that there should be a collection of flows on G' that allows any edge in G to send a unit of flow to any other edge in G while this collection has small congestion. In this case X_E is representing all edges in G , and embedding K_E into G' is finding the collection of flows that we described above. To show this perspective is true, we state the following lemma formally,

Lemma 6.1. *We have:*

1. If G is a ϕ -expander, then $\frac{1}{m-1}K_E \leq^{\text{flow}} O(\frac{\log n}{\phi}) \cdot G'$.
2. If G is not a ϕ -expander, then $\frac{1}{m-1}K_E \not\leq^{\text{flow}} \frac{1}{100\phi} \cdot G'$.

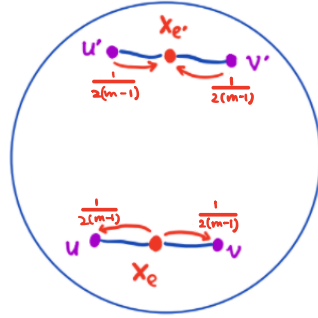
Corollary 6.2. *For any 1-restricted graph H_E where $V(H_E) = X_E$, we have $H_E \leq^{\text{flow}} O(\frac{\log n}{\phi}) \cdot G'$. That is, every edge can exchange 1 unit of flow between any other edges.*

Now, we prove the lemma.

Proof. There are two directions.

1. Suppose G is a ϕ -expander.

- You just to construct a $\frac{1}{m-1}K_E$ -concurrent flow \mathcal{F} with small congestion.
- We will construct \mathcal{F} in several steps.
 - (a) Fix a demand $(x_e, x_{e'})$ where $e = (u, v)$ and $e' = (u', v')$,
 - (b) Think of this demand as directed demand so that we can draw the flow, but all we need is just a undirected flow.
 - (c) x_e sends flow to u, v (each of value $1/2(m-1)$)
 - (d) $x_{e'}$ receive flow from u', v' (each of value $1/2(m-1)$)



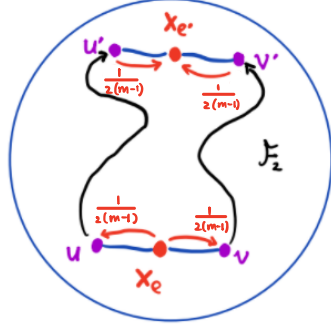
$(x_e, x_{e'})$ is a demand.



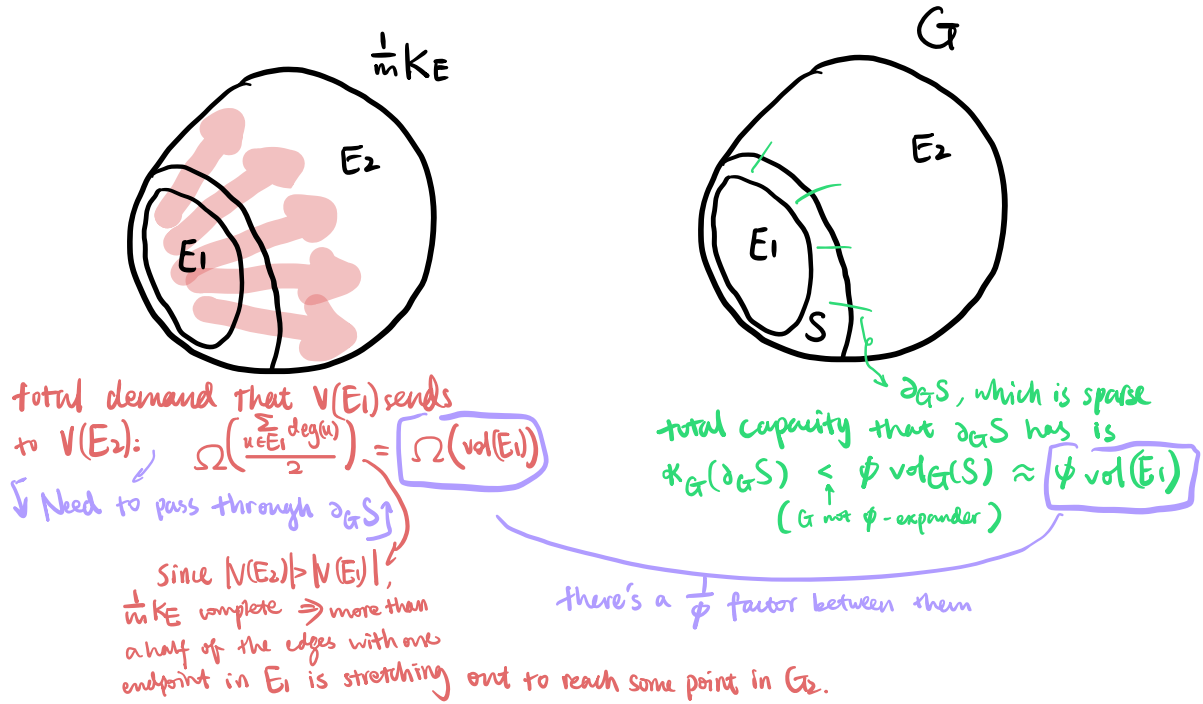
Think of $(u, u'), (v, v')$ as new demands with value $\frac{1}{2(m-1)}$

- (e) This induces new demand. Indeed, consider the demand graph $H = (V, E_H, \kappa_H)$ where $E_H = \{(u, v)\}_{u, v \in V(E)}$. Essentially H is a clique on V , and we will discuss the construction of κ_H in the next step.
- (f) We still look at the fixed demand $(x_e, x_{e'})$ first. For this demand $(x_e, x_{e'})$, we can treat sending $\frac{1}{m-1}$ unit of demand between x_e and $x_{e'}$ as a two-step process where firstly (1) x_e sending $\frac{1}{2(m-1)}$ unit of demand between x_e and u , x_e and v respectively; $x_{e'}$ sending $\frac{1}{2(m-1)}$ unit of demand between $x_{e'}$ and u' , $x_{e'}$ and v' respectively, then (2) sending $\frac{1}{2(m-1)}$ unit of demand between u, u' and v, v' respectively. In this way, we separate the demand sending process for all possible pairs $(x_{e_1}, x_{e_2}), x_{e_1}, x_{e_2} \in X_E$, and accumulate the unit of demand on each edge in E_H to form κ_H . Notice that the total demand each original node $u \in V$ needs to send/receive is $\deg_G(u)/2$, i.e. $\deg_H(u) = \deg_G(u)/2$. This is because u is adjacent to $\deg_G(u)$ vertices in X_E where each vertex is involved in $(m-1)$ edges in demand graph K_E , and each edge (representing $1/(m-1)$ unit of demand need to pass through) induces $\frac{1}{2(m-1)}$ unit of demand got send between u and some other vertex.

- (g) Therefore, H is a $\frac{\deg_G}{2}$ -restricted demand graph. We know $H \preceq^{\text{flow}} O(\frac{\log n}{\phi})G$. As $G \preceq^{\text{flow}} G'$, we have $H \preceq^{\text{flow}} O(\frac{\log n}{\phi}) \cdot G'$.
- (h) So there is an H -concurrent flow \mathcal{F}_2 that embeds H into G' with congestion $q_2 = O(\frac{\log n}{\phi})$.



- (i) Concatenate the \mathcal{F}_2 with the “red” flow for every demand. That would satisfy all demand of $\frac{1}{m-1}K_E$.
- (j) What is the congestion? It is $1/2 + q_2 = O(\frac{\log n}{\phi})$. It is not just q_2 ! Why? This is because it's possible that there might be a flow in \mathcal{F}_2 with the largest congestion overlap with the flow between an endpoint in $V(G)$ and a split point in X_E on the edge with congestion q_2 . In that case, the congestion on that edge should be $q_2 + 1/2$ where $1/2$ is the congestion generated by the flow between one endpoint to its adjacent split node.
- (k) We have a $\frac{1}{m-1}K_E$ -concurrent flow with congestion $O(\frac{\log n}{\phi})$ in G' .
2. Suppose G is not a ϕ -expander, where $\phi < 1/10$.
- Let (S, \bar{S}) be a cut such that $\text{vol}_G(S) \leq \text{vol}_G(\bar{S})$ and $\kappa_G(\partial_G S) < \phi \text{vol}_G(S)$.
 - Consider $E_1 = E_G(S, S)$ and $E_2 = E_G(\bar{S}, \bar{S})$. Note that $|E_1| = \Theta(\text{vol}_G(S))$.
 - The total demand that E_1 need to send to E_2 is $\Theta(|E_1|)$. Any concurrent flow for this demand must go through the cut S which has capacity $\kappa_G(\partial_G S)$.
 - So the congestion must be least $\Theta(|E_1|)/\kappa_G(\partial_G S) = \Omega(1/\phi)$ (or $1/100\phi$ if we compute the constant explicitly).



□

- The proof above is quite simple for flow.
- You can give a tighter bound for cut. But it is a bit more tedious. This is an example where it can be easier to think about flow instead of cut (although they are essentially equivalent).

Exercise 6.3. Show that $\Phi(G) = \Theta(\Phi(G'))$.

Exercise 6.4. We have:

- If G is a ϕ -expander, then $K_E \leq^{\text{cut}} \frac{100}{\phi} \cdot G'$.
- If G is not a ϕ -expander, then $K_E \not\leq^{\text{cut}} \frac{1}{100\phi} \cdot G'$.

7 Optional: Computing Quality of Cut/Flow Equivalence

- Question: Given G and H , what is the smallest α where H and G are α -cut/flow-equivalent?

7.1 Cut

For any graph G and H ,

- H and G are α -cut-equivalent where $\alpha = 1/(\Phi(G, H) \cdot \Phi(H, G)) \geq 1$.
- H and G are not α' -cut-equivalent for any $\alpha' < \alpha$.

Proof. Verify that $\Phi(G, H) \cdot H \preceq^{\text{cut}} G$ and $\Phi(H, G) \cdot G \preceq^{\text{cut}} H$. \square

- As we know how to $O(\log n)$ -approximate both $\Phi(G, H)$ and $\Phi(H, G)$ (even $O(\sqrt{\log n} \log \log n)$ -approximation algorithm is known), we have the following:

Corollary 7.1. *Given graphs G and H , there is a polynomial time algorithm that $\text{poly} \log(n)$ -approximates the minimum α cut-equivalence factor of G and H .*

Question 7.2 (Very Interesting Open Problem). *Is there a polynomial time or even subexponential time algorithm that $O(1)$ -approximates the minimum α cut-equivalence factor between G and H ?*

7.2 Flow

For any graph G and H ,

- H is a α -flow sparsifier of G where $\alpha = 1/(\text{mcf}(G, H) \cdot \text{mcf}(H, G)) \geq 1$.
- Moreover, H is not a α' -flow sparsifier of G for any $\alpha' < \alpha$.

Proof. Verify that $\text{mcf}(G, H) \cdot H \preceq^{\text{cut}} G$ and $\text{mcf}(H, G) \cdot G \preceq^{\text{cut}} H$. \square

- In contract to the cut case, we can compute the embeddability factor α exactly because $\text{mcf}(G, H)$ and $\text{mcf}(H, G)$ are computable via LPs.

Corollary 7.3. *Given graphs G and H , there is an polynomial algorithm that exactly computes the minimum α flow-equivalence factor between G and H .*