

# CS5275 – The Algorithm Designer’s Toolkit (S2 AY2025/26)

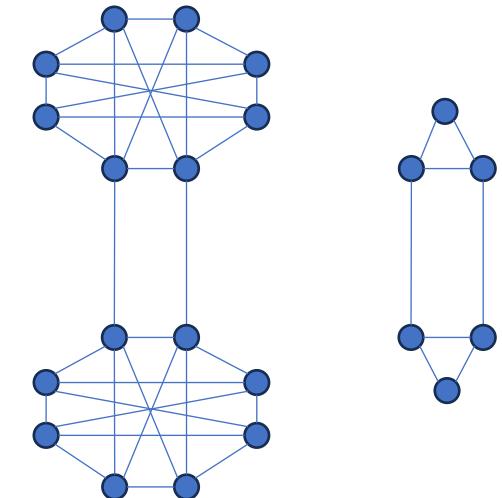
Lecture 1:  
Expanders – robustness against failures

# Measuring connectivity

- **Edge Connectivity**
  - Minimum number of edges whose removal disconnects the graph.
  - Captures global robustness against edge failures.
- **Vertex Connectivity**
  - Minimum number of vertices whose removal disconnects the graph.
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## Key limitations of Edge & Vertex Connectivity:

- Ignore the size of the affected region.
- Larger cities → more roads

# Conductance

**Note:** We focus on **simple unweighted graphs**, but the definitions also extend to weighted graphs with multi-edges and self-loops.

Consider a graph  $G = (V, E)$ .

**Volume** of a vertex set  $S$ :

- $\text{vol}(S) = \sum_{v \in S} \deg(v)$ .

It measures the size of a region.

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**Some facts:**

- If  $A \cap B = \emptyset$ , then  $\text{vol}(A \cup B) = \text{vol}(A) + \text{vol}(B)$ .
- $0 \leq \Phi(S) \leq 1$ .

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**Conductance** of a graph  $G$ :

- $\Phi(G) = \min_{S \subseteq V : S \neq V \text{ and } S \neq \emptyset} \Phi(S)$ .

The minimum conductance over all cuts in the graph.

**Intuition:**

- What is the weakest cut in the network?
- Which region is easiest to separate relative to its size?

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**Volume** of a vertex set  $S$ :

- $\text{vol}(S) = \sum_{v \in S} \deg(v)$ .

**Some facts:**

- $0 \leq \Phi(G) \leq 1$ .
- If  $G = (V, E)$  is connected, then  $\Phi(G) \geq 1/|E|$ .

**Conductance** of a cut  $(S, V \setminus S)$ :

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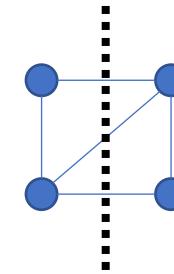
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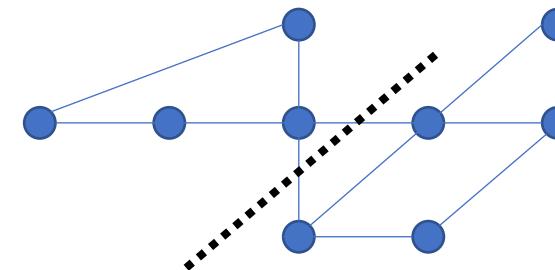
$$\text{Conductance} = \frac{3}{\min\{2+3, 2+3\}} = \frac{3}{5} = 0.6$$

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$$\text{Conductance} = \frac{2}{\min\{4+2+2+2, 4+3+3+2+2\}} = \frac{2}{10} = 0.2$$

# Expanders

- A graph  $G$  is a  **$\phi$ -expander**  $\leftrightarrow \Phi(G) \geq \phi$ .

## Key takeaways:

- Conductance provides a size-sensitive notion of expansion.
- It better captures robustness and connectivity in real networks.

Comparing with vertex & edge connectivity

# Expanders

Two  $(n/2)$ -cliques connected by an edge →

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Graph	Conductance
Dumbbells	$\Theta(1/n^2)$
Paths	$\Theta(1/n)$
Cycles	$\Theta(1/n)$
$(\sqrt{n} \times \sqrt{n})$ -grids	$\Theta(1/\sqrt{n})$
Hypercubes	$\Theta(1/\log n)$
Stars	$\Theta(1)$
Complete graphs / cliques	$\Theta(1)$

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Exercises!

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# Robustness against edge deletions

- Suppose we remove a subset of edges  $D \subseteq E$  from a  $\phi$ -expander  $G = (V, E)$ .
  - The resulting graph is  $G' = (V, E \setminus D)$ .
  - We say that a connected component  $S$  of  $G'$  is **small** if  $\text{vol}_G(S) \leq \frac{\text{vol}_G(V)}{2}$ .

**Note:** There can be at most one component that is not small.

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- **Claim:** The total volume of small connected components of  $G'$  is  $O\left(\frac{|D|}{\phi}\right)$ .

In other words, deleting the edges  $D \subseteq E$  can only disconnect a small subset of volume  $O\left(\frac{|D|}{\phi}\right)$ .

- Therefore, expanders are **robust against edge deletions**.

Useful in fault-tolerant distributed computing

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- **Claim:** The total volume of small connected components of  $G'$  is  $O\left(\frac{|D|}{\phi}\right)$ .

**Proof of the claim – Case 1:** There is a connected component that is not small.

- Let  $X$  be the union of all small connected components.

$$\begin{array}{c} \text{vol}_G(X) \leq \frac{\text{vol}_G(V)}{2} \\ \xrightarrow{\hspace{1cm}} \phi \leq \frac{|E(X, V \setminus X)|}{\text{vol}_G(X)} \leq \frac{|D|}{\text{vol}_G(X)} \\ \xrightarrow{\hspace{1cm}} \text{vol}_G(X) \leq \frac{|D|}{\phi} \end{array}$$

# Robustness against edge deletions

**Proof of the claim – Case 2:** All connected components are small.

- We show that there is  $X = \text{union of } \underline{\text{some}}$  small connected components such that:
  - $\frac{\text{vol}_G(V)}{3} \leq \text{vol}_G(X) \leq \frac{2\text{vol}_G(V)}{3}$

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$$\phi \leq \frac{|E(X, V \setminus X)|}{\min\{\text{vol}_G(X), \text{vol}_G(V \setminus X)\}} \leq \frac{|D|}{\text{vol}_G(V)/3}$$

$$\min\{\text{vol}_G(X), \text{vol}_G(V \setminus X)\} \geq \frac{\text{vol}_G(V)}{3}$$

$$\text{The total volume of small connected components} = \text{vol}_G(V) \leq \frac{3|D|}{\phi}$$

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**Selection of  $X$ :**

- If there is a small connected component  $S$  with  $\text{vol}_G(S) \geq \frac{\text{vol}_G(V)}{3}$ :
  - We may select  $X \leftarrow S$ .
- Otherwise:
  - Initialize  $X \leftarrow \emptyset$ .
  - While ( $\text{vol}_G(X) < \frac{\text{vol}_G(V)}{3}$ )
    - $X \leftarrow \text{union of } X \text{ and some small connected component.}$

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**Next:** We will demonstrate an application of this result to **sublinear-time algorithms**.

# Fast connectivity check under edge failures

- **Input:**

- A  $\phi$ -expander  $G = (V, E)$ .
- A subset of edges  $D \subseteq E$ .
- Two vertices  $s$  and  $t$ .

- **Goal:**

- Decide whether  $s$  and  $t$  are connected in  $G' = (V, E \setminus D)$ .



- Of course, the problem can be solved in linear time using BFS.
- Here, however, our goal is to solve it in **sublinear time**.

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- **Claim:**

- The problem can be solved in  $O\left(\frac{|D|}{\phi}\right)$  time.

If both  $s$  and  $t$  are not in small connected components, then the algorithm returns YES.

Otherwise, at least one of  $s$  and  $t$  belongs to a small connected component, and the volume is sufficient for us to search the entire component to decide whether  $s$  and  $t$  are connected.

# Expanders have small diameter

- **Claim:** The diameter of any  $n$ -vertex  $\phi$ -expander is  $O\left(\frac{\log n}{\phi}\right)$ .



Maximum shortest-path distance between any two vertices

## Intuition:

- In a well-connected network:
  - Everything is only a few steps away.

## Algorithmic Advantages:

- Quick broadcast and aggregation:
  - Messages reach the whole network rapidly.

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We prove the claim using a **ball-growing** argument:

- $B(v, r)$  = the ball of radius  $r$  around  $v$ .
- It suffices to show that:
  - There exists  $r^* \in O\left(\frac{\log n}{\phi}\right)$  such that  $\text{vol}(B(v, r^*)) > \frac{\text{vol}(V)}{2}$  for every vertex  $v$ .



$B(u, r^*) \cap B(v, r^*) \neq \emptyset$  for any  $u$  and  $v$ .



Graph diameter  $\leq 2r^* \in O\left(\frac{\log n}{\phi}\right)$ .

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If  $\text{vol}(B(v, r)) \leq \frac{\text{vol}(V)}{2}$ , then  $\text{vol}(B(v, r + 1)) \geq (1 + \phi)\text{vol}(B(v, r))$ .

$$\begin{aligned} & \text{vol}(B(v, r + 1)) \\ & \geq \text{vol}(B(v, r)) + E(B(v, r), V \setminus B(v, r)) \\ & \geq (1 + \phi)\text{vol}(B(v, r)) \end{aligned}$$

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If  $\text{vol}(B(v, r)) \leq \frac{\text{vol}(V)}{2}$ , then  $\text{vol}(B(v, r)) > e^{\frac{r\phi}{2}}$ .

$$\begin{aligned} \text{vol}(B(v, r)) &\geq (1 + \phi)^r \text{vol}(B(v, 0)) \geq (1 + \phi)^r \\ &> e^{\frac{r^2\phi}{2}} \geq e^{\frac{r^2\phi}{2r}} = e^{\frac{r\phi}{2}} \end{aligned}$$

$$\left(1 + \frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}$$

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There exists  $r^* \in O\left(\frac{\log n}{\phi}\right)$  such that  $\text{vol}(B(v, r^*)) > \frac{\text{vol}(V)}{2}$ .

# Construction of expanders

- **Stars** and **cliques** already have very good conductance:
  - $\Phi(G) \in \Omega(1)$
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- **Stars** and **cliques** already have very good conductance:
  - $\Phi(G) \in \Omega(1)$
- However, they are undesirable in that they have high-degree vertices.
- Can we simultaneously achieve both of the following?
  - $\Phi(G) \in \Omega(1)$
  - Maximum degree  $\in O(1)$
- **Hypercubes** nearly achieve this goal, up to a factor of  $O(\log n)$ .

# Randomized construction 1

- Erdős-Renyi random graph  $\mathcal{G}(n, p)$ :
  - An  $n$ -vertex graph  $G = (V, E)$  such that:
    - for each pair  $\{u, v\}$  of vertices in  $V$ ,  $\{u, v\} \in E$  with probability  $p$  independently.

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  - There is a choice of sampling probability  $p \in \Theta\left(\frac{\log n}{n}\right)$  such that:
    - With probability  $1 - 1/\text{poly}(n)$ ,
      - The maximum degree of  $G$  is  $O(\log n)$ .
      - $\Phi(G) \in \Omega(1)$ .
- Intuitively, this means that almost all graphs are good expanders!

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This is an **exercise**:

- Apply a Chernoff bound for every cut.
- Sum up the error probability.

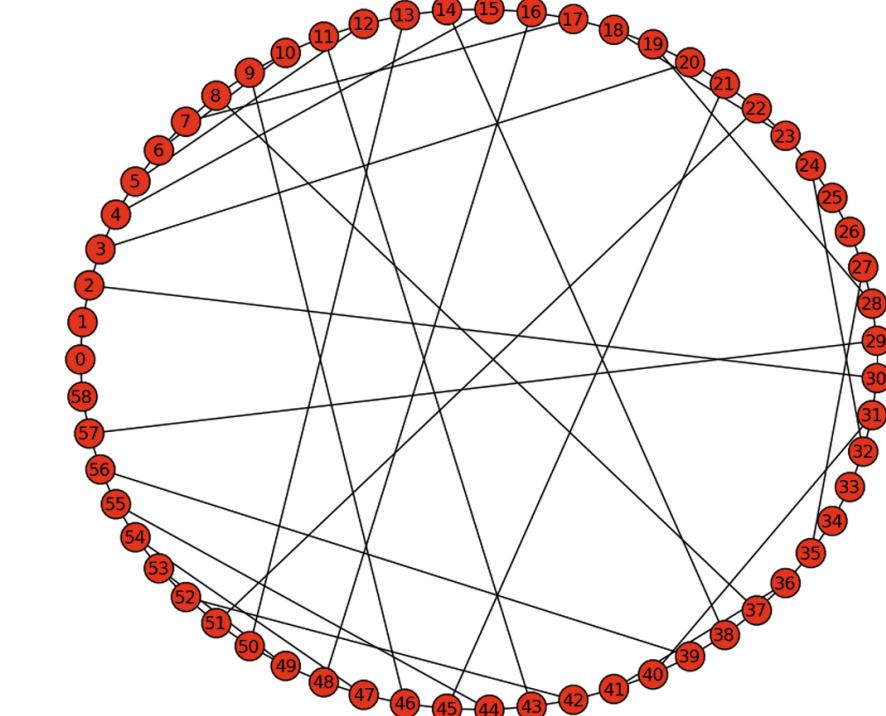
# Randomized construction 2

- Generate  $d$  random perfect matching over  $n$  vertices.
- Take the union of all edges in the matchings.
- There exists a constant  $d$  such that the resulting graph  $G$  satisfies  $\Phi(G) \in \Omega(1)$  with probability  $1 - 1/\text{poly}(n)$ .

<https://lucatrevisan.github.io/teaching/expanders2016/lecture19.pdf>

# Deterministic construction

- Let  $p$  be a prime.
- Define the graph  $G_p = (V_p, E_p)$  as follows.
  - $V_p = \{0, 1, \dots, p - 1\}$
  - Each vertex  $a \in V_p \setminus \{0\}$  is connected to:
    - $a - 1 \bmod p$
    - $a + 1 \bmod p$
    - $a^{-1}$  (multiplicative inverse: the unique element  $a^{-1} \in V_p$  such that  $a \cdot a^{-1} \bmod p = 1$ )
  - The vertex 0 is connected to 1, to  $p - 1$ , and has a self-loop.
- $\Phi(G_p) \in \Omega(1)$
- The degree of every vertex in  $G_p$  equals 3.



# Variations

- Given a vertex set  $S \subseteq V$ , define:
  - $\partial_{\text{in}}(S)$  = the set of vertices in  $S$  adjacent to  $V \setminus S$ .
  - $\partial_{\text{out}}(S)$  = the set of vertices in  $V \setminus S$  adjacent to  $S$ .
- **Edge expansion:**
  - $$h(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|E(V, V \setminus S)|}{|S|}$$
- **Vertex expansion:**
  - $$h_{\text{in}}(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|\partial_{\text{in}}(S)|}{|S|}$$
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# Variations

**Claim:** If the maximum degree  $\Delta$  of  $G$  is  $O(1)$ , then  $\Phi(G), h(G), h_{\text{in}}(G), h_{\text{out}}(G)$  are within a constant factor of each other.

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What about  $\Phi(G)$ ?

$h(G), h_{\text{in}}(G), h_{\text{out}}(G)$  are within a constant factor of each other.

- $|\partial_{\text{in}}(S)| \leq |E(V, V \setminus S)| \leq \Delta |\partial_{\text{in}}(S)|$
- $|\partial_{\text{out}}(S)| \leq |E(V, V \setminus S)| \leq \Delta |\partial_{\text{out}}(S)|$

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**Claim:** If the maximum degree  $\Delta$  of  $G$  is  $O(1)$ , then  $\Phi(G), h(G), h_{\text{in}}(G), h_{\text{out}}(G)$  are within a constant factor of each other.

- It remains to show that the two parameters are within an  $O(1)$ -factor:

$$\bullet \quad h(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|E(S, V \setminus S)|}{|S|}$$

$$\bullet \quad \Phi(G) = \min_{S \subseteq V : S \neq V \text{ and } S \neq \emptyset} \frac{|E(S, V \setminus S)|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}} = \min_{0 < |S| \leq \frac{n}{2}} \frac{|E(S, V \setminus S)|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}}$$

# Variations

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- To do so, it suffices to show that for any  $0 < |S| \leq \frac{n}{2}$ , the two parameters are within an  $O(1)$ -factor:
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- Indeed,  $|S| \leq \min\{|S|, |V \setminus S|\} \leq \min\{\text{vol}(S), \text{vol}(V \setminus S)\} \leq \text{vol}(S) \leq \Delta|S|$ .



- This step requires that there is no isolated vertex.
- If there is an isolated vertex, then all the parameters  $\Phi(G), h(G), h_{\text{in}}(G), h_{\text{out}}(G)$  are zero.

# Vertex expansion vs. conductance

- We have shown the following results for  $\phi$ -expanders:
  - Robustness against edge deletions.
  - Small diameter.
- **Exercise:** Extend the above results to graphs with high vertex expansion:
  - Robustness against vertex deletions.
  - Small diameter.

# Vertex expansion vs. conductance

- Vertex expansion and conductance can be very different!

There is an  $n$ -vertex graph  $G$  with

- $\Phi(G) \in \Omega(1)$
- $h_{\text{in}}(G) \in O\left(\frac{1}{n}\right)$
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Star graphs

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Two  $(n/2)$ -cliques with a perfect matching between them

# Outlook

- So far, we mostly talk about nice properties of expanders, focusing on the robustness against failures.
- **Next:**
  - How to compute the conductance of a graph efficiently? ← We will do this first.
  - What are the applications of expanders in algorithm design?

# References

- **Main reference:**
  - Lecture 1.2 of <https://sites.google.com/site/thsaranurak/teaching/Expander>
- **Additional/optional reading:**
  - More about expander graphs:
    - [https://en.wikipedia.org/wiki/Expander\\_graph](https://en.wikipedia.org/wiki/Expander_graph)
  - An application in distributed computing:
    - Ghosh, Bhaskar, et al. “Tight analyses of two local load balancing algorithms.” SIAM Journal on Computing 29.1 (1999): 29-64.
    - <https://pubs.siam.org/doi/10.1137/S0097539795292208>