CS5275 – The Algorithm Designer’s Toolkit (S2 AY2025/26)

Lecture 8:

Expanders – Ramanujan graphs

Quasirandomness

- **Observation:**
  - For n-vertex d-regular graphs, the following two statements are equivalent.

For any cut $(S, V \setminus S)$, we have:
$$|E(S, V \setminus S)| \in \Theta\left(\frac{d}{n} \cdot |S||V \setminus S|\right)$$

$G = (V, E)$ is an $\Omega(1)$-expander.

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This is the expected number of edges in $E(S, V \setminus S)$ if G is a random n-vertex d-regular graph.

This can be seen as a **quasirandomness** property.

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Question: Can we extend this to $E(S, T)$ for any two subsets $S \subseteq V$ and $T \subseteq V$?

Notation

- For simplicity, we restrict ourselves to n-vertex d-regular graphs.

$e(A, B) = |(a, b) \in A \times B : \{a, b\} \in E|$

- This is the number of edges between A and B.
- If $\{u, v\} \in E$ satisfies $\{u, v\} \subseteq A \cap B$, then $\{u, v\}$ is counted twice.

$\sigma_2 = \max\{|\lambda_2|, |\lambda_3|, \dots, |\lambda_n|\}$,

- where $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq -d$ are the eigenvalues of the adjacency matrix $\boldsymbol{A}$.

Expander mixing lemma

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Expander mixing lemma:

$$\left| e(S, T) - \frac{d}{n} \cdot |S||T| \right| \leq \sigma_2 \cdot \sqrt{|S||T|}$$

The number of edges between S and T is close to the expected number of edges between them in a random d-regular graph.

Proof of the lemma

- $e(S,T) = \mathbf{1}_S^\top \boldsymbol{A} \mathbf{1}_T$
- $|S||T| = \mathbf{1}_S^\top \boldsymbol{J} \mathbf{1}_T$, where $\boldsymbol{J}$ is the all-1 $(n \times n)$ matrix.

$$\left| e(S,T) - \frac{d}{n} \cdot |S||T| \right| = \left| \mathbf{1}_S^\top \boldsymbol{A} \mathbf{1}_T - \frac{d}{n} \cdot \mathbf{1}_S^\top \boldsymbol{J} \mathbf{1}_T \right| = \left| \mathbf{1}_S^\top \left(\boldsymbol{A} - \frac{d}{n} \boldsymbol{J} \right) \mathbf{1}_T \right|$$

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	$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{}$ **Eigenvalues**		
Eigenvectors	$\boldsymbol{A}$	$\frac{d}{n}\boldsymbol{J}$	$\boldsymbol{A} - \frac{d}{n}\boldsymbol{J}$
$\boldsymbol{v}_1 = \frac{1}{\sqrt{n}} \mathbf{1}$	$\lambda_1 = d$	d	0
$\boldsymbol{v}_2$	λ_2	0	λ_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
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$$\left\| \left(\boldsymbol{A} - \frac{d}{n} \boldsymbol{J} \right) \mathbf{1}_T \right\| = \left\| \sum_{i=2}^{n} \lambda_i \langle \mathbf{1}_T, \boldsymbol{v}_i \rangle \boldsymbol{v}_i \right\|$$

$$= \sqrt{\sum_{i=2}^{n} \lambda_i^2 \langle \mathbf{1}_T, \boldsymbol{v}_i \rangle^2} \leq \sigma_2 \cdot \sqrt{\sum_{i=2}^{n} \langle \mathbf{1}_T, \boldsymbol{v}_i \rangle^2} \leq \sigma_2 \cdot \|\mathbf{1}_T\|$$

$$\sigma_2 = \max\{|\lambda_2|, |\lambda_3|, \dots, |\lambda_n|\}$$

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$$\leq \|\mathbf{1}_S\| \cdot \left\| \left(\boldsymbol{A} - \frac{d}{n} \boldsymbol{J} \right) \mathbf{1}_T \right\| \leq \|\mathbf{1}_S\| \cdot \sigma_2 \cdot \|\mathbf{1}_T\| = \sigma_2 \cdot \sqrt{|S||T|}$$

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Ramanujan graphs

- How small σ_2 can be?
  - **Answer:** $2\sqrt{d-1}$

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Alon–Boppana bound: For every d and $\epsilon > 0$, there exists n_0 such that all graphs with $\geq n_0$ vertices have $\sigma_2 > 2\sqrt{d-1} - \epsilon$.

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Proof of a weaker bound:

$\sigma_2 \in \Omega(\sqrt{d})$ when $d \leq 0.99 \cdot n$

$\mathrm{tr}(\boldsymbol{A}^2) = nd$

The trace of a matrix is the <u>sum of its diagonal entries</u> and equals the <u>sum of its eigenvalues</u>.

$$\mathrm{tr}(\boldsymbol{A}^2) = \sum_{i=1}^{n} \lambda_i^2 \leq d^2 + (n-1)\sigma_2^2$$

$$\sigma_2 \geq \sqrt{\frac{nd - d^2}{n-1}} \in \Omega\left(\sqrt{d}\right)$$

Ramanujan graphs

The name comes from the Ramanujan–Petersson conjecture, which was used in a construction of some of these graphs.

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The graphs with $\sigma_2 \leq 2\sqrt{d-1}$ are called **Ramanujan graphs**.

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Ramanujan graphs are $\Omega(1)$-expanders.

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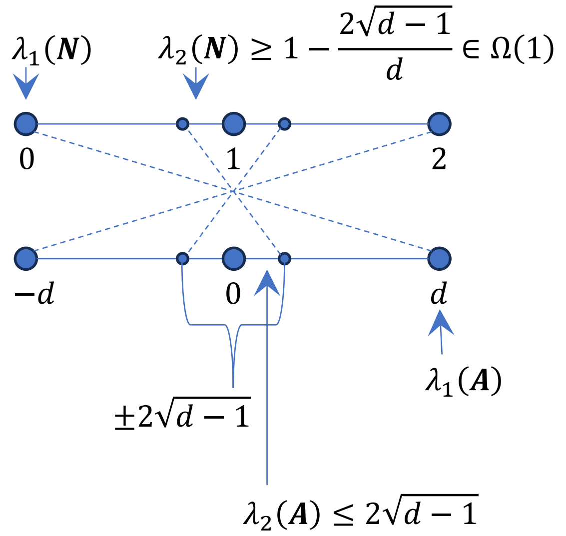

$$\boldsymbol{N} = \boldsymbol{I} - \frac{1}{d}\boldsymbol{A}$$

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Some $\Omega(1)$-expanders are not Ramanujan.

$\sigma_2 = d$ for any bipartite graph.

Random graphs are nearly Ramanujan

- For every d and $\epsilon > 0$, the probability that a random d-regular graph satisfies $\sigma_2 < 2\sqrt{d-1} + \epsilon$ tends to 1 as $n \to \infty$.

- Joel Friedman (Duke Mathematical Journal, 2003)

Algebraic constructions

- There is an infinite family of d-regular Ramanujan graphs, whenever $d - 1$ is a prime power.

  - Alexander Lubotzky, Ralph Phillips, and Peter Sarnak (Combinatorica, 1988)
  - Moshe Morgenstern (Journal of Combinatorial Theory, Series B, 1994).

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Open problem: Show this for all $d \geq 3$.

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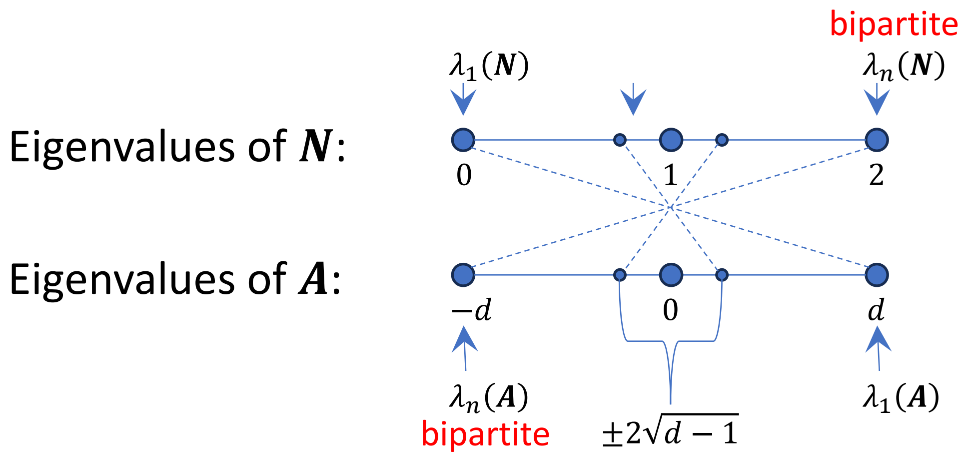
Open problem: Show this for all $d \geq 3$.

Solved for <u>bipartite</u> Ramanujan graphs:

$$\max\{|\lambda_2|, |\lambda_3|, \dots, |\lambda_{n-1}|\} \leq 2\sqrt{d-1}$$



> - Adam Marcus, Daniel Spielman and Nikhil Srivastava (Annals of Mathematics, 2015)

Properties of Ramanujan graphs

- **Claim:** For any $S, T \subseteq V$, if $|S| \cdot |T| \cdot d \geq 4n^2$, then $e(S,T) > 0$.

Expander mixing lemma:

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Ramanujan graphs: $\sigma_2 \leq 2\sqrt{d-1}$

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$$\sqrt{d|S||T|} \geq 2n$$

$$\downarrow$$

$$\frac{d}{n} \cdot |S||T| \geq 2\sqrt{d|S||T|} > \sigma_2 \cdot \sqrt{|S||T|}$$

$$\downarrow$$

$$e(S,T) \geq \frac{d}{n} \cdot |S||T| - \sigma_2 \cdot \sqrt{|S||T|} > 0$$

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Exercise: The result is asymptotically the best possible.
- If you change $4n^2$ to ϵn^2 for some sufficiently small constant $\epsilon > 0$,
  - then this claim is false for every large enough d-regular graph.

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Corollary 1: Any independent set S has size $|S| < \frac{2n}{\sqrt{d}}$.

If $|S| \geq \frac{2n}{\sqrt{d}}$, then $|S|^2 d \geq 4n^2$, so $e(S,S) > 0$.

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Corollary 2: Chromatic number $> \frac{\sqrt{d}}{2}$.

If a proper coloring with $\leq \frac{\sqrt{d}}{2}$ colors exists, then $n < \frac{\sqrt{d}}{2} \cdot \frac{2n}{\sqrt{d}} = n$, which is impossible.

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Corollary 2: Chromatic number $> \frac{\sqrt{d}}{2}$.

These results have been used to give lower bounds on the number of communication rounds needed to compute certain colorings and independent sets in a distributed network.

- Marijke H.L. Bodlaender, Magnús M. Halldórsson, Christian Konrad, and Fabian Kuhn. "Brief announcement: Local independent set approximation." *Proceedings of the ACM Symposium on Principles of Distributed Computing* (PODC 2016).
- Nathan Linial. "Locality in distributed graph algorithms." *SIAM Journal on computing* (1992).

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Success probability amplification

- Consider a $\overbrace{\text{randomized algorithm}}^{\text{It uses } r \text{ random bits and takes } t \text{ time.}}$ that succeeds with probability $\frac{1}{2}$.
- **Goal:** Amplify the success probability to $1 - f$.

Success probability amplification

It uses r random bits and takes t time.

- Consider a randomized algorithm that succeeds with probability $\frac{1}{2}$.
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- **Standard solution:** Repeat for $x = \left\lceil \log \frac{1}{f} \right\rceil$ times.

Probability of no success is $\frac{1}{2^x} \le f$.

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This requires $rx = r \left\lceil \log \frac{1}{f} \right\rceil$ random bits.

Question: Is it possible to reduce the failure probability without using more than r random bits?

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It uses r random bits and takes t time.

- Consider a randomized algorithm that succeeds with probability $\frac{1}{2}$.
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- Set $d = \left\lceil \frac{8}{f} \right\rceil$.
- Take a d-regular Ramanujan graph with $n = 2^r$ vertices.
  - Each vertex corresponds to an r-bit string.

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- Take a d-regular Ramanujan graph with $n = 2^r$ vertices.
  - Each vertex corresponds to an r-bit string.
  - $S =$ the set of vertices that lead to a successful execution of the algorithm.
    - $|S| = \frac{n}{2}$.
  - $T =$ the set of vertices that do not have a neighbor in S.
    - $|S| \cdot |T| \cdot d < 4n^2 \rightarrow |T| < fn$.

Recall: For any $S, T \subseteq V$, if $|S| \cdot |T| \cdot d \geq 4n^2$, then $e(S,T) > 0$.

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- Sample a vertex v uniformly at random, for each neighbor u of v, run the algorithm using the r-bit string of u.
  - Successful $\leftrightarrow v \notin T$
  - This happens with probability at least $1 - f$.

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Standard	$O\left(t \log \frac{1}{f}\right)$	$O\left(r \log \frac{1}{f}\right)$
Ramanujan	$O\left(\frac{t}{f}\right)$	r

Here we omit the cost for simulating a Ramanujan graph.

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Next	$O\left(t \log \frac{1}{f}\right)$	$O\left(r + \log \frac{1}{f}\right)$

A Chernoff Bound for random walks

- Let $G = (V, E)$ be an n-vertex d-regular graph.
- Let $(v_1, v_2, \ldots, v_\ell)$ be an $(\ell - 1)$-step random walk starting from a uniformly random vertex.
- Let $f : V \to [0, 1]$ be a function.
- Let $\mu = \frac{1}{n} \sum_{v \in V} f(v)$.
- Let $\epsilon > 0$.

$$\Pr\left[\frac{1}{\ell}\sum_{i=1}^{\ell} f(v_i) \geq \mu + \epsilon + \frac{\sigma_2}{d}\right] \in e^{-\Omega(\epsilon^2 \ell)}$$

The proof is omitted.

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Next	$O\left(t \log \frac{1}{f}\right)$	$O\left(r + \log \frac{1}{f}\right)$

New approach:

- Take $d \in O(1)$ and $n = 2^r$.
- Construct a random walk $(v_1, v_2, \ldots, v_\ell)$ with $\ell \in O\left(\log \frac{1}{f}\right)$.
- Run the algorithm with these bit strings.

A Chernoff Bound for random walks

It uses r random bits and takes t time.

- Consider a randomized algorithm that succeeds with probability $\frac{1}{2}$.
- **Goal:** Amplify the success probability to $1 - f$.

$n = 2^r$ $\quad$ $d \in O(1)$ is large enough so that $\frac{\sigma_2}{d} \leq \frac{1}{8}$.

- Let $G = (V, E)$ be an n-vertex d-regular graph.
- Let $(v_1, v_2, \dots, v_\ell)$ be an $(\ell - 1)$-step random walk starting from a uniformly random vertex.
- Let $f : V \to [0, 1]$ be a function.
- Let $\mu = \frac{1}{n} \sum_{v \in V} f(v)$.
- Let $\epsilon > 0$.

0 : the algorithm succeeds.
1 : the algorithm fails.

$$\Pr\left[\frac{1}{\ell}\sum_{i=1}^{\ell} f(v_i) \geq \underbrace{\overset{\frac{1}{2}}{\mu} + \overset{\frac{1}{8}}{\epsilon} + \overset{\leq \frac{1}{8}}{\frac{\sigma_2}{d}}}_{\leq \frac{3}{4}}\right] \in e^{-\Omega(\epsilon^2 \ell)}$$

$\ell \in O\left(\log \frac{1}{f}\right)$ is large enough so that the error probability $e^{-\Omega(\epsilon^2 \ell)}$ is at most f.

$\frac{1}{\ell}\sum_{i=1}^{\ell} f(v_i) < 1 \to$ some of $v_1, v_2, \dots, v_\ell$ make the algorithm successful.

Further applications

- Ramanujan graphs can be used to construct <u>error correcting codes</u>.

  Michael Sipser and Daniel A. Spielman.
  “Expander codes.”
  IEEE transactions on Information Theory, 2002.

- Ramanujan graphs can be used to construct <u>cryptographic hash functions</u>.

  Denis X. Charles, Eyal Z. Goren, and Kristin E. Lauter.
  “Cryptographic hash functions from expander graphs.”
  Journal of Cryptology, 2009.

Outlook

- **Next:**
  - An interesting application of Ramanujan graphs for $\underbrace{\text{graph algorithm design}}_{\text{for general graphs}}$.

References

- **Main reference:**
  - Lecture 5.1 of https://sites.google.com/site/thsaranurak/teaching/Expander
  - Chapter 21 of https://lucatrevisan.github.io/books/expanders-2016.pdf

- **Additional/optional reading:**
  - https://en.wikipedia.org/wiki/Expander_mixing_lemma
  - https://en.wikipedia.org/wiki/Ramanujan_graph
  - Shlomo Hoory, Nathan Linial, and Avi Wigderson. "Expander graphs and their applications." *Bulletin of the American Mathematical Society* 43.4 (2006): 439-561.