

CS5275 – The Algorithm Designer's Toolkit (S2 AY2025/26)

Lecture 1:

Expanders – robustness against failures

Measuring connectivity

- **Edge Connectivity**

- Minimum number of edges whose removal disconnects the graph.
- Captures global robustness against edge failures.

- **Vertex Connectivity**

- Minimum number of vertices whose removal disconnects the graph.
- Captures global robustness against vertex failures.

Measuring connectivity

- **Edge Connectivity**

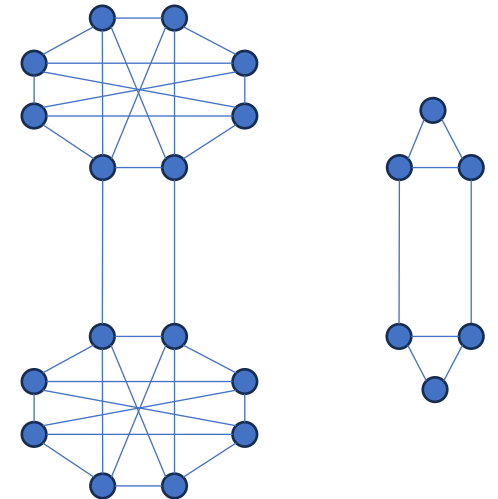
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- **Vertex Connectivity**

- Minimum number of vertices whose removal disconnects the graph.
- Captures global robustness against vertex failures.

Key limitations of Edge & Vertex Connectivity:

- Ignore the size of the affected region.
- Larger cities → more roads



Conductance

Volume of a vertex set S :

- $\text{vol}(S) = \sum_{v \in S} \deg(v)$.

Note: We focus on **simple unweighted graphs**, but the definitions also extend to weighted graphs with multi-edges and self-loops.

Consider a graph $G = (V, E)$.

It measures the size of a region.

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Conductance of a cut $(S, V \setminus S)$:

- $\Phi(S) = \frac{|E(S, V \setminus S)|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}},$ where $E(A, B) = \{ \{u, v\} \in E \mid u \in A \text{ and } v \in B \}.$

It measures the ratio between the size of a cut and the size of the region it separates.

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It measures the ratio between the size of a cut and the size of the region it separates.

Some facts:

- If $A \cap B = \emptyset$, then $\text{vol}(A \cup B) = \text{vol}(A) + \text{vol}(B)$.
- $0 \leq \Phi(S) \leq 1$.

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Conductance of a graph G :

- $\Phi(G) = \min_{S \subseteq V : S \neq V \text{ and } S \neq \emptyset} \Phi(S)$.

The minimum conductance over all cuts in the graph.

Intuition:

- What is the weakest cut in the network?
- Which region is easiest to separate relative to its size?

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Conductance

Consider a graph $G = (V, E)$.

Some facts:

- $0 \leq \Phi(G) \leq 1$.
- If $G = (V, E)$ is connected, then $\Phi(G) \geq 1/|E|$.

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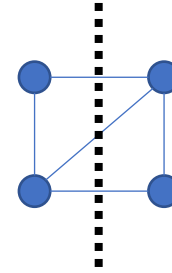
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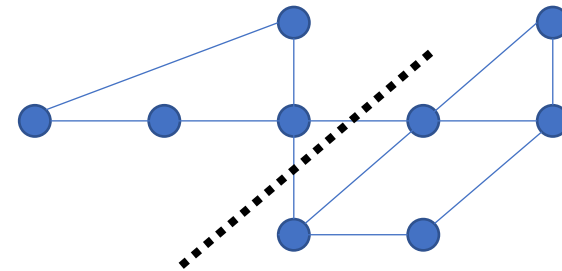
$$\text{Conductance} = \frac{3}{\min\{2+3, 2+3\}} = \frac{3}{5} = 0.6$$

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Conductance of a graph G :

- $\Phi(G) = \min_{S \subseteq V : S \neq V \text{ and } S \neq \emptyset} \Phi(S)$.



$$\text{Conductance} = \frac{2}{\min\{4+2+2+2, 4+3+3+2+2\}} = \frac{2}{10} = 0.2$$

Expanders

- A graph G is a **ϕ -expander** $\leftrightarrow \Phi(G) \geq \phi$.

Key takeaways:

- Conductance provides a size-sensitive notion of expansion.
- It better captures robustness and connectivity in real networks.

Comparing with vertex & edge connectivity

Expanders

Two $(n/2)$ -cliques connected by an edge →

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Graph	Conductance
Dumbbells	$\Theta(1/n^2)$
Paths	$\Theta(1/n)$
Cycles	$\Theta(1/n)$
$(\sqrt{n} \times \sqrt{n})$ -grids	$\Theta(1/\sqrt{n})$
Hypercubes	$\Theta(1/\log n)$
Stars	$\Theta(1)$
Complete graphs / cliques	$\Theta(1)$

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Exercises!

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Comparing with vertex & edge connectivity

Robustness against edge deletions

- Suppose we remove a subset of edges $D \subseteq E$ from a ϕ -expander $G = (V, E)$.
 - The resulting graph is $G' = (V, E \setminus D)$.
 - We say that a connected component S of G' is **small** if $\text{vol}_G(S) \leq \frac{\text{vol}_G(V)}{2}$.

Note: There can be at most one component that is not small.

Robustness against edge deletions

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 - The resulting graph is $G' = (V, E \setminus D)$.
 - We say that a connected component S of G' is **small** if $\text{vol}_G(S) \leq \frac{\text{vol}_G(V)}{2}$.
- **Claim:** The total volume of small connected components of G' is $O\left(\frac{|D|}{\phi}\right)$.

In other words, deleting the edges $D \subseteq E$ can only disconnect a small subset of volume $O\left(\frac{|D|}{\phi}\right)$.

- Therefore, expanders are **robust against edge deletions**.

Useful in fault-tolerant distributed computing

Robustness against edge deletions

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- **Claim:** The total volume of small connected components of G' is $O\left(\frac{|D|}{\phi}\right)$.

Proof of the claim – Case 1: There is a connected component that is not small.

- Let X be the union of all small connected components.

$$\begin{array}{ccccc} \boxed{\text{vol}_G(X) \leq \frac{\text{vol}_G(V)}{2}} & \longrightarrow & \boxed{\phi \leq \frac{|E(X, V \setminus X)|}{\text{vol}_G(X)} \leq \frac{|D|}{\text{vol}_G(X)}} & \longrightarrow & \boxed{\text{vol}_G(X) \leq \frac{|D|}{\phi}} \end{array}$$

Robustness against edge deletions

Proof of the claim – Case 2: All connected components are small.

- We show that there is $X =$ union of some small connected components such that:
 - $\frac{\text{vol}_G(V)}{3} \leq \text{vol}_G(X) \leq \frac{2\text{vol}_G(V)}{3}$

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$$\phi \leq \frac{|E(X, V \setminus X)|}{\min\{\text{vol}_G(X), \text{vol}_G(V \setminus X)\}} \leq \frac{|D|}{\text{vol}_G(X)/2}$$

$$\text{vol}_G(X) \leq \frac{2|D|}{\phi}$$

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Selection of X :

- If there is a small connected component S with $\text{vol}_G(S) \geq \frac{\text{vol}_G(V)}{3}$:
 - We may select $X \leftarrow S$.
- Otherwise:
 - Initialize $X \leftarrow \emptyset$.
 - While $(\text{vol}_G(X) < \frac{\text{vol}_G(V)}{3})$
 - $X \leftarrow \text{union of } X \text{ and some small connected component.}$

Robustness against edge deletions

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Next: We will demonstrate an application of this result to **sublinear-time algorithms**.

Fast connectivity check under edge failures

- **Input:**

- A ϕ -expander $G = (V, E)$.
- A subset of edges $D \subseteq E$.
- Two vertices s and t .

- **Goal:**

- Decide whether s and t are connected in $G' = (V, E \setminus D)$.



- Of course, the problem can be solved in linear time using BFS.
- Here, however, our goal is to solve it in **sublinear time**.

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- **Claim:**

- The problem can be solved in $O\left(\frac{|D|}{\phi}\right)$ time.

Fast connectivity check under edge failures

It suffices to do BFS to explore up to $O\left(\frac{|D|}{\phi}\right)$ volume from both s and t .

- **Input:**

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The volume is sufficient for us to decide whether s and t belong to small connected components.

- **Goal:**

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- **Claim:**

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It suffices to do BFS to explore up to $O\left(\frac{|D|}{\phi}\right)$ volume from both s and t .

The volume is sufficient for us to decide whether s and t belong to small connected components.

If both s and t are not in small connected components, then the algorithm returns **YES**.

Otherwise, at least one of s and t belongs to a small connected component, and the volume is sufficient for us to search the entire component to decide whether s and t are connected.

Expanders have small diameter

- **Claim:** The diameter of any n -vertex ϕ -expander is $O\left(\frac{\log n}{\phi}\right)$.



Maximum shortest-path distance between any two vertices

Intuition:

- In a well-connected network:
 - Everything is only a few steps away.

Algorithmic Advantages:

- Quick broadcast and aggregation:
 - Messages reach the whole network rapidly.

Expanders have small diameter

- **Claim:** The diameter of any n -vertex ϕ -expander is $O\left(\frac{\log n}{\phi}\right)$.

We prove the claim using a **ball-growing** argument:

- $B(v, r)$ = the ball of radius r around v .
- It suffices to show that:
 - There exists $r^* \in O\left(\frac{\log n}{\phi}\right)$ such that $\text{vol}(B(v, r^*)) > \frac{\text{vol}(V)}{2}$ for every vertex v .



$B(u, r^*) \cap B(v, r^*) \neq \emptyset$ for any u and v .



Graph diameter $\leq 2r^* \in O\left(\frac{\log n}{\phi}\right)$.

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If $\text{vol}(B(v, r)) \leq \frac{\text{vol}(V)}{2}$, then $\text{vol}(B(v, r + 1)) \geq (1 + \phi)\text{vol}(B(v, r))$.

$$\begin{aligned} & \text{vol}(B(v, r + 1)) \\ & \geq \text{vol}(B(v, r)) + E(B(v, r), V \setminus B(v, r)) \\ & \geq (1 + \phi)\text{vol}(B(v, r)) \end{aligned}$$

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$$\text{If } \text{vol}(B(v, r)) \leq \frac{\text{vol}(V)}{2}, \text{ then } \text{vol}(B(v, r+1)) \geq (1 + \phi) \text{vol}(B(v, r)).$$



$$\text{If } \text{vol}(B(v, r)) \leq \frac{\text{vol}(V)}{2}, \text{ then } \text{vol}(B(v, r)) > e^{\frac{r\phi}{2}}.$$

$$\begin{aligned} \text{vol}(B(v, r)) &\geq (1 + \phi)^r \text{vol}(B(v, 0)) \geq (1 + \phi)^r \\ &> e^{\frac{r^2\phi}{r\phi+r}} \geq e^{\frac{r^2\phi}{2r}} = e^{\frac{r\phi}{2}} \end{aligned}$$

$$\left(1 + \frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}$$

Expanders have small diameter

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If $\text{vol}(B(v, r)) \leq \frac{\text{vol}(V)}{2}$, then $\text{vol}(B(v, r+1)) \geq (1 + \phi)\text{vol}(B(v, r))$.

If $\text{vol}(B(v, r)) \leq \frac{\text{vol}(V)}{2}$, then $\text{vol}(B(v, r)) > e^{\frac{r\phi}{2}}$. $\rightarrow e^{\frac{r\phi}{2}} < \frac{\text{vol}(V)}{2} \rightarrow r < \frac{2 \ln \frac{\text{vol}(V)}{2}}{\phi}$

There exists $r^* \in O\left(\frac{\log n}{\phi}\right)$ such that $\text{vol}(B(v, r^*)) > \frac{\text{vol}(V)}{2}$.

Construction of expanders

- **Stars** and **cliques** already have very good conductance:
 - $\Phi(G) \in \Omega(1)$
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Construction of expanders

- **Stars** and **cliques** already have very good conductance:
 - $\Phi(G) \in \Omega(1)$
- However, they are undesirable in that they have high-degree vertices.
- Can we simultaneously achieve both of the following?
 - $\Phi(G) \in \Omega(1)$
 - Maximum degree $\in O(1)$
- **Hypercubes** nearly achieve this goal, up to a factor of $O(\log n)$.

Randomized construction 1

- Erdős-Renyi random graph $\mathcal{G}(n, p)$:
 - An n -vertex graph $G = (V, E)$ such that:
 - for each pair $\{u, v\}$ of vertices in V , $\{u, v\} \in E$ with probability p independently.

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- There is a choice of sampling probability $p \in \Theta\left(\frac{\log n}{n}\right)$ such that:
 - With probability $1 - 1/\text{poly}(n)$,
 - The maximum degree of G is $O(\log n)$.
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Intuitively, this means that almost all graphs are good expanders!

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This is an **exercise**:

- Apply a Chernoff bound for every cut.
- Sum up the error probability.

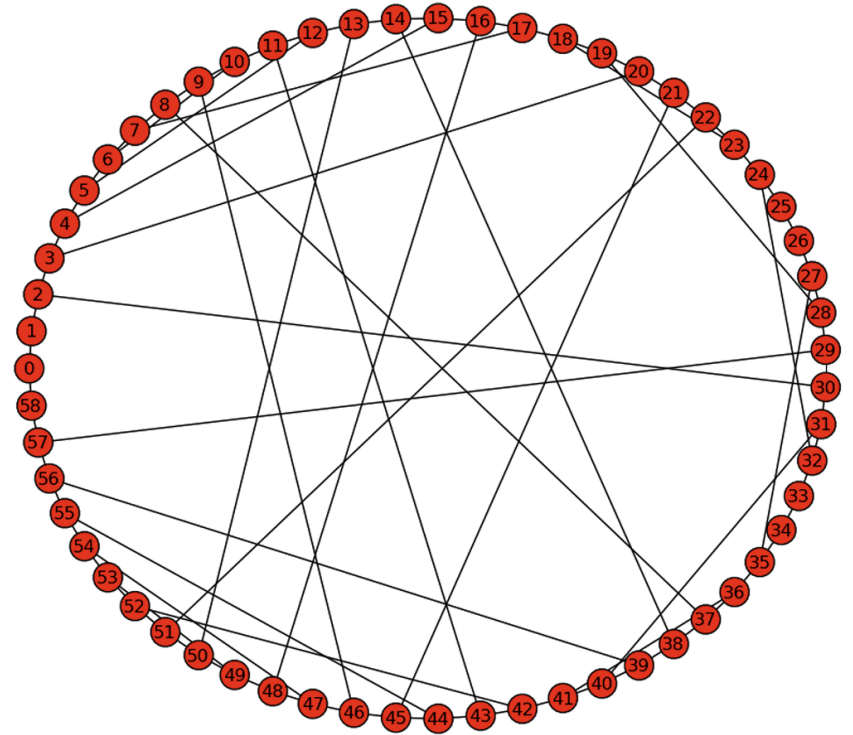
Randomized construction 2

- Generate d random perfect matching over n vertices.
- Take the union of all edges in the matchings.
- There exists a constant d such that the resulting graph G satisfies $\Phi(G) \in \Omega(1)$ with probability $1 - 1/\text{poly}(n)$.

<https://lucatrevisan.github.io/teaching/expanders2016/lecture19.pdf>

Deterministic construction

- Let p be a prime.
- Define the graph $G_p = (V_p, E_p)$ as follows.
 - $V_p = \{0, 1, \dots, p-1\}$
 - Each vertex $a \in V_p \setminus \{0\}$ is connected to:
 - $a - 1 \bmod p$
 - $a + 1 \bmod p$
 - a^{-1} (multiplicative inverse: the unique element $a^{-1} \in V_p$ such that $a \cdot a^{-1} \bmod p = 1$)
 - The vertex 0 is connected to 1, to $p-1$, and has a self-loop.
- $\Phi(G_p) \in \Omega(1)$
- The degree of every vertex in G_p equals 3.



Variations

- Given a vertex set $S \subseteq V$, define:
 - $\partial_{\text{in}}(S)$ = the set of vertices in S adjacent to $V \setminus S$.
 - $\partial_{\text{out}}(S)$ = the set of vertices in $V \setminus S$ adjacent to S .
- **Edge expansion:**
 - $$h(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|E(V, V \setminus S)|}{|S|}$$
- **Vertex expansion:**
 - $$h_{\text{in}}(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|\partial_{\text{in}}(S)|}{|S|}$$
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Variations

Claim: If the maximum degree Δ of G is $O(1)$, then $\Phi(G)$, $h(G)$, $h_{\text{in}}(G)$, $h_{\text{out}}(G)$ are within a constant factor of each other.

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What about $\Phi(G)$?

$h(G)$, $h_{\text{in}}(G)$, $h_{\text{out}}(G)$ are within a constant factor of each other.

- $|\partial_{\text{in}}(S)| \leq |E(V, V \setminus S)| \leq \Delta |\partial_{\text{in}}(S)|$
- $|\partial_{\text{out}}(S)| \leq |E(V, V \setminus S)| \leq \Delta |\partial_{\text{out}}(S)|$

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- It remains to show that the two parameters are within an $O(1)$ -factor:

- $h(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|E(V, V \setminus S)|}{|S|}$

- $\Phi(G) = \min_{S \subseteq V : S \neq V \text{ and } S \neq \emptyset} \frac{|E(S, V \setminus S)|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}} = \min_{0 < |S| \leq \frac{n}{2}} \frac{|E(S, V \setminus S)|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}}$

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- To do so, it suffices to show that for any $0 < |S| \leq \frac{n}{2}$, the two parameters are within an $O(1)$ -factor:
 - $|S|$
 - $\min\{\text{vol}(S), \text{vol}(V \setminus S)\}$

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 - $|S|$
 - $\min\{\text{vol}(S), \text{vol}(V \setminus S)\}$
- Indeed, $|S| \leq \min\{|S|, |V \setminus S|\} \leq \min\{\text{vol}(S), \text{vol}(V \setminus S)\} \leq \text{vol}(S) \leq \Delta|S|$.
 - This step requires that there is no isolated vertex.
 - If there is an isolated vertex, then all the parameters $\Phi(G)$, $h(G)$, $h_{\text{in}}(G)$, $h_{\text{out}}(G)$ are zero.

Vertex expansion vs. conductance

- We have shown the following results for ϕ -expanders:
 - Robustness against edge deletions.
 - Small diameter.
- **Exercise:** Extend the above results to graphs with high vertex expansion:
 - Robustness against vertex deletions.
 - Small diameter.

Vertex expansion vs. conductance

- Vertex expansion and conductance can be very different!

There is an n -vertex graph G with

- $\Phi(G) \in \Omega(1)$
- $h_{\text{in}}(G) \in O\left(\frac{1}{n}\right)$
- $h_{\text{out}}(G) \in O\left(\frac{1}{n}\right)$

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Star graphs

There is an n -vertex graph G with

- $\Phi(G) \in O\left(\frac{1}{n}\right)$
- $h_{\text{in}}(G) \in \Omega(1)$
- $h_{\text{out}}(G) \in \Omega(1)$

Two $(n/2)$ -cliques with a perfect matching between them

Outlook

- So far, we mostly talk about nice properties of expanders, focusing on the robustness against failures.
- **Next:**
 - How to compute the conductance of a graph efficiently?  We will do this first.
 - What are the applications of expanders in algorithm design?

References

- **Main reference:**
 - Lecture 1.2 of <https://sites.google.com/site/th saranurak/teaching/Expander>
- **Additional/optional reading:**
 - More about expander graphs:
 - https://en.wikipedia.org/wiki/Expander_graph
 - An application in distributed computing:
 - Ghosh, Bhaskar, et al. "Tight analyses of two local load balancing algorithms." SIAM Journal on Computing 29.1 (1999): 29-64.
 - <https://epubs.siam.org/doi/10.1137/S0097539795292208>