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A Quick Guide to Lagrange Multipliers

What is the point (x, y) on the unit circle that maximizes the product $x \cdot y$? This can be formulated as the following problem: maximize $f(x, y) = x \cdot y$, subject to $\sqrt{x^2 + y^2} = 1$ (or equivalently $x^2 + y^2 - 1 = 0$). More generally, suppose that we want to maximize/minimize a function $f : \mathbb{R}^n \to \mathbb{R}$, subject to some constraint encoded by equating some function $g : \mathbb{R}^n \to \mathbb{R}$ to 0. Thus we get

$$\begin{aligned} \text{maximize: } & f(\vec{x}) \\ \text{such that: } & g(\vec{x}) = 0 \end{aligned}$$

One method of doing so is using Lagrange multipliers. We first write

$$L(\vec{x}, \lambda) = f(\vec{x}) - \lambda g(\vec{x})$$

and then differentiate L with respect to all n variables as well as λ. Hence, we get

$$\begin{aligned} \frac{\partial L(\vec{x}, \lambda)}{\partial x_i} &= \frac{\partial f(\vec{x})}{\partial x_i} - \lambda \frac{\partial g(\vec{x})}{\partial x_i} \\ \frac{\partial L(\vec{x}, \lambda)}{\partial \lambda} &= -g(\vec{x}) \end{aligned}$$

Here we assume that both g and f are differentiable. Equating all of the differentials to 0 and solving the resulting set of equations will give us the extreme points of f over the space $g = 0$. Returning to our example, we get

$$\begin{aligned} L(x, y, \lambda) &= xy - \lambda(x^2 + y^2 - 1) \\ \frac{\partial L(x, y, \lambda)}{\partial x} &= y - 2\lambda x \\ \frac{\partial L(x, y, \lambda)}{\partial y} &= x - 2\lambda y \\ \frac{\partial L(x, y, \lambda)}{\partial \lambda} &= -(x^2 + y^2 - 1) \end{aligned}$$

Setting all differentials to 0 we get

$$\begin{aligned} y - 2\lambda x &= 0 \\ x - 2\lambda y &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$$

If $x = 0$ or $y = 0$ then $xy = 0$ which is clearly not a maximum. We then assume $x, y \neq 0$.

$$y - 2\lambda x = 0 \iff y = 2\lambda x \iff \lambda = \frac{y}{2x}$$
$$x - 2\lambda y = 0 \iff x = 2\lambda y \iff \lambda = \frac{x}{2y}$$
$$\implies \frac{y}{2x} = \frac{x}{2y} \iff x^2 = y^2$$

Thus, $x = \pm y$ in an optimal solution. Plugging $x^2 = y^2$ into $x^2 + y^2 = 1$ we get that there are four points to consider, of the form: $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Evaluating them on $f(x, y) = xy$ we get that the two optimal points must be $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Note that the other two points are also extreme points, but they are minima of $f(x, y) = xy$ over $g(x, y) = 0$.

Note There are many good primers on Lagrange multipliers online. See, for example, `http://tutorial.math.lamar.edu/classes/calciii/lagrangemultipliers.aspx`.