CS4261/5461 Algorithmic Mechanism Design

Instructor: Warut Suksompong

2025

Response to mid-semester survey

“Good course, can release the midterm practice questions slightly earlier so that we have more time to practice.”

I will be releasing midterm practice questions along with the lecture for each topic. Following some requests, I will also increase the number of practice questions.

Response to mid-semester survey

“Better venue for quizzes. Too squeezy and the person beside was shaking his leg throughout the quiz. Couldn't concentrate at all.”

There isn’t enough space in the lecture venue to seat everyone non-adjacently. If you sit next to someone else, please be mindful of your neighbor(s).

Response to mid-semester survey

“For the midterm exam, I felt that the time was a bit too tight, which caused a lot of pressure. It would be really helpful if the exam time could be made a bit more relaxed.”

I will continue to be mindful that the exam questions are appropriate for the given amount of time. For time management, it may be a good idea to start with questions that are less time-consuming.

Response to mid-semester survey

“The course is very interesting. However, the slides and the content taught in class are quite inadequate for me. This is my first time taking a game theory class, and sometimes the professor skips important parts, such as how to calculate the cost of routing games.”

Specifically, there were examples of calculating the cost of routing games in the lecture, assignment, and practice questions.

More generally, this is a large course, and students have varying backgrounds/preferences. Whenever you have questions, I am happy to answer them during/after lecture or on the discussion forum (anonymously), so feel free to ask!

Stable Matching

Markets without Money

Two-sided markets

- Residency Matching
- Dating Websites
- College Admission
- Job Markets

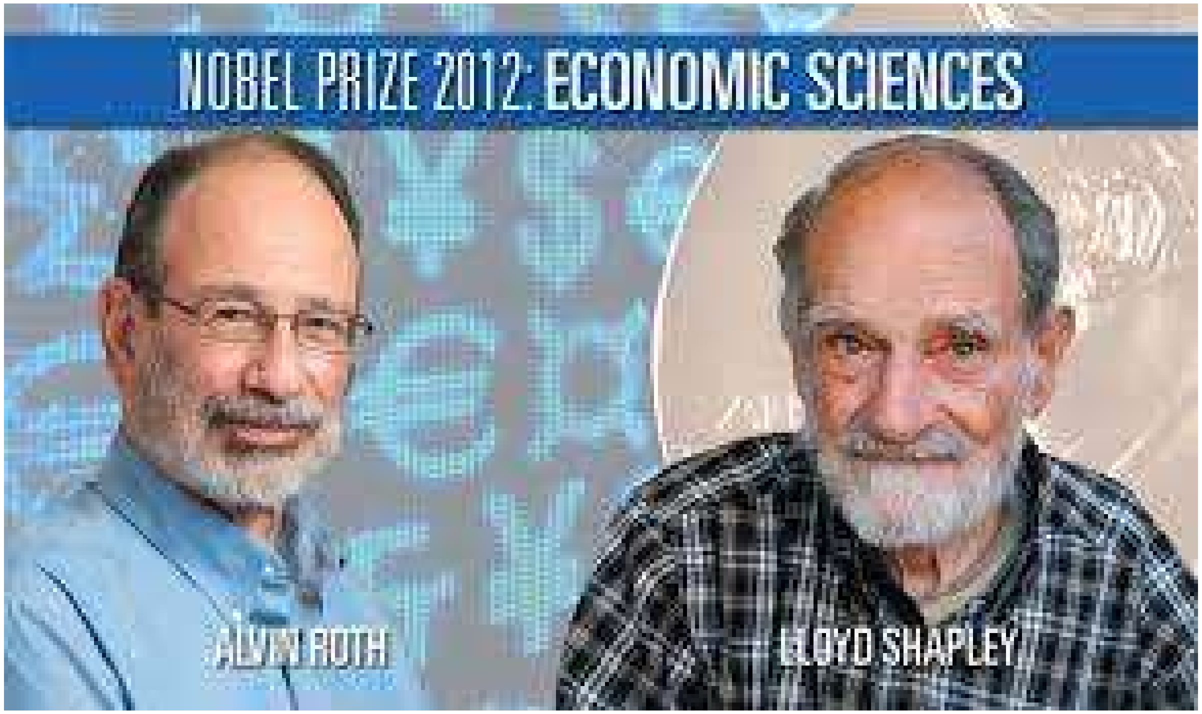
One-sided markets

- Students/Courses
- Dormitory Room Assignment

Barter Exchanges

- Kidney Exchange
- Barter Markets

2012 Nobel Prize: Market Design



What Makes for a Good Market?

Thick

- Lots of buyers, lots of sellers, lots of options…
- Everyone is aware of their options

Timely

- Not too fast (time to weigh out decisions)
- Not too slow (offers are processed quickly, new offers arrive quickly)

Safe

- People cannot be hurt by revealing preferences
- Outcomes are fair
- People are better off by participating

Good Markets Gone Bad: Unravelling

- Law firms want to get good law students
- It is best to make offers once student graduates – strongest signal of their quality
- However, companies prefer making earlier offers – snap up good 2nd year (or 1st year) students (less likely to be good, but still).
- An example of bad equilibrium

Good Markets Gone Bad: Unravelling

Bad for companies	Bad for students	Makes market thin
• hiring based on projected needs 2-3 years in advance • riskier prospects	• need to make life-changing decisions after only one year • don't know what else is out there	• both sides have less access to all good options before making a final decision

Matching Hospitals to Doctors

- Since the 1900s, newly graduated doctors in the US must take on a residency (training period)
- Important for both doctors and hospitals to make a good match!

Matching Hospitals to Doctors

- Hospitals started making offers to students increasingly early
- Students had to consider each offer separately – no exposure to the entire market!
- By 1940 students were hired two years before graduation
- Bad for hospitals and students!

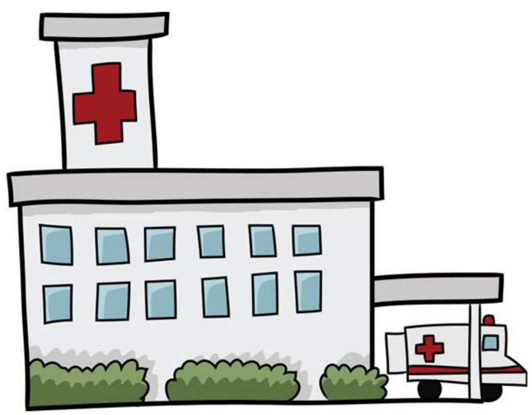
Matching Hospitals to Doctors

- In an attempt to control the market, medical schools stopped releasing data on students until a commonly agreed date.
- This led to “exploding offers”, shorter market duration and chaos.

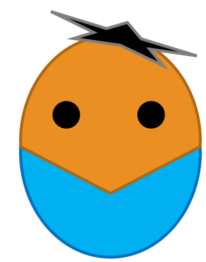
Market Design

- Can we **(re)design** a market that is thick, safe and timely?
- National Resident Matching Program – a (mostly) successful story of market design

Matching Hospitals to Doctors – a Clearinghouse



$Alice \succ Bob \succ Claire \succ \cdots$



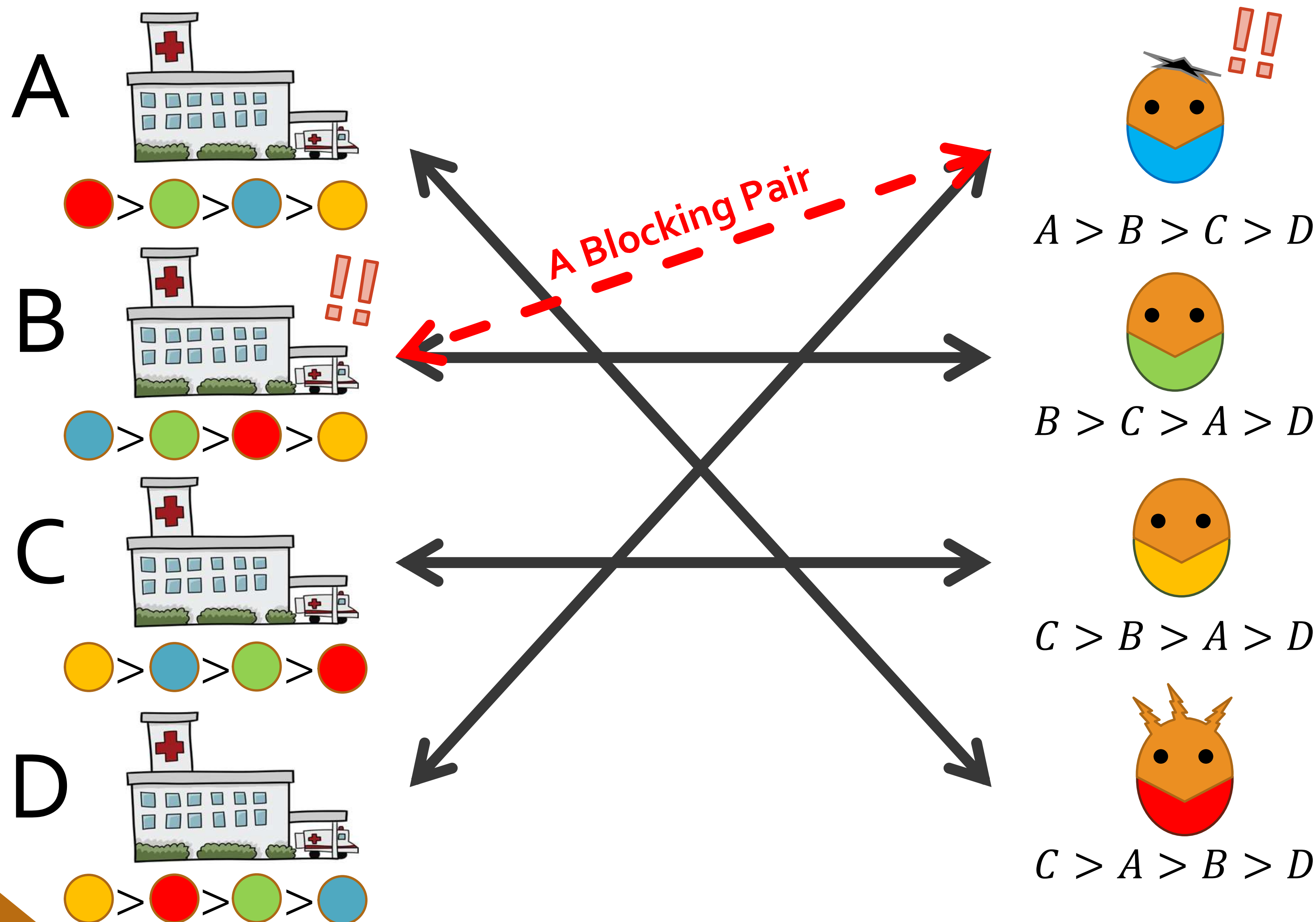
$$\begin{aligned} &\text{KKH} \\ \succ\ &\text{NUH} \\ \succ\ &\text{Mt. Alvernia} \\ \succ\ &\cdots \end{aligned}$$

Matching Hospitals to Doctors – a Clearinghouse

- A set of students $S = \{s_1, \dots, s_n\}$
- A set of hospitals $H = \{h_1, \dots, h_m\}$
- Each student $s \in S$ has a strict preference order over H, denoted by $\succ_s$
- Each hospital $\mathrm{h} \in H$ has a strict preference order over S, denoted by $\succ_h$
- A matching $M: S \to H$ maps each student to a hospital.
- Assume for simplicity that $n = m$ (otherwise, can insert dummy students/hospitals that are least preferred by the other side).

An Initial Matching Mechanism

- Let M_{ij} be the set of all student-hospital pairs (s, h) such that s ranks h in the i-th place, and h ranks s in the j-th place.
- Match all pairs in the order $M_{11}, M_{12}, M_{21}, M_{22}, M_{23}, M_{32}, M_{13}, \ldots$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A Blocking Pair
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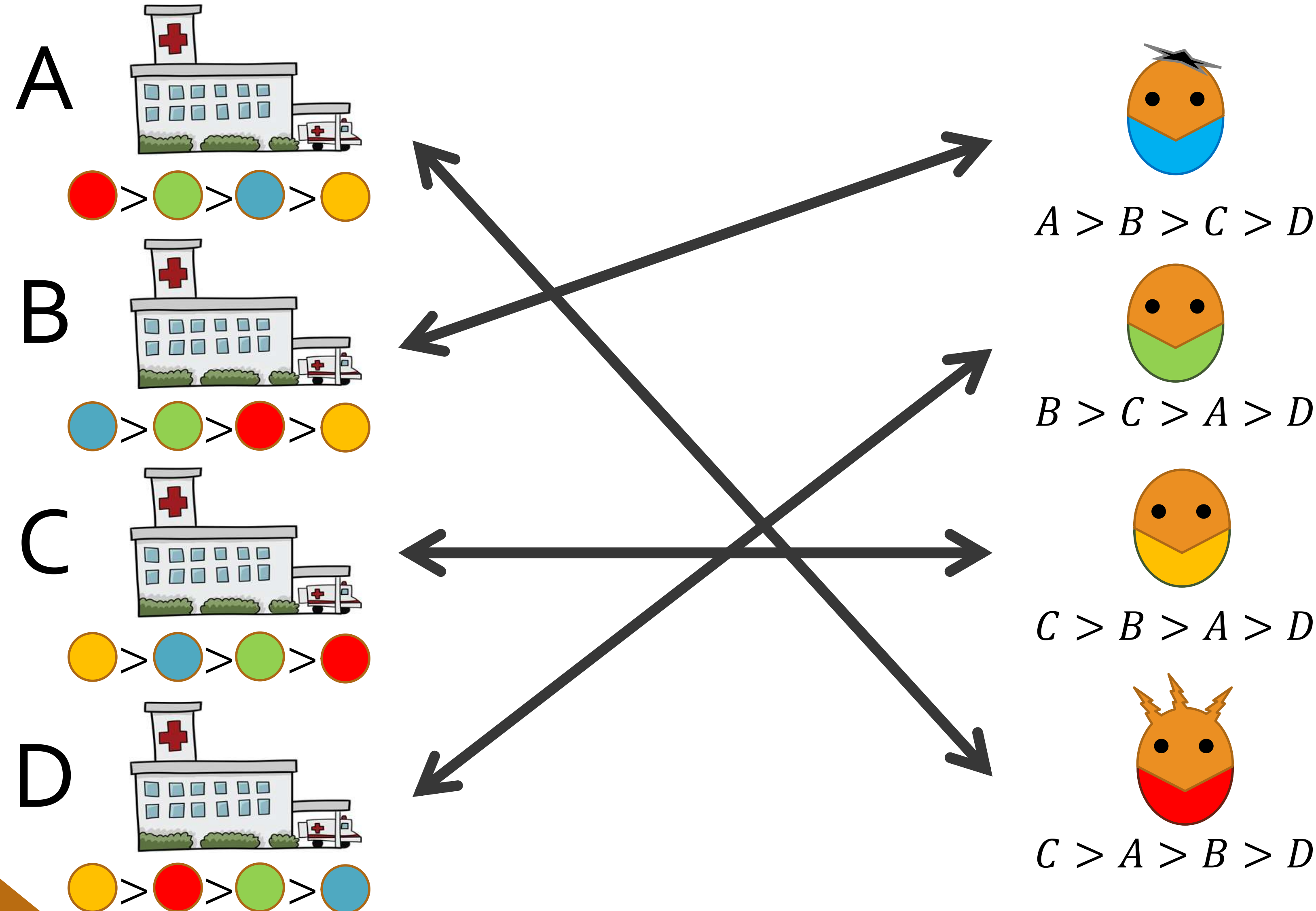

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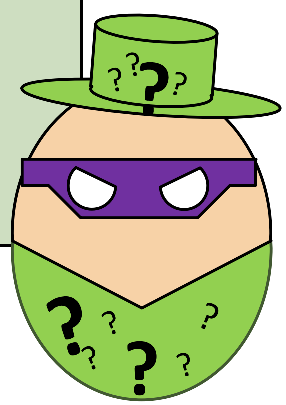
Stable Matchings

- A pair $(s, h) \in S \times H$ blocks M if $h \succ_s M(s)$ and $s \succ_h M^{-1}(h)$.
- A matching M is called stable if there are no blocking pairs.

A Stable Matching

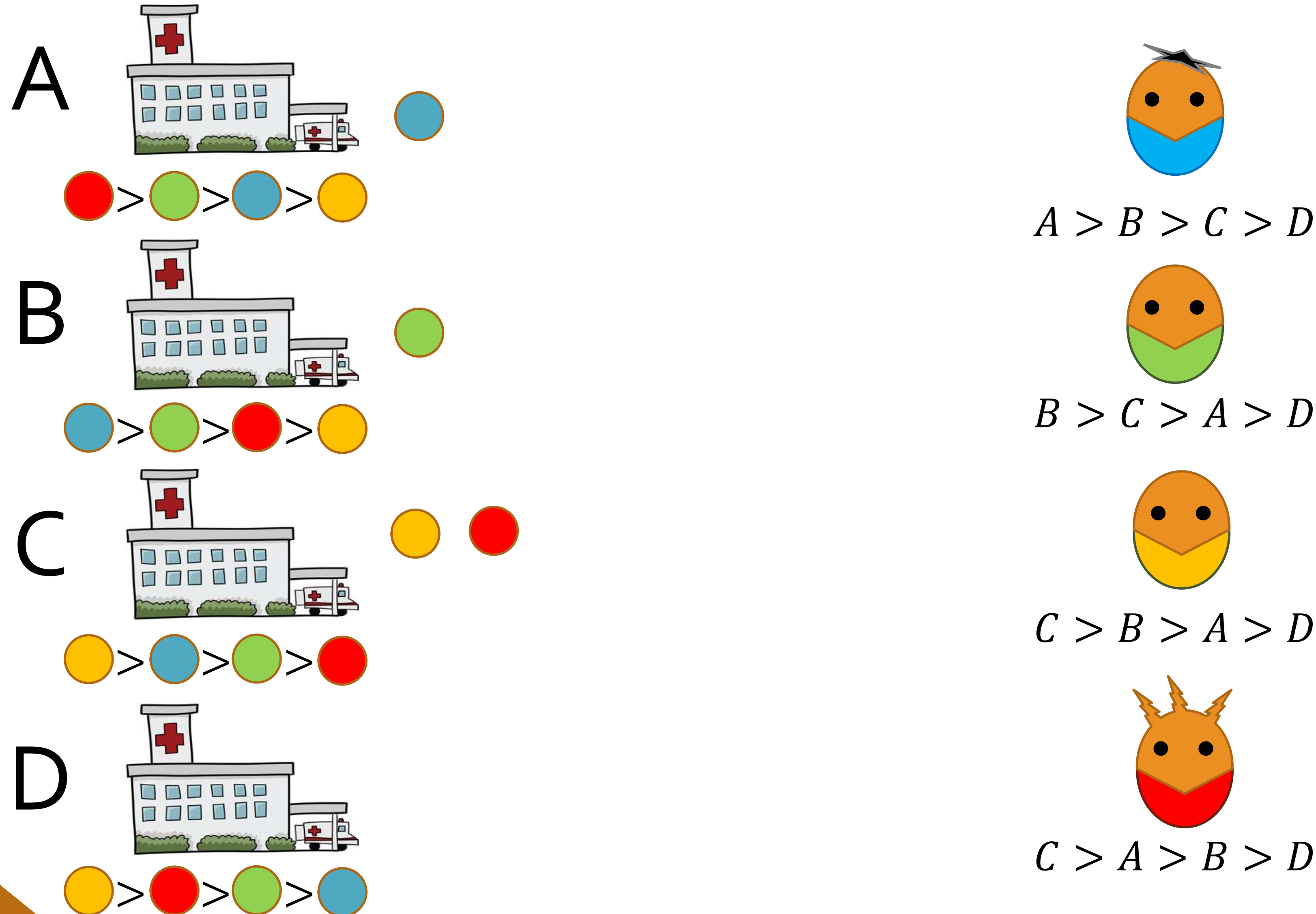
1. Always exists?

2. Can be found in polynomial time? (if exists)



Gale-Shapley Deferred Acceptance Algorithm (1962)

- Start with all students unassigned
- While there are unassigned students
  - Each unassigned student proposes to his/her favorite not-yet-proposed-to hospital.
  - Each hospital looks at the list of students that proposed to it in this round + whoever is assigned to it now, picks its most preferred student; all others remain unassigned.
- Return the resulting matching



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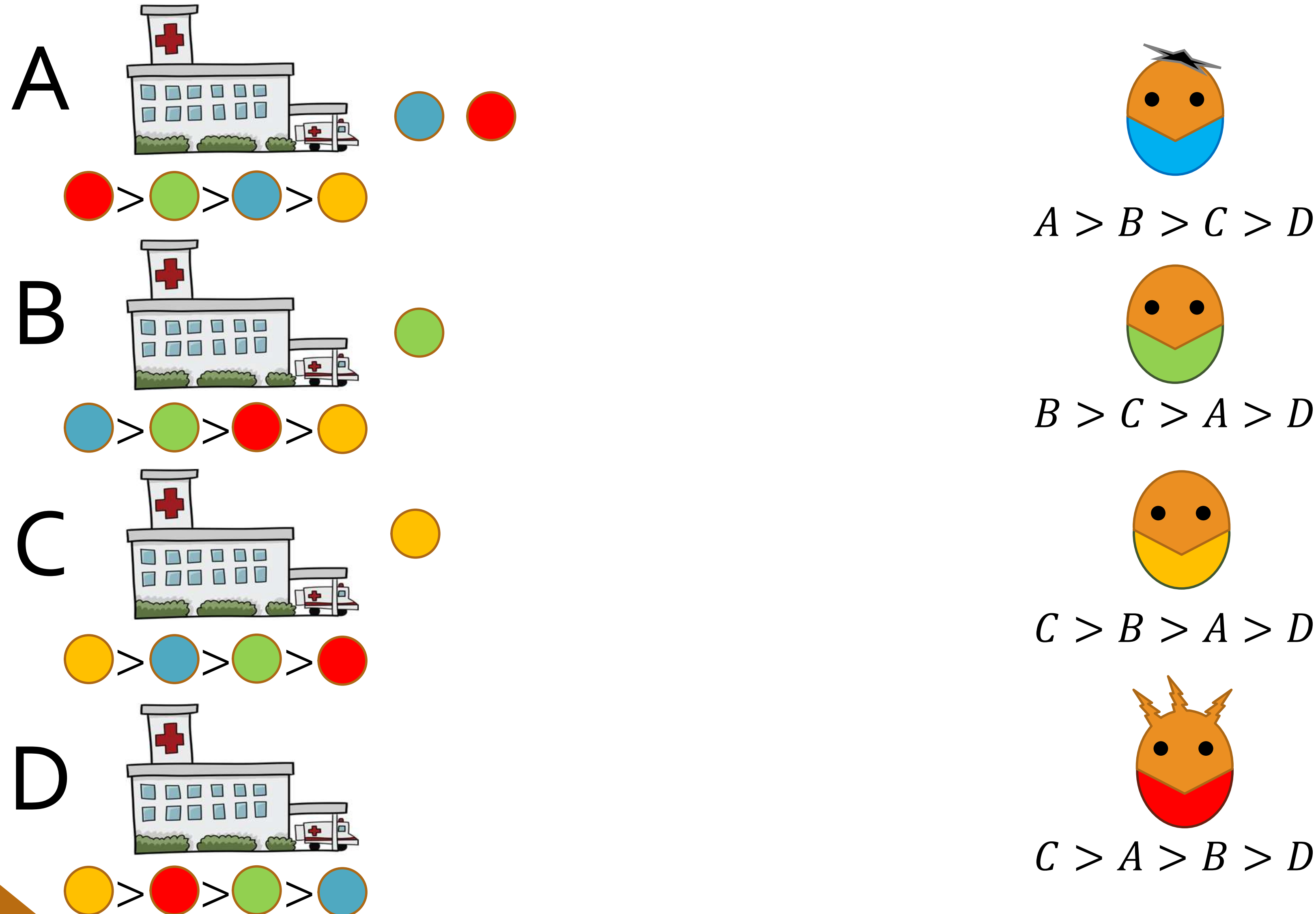
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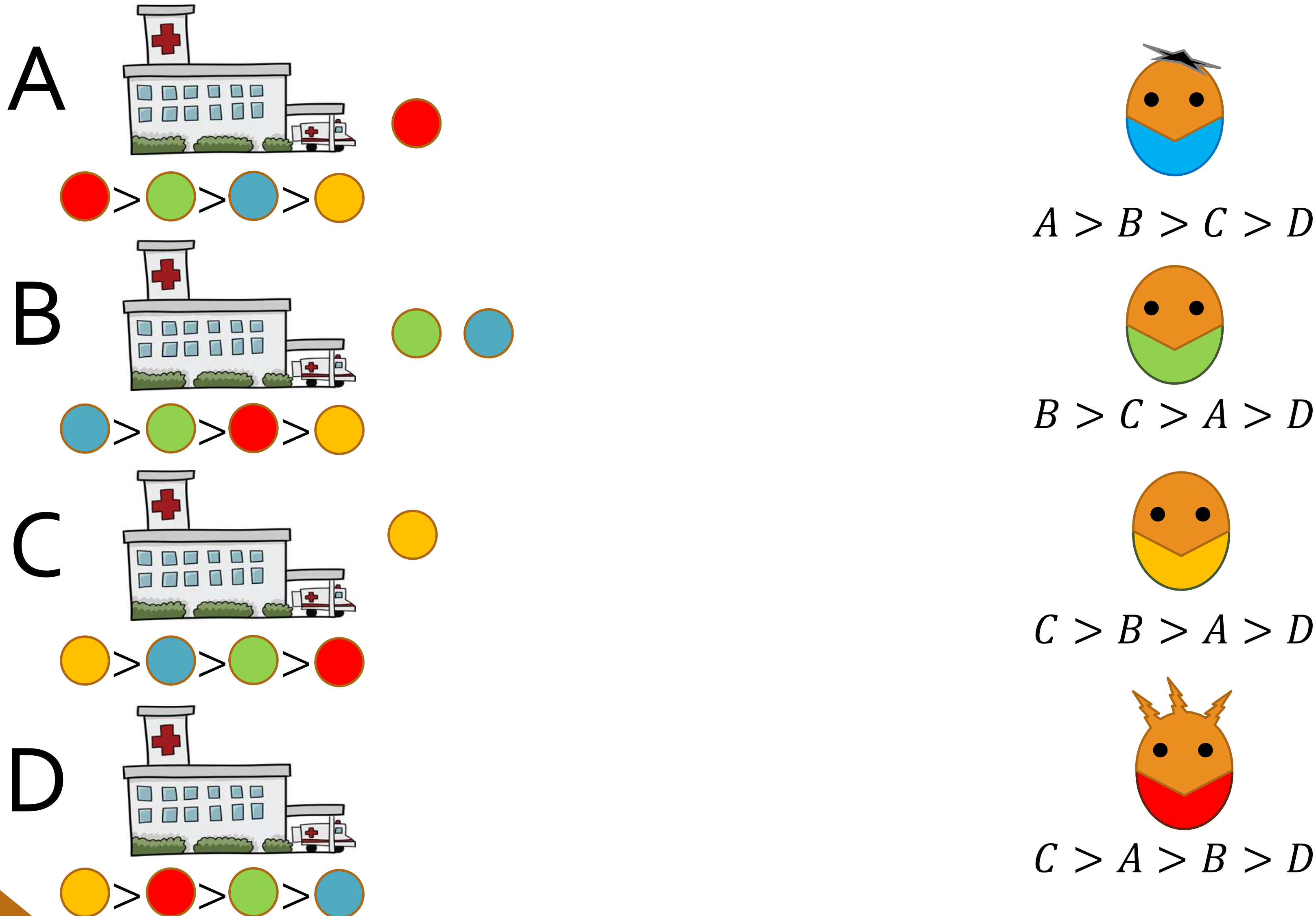
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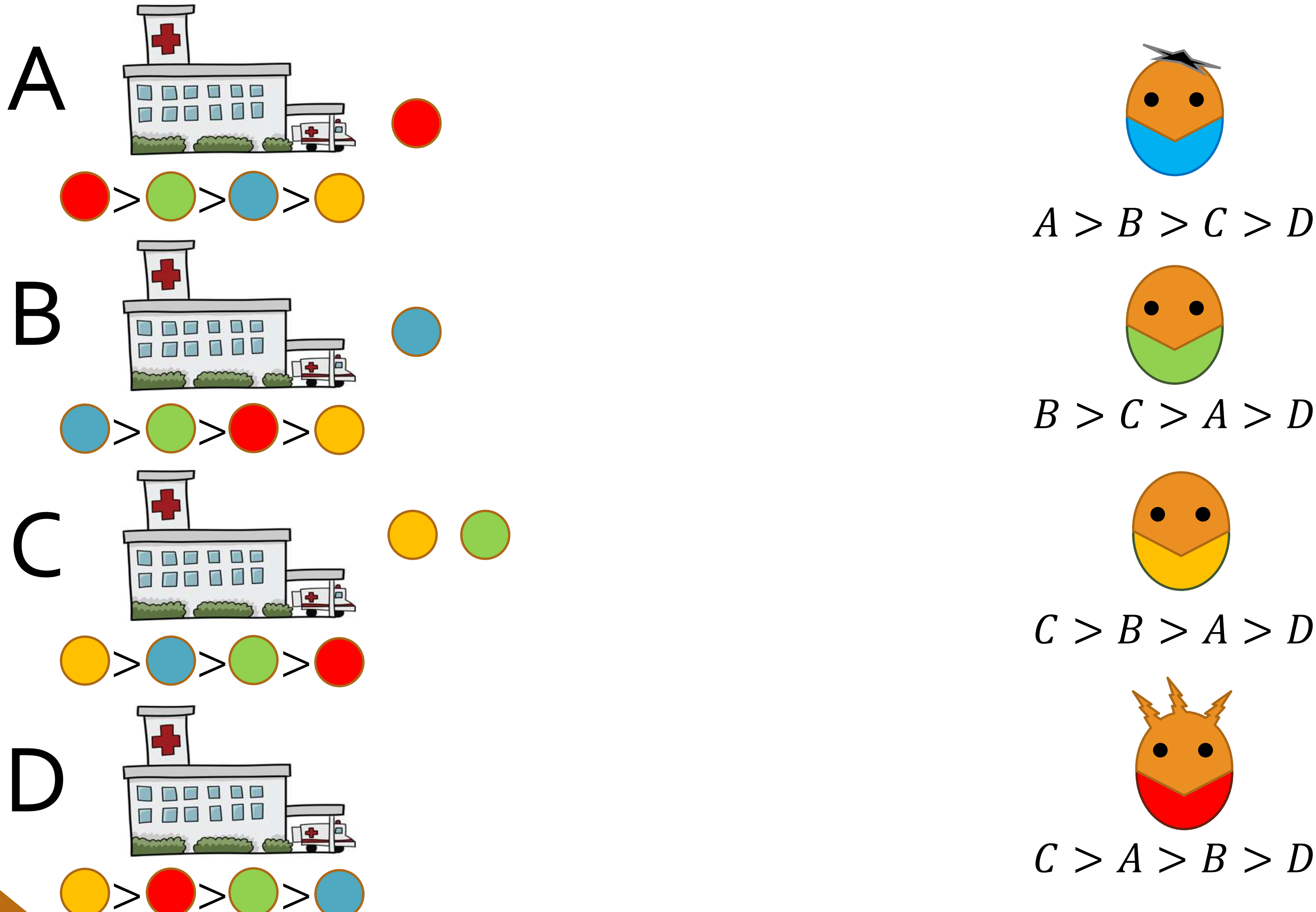
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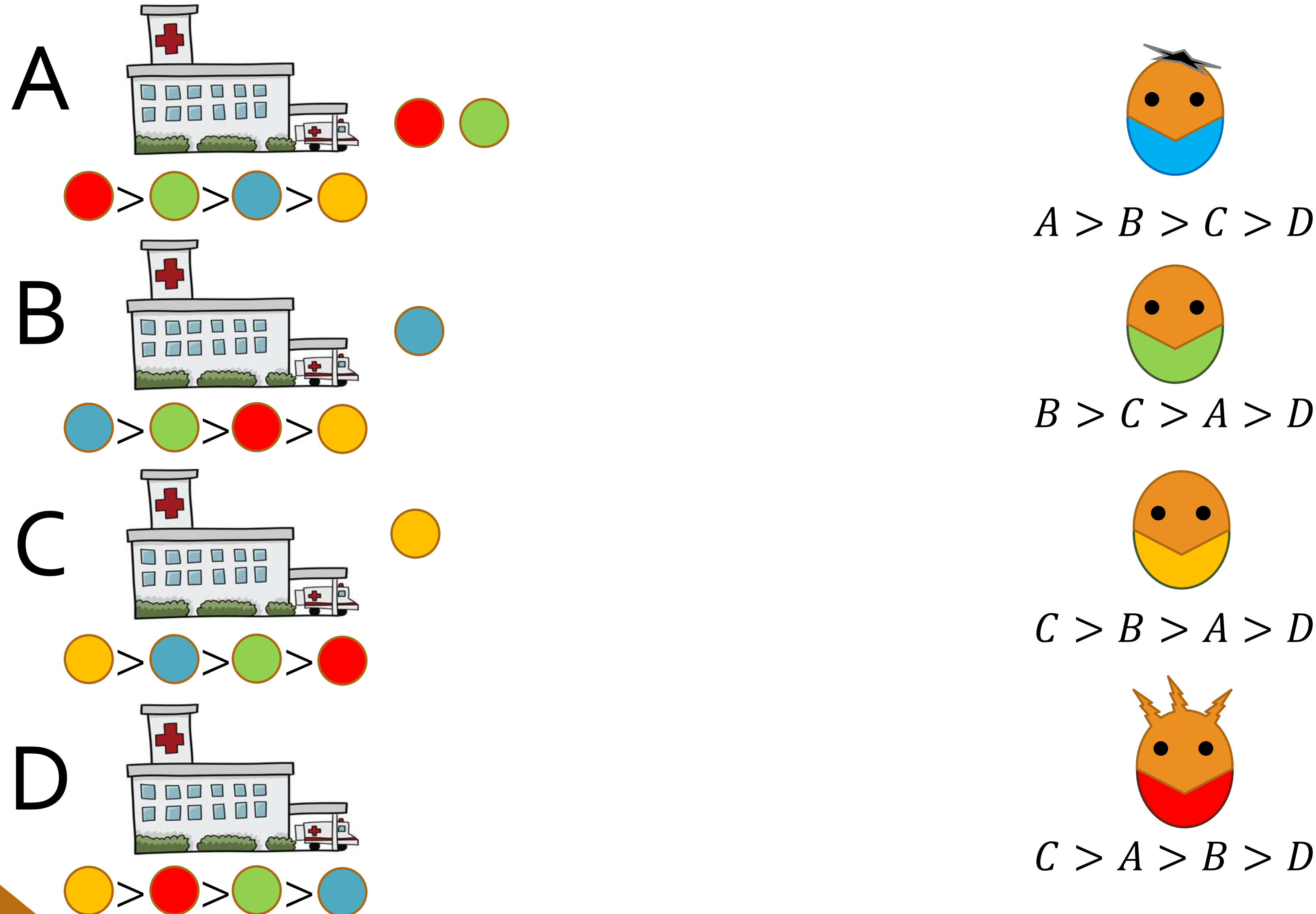
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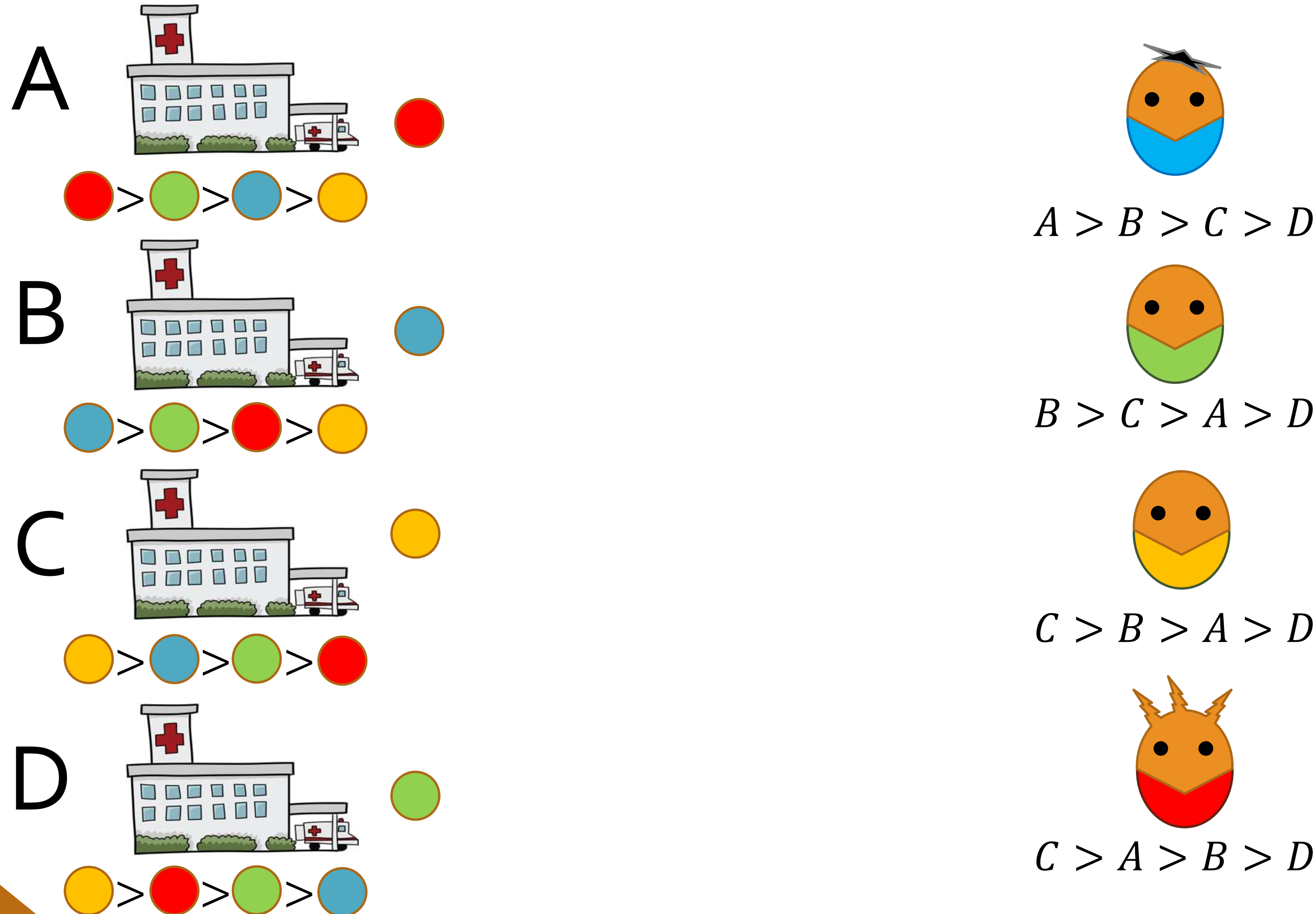

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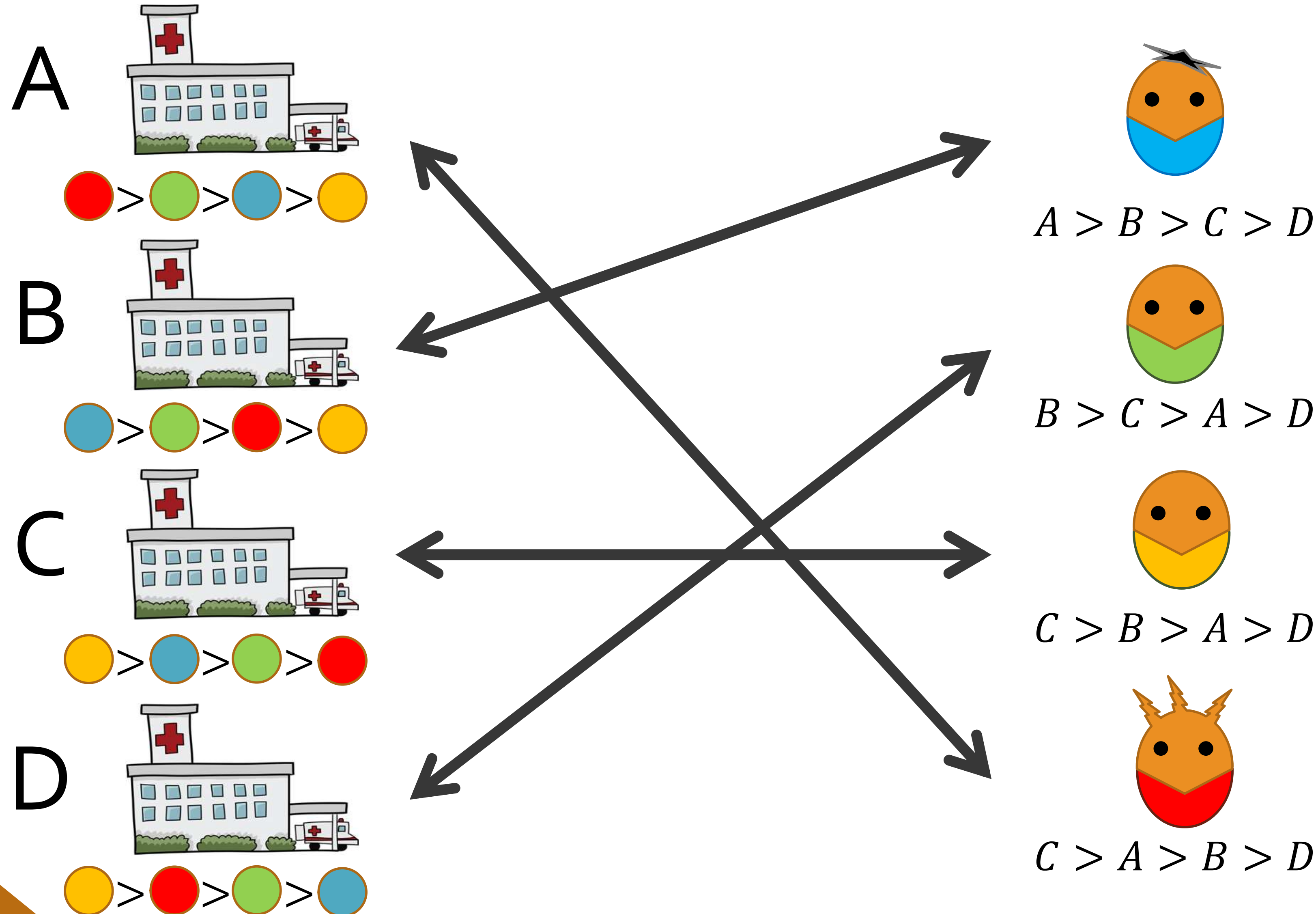

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Sample website for G-S algorithm:

http://www.matchu.ai/GaleShapley

The algorithm implemented there is a variant where one unassigned student proposes at a time, instead of all unassigned students proposing at once.

Theorem: The G-S algorithm terminates in at most n^2 iterations with a stable matching.

- n^2 iterations: No student proposes to the same hospital twice.
- Terminates with a perfect matching (i.e., everyone is matched):
  - If not, some student is rejected from all n hospitals.
  - A hospital only rejects a student in favor of a better student.
  - Once a hospital is matched, it remains matched throughout.
  - All n hospitals are matched.
  - Contradiction!

Theorem: The G-S algorithm terminates in at most n^2 iterations with a stable matching.

- The perfect matching is stable:
  - Consider a student s and a hospital h not matched to each other.
  - Case 1: s never proposed to h.
  - Since s goes down his/her list, s is matched to a better hospital than h.
  - Case 2: s proposed to and is rejected by h.
  - This means that h is matched to a better student than s.

How Fair is the G-S Algorithm?

Suppose that students' preferences are such that each student ranks a different hospital first.

What would the outcome be?

How Fair is the G-S Algorithm?

- Given a student $s \in S$, a hospital $h \in H$ is called **valid** if there exists some **stable** matching M such that $M(s) = h$.
- Let $best(s)$ be the most highly-ranked valid hospital for s.

- Define a **valid** student for a hospital similarly.
- Let $worst(h)$ be the least highly-ranked valid student for h.

How Fair is the G-S Algorithm?

Theorem: the G-S algorithm (with students proposing) assigns

- each student $s \in S$ to the hospital $best(s)$, and
- each hospital $h \in H$ to the student $worst(h)$

Theorem: the G-S algorithm (with students proposing) assigns each student $s \in S$ to $best(s)$

- Assume for contradiction that this is not the case
- Let s be the first student rejected by her best valid hospital, i.e., $best(s)$. Let $h = best(s)$.
- Suppose that s ends up matched to h', where $h \succ_s h'$
- h rejected s in favor of a better student, s' (i.e., $s' \succ_h s$)
- Since s is the first student to be rejected by her best valid hospital, $h \succcurlyeq_{s'} best(s')$

- Let s be the first student rejected by $best(s) = h$
- Suppose that s ends up matched to h', where $h \succ_s h'$
- h rejected s in favor of a better student, s' (i.e., $s' \succ_h s$)
- Since s is the first student to be rejected by her best valid hospital, $h \succcurlyeq_{s'} best(s')$
- There exists another stable matching M' where s is matched to h
- Suppose that s' is matched to $h'' \neq h$ in M'
- We have $h'' \preccurlyeq_{s'} best(s') \preccurlyeq_{s'} h$, so $h'' \prec_{s'} h$
- However, $s' \succ_h s$ as well, so (h, s') form a blocking pair in M'
- Contradiction!

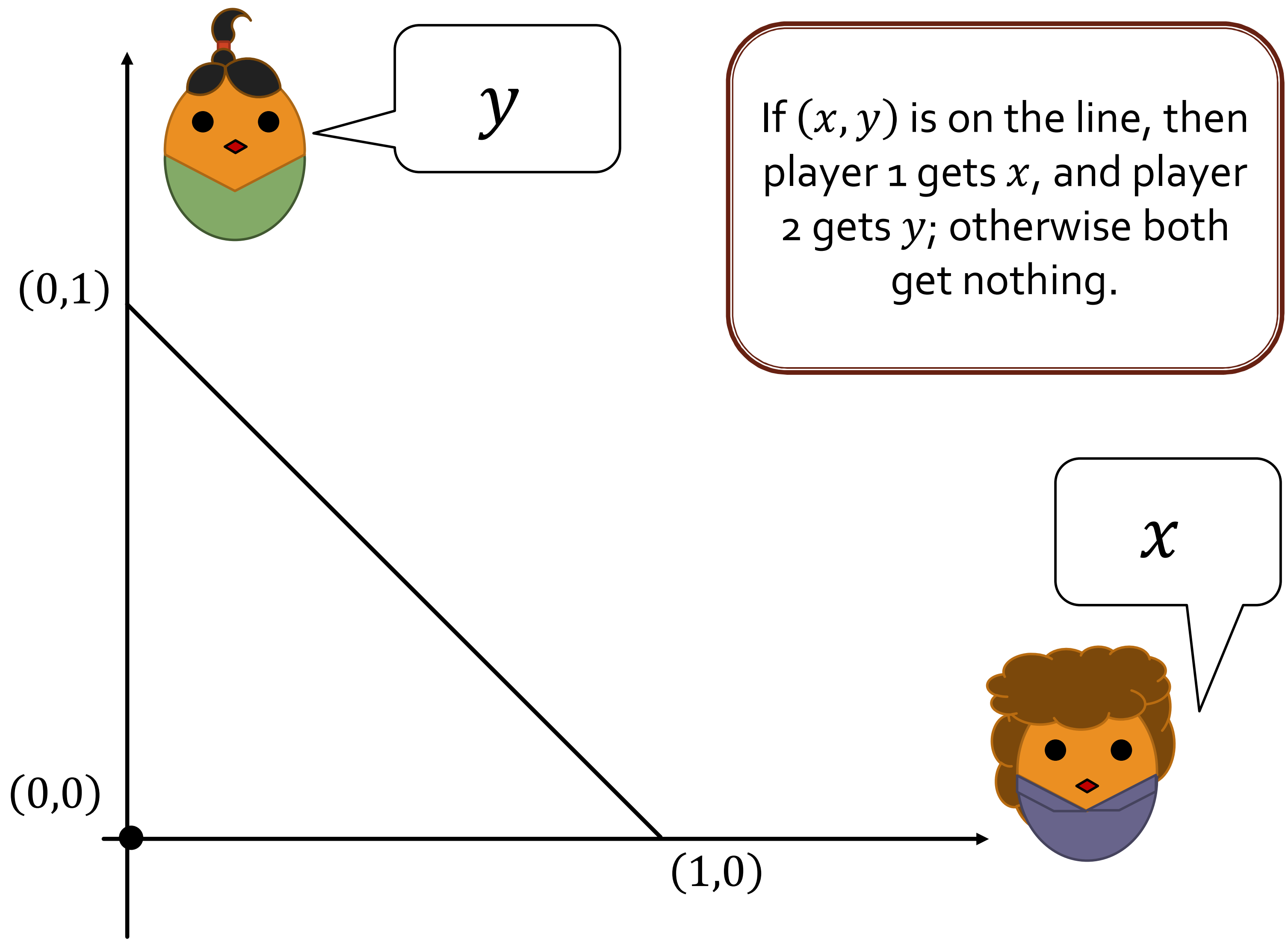
Theorem: the G-S algorithm (with students proposing) assigns each hospital $h \in H$ to $worst(h)$

- Assume for contradiction that this is not the case
- Let h be matched to $s \neq worst(h)$
- We know that $h = best(s)$ and $s \succ_h worst(h)$
- Consider a stable matching M' where h is matched to $worst(h)$
- Suppose that s is matched to h' in M', where $h' \neq h$
- Since $h = best(s)$, we have $h \succ_s h'$
- (s, h) form a blocking pair in M', contradiction!

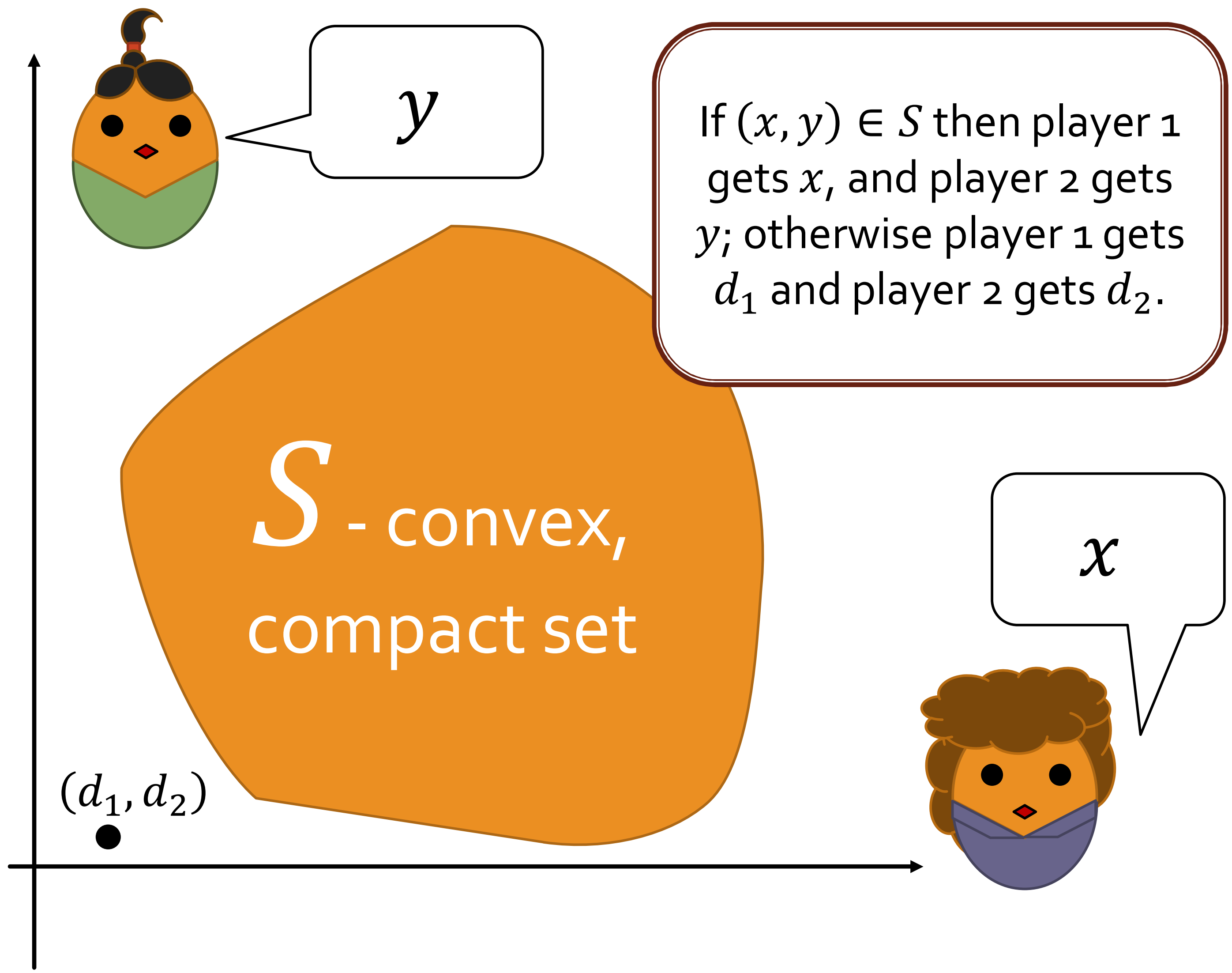
How the National Resident Matching Program (NRMP) matching algorithm works:

http://www.youtube.com/watch?v=kvgfgGmemdA

Nash Bargaining Solution



If (x, y) is on the line, then player 1 gets x, and player 2 gets y; otherwise both get nothing.

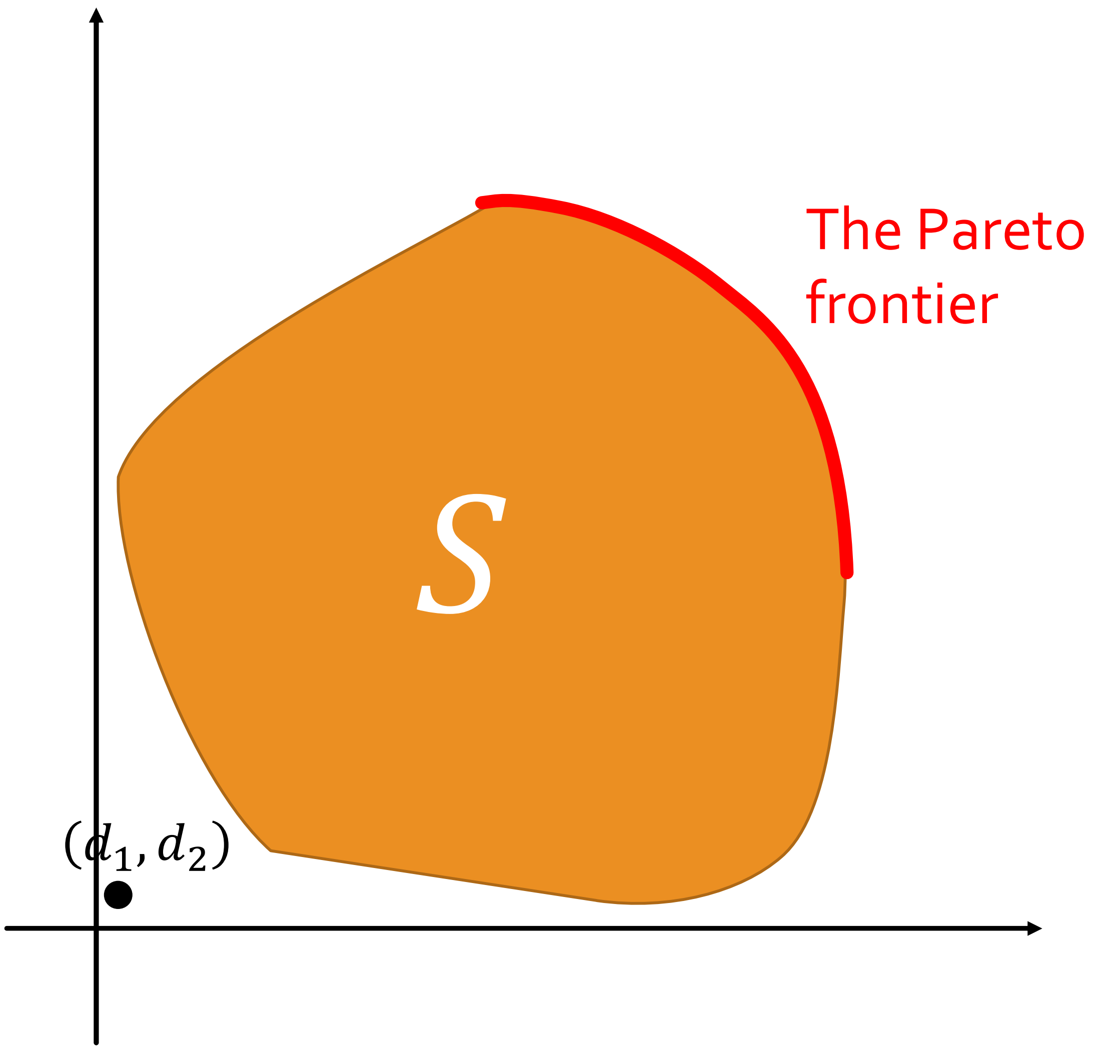

y
If $(x, y) \in S$ then player 1 gets x, and player 2 gets y; otherwise player 1 gets d_1 and player 2 gets d_2.
S - convex, compact set
x
(d_1, d_2)

Pareto optimality

An outcome (x_1, y_1) Pareto dominates another outcome (x_2, y_2) if $x_1 \geq x_2$ and $y_1 \geq y_2$, and at least one of these two inequalities is strict.

In this case, (x_1, y_1) is said to be a Pareto improvement of (x_2, y_2).

An outcome (x, y) is Pareto optimal if it is not Pareto dominated by any other outcome.



The Pareto frontier
S
(d_1, d_2)

Efficiency

- No outcome (v_1, v_2) Pareto dominates $\left(f_1(S,\vec{d}), f_2(S,\vec{d})\right)$

Symmetry

- Let $S^T = \{(y,x) : (x,y) \in S\}$ and $\vec{d}^T = (d_2, d_1)$; then $\left(f_1(S^T,\vec{d}^T), f_2(S^T,\vec{d}^T)\right) = \left(f_2(S,\vec{d}), f_1(S,\vec{d})\right)$

Independence of Irrelevant Alternatives (IIA)

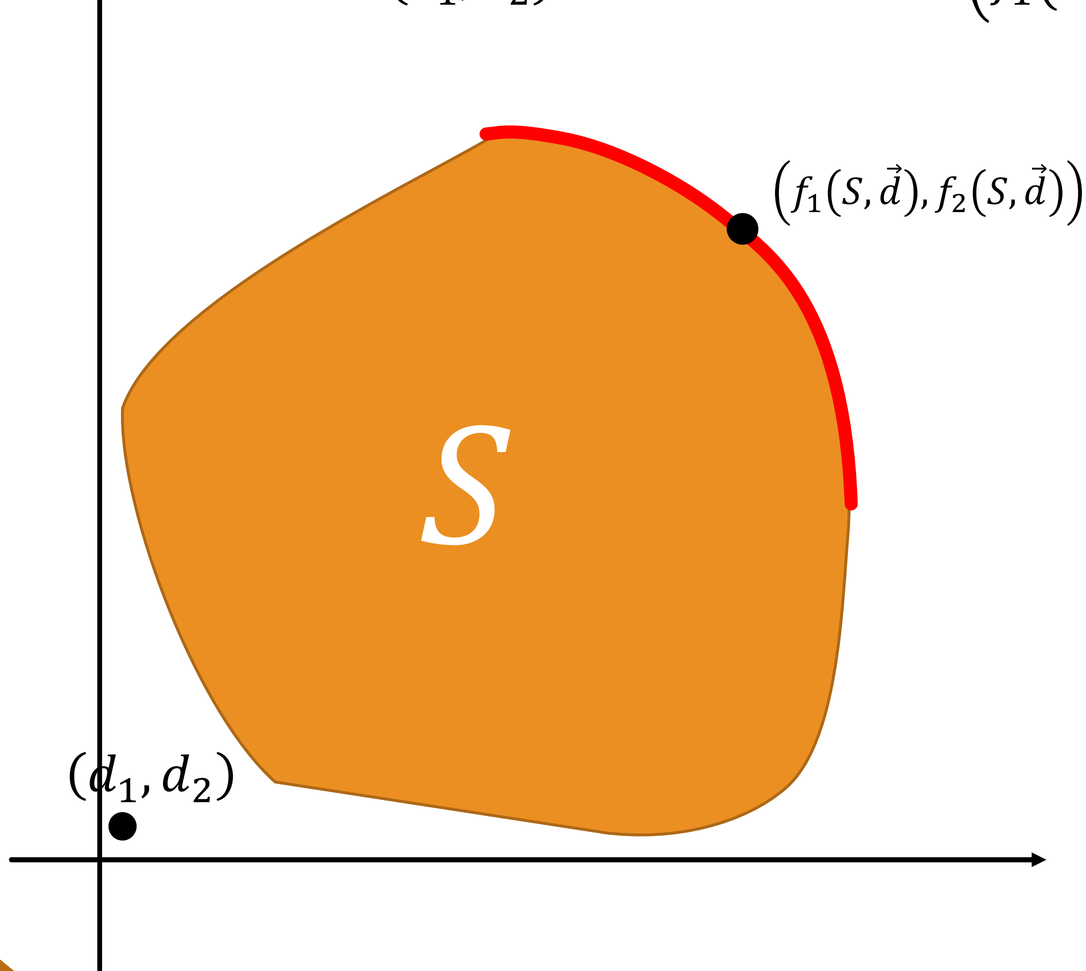
- Let $S' \subseteq S$ be such that $\left(f_1(S,\vec{d}), f_2(S,\vec{d})\right) \in S'$;
  then $\left(f_1(S',\vec{d}), f_2(S',\vec{d})\right) = \left(f_1(S,\vec{d}), f_2(S,\vec{d})\right)$

Invariance under Equivalent Representations (IER)

- For any $\alpha_1, \alpha_2 \in \mathbb{R}, \vec{\beta} \in \mathbb{R}^2$:
  $$f_i\left((\alpha_1,\alpha_2)S + \vec{\beta}, (\alpha_1,\alpha_2)\vec{d} + \vec{\beta}\right) = \alpha_i f_i(S,\vec{d}) + \beta_i$$

Efficiency

No outcome (v_1, v_2) Pareto dominates $\left(f_1(S, \vec{d}), f_2(S, \vec{d})\right)$



Symmetry

Let $S^T = \{(y, x) : (x, y) \in S\}$ and $\vec{d}^T = (d_2, d_1)$;
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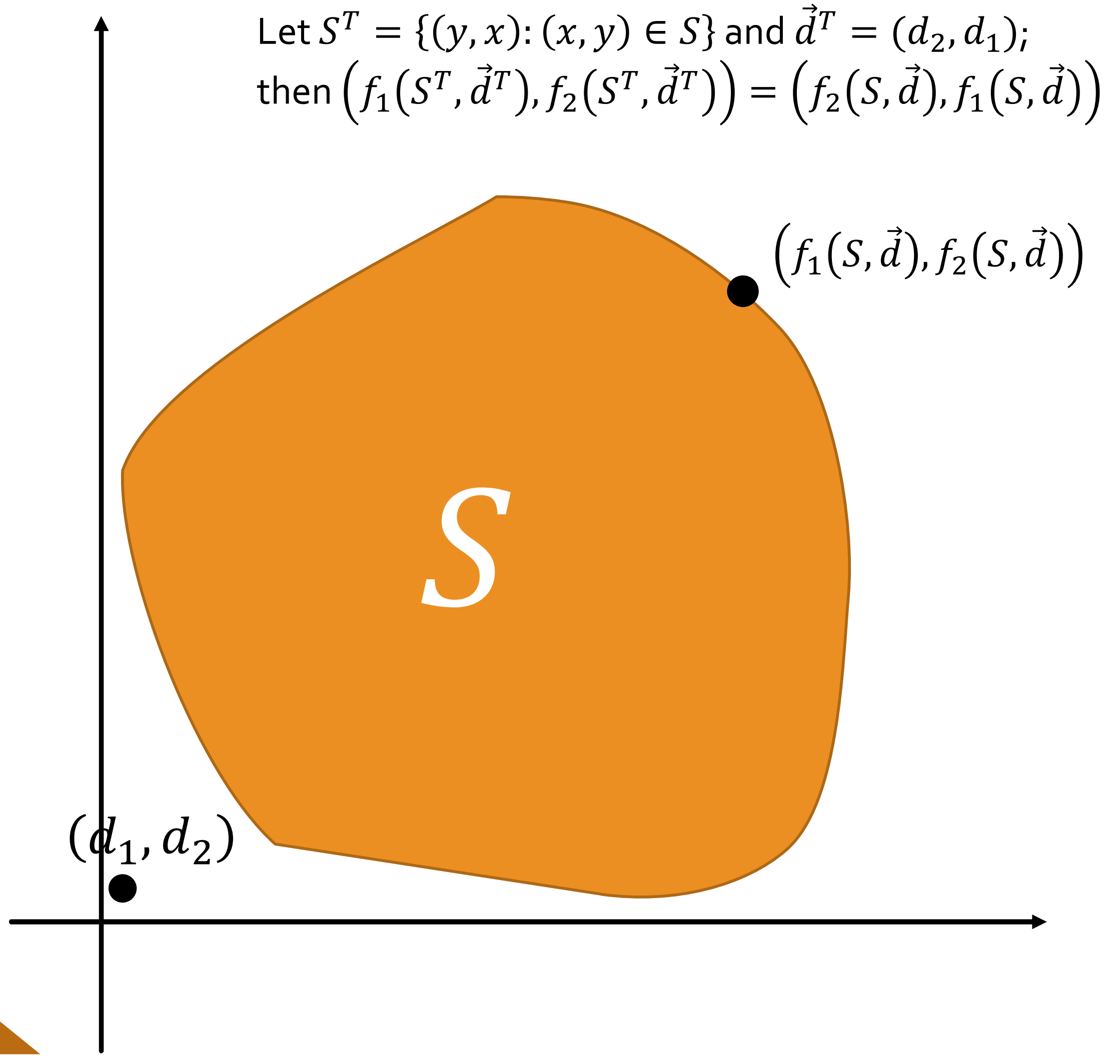

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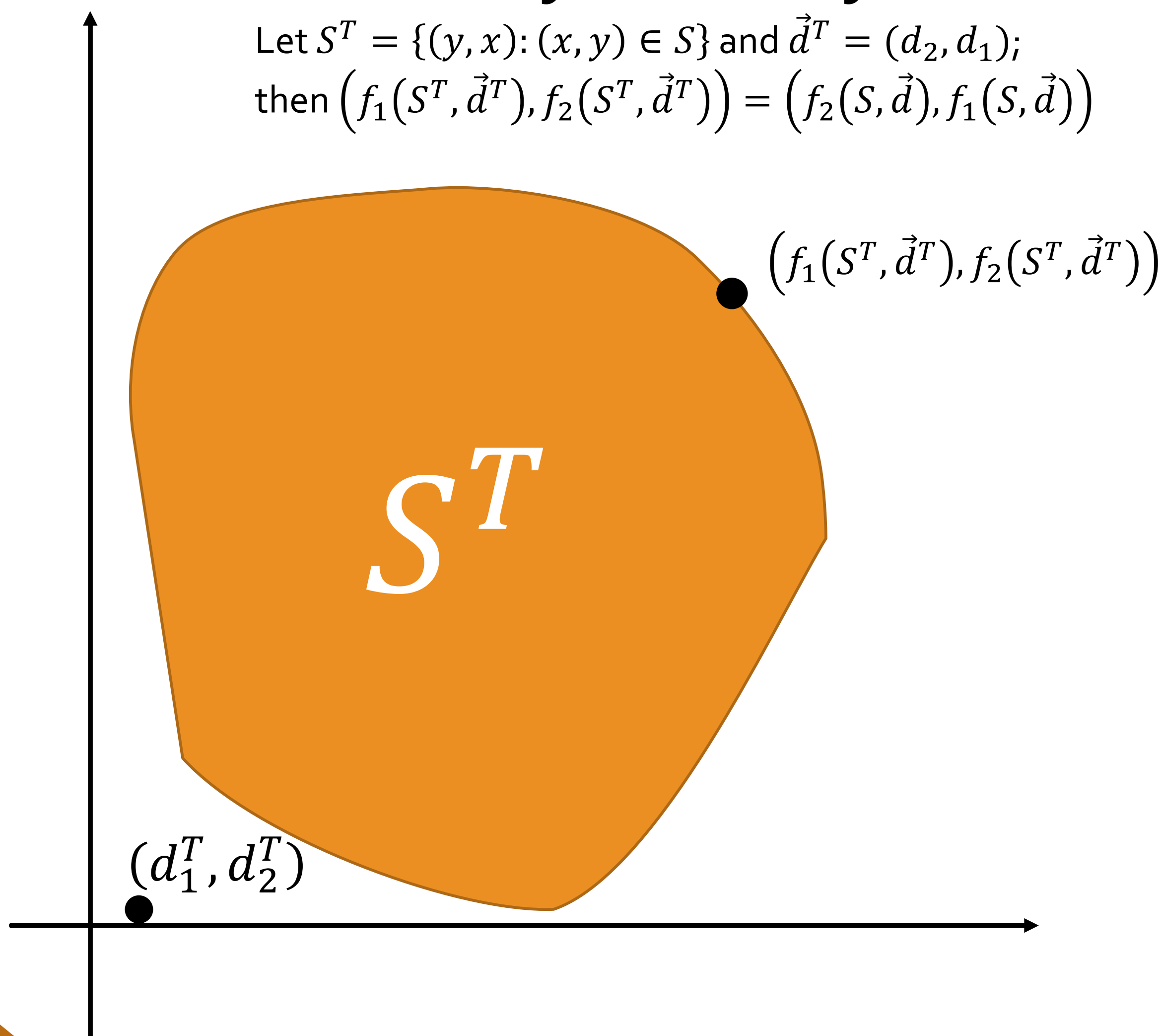
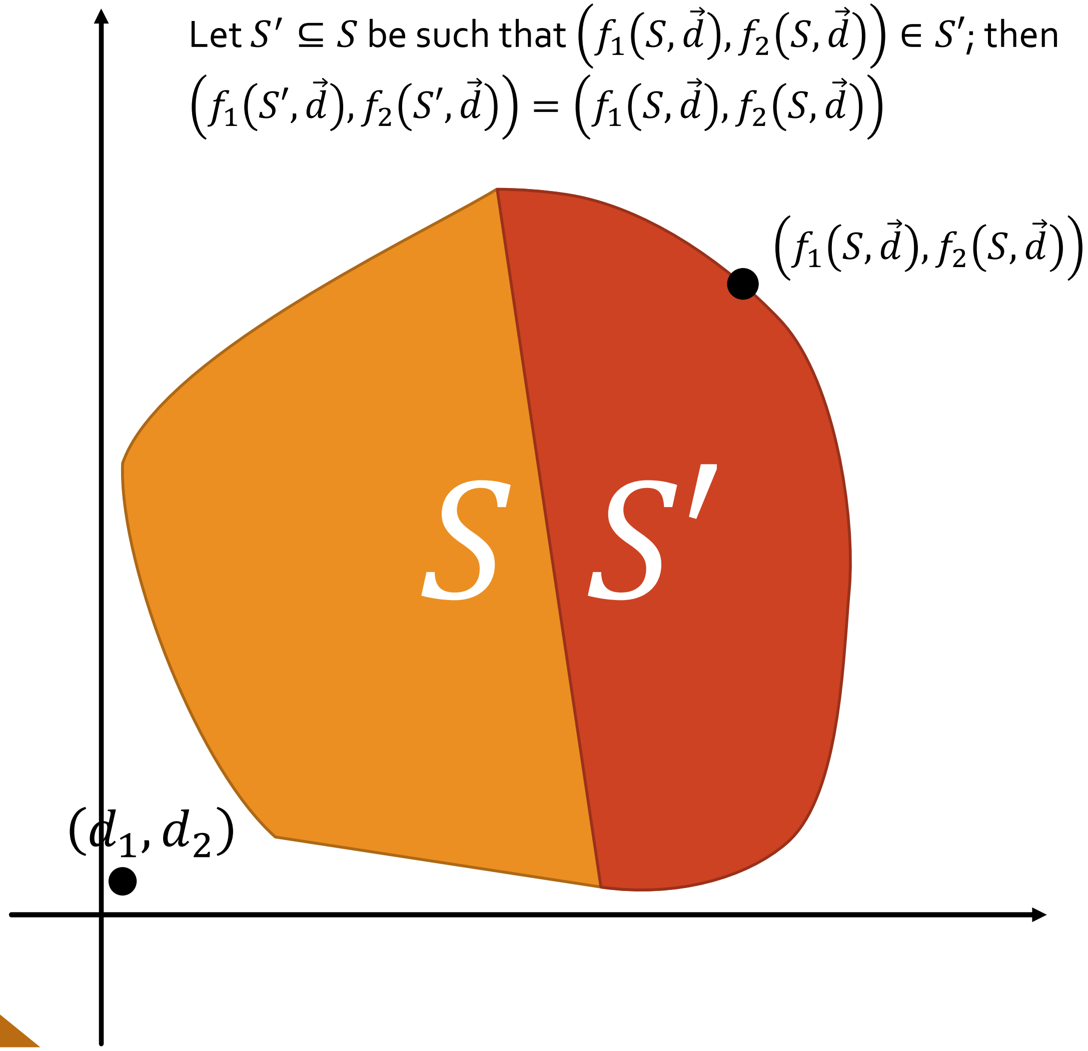


Independence of Irrelevant Alternatives (IIA)

Let $S' \subseteq S$ be such that $\left(f_1(S,\vec{d}), f_2(S,\vec{d})\right) \in S'$; then $\left(f_1(S',\vec{d}), f_2(S',\vec{d})\right) = \left(f_1(S,\vec{d}), f_2(S,\vec{d})\right)$



The Nash Bargaining Solution

$$\max\,(v_1 - d_1)(v_2 - d_2)$$

$$s.t. (v_1, v_2) \in S$$

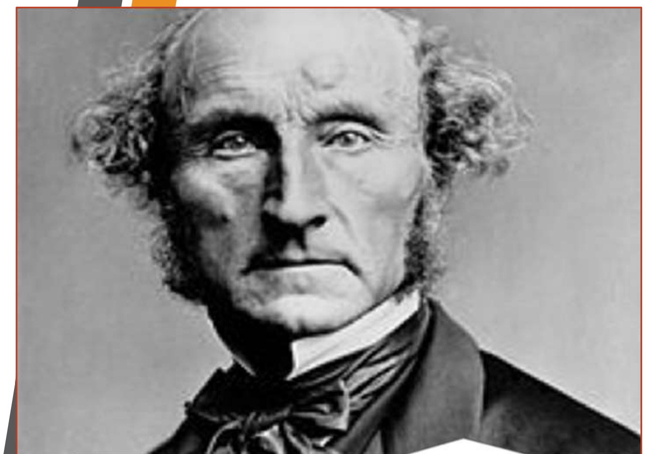
Theorem

The Nash bargaining solution satisfies efficiency, symmetry, IIA and IER

Theorem

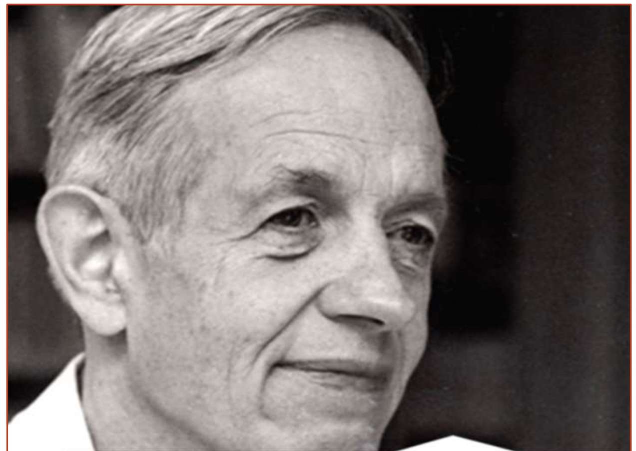
The Nash bargaining solution is the **only** solution that satisfies efficiency, symmetry, IIA and IER.

See proof on Canvas folder "Week 7" (optional)



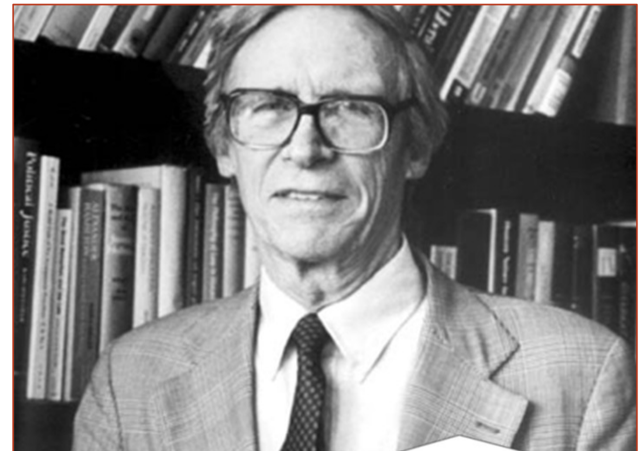
Utilitarian

- $\max \sum_i u_i(A)$



Nash

- $\max \prod_i u_i(A)$



Egalitarian

- $\max \min_i u_i(A)$

Maximize a function of several variables subject to a constraint

Use Lagrange multipliers. See notes on Canvas folder "Week 7" (required)

Check the boundaries

- Maximize x^3 in the range $x \in [-1,1]$
- Taking the derivative and setting it to zero: $3x^2 = 0 \rightarrow x = 0$
- This is not the maximum (nor the minimum!)
- The maximum occurs at $x = 1$ (and the minimum at $x = -1$)