CS4261/5461 Algorithmic Mechanism Design

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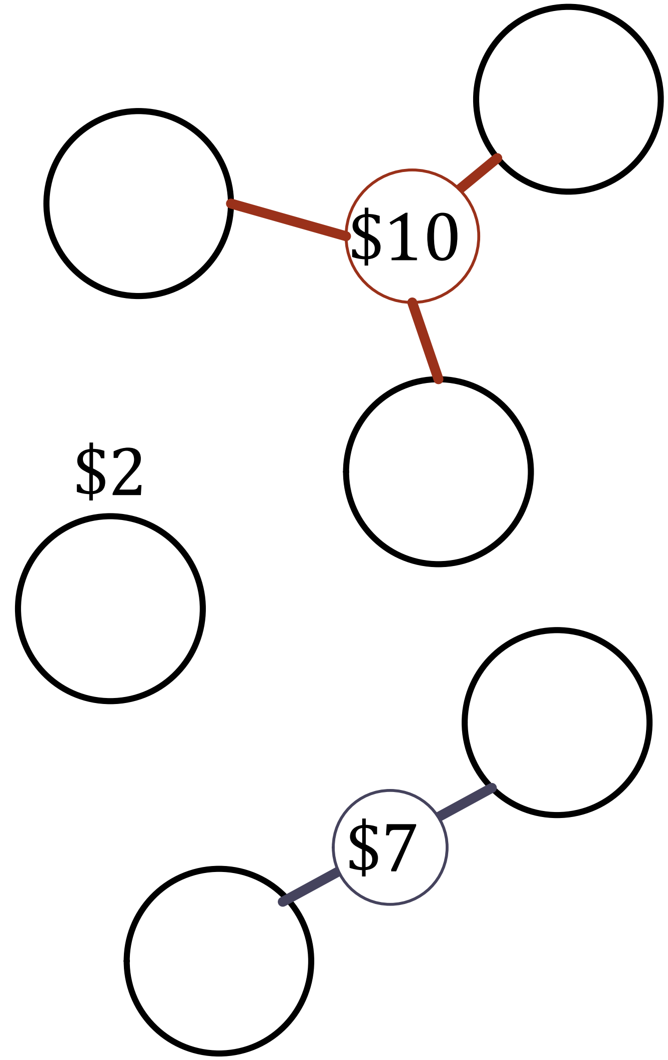
2025

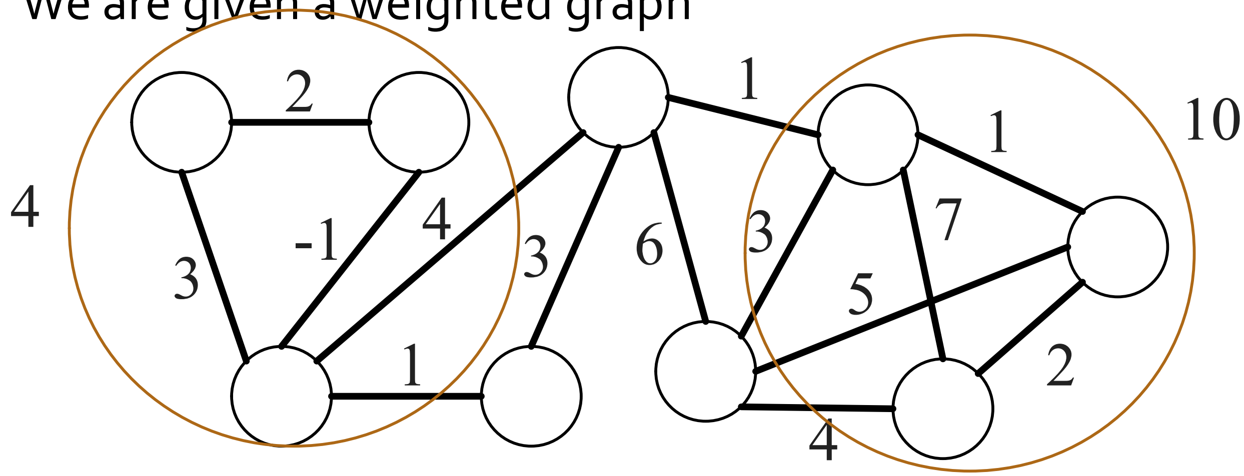
Cooperative Games

- Players divide into coalitions to perform tasks
- Coalition members can freely divide profits
- How should profits be divided?



Induced Subgraph Games

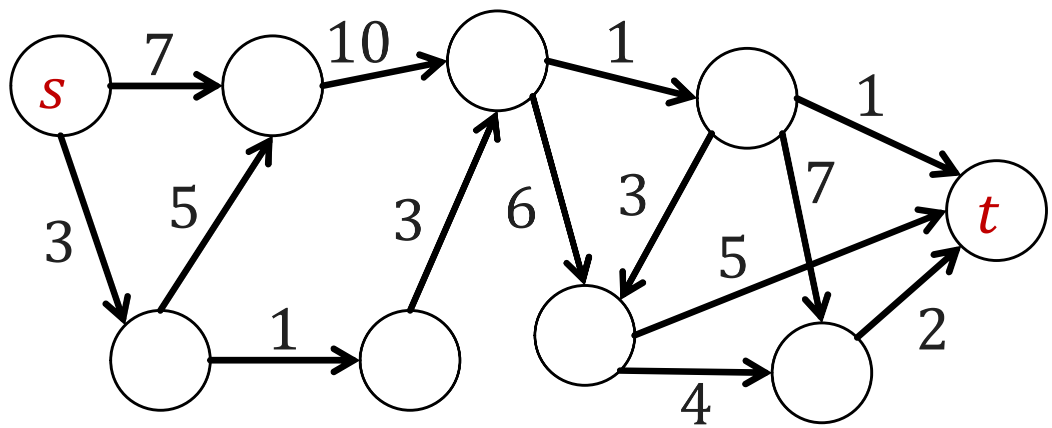
- We are given a weighted graph



- Players are nodes; value of a coalition is the value of the total edge weights in the subgraph.
- Models: collaboration networks, group formation.

Network Flow Games

- We are given a weighted, directed graph



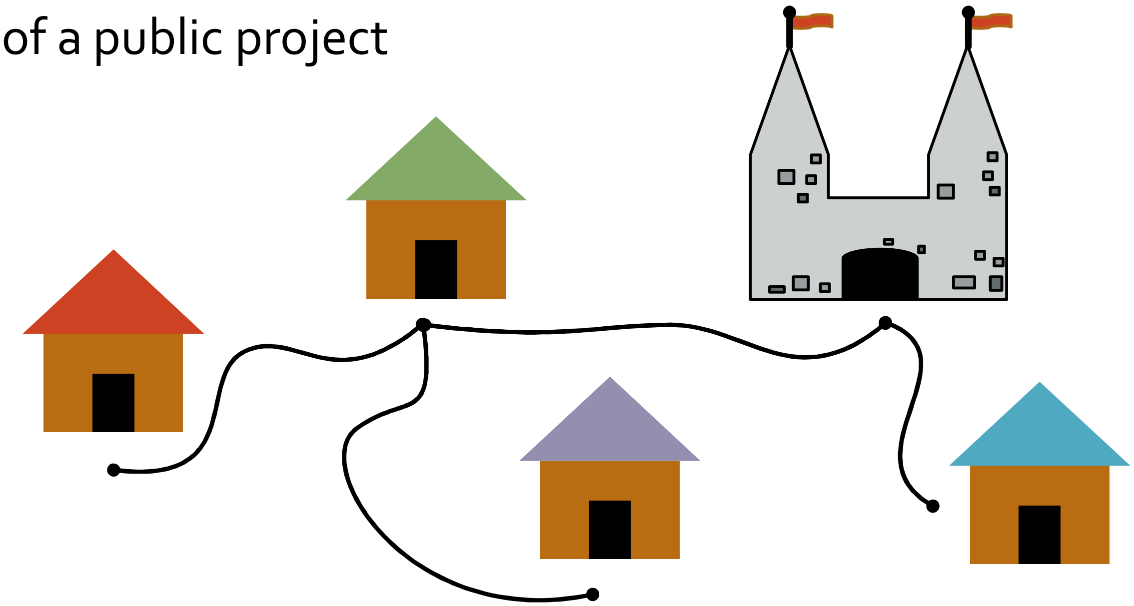
- Players are edges; value of a coalition is the value of the max. flow it can pass from s to t.
- Applications: computer networks, traffic flow, transport networks.

Weighted Voting Games

- We are given a list of weights and a threshold.
- $(w_1, \ldots, w_n; q)$, all of which are positive integers
- Each player i has a weight w_i; value of a coalition is 1 if its total weight is at least q (winning), and 0 otherwise (losing).
- Applications: parliaments, UN security council, EU council of members.

Cost Sharing

- Splitting a taxi fare or a bill
- Sharing the cost of a public project



Cooperative Games

- A set of players - $N = \{1, \ldots, n\}$
- Valuation function - $v: 2^N \to \mathbb{R}_{\geq 0}$
- $v(S)$ – value of a coalition S, $v(\emptyset) = 0$
- CS – a partition of N; a coalition structure.
- $OPT(\mathcal{G}) = \max_{CS} \sum_{S \in CS} v(S)$
- Efficiency: a vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ satisfying $\sum_{i \in N} x_i = v(N)$
- Individual rationality: $x_i \geq v(i)$ for all $i \in N$
- Imputation: a vector satisfying efficiency + ind. rationality

Cooperative Game Properties

A game $\mathcal{G} = \langle N, v \rangle$ is called **monotone** if for any $S \subseteq T \subseteq N$:

$$v(S) \le v(T)$$

$\mathcal{G}$ is **simple** if it is monotone and

$$v(S) \in \{0,1\} \text{ for all } S$$

$\mathcal{G}$ is **superadditive** if for disjoint $S, T \subseteq N$:

$$v(S) + v(T) \le v(S \cup T)$$

$\mathcal{G}$ is **convex** if for $S \subseteq T \subseteq N$ & $i \in N \setminus T$:

$$v(S \cup \{i\}) - v(S) \le v(T \cup \{i\}) - v(T)$$

Example

Here is an example 3-player game

$$v(1) = v(2) = v(3) = 0;$$
$$v(1,2) = 4, v(1,3) = 5, v(2,3) = 6$$
$$v(1,2,3) = 7$$

Monotone? Simple? Superadditive? Convex?

Dividing Payoffs in Cooperative Games

The core and the Shapley value

The core

- An imputation $\vec{x}$ is in the core if

$$\sum_{i \in S} x_i = x(S) \geq v(S), \forall S \subseteq N$$

- Each subset of players is getting at least what it can make on its own.
- A notion of stability; no subset can deviate.

The core is a set of vectors in $\mathbb{R}^n$ that satisfy linear constraints

$$\sum_{i \in N} x_i = v(N)$$

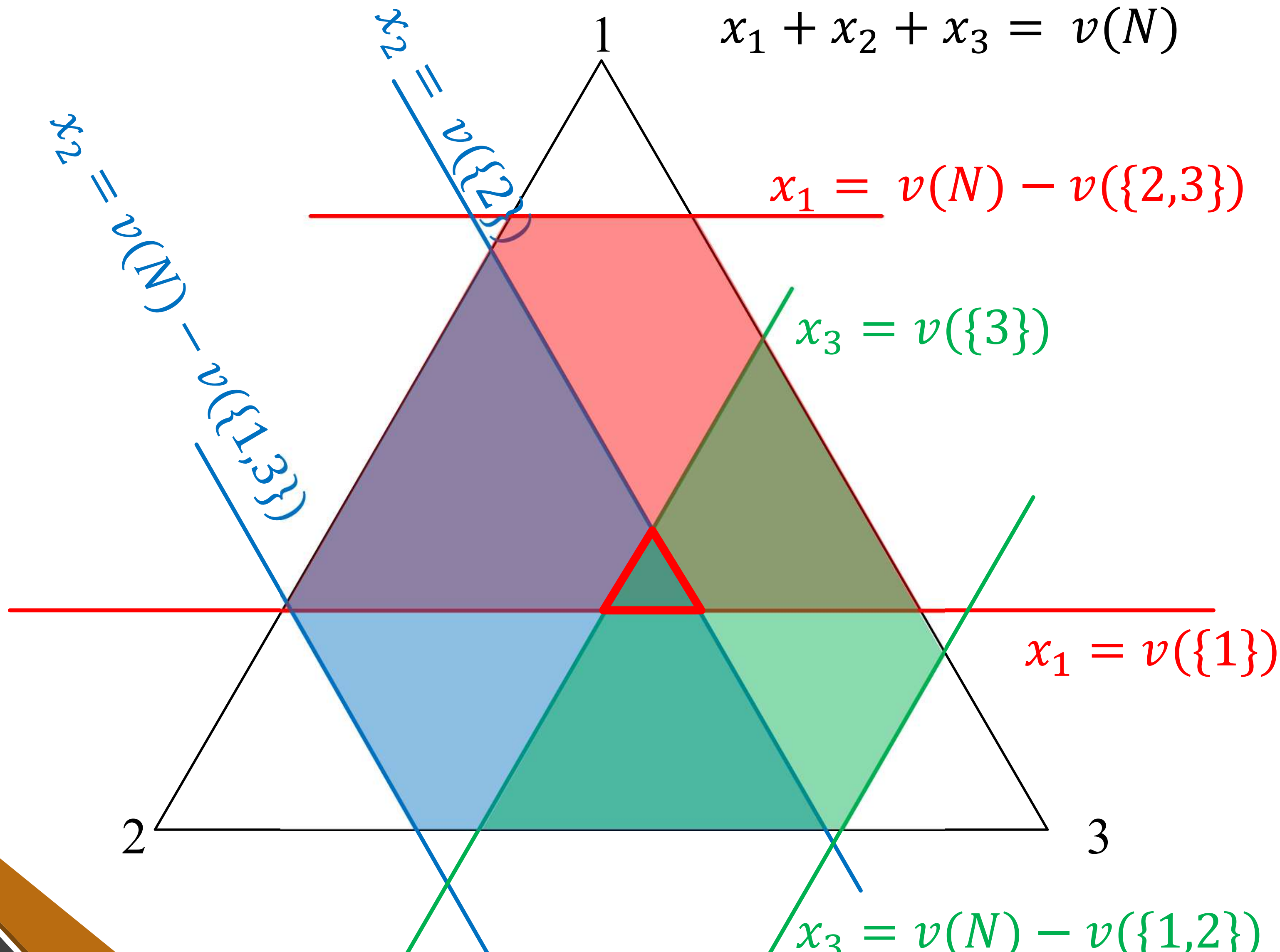
$$\sum_{i \in S} x_i \geq v(S), \forall S \subseteq N$$

For three players, $N = \{1,2,3\}$

$$x_1 + x_2 + x_3 = v(N)$$

$$x_i \geq v(\{i\}) \quad \forall i \in \{1,2,3\}$$

$$x_i + x_j \geq v(\{i,j\}) \quad \forall i,j \in N$$



$x_1 + x_2 + x_3 = v(N)$
1
2
3
$x_1 = v(N) - v(\{2,3\})$
$x_1 = v(\{1\})$
$x_2 = v(\{2\})$
$x_2 = v(N) - v(\{1,3\})$
$x_3 = v(\{3\})$
$x_3 = v(N) - v(\{1,2\})$

Example

Here is an example 3-player game

$$v(1) = v(2) = v(3) = 0;$$
$$v(1,2) = 4, v(1,3) = 5, v(2,3) = 6$$
$$v(1,2,3) = 7$$

Is the core empty?

How much higher does $v(1,2,3)$ need to be for it not to be empty?

Is the Core Empty?

Simple Games: a game is called simple if $v(S) \in \{0,1\}$ for all $S \subseteq N$ and it is monotone (adding players to a coalition does not decrease the value).

Coalitions with value 1 are winning;

those with value 0 are losing.

A player is called a veto player if she is a member of every winning coalition (can't win without her).

Example

Here is an example 3-player game

$$v(1) = v(2) = v(3) = 0;$$
$$v(1,2) = 0, v(1,3) = 1, v(2,3) = 1$$
$$v(1,2,3) = 1$$

Are there veto players?

Theorem: let $\mathcal{G} = \langle N, v \rangle$ be a simple game; then $Core(\mathcal{G}) \neq \emptyset$ iff $\mathcal{G}$ has veto players.

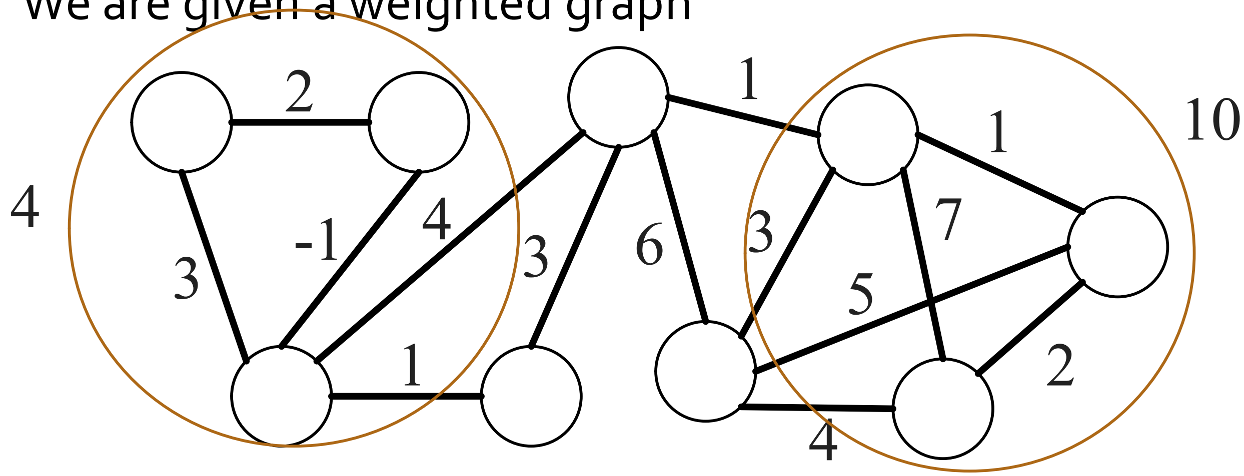
In fact, the core has precisely the vectors that distribute the payoff only among veto players.

If we give positive payoff to a non-veto player, then the remaining players can deviate together.

If we give positive payoff only to veto players, then since any winning coalition must contain all veto players, there is no beneficial deviation.

Induced Subgraph Games

- We are given a weighted graph



- Players are nodes; value of a coalition is the value of the total edge weights in the subgraph.
- Models: collaboration networks, group formation.

Theorem: the core of an induced subgraph game is not empty iff the graph has no negative cut.

Proof: We will show first that if there is no negative cut, then the core is not empty.

Consider the payoff division that assigns each node half the value of the edges connected to it

$$\phi_i = \frac{1}{2}\sum_{j \in N} w(i, j)$$

Need to show that $\phi(S) \geq v(S)$ for all $S \subseteq N$.

$$\phi(S) = \sum_{i \in S} \phi_i = \sum_{i \in S} \sum_{j \in N} \frac{1}{2} w(i,j)$$

$$= \sum_{i \in S} \sum_{j \in S} \frac{1}{2} w(i,j) + \sum_{i \in S} \sum_{j \in N \setminus S} \frac{1}{2} w(i,j)$$

$$= v(S) + \frac{1}{2} Cut(S, N \setminus S)$$

Since there are no negative cuts, the last expression is at least $v(S)$.

Note: Efficiency, i.e., $\phi(N) = v(N)$, is trivial from the above (just set $S = N$). Also, non-negativity of payoff vector holds because there are no negative cuts.

Now, suppose that there is some negative cut; i.e., there is some $S \subseteq N$ such that

$$\sum_{i \in S} \sum_{j \in N \setminus S} w(i,j) < 0$$

Take any $\vec{x}$ satisfying efficiency; then

$$\begin{aligned} x(S) + x(N \setminus S) = \sum_{i \in N} x_i &= v(N) \\ &= \phi(N) \\ &= \phi(S) + \phi(N \setminus S) \end{aligned}$$

where again $\phi_i = \sum_{j \in N} \frac{1}{2} w(i,j)$

Therefore:

$$\begin{aligned}
x(S) - v(S) + x(N \setminus S) - v(N \setminus S) = \\
\phi(S) - v(S) + \phi(N \setminus S) - v(N \setminus S) = \\
\sum_{i \in S} \sum_{j \in N \setminus S} \frac{1}{2} w(i,j) + \sum_{i \in N \setminus S} \sum_{j \in S} \frac{1}{2} w(i,j) = \\
Cut(S, N \setminus S) < 0
\end{aligned}$$

So, it is either the case that $x(S) < v(S)$ or $x(N \setminus S) < v(N \setminus S)$; hence x cannot be in the core.

The payoff division ϕ_i is special: it is in fact the **Shapley value** for induced subgraph games.

Axiomatic Approaches to Fairness

A Normative Approach to Justice

Strategic Considerations

- Nash equilibria
- Core outcomes
- Mechanism design

Fairness Considerations

- Shapley value
- Nash bargaining solution
- Fair allocation of goods

"I want what's fair!"
– Harvey Dent

Question: given a cooperative game $v: 2^N \to \mathbb{R}_+$, how should the revenue $v(N)$ be **fairly** divided?

What properties should such a method satisfy?

Efficiency

- $\sum_{i \in N} \phi_i = v(N)$

Symmetry

- Symmetric players are paid equally.

Dummy

- Dummy players aren't paid.

Linearity

- $\phi_i(\mathcal{G}_1) + \phi_i(\mathcal{G}_2) = \phi_i(\mathcal{G}_1 + \mathcal{G}_2)$
- $\phi_i(a \cdot \mathcal{G}_1) = a \cdot \phi_i(\mathcal{G}_1)$

The Shapley Value

The Shapley Value

Given a player i, and a set $S \subseteq N$, the marginal contribution of i to S is

$$m_i(S) = v(S \cup \{i\}) - v(S)$$

How much does i contribute by joining S?

Given a permutation $\sigma \in \Pi(N)$ of players, let the predecessors of i in σ be

$$P_i(\sigma) = \{j \in N \mid \sigma(j) < \sigma(i)\}$$

We write $m_i(\sigma) = m_i(P_i(\sigma))$

The Shapley Value

Suppose that we choose an ordering of the players uniformly at random. The Shapley value of player i is

$$Sh_i = \mathbb{E}[m_i(\sigma)] = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i(\sigma)$$

$q = 50$

$$\overbrace{14 \quad 6 \quad 1 \quad 9 \quad 12 \quad 7}^{49} \quad 4$$

Computing the Shapley Value

We are given a WVG $w_1 = 1, w_2 = 1, w_3 = 3, w_4 = 4; q = 5$
Compute the Shapley value of all players

Player 1 can only be pivotal when he is preceded by players whose combined weight is exactly 4.

Either preceded by $\{2,3\}$ or $\{4\}$

$$\left.\begin{array}{cccc} 2 & 3 & 1 & 4 \\ 3 & 2 & 1 & 4 \\ 4 & 1 & 2 & 3 \\ 4 & 1 & 3 & 2 \end{array}\right\} = 4 \text{ permutations out of } 4! = 24 \text{ total} \Rightarrow \frac{4}{24} = \frac{1}{6}$$

Computing the Shapley Value

We are given a WVG $w_1 = 1, w_2 = 1, w_3 = 3, w_4 = 4; q = 5$
Compute the Shapley value of all players

By symmetry, $Sh_2 = Sh_1 = \frac{1}{6}$

Computing the Shapley Value

We are given a WVG $w_1 = 1, w_2 = 1, w_3 = 3, w_4 = 4; q = 5$
Compute the Shapley value of all players

Player 4 is pivotal always, unless she is last or first.

$$\Pr_{\sigma \sim U(\Pi(N))}[4 \text{ is 2nd or 3rd in } \sigma] = \frac{1}{2}$$

So $Sh_4 = \frac{1}{2}$

Computing the Shapley Value

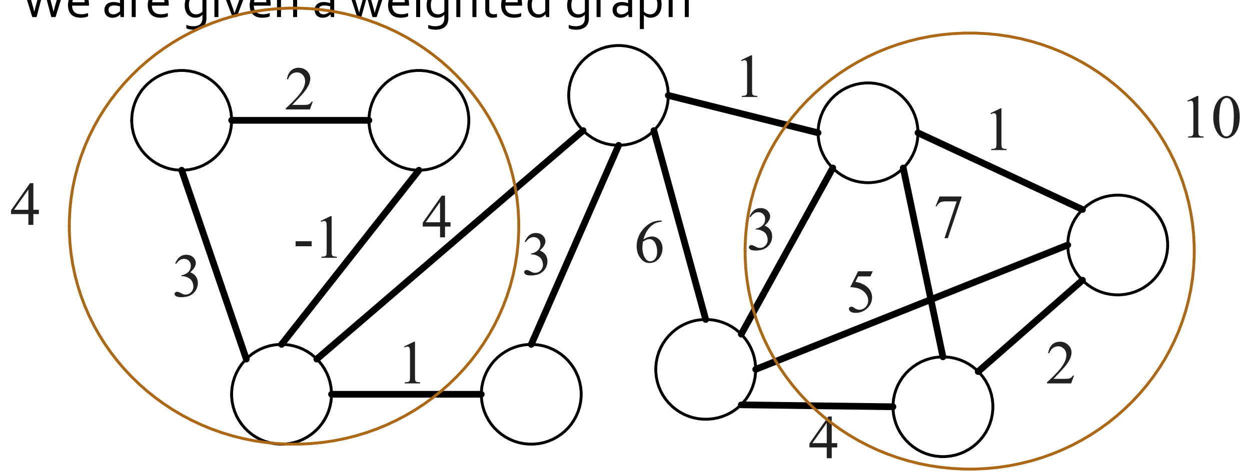
We are given a WVG $w_1 = 1, w_2 = 1, w_3 = 3, w_4 = 4; q = 5$
Compute the Shapley value of all players

By efficiency

$$Sh_3 = 1 - Sh_1 - Sh_2 - Sh_4 = 1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{2} = \frac{1}{6}$$

Induced Subgraph Games

- We are given a weighted graph



- Players are nodes; value of a coalition is the value of the total edge weights in the subgraph.
- Models: collaboration networks, group formation.

Shapley Value in Induced Subgraph Games

Theorem: In an induced subgraph game, $\phi_i = \frac{1}{2}\sum_{j \in N} w(i,j)$.

For every player i, her marginal contribution to a set $S \subseteq N \setminus \{i\}$ equals $\sum_{j \in S} w(i,j)$.

For every player $j \neq i$, let $I(j \in P_i(\sigma))$ be an indicator random variable that equals 1 if j appears before i in σ, and 0 otherwise.

Shapley Value in Induced Subgraph Games

The Shapley value of i is

$\mathbb{E}_\sigma[v(P_i(\sigma) \cup \{i\}) - v(P_i(\sigma))] = \mathbb{E}_\sigma[\sum_{j \in P_i(\sigma)} w(i,j)]$

$= \mathbb{E}_\sigma[\sum_{j \in N \setminus \{i\}} I\big(j \in P_i(\sigma)\big) \cdot w(i,j)]$

$= \sum_{j \in N \setminus \{i\}} \mathbb{E}_\sigma[I\big(j \in P_i(\sigma)\big) \cdot w(i,j)]$ (by linearity of expectation)

$= \sum_{j \in N \setminus \{i\}} w(i,j) \cdot \mathbb{E}_\sigma[I\big(j \in P_i(\sigma)\big)] = \sum_{j \in N \setminus \{i\}} w(i,j) \cdot \frac{1}{2}$

Example

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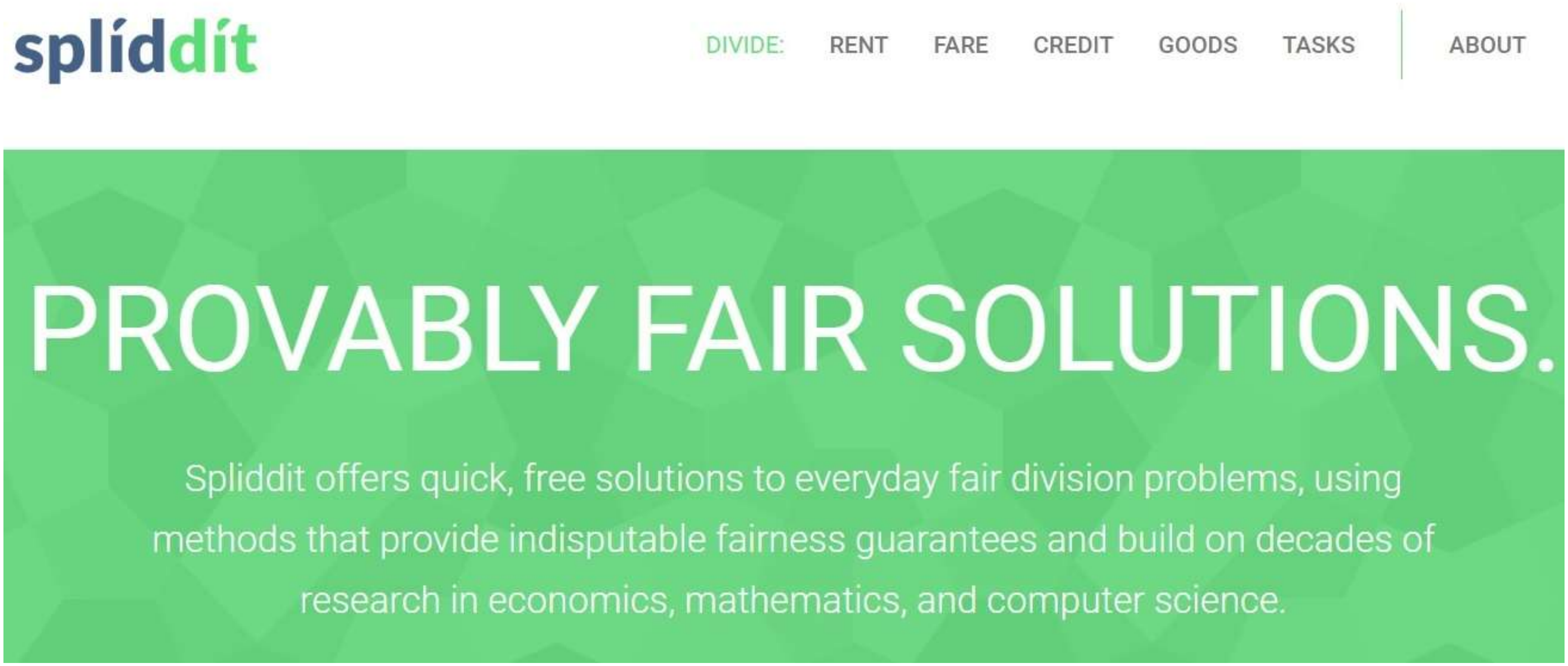
What are the Shapley values?

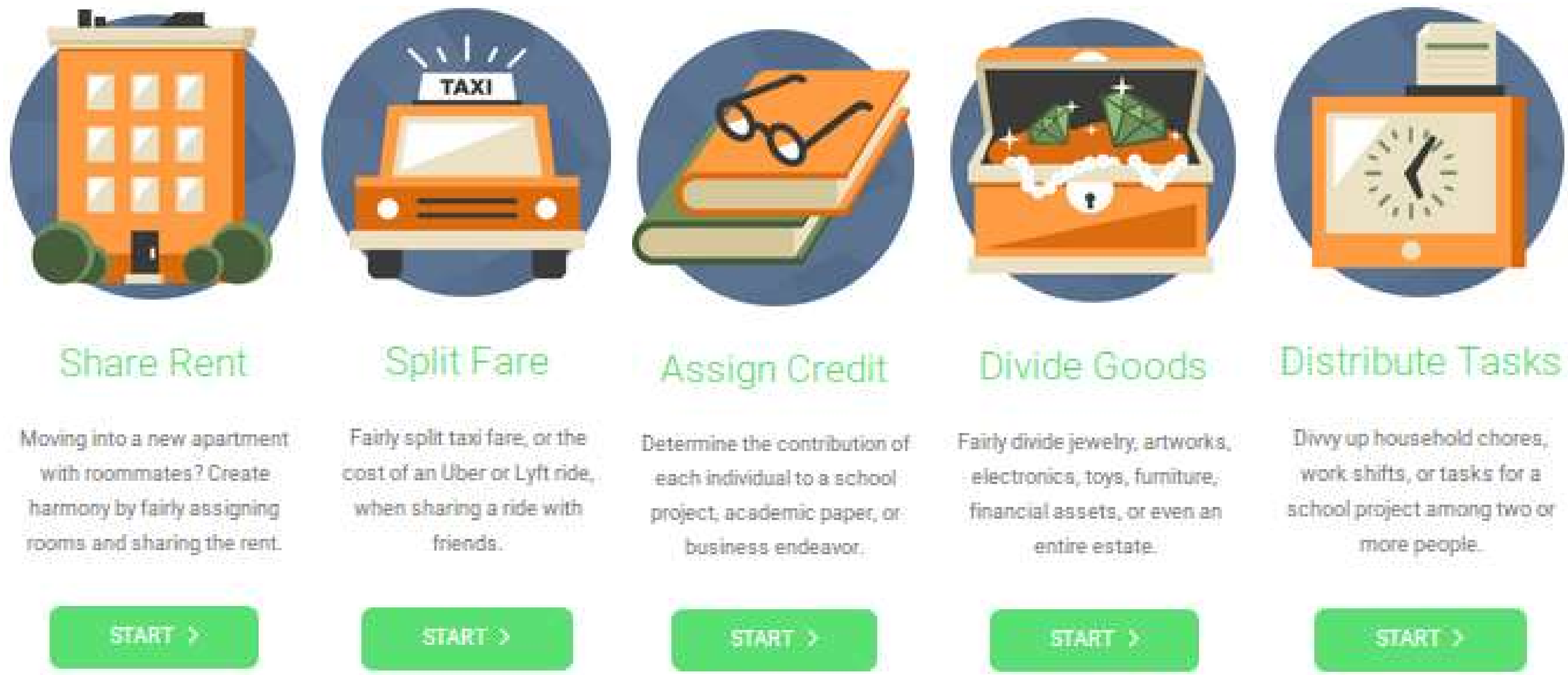
The Shapley Value

Efficiency	• $\sum_{i \in N} \phi_i = v(N)$
Symmetry	• Symmetric players are paid equally.
Dummy	• Dummy players aren't paid.
Linearity	• $\phi_i(\mathcal{G}_1) + \phi_i(\mathcal{G}_2) = \phi_i(\mathcal{G}_1 + \mathcal{G}_2)$ • $\phi_i(a \cdot \mathcal{G}_1) = a \cdot \phi_i(\mathcal{G}_1)$

The Shapley value satisfies all of the above.



http://spliddit.org


TAXI
Share Rent
Moving into a new apartment with roommates? Create harmony by fairly assigning rooms and sharing the rent.
START
Split Fare
Fairly split taxi fare, or the cost of an Uber or Lyft ride, when sharing a ride with friends.
START
Assign Credit
Determine the contribution of each individual to a school project, academic paper, or business endeavor.
START
Divide Goods
Fairly divide jewelry, artworks, electronics, toys, furniture, financial assets, or even an entire estate.
START
Distribute Tasks
Divvy up household chores, work shifts, or tasks for a school project among two or more people.
START

Spliddit's fare calculator helps to fairly split taxi fares between multiple passengers. The fare calculator can also be used to divide the fares for ride-sharing services including Uber and Lyft. Simply enter the address you'd like to be picked up from, as well as the addresses of all the passengers, and we'll compute the results in seconds. You can use the fare calculator before riding, and Spliddit will use TaxiFareFinder to estimate fares. For the most accurate split, use the calculator at the end of your ride and enter the actual total fare.

http://spliddit.org/apps/fare

(unfortunately, this app currently does not work)

Algorithm Overview

We first compute the fares between each pair of addresses using TaxiFareFinder. We use these estimates to calculate the fare of a hypothetical ride to every subset of destinations, thereby obtaining the cost of every subset of passengers. The payment of an individual passenger is her Shapley value: her average (roughly speaking) marginal contribution to the cost of any subset of other passengers. Assuming the entire fare must be split, the Shapley Value is provably the unique method that satisfies marginalism together with another basic property called anonymity. When the estimated fare is different from the actual fare, we simply scale the individual payments to match the actual fare; this provably preserves all the guarantees.

Reference: Shapley Value (Wikipedia)

http://spliddit.org/apps/fare

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The Shapley Value

Theorem
The only value satisfying efficiency, linearity, dummy, and symmetry is the Shapley value.