Week 11: Committee Voting

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Voting

Selecting a public outcome, shared by everyone.

Voting

- Types of ballots (i.e., input):
  - Ranking: Submit an ordering of the candidates
  - Score: Submit a number for each candidate
  - Approval: Submit a subset of approved candidates
    - Advantages: Simple, less cognitive effort required
    - Disadvantages: Does not allow refined or combinatorial preferences

- Output of voting:
  - Single-winner (e.g., president of student council)
  - Multiwinner/committee (e.g., student council committee, places to visit on a family trip, dishes to serve in a reunion party)
  - Ranked committee (e.g., president, vice president, and secretary)

Approval Committee Voting

- Voters $N = \{1, 2, \ldots, n\}$
- Candidates C, where $|C| = m$
- Voter i approves a set of candidates $A_i \subseteq C$
- We want to choose a committee W of size k, where $k \leq m$ is given
- Voter i's utility is $u_i(W) = |A_i \cap W|$

- Social welfare: Total number of approvals obtained by members of the committee ("excellence")
- Coverage: Number of voters who approve at least one committee member ("diversity")
- If a voter approves a candidate, we say that the candidate covers the voter.

AV and CC

- Approval Voting (AV): Select a committee maximizing social welfare
- Chamberlin–Courant (CC): Select a committee maximizing coverage
- **Example:**
  - $n = m = 6$, $k = 3$
  - $A_1 = \{a, b, c, d, e\}$
  - $A_2 = \{b, c, d, e, f\}$
  - $A_3 = \{a, b, c, d\}$
  - $A_4 = \{a, b, c\}$
  - $A_5 = A_6 = \{e, f\}$

  - b, c, e: 4 votes, a, d, f: 3 votes
  - AV returns $\{b, c, e\}$
  - CC returns, for example, $\{a, b, e\}$ (covers all voters)

Beyond AV/CC

- Let's consider a more extreme example:
  - $n = 301$, $m = 5$, $k = 3$
  - $A_1 = A_2 = \cdots = A_{200} = \{a, b, c\}$
  - $A_{201} = A_{202} = \cdots = A_{300} = \{d\}$
  - $A_{301} = \{e\}$

  - AV returns $\{a, b, c\}$
  - CC returns, e.g., $\{a, d, e\}$
  - Neither feels "proportional"...
  - ... a more proportional committee would be, e.g., $\{a, b, d\}$

- Intuition of proportionality: A sufficiently large group of voters that agrees on sufficiently many candidates should be correspondingly satisfied in the committee.

Justified Representation

- There are n voters and k committee slots, so a group of n/k voters "deserves" one slot.
- **First attempt:** For a group of voters $S \subseteq N$ such that $|S| \geq n/k$ and $|\bigcap_{i \in S} A_i| \geq 1$, we have $|\left(\bigcap_{i \in S} A_i\right) \cap W| \neq \emptyset$.
- Unfortunately, this cannot always be satisfied ...
  - $n = m = 4$, $k = 2$, so $n/k = 2$
  - $A_1 = \{a, b\}, A_2 = \{b, c\}, A_3 = \{c, d\}, A_4 = \{d, a\}$
  - $S = \{1, 2\}$ demands that we pick b
  - Similarly, we must pick c, d, a, but this exceeds the committee size.
- Call a group of voters $S \subseteq N$ such that $|S| \geq n/k$ and $|\bigcap_{i \in S} A_i| \geq 1$ a cohesive group.
- Justified representation (JR): For any cohesive group of voters $S \subseteq N$, there exists $i \in S$ such that $|A_i \cap W| \neq \emptyset$.
- No cohesive group should go unrepresented!

Justified Representation

- AV may fail JR.
  - $n = 300$, $k = 3$
  - $A_1 = A_2 = \cdots = A_{200} = \{a, b, c\}$
  - $A_{201} = A_{202} = \cdots = A_{300} = \{d\}$
  - AV chooses $\{a, b, c\}$
  - $n/k = 100$, so the group of voters $\{201, 202, \ldots, 300\}$ is cohesive, but goes unrepresented in the AV committee!
- CC always satisfies JR.
  - Suppose for contradiction that it does not.
  - Let S be a cohesive group of voters that is unrepresented by the CC committee W, and let x be a candidate approved by all voters in S.
  - Consider the marginal contribution of each $w \in W$ to the coverage. This is the number of voters who approve w but no one else in W.
  - Since the coverage of W is $< n$, the marginal contribution of some $w^* \in W$ is less than n/k.
  - Remove w^* and add x to obtain higher coverage.

GreedyCC

- Computing a CC committee is NP-hard (by a reduction from SET COVER)
- Fortunately, there is a greedy variant, which runs in polynomial time.
- GreedyCC:
  - Start with an empty set of candidates.
  - In each step, choose a candidate that covers as many uncovered voters as possible.
  - Repeat this until k candidates have been chosen.
- **Example:**
  - $n = 6,\ k = 3$
  - $A_1 = \{a, d\}, A_2 = \{b, d\}, A_3 = \{c, d\}, A_4 = \{a\}, A_5 = \{b\}, A_6 = \{c\}$
  - GreedyCC returns, e.g., $\{a, b, d\}$.
  - Coverage is worse than CC committee $\{a, b, c\}$.
- GreedyCC satisfies JR (see assignment)

Is JR Sufficient?

- Is JR sufficient?
  - $n = 100$, $k = 10$
  - $A_1 = A_2 = \cdots = A_{50} = \{a_1, a_2, \ldots, a_{10}\}$
  - $A_{51} = A_{52} = \cdots = A_{100} = \{b_1, b_2, \ldots, b_{10}\}$
- Does $W = \{a_1, a_2, \ldots, a_{10}\}$ satisfy JR?
- No. Voters $51, 52, \ldots, 60$ form a cohesive group that is unrepresented.
- Does $W = \{a_1, a_2, \ldots, a_9, b_1\}$ satisfy JR?
- Yes.
- But $\{a_1, a_2, \ldots, a_5, b_1, b_2, \ldots, b_5\}$ feels much more "proportional"!

Extended Justified Representation

- For a positive integer t, call a group of voters $S \subseteq N$ such that $|S| \geq t \cdot n/k$ and $|\bigcap_{i \in S} A_i| \geq t$ a t-cohesive group.
- The previous definition of cohesive group is when $t = 1$.

- Extended Justified representation (EJR): For any positive integer t and any t-cohesive group of voters $S \subseteq N$, there exists $i \in S$ such that $|A_i \cap W| \geq t$.

- This fixes the problem with JR.
  - $n = 100$, $k = 10$
  - $A_1 = A_2 = \cdots = A_{50} = \{a_1, a_2, \ldots, a_{10}\}$
  - $A_{51} = A_{52} = \cdots = A_{100} = \{b_1, b_2, \ldots, b_{10}\}$
  - $W = \{a_1, a_2, \ldots, a_9, b_1\}$ fails EJR: Take $S = \{51, 52, \ldots, 100\}$ and $t = 5$
  - $W = \{a_1, a_2, \ldots, a_5, b_1, b_2, \ldots, b_5\}$ satisfies EJR.

Proportional Approval Voting

- Fix an infinite nonincreasing sequence $s_1, s_2, \ldots$
- Thiele methods: Choose a committee W maximizing the score

$$\sum_{i \in N} \left(s_1 + s_2 + \cdots + s_{u_i(W)}\right)$$

- AV: $s_i = 1$ for all i
- CC: $s_1 = 1$, $s_2 = s_3 = \cdots = 0$
- Proportional Approval Voting (PAV): $s_i = 1/i$ for all i
  - If a voter approves r candidates in the committee, the voter contributes $1 + \frac{1}{2} + \cdots + \frac{1}{r}$ to the score of the committee.
  - $1 + \frac{1}{2} + \cdots + \frac{1}{r}$ is the r-th harmonic number, usually denoted by H_r

Harmonic Numbers

- Why harmonic numbers?
- Harmonic numbers result in a roughly "proportional" committee.
  - $n = 12$, $k = 6$
  - $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = \{a_1, \ldots, a_6\}$
  - $A_7 = A_8 = A_9 = A_{10} = \{b_1, \ldots, b_6\}$
  - $A_{11} = A_{12} = \{c_1, \ldots, c_6\}$
  - AV: $\{a_1, \ldots, a_6\}$
  - CC: Any committee containing at least one candidate from each of the three "categories"
  - Marginal contribution to the score of candidates from each category
  - a_i : 6, 3, 2, 1.5, 1.2, 1
  - b_i : 4, 2, 1.3, 1, 0.8, 0.7
  - c_i : 2, 1, 0.7, 0.5, 0.4, 0.3
  - PAV: Choose 3 of the a_i's, 2 of the b_i's, and 1 of the c_i's

PAV and Nash

- AV = utilitarian
- CC ≈ egalitarian
- PAV ≈ Nash
  - Maximize $\sum_{i \in N} H_{u_i(W)}$
  - **Fact:** $H_r \approx \ln r$ for each positive integer r
  - Maximizing $\sum_{i \in N} \ln(u_i(W))$ is the same as maximizing $\prod_{i \in N} u_i(W)$, i.e., maximizing the Nash welfare!

- Can we use Nash instead of PAV?
- Not a good idea!
  - $n = 301$, $k = 3$
  - $A_1 = A_2 = \cdots = A_{200} = \{a, b, c\}$
  - $A_{201} = A_{202} = \cdots = A_{300} = \{d\}$
  - $A_{301} = \{e\}$
  - Nash returns, e.g., $\{a, d, e\}$

PAV and EJR

- **Theorem:** PAV satisfies EJR.
- **Proof idea:**
  - Suppose for contradiction that $u_i(W) < t$ for all voters i belonging to some t-cohesive group S.
  - There is a candidate x which is approved by all members of S but not included in W.
  - By adding x to W, the PAV score increases by at least $(1/t) \cdot (tn/k) = n/k$.
  - Need to show that there is a candidate $y \in W$ such that by removing y from $W \cup \{x\}$, the PAV score decreases by less than n/k.
- PAV is NP-hard to compute.
- A greedy variant of PAV does not even satisfy JR.
- Can we satisfy EJR in polynomial time?

Method of Equal Shares

- Method of Equal Shares (MES):
  - Each voter has a budget of k/n.
  - Each candidate costs 1; the voters who approve this candidate have to “pool” their money to add this candidate to the committee.
  - Start with an empty committee.
  - In each round, we want to add a candidate whose approved voters have a total budget of ≥ 1 left.
  - If there are several such candidates, choose one such that the maximum amount that any agent has to pay is minimized.
  - If no more candidate can be afforded but the committee still has size $< k$, fill in the rest of the committee using some tie-breaking criterion (e.g., by maximizing approval score).

Method of Equal Shares

- **Example:**
  - $n = 8, k = 3$
  - $A_1 = A_2 = A_3 = \{a, b\}$
  - $A_4 = A_5 = \{c, d\}$
  - $A_6 = A_7 = \{a, c\}$
  - $A_8 = \{b, d\}$
- Each voter starts with a budget of $3/8$.
- a is chosen first. Each of voters $1, 2, 3, 6, 7$ pays $1/5$, and has budget $3/8 - 1/5 = 7/40$ left.
- b is not affordable, c would require some voter to pay $13/40$, while d would require some voter to pay $1/3$.
- Since $13/40 < 1/3$, c is chosen second. Voters $6, 7$ pay $7/40$ each (and have 0 left), while voters $4, 5$ pay $13/40$ each (have $1/20$ left).
- No more candidate is affordable, so b is chosen third by if we do the tie-breaking by approval score.

Method of Equal Shares

- MES never chooses more than k candidates.
  - Total budget of all n voters is $(k/n) \cdot n = k$, and each candidate costs 1.
- **Theorem:** MES satisfies EJR (and can be implemented in polytime).
- **Proof:**
  - Suppose for contradiction that $u_i(W) < t$ for all voters i belonging to some t-cohesive group S.
  - When MES stops, some voter $i \in S$ must have budget $< \frac{k}{tn}$ left. (Otherwise, the voters in S have budget $\geq |S| \cdot \frac{k}{tn} \geq 1$ and should have bought a candidate that they all approve.)
  - i has used budget $> \frac{k}{n} - \frac{k}{tn} = \frac{(t-1)k}{tn}$, so for some chosen committee member, i paid more than $\frac{1}{t-1} \cdot \frac{(t-1)k}{tn} = \frac{k}{tn}$.

Method of Equal Shares

- **Proof (cont.):**
  - Consider the first committee member x such that some voter in S paid more than $\frac{k}{tn}$ for it.
  - Before x was added, each voter in S has $\leq t-1$ approved candidates, and paid $\leq \frac{k}{tn}$ for each of them.
  - Thus, each voter in S has budget at least $\frac{k}{n} - (t-1) \cdot \frac{k}{tn} = \frac{k}{tn}$ remaining.
  - Since $|S| \geq \frac{tn}{k}$, the voters in S could afford a commonly approved candidate by paying $\leq \frac{k}{tn}$ each.
  - No voter in S should have paid more than $\frac{k}{tn}$ for x, a contradiction!

Summary

	JR	EJR	Polytime
AV (maximizes welfare)	✗	✗	✓
CC (maximizes coverage)	✓	✗	✗
GreedyCC	✓	✗	✓
PAV	✓	✓	✗
MES	✓	✓	✓

- Participatory budgeting: A generalization of committee voting where the "candidates" (i.e., projects) may have unequal costs.

Participatory Budgeting

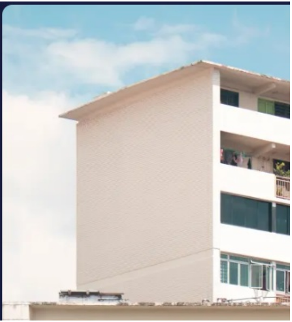
Participatory Budgeting & Citizen Design in Town Councils

WHAT IS IT?

Participatory Budgeting (PB) is a process whereby a community decides how to spend a portion of public budget.[1]

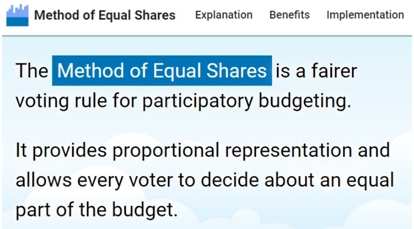
The process can be used by a Town Council to engage its citizens in developing ideas, deliberate on them, and vote on how the budget is used. A portion of the discretionary budget for estate improvement can be earmarked for this purpose as a social experiment.

We can start with a specific set of blocks or HDB estate within a Town Council if we can secure the support of a sponsoring MP.



http://futurereadysociety.sg/participatory-budgeting-citizen-design-in-town-councils

MES in Participatory Budgeting



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http://equalshares.net/