CS5461 Assignment 5

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1. (a) Yes.
   (b) Yes.
   (c) No.
   (d) No.
2. Recall that the core is defined as a set of vectors $\mathbf{x}$ such that

$$\sum_{i\in N} x_i = v(N),$$
$$\sum_{i\in S} x_i \geq v(S), \quad \forall S \subseteq N.$$

   (a) Given the weighted voting game $(1, 2, 3; 4)$, we can rewrite the constraints as

$$x_1 + x_3 \geq 1, \tag{1}$$
$$x_2 + x_3 \geq 1, \tag{2}$$
$$x_1 + x_2 + x_3 = 1, \tag{3}$$
$$x_1, x_2, x_3 \geq 0, \tag{4}$$

   since the winning coalitions are $(1, 3)$, $(2, 3)$ and $(1, 2, 3)$. By (1) we have $x_1 + x_2 + x_3 \geq x_2 + 1$, so by (3) we get $1 \geq x_2 + 1 \implies x_2 \leq 0$, and by (4) we must have $x_2 = 0$. Therefore by (2) we require $x_3 \geq 1$, and by efficiency we need $x_3 = 1$. Finally by (3) we can see that $x_1 = 0$. Therefore the only element in the core is $\boxed{(0, 0, 1)}$.

   (b) The only winning coalitions are $(2, 3)$ and $(1, 2, 3)$, so we can rewrite the constraints as

$$x_2 + x_3 \geq 6,$$
$$x_1 + x_2 + x_3 = 6,$$
$$x_1, x_2, x_3 \geq 0.$$

   Therefore by efficiency we must have $x_1 = 0$, and the core is the set of elements $\boxed{\{(0, t, 6 - t) | 0 \leq t \leq 6\}}$.

   (c) Here we can rewrite the constraints as

$$x_1 + x_2 + x_i \geq 1, \quad i \in \{3, 4, \ldots, 10\},$$
$$x_1 + x_2 + \cdots + x_{10} = 1,$$
$$x_1, x_2, \ldots, x_{10} \geq 0.$$

   Again by efficiency we require all $x_i = 0$ for $i \in \{3, 4, \ldots, 10\}$, and the core is the set of elements $\boxed{\{(t, 1 - t, 0, 0, \ldots, 0) | 0 \leq t \leq 1\}}$.

   (d) Note that $v(\{i\}) = |\{i\}| + 1 = 2$ for any single player i. But we have $x_i \geq v(\{i\})$, so $v(N) = \sum_{i\in N} x_i \geq \sum_{i\in N} v(\{i\}) = 10 \times 2 = 20$, contradicting with $v(N) = |N| + 1 = 11$. Therefore the core is empty.

3. (a) Consider the following game such that for each subset $S \subseteq N = \{1, 2, 3, 4\}$,

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 3, \\ 0 & \text{otherwise,} \end{cases}$$

which is monotone and superadditive. We now justify that its core is empty.

By definition, the winning coalitions are all the sets with 3 or 4 players. Consider any set of 3 players T. There are 4 of them, each of which must satisfy $x(T) \geq v(T) = 1$. We then have

$$\sum_T x(T) = 4x(T) \geq 4,$$

but the left hand side is just $3(x_1 + x_2 + x_3 + x_4) = 3v(N)$ since each player occurs exactly 3 times. Therefore $v(N) \geq 4/3$, contradicting with efficiency $v(N) = 1$. Thus the core is empty, as required.

(b) Consider the following game such that with $N = \{1, 2, 3\}$, we have

$$\begin{aligned} v(\{1\}) = v(\{2\}) = v(\{3\}) &= 0, \\ v(\{1, 2\}) = v(\{1, 3\}) = 1, \quad v(\{2, 3\}) &= 0, \\ v(N) = 1, \quad v(\varnothing) &= 0, \end{aligned}$$

which is monotone and superadditive. We now justify that it is not convex.

Recall that a game is convex if for $S \subseteq T \subseteq N$ and $i \in N \backslash T$, we have

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

However, if we take $S = \varnothing$, $T = \{2\}$, $i = 3$, then by definition,

$$v(S) = 0, \quad v(T) = 1, \quad v(S \cup \{i\}) = 1, \quad v(T \cup \{i\}) = 0,$$

so that

$$v(S \cup \{i\}) - v(S) = 1 - 0 = 1, \quad v(T \cup \{i\}) - v(T) = 0 - 1 = -1.$$

Therefore, this game is not convex, as required.