Week 12: Tournaments

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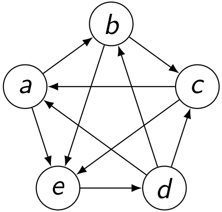
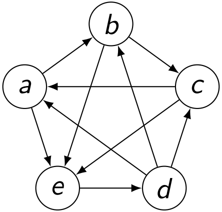
Tournaments



- Tournament $T = (A, \succ)$
- A is the set of alternatives and $\succ$ is the dominance relation.
- For every pair of alternatives, exactly one dominates the other.
- Here, $A = \{a, b, c, d, e\}$ and $a \succ b$, $b \succ c$, $d \succ b$, $e \succ d$, etc.
- Applications: Sports, elections, webpage ranking, biological interactions, ...

Tournaments



- Suppose there are n alternatives in A (here, $n = 5$)
- Outdegree of $x \in A$: Number of alternatives dominated by x
  - a: 2, b: 2, c: 2, d: 3, e: 1
- Condorcet winner: Alternative that dominates all other alternatives (i.e., has outdegree $n - 1$)
- Condorcet loser: Alternative that is dominated by all other alternatives (i.e., has outdegree 0)

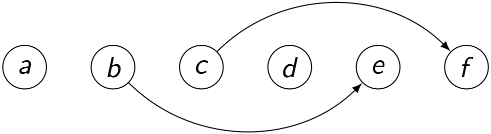
Tournament Solutions

- A tournament solution is a method for choosing the "winners" of any tournament. Formally, it returns a nonempty subset of alternatives from any tournament.
- We require tournament solutions to be invariant under isomorphisms:
  - If $h : A \to A'$ is an isomorphism between two tournaments $T = (A, \succ)$ and $T' = (A', \succ')$, then the subset that a tournament solution S chooses from T' is the image (with respect to h) of the subset that S chooses from T.
  - Let $A = \{a, b, c\}$ and $a \succ b$, $b \succ c$, $a \succ c$.
  - Let $A' = \{d, e, f\}$ and $d \succ e$, $e \succ f$, $d \succ f$.
  - If $S(A) = \{b, c\}$, then $S(A') = \{e, f\}$
  - **Consequence:** In a cyclic tournament of size 3, every tournament solution must select all alternatives.

Tournament Solutions

- Copeland set (CO): Alternatives with the highest outdegree
- Top cycle (TC): Alternatives that can reach every other alternative via a directed path (of any length)
- Uncovered set (UC): Alternatives that can reach every other alternative via a directed path of length ≤ 2
- Banks set (BA): Alternatives that appear as the maximal (i.e., strongest) element of some transitive subtournament that cannot be extended
  - Transitive tournament: The alternatives can be ordered as $a_1, \ldots, a_k$ so that a_i dominates a_j for all $i < j$

Example

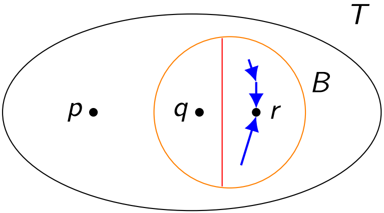


All omitted edges point from right to left.

- Outdegrees:
  - a: 0, b: 2, c: 3, d: 3, e: 3, f: 4
- $CO = \{f\}$
- $TC = \{b, c, d, e, f\}$
- $UC = \{c, d, e, f\}$
- $BA = \{c, d, e, f\}$

Top Cycle

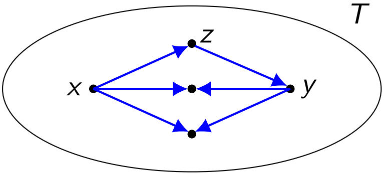
- Equivalent definition of TC:
  - (Unique) smallest nonempty set B of alternatives such that all alternatives in B dominate all alternatives outside B.



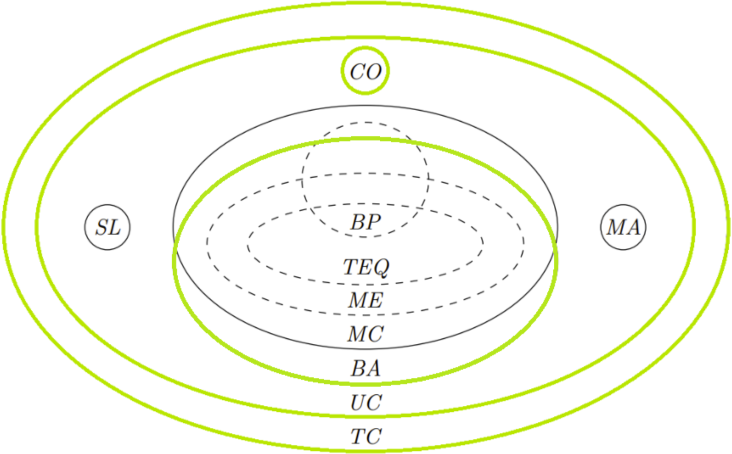
- Proof of equivalence:
  - $p \notin B$ cannot reach $q \in B$, so p does not belong to TC.
  - $q \in B$ can reach $p \notin B$ directly.
  - If $q \in B$ could not reach $r \in B$, all alternatives that could reach r would form a smaller subset in the definition of B, contradiction.

Uncovered Set

- Covering relation: An alternative x covers another alternative y if
  - x dominates y.
  - For any z, if y dominates z, then x also dominates z.
- Strong indicator that x is better than y.
- Equivalent definition of UC:
  - The set of all uncovered alternatives.
  - **Proof:** x can reach y in ≤ 2 steps $\iff$ y does not cover x.



Zoo of Tournament Solutions



Brandt et al. (2016), "Tournament solutions"

Containment Relations

1. $UC \subseteq TC$
   - By definition, if x can reach every other alternative via a path of length ≤ 2, it can reach every other alternative.
2. $CO \subseteq UC$
   - Let $x \in CO$, so x has the highest outdegree among all alternatives.
   - Suppose for contradiction that y covers x.
   - The outdegree of y must be higher than that of x, a contradiction.
3. $BA \subseteq UC$
   - Let $x \in BA$. There is a transitive subtournament T', with x as the maximal element, that cannot be extended.
   - Suppose for contradiction that y covers x.
   - y can extend T', a contradiction.

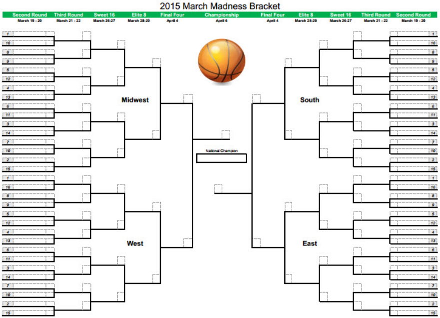
Axioms

- Condorcet-consistency: If there is a Condorcet winner x, then x is uniquely chosen.
- Monotonicity: If x is chosen, then it should remain chosen when it is strengthened against another alternative y (and everything else stays the same).
- CO, TC, UC, and BA all satisfy both of these axioms.

Tournament Solutions

- Trivial ($TRIV$): All alternatives.
- Slater set (SL): Alternatives that are maximal elements in some transitive tournament that can be obtained by inverting as few edges as possible.
- Bipartisan set (BP): Alternatives that are chosen with nonzero probability in the (unique) Nash equilibrium of the zero-sum game formed by the tournament matrix.
- Markov set (MC): Alternatives that stay most often in the "winner-stays" competition corresponding to the tournament.

Knockout Tournaments



- An alternative is said to be a knockout winner if it wins a (balanced) knockout tournament under some bracket.
- Assume that n is a power of 2.

Tournament Fixing Problem

The winner of a given knockout tournament can depend significantly on the initial bracket!

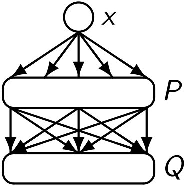
The Tournament Fixing Problem (TFP): Given

- A set of alternatives A
- Information for each pair of alternatives (x, y) about whether x or y would win in a head-to-head matchup ("tournament graph")
- Our favorite alternative

Is there a bracket for a balanced knockout tournament where our favorite alternative wins?

Strong Kings

Let x be a king (i.e., belongs to UC).
Suppose x beats P and loses to Q.



- **Theorem:** If $|P| \geq n/2$, then x is a knockout winner.
- Proof by induction. We will construct the pairing for the first round, so that x survives, and the king and outdegree conditions hold in the next round.
- Base case $n = 2$ is trivial.

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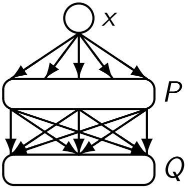

- Inductive step:
  - Find a maximum matching from P to Q.
  - Match x to an arbitrary player in P.
  - Match players within P arbitrarily, and same within Q.
  - If necessary, match the remaining pair from Q to P.
- **Consequence:** Any alternative in CO is a knockout winner.

Midterm 3: Tournaments

- Midterm 3 next week (Nov 13, 6:30–7:50pm)
- Possible questions: For a given tournament
  - Determine the outdegree of each alternative.
  - Is there a Condorcet winner? A Condorcet loser?
  - Determine the Copeland set.
  - Determine the top cycle.
  - Determine the uncovered set.
  - Determine the Banks set.
- Practice questions for tournaments posted on Canvas

Please fill in the student feedback survey!