NATIONAL UNIVERSITY OF SINGAPORE
SCHOOL OF COMPUTING

2nd Midterm Assessment for CS4261/5461

October 23, 2025 Time Allowed: 45 minutes (6:30–7:15pm)

INSTRUCTIONS:

- This paper consists of **four** parts for a total of 40 points.
- This is a **closed book/notes** examination. No calculators or other electronic devices are allowed.
- Write your answers **clearly** in the given space. Justification is required only for question 3(d).
- For questions with a "Don't know" option, the answer "Don't know" guarantees 1 point.

Name: _____

Student Number: _____

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Grader's use only

Part	**Points**
1	
2	
3	
4	
Total	

1. (12 points)

For questions (a), (b), (c), and (d), consider the cooperative game with four players, $N = \{1, 2, 3, 4\}$, such that

$$v(\{1, 2\}) = v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = v(\{2, 3, 4\}) = v(\{1, 2, 3, 4\}) = 1$$

and $v(S) = 0$ for all other coalitions $S \subseteq N$.

(a) (2 points) Recall that a game is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all disjoint $S, T \subseteq N$. Is this game superadditive? (Write "Yes", "No", or "Don't know".)

Answer:

(b) (2 points) Recall that a game is *convex* if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for all $S \subseteq T \subseteq N$ and $i \in N \setminus T$. Is this game convex? (Write "Yes", "No", or "Don't know".)

Answer:

(c) (2 points) Compute the Shapley value of player 1.

Answer:

(d) (2 points) Determine **all** payoff vectors in the core.

Answer:

For question (e), consider the cooperative game with three players, $N = \{1, 2, 3\}$, such that $v(\emptyset) = 0$ and

$$\begin{aligned} &v(\{1\}) = 1, \quad v(\{2\}) = 2, \quad v(\{3\}) = 3, \\ &v(\{1, 2\}) = 4, \quad v(\{1, 3\}) = 5, \quad v(\{2, 3\}) = c, \\ &v(\{1, 2, 3\}) = 7, \end{aligned}$$

where c is a positive real number.

(e) (4 points) Determine **all** (positive real) values of c such that the game is monotone and the core contains exactly one element. (Recall that a game is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$.)

Answer:

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2. (8 points)

(a) (2 points) Find a Nash bargaining solution (i.e., a solution that maximizes the **product** of the two agents' utilities) of the following problem:

$$S = \{(x, y) \in \mathbb{R}^2_{\geq 0} \mid x^2 + 4y^2 = 50\},$$

where $u_1(x) = x$ and $u_2(y) = y$. Answer with the values of x and y.
Answer:

(b) (2 points) Find an egalitarian optimal solution (i.e., a solution that maximizes the **minimum** of the two agents' utilities) of the problem in question (a). Answer with the values of x and y.
Answer:

(c) (2 points) Find a Nash bargaining solution (i.e., a solution that maximizes the **product** of the two agents' utilities) of the following problem:

$$S = \{(x, y) \in \mathbb{R}^2_{\geq 0} \mid x^2 + 4y(x + y) = 1\},$$

where $u_1(x) = x^2$ and $u_2(y) = y$. Answer with the values of x and y.
Answer:

(d) (2 points) Find a utilitarian optimal solution (i.e., a solution that maximizes the **sum** of the two agents' utilities) of the problem in question (c). Answer with the values of x and y.
Answer:

Scratch paper area

3. (12 points) For questions (a), (b), and (c), consider the following matching instance with four students (A, B, C, D) and four hospitals (W, X, Y, Z). The students' preferences are:

$$A : Z \succ W \succ Y \succ X \qquad B : Z \succ Y \succ W \succ X$$

$$C : X \succ Y \succ Z \succ W \qquad D : W \succ Y \succ X \succ Z$$

The hospitals' preferences are:

$$W : A \succ B \succ C \succ D \qquad X : B \succ D \succ A \succ C$$

$$Y : B \succ A \succ C \succ D \qquad Z : D \succ C \succ B \succ A$$

Suppose we run Gale-Shapley with **students proposing** on the instance above.

(a) (2 points) Which hospital is student D matched to?
Answer:

(b) (2 points) Which student is hospital X matched to?
Answer:

(c) (2 points) Is hospital Z matched to its **best** match among all stable matchings? (Write "Yes", "No", or "Don't know".)
Answer:

For question (d), we refer to either a student or a hospital as a **party**. Consider **all** possible instances with four students and four hospitals such that each party has a strict preference over the four parties on the other side. (The instance shown above is one such instance.)

(d) (6 points) Determine the smallest integer k with the following property: For every such instance, every stable matching $\mathcal{P}$, and every (not necessarily stable) matching $\mathcal{Q}$, at most k out of the eight parties strictly prefer its match in $\mathcal{Q}$ to its match in $\mathcal{P}$.
Either give an answer **with justification** or write "Don't know".
Answer:

Scratch paper area. You may continue your answer for question (d) here (or on the back side if needed).

4. (8 points) Consider an instance with three players and five goods, and the following additive valuations:

Player	g_1	g_2	g_3	g_4	g_5
1	0	3	0	6	1
2	3	5	2	0	0
3	0	1	6	2	1

(a) (2 points) What is player 3's maximin share?
Answer:

(b) (2 points) What is the maximum **Nash** welfare among all allocations? (Recall that the Nash welfare is the product of all agents' utilities.)
Answer:

(c) (2 points) Is there an EF1 allocation that gives **only** g_1 to player 1? (Write "Yes", "No", or "Don't know".)
Answer:

(d) (2 points) Is there **more than one** Pareto optimal allocation? (Write "Yes", "No", or "Don't know".)
Answer:

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