CS5461 Assignment 11

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1. (a) The approval counts for candidates a, b, c, d are $3, 4, 3, 2$ respectively, so AV returns the committee $\boxed{\{a, b, c\}}$ only.

   (b) To cover voters 4 and 6 we must include c and d respectively, and adding either a or b could now cover everyone. Thus CC returns either $\boxed{\{a, c, d\}}$ or $\boxed{\{b, c, d\}}$.

   (c) The utilities for voters 1 to 6 are $3, 3, 2, 1, 1, 0$ respectively, so the PAV score is $2H_3 + H_2 + 2H_1 = 2(1 + 1/2 + 1/3) + (1 + 1/2) + 2 = \boxed{43/6}$.

   (d) The initial budget for each voter is $k/n = 3/6 = 1/2$. We pick b first who has the highest approval count. Voters $1, 2, 3, 5$ each pays $1/4$, leaving each of them $1/4$. Then a is not affordable $(3 \times 1/4 < 1)$, c would require voters 1 and 2 to pay $1/4$ each and voter 4 to pay $1/2$, and d is not affordable $(1/4 + 1/2 < 1)$. Thus we pick c. Now neither a nor d is affordable, so we fill the one remaining seat by approval score, and we pick a since $3 > 2$. Thus MES returns the committee $\boxed{\{a, b, c\}}$.

2. We prove by contradiction using a similar argument for CC satisfying JR as in lecture.

   Let S be a cohesive group of voters that is unrepresented by the GreedyCC committee W, and let x be a candidate approved by all voters in S.

   Since GreedyCC always picks a candidate that maximises the number of uncovered voters, the candidate it picks in each round must have at least $|S|$ uncovered supporters, none of which is in S.

   Thus after k rounds, it will cover at least $k|S|$ voters outside S, but there are only $n - |S|$ voters outside S in total.

   By definition of a cohesive group, we require $|S| \geq n/k$, so $k|S| \geq n > n - |S|$, giving a contradiction.

3. Let $q := n/k$ which is an integer as n is divisible by k. As S is t-cohesive, by definition we have $|S| \geq tq$.

   Define $S_j := \{i \in S \mid u_i(W) < j\}$ for each $j \in \{1, 2, \ldots, t\}$. We claim that $|S_j| < jq$. Otherwise, we have $|S_j| \geq jq$ and each S_j shares at least the same $t \geq j$ common approved candidates as S, so S_j is a j-cohesive group. But as W is EJR, that requires $u_i(W) \geq j$ for some $i \in S_j$, contradicting with the definition.

   Recall that the *Iverson bracket* $[P]$ is defined as $[P] = 1$ if P is true and 0 if P is false. By definition, for $x, t \in \mathbb{N}$, we have the 'layer cake representation'

   $$\min(x, t) = \sum_{j=1}^{t} [x \geq j],$$

   since we add 1 for each 'level' $j = 1, 2, \ldots, t$ that x reaches and stop at t.

   By definition, $u_i(W) \in \mathbb{N}$ and $u_i(W) \geq \min(u_i(W), t)$, so we can rewrite the LHS of the required inequality as

$$\begin{aligned}
\frac{1}{|S|}\sum_{i\in S} u_i(W) &\geq \frac{1}{|S|}\sum_{i\in S} \min(u_i(W), t) \\
&= \frac{1}{|S|}\sum_{i\in S}\sum_{j=1}^{t}[u_i(W) \geq j] \\
&= \frac{1}{|S|}\sum_{j=1}^{t}\sum_{i\in S}[u_i(W) \geq j],
\end{aligned}$$

where we exchanged summation which is allowed since both sums are finite.

Now define the set $S'_j := \{i \in S \mid u_i(W) \geq j\}$, which has cardinality precisely

$$|S'_j| = |\{i \in S \mid u_i(W) \geq j\}| = \sum_{i\in S}[u_i(W) \geq j],$$

and by definition we also have $S'_j = S \setminus S_j$, so $|S'_j| = |S| - |S_j|$.

Thus we have

$$\begin{aligned}
\frac{1}{|S|}\sum_{j=1}^{t}\sum_{i\in S}[u_i(W) \geq j] &= \frac{1}{|S|}\sum_{j=1}^{t}|\{i \in S \mid u_i(W) \geq j\}| \\
&= \frac{1}{|S|}\sum_{j=1}^{t}(|S| - |S_j|) \\
&= \frac{1}{|S|}\left(t|S| - \sum_{j=1}^{t}|S_j|\right),
\end{aligned}$$

and since $|S_j| < jq$ and $|S| \geq tq \implies q/|S| \leq 1/t$ as shown before, we finally get

$$\begin{aligned}
\frac{1}{|S|}\left(t|S| - \sum_{j=1}^{t}|S_j|\right) &> t - \frac{1}{|S|}\sum_{j=1}^{t} jq \\
&= t - \frac{q}{|S|}\frac{t(t+1)}{2} \\
&\geq t - \frac{1}{t}\frac{t(t+1)}{2} \\
&= \frac{2t}{2} - \frac{t+1}{2} = \frac{t-1}{2},
\end{aligned}$$

which is the required inequality.

[**AI Tool Declaration**: I used ChatGPT 5 to formalise my proof idea and improve the expression for Question 3 only. I am responsible for the content and quality of the submitted work.]