CS4261/5461: Assignment for Week 9

Due: Sunday, 26th Oct 2025, 11:59 pm SGT.

Please upload PDFs containing your solutions (hand-written & scanned, or typed) by 26th Oct, 11:59 pm to `Assignments/Assignment9/Submissions`. Name the file **Assignment9_SID.pdf**, where SID should be replaced by your student ID.

You may discuss the problems with your classmates or read material online, but you should write up your solutions on your own. Please note the names of your collaborators or online sources in your submission; failure to do so would be considered plagiarism.

Note: For this assignment, justification is required for all questions.

1. (7 points, graded for correctness) Consider a cake-cutting instance with four agents and the following density functions:

$$f_1(x) = 1 \text{ for all } x \in [0,1];$$

$$f_2(x) = \begin{cases} 2 & \text{if } x \in [0,1/2]; \\ 0 & \text{otherwise;} \end{cases}$$

$$f_3(x) = \begin{cases} 5 & \text{if } x \in [0,1/5]; \\ 0 & \text{otherwise;} \end{cases}$$

$$f_4(x) = \begin{cases} 8x-4 & \text{if } x \in [1/2,1]; \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (2 points) What is agent 4's value for the interval $[1/3, 2/3]$?

   (b) (2 points) What is the output of the Dubins–Spanier protocol?

   (c) (3 points) What is the output of the Even–Paz protocol?

2. (1 point) Is the output of the cut-and-choose protocol for two agents always Pareto optimal? Prove or give a counterexample.

3. (1 point) Consider the following approximate envy-free protocol, where we replace the value 1/3 from the protocol covered in lecture by 1/6. Specifically:

   - Step 1: The referee moves a knife over the remaining cake starting from the left.
   - Step 2: When the piece of cake to the left of the knife is worth 1/6 to some agent, that agent shouts "Stop!" and leaves the procedure with that piece. Go back to Step 1.
   - Step 3: Suppose the knife reaches the end of the cake.
     - If all of the cake has been allocated, any remaining agent receives no cake.
     - If some cake is still unallocated:
       * If there are still remaining agents, the remaining cake is given to one of the remaining agents (chosen arbitrarily).
       * Otherwise, the remaining cake is given to the agent who received the last piece.

   Determine the smallest constant c such that, in the output of this protocol, any agent always envies any other agent by at most c.