# CS4261/5461: Assignment for Week 8 Solutions

Due: Sunday, 19th Oct 2025, 11:59 pm SGT.

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1. (a) To maximize utilitarian welfare, each good must be allocated to a player with the highest value for it. Hence, player 1 receives $g_1$, player 2 receives $g_2$, $g_3$, and player 3 receives $g_4$. The maximum utilitarian welfare is $40 + 40 + 30 + 50 = \boxed{160}$.

   (b) By giving $g_1$ to player 1, $g_2$ and $g_3$ to player 2, and $g_4$ to player 3, the egalitarian welfare is 40. In order to achieve a higher egalitarian welfare, player 1 needs at least two goods, player 2 needs at least two goods, and player 3 needs at least one good. This means that five goods are needed in total, which is impossible because there are only four goods. Hence, the maximum egalitarian welfare is $\boxed{40}$.

   (c) $\boxed{\text{No.}}$ For example, giving $g_2$ to player 2 instead makes player 2 better off and no other player worse off.

   (d) $\boxed{\text{Yes.}}$ Player 1's maximin share is 30 (from the partition $(\{g_1\}, \{g_2\}, \{g_3, g_4\})$), player 2's maximin share is 30 (from the partition $(\{g_1, g_2\}, \{g_3\}, \{g_4\})$), and player 3's maximin share is 20 (from the partition $(\{g_1, g_2\}, \{g_3\}, \{g_4\})$). In the allocation $A$, player 1 gets utility 40, player 2 gets utility 30, and player 3 gets utility 20. Hence, all players get at least their maximin share.

2. Start with an MMS allocation $A = (A_1, \ldots, A_n)$. If $A$ is Pareto-optimal, then we are done; if not, then there exists some allocation $A'$ that Pareto-dominates $A$. Note that under $A'$ all players have weakly higher utilities, and at least one player has a strictly higher utility. Thus this player is still guaranteed at least his/her MMS, and $A'$ remains an MMS allocation. We can repeat this operation until we obtain a Pareto-optimal allocation—the process must end since there are only a finite number of allocations and the utilitarian welfare strictly increases with each operation.

3. Yes. Recall from the EF1 proof that if $i$ is ahead of $j$ in the round-robin ordering, then $i$ does not envy $j$, while if $i$ is behind $j$, then $i$'s envy towards $j$ can be eliminated by removing the first good that $j$ picks. Hence, for each $j \in N$, if $A_j$ is nonempty, we can choose $g_j$ to be the first good that $j$ picks.