CS4261/5461: Assignment for Week 6 Solutions

Due: Wednesday, 1st Oct 2025, 11:59 pm SGT.

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1. (a)
   - The player with weight 4 is pivotal unless she is in the first or last position, so her Shapley value is $2/4 = 1/2$.
   - The player with weight 1 is pivotal if she is in the second position and the player with weight 4 is in the first position (2 permutations), or she is in the third position and the player with weight 4 is in the last position (2 permutations), so her Shapley value is $4/24 = 1/6$.
   - By efficiency and symmetry, the Shapley value of each player with weight 2 is $\frac{1}{2}(1 - 1/2 - 1/6) = 1/6$.

   Hence, the Shapley value is $\boxed{(1/6, 1/6, 1/6, 1/2)}$ for the players with weights $1, 2, 2, 4$, respectively.

   (b) Note that each boy always contributes 1 to any coalition and each girl always contributes 2 to any coalition. Hence, the Shapley value is $\boxed{\text{1 for each boy and 2 for each girl}}$.

   (c) Observe that if players 1 and 2 are in the first two positions (in some order), the pivotal player is the player in the third position; otherwise, the pivotal player is either player 1 or 2, whoever comes later.

   The fraction of permutations such that players 1 and 2 are in the first two positions is $\frac{1}{\binom{7}{2}} = \frac{1}{21}$. So by symmetry, the Shapley value of each player besides 1 and 2 is $\frac{1}{5} \cdot \frac{1}{21} = \frac{1}{105}$. By efficiency and symmetry, the Shapley value of players 1 and 2 is $\frac{1}{2}\left(1 - \frac{1}{21}\right) = \frac{10}{21}$ each.

   Hence, the Shapley value is $\boxed{\frac{10}{21} \text{ for players 1 and 2, and } \frac{1}{105} \text{ for each remaining player}}$.

2. (a) No. For example, the players with weight 1 and 4 can together get a payoff of 1 by themselves. In fact, there are no veto players in this game, so the core is empty.

   (b) Yes. Any coalition of a boys and b girls can make $a + 2b$ on their own, and they also get $a + 2b$ from the Shapley value.

   (c) No. For example, players $1, 2, 3$ can together get a payoff of 1 on their own, which is more than the Shapley value gives them.

3. (a) False. In the weighted voting game $\langle 1,2,3;4\rangle$, player 3 is the only veto player, so the only core payoff vector is $(0,0,1)$, but the Shapley value vector is $(1/6,1/6,2/3)$.

   (b) True. If $\vec{x},\vec{y}\in Core(\mathcal{G})$, then $\sum_{i=1}^{n}(\alpha x_i+(1-\alpha)y_i)=\alpha(\sum_{i=1}^{n}x_i)+(1-\alpha)(\sum_{i=1}^{n}y_i)=\alpha v(N)+(1-\alpha)v(N)=v(N)$, so the resulting vector is efficient. Since $\alpha\in[0,1]$, the vector is also non-negative as both $\vec{x}$ and $\vec{y}$ are. Finally, for any $S\subseteq N$,

$$\sum_{i\in S}(\alpha x_i+(1-\alpha)y_i)\geq \alpha v(S)+(1-\alpha)v(S)=v(S),$$

   so the resulting payoff division is stable as well.

   (c) False. From part (b), if $\vec{x}$ and $\vec{y}$ belong to the core, then so does every convex combination of them.