CS4261/5461: Algorithmic Mechanism Design

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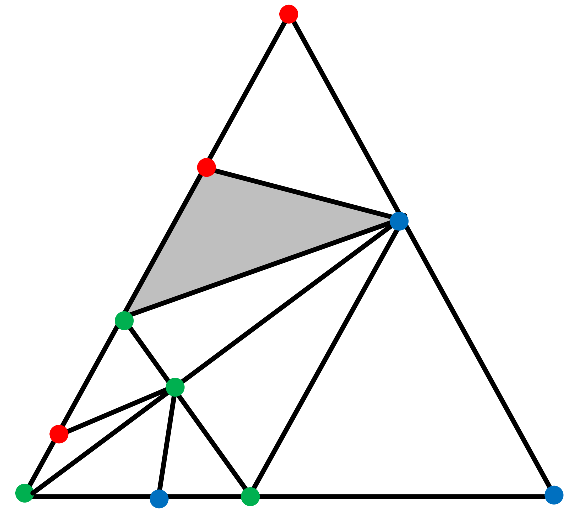
Sperner’s Lemma and Its Applications

Brouwer's Fixed Point Theorem

Let $\Delta_n = \{\vec{x} \in \mathbb{R}^n_{\geq 0}: \sum_i x_i = 1\}$ be a simplex.

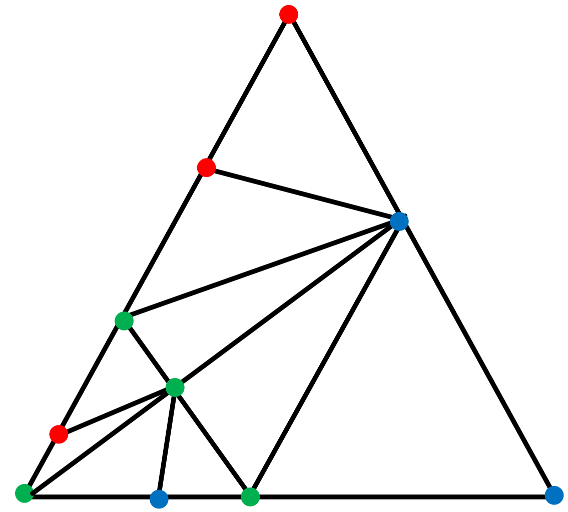
If $f: \Delta_n \to \Delta_n$ is continuous, then there is some point $\vec{x}^* \in \Delta_n$ such that $f(\vec{x}^*) = \vec{x}^*$

Sperner's Lemma



- Let $\Delta_n = \{\vec{x} \in \mathbb{R}^n_{\geq 0}: \sum_i x_i = 1\}$ be a simplex.
- **Triangulation:** cutting the simplex to smaller simplices that share faces of lower dimension.
- **Coloring:** color each vertex in one of n colors such that the vertices of the main simplex are all of different colors. Colors of vertices on a face must be a subset of colors on the face's endpoints.
- There are an **odd** number of rainbow simplices (ones that have all points a different color)!

Sperner's Lemma



Proof of the case $n = 3$

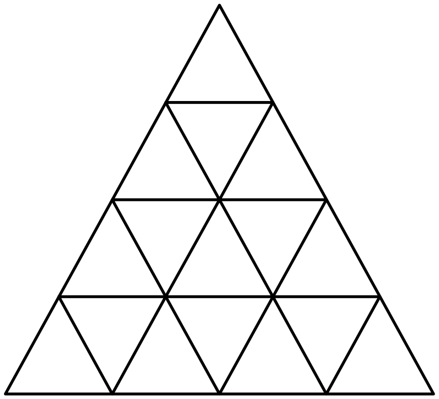
- Let Q be the # of triangles colored (G, B, B) or (G, G, B)
- Let R be the # of triangles colored (R, G, B)
- Let X be the # of edges colored (G, B) on the boundary
- Let Y be the # of edges colored (G, B) on the interior

Count the ordered pairs (edge (G, B), triangle to which this edge belongs)

What is the relation between Q, R and X, Y?

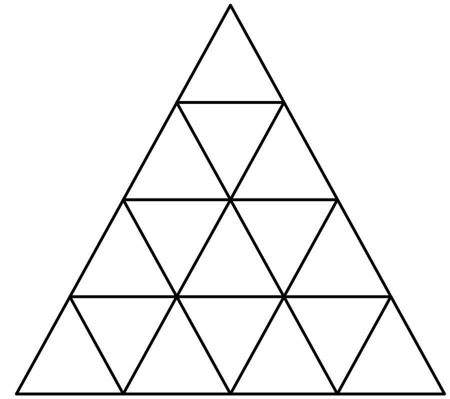
What can you say about the number X?

From Sperner to Brouwer

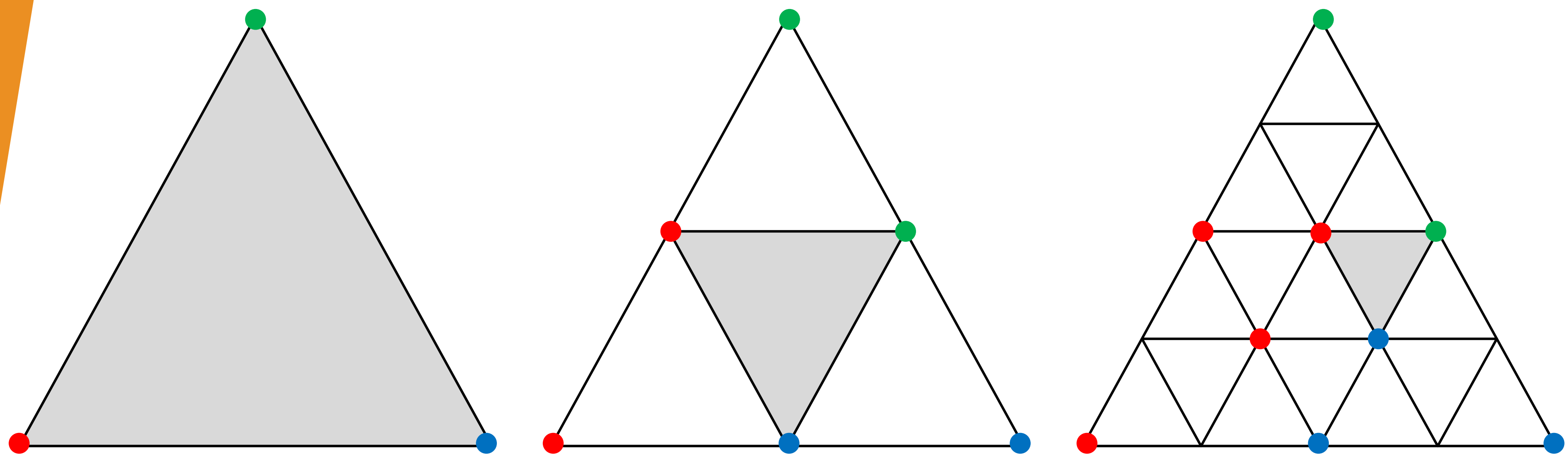


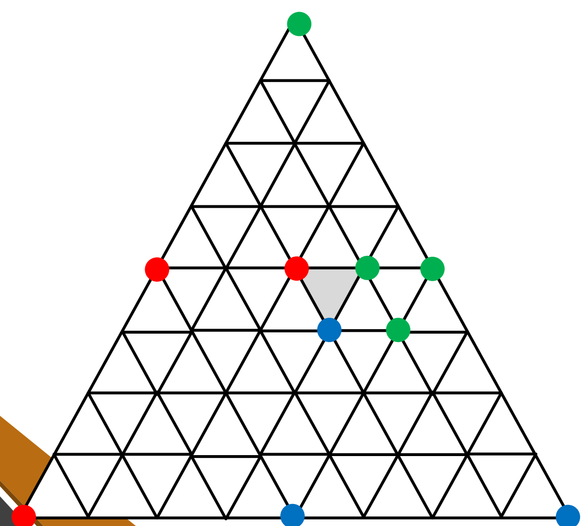
- Assume w.l.o.g. that the vertices of our simplex are $(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,1)$
- We take triangulations $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\ldots$ so that each $\mathcal{S}_j$ subdivides $\mathcal{S}_{j-1}$, and that the size of each triangle in $\mathcal{S}_j$ tends to 0.
- Assume for contradiction that f has no fixed point.

From Sperner to Brouwer

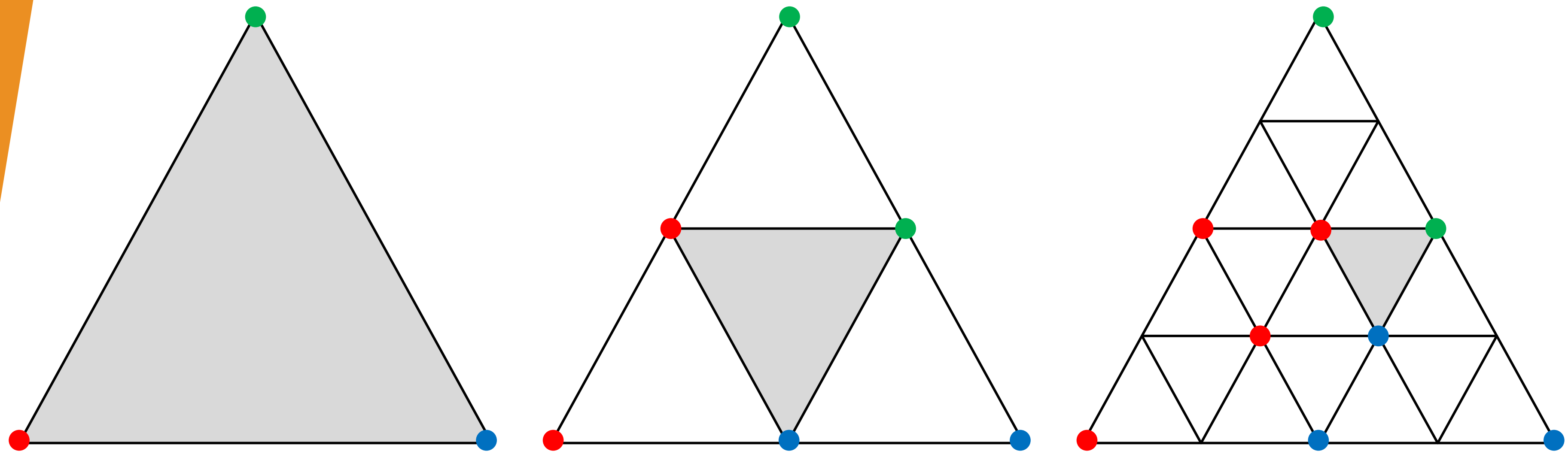


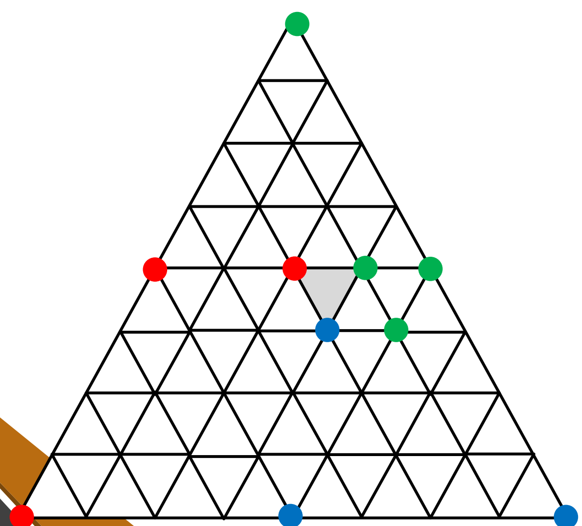
- Assume for contradiction that f has no fixed point.
- For a triangulation $\mathcal{S}_j$, define a coloring $c_j: \mathcal{S}_j \to \{1,2, \dots, n\}$
  - For every $\vec{x} \in \mathcal{S}_j$, set $c_j(\vec{x})$ such that $f(\vec{x})_{c_j(\vec{x})} < x_{c_j(\vec{x})}$ (why is this always possible?)
  - What are the colors of the main simplex vertices?
- This is a proper coloring:
  - For a vertex on some face of $\mathcal{S}_j$, the only coordinates for which $f(\vec{x})_i < x_i$ can hold are those that belong to that face (otherwise $x_i = 0$, so $f(\vec{x})_i$ cannot be less than x_i).





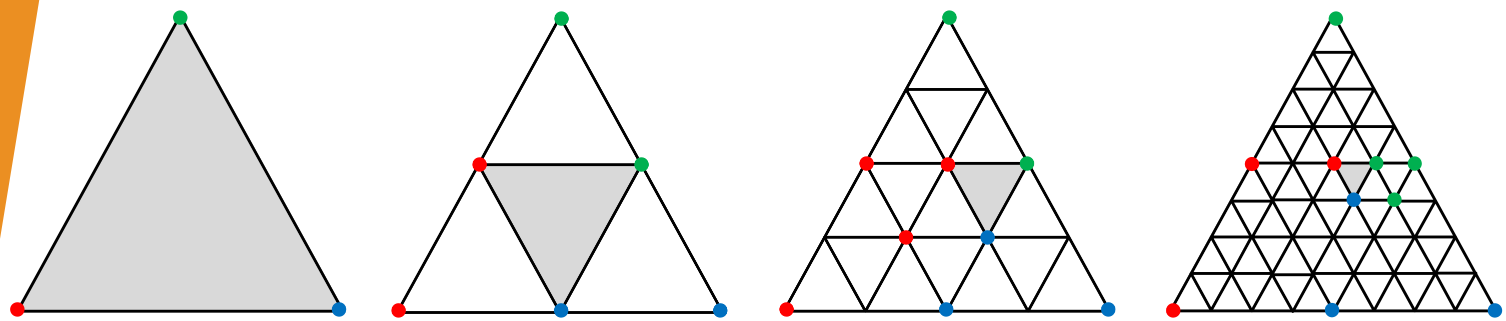
For every triangulation $\mathcal{S}_j$, there is some rainbow simplex R_j (Sperner's Lemma).





The sequence of points of the same color has a convergent subsequence (why?); since the triangulations cell size vanishes, they all converge to the same point!

Call that point $\vec{x}^*$.



Call that point $\vec{x}^*$. **We argue that $\vec{\boldsymbol{x}}^*$ is a fixed point, $\boldsymbol{f}(\vec{\boldsymbol{x}}^*) = \vec{\boldsymbol{x}}^*$.**

All convergent points of color k satisfy $f\left(\vec{x}^{(j,k)}\right)_k < x_k^{(j,k)}$

By continuity: $f(\vec{x}^*)_k = \lim_{j\to\infty} f\left(\vec{x}^{(j,k)}\right)_k \leq \lim_{j\to\infty} x_k^{(j,k)} = x_k^*$ for all k

Since $\sum_k f(\vec{x}^*)_k = 1 = \sum_k x_k^*$ equality holds at every coordinate!

From Brouwer to Nash

- Let $\Delta(A) = \Delta(A_1) \times \cdots \times \Delta(A_n)$ be the space of all players' mixed strategies
- **Fact:** $\Delta(A)$ is nonempty, convex, and compact
- Given $\vec{p} \in \Delta(A)$ and $a \in A_i$, define $G_i(\vec{p}, a) = \max\{u_i(\vec{p}_{-i}, a) - u_i(\vec{p}), 0\}$
- Let $s_i(\vec{p}, a) = \frac{p_i(a)+G_i(\vec{p},a)}{1+\sum_{a' \in A_i} G_i(\vec{p},a')}$
- $s_i(\vec{p}) = \left(s_i(\vec{p}, a)\right)_{a \in A_i}$ is a mixed strategy over A_i
  - $s_i(\vec{p}, a) \geq 0$ for each action a
  - $\sum_{a \in A_i} s_i(\vec{p}, a) = 1$
- Hence, $\beta(\vec{p}) = (s_1(\vec{p}), \dots, s_n(\vec{p}))$ is a mapping from $\Delta(A)$ to itself

From Brouwer to Nash

- Hence, $\beta(\vec{p}) = (s_1(\vec{p}), \dots, s_n(\vec{p}))$ is a mapping from $\Delta(A)$ to itself
- β is continuous (as a function of $\vec{p}$)
- Brouwer implies that there exists $\vec{p}$ such that $\beta(\vec{p}) = \vec{p}$
- **Claim:** $\vec{p}$ is a Nash equilibrium
- Since $\beta(\vec{p}) = \vec{p}$, we have $s_i(\vec{p}) = p_i$, so $s_i(\vec{p}, a) = p_i(a)$
- This means $\frac{p_i(a)+G_i(\vec{p},a)}{1+\sum_{a' \in A_i} G_i(\vec{p},a')} = p_i(a)$
- Equivalently, $p_i(a) \cdot \sum_{a' \in A_i} G_i(\vec{p}, a') = G_i(\vec{p}, a)$

From Brouwer to Nash

- Equivalently, $p_i(a) \cdot \sum_{a' \in A_i} G_i(\vec{p}, a') = G_i(\vec{p}, a)$
- If $G_i(\vec{p}, a') = 0$ for all actions a', then no pure strategy improves player i's utility, and we are done
- Assume that $\sum_{a' \in A_i} G_i(\vec{p}, a') > 0$, so $G_i(\vec{p}, a) > 0$ iff $p_i(a) > 0$
- Player i's expected utility under $\vec{p}$ is

$$u_i(\vec{p}) = \sum_{a \in A_i} p_i(a) \cdot u_i(\vec{p}_{-i}, a) = \sum_{a \in A_i : p_i(a) > 0} p_i(a) \cdot u_i(\vec{p}_{-i}, a)$$

$$= \sum_{a \in A_i : G_i(\vec{p}, a) > 0} p_i(a) \cdot u_i(\vec{p}_{-i}, a)$$

From Brouwer to Nash

- Player i's expected utility under $\vec{p}$ is

$$u_i(\vec{p}) = \sum_{a \in A_i : G_i(\vec{p},a) > 0} p_i(a) \cdot u_i(\vec{p}_{-i}, a) > \sum_{a \in A_i : G_i(\vec{p},a) > 0} p_i(a) \cdot u_i(\vec{p})$$

$$= u_i(\vec{p}) \cdot \sum_{a \in A_i : G_i(\vec{p},a) > 0} p_i(a) = u_i(\vec{p}) \cdot \sum_{a \in A_i : p_i(a) > 0} p_i(a) = u_i(\vec{p})$$

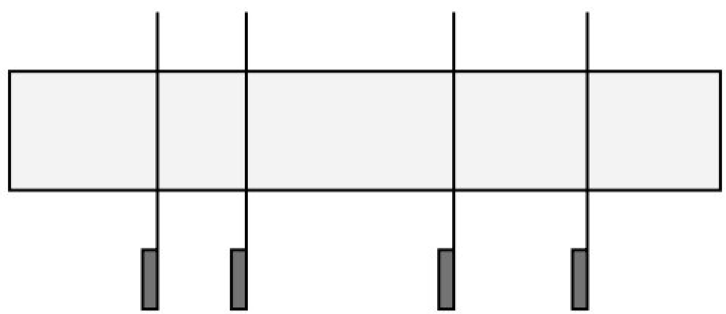
- Contradiction!

Envy-Free Cake Cutting

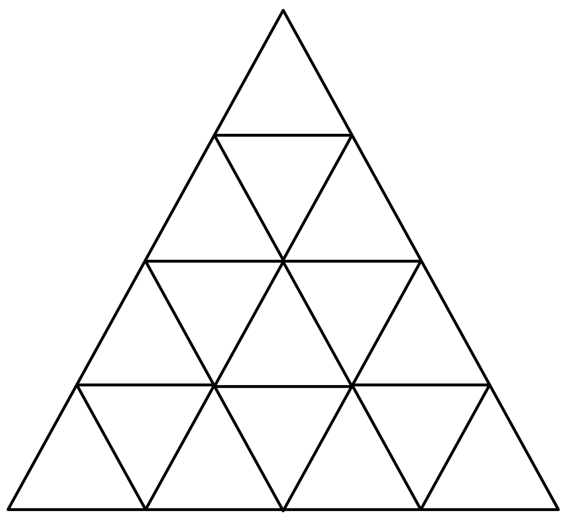
- **Theorem:** Suppose that the players' preferences over the cake satisfy the following:

  1) Hungry assumption: Any nonempty piece is always preferable to an empty piece.
  2) Preferences are continuous.

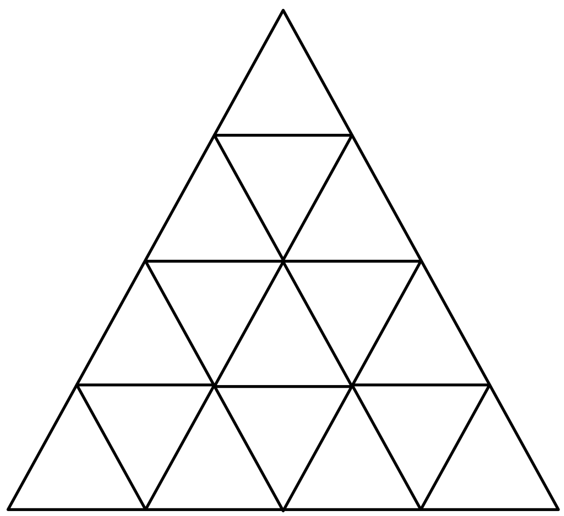
  Then there is always a connected envy-free allocation of the cake.



- Consider the simplex $\Delta_n = \{\vec{x} \in \mathbb{R}^n_{\geq 0}: \sum_i x_i = 1\}$
- Associate each point $(x_1, x_2, \ldots, x_n)$ of Δ_n with the partition of the cake where the pieces have length $x_1, x_2, \ldots, x_n$
- Triangulate Δ_n, and label the vertices with players in such a way that each subsimplex has distinct labels at all vertices



- At each vertex, ask the corresponding agent to color it with the part of the partition that the agent prefers most.
- This coloring satisfies the conditions of Sperner's lemma.
- Thus, there exists a rainbow simplex!
- Make finer and finer triangulations, and use continuity.



Envy-Free Rent Division

- Consider the simplex $\Delta_n = \{\vec{x} \in \mathbb{R}^n_{\geq 0}: \sum_i x_i = 1\}$
- Assume without loss of generality that the total rent of all rooms is 1
- Associate each point $(x_1, x_2, \ldots, x_n)$ of Δ_n with the outcome where the rent of room i is x_i
- At each vertex of the triangulation, ask the vertex owner which room he/she prefers given the prices
- Twist: Not a Sperner labeling!
- May assume that players always prefer a free room to non-free room, and prove a Sperner-like lemma

Reference: "Rental harmony: Sperner's lemma in fair division" by Francis E. Su

That's all for CS4261/5461!

Good luck with the last exam (and beyond) ☺