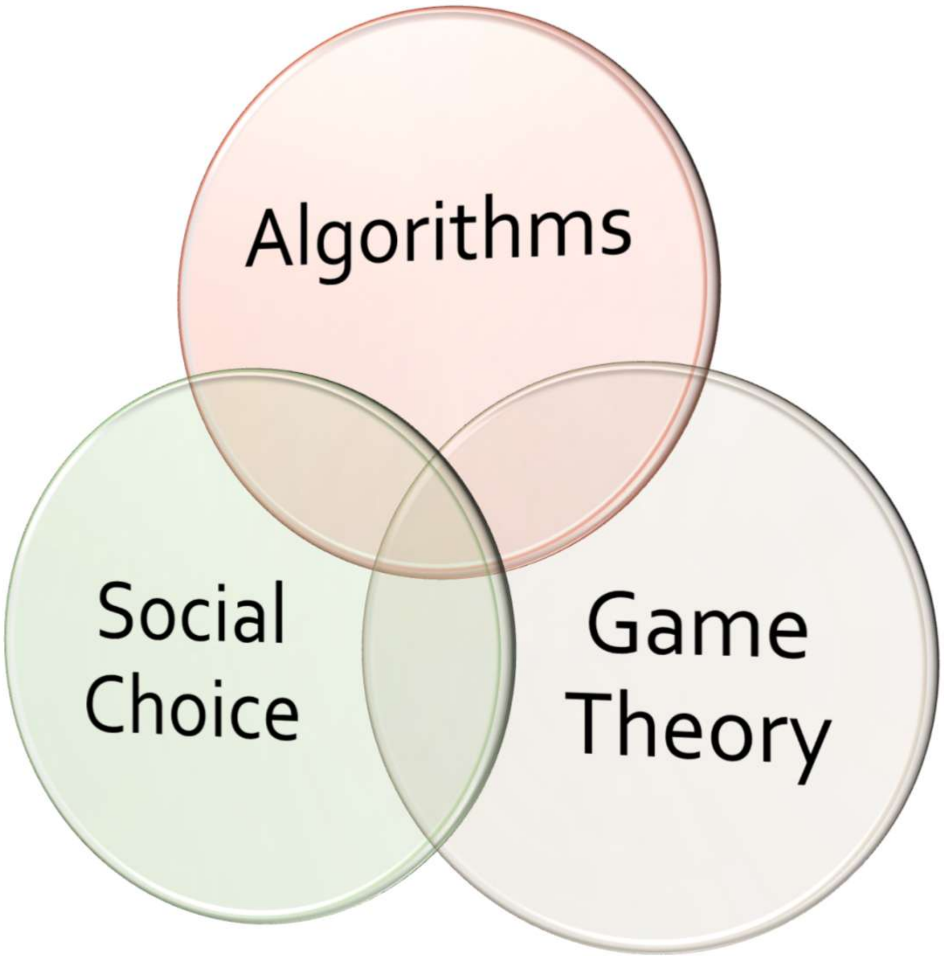
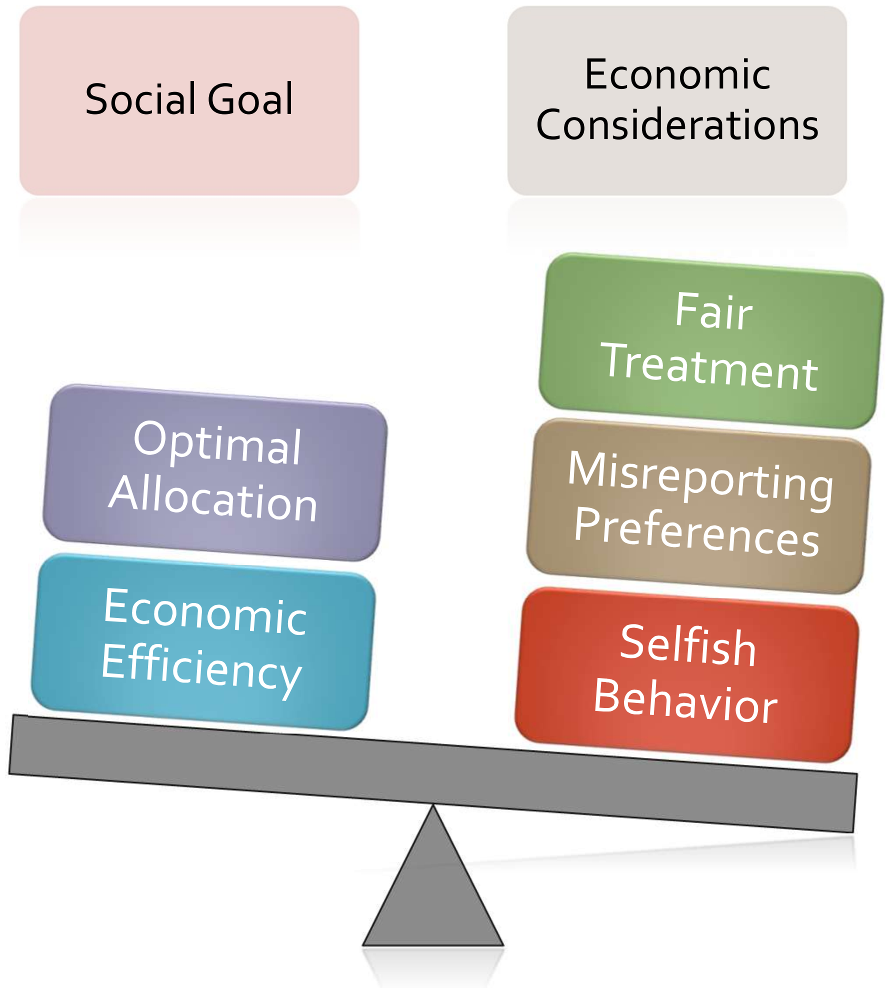
CS4261/5461: Algorithmic Mechanism Design

Instructor: Warut Suksompong

2025


Algorithms
Social Choice
Game Theory


Social Goal
Economic Considerations
Optimal Allocation
Economic Efficiency
Fair Treatment
Misreporting Preferences
Selfish Behavior

Example: Allocating Goods

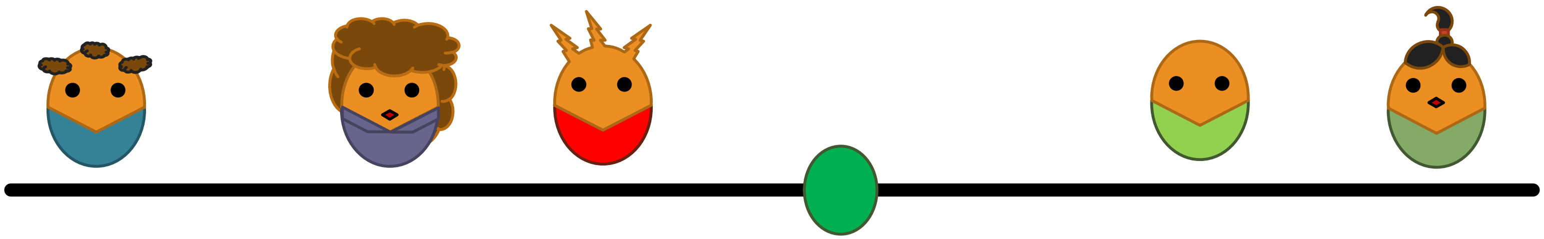


$220	$140	$240	$400
$280	$200	$120	$400
$260	$260	$260	$220




Find an allocation that is:
socially optimal? Envy free?

Example: Facility Location



Need to place a shared facility that serves a bunch of people; placement rule (mechanism) needs to satisfy

- High social welfare
- Agents report their locations truthfully

Tentative Schedule

Week	Date	Topic
1	14 Aug	Intro and Nash equilibrium
2	21 Aug	No class (but still an assignment!)
3	28 Aug	Auctions
4	4 Sep	Facility location + Routing games
5	11 Sep	Cooperative games
6	18 Sep	**Midterm 1** + Cooperative games (cont.)
7	2 Oct	Stable matching + Nash bargaining solution
8	9 Oct	Fair allocation of indivisible goods
9	16 Oct	Cake cutting
10	23 Oct	**Midterm 2** + Rent division
11	30 Oct	Committee voting
12	6 Nov	Tournaments + Optional material
13	13 Nov	**Midterm 3**

Logistics

- Lecture: 18:30 – 20:30 Thursdays @LT19
  - I will stick around to answer questions after lecture
  - Lecture slides posted on Canvas
  - Lectures will be recorded and uploaded to Canvas afterwards. (However, in case of technical issues, I will not redo the lecture.)
- Canvas discussions: Ask questions here!
  - I have enabled the option to create an anonymous discussion.
  - Please don't email me questions about course material (or matters of interest to everyone in the course)

Assessment

- Weekly assignments (30%)
  - 11 assignments (Week 1-11), 3% each. Also 3% free marks
  - Can earn up to 36% but capped at 30%
  - One question per assignment graded for correctness, the rest for effort
  - Released on lecture day (Thu), due 11:59pm Sunday of the following week
  - Assignment 1 already posted, due on Sunday, Aug 24
  - No late assignment accepted. This covers the vast majority of “reasons”:
    - Lost internet connection 5 mins before the deadline
    - Already wrote up the assignment but forgot to submit
    - Submitted the wrong file
    - Have a cold/flu, etc.

Assessment

- Three midterm exams (70% = 20%+20%+30%)
  - Week 6 (18 Sep): 6:30-7:15pm
  - Week 10 (23 Oct): 6:30-7:15pm
  - Week 13 (13 Nov): 6:30-7:**50**pm
- Held in lecture venue (LT19)

Other notes

- Prerequisites:
  - Mathematical maturity (experience with reading and writing mathematical proofs; knowledge in calculus and linear algebra)
  - Knowledge in theoretical computer science (algorithms and NP-hardness)
- TAs:
  - Yuhong Deng (yuhongdeng@u.nus.edu): Assignments 1, 4, 7, 10
  - Karen Frilya Celine (karenfc@nus.edu.sg): Assignments 2, 5, 8, 11
  - Haoyun Tang (e1154532@u.nus.edu): Assignments 3, 6, 9
- This is a theory course

Nash Equilibria and Game Theory basics

What is a Game?

- Players $N = \{1, \ldots, n\}$
- Actions (players can do something to affect the world)
- Preferences over outcomes
- A general, abstract, framework for **strategic interaction**

Not Just Child's Play

- Any distributed system, where individual actors may have preferences over outcomes.
- Some system behaviors simply cannot be explained without game-theoretic language
  - Auctions
  - Course allocation
  - Traffic flow

Prisoner's Dilemma – a classic puzzle

Two criminals are arrested. Interrogators do not have enough evidence to convict them, but can convict them for a minor offense.

Each suspect is offered the same deal:
Implicate your friend, and we'll let you go!

- Both confess: get a sentence of 2 years
- If one confesses and the other does not, the confessing party goes free while the other party serves 3 years.
- Both stay quiet: both go to prison for 1 year (for the minor offense)

Prisoner's Dilemma – a classic puzzle

	Stay Quiet	Confess
Stay Quiet	-1,-1	-3,0
Confess	0,-3	-2,-2

"Normal-form games": Matrix representation

Normal-Form Games

- A set of players $N = \{1, \dots, n\}$
- Each player $i \in N$ has a set of possible *actions* A_i
- An action profile: a vector $\vec{a} \in A_1 \times A_2 \times \cdots \times A_n = A$
- Utility of player i from $\vec{a} \in A$ is the value $u_i(\vec{a})$.

$$u_i : A \to \mathbb{R}$$

Normal Form Games – Pure Nash Equilibria

Given everyone else's actions $\vec{a}_{-i}$, the best response set of i is

$$BR_i(\vec{a}_{-i}) = \{\, b \in A_i \mid b \in \operatorname{argmax} u_i(\vec{a}_{-i}, b) \,\}$$

An action profile is a (pure) Nash equilibrium if:

$$\forall i \in N, a_i \in BR_i(\vec{a}_{-i})$$

"I'm doing the best I can, given everyone else's actions!"

Prisoner's Dilemma

	Stay Quiet	Confess
Stay Quiet	-1,-1	-3,0
Confess	0,-3	-2,-2

Pure NE: (Confess, Confess)

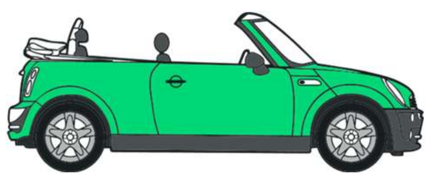
Work Hard or Have Fun?

	Chill	Work hard
Chill	7,7	4,9
Work hard	9,4	6,6

Pure NE: (Work hard, Work hard)

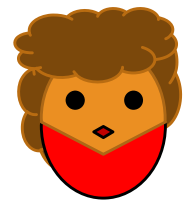
The Game of Chicken

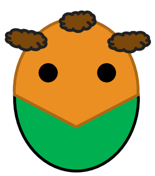
	Drive Straight	Swerve
Drive Straight	-10,-10	1,-1
Swerve	-1,1	0,0



Pure NE: (Swerve, Drive straight), (Drive straight, Swerve)

Coordination Game





	Chinese	Indian
Chinese	5,4	1,1
Indian	0,0	4,5

Pure NE: (Chinese, Chinese), (Indian, Indian)

Mixed Nash Equilibrium

- Playing a single strategy may be foolish – an opponent who knows you well can always beat you with a best response!
- It is often better to be unpredictable

Randomized Actions

- Instead of choosing a single action, one can play a random mix of them: $\vec{p}_i \in \Delta(A_i)$ is a probability distribution over player i's actions.
- A (not necessarily pure) strategy profile:

$$\vec{p} = (\vec{p}_1, \ldots, \vec{p}_n) \in \Delta(A_1) \times \cdots \times \Delta(A_n)$$

Mixed Nash Equilibrium

Players are risk neutral!

- Player Utility: $u_i(\vec{p}) = \sum_{\vec{a} \in A} u_i(\vec{a}) \Pr[\vec{a}] = \mathbb{E}_{\vec{a} \sim \vec{p}}[u_i(\vec{a})]$
- Nash equilibrium: for all $i \in N$, and all $\vec{q}_i \in \Delta(A_i)$,

$$u_i(\vec{p}) \geq u_i(\vec{p}_{-i}, \vec{q}_i)$$

- Unlike pure NE, a (not necessarily pure) NE *always exists* (Nash's Theorem)

Computing Nash Equilibria in 2x2 Games

- **Case 1:** Compute all NE in which at least one player plays a pure strategy

- **Case 2:** Compute all NE in which both players strictly mix between both actions
  - In this case, each player must be indifferent between the two actions!

Coordination Game

	Chinese	Indian
Chinese	5,4	1,1
Indian	0,0	4,5

Case 1: At least one player plays a pure strategy

- Row = Chinese --> Col = Chinese --> Row = Chinese is a best response --> (Chinese, Chinese)
- Row = Indian --> Col = Indian --> Row = Indian is a best response --> (Indian, Indian)
- Col = Chinese --> Row = Chinese --> Col = Chinese is a best response --> (Chinese, Chinese)
- Col = Indian --> Row = Indian --> Col = Indian is a best response --> (Indian, Indian)

Coordination Game

		q	1-q
		Chinese	Indian
p	Chinese	5,4	1,1
1-p	Indian	0,0	4,5

Case 2: Both players mix between both actions

- Assume Row plays (Chinese, Indian) with probability (p, 1-p), and Col with probability (q, 1-q), where $0 < p, q < 1$
- Row player indifferent --> $5*q + 1*(1-q) = 0*q + 4*(1-q)$ --> $q = 3/8$
- Col player indifferent --> $4*p + 0*(1-p) = 1*p + 5*(1-p)$ --> $p = 5/8$
- (5/8 Chinese + 3/8 Indian, 3/8 Chinese + 5/8 Indian)

Dominant Strategies

- We say that a strategy $\vec{p} \in \Delta(A_i)$ **dominates** $\vec{q} \in \Delta(A_i)$ if
$$\forall \vec{p}_{-i} \in \Delta(A_{-i}) \colon u_i(\vec{p}_{-i}, \vec{p}) \geq u_i(\vec{p}_{-i}, \vec{q})$$
- **Strict domination:** $u_i(\vec{p}_{-i}, \vec{p}) > u_i(\vec{p}_{-i}, \vec{q})$
- No matter what the other players do, playing $\vec{p}$ is better than playing $\vec{q}$ for player i.

Dominant Strategies

Theorem: if an action $a \in A_i$ is **strictly dominated** by some strategy $\vec{p} \in \Delta(A_i)$, then action a is **never played** with any positive probability in any Nash equilibrium.

Iterated Removal of Dominated Strategies

	L	M	R
~~T~~	~~0,11~~	~~1,17~~	~~2,20~~
C	10, −1	~~0,0~~	~~3,1~~
B	0,1	10,2	4,0

$u_1(T,\vec{q}) < \frac{1}{2}u_1(C,\vec{q}) + \frac{1}{2}u_1(B,\vec{q})$

$u_1(B,\vec{q}) > u_1(C,\vec{q})$

$u_2(\vec{p},M) > u_2(\vec{p},L)$

$u_2(\vec{p},M) > u_2(\vec{p},R)$

Useful for reducing the search space when computing Nash equilibria!

Prisoner’s Dilemma

	Stay Quiet	Confess
Stay Quiet	-1,-1	-3,0
Confess	0,-3	-2,-2

“Stay Quiet” is strictly dominated by “Confess”

All NE: (Confess, Confess)

Work Hard or Have Fun?

	Chill	Work hard
Chill	7,7	4,9
Work hard	9,4	6,6

"Chill" is strictly dominated by "Work hard"
All NE: (Work hard, Work hard)

Golden Balls

Each player is given a set of two balls, one marked “Split” and the other marked “Steal”. There is a jackpot of, say, $1000.

- If both players choose “Split”, each of them gets half of the jackpot.
- If one player chooses “Split” and the other player chooses “Steal”, the player who chooses “Steal” gets the entire jackpot, while the player who chooses “Split” gets nothing.
- If both players choose “Steal”, both of them get nothing.

Golden Balls

www.youtube.com/watch?v=SoqjK3TWZE8

Golden Balls

	Split	Steal
Split	500,500	0,1000
Steal	1000,0	0,0

"Split" is not strictly dominated by "Steal"

Cannot remove "Split"!

Golden Balls

	Split	Steal
Split	500,500	0,1000
Steal	1000,0	0,0

Case 1: At least one player plays a pure strategy

- Row = Split --> Col = Steal --> Row = Split is a best response --> (Split, Steal)
- Row = Steal --> Col = q*Split + (1-q)*Steal for any 0 <= q <= 1 -->
  Row playing Steal is a best response for every q --> (Steal, q*Split + (1-q)*Steal)
- Similar reasoning starting with Col --> (Steal, Split), (p*Split + (1-p)*Steal, Steal)

Summary: (Steal, q*Split + (1-q)*Steal) for any 0 <= q <= 1
(p*Split + (1-p)*Steal, Steal) for any 0 <= p <= 1

Golden Balls

		q	1-q
		Split	Steal
p	Split	500,500	0,1000
1-p	Steal	1000,0	0,0

Case 2: Both players mix between both actions

- Assume Row plays (Split, Steal) with probability (p, 1-p), and Col with probability (q, 1-q) for $0 < p, q < 1$
- Row player indifferent --> $500*q + 0*(1-q) = 1000*q + 0*(1-q)$ --> $q = 0$
- Contradiction with the assumption of this case!