

## CS4261/5461: Assignment for Week 10 Solutions

Due: Sunday, 2nd Nov 2025, 11:59 pm SGT.

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1. (a) First room to player 1 and second room to player 2. (The total value is 1200.)  
(b) Suppose that the price vector is  $(p, 1000 - p)$ . Player 1's utility is  $900 - p$ , and player 2's utility is  $300 - (1000 - p) = p - 700$ . Since we want to maximize the minimum utility, we set  $900 - p = p - 700$ , which results in  $p = 800$ . Each player gets utility 100, and would have gotten a utility of  $-100$  in the other player's position, so there is no envy. Hence, the price vector is  $\boxed{(800, 200)}$ .
2. (a) No. In the instance from Question 1, if player 1 reports  $(800, 200)$ , one can check in a similar way that the room allocation remains the same while the price vector becomes  $(750, 250)$ . Hence, player 1's utility increases from 100 to 150 by misreporting.  
(b) Yes. Since the mechanism maximizes the sum of players' values, while the total payment is fixed, the mechanism also maximizes the sum of players' utilities among all possible outcomes. Hence, it cannot be the case that some other outcome increases one player's utility and does not decrease any other player's utility.  
(c) Yes. Suppose that the total rent is  $r$ , and the two players report  $(a, r - a)$  and  $(b, r - b)$ , where we assume without loss of generality that  $a \geq b$ . The welfare-maximizing allocation assigns the first room to player 1 and the second room to player 2. Suppose that the price vector is  $(p, r - p)$ . Player 1's utility is  $a - p$ , and player 2's utility is  $(r - b) - (r - p) = p - b$ . Since we want to maximize the minimum utility, we set  $a - p = p - b$ , which results in  $p = \frac{a+b}{2}$ . Each player gets utility  $\frac{a-b}{2}$ , and would have gotten a utility of  $-\frac{a-b}{2}$  in the other player's position, so there is no envy, and this outcome is chosen by the mechanism. In particular, both players obtain the same utility of  $\frac{a-b}{2}$ .
3. (a) True. We will show that the price of each room must be equal to its (common) value, i.e.,  $p_j = v_j$  for all  $j$ .  
Suppose for contradiction that in an envy-free price vector,  $p_\ell \neq v_\ell$  for some room  $\ell$ . Since  $\sum_{i=1}^n p_i = \sum_{i=1}^n v_i$ , there is at least one room  $j$  for which  $p_j > v_j$  and there is also

a room  $k$  for which  $p_k < v_k$ . The player who gets room  $j$  gets a negative utility from room  $j$ —her utility is  $v_j - p_j < 0$ ; however, her utility from room  $k$  would be strictly positive, since  $v_k > p_k$ . Hence, she envies the player who got room  $k$ , a contradiction.

- (b) False. Suppose there are two players and two rooms, and each player has value 100 for each room. From part (a), the unique envy-free price vector is  $(100, 100)$ . However, either of the two room allocations is envy-free.