CS4261/5461: Assignment for Week 7 Solutions

Due: Sunday, 12th Oct 2025, 11:59 pm SGT.

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1. (a) No. (D, Z) is a blocking pair, since D prefers Z to its current match W, while Z prefers D to its current match A. It is also the only blocking pair—indeed, A, B, and C are already matched to their top choices, and D can only improve by being matched to Z.

   (b) The algorithm runs as follows:
   - A and D propose to Z, B to Y, and C to X.
   - Z rejects A.
   - A proposes to W.
   - The final matching is $\boxed{\{(A, W), (B, Y), (C, X), (D, Z)\}}$.

   (c) The algorithm runs as follows:
   - W and Y propose to A, X to D, and Z to B.
   - A rejects Y.
   - Y proposes to B.
   - B rejects Z.
   - Z proposes to D.
   - D rejects X.
   - X proposes to A.
   - A rejects X.
   - X proposes to B.
   - B rejects X.
   - X proposes to C.
   - The final matching is $\boxed{\{(A, W), (B, Y), (C, X), (D, Z)\}}$, the same as in (b).

   (d) There is $\boxed{1}$ stable matching. Recall that in Gale–Shapley with students proposing, each student gets the best match among all stable matchings, whereas in Gale–Shapley with hospitals proposing, each student gets the worst match among all stable matchings. From (b) and (c), we have that the best match and the worst match of every student are the same. This means that there is only one possible match for each student across all stable matchings, and therefore only one stable matching.

2. Write the students from top ranked to bottom ranked as $s_1 > \cdots > s_n$. Let h_1 be the top choice of s_1. If s_1 is not assigned to h_1 then the matching is not stable: h_1 prefers s_1, and s_1 wants h_1 the most. Similarly, s_2 must be assigned to his/her top choice besides h_1. By repeating the same argument, we find that there is only one possibility for a stable matching.

3. (a) If $x = 0$ then $xy = 0$, which is clearly not the maximum, so assume that $x > 0$. Similarly, we may assume that $y > 0$. Using the Lagrangian, we get

$$L(x, y, \lambda) = xy - \lambda(4x + \sqrt{y} - 1)$$
$$\frac{\partial L}{\partial x} = y - 4\lambda$$
$$\frac{\partial L}{\partial y} = x - \frac{\lambda}{2\sqrt{y}}$$

Equating the partial derivatives to 0 yields $\frac{y}{4} = \lambda = 2x\sqrt{y}$, so $\sqrt{y} = 8x$. Plugging this back into $4x + \sqrt{y} = 1$ yields $4x + 8x = 1$, so $x = 1/12$ and $y = 4/9$. The Nash bargaining solution is then $(x, y) = \boxed{(1/12,\ 4/9)}$.

(b) If $x = 0$, then $y = 1$ and $x + y = 1$. If $y = 0$, then $x = 1/4$ and $x + y = 1/4$. Assume from now on that $x, y > 0$. Using the Lagrangian, we get

$$L(x, y, \lambda) = x + y - \lambda(4x + \sqrt{y} - 1)$$
$$\frac{\partial L}{\partial x} = 1 - 4\lambda$$
$$\frac{\partial L}{\partial y} = 1 - \frac{\lambda}{2\sqrt{y}}$$

Equating the partial derivatives to 0 yields $8\sqrt{y} = 1$, so $\sqrt{y} = 1/8$. Plugging this back into $4x + \sqrt{y} = 1$ yields $4x = 7/8$, so $x = 7/32$ and $y = 1/64$. We have $x + y = 15/64 < 1$. So the maximal utilitarian welfare is obtained at $(x, y) = \boxed{(0,1)}$.

(Note that $(x, y) = (7/32, 1/64)$ yields the **minimum**, not the maximum.)