Week 9: Cake Cutting

Instructor: Warut Suksompong

National University of Singapore

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Cake Cutting

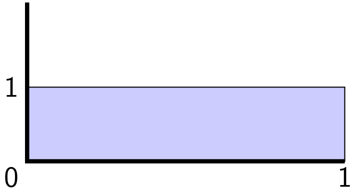
- How to fairly divide a heterogeneous divisible good among interested agents with different preferences?
- The cake could represent land, machine processing time, etc.

Setting

- The cake is represented by the interval $[0, 1]$.
- Set of agents $N = \{1, \ldots, n\}$
- Agents have valuation functions $v_1, \ldots, v_n$ over the cake.
- Valuation functions are
  - Nonnegative: No "bad" cake.
  - Additive: Values of disjoint pieces add up.
  - Nonatomic: The value of any single point is 0. (For example, we do not allow a cherry that cannot be cut.)
  - Normalized: The value of the whole cake is 1.
- Allocation $A = (A_1, \ldots, A_n)$
- Each A_i is a union of finitely many intervals.
  - The allocation A is connected if each A_i is a single interval.

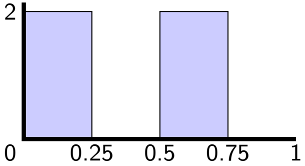
Valuation Functions

- For convenience, write $v_i(x, y)$ instead of $v_i([x, y])$ for $0 \leq x \leq y \leq 1$.
- v_i often specified through density function f_i:
  - $v_i(B) = \int_B f_i(x)\,\mathrm{d}x$ for $B \subseteq [0, 1]$
  - Normalization: $\int_{x=0}^{1} f_i(x)\,\mathrm{d}x = 1$
  - Additivity and nonatomicity follow directly from properties of integration!
- **Example:** $f(x) = 1$ for all $x \in [0, 1]$

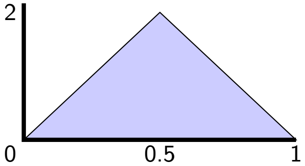


Valuation Functions

- $f(x) = \begin{cases} 2 & \text{if } x \in [0, 0.25] \cup [0.5, 0.75] \\ 0 & \text{otherwise} \end{cases}$



- $f(x) = \begin{cases} 4x & \text{if } x \in [0, 0.5] \\ 4(1 - x) & \text{if } x \in [0.5, 1] \end{cases}$



Fairness Notions

- When is an allocation fair?
- Proportionality: $v_i(A_i) \geq \frac{1}{n}$ for all $i \in N$
- Envy-freeness: $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$
- For $n = 2$, envy-freeness and proportionality are equivalent.
- For $n \geq 3$, envy-freeness is stronger than proportionality.

Two Agents: Cut-and-Choose

- Agent 1 cuts the cake into two equal pieces according to her opinion. Agent 2 chooses the piece that he prefers.

- Abraham: "If you prefer the left, I will go to the right; if you prefer the right, I will go to the left." [Book of Genesis, Chapter 13]
- Lot saw how abundantly watered the Jordan Plain was, and chose for himself the Jordan Plain. Abraham settled in the land of Canaan.
- Both Abraham and Lot are envy-free (and therefore proportional).

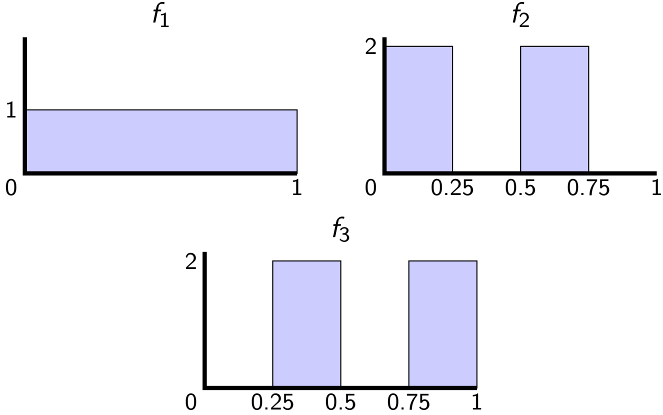
Robertson–Webb Model

- Cut-and-choose is clearly a "simple" protocol...
- But how can we reason about the complexity of cake-cutting algorithms? The input is continuous rather than discrete.
- The Robertson–Webb model allows two types of queries:
  - $\mathrm{Eval}_i(x, y)$: Return $v_i(x, y)$—the value of agent i for the interval $[x, y]$
  - $\mathrm{Cut}_i(x, \alpha)$: Return the leftmost point y such that $v_i(x, y) = \alpha$, or state that no such point exists.
- **Question:** How many Robertson–Webb queries do we need to implement the cut-and-choose protocol?
- **Answer:** 2 queries

Dubins–Spanier Protocol

- Achieves proportionality for any number of agents
- The referee moves a knife over the cake starting from the left.
- Repeat: When the piece of cake to the left of the knife is worth $1/n$ to some agent, that agent shouts "Stop!" and leaves the procedure with that piece.
- The last agent gets the remaining cake.
- **Question:** How many Robertson–Webb queries do we need to implement the Dubins–Spanier protocol?
- **Answer:** $O(n^2)$ queries

Dubins–Spanier Protocol

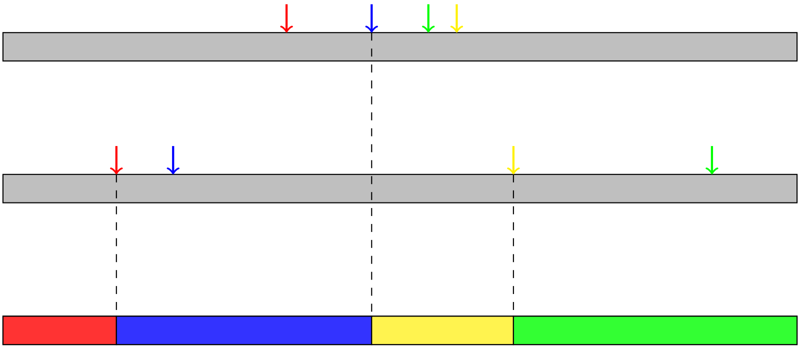


- Agent 2 gets $[0, 1/6]$, agent 3 gets $[1/6, 5/12]$, agent 1 gets $[5/12, 1]$

Even–Paz Protocol

- Achieves proportionality for any number of agents, with fewer queries!
- Assume for simplicity that n is a power of 2.
- **Idea:** Use divide-and-conquer.
- Each agent marks the point that divides the cake into two halves of equal value, according to his/her own opinion.
- Let t be mark number $n/2$ from the left.
- Recurse on $[0, t]$ with the left $n/2$ agents, and on $[t, 1]$ with the right $n/2$ agents.
- When we are down to one agent, that agent gets the whole cake.

Even–Paz Protocol



Even–Paz Protocol

- **Correctness:** Why does the Even–Paz protocol produce a proportional allocation?
- Suppose $n = 2^k$.
- At the beginning, n agents are sharing a cake for which each of them has value 1.
- After the first cut, $n/2$ agents are sharing a cake for which each of them has value at least $1/2$.
- ...
- Eventually, $n/2^k = 1$ agent has a cake for which he/she has value at least $1/2^k$.
- This is the definition of proportionality!

Even–Paz Protocol

- **Question:** How many Robertson–Webb queries do we need to implement the Even–Paz protocol?
- First stage: Ask each agent to mark the "midpoint" $\rightarrow$ n queries.
- Second stage: Ask each agent to evaluate the piece of cake that the agent ends up sharing from the previous stage, then ask each agent to mark the "midpoint" $\rightarrow 2n$ queries.
- Third stage: Same thing! $\rightarrow 2n$ queries
- There are $\log_2 n$ stages, so $O(n \log n)$ queries in total.
- This is optimal among all proportional protocols, even if the allocation is not required to be connected! [Edmonds/Pruhs, 2011]

Selfridge–Conway Protocol

- Achieves envy-freeness for three agents.
- Agent 1 divides the cake into three equal pieces according to her opinion.
- If agents 2 and 3 prefer different pieces, we are done.
- If both agents 2 and 3 prefer the same piece, say the first piece, agent 2 is asked to trim the first piece so that it has equal value as the second piece.
- The trimmed piece is cut further and allocated carefully. (We will not go into the details.)
- The Selfridge-Conway protocol needs 5 cuts and 9 queries.

Complexity of Envy-Freeness

# Agents	# Cuts	# Queries
2	1	2
3	5	9
4	203	584
n	$n^{n^{n^{n^{n^{n}}}}}$	$n^{n^{n^{n^{n^{n}}}}}$

- Brams and Taylor (1995) came up with a finite protocol for any number of agents.
- But even for four agents, their protocol is unbounded: given any integer k, there are valuations of the four agents such that the protocol makes more than k queries.
- Aziz and Mackenzie (2016) proposed the first bounded envy-free protocol for any number of agents.
- The current best lower bound is $\Omega(n^2)$ [Procaccia, 2011]

Envy-Freeness Approximation

- Simple approximate envy-free protocol: Follow Dubins–Spanier.
- The referee moves a knife over the cake starting from the left.
- When the piece of cake to the left of the knife is worth $1/3$ to some agent, that agent shouts "Stop!" and leaves the procedure with that piece.
- Suppose the knife reaches the right end of the cake, but some cake is still unallocated:
  - Case 1: There are still agents left. The remaining cake is given to one of them arbitrarily.
  - Case 2: There is no agent left. The remaining cake is given to the agent who received the last piece (to ensure connectivity).
- **Claim:** Any agent envies any other agent by at most $1/3$.

Truthfulness

- Is the cut-and-choose protocol truthful?
- Yes for the chooser, no for the cutter.
- Here is a truthful mechanism for piecewise uniform valuations (i.e., the density function takes on only the values 0 or some s_i for each agent i).
- Agent 1 starts at the left end of the cake, agent 2 at the right end.
  - Step 1: Each agent eats her desired part of the cake at the same speed until they meet.
  - Step 2: Any part that an agent jumps over goes to the other agent.