CS4261/5461: Assignment for Week 9 Solutions

Due: Sunday, 26th Oct 2025, 11:59 pm SGT.

---

1. (a) Agent 4's value for the interval $[1/3, 2/3]$ is

$$\begin{aligned}\int_{1/3}^{2/3} f_4(x)\,\mathrm{d}x &= \left(\int_{1/3}^{1/2} f_4(x) + \int_{1/2}^{2/3} f_4(x)\right)\mathrm{d}x \\ &= \int_{1/3}^{1/2} f_4(x)\,\mathrm{d}x + \int_{1/2}^{2/3} f_4(x)\,\mathrm{d}x \\ &= \int_{1/3}^{1/2} 0\,\mathrm{d}x + \int_{1/2}^{2/3} (8x-4)\,\mathrm{d}x \\ &= 0 + (4x^2 - 4x)\Big|_{1/2}^{2/3} \\ &= \frac{1}{9}.\end{aligned}$$

   (b) First, each agent i marks the point x_i such that $v_i(0, x_i) = 1/4$. We have $x_1 = 1/4$, $x_2 = 1/8$, $x_3 = 1/20$, and $x_4 > 1/2$, so the smallest x_i is x_3, and agent 3 receives the piece $[0, 1/20]$.

   Second, each remaining agent i marks the point y_i such that $v_i(1/20, y_i) = 1/4$. We have $y_1 = 3/10$, $y_2 = 7/40$, and $y_4 > 1/2$, so the smallest y_i is y_2, and agent 2 receives the piece $[1/20, 7/40]$.

   Third, each remaining agent i marks the point z_i such that $v_i(7/40, z_i) = 1/4$. We have $z_1 = 17/40$ and $z_4 > 1/2 > 17/40$, so the smallest z_i is z_1, and agent 1 receives the piece $[7/40, 17/40]$.

   Finally, agent 4 receives the remaining piece $[17/40, 1]$.

   (c) First, each agent i marks the point a_i such that $v_i(0, a_i) = 1/2$. We have $a_1 = 1/2$, $a_2 = 1/4$, $a_3 = 1/10$, and $a_4 > 1/2$. The second smallest a_i is a_2, so we recurse on $[0, 1/4]$ with agents 2 and 3, and recurse on $[1/4, 1]$ with agents 1 and 4.

   On $[0, 1/4]$, each agent $i \in \{2, 3\}$ marks the point b_i such that $v_i(0, b_i) = \frac{1}{2} \cdot v_i(0, 1/4)$. We have $b_2 = 1/8$ and $b_3 = 1/10$, so the smallest b_i is $b_3 = 1/10$. Hence, agent 3 receives the piece $[0, 1/10]$ and agent 2 receives the piece $[1/10, 1/4]$.

   On $[1/4, 1]$, each agent $i \in \{1, 4\}$ marks the point c_i such that $v_i(1/4, c_i) = \frac{1}{2} \cdot v_i(1/4, 1)$. We have $c_1 = 5/8$, and since agent 4 has positive value only for $[1/2, 1]$ and this value is skewed to the right, $c_4 > 3/4 > 5/8$. So the smallest c_i is $c_1 = 5/8$. Hence, agent 1 receives the piece $[1/4, 5/8]$ and agent 4 receives the piece $[5/8, 1]$.

2. No. For example, let

$$f_1(x) = \begin{cases} 2 & \text{if } x \in [0, 1/2]; \\ 0 & \text{otherwise}; \end{cases}$$

$$f_2(x) = \begin{cases} 2 & \text{if } x \in [1/2, 1]; \\ 0 & \text{otherwise}. \end{cases}$$

The cut-and-choose protocol gives $[0, 1/4]$ to agent 1 and $[1/4, 1]$ to agent 2, but this is Pareto dominated by the allocation that gives $[0, 1/2]$ to agent 1 and $[1/2, 1]$ to agent 2.

3. We claim that $c = 2/3$.

First, we show that in the output of this protocol, no agent envies another agent by more than $2/3$. Consider any agent i.

- If i receives a piece from Step 2, then i gets value at least $1/6$, and values any other agent's piece at most $1 - 1/6 = 5/6$. Hence, i's envy toward any other agent is at most $5/6 - 1/6 = 2/3$.
- If i does not receive a piece from Step 2, i still remains when the knife reaches the end of the cake. We will show that i has value at most $1/6$ for any other agent's piece, and therefore envy at most $1/6$.

  Note that i has value at most $1/6$ for a piece of cake allocated to any agent in Step 2 (otherwise i should have shouted "Stop!"). Since i remains after Step 2, any agent who receives cake in Step 2 does not receive more cake in Step 3. Now, consider an agent who receives cake (only) in Step 3. Since the knife reached the end of the cake without i having shouted "Stop!", i also has value at most $1/6$ for the piece allocated in Step 3.

Next, we give an example with two agents such that, in the output of this protocol, one agent envies the other by exactly $2/3$. Suppose that $f_1(x) = f_2(x) = x$ for all $x \in [0, 1]$. The protocol first allocates $[0, 1/6]$ to either agent, say, agent 1. It then allocates $[1/6, 1/3]$ to agent 2. Finally, the remaining cake $[1/3, 1]$ is added to agent 2's piece. As a result, agent 1 envies agent 2 by $5/6 - 1/6 = 2/3$.