CS5461 Assignment 8

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1. (a) To achieve the maximum utilitarian welfare, we allocate each good to the player with the highest valuation. Thus the allocation is $A_1 = (g_1), A_2 = (g_2, g_3), A_3 = (g_4)$ with utilitarian welfare $40 + 40 + 30 + 50 = \boxed{160}$.

   (b) To achieve the maximum egalitarian welfare, note that the maximum of the minimum utility among players is 40, since making this any higher will require players 1 and 2 to take at least 2 goods but then player 3 would receive nothing. Thus the maximum egalitarian welfare is $\boxed{40}$ with one possible allocation being the same as in (a).

   (c) No, since if we instead give g_2 to player 1 or 2 then they would receive a higher utiity while the utility of player 3 remains the same.

   (d) Yes. Indeed we have $\text{MMS}_1 = 30$ (e.g., $\{g_1\}, \{g_2\}, \{g_3, g_4\}$) for player 1, $\text{MMS}_2 = 30$ (e.g., $\{g_1, g_2\}, \{g_3\}, \{g_4\}$) for player 2, $\text{MMS}_3 = 20$ (e.g., $\{g_1\}, \{g_2, g_3\}, \{g_4\}$) for player 3. The allocation A gives utilities $(40, 30, 20)$ which are no lower than the maximum share $(30, 30, 20)$.

2. Denote the set of all MMS allocations in that instance as S. We claim that an MMS allocation $A \in S$ that maximises the utilitarian welfare must also be Pareto optimal.

   Otherwise, by definition of Pareto optimality there must exist another allocation B such that the utilities of all players under B must be greater than or equal to those under A, with at least one of the inequalities being strict.

   But then we also have $B \in S$ because as the utilities do not become lower they must remain at least the MMS for each player.

   However, B now attains a strictly higher utilitarian welfare than A since at least one of the inequalities is strict. This is a contradiction, so A is indeed Pareto optimal.

3. The answer is true. Intuitively, s-EF1 requires the existence of a single good g_j per bundle A_j that simultaneously 'frees everyone's envy' of j on removal. We use a similar proof as the EF1 proof given in the lecture.

   Indeed, in the round-robin algorithm, any player ahead of j in the round-robin ordering does not envy j at all since they get to choose before j, so s-EF1 must hold due to the stronger envy-freeness condition.

   For any player i behind j, we consider the first round to start with i's first pick. Then as in shown in the lecture, i does not envy j up to j's first good g_j. But note that this good is the same for each i, since from that perspective j is always the first one to choose: in other words, it does not depend on i (which might not be true in EF1). This is precisely the definition of s-EF1 where we show the existence of a $g_j \in A_j$.

   Therefore s-EF1 holds for all the players, and so in conclusion the algorithm is always s-EF1.