CS4261/5461: Assignment for Week 5 Solutions

Due: Sunday, 21st Sep 2025, 11:59 pm SGT.

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1. (a) Yes. Consider any $S \subseteq T \subseteq N$. If $S = T$, clearly $v(S) = v(T)$. Otherwise, $|S| < |T|$, and one can observe that $v(S) \leq v(T)$ always holds.

   (b) Yes. The game is monotone and all payoffs are either 0 or 1.

   (c) No. We have $v(\{3\}) + v(\{1,2\}) = 1 + 1 > 1 = v(\{1,2,3\})$.

   (d) No. We have $v(\{3\}) - v(\varnothing) = 1 - 0 > 1 - 1 = v(\{1,2,3\}) - v(\{1,2\})$.

2. (a) This is a simple game, and player 3 is the only veto player. Since the core of a simple game consists exactly of the vectors that distribute only among veto players, the core of this game consists only of $\boxed{(0,0,1)}$.

   (b) This is a scaled version of a weighted voting game where the payoffs are multiplied by 6. Players 2 and 3 are veto players, while player 1 is not, so the payoff vectors in the core are $\boxed{(0, x, 6-x) \text{ for any } 0 \leq x \leq 6}$.

   (c) This is a simple game where players 1 and 2 are veto players, so the payoff vectors in the core are $\boxed{(x, 1-x, 0, 0, \ldots, 0) \text{ for any } 0 \leq x \leq 1}$.

   (d) We claim that the core is $\boxed{\text{empty}}$. Suppose for contradiction that there is a payoff vector $(x_1, \ldots, x_{10})$ in the core. By efficiency, we have $x_1 + \cdots + x_{10} = 11$. Now, any coalition of nine players can make a value of 10 on their own, so for each $i \in \{1, 2, \ldots, 10\}$, we must have

   $$11 - x_i = (x_1 + \cdots + x_{10}) - x_i \geq 10,$$

   which means that $x_i \leq 1$. However, this implies that $x_1 + \cdots + x_{10} \leq 10$, a contradiction.

3. (a) Consider the weighted voting game with four players, each having weight 1, and the threshold is 3. This is a simple game with no veto player, so the core is empty. (It is not hard to verify that the game is monotone and superadditive.)

(b) Consider the weighted voting game with three players, each having weight 1, and the threshold is 2. This game is not convex because $v(\{1,2\}) - v(\{1\}) = 1 - 0 > 1 - 1 = v(\{1,2,3\}) - v(\{1,3\})$. (It is not hard to verify that the game is monotone and super-additive.)