CS4261/5461 - Algorithmic Mechanism Design
Nash Bargaining Solution Uniqueness Proof

1 The Bargaining Problem

A *bargaining problem* is defined as a pair $\langle S, \vec{d} \rangle$ where $S \subseteq \mathbb{R}^2$ and $\vec{d} \in \mathbb{R}^2$. We assume that there exists at least one point $\vec{x} \in S$ such that $x_1 \geq d_1$ and $x_2 \geq d_2$. Furthermore, we assume that S is compact and convex. In this game, player 1 chooses a value $x \in \mathbb{R}$ and player 2 chooses a value $y \in \mathbb{R}$. If $(x, y) \in S$, then player 1 receives x and player 2 receives y. Otherwise, player 1 receives d_1 and player 2 receives d_2.
What is a 'good' solution to the bargaining problem?

Formally, a solution is a function $\vec{f}(S, \vec{d}) \in \mathbb{R}^2$ that takes as input a bargaining problem $\langle S, \vec{d} \rangle$ and outputs two values $f_1(S, \vec{d})$ and $f_2(S, \vec{d})$ that are paid to players 1 and 2 respectively.

2 Axioms

Efficiency: $\vec{f} \in S$; for every $x > f_1$, $(x, f_2) \notin S$, and for every $y > f_2$, $(f_1, y) \notin S$. This is also known as Pareto efficiency.

Symmetry: $\vec{f}(S^T, (d_2, d_1)) = (f_2(S, \vec{d}), f_1(S, \vec{d}))$

Independence of Irrelevant Alternatives: Suppose that $S' \subseteq S$, and $\vec{f}(S, \vec{d}) \in S'$; then $\vec{f}(S', \vec{d}) = \vec{f}(S, \vec{d})$.

Invariance Under Equivalent Representations: This is basically scale and shift invariance. Given vectors $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ and $\vec{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$, let

$$\vec{\alpha} S + \vec{\beta} = \left\{ (\alpha_1 x + \beta_1, \alpha_2 y + \beta_2) \in \mathbb{R}^2 \mid (x, y) \in S \right\}.$$

Note that we can multiply the x coordinates and the y coordinates by different values! Similarly for a given vector $\vec{v} \in \mathbb{R}^2$, let $\vec{\alpha}\vec{v} + \vec{\beta} = (\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2)$. Then the IUER axiom states that

$$\vec{f}(\vec{\alpha} S + \vec{\beta}, \vec{\alpha}\vec{d} + \vec{\beta}) = \vec{\alpha} f(S, \vec{d}) + \vec{\beta}.$$

3 The Nash Bargaining Solution

Let the Nash bargaining solution be

$$\begin{aligned} &\text{maximize } (x - d_1)(y - d_2) \\ &\text{such that } (x, y) \in S \\ &\qquad\qquad x \geq d_1; y \geq d_2 \end{aligned} \tag{1}$$

Note that since (1) is convex and S is convex, the solution to the above optimization problem is unique. The solution to the optimization problem above is also known as the *Nash bargaining solution*.

Lemma 3.1. *The Nash bargaining solution satisfies all four axioms.*

Proof. We just go over the axioms one by one.

Efficiency: This follows immediately from the fact that the objective (1) is a monotone increasing function in x and y.

Symmetry: Again, this is immediate from the definition of the optimization problem (switching the roles of x and y should lead to the same result)

IIA: Note that if we optimize (1) over $S' \subseteq S$, we can only get a point with a (weakly) worse value (when we optimize over a smaller set with less choices we can only get a worse solution, not a better one). Since the optimal point is unique under S, all other points, those in S' in particular, must have a worse value. Therefore, if the optimal solution to (1) is in S', it is still optimal for S'.

IUER: By the variation operation, we are maximizing $(x - \alpha_1 d_1 - \beta_1)(y - \alpha_2 d_2 - \beta_2)$, under the constraint that $(x, y) \in \vec{\alpha} S + \vec{\beta}$. Thus, we can write $x = \alpha_1 x' + \beta_1; y = \alpha_2 y' + \beta_2$, where $(x', y') \in S$. Plugging this into the target we get

$$\begin{aligned}(x - \alpha_1 d_1 - \beta_1)(y - \alpha_2 d_2 - \beta_2) &= (\alpha_1 x' + \beta_1 - \alpha_1 d_1 - \beta_1)(\alpha_2 y' + \beta_2 - \alpha_2 d_2 - \beta_2) \\ &= \alpha_1 \alpha_2 (x' - d_1)(y' - d_2)\end{aligned}$$

where $(x', y') \in S$. Since $\alpha_1 \alpha_2$ is a constant, the solution is the same as the optimal solution for $\langle S, \vec{d} \rangle$.

This completes the proof. $\square$

To show uniqueness of the Nash bargaining solution, we will need the following technical lemma.

Lemma 3.2. *Given two vectors $\vec{v}^*$ and $\vec{d}$ such that $v_1^* > d_1$ and $v_2^* > d_2$, there exist vectors $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}_+^2$ and $\vec{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$ such that*

$$\begin{aligned}\alpha_1 v_1^* + \beta_1 = \frac{1}{2}; \quad \alpha_2 v_2^* + \beta_2 &= \frac{1}{2} \\ \alpha_1 d_1 + \beta_1 = 0; \quad \alpha_2 d_2 + \beta_2 &= 0.\end{aligned}$$

Proof. We can actually solve this analytically since there are four variables $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and four equalities. The explicit values are

$$\begin{aligned}\alpha_1 = \frac{1}{2(v_1^* - d_1)}; \quad \alpha_2 &= \frac{1}{2(v_2^* - d_2)} \\ \beta_1 = -\frac{d_1}{2(v_1^* - d_1)}; \quad \beta_2 &= -\frac{d_2}{2(v_2^* - d_2)}.\end{aligned}$$

Note that since $v_1^* > d_1$ and $v_2^* > d_2$, both α_1 and α_2 are strictly positive, as desired. $\square$

Theorem 3.3. *The Nash bargaining solution is the only solution satisfying efficiency, symmetry, IIA and IUER.*

Proof. Given a bargaining problem $\langle S, \vec{d} \rangle$ let $\vec{v}^* = (v_1^*, v_2^*)$ be the Nash bargaining solution. Let

$$S' = \left\{ \vec{\alpha}(x, y) + \vec{\beta} \,\middle|\, \vec{\alpha}\vec{v}^* + \vec{\beta} = \left(\frac{1}{2}, \frac{1}{2}\right); \vec{\alpha}\vec{d} + \beta = (0, 0); (x, y) \in S \right\}.$$

Such vectors $\vec{\alpha}$ and $\vec{\beta}$ exist by Lemma 3.2. We observe that since the Nash bargaining solution satisfies IUER, $\alpha\vec{v}^* + \vec{\beta} = (\frac{1}{2}, \frac{1}{2})$ is the Nash bargaining solution for $\langle S', (0, 0) \rangle$. Note that if we show that any

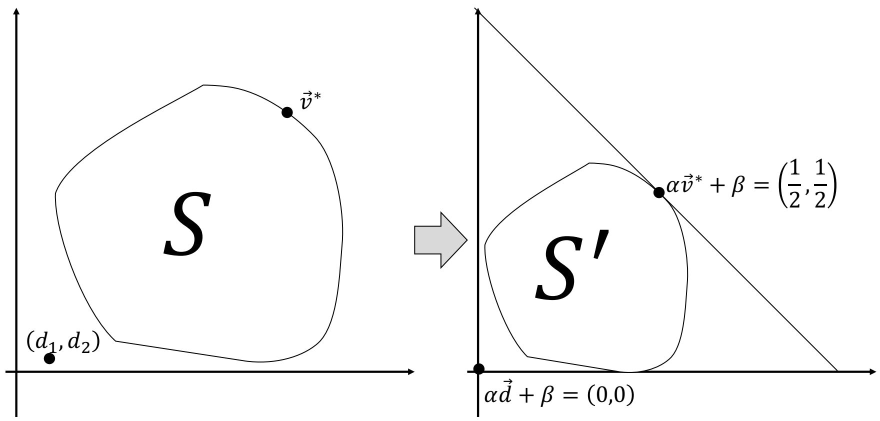

Figure 1: A Visualization of the Nash Bargaining Solution Transformation

solution that satisfies the four axioms coincides with the Nash solution on $\langle S', (0,0)\rangle$, then by IUER the claim will hold for $\langle S, \vec{d}\rangle$ and we will be done.

Next, S' cannot contain any point $(a, b) \in \mathbb{R}^2_+$ such that $a + b > 1$. Suppose that such a point exists, and assume without loss of generality that $a > \frac{1}{2}$. We can find a value $\lambda \in (0, 1)$ such that

$$\left(\lambda \cdot \frac{1}{2} + (1-\lambda)a\right) \cdot \left(\lambda \cdot \frac{1}{2} + (1-\lambda)b\right) > \frac{1}{4}$$

for a sufficiently large λ (check!). Since S' is convex, the point $(\lambda \cdot \frac{1}{2} + (1-\lambda)a, \lambda \cdot \frac{1}{2} + (1-\lambda)b)$ is in S' as well, and has a higher value for (1) than $(\frac{1}{2}, \frac{1}{2})$, a contradiction to $(\frac{1}{2}, \frac{1}{2})$ being the Nash bargaining solution for S'.

We conclude that the set of all non-negative points in S' (i.e., $\mathbb{R}^2_+ \cap S'$) is contained in a triangle whose endpoints are $(1, 0), (0, 1), (0, 0)$; let us call this triangle T. This is visualized in Figure 1.

Any solution that satisfies efficiency and symmetry must choose the point $(\frac{1}{2}, \frac{1}{2})$ on this triangle. We get that if a solution f satisfies all four axioms, then

$$\left(\frac{1}{2}, \frac{1}{2}\right) = f_{Nash}(S', (0,0)) = f(T, (0,0)) = f(S', (0,0)),$$

which concludes the proof. $\square$