Week 11: Committee Voting

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Voting

Selecting a public outcome, shared by everyone.

Voting

- Types of ballots (i.e., input):
  - Ranking: Submit an ordering of the candidates
  - Score: Submit a number for each candidate
  - Approval: Submit a subset of approved candidates
    - Advantages: Simple, less cognitive effort required
    - Disadvantages: Does not allow refined or combinatorial preferences

- Output of voting:
  - Single-winner (e.g., president of student council)
  - Multiwinner/committee (e.g., student council committee, places to visit on a family trip, dishes to serve in a reunion party)
  - Ranked committee (e.g., president, vice president, and secretary)

Approval Committee Voting

- Voters $N = \{1, 2, \ldots, n\}$
- Candidates C, where $|C| = m$
- Voter i approves a set of candidates $A_i \subseteq C$
- We want to choose a committee W of size k, where $k \leq m$ is given
- Voter i's utility is $u_i(W) = |A_i \cap W|$

- Social welfare: Total number of approvals obtained by members of the committee ("excellence")
- Coverage: Number of voters who approve at least one committee member ("diversity")
- If a voter approves a candidate, we say that the candidate covers the voter.

AV and CC

- Approval Voting (AV): Select a committee maximizing social welfare
- Chamberlin–Courant (CC): Select a committee maximizing coverage

- **Example:**
  - $n = m = 6$, $k = 3$
  - $A_1 = \{a, b, c, d, e\}$
  - $A_2 = \{b, c, d, e, f\}$
  - $A_3 = \{a, b, c, d\}$
  - $A_4 = \{a, b, c\}$
  - $A_5 = A_6 = \{e, f\}$

  - b, c, e: 4 votes, a, d, f: 3 votes
  - AV returns $\{b, c, e\}$
  - CC returns, for example, $\{a, b, e\}$ (covers all voters)

Beyond AV/CC

- Let's consider a more extreme example:
  - $n = 301$, $m = 5$, $k = 3$
  - $A_1 = A_2 = \cdots = A_{200} = \{a, b, c\}$
  - $A_{201} = A_{202} = \cdots = A_{300} = \{d\}$
  - $A_{301} = \{e\}$

  - AV returns $\{a, b, c\}$
  - CC returns, e.g., $\{a, d, e\}$
  - Neither feels "proportional" . . .
  - . . . a more proportional committee would be, e.g., $\{a, b, d\}$

- Intuition of proportionality: A sufficiently large group of voters that agrees on sufficiently many candidates should be correspondingly satisfied in the committee.

Justified Representation

- There are n voters and k committee slots, so a group of n/k voters "deserves" one slot.
- **First attempt:** For a group of voters $S \subseteq N$ such that $|S| \geq n/k$ and $|\bigcap_{i \in S} A_i| \geq 1$, we have $|\left(\bigcap_{i \in S} A_i\right) \cap W| \neq \emptyset$.
- Unfortunately, this cannot always be satisfied ...
  - $n = m = 4$, $k = 2$, so $n/k = 2$
  - $A_1 = \{a, b\}, A_2 = \{b, c\}, A_3 = \{c, d\}, A_4 = \{d, a\}$
  - $S = \{1, 2\}$ demands that we pick b
  - Similarly, we must pick c, d, a, but this exceeds the committee size.
- Call a group of voters $S \subseteq N$ such that $|S| \geq n/k$ and $|\bigcap_{i \in S} A_i| \geq 1$ a cohesive group.
- Justified representation (JR): For any cohesive group of voters $S \subseteq N$, there exists $i \in S$ such that $|A_i \cap W| \neq \emptyset$.
- No cohesive group should go unrepresented!

Justified Representation

- AV may fail JR.
  - $n = 300$, $k = 3$
  - $A_1 = A_2 = \cdots = A_{200} = \{a, b, c\}$
  - $A_{201} = A_{202} = \cdots = A_{300} = \{d\}$
  - AV chooses $\{a, b, c\}$
  - $n/k = 100$, so the group of voters $\{201, 202, \ldots, 300\}$ is cohesive, but goes unrepresented in the AV committee!
- CC always satisfies JR.
  - Suppose for contradiction that it does not.
  - Let S be a cohesive group of voters that is unrepresented by the CC committee W, and let x be a candidate approved by all voters in S.
  - Consider the marginal contribution of each $w \in W$ to the coverage. This is the number of voters who approve w but no one else in W.
  - Since the coverage of W is $< n$, the marginal contribution of some $w^* \in W$ is less than n/k.
  - Remove w^* and add x to obtain higher coverage.

GreedyCC

- Computing a CC committee is NP-hard (by a reduction from SET COVER)
- Fortunately, there is a greedy variant, which runs in polynomial time.
- GreedyCC:
  - Start with an empty set of candidates.
  - In each step, choose a candidate that covers as many uncovered voters as possible.
  - Repeat this until k candidates have been chosen.
- **Example:**
  - $n = 6,\ k = 3$
  - $A_1 = \{a, d\}, A_2 = \{b, d\}, A_3 = \{c, d\}, A_4 = \{a\}, A_5 = \{b\}, A_6 = \{c\}$
  - GreedyCC returns, e.g., $\{a, b, d\}$.
  - Coverage is worse than CC committee $\{a, b, c\}$.
- GreedyCC satisfies JR (see assignment)

Is JR Sufficient?

- Is JR sufficient?
  - $n = 100$, $k = 10$
  - $A_1 = A_2 = \cdots = A_{50} = \{a_1, a_2, \ldots, a_{10}\}$
  - $A_{51} = A_{52} = \cdots = A_{100} = \{b_1, b_2, \ldots, b_{10}\}$
- Does $W = \{a_1, a_2, \ldots, a_{10}\}$ satisfy JR?
- No. Voters $51, 52, \ldots, 60$ form a cohesive group that is unrepresented.
- Does $W = \{a_1, a_2, \ldots, a_9, b_1\}$ satisfy JR?
- Yes.
- But $\{a_1, a_2, \ldots, a_5, b_1, b_2, \ldots, b_5\}$ feels much more "proportional"!

Extended Justified Representation

- For a positive integer t, call a group of voters $S \subseteq N$ such that $|S| \geq t \cdot n/k$ and $|\bigcap_{i \in S} A_i| \geq t$ a t-cohesive group.
- The previous definition of cohesive group is when $t = 1$.

- Extended Justified representation (EJR): For any positive integer t and any t-cohesive group of voters $S \subseteq N$, there exists $i \in S$ such that $|A_i \cap W| \geq t$.

- This fixes the problem with JR.
  - $n = 100$, $k = 10$
  - $A_1 = A_2 = \cdots = A_{50} = \{a_1, a_2, \ldots, a_{10}\}$
  - $A_{51} = A_{52} = \cdots = A_{100} = \{b_1, b_2, \ldots, b_{10}\}$
  - $W = \{a_1, a_2, \ldots, a_9, b_1\}$ fails EJR: Take $S = \{51, 52, \ldots, 100\}$ and $t = 5$
  - $W = \{a_1, a_2, \ldots, a_5, b_1, b_2, \ldots, b_5\}$ satisfies EJR.

Proportional Approval Voting

- Fix an infinite nonincreasing sequence $s_1, s_2, \ldots$
- Thiele methods: Choose a committee W maximizing the score

$$\sum_{i \in N} \left(s_1 + s_2 + \cdots + s_{u_i(W)} \right)$$

- AV: $s_i = 1$ for all i
- CC: $s_1 = 1$, $s_2 = s_3 = \cdots = 0$
- Proportional Approval Voting (PAV): $s_i = 1/i$ for all i
  - If a voter approves r candidates in the committee, the voter contributes $1 + \frac{1}{2} + \cdots + \frac{1}{r}$ to the score of the committee.
  - $1 + \frac{1}{2} + \cdots + \frac{1}{r}$ is the r-th harmonic number, usually denoted by H_r

Harmonic Numbers

- Why harmonic numbers?
- Harmonic numbers result in a roughly "proportional" committee.
  - $n = 12$, $k = 6$
  - $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = \{a_1, \ldots, a_6\}$
  - $A_7 = A_8 = A_9 = A_{10} = \{b_1, \ldots, b_6\}$
  - $A_{11} = A_{12} = \{c_1, \ldots, c_6\}$
  - AV: $\{a_1, \ldots, a_6\}$
  - CC: Any committee containing at least one candidate from each of the three "categories"
  - Marginal contribution to the score of candidates from each category
  - a_i : 6, 3, 2, 1.5, 1.2, 1
  - b_i : 4, 2, 1.3, 1, 0.8, 0.7
  - c_i : 2, 1, 0.7, 0.5, 0.4, 0.3
  - PAV: Choose 3 of the a_i's, 2 of the b_i's, and 1 of the c_i's

PAV and Nash

- AV = utilitarian
- CC $\approx$ egalitarian
- PAV $\approx$ Nash
  - Maximize $\sum_{i \in N} H_{u_i(W)}$
  - **Fact:** $H_r \approx \ln r$ for each positive integer r
  - Maximizing $\sum_{i \in N} \ln(u_i(W))$ is the same as maximizing $\prod_{i \in N} u_i(W)$, i.e., maximizing the Nash welfare!

- Can we use Nash instead of PAV?
- Not a good idea!
  - $n = 301$, $k = 3$
  - $A_1 = A_2 = \cdots = A_{200} = \{a, b, c\}$
  - $A_{201} = A_{202} = \cdots = A_{300} = \{d\}$
  - $A_{301} = \{e\}$
  - Nash returns, e.g., $\{a, d, e\}$

PAV and EJR

- **Theorem:** PAV satisfies EJR.
- **Proof idea:**
  - Suppose for contradiction that $u_i(W) < t$ for all voters i belonging to some t-cohesive group S.
  - There is a candidate x which is approved by all members of S but not included in W.
  - By adding x to W, the PAV score increases by at least $(1/t) \cdot (tn/k) = n/k$.
  - Need to show that there is a candidate $y \in W$ such that by removing y from $W \cup \{x\}$, the PAV score decreases by less than n/k.
- PAV is NP-hard to compute.
- A greedy variant of PAV does not even satisfy JR.
- Can we satisfy EJR in polynomial time?

Method of Equal Shares

- Method of Equal Shares (MES):
  - Each voter has a budget of k/n.
  - Each candidate costs 1; the voters who approve this candidate have to "pool" their money to add this candidate to the committee.
  - Start with an empty committee.
  - In each round, we want to add a candidate whose approved voters have a total budget of ≥ 1 left.
  - If there are several such candidates, choose one such that the maximum amount that any agent has to pay is minimized.
  - If no more candidate can be afforded but the committee still has size $< k$, fill in the rest of the committee using some tie-breaking criterion (e.g., by maximizing approval score).

Method of Equal Shares

- **Example:**
  - $n = 8, k = 3$
  - $A_1 = A_2 = A_3 = \{a, b\}$
  - $A_4 = A_5 = \{c, d\}$
  - $A_6 = A_7 = \{a, c\}$
  - $A_8 = \{b, d\}$
- Each voter starts with a budget of $3/8$.
- a is chosen first. Each of voters $1, 2, 3, 6, 7$ pays $1/5$, and has budget $3/8 - 1/5 = 7/40$ left.
- b is not affordable, c would require some voter to pay $13/40$, while d would require some voter to pay $1/3$.
- Since $13/40 < 1/3$, c is chosen second. Voters $6, 7$ pay $7/40$ each (and have 0 left), while voters $4, 5$ pay $13/40$ each (have $1/20$ left).
- No more candidate is affordable, so b is chosen third by if we do the tie-breaking by approval score.

Method of Equal Shares

- MES never chooses more than k candidates.
  - Total budget of all n voters is $(k/n) \cdot n = k$, and each candidate costs 1.
- **Theorem:** MES satisfies EJR (and can be implemented in polytime).
- **Proof:**
  - Suppose for contradiction that $u_i(W) < t$ for all voters i belonging to some t-cohesive group S.
  - When MES stops, some voter $i \in S$ must have budget $< \frac{k}{tn}$ left. (Otherwise, the voters in S have budget $\geq |S| \cdot \frac{k}{tn} \geq 1$ and should have bought a candidate that they all approve.)
  - i has used budget $> \frac{k}{n} - \frac{k}{tn} = \frac{(t-1)k}{tn}$, so for some chosen committee member, i paid more than $\frac{1}{t-1} \cdot \frac{(t-1)k}{tn} = \frac{k}{tn}$.

Method of Equal Shares

- **Proof (cont.):**
  - Consider the first committee member x such that some voter in S paid more than $\frac{k}{tn}$ for it.
  - Before x was added, each voter in S has $\leq t-1$ approved candidates, and paid $\leq \frac{k}{tn}$ for each of them.
  - Thus, each voter in S has budget at least $\frac{k}{n} - (t-1) \cdot \frac{k}{tn} = \frac{k}{tn}$ remaining.
  - Since $|S| \geq \frac{tn}{k}$, the voters in S could afford a commonly approved candidate by paying $\leq \frac{k}{tn}$ each.
  - No voter in S should have paid more than $\frac{k}{tn}$ for x, a contradiction!

Summary

	JR	EJR	Polytime
AV (maximizes welfare)	✗	✗	✓
CC (maximizes coverage)	✓	✗	✗
GreedyCC	✓	✗	✓
PAV	✓	✓	✗
MES	✓	✓	✓

- Participatory budgeting: A generalization of committee voting where the “candidates” (i.e., projects) may have unequal costs.

Participatory Budgeting

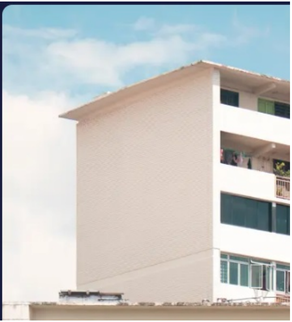
Participatory Budgeting & Citizen Design in Town Councils

WHAT IS IT?

Participatory Budgeting (PB) is a process whereby a community decides how to spend a portion of public budget.[1]

The process can be used by a Town Council to engage its citizens in developing ideas, deliberate on them, and vote on how the budget is used. A portion of the discretionary budget for estate improvement can be earmarked for this purpose as a social experiment.

We can start with a specific set of blocks or HDB estate within a Town Council if we can secure the support of a sponsoring MP.



http://futurereadysociety.sg/participatory-budgeting-citizen-design-in-town-councils

MES in Participatory Budgeting

Method of Equal Shares Explanation Benefits Implementation

The **Method of Equal Shares** is a fairer voting rule for participatory budgeting.

It provides proportional representation and allows every voter to decide about an equal part of the budget.

CH Winterthur

NL Assen

PL Świecie

CH Aarau

PL Wieliczka

http://equalshares.net/