CS5461 Assignment 9

Li Jiaru (SID: A0332008U)

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1. (a) Agent 4's value is calculated as

$$\begin{aligned} v_4\left(\frac{1}{3},\frac{2}{3}\right) &= \int_{\frac{1}{3}}^{\frac{2}{3}} f_4(x)\mathrm{d}x \\ &= \int_{\frac{1}{3}}^{\frac{1}{2}} f_4(x)\mathrm{d}x + \int_{\frac{1}{2}}^{\frac{2}{3}} f_4(x)\mathrm{d}x \\ &= 0 + \int_{\frac{1}{2}}^{\frac{2}{3}} (8x-4)\mathrm{d}x = \boxed{\frac{1}{9}}. \end{aligned}$$

   (b) For the Dubins–Spanier protocol, we consider, in each round, which agent shouts first (i.e., first attains a value of 1/4).

   - Round 1: agents 1 to 4 will shout at $x = \frac{1}{4}, \frac{1}{8}, \frac{1}{20}, \frac{3}{4}$ respectively. Therefore we give $\left[0, \frac{1}{20}\right]$ to agent 3 and continue.
   - Round 2: agents 1, 2, 4 will shout at $x = \frac{3}{10}, \frac{7}{40}, \frac{3}{4}$ respectively. Therefore we give $\left[\frac{1}{20}, \frac{7}{40}\right]$ to agent 2 and continue.
   - Round 3: agents 1 and 4 will shout at $x = \frac{17}{40}, \frac{3}{4}$ respectively. Therefore we give $\left[\frac{7}{40}, \frac{17}{40}\right]$ to agent 1 and the rest to agent 4.

   In conclusion, agents 1 to 4 get $\boxed{\left[\frac{7}{40}, \frac{17}{40}\right], \left[\frac{1}{20}, \frac{7}{40}\right], \left[0, \frac{1}{20}\right], \left[\frac{17}{40}, 1\right]}$ respectively.

   (c) For the Even–Paz protocol, we consider, in each round, where each agent marks the 'midpoint' for them.

   - Round 1: agents 1 to 4 mark at $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{10}, \frac{\sqrt{2}+2}{2}$ respectively. Therefore the left group consists of agents 2, 3 on $\left[0, \frac{1}{4}\right]$ and the right group consists of agents 1, 4 on $\left[\frac{1}{4}, 1\right]$.
   - Round 2: for the left group, agents 2 and 3 mark at $x = \frac{1}{8}, \frac{1}{10}$ respectively. Therefore we give $\left[0, \frac{1}{10}\right]$ to agent 3 and the rest to agent 2.
   - Round 3: for the right group, agents 1 and 4 mark at $x = \frac{5}{8}, \frac{\sqrt{2}+2}{2}$ respectively. Therefore we give $\left[\frac{1}{4}, \frac{5}{8}\right]$ to agent 1 and the rest to agent 4.

   In conclusion, agents 1 to 4 get $\boxed{\left[\frac{1}{4}, \frac{5}{8}\right], \left[\frac{1}{10}, \frac{1}{4}\right], \left[0, \frac{1}{10}\right], \left[\frac{5}{8}, 1\right]}$ respectively.

2. No. Consider a counterexample where agents 1 and 2 have density functions

$$f_1(x) = \begin{cases} 2, & x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right], \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 2, & x \in \left[\frac{1}{4}, \frac{1}{2}\right] \cup \left[\frac{3}{4}, 1\right], \\ 0, & \text{otherwise.} \end{cases}$$

Under the cut-and-choose protocol, agent 1 will cut at $x = \frac{1}{2}$, leaving a value of $\frac{1}{2}$ for both agents. However, we could have allocated $\left[0, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right]$ to agent 1 and $\left[\frac{1}{4}, \frac{1}{2}\right] \cup \left[\frac{3}{4}, 1\right]$ to agent 2, giving each a value of 1. Therefore the protocol is not always Pareto optimal.

3. The answer is $c = \boxed{2/3}$.

We first show that the envy is always at most 2/3. Notice that whenever an agent takes a piece, every other remaining agent values that piece by at most 1/6, since otherwise they would have shouted first.

For any two agents, we always have one of the three cases:

- Both get their pieces by shouting. Then both are envy-free since each gets a piece with value at least 1/6 and values the other's piece at most 1/6.
- One gets by shouting and another gets the remaining cake (or nothing), say i and j respectively. Then i values j's piece by at most $1 - 1/6 = 5/6$, so i will envy j by at most $5/6 - 1/6 = 2/3$.
- Both remained till the end. Then one's envy on another is at most 1/6 since whatever they get has value at most 1/6.

In all cases the envy is indeed always at most 2/3. We now show an example where the envy is at least 2/3. Consider an example where agents 1, 2 and 3 have density functions

$$f_1(x) = \begin{cases} \frac{1}{2}, & x \in \left[0, \frac{1}{3}\right], \\ \frac{5}{4}, & x \in \left(\frac{1}{3}, 1\right]; \end{cases} \quad f_2(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{3}\right], \\ \frac{1}{2}, & x \in \left(\frac{1}{3}, \frac{2}{3}\right], \\ \frac{5}{2}, & x \in \left(\frac{2}{3}, 1\right]; \end{cases} \quad f_3(x) = \begin{cases} 0, & x \in \left[0, \frac{2}{3}\right], \\ 3, & x \in \left(\frac{2}{3}, 1\right]. \end{cases}$$

Note that agents 1, 2 and 3 will shout at $x = 1/3, 2/3, 1$ respectively, so each agent gets exactly 1/3 of the original cake. Note also that agent 1 envies agent 3 by $5/4 \times 1/3 - 1/2 \times 1/3 = 2/3$.

In conclusion, we have shown that, in general, in the output of this protocol, any agent always envies any other agent by at most 2/3, and this constant is indeed optimal.

(In fact, by the same argument, we could show that $c = \max(t, 1 - 2t)$ for any 'threshold' t; here $t = 1/6$. Since $0 \le t \le 1$, c is minimised at $t = 1/3$ with $c_{\min} = 1/3$. Therefore, the constant 1/3 used in lecture is indeed optimal in some sense.)