Optional Material: Mixed Indivisible + Divisible Goods

Model: Mixed Indivisible and Divisible Goods

- Agents $N = \{1, 2, \dots, n\}$
- m indivisible goods and a cake
- Each agent has
  utility function for the indivisible goods;
  density function for the cake.
- Allocation $A = (A_1, A_2, \dots, A_n)$, where $A_i = M_i \cup C_i$
  Indivisible goods: $(M_1, M_2, \dots, M_n)$
  Cake: $(C_1, C_2, \dots, C_n)$
- Utility $u_i(A_i) = u_i(M_i) + u_i(C_i)$

Candidate Fairness Notions

- Envy-freeness (EF): No agent envies another.

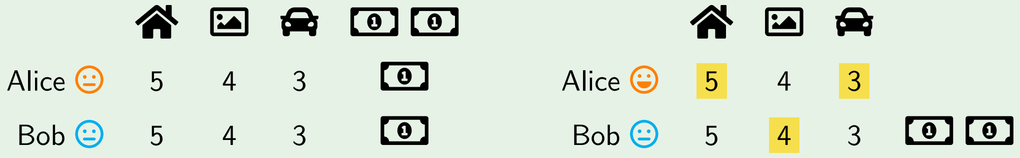
$$\forall i, j \in N, u_i(A_i) \geq u_i(A_j)$$

- Envy-freeness up to one (indivisible) good (EF1): Any envy that an agent has towards another agent can be eliminated by removing *some* good from the latter agent's bundle.

$$\forall i, j \in N, \exists g \in A_j \text{ such that } u_i(A_i) \geq u_i(A_j \setminus \{g\})$$

- EF for divisible goods + EF1 for indivisible goods.

Alice and Bob divide three indivisible goods and two dollars



Envy-freeness for Mixed Goods (EFM)

Definition (EFM)

For all agents i, j,

- if agent j's bundle consists of *only* indivisible goods, there exists $g \in A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{g\})$;
- otherwise, $u_i(A_i) \geq u_i(A_j)$.

With only divisible goods: EFM reduces to EF.

With only indivisible goods: EFM reduces to EF1.

EFM Existence

Theorem

EFM allocations always exist for any number of agents and can be found in polynomial time.

Proof Sketch.

- Start with an EF1 allocation of indivisible goods.
- Iteratively (and carefully) add some cake.
- Maintain EFM throughout the process. $\square$

Envy Graph

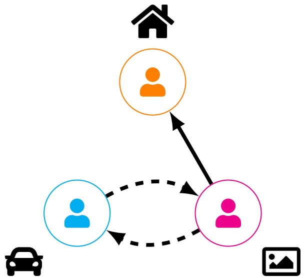
Definition

A directed graph of agents with

Envy edge: $i \longrightarrow j$ if $u_i(A_i) < u_i(A_j)$;

Equality edge: $i \dashrightarrow j$ if $u_i(A_i) = u_i(A_j)$.

	5	4	1	5
	3	4	4	5
	4	3	3	5

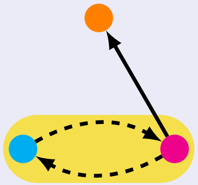


Addable Set

Definition

A subset of agents $S \subseteq N$ such that

- no envy edge in S;
- no edge from $N \setminus S$ to S.



Maximal addable set

There does not exist any other addable set $S' \subseteq N$ such that $S \subsetneq S'$.

- If exists, is unique.
- Can be found in polynomial time.

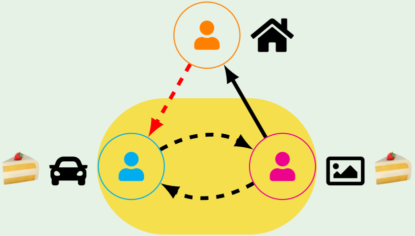
Intuition

Add some cake to the maximal addable set (in a "perfect" manner).

Cake-Adding Phase

Add some cake to the maximal addable set S

5	4	1	5
3	**4**	4	5
4	3	**3**	5



Perfect allocation [Alon, 1987]

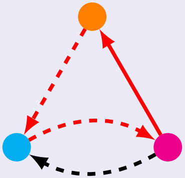
Every agent in N values all $|S|$ pieces equally.

Given an EFM allocation, after a cake-adding phase, the resulting allocation is still EFM.

Envy Cycle

Definition

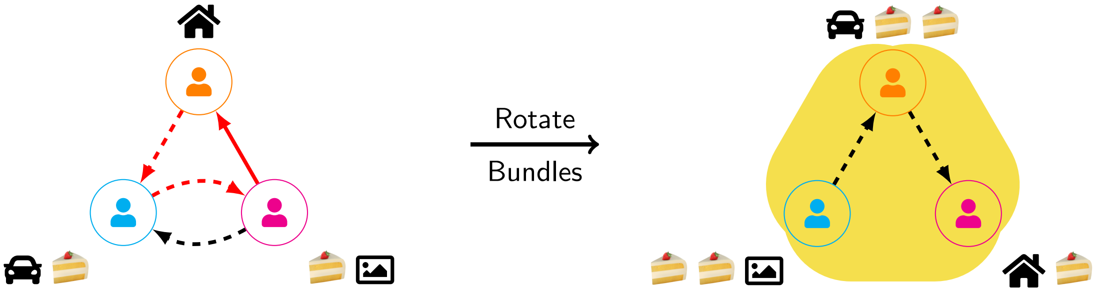
A cycle in the envy graph with at least one *envy* edge.



Intuition

Eliminate an envy cycle by rotating bundles.

Envy-Cycle-Elimination Phase



Given an EFM allocation, after an envy-cycle-elimination phase, the allocation is still EFM.

What can we do now?

Connection Between Addable Set and Envy Cycle

Key Lemma

At any time, there exists either an addable set or an envy cycle.

- Always make progress.
- The partial allocation is always EFM.
- The process always terminates.

Caveat

- A polynomial-time algorithm if we have a perfect allocation oracle for cake cutting.
- The perfect allocation oracle cannot be implemented in a bounded time in the Robertson–Webb model.

Open Question

A bounded (or even finite) EFM protocol in the Robertson–Webb model?

EFM Relaxation

ε-Envy-freeness for mixed goods (ε-EFM)

For all agents i, j,

- if agent j's bundle consists of only indivisible goods, there exists $g \in A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{g\})$;
- otherwise, $u_i(A_i) \geq u_i(A_j) - \varepsilon$.

Theorem

An ε-EFM allocation can be found in time $\text{poly}(n, m, \frac{1}{\varepsilon})$ in the Robertson–Webb model.