Instructions

This homework comprises 5 problems. Attempt all questions and submit your solutions on Canvas. You may scan it (if handwritten), though you are highly encouraged to type up your solutions. Upload your solutions in a zip file containing (i) a pdf document containing your solutions, and (ii) code for question 4. I will not grade your code explicitly, but you are still required to upload them. The preferred language of coding is Python. You are allowed to use numpy, as well as any other linear programming or optimization libraries. There are a few bonus problems. These are worth a few points each and are only recommended if you are very interested in the topic. It is best to **attempt problem 4 only after attending Lecture 5**, where we review regret minimization. It may also help to **review Lecture 2 and 3**, and in particular the slide on dominated strategies .

This homework looks lengthy, but do not be intimidated. Most of the text is simply to describe the problem (or give hints). To make things slightly clearer, the parts where you are required to submit solutions are in highlighted in blue (excluding bonus problems). Nonetheless, be sure to read the full question thoroughly. Erratas (if any) will be given in red.

This homework is to be completed in **teams of 2-3** and is due in ~ 3 weeks. Only one group member needs to submit the assignment. Make sure to **include the names of all team members**. Do not work on the assignment as an individual without my explicit permission. Please start early and attend office hours if you are facing difficulties. You are allowed to utilize generative AI (at your own risk). However, you are required to write the solutions yourself and acknowledge any external help received.

1 Can having more actions hurt? [20 points]

Let $n, m \in \{1, 2, \dots\}$. Consider a 2-player game G, where $A, B \in \mathbb{R}^{n\times m}$ are payoff matrices for players 1 and 2 respectively. Now consider another game $\hat{G}$ with payoff matrices $\hat{A}, \hat{B} \in \mathbb{R}^{(n+1)\times m}$ obtained from A, B by adding a single action to player 1, with payoffs $a, b \in \mathbb{R}^m$ associated to this new $(n + 1)$-th action such that

$$\hat{A} = \begin{bmatrix} A \\ a_1, a_2, \dots, a_m \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} B \\ b_1, b_2, \dots, b_m \end{bmatrix}. \tag{1}$$

Here, a_i and b_j are the i-th and j-th entries of vectors a and b.

For this question, **assume both players seek to maximize their expected payoffs**. Our objective is to examine if (and when) the addition of this new $(n + 1)$-th action can possibly hurt harm or harm either/both players. To avoid complications with equilibrium selection, we restrict ourselves to cases where G has a unique NE (x^*, y^*). Thus, the expected utilities at equilibrium for G are $U_1^* = {x^*}^\top A y^*$ and $U_2^* = {x^*}^\top B y^*$ respectively.

(a) Zero-sum games [10 points] Assume that $\hat{A} = -\hat{B}$ (i.e., $A = -B$ and $a = -b$). Recall that $U_1^* = -U_2^*$. Let $(\hat{x}^*, \hat{y}^*)$ be some NE of $\hat{G}$ such that $\hat{U}_1^* = {\hat{x}^*}^\top \hat{A} \hat{y}^*$. Prove that $U_1^* \leq \hat{U}_1^*$ and consequently, $U_2^* \geq \hat{U}_2^*$.

Hint: recall from class that the value of a zero-sum game is given by the min-max or max-min problem. You may also use the fact that $\max_{x \in \mathcal{X}} f(x) \leq \max_{x' \in \mathcal{X}'} f(x')$ for sets $\mathcal{X} \subseteq \mathcal{X}' \subseteq \mathbb{R}^z$, where z is some positive integer and assuming of course that these maximums exist.

This shows that having additional actions — regardless of what $\hat{A}$ is — cannot strictly hurt a player when the game is 2-player zero-sum. Conversely, the other player cannot strictly benefit from the other player having more actions.

(b) General-sum games [10 points] We will construct examples showing that the above rule does not always hold in general-sum games. For parts (i)-(iii), your solution must hold for *any* A, B that has a unique Nash. You may construct the added row a, b in any way you wish, possibly with reference to the entries of payoff matrices A_{ij} and B_{ij}. Since $\hat{G}$ may have multiple Nash, you are free to choose between them when referencing $\hat{U}_1^*$ and $\hat{U}_2^*$.

(i) Show that there exists some a, b such that $U_1^* < \hat{U}_1^*$ and $U_2^* < \hat{U}_2^*$, i.e., both players benefit from this new row. Hint: consider the case that the row added is a strictly dominant action of player 1.

(ii) Show there exists some a, b such that $U_1^* < \hat{U}_1^*$ but $U_2^* > \hat{U}_2^*$, i.e., the row player benefits, but column player is hurt. Hint: again, consider the case that the row added is a strictly dominant action of player 1.

(iii) Show there exists some a, b such that $U_1^* = \hat{U}_1^*$ and $U_2^* = \hat{U}_2^*$. Hint: consider the case that the row added is a strictly *dominated* action of player 1.

We now move on to the counter-intuitive yet important case. We will specify A and B such that player 1 is harmed by the presence of *its own* additional action.

$$A = \begin{bmatrix} 10 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 10 & 0 \end{bmatrix}. \tag{2}$$

Clearly this game G is cooperative, with a unique deterministic Nash yielding a utility of 10 for each player. Now, let $c \in \mathbb{R}$ be some constant and consider the new game $\hat{G}$ given by

$$a = \begin{bmatrix} 100 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} -1 & c \end{bmatrix} \tag{3}$$

such that

$$\hat{A} = \begin{bmatrix} 10 & 0 \\ 100 & 1 \end{bmatrix}, \hat{B} = \begin{bmatrix} 10 & 0 \\ -1 & c \end{bmatrix} \tag{4}$$

(iv) Suppose $c = 1$. Show that with this choice of $\hat{A}, \hat{B}$, the *unique* NE is for both players to play the second action (i.e., the pure NE is the bottom right outcome). Observe that player 1's payoff under Nash in $\hat{G}$ is 1, which is strictly less than what it got in G. Furthermore, player 2 suffered as well — it now got a payoff of $c = 1$ instead of 10 in G.

(v) We will now relax the assumption that $c = 1$. Write down the expected utility for each player under Nash in $\hat{G}$ for every value of $c \in \mathbb{R}$. If there are multiple NE for certain values of c, select your favourite one. Your answer should be of the form

$$\hat{U}_1^*(c) = \ldots, \qquad \text{and} \qquad \hat{U}_2^*(c) = \ldots$$

Hint: is player 1 ever incentivized to play its second action?

2 NE under transformations on player utilities [20 points]

Let $n, m \in \{1, 2, \dots\}$. Consider a 2-player game G, where $A, B \in \mathbb{R}^{n\times m}$ are payoff matrices for players 1 and 2 respectively. Likewise, let consider the game $\hat{G}$ with payoff matrices $\hat{A}, \hat{B} \in \mathbb{R}^{n\times m}$.

(i) Show that if $\hat{A} = \alpha_1 A + \beta_1 \mathbb{1}^{n\times m}$, where $\mathbb{1}^{n\times m}$ is the all-ones matrix of size $n \times m$, $\alpha_1 \in \mathbb{R}_{++}$ (i.e., strictly positive), $\beta_1 \in \mathbb{R}$, $\hat{B} = B$ then *the set* of NE of G is equal to that of $\hat{G}$, i.e., the set of NE is invariant to non-negative scaling and translations of a player's payoff matrices.[1] The proof for this fact can be easily found online, but I encourage you to go through the thought process yourself.

(ii) Give an example (e.g., using a dummy second player with a single action) where the statement in (i) is untrue if α_1 is allowed to be negative.

Consider a modified game of matching pennies (the soccer game) with fair coins with stakes \$$s$, i.e., the winner gets \$$s$ and the loser loses \$$s$, where $s > 0$. However, you (the row player) are allowed to surrender at the cost of paying \$5 to the opponent. The payoff matrices A, B in dollars are

$$A = \begin{bmatrix} s & -s \\ -s & s \\ -5 & -5 \end{bmatrix}, \qquad B = -A = \begin{bmatrix} -s & s \\ s & -s \\ 5 & 5 \end{bmatrix}. \tag{5}$$

Suppose each player has utility equal to the number of dollars obtained (losing 1 dollar means -1 utility).

(iii) Show that for the modified matching pennies game, the unique NE is (x^*, y^*) is given by $x^* = [1/2, 1/2, 0], y^* = [1/2, 1/2]$, i.e., you would never surrender, and that the value of this zero-sum game is 0.

Now suppose that your utility is non-linear in the number of dollars gained/lost. After all, if you lose more than a certain amount, you'd not be able to pay your bills, rent, nor your PhD students' stipend; this amounts to a much greater deal of pain. We model it as such: for every dollar past \$10 that you lose, you would incur a utility of -2 instead of -1. So, if you lose 10 dollars, you incur a utility of -10, but if you lose 11 dollars, you incur a utility of -12. However, you always incur a utility of 1 for every dollar you earn. This non-linear utility is given by the piecewise linear function

$$h(d) = \begin{cases} 10 + 2d & d \le -10 \\ d & d > -10 \end{cases}. \tag{6}$$

This sort of utility is characteristic of *risk-adverse* utility functions. Suppose $\hat{G}$ is another game with payoff matrices $\hat{A}, \hat{B}$ for players 1 and 2 respectively. Let $\hat{A} \in \mathbb{R}^{3\times 2}$ such that $\hat{A}_{ij} = h(A_{ij})$

[1]Of course, $\hat{B}$ may be subject to the similar scaling and translations. But we keep things simple for now.

and $\hat{B} = B$, i.e.,

$$\hat{A} = \begin{bmatrix} h(s) & h(-s) \\ h(-s) & h(s) \\ h(-5) & h(-5) \end{bmatrix}, \qquad \hat{B} = B = \begin{bmatrix} -s & s \\ s & -s \\ 5 & 5 \end{bmatrix} \tag{7}$$

That is, player 1 suffers from risk adversity, while player 2 continues to enjoy payoffs that are linear in number of dollars obtained. Note that $\hat{G}$ is no longer zero-sum, therefore it may have multiple equilibrium, or at least, equilibria with different utilities for either player.

(iv) Suppose that $\hat{y}^* = [1/2, 1/2]$. Show that for every value of $s > 0$, there exists some $\hat{x}^*$ for which $(\hat{x}^*, \hat{y}^*)$ is a NE. Plot (or draw) a graph of the utility (under $\hat{A}$) of player 1's utility at equilibrium $\hat{x}^{*\top} \hat{A} \hat{y}^*$ as a function of s, where $s \in (0, 30]$. Plot in another graph a similar plot for the utility of player 2 for the same range of s. For values of s where there are multiple $\hat{x}^*$ that constitute an equilibrium, choose your favourite. Hint 1: compute $\hat{A}\hat{y}^*$ and recall what it means from player 1's perspective. Hint 2: what would you expect if $s \to \infty$?

This problem shows that despite the game being fair in terms of dollar value (i.e., in part (iii)), the relative risk appetite between the players can have an impact on the resultant equilibrium, and in more complicated cases possibly exploited by an opponent.

3 Are NE in nested sub-matrix games transitive? [20 points]

Consider the standard Chicken game. We use the following values (taken from Wikipedia). The story is as follows: you and your rival are driving to a head-on collision. Each of you can decide to swerve to avoid the other player (i.e., action 1: **C**hicken) or continue driving ahead (i.e., action 2: **D**are).

$$A = \begin{bmatrix} 6 & 2 \\ 7 & 0 \end{bmatrix}, \qquad B = A^\top = \begin{bmatrix} 6 & 7 \\ 2 & 0 \end{bmatrix} \tag{8}$$

If both players swerve (top left), then the whole interaction is safe. Moreover, both of your egos are kept intact — even though you caved in, so did your rival! This gives a payoff of $(6, 6)$. If neither player swerves (bottom right), then a crash occurs, and both players are left dead with a payoff $(0, 0)$. If exactly one player swerves, then both players survive; the player who swerved is left with a bruised ego (payoff of 2), while the player which continued is euphoric that his rival caved in (payoff of 7). Recall that we showed in class that (Dare, Chicken) and (Chicken, Dare) are pure NE.

(i) Show that there is a mixed strategy NE where each player plays **C**hicken with probability 2/3 and **D**are with probability 1/3.

Now, we will augment the game by allowing the use of lethal **W**eapons. Instead of swerving or continuing, players may also activate their hi-tech laser cannon, which will disintegrate their rival *if they do not swerve.* If this happens, not only are you kept safe, your rival is also eliminated, deriving you great pleasure. However, if your rival does indeed swerve, then not only do they survive, they will file a lawsuit against you for the improper use of weapons, earning you life imprisonment — a fate apparently worse than death. Finally, if both of you activate your laser cannons, then both will die. The payoff matrix is given by

$$\hat{A} = \begin{bmatrix} 6 & 2 & 10 \\ 7 & 0 & 0 \\ -5 & 10 & 0 \end{bmatrix}, \qquad \hat{B} = \hat{A}^{\top} = \begin{bmatrix} 6 & 7 & -5 \\ 2 & 0 & 10 \\ 10 & 0 & 0 \end{bmatrix}, \tag{9}$$

where the third row and column contain payoffs where at least one player utilizes **W**eapons.

(ii) Show that the mixed NE in the original Chicken game still holds, i.e., one where each player plays **C**hicken with probability 2/3 and **D**are with probability 1/3, while no player ever uses their **W**eapon, i.e., $x^* = y^* = [\mathbf{C} = 2/3, \mathbf{D} = 1/3, \mathbf{W} = 0.0]$

(iii) Show that the other 2 pure NE in the original Chicken game $x = [\mathbf{C} = 0.0, \mathbf{D} = 1.0, \mathbf{W} = 0.0]$, $y = [\mathbf{C} = 1.0, \mathbf{D} = 0.0, \mathbf{W} = 0.0]$ (and vice versa) are *not* NE in this augmented game.

For parts (iv)-(vi), we will use the following setting. Let $1 \le n < \hat{n}$ and $1 \le m < \hat{m}$. Let $\hat{G}$ be a game with payoff matrices $\hat{A}, \hat{B} \in \mathbb{R}^{\hat{n} \times \hat{m}}$. Suppose $(\hat{x}^*, \hat{y}^*)$ is an equilibrium of $\hat{G}$, and that both $\hat{x}^*$ and $\hat{y}^*$ have support sizes of n and m respectively (i.e., the NE strategies not have full support). Without loss of generality, assume that the supports of $\hat{x}^*$ and $\hat{y}^*$ are the first n and m actions respectively. Let G' be a new game where players are restricted to playing the first n and m actions respectively, i.e., G' has payoff matrices corresponding to the "top-left rectangle" of those of $\hat{G}$.

(iv) Let $x^* = [\hat{x}^*_1, \dots \hat{x}^*_n] \in \Delta_n$ and Let $y^* = [\hat{y}^*_1, \dots \hat{y}^*_m] \in \Delta_m$, i.e., x^* is equal to $\hat{x}^*$ for all actions played with positive probability. Show that (x^*, y^*) is a NE of G'.

(v) Consider the following conjecture.

Conjecture 1. *Let $x' \in \Delta_n$, $y' \in \Delta_m$. Further assume that (x', y') is some NE of G' (note that x' and y' may not be equal to x^* and y^*). Then, the strategies*

$$\hat{x}' = [x', \underbrace{0, \dots, 0}_{\hat{n}-n \text{ times}}] \in \Delta_{\hat{n}}, \qquad \hat{y}' = [y', \underbrace{0, \dots, 0}_{\hat{m}-m \text{ times}}] \in \Delta_{\hat{m}}$$

is guaranteed to be a NE of the larger game $\hat{G}$.

Informally, what Conjecture 1 says is this. We start with a large game $\hat{G}$, solved it and got a NE with medium-sized support. Then, we take the subgame G' restricted to actions in that medium sized support, solved it, and got a new NE with an even smaller support. This conjecture asserts that this new NE with an even smaller support (which is a NE of the medium game G') is also a NE of the original large game $\hat{G}$.

Using parts (i)-(iv), deduce that Conjecture 1 is False.

(vi) BONUS: does Conjecture 1 hold if we are restricted to 2-player zero-sum games? Prove it, or give a counterexample.

4 Extensions of Tian Ji's horse racing problem [20 points]

This question is an extension of a classic story[2] originating from the Warring States period in Chinese history, and is closely related to the Colonel Blotto game that we briefly discussed in class. Both are classic problems in game theory. We will use this opportunity to implement some of the two-player zero-sum game solvers covered in class.

Our modified game setting is as such. There are $q > 1$ horse races to be held. Each of them is worth $(v_1, \ldots, v_q) \in \mathbb{R}^q_+$ if won (and $-v_1, \ldots, -v_q$ if lost). There are two players, each owns q horses which they will send to the q races. Each horse participates in *exactly* one race; which race it is assigned to is it's owner's decision. The horses have different speeds, $(\alpha_1, \ldots, \alpha_q) \in \mathbb{R}^q_+$ for horses belonging to the first player, and $(\beta_1, \ldots, \beta_q) \in \mathbb{R}^q_+$ for the second. For the j-th race, the faster horse will win the race, and the winner's owner will obtain a *race value* of v_j, while the loser obtained a race value of $-v_j$. If there are ties, both players get a race value of 0 for that race.[3] The *cumulative race value* is the sum of the race values obtained over all q races.

Variant 1 [Utility = sum of race values] The utility that each player gets is its cumulative race value. For example, if player 1 wins race 1, ties in race 2, and loses race 3, then its utility is $v_1 - v_3$.

Variant 2 [Winner takes all] The utility is based on which player has the greater cumulative race value. Since the cumulative race value is zero-sum, the utility is the *sign* of the cumulative race value (i.e., $+1$, -1, and 0), which is itself zero-sum.

Clearly, both variants are zero-sum games and can be analyzed and solved numerically using the tools learned in class.

(a) Specifying the payoff matrix [5 points]

(i) [NO NEED TO INCLUDE] Write a function to generate the payoff matrix given inputs α, β, v. Do this for both Variant 1 and Variant 2. Hint: Each player should have $q!$ actions.

(ii) Let $q = 3$, $\alpha = [0, 1, 2]$, $\beta = [1, 2, 3]$, $v = [1, 2, 3]$. Using your code in part (i) to write down the payoff matrix for both Variant 1 and Variant 2, making sure to label the actions in the payoff matrix, showing clearly the assignment of horses to races.

(iii) [NO NEED TO INCLUDE] Write a function computing player 1's utility (or loss) given that player 2 best responds. It should take in the payoff matrix and player 1's mixed strategy x. Write a similar function for player 2's strategy y. This will be a useful subroutine for computing the saddle point residual for any pair of strategies x, y.

(b) Implementing 2-player zero-sum game solvers [15 points]

For this part, we will use the following race times, obtained from the top 6 finalists in the Men's 100 metre Summer Olympics for the years 2020 and 2016.

[2] `https://en.wikipedia.org/wiki/Tian_Ji#The_game`

[3] The original story assumes that ties were broken in favour of one of the players.

```
T1 = [9.80, 9.84, 9.89, 9.93, 9.95, 9.98]  # 2020
T2 = [9.81, 9.89, 9.91, 9.93, 9.94, 9.96]  # 2016
```

We can use this to compute their average speeds,

```
S1 = 1/np.array(T1) # alpha
S2 = 1/np.array(T2) # beta
```

while remembering that it is only their *relative speeds* (faster, equal, or slower) that matter. The races are worth $v = [1, 2, 3, 4, 5, 6]$ respectively.

(i) [OPTIONAL] Implement the linear programming based solver (for any payoff matrix A) that we introduced in class. I recommend using CVXPY for its simple API (I use Gurobi as the backend, for small problems other free solvers should suffice). You can verify using part (a) that the saddle-point residual of the solution should be very small.

(ii) Implement solvers that utilize online learning with self-play. Do it for Fictitious Play (FP), Regret Matching (RM) and Regret Matching Plus (RM+). You may use alternating or simultaneous updates, though you should be consistent. Test your solvers on **Variant 1** using the race speeds specified above. For all three methods, **plot clearly on a single graph** the saddle point residual against the number of iterations (it may be useful to use a log scale for the y-axis). You should not terminate before obtaining a saddle-point residual of less than 0.05 for *at least one of the three methods*. Write down the value of the game, stating clearly whether it is from player 1 or 2's perspective, and whether these are utilities or losses.

Hint: Be extra careful with sign errors, and which player is minimizing or maximizing. Any convention is okay, but you need to be consistent.
Hint: As a sanity check, test your implementation on problems which we know the equilibrium of exactly (e.g., the 2×2 games we discussed during lectures).

(iii) Repeat subproblem (ii) on **Variant 2**.

(iv) OPTIONAL: scaling up your solver to large q. Observe that our current approach of explicitly constructing the payoff matrix does not perform well when q is large. For example, when $q = 100$, the payoff matrix is of size $100! \times 100!$. Construct a linear programming based solver that scales to such sizes of q for **Variant 1**. [4]

Hint: Consider the assignment polytope, which has vertices precisely corresponding to matchings. By Birkhoff's Theorem, every mixture of matchings lies in the assignment polytope. Decompose utilities into the sum of race values across all races and use this to rewrite the utilities for each player in bilinear form $\hat{x}^\top \hat{A} \hat{y}$, where $\hat{x}$ and $\hat{y}$ are in the assignment polytope. Deduce that the minimax theorem holds, and convert the underlying min-max problem into a min-min problem by dualization of the inner maximization problem.

(v) BONUS: suppose $v_1 = v_2 = \cdots = v_q = 1$. Show that playing uniformly at random is a NE (i.e., optimal) for both players, regardless of α and β.

[4] Solvers based on self play are still possible, but are slightly trickier.

5 Minimax Beyond Simplexes [20 points]

(a) Conditions for the minimax theorem to hold [10 points]

In class, we learned about Von Neumann's Minimax Theorem, which gives us conditions for $\mathcal{X}, \mathcal{Y}$ and f such that

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} f(x, y).$$

In particular, these conditions are satisfied for $n \times m$ matrix games, given the simplexes over actions $\mathcal{X} = \Delta_n$, $\mathcal{Y} = \Delta_m$ and $f(x, y) = x^T Ay$ for some payoff matrix $A \in \mathbb{R}^{n \times m}$.

Now consider an alternative domain $\hat{\mathcal{X}}$. For which of the following $\hat{\mathcal{X}} \subseteq \mathcal{X}$ does the Von Neumann minimax theorem hold for $(\hat{\mathcal{X}}, \mathcal{Y})$? *Briefly* justify your answer. If it is true, explain why the conditions required hold. If it is false, explain why the conditions do not hold — you do *not* have to give a counterexample.

(i) $\hat{\mathcal{X}} = \{x \in \Delta_n | x_i = 0.1 \cdot k_i \text{ for some } k_i \in \mathbb{Z}\}$. That is, $\hat{\mathcal{X}}$ contains probability distributions x where each probability x_i is an integer multiple of 0.1.

(ii) $\hat{\mathcal{X}} = \{x \in \Delta_n | x_{i+1} \geq x_i \text{ for all } 1 \leq i \leq n-1\}$, i.e., $\hat{\mathcal{X}}$ contains probability distributions where x_i are non-increasing in i. Hint: nonempty intersections of halfspaces (i.e., constraints of the form $c^T x \leq d$) are convex.

(b) Minimax on directed acyclic graphs [10 points]

Consider a directed acyclic graph $G = (\mathcal{V}, \mathcal{E})$ with a single source and sink $s, t \in \mathcal{V}$. You may assume that all vertices (except for s) has at least one parent and all vertices (except for t) has at least one child. Let $\mathcal{P}$ be the set of all s-t paths in $\mathcal{G}$. Consider the following games.

Game 1: Network-Interdiction You, the *minimizing* player plays the role of a fugitive escaping from s to t. Your action set $\mathcal{A}_1$ is equal to the set of all paths $\mathcal{P}$. Your opponent, the maximizing player is the police and sets up a roadblock at one of the edges, i.e., $\mathcal{A}_2 = \mathcal{E}$. Suppose your path chosen is $a_1 = p \in \mathcal{P}$ and your opponent's edge chosen is $a_2 = e \in \mathcal{E}$. Then you win if the roadblock does not intersect with p, i.e., $e \notin p$ and vice versa if $e \in p$.

Game 2: Longest-Shortest Path You, the *maximizing* player are a sadistic professor. You have 1 unit of homework to distribute over all $\mathcal{E}$ edges (the homework is divisible into nonnegative real portions). Your opponent is a student at s, trying to get home at t. The student's total workload is the *sum* of the homework that he collects along his path from s to t, which he chooses after observing how the homework is distributed. You may assume that homework accumulates additively. In other words, if the length of each edge is the amount of homework on each edge, the student selects the shortest s-t path.

Observe that the optimal distribution over paths for Game 1 can be written as a min-max problem of the form

$$\min_{x \in \Delta_{\mathcal{P}}} \max_{e \in \mathcal{E}} \sum_{p \in \mathcal{P}} x(p) \cdot 1[e \in p] \tag{10}$$

where $\Delta_{\mathcal{P}}$ is the set of distributions over paths. Notice that the inner maximization player is choosing a single edge to interdict.

Likewise, in Game 2 the optimal distribution over edges can be written as

$$\max_{y \in \Delta_{\mathcal{E}}} \min_{p \in \mathcal{P}} \sum_{e \in p} y(e) = \max_{y \in \Delta_{\mathcal{E}}} \min_{p \in \mathcal{P}} \sum_{e \in \mathcal{E}} y(e) \cdot 1[e \in p]. \tag{11}$$

(i) Using the minimax theorem, show that the two problems (10) and (11) yield the same solution; that is, their values are identical. Make sure to justify why the minimax theorem holds.

Hint: you may have to do some massaging of the min-max and max-min problems before applying the minimax theorem.

(ii) Use the dualization trick described in class to convert the min-max problem in (10) to a min-min problem, yielding a linear program that solves the game. The size of your linear program must be polynomial in $|\mathcal{V}|, |\mathcal{E}|$. Write this linear program down, stating clearly the dimensionality of all the variables, the objective, and (linear) constraints.

Hint: to represent $\Delta_{\mathcal{P}}$ compactly, we can use unit *s-t flows* as the strategy space as opposed to distribution over paths. First, observe that despite choosing a distribution over paths, the expected payoffs only depend on the *marginal probabilities* that any edge is traversed. Consider any distribution over paths $x \in \Delta_{\mathcal{P}}$. Define $q_x(e)$ to be the marginal probability that edge e is traversed (possibly over different paths), i.e.,

$$q_x(e) = \sum_{p \in \mathcal{P}} x(p) \cdot 1[e \in p].$$

Then the set of achievable edge traversal probabilities by P1 is:

$$Q = \{q \in [0,1]^{\mathcal{E}} \,|\, \exists p \in \Delta_{\mathcal{P}} \text{ such that } q = q_x(e)\}$$

which can be shown to be *equivalent* to the feasible set of unit s-t flows, which in turn is given by the polytope:

$$\begin{aligned} q \in \mathbb{R}^{\mathcal{E}}, q &\geq 0 \\ \sum_{e \in \mathcal{E}^-(s)} q(e) &= 1 \\ \sum_{e \in \mathcal{E}^+(t)} q(e) &= 1 \\ \sum_{e \in \mathcal{E}^-(v)} q(e) - \sum_{e \in \mathcal{E}^+(v)} q(e) &= 0 \qquad \forall v \in \mathcal{V} \backslash \{s,t\} \end{aligned}$$

where $\mathcal{E}^+(v) \subseteq \mathcal{E}$ is the set of edges pointing towards vertex v and $\mathcal{E}^-(v) \subseteq \mathcal{E}$ is the set of edges pointing away from v. The last equality are flow conservation constraints and the other two fix the total system in and outflow (technically, one of them can be omitted). **You can use this flow representation without proof.**

Remark 1. *Technically, we can also do the same starting from* (11)*; that however requires us to take the dual of a shortest path problem, which is slightly more involved (but definitely still doable!).*