Practice Questions

August 30, 2025

1 A game of pure coordination

1.1 Uniform case

Consider the fully cooperative 2-player coordination game. Each player has action set $\mathcal{A} = \{1, \ldots, K\}$ for some integer $K > 1$. The payoff function is

$$u_1(a_1, a_2) = u_2(a_1, a_2) = \begin{cases} 0 & a_1 \neq a_2 \\ 1 & a_1 = a_2 \end{cases},$$

for all $a_1, a_2 \in \mathcal{A}$. That is, both players must play the same action to win.

How many pure Nash equilibria are there? How many Nash equilibria are there (including both pure and mixed)? Write them all down.

1.2 Nonuniform case

Consider the same setting as above, but where the payoff function is

$$u_1(a_1, a_2) = u_2(a_1, a_2) = \begin{cases} 0 & a_1 \neq a_2 \\ a_1 & a_1 = a_2 \end{cases},$$

Find expressions for all the Nash equilibria in this modified game.

1.3 Multiplayer setting (optional)

Consider a further modification to Q1.2. Now there are $n > 2$ players each with action space $\mathcal{A}$ and the payoff function continues to be such that the game is cooperative

$$u_1(a_1, \ldots, a_n) = u_2(a_1, \ldots, a_n) = \cdots = a_n(a_1, \ldots, a_n)$$

and

$$u_i(a_1, \ldots, a_n) = \begin{cases} 0 & \text{there exists } i, j \in [n] \text{ such that } a_i \neq a_j \\ a_1 & \text{otherwise} \end{cases}.$$

How many pure Nash equilibria are there? How many Nash equilibria are there (including both pure and mixed)? Write them all down.

2 Battle of the sexes

Consider the following 2-player matrix game with K actions per player, $\{1, 2, \ldots, K\}$. This is an extension of the classic *Battle of the Sexes*, except that there are now K possible actions. The payoff to both players is 0 if they do not take the same action. However, if they both choose action i, then Player 1 obtains a utility of i and Player 2 obtains $K + 1 - i$.

$$u_1(a_1, a_2) = \begin{cases} 0 & a_1 \neq a_2 \\ a_1 & a_1 = a_2 \end{cases}$$
$$u_2(a_1, a_2) = \begin{cases} 0 & a_1 \neq a_2 \\ K + 1 - a_1 & a_1 = a_2 \end{cases}$$

How many pure Nash equilibria are there? How many Nash equilibria are there (including both pure and mixed)? Write them all down. Find all the Nash equilibrium that maximize expected social welfare (i.e., sum of players' playoffs).

3 Zero-sum games

Consider the game of *Rock, Paper, Scissors minus one* shown in the TV series *Squid Game 2*. The game is played in two stages:

Phase 1 Players simultaneously display two symbols with their hands, e.g., rock-rock (RR) or paper-scissors(PS). The order doesn't really matter. The players observe what symbols their opponent played at the end of the phase.

Phase 2 Players simultaneously put down one of their hands, leaving one symbol behind. The winner is based on regular rock-paper-scissors on the remaining symbol. For example, if Player 1 chose RP, he can drop R and leave P remaining. If Player 2 chose SP and dropped P, then Player 2 wins.

We will analyze how to play the game optimally using the tools we've developed in class.

1. Argue that in Phase 1, choosing PP, SS and RR are at least weakly dominated, so there exist Nash equilibria that never choose such actions.
2. Argue that by symmetry, there exists a Nash such that PS, PR and SR are played with equal probability in Phase 1.
3. Without loss of generality, the only three meaningful "initial states" in Phase 2 is (i) where P1 plays PS and P2 plays PR in phase 1, and (ii) vice versa, i.e., P1 plays PR and P2 plays PS, and (iii) both players play PS. Case (i) and (ii) are just swapping player ids around. So, there are really two cases to analyze. Solve both of these games to get the Nash!

Remark. *In the actual series, losing meant that you lose your life. So, it might not be a zero-sum game unless you hate your opponent a lot. How would you model the general-sum game then? What do you think some of the NE look like?*

Suppose the game is a tie, the game repeats itself again until one player loses. One may think that this makes the game amenable to game theoretic analysis. However the situation is a bit more complicated since it is possible for the game to continue further. In fact, repeated games without discounting are quite a bit more challenging to analyze.