

Almost Envy-Freeness in Group Resource Allocation

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Abstract

We study the problem of fairly allocating indivisible goods between groups of agents using the recently introduced relaxations of envy-freeness. We consider the existence of fair allocations under different assumptions on the valuations of the agents. In particular, our results cover cases of arbitrary monotonic, responsive, and additive valuations, while for the case of binary valuations we fully characterize the cardinalities of two groups of agents for which a fair allocation can be guaranteed with respect to both envy-freeness up to one good (EF1) and envy-freeness up to any good (EFX). Moreover, we introduce a new model where the agents are not partitioned into groups in advance, but instead the partition can be chosen in conjunction with the allocation of the goods. In this model, we show that for agents with arbitrary monotonic valuations, there is always a partition of the agents into two groups of any given sizes along with an EF1 allocation of the goods. We also provide an extension of this result to any number of groups.

1 Introduction

Fairness is one of the primary desiderata in decision-making procedures involving multiple agents. For instance, machine learning researchers have recently studied how to design classification systems that do not discriminate based on sensitive attributes such as race or gender [Dwork *et al.*, 2012]. Another problem that is ubiquitous in every society is that of allocating resources among its members. The study of how the allocation can be done fairly, commonly known as *fair division*, has found applications ranging from settling divorce disputes [Brams and Taylor, 1996] to sharing apartment rent [Su, 1999; Gal *et al.*, 2017].

The vast majority of the fair division literature has focused on allocating resources among *individual* agents. However, in many practical situations the resources need to be allocated among *groups* of agents. The agents in each group share the same set of resources, but may have different preferences over them. For instance, the books allocated to a library can be enjoyed by all of its members, and it may be the case that some members prefer detective novels while others would

rather read science fiction. Another example is the division of household goods between families; different members of a family may have contrasting opinions on the television or the sofa in their apartment.

The group aspect of fair division was introduced independently by Segal-Halevi and Nitzan [2016] and Manurangsi and Suksompong [2017]. Segal-Halevi and Nitzan investigated the allocation of *divisible* goods such as cake or land. In contrast, Manurangsi and Suksompong studied the group allocation of *indivisible* goods like books and cars. Both of these works used the fairness notion of *envy-freeness*—an agent is said to be *envy-free* if she finds her group’s share to be as least as good as the share of any other group. While envy-freeness cannot be guaranteed even when allocating indivisible goods among individuals (consider two agents who quarrel over a single valuable good), Manurangsi and Suksompong showed that if the agents’ utilities for the goods are drawn at random, an envy-free allocation exists with high probability in the group setting as the number of agents and goods grows. Segal-Halevi and Suksompong [2018] then introduced the concept of *democratic fairness*, which aims to satisfy a certain fraction of the agents in each group. Among other fairness notions, they considered *envy-freeness up to one good (EF1)*, which means that while some agent may envy another group under the given allocation, the envy can be eliminated by removing a single good from the other group’s share. Segal-Halevi and Suksompong showed that for two groups, there always exists an allocation that is EF1 for at least 1/2 of the agents in each group, and the factor 1/2 cannot be improved.

While the aforementioned works provide different methods for extending individual fair division to the group setting, in some situations it may be important that all agents receive a fairness guarantee with certainty regardless of their valuations. Suksompong [2018] showed the possibilities and limitations of using the *maximin share* notion to guarantee every agent a fair share. In this work, we study the extent to which the recently introduced relaxations of envy-freeness, most notably EF1 and another notion called EFX, can be used for the same purpose. We show that while EF1 is surprisingly robust and can be guaranteed in a number of group settings, this is not the case for EFX. In addition, we introduce a new model in which the partition of the agents into groups is not fixed in advance, but instead can be chosen in conjunction with the

allocation of the goods. This model captures settings where agents (or a central authority) can choose the group that they want to be part of, such as membership in a library or gym.

1.1 Our Results

With the exception of Section 5.2, we assume that the goods are allocated between two groups of agents. While this may seem restrictive at first glance, we remark here that fair division between two *individual* agents, which is much more restrictive, has received a significant amount of attention in the literature (e.g., [Brams and Fishburn, 2000; Brams *et al.*, 2014; Kilgour and Vetschera, 2018]). Indeed, as we will see, the setting of two groups is quite rich and already allows for many interesting, non-trivial results.

In Section 3, we assume that the agents in the two predetermined groups have binary valuations, i.e., each agent either desires each good or not. We characterize the cardinalities of the groups for which an EF1 or EFX allocation always exists. Additionally, we consider a stronger variant of EFX introduced by Plaut and Roughgarden [2018], which we refer to as EFX₀. We prove a very strong negative result for the group fairness setting, implying that this fairness notion can only be guaranteed when both groups are singletons.

Next, in Section 4, we consider more general classes of valuations. If one group is a singleton and the other group consists of two agents, we show that a *balanced* EF1 allocation always exists provided that the agents are endowed with responsive valuations, a general class that contains the well-studied class of additive valuations. Balancedness means that the sizes of the two bundles differ by at most one. Moreover, we establish a surprising connection between our group fair division problem and a class of graphs known as *generalized Kneser graphs*. We show that if a conjecture by Jafari and Alipour [2017] on the chromatic number of particular graphs from this class is true, it would imply that a balanced EF1 allocation exists whenever the two groups contain a total of at most five agents with arbitrary monotonic valuations. This bound would be tight due to our results in Section 3.

Finally, in Section 5 we examine the newly introduced setting where we assume that the partition of the agents into groups is no longer fixed and can be chosen along with the allocation of the goods. Our results indicate that if a central authority or the agents themselves have the power to decide which group to join, then fair allocations are much easier to achieve. In particular, we show that for two groups of agents with arbitrary monotonic valuations, it is always possible to simultaneously obtain a balanced partition of the agents and a balanced EF1 allocation of the goods. In addition, for any given sizes of the two groups, there is a partition of the agents conforming to those sizes together with an EF1 allocation of the goods. We also present an extension of this result to any number of groups.

1.2 Further Related Work

The fairness notions EF1 and EFX were introduced by Lipton *et al.* [2004] and Caragiannis *et al.* [2016], and studied in several papers in the last few years [Plaut and Roughgarden, 2018; Amanatidis *et al.*, 2018; Biswas and Barman, 2018; Bilò *et al.*, 2019; Oh *et al.*, 2019]. For individual fair division,

it is known that an EF1 allocation is guaranteed to exist for any number of agents with monotonic valuations, while the question remains open for EFX.

A recent paper by Ghodsi *et al.* [2018] addressed the problem of *rent division* for groups. In addition to determining the allocation of the rooms, the rent of the apartment must be divided among the agents. Like us, Ghodsi et al. also considered a model where the groups are not predetermined.

Another line of research has also considered group fairness in resource allocation but using a different kind of fairness notions than ours [Berliant *et al.*, 1992; Husseinov, 2011; Todo *et al.*, 2011; Aleksandrov and Walsh, 2018; Conitzer *et al.*, 2019]. In these papers, the resources are allocated to *individual* agents, and the aim is to minimize the envy that arises between groups of these agents. In contrast, in our work the resources are allocated to *groups* of agents and shared as public goods among the agents within each group.

Recently, Biswas and Barman [2018] examined cardinality constraints in individual fair division, where the goods are categorized and there is a limit on the number of goods from each category that can be allocated to each agent. Our balancedness notion can be seen as a special case of these constraints.

The group resource allocation problem with variable groups is similar to coalition formation problems [Drèze and Greenberg, 1980; Bogomolnaia and Jackson, 2002] in the sense that there is flexibility in how the agents form groups. However, in coalition formation the utilities of an agent depend on how the agents are grouped and there are no goods involved, whereas in our setting these utilities depend on how the goods are distributed and not how the agents are grouped.

2 Preliminaries

Let $G = \{g_1, \dots, g_m\}$ denote the set of goods, and A the set of n agents. A *bundle* is a subset of G . The agents are partitioned into $k \geq 2$ groups. We assume in Sections 3 and 4 that this partition is fixed in advance, and in Section 5 that the partition is variable and can be chosen. Denote by n_i the size of group i (so $n = \sum_{i=1}^k n_i$), and a_{ij} the j th agent of group i . The agents in each group will be collectively allocated a subset of G ; denote by B_i the bundle allocated to group i so that $B_i \cap B_j = \emptyset$ for any $i \neq j$ and $\cup_{i=1}^k B_i = G$. A partition of the agents is called *balanced* if $|n_i - n_j| \leq 1$ for any i, j . Similarly, an allocation of the goods is called *balanced* if $||B_i| - |B_j|| \leq 1$ for any i, j .

Each agent a_{ij} has some nonnegative valuation $u_{ij}(G')$ for each set of goods $G' \subseteq G$; for convenience we write $u_{ij}(g)$ instead of $u_{ij}(\{g\})$ for a good $g \in G$. Let $\mathbf{u}_{ij} := (u_{ij}(g_1), \dots, u_{ij}(g_m))$ be the valuation vector of agent a_{ij} for individual goods. When the identity of the agent is not important, we will drop the indices i, j from the valuation u_{ij} . We assume that the valuations are *monotonic*, i.e., $u(G_1) \leq u(G_2)$ for any $G_1 \subseteq G_2 \subseteq G$, and normalized, i.e., $u(\emptyset) = 0$. A valuation function u is said to be *responsive* if $u(G' \cup \{g\}) \geq u(G' \cup \{\bar{g}\})$ for any $G' \subseteq G$ and any goods $g, \bar{g} \notin G'$ such that $u(g) \geq u(\bar{g})$. It is said to be *additive* if $u(G') = \sum_{g \in G'} u(g)$ for any $G' \subseteq G$, and *binary* if it is additive and $u(g) \in \{0, 1\}$ for all $g \in G$. Note that every

additive valuation is responsive. Additive valuations are often assumed in recent fair division literature [Caragiannis *et al.*, 2016; Amanatidis *et al.*, 2018; Biswas and Barman, 2018; Conitzer *et al.*, 2019]. An *instance* consists of agents, goods, and utility functions (and in the model of Sections 3 and 4, the partition of agents into groups). In Section 5, we simply denote the agents by a_1, \dots, a_n and their valuations by u_1, \dots, u_n .

We are now ready to define the fairness notions that we consider in this paper.

Definition 2.1. An allocation (B_1, \dots, B_n) is said to be

- *envy-free for agent a_{ij}* if $u_{ij}(B_i) \geq u_{ij}(B_{i'})$ for any i' ;
- *envy-free up to any good (EFX_0) for agent a_{ij}* if for any i' and any good $g \in B_{i'}$, we have $u_{ij}(B_i) \geq u_{ij}(B_{i'} \setminus \{g\})$;
- *envy-free up to any positively valued good (EFX) for agent a_{ij}* if for any i' and any good $g \in B_{i'}$ such that $u_{ij}(g) > 0$, we have $u_{ij}(B_i) \geq u_{ij}(B_{i'} \setminus \{g\})$;
- *envy-free up to c goods (EFc) for agent a_{ij}* , for a given positive integer c , if for any i' there is a set $B \subseteq B_{i'}$ with $|B| \leq c$ such that $u_{ij}(B_i) \geq u_{ij}(B_{i'} \setminus B)$.

An allocation is said to be *envy-free* if it is envy-free for every agent. When there are two groups, we say that an agent finds a bundle to be *envy-free* if the allocation that assigns the bundle to her group and the complement bundle to the other group is envy-free for her. Analogous definitions hold for EFX_0 , EFX , and EFc .

EFX_0 is a variant of EFX introduced by Plaut and Roughgarden [2018].¹ For additive valuations, it is clear that each property in the list is stronger than the next, with EFX implying $EF1$. We will only consider EFX and EFX_0 in the context of additive valuations. In Sections 4 and 5 we only state results for EFX , but all of these results also hold for EFX_0 .

All omitted proofs can be found in the full version of this paper.²

3 Fixed Groups with Binary Valuations

In this section, we assume that the agents have binary valuations and are partitioned in advance into two groups of size n_1 and n_2 . Note that any nonexistence result for (n_1, n_2) yields an analogous result for (n'_1, n'_2) with $n'_1 \geq n_1$ and $n'_2 \geq n_2$, since in the latter case a subset of n_1 agents from the first group and a subset of n_2 agents from the second group still need to consider the allocation fair. Similarly, an existence result for (n_1, n_2) yields a corresponding result for (n'_1, n'_2) with $n'_1 \leq n_1$ and $n'_2 \leq n_2$.

We begin by considering the notions EFX and $EF1$. In fact, for binary valuations one can easily verify that EFX and $EF1$ are equivalent, so it suffices to consider only $EF1$.

We first present two results that establish the existence of an $EF1$ allocation for the cases $(n_1, n_2) = (5, 1)$ and $(3, 2)$.

¹In their paper this property is simply called EFX ; we rename it to avoid confusion with the original definition of Caragiannis *et al.* [2016], which we refer to as EFX .

²<https://arxiv.org/abs/1901.08463>

While the proofs are quite lengthy and left to the full version, we give here a high-level overview. First, observe that since the valuations are binary, each good can be described by the set of agents who desire it. If two goods are desired by the same set of agents, we can allocate one to each group and then search for an $EF1$ allocation of the reduced instance with the remaining goods. Hence we may assume that every good is desired by a distinct subset of agents. This reduces the problem to a finite (but still large) number of possible instances. We then perform other preprocessing steps to reduce the number of cases even further. For example, if an agent desires an odd number of goods, the requirement that $EF1$ imposes on the agent remains the same when we perturb the valuation of the agent so that she no longer desires an arbitrary good. Consequently, we may assume that every agent desires an even number of goods.

Theorem 3.1. For $(n_1, n_2) = (5, 1)$ and binary valuations, an $EF1$ allocation always exists.

Theorem 3.2. For $(n_1, n_2) = (3, 2)$ and binary valuations, an $EF1$ allocation always exists.

The following result completes the characterization for $EF1$ (and EFX) by proving that existence is not guaranteed for larger sets.

Proposition 3.3. For $(n_1, n_2) = (6, 1)$, $(4, 2)$, or $(3, 3)$ and binary valuations, an $EF1$ allocation does not always exist.

Before addressing EFX_0 , we show that for two groups of arbitrary sizes, determining whether an $EF1$ (and EFX) allocation exists is computationally hard. Our reduction is similar to the one used by Segal-Halevi and Suksompong [2018] to show the hardness of deciding the existence of an allocation that gives every agent a positive utility.

Proposition 3.4. For two groups of agents with binary valuations, it is NP-complete to decide whether there exists an $EF1$ allocation.

We now turn to the stronger notion of EFX_0 , and show a negative result.

Proposition 3.5. For $(n_1, n_2) = (2, 1)$ and binary valuations, an EFX_0 allocation does not always exist.

In contrast, Plaut and Roughgarden [2018] showed that an EFX_0 allocation always exists if $(n_1, n_2) = (1, 1)$, even for arbitrary monotonic valuations. Combined with Proposition 3.5, this yields a complete characterization of EFX_0 for every class of valuations between binary and monotonic.

4 Fixed Groups with General Valuations

In this section, we again assume that the partition of the agents into two groups is predetermined, but allow them to have more general valuations.

We first show that the existence of $EF1$ allocations is guaranteed for $(n_1, n_2) = (2, 1)$. The proof relies on the following lemma which may be of independent interest. A partition of the goods in G into two bundles is said to be *exact up to one good (Exact1)* for an agent if the agent views each bundle to be $EF1$. As with allocations, we call a partition of the goods *balanced* if the sizes of the two bundles differ by at most one.

Lemma 4.1. *For two agents with responsive valuations, there always exists a balanced partition of G into two bundles that is Exact1 for both agents.*

Proof. Suppose first that the number of goods m is even, say $m = 2t$. Assume without loss of generality that the valuation u_1 of the first agent is such that $u_1(g_1) \geq u_1(g_2) \geq \dots \geq u_1(g_{2t})$. Construct an undirected graph H with $2t$ vertices corresponding to the goods, and add t red edges $(g_1, g_2), (g_3, g_4), \dots, (g_{2t-1}, g_{2t})$. Similarly, add t blue edges according to the valuation u_2 of the second agent. Since no two edges of the same color are adjacent, the graph cannot contain an odd cycle, which means that H is bipartite. Therefore, its vertices can be partitioned into disjoint independent sets V_1 and V_2 . If $|V_1| \geq t+1$, there is an edge among the vertices in V_1 , a contradiction. An analogous statement holds for V_2 . It follows that $|V_1| = |V_2| = t$.

The partition (V_1, V_2) is balanced; it remains to show that it is Exact1 for both agents. By symmetry, it suffices to prove this for the first agent. By construction, each of V_1 and V_2 contains exactly one good from each of the pairs $(g_1, g_2), (g_3, g_4), \dots, (g_{2t-1}, g_{2t})$. For $i = 1, 2, \dots, t-1$, the i th best good in V_1 according to the first agent's valuation is no worse than the $(i+1)$ st best good in V_2 . Responsiveness then implies that the agent values V_1 at least as much as V_2 when the best good in V_2 is removed. This means that she regards V_1 to be EF1. Similarly, she regards V_2 to be EF1; hence the partition is Exact1 for her.

The case where m is odd can be handled similarly; we leave the details to the full version. \square

Lemma 4.1 yields the following EF1 existence result.

Theorem 4.2. *For $(n_1, n_2) = (2, 1)$ and responsive valuations, a balanced EF1 allocation always exists.*

Proof. Choose two arbitrary agents and consider a balanced partition of G into two bundles that is Exact1 for both agents; such a partition exists by Lemma 4.1. Let the remaining agent choose for her group the bundle that she prefers, and allocate the other bundle to the other group. It is clear that the resulting allocation is balanced and EF1. \square

In light of Theorem 4.2 and our characterization for binary valuations in Section 3, it is natural to ask whether EF1 can also be guaranteed for larger groups with additive valuations and beyond. While we were unable to settle this question, we show that the existence of EF1 allocations would be guaranteed for almost all of the remaining cases provided that a graph-theoretic conjecture of Jafari and Alipour [2017] is true. To describe the conjecture and its implications in our setting, we need to introduce a class of graphs called *generalized Kneser graphs*.³

Definition 4.3. Let $b \geq r \geq s$ be positive integers and consider an underlying set of elements \mathcal{U} such that $|\mathcal{U}| = b$. The *generalized Kneser graph* $K(b, r, s)$ is an undirected graph with all r -element subsets of \mathcal{U} as its vertices. Two vertices

³Kneser graphs have previously been used in the context of fair division and resource allocation by Suksompong [2016] and Plaut and Roughgarden [2018].

are connected by an edge if and only if the corresponding subsets intersect in at most $s-1$ elements.

Recall that the *chromatic number* of a graph H , denoted by $\chi(H)$, is the minimum number of colors needed to color the vertices of H so that any two adjacent vertices have different colors. For example, $K(4, 2, 2)$ is a clique of size 6, so its chromatic number is 6.

We are now ready to establish the connection between the generalized Kneser graph and our fair division problem.

Theorem 4.4. *Let $z := \min_{r \geq 2} \chi(K(2r, r, 2))$. For two groups with at most $z-1$ agents in total and arbitrary monotonic valuations, a balanced EF1 allocation always exists.*

Proof. Suppose first that the number of goods m is even, say $m = 2t$. If $t = 1$, any allocation that gives one good to each group is balanced and EF1, so we may assume that $t \geq 2$. Consider the graph $K(2t, t, 2)$ with the vertices corresponding to all balanced allocations, where we identify each vertex by the set of goods allocated to the first group.

Give each agent a distinct color, and let her color all allocations that she *does not* regard as EF1. We claim that no agent can color two adjacent vertices. Consider an agent in the first group with valuation u , and suppose for contradiction that she colors two adjacent vertices corresponding to the sets G_1 and G_2 . Since the two vertices are adjacent, $|G_1 \cap G_2| \leq 1$. If $G_1 \cap G_2 = \emptyset$, since $|G_1| = |G_2| = t$ it holds that $G_2 = G \setminus G_1$. So the agent should consider one of the two allocations as EF, which is a contradiction to her coloring both vertices. Therefore $|G_1 \cap G_2| = 1$. Let g be the common good of G_1 and G_2 . Since the agent does not view G_1 to be EF1, we have $u(G_1) < u(G_2 \setminus \{g\}) = u((G \setminus G_1) \setminus \{g'\})$, where g' is the unique good that does not belong to $G_1 \cup G_2$. Similarly, since the agent does not view G_2 to be EF1, we have $u(G_2) < u(G_1 \setminus \{g\})$. Monotonicity then implies that

$$u(G_1) < u(G_2 \setminus \{g\}) \leq u(G_2) < u(G_1 \setminus \{g\}) \leq u(G_1),$$

a contradiction. The claim can be proven similarly for agents in the second group by observing that for any two balanced allocations, the bundles allocated to the first group intersect in at most one good if and only if the same condition holds for the bundles allocated to the second group.

Since there are at most $z-1$ agents, the number of colors is at most $z-1 \leq \chi(K(2t, t, 2)) - 1$. Hence there is a vertex that does not receive any color. By definition, this vertex corresponds to a balanced allocation that is EF1 for all agents. This completes the proof for the case where m is even.

The case where m is odd can be handled similarly; we leave the details to the full version. \square

Jafari and Alipour [2017] proved that $K(2r, r, 2) \leq 6$ for all $r \geq 2$, and conjectured that this bound is always tight.

Conjecture 4.5 ([Jafari and Alipour, 2017]). *For any $r \geq 2$, we have $\chi(K(2r, r, 2)) = 6$.*

If Conjecture 4.5 is true, then together with Theorem 4.4, it would imply that a balanced EF1 allocation is guaranteed to exist for two groups with at most 5 agents in total and

arbitrary monotonic valuations.⁴ The bound of 5 cannot be improved to 6 due to Proposition 3.3. Moreover, this result would answer the EF1 existence question in the affirmative for all of the remaining group sizes except for the case $(5, 1)$. We remark that for this case, a *balanced* EF1 allocation might not exist, even when valuations are binary.

Proposition 4.6. *For $(n_1, n_2) = (5, 1)$ and binary valuations, a balanced EF1 allocation does not always exist.*

We note that our techniques in Theorem 4.4 can be extended to weaker relaxations of envy-freeness. Indeed we can similarly derive a corresponding result for EF c , for any positive integer c , if we consider graphs $K(2r, r, c+1)$. See the full version for the relevant discussion.

We conclude this section by showing that existence can no longer be guaranteed if we strengthen the fairness requirement from EF1 to EFX. Recall that for binary valuations, an EFX allocation always exists when one group contains at most five agents and the other group is a singleton. We show that this is not the case for additive valuations, even when the first group contains only two agents.

Proposition 4.7. *For $(n_1, n_2) = (2, 1)$ and additive valuations, an EFX allocation does not always exist.*

Since an EFX allocation always exists when $(n_1, n_2) = (1, 1)$ for arbitrary monotonic valuations [Plaut and Roughgarden, 2018], we have a complete characterization of EFX for every class of valuations between additive and monotonic.

5 Variable Groups

Thus far, we have worked under the assumption that the partition of agents into groups is determined in advance. This assumption is appropriate when we consider, for example, membership in a family or citizenship of a country. In other settings, however, the choice of the group to which the agents belong can be made by a central authority or by the agents themselves. This applies to membership in a library, gym, or other facilities.

With this motivation in mind, we depart from the framework of fixed groups in this section, and instead assume that the partition of the agents into groups can be chosen along with the allocation of the goods. Under this assumption, finding an EF1, EFX, or even envy-free allocation is trivial: simply put all agents in one group and allocate all goods to that group. However, this may lead to undesirable situations where a gym is overcrowded or a library does not have enough space to hold all of its books. As we will show, it is nevertheless possible to obtain a fair outcome that is moreover balanced with respect to both the agents and the goods, for any number of agents with general valuations.

⁴Jafari and Alipour [2017] also claimed that $\chi(K(2r, r, 2)) \geq 4$ for all $r \geq 2$. In combination with our Theorem 4.4, this would imply that our Theorem 4.2 can be generalized to arbitrary monotonic valuations. However, the proof of their Theorem 5.1 contains an error when they claim that the intersection of the two k -subsets of color j has at most $i - 1$ elements. It is only true that the intersection has at most $i - 1$ elements *in each of the two hemispheres*, and therefore at most $2i - 2$ elements in total. It is not clear whether the proof can be recovered in light of this error.

5.1 Two Groups

We start with two groups and show that EF1 can be guaranteed for any desired sizes of these groups. Our algorithm generalizes the discrete “cut-and-choose” algorithm for allocating indivisible goods between two individual agents [Bilò *et al.*, 2019; Oh *et al.*, 2019].

Theorem 5.1. *Let n be any positive integer. Suppose that there are n agents with arbitrary monotonic valuations, and let n_1 and n_2 be nonnegative integers with $n_1 + n_2 = n$. There always exists a partition of the agents into two groups such that group $i \in \{1, 2\}$ contains n_i agents, along with an EF1 allocation of the goods to the two groups.*

Proof. Arrange the goods in a line. Starting with an empty bundle, we add one good at a time from the left until at least n_1 agents find the bundle to be EF1. If this condition is met before we add any good, we give n_1 of these agents an empty bundle and the remaining n_2 agents the entire set G . Otherwise, denote by g the last good added to the bundle. We assign to the first group all agents who view the bundle as EF1 before the addition of g , along with an arbitrary subset of those who find it EF1 after g is added so that the first group has size n_1 . We allocate this bundle to the first group, and the remaining goods to the second group, which consists of the remaining agents.

Since the entire set G is EF1 for all agents, the process terminates. By construction, the agents in the first group regard the allocation as EF1, so we only need to show that the same holds for the second group. This is trivial if the second group receives the entire set G . Otherwise, let G_L and G_R be the bundles to the left and right of g , respectively (not containing g). Every agent in the second group does not find G_L to be EF1, which means that she has more value for G_R than G_L . This implies that the agent finds G_R to be EF1, as desired. \square

For the case where a balanced allocation of the goods is required, we prove that this can also be achieved and, in fact, it can always be combined with a balanced partition of the agents. This means that in our gym and library applications, it is possible to reach a balance in terms of the users as well as the resources.

Theorem 5.2. *Let n be any positive integer, and suppose that there are n agents with arbitrary monotonic valuations. There always exists a balanced partition of the agents into two groups along with a balanced EF1 allocation of the goods to the two groups.*

Proof. Suppose first that both the number of agents n and the number of goods m are even, say $n = 2s$ and $m = 2t$; we leave the case where either or both are odd to the full version. Assume for contradiction that there is no balanced partition along with a balanced allocation.

Arrange the goods around a circle with equal spacing between adjacent goods. Imagine a knife that cuts through the center of the circle, dividing the goods into two bundles G_1 and G_2 , each of size t . By our assumption, any balanced assignment of the agents to G_1 and G_2 does not result in an EF1 allocation. On the other hand, there does exist an assignment

such that the resulting allocation is EF1 (e.g., an assignment that gives every agent her favorite bundle). Consider such an assignment that moreover minimizes the difference between the numbers of agents in the two groups. Assume without loss of generality that more than half of the agents are assigned to G_1 . If one of these agents finds G_2 to be EF1, we can reassign her to G_2 and reduce the discrepancy between the two groups. Hence all of these agents do not find G_2 to be EF1.

Next, we rotate the knife clockwise by one position, thereby moving a good g from G_1 to G_2 and another good \bar{g} from G_2 to G_1 . Call the resulting bundles $H_1 = (G_1 \cup \{\bar{g}\}) \setminus \{g\}$ and $H_2 = (G_2 \cup \{g\}) \setminus \{\bar{g}\}$, respectively. We claim that the agents who do not find G_2 to be EF1 regard H_1 as EF1. Denoting the valuation of an arbitrary such agent by u , we have

$$\begin{aligned} u(H_1) &\geq u(G_1 \setminus \{g\}) \\ &> u(G_2) \geq u(G_2 \setminus \{\bar{g}\}) = u(H_2 \setminus \{g\}), \end{aligned}$$

where the first and third inequalities follow from monotonicity. So the agent indeed finds H_1 to be EF1. Since more than half of the agents do not find G_2 to be EF1, more than half of the agents regard H_1 as EF1. By our assumption, any balanced assignment of the agents to H_1 and H_2 does not result in an EF1 allocation. It follows that in any assignment such that the resulting allocation is EF1, more than half of the agents are assigned to H_1 . Additionally, more than half of the agents do not find H_2 to be EF1.

If we rotate the knife clockwise repeatedly, the same argument tells us that more than half of the agents do not find the second bundle (i.e., G_2 , H_2 , and so on) to be EF1. After t rotation steps, the knife has rotated halfway around the circle, and the second bundle coincides with the original first bundle G_1 . However, we know from earlier that more than half of the agents find G_1 to be EF1. This yields the desired contradiction. \square

Theorem 5.2 yields the following result on individual fair division, which is new to the best of our knowledge.

Corollary 5.3. *For two individual agents with arbitrary monotonic valuations, there always exists a balanced EF1 allocation.*

When valuations are responsive, a balanced EF1 allocation (for arbitrarily many agents) can also be obtained by the round-robin algorithm, which lets the agents take turns choosing their favorite good from the remaining goods until all goods are taken (see, e.g., [Caragiannis *et al.*, 2016]). However, the round-robin algorithm does not work for arbitrary monotonic valuations.

Turning our attention to EFX, we show that if we require the partition of the agents to be balanced, an EFX allocation might not exist; this complements Theorems 5.1 and 5.2 above. In addition, a balanced EFX allocation does not necessarily exist for two individual agents even when the agents have identical additive valuations, which complements Corollary 5.3.

Proposition 5.4. *There does not always exist a balanced partition of the agents into two groups along with an EFX allocation of the goods to the two groups, even when valuations are additive.*

Proposition 5.5. *Let m be a positive integer. There exists an instance with two individual agents who have identical additive valuations and m goods, such that in any EFX allocation, one of the agents receives exactly one good.*

5.2 Any Number of Groups

We now consider any number of groups and show how Theorem 5.1 can be partially extended to this setting. While we do not know whether EF1 or other relaxations of envy-freeness can be achieved in this general setting, we show that we can obtain a positive result for a weaker fairness notion called *proportionality*. An allocation of the goods in G to k groups is said to be *proportional* if every agent receives value at least $1/k$ of her value for the whole set G . As with envy-freeness, a proportional allocation does not always exist (e.g., when there are two individual agents and one valuable good), so it is necessary to consider a relaxation. Let $u_{j,\max} := \max_{t=1}^m u_j(g_t)$ denote the maximum value of agent a_j for a single good.

Theorem 5.6. *Let n and k be any positive integers. Suppose that there are n agents with additive valuations, and let n_1, \dots, n_k be nonnegative integers with $\sum_{i=1}^k n_i = n$. There always exists a partition of the agents into k groups such that group i contains n_i agents, along with an allocation of the goods to the k groups such that each agent a_j receives utility at least $\frac{1}{k} \cdot u_j(G) - \frac{k-1}{k} \cdot u_{j,\max}$.*

Theorem 5.6 generalizes a result of Suksompong [2019], which holds for individual fair division (i.e., $n_i = 1$ for all $i = 1, \dots, k$). The factor $\frac{k-1}{k}$ in the approximation cannot be improved even in this special case.

6 Discussion

In this paper, we examine the fairness guarantees that can be obtained in the allocation of indivisible goods among groups of agents. For two fixed groups of agents, we provide a complete picture for EF1 and EFX when agents have binary valuations, and we present further positive and negative results for more general valuations. We also introduce a new model where the partition of the agents into groups can be determined along with the allocation of the goods, and show that it is possible to attain a balance in both the agents and the goods simultaneously.

Our work leaves many open questions for future study. For two groups, one could try to establish the existence of EF1 allocations for larger group sizes, either by settling the graph-theoretic conjecture of Jafari and Alipour [2017] or via other means. In addition, the questions that we study in this paper can also be asked for multiple groups; our techniques do not seem to extend easily to more than two groups in most cases. For individual fair division, we also leave the question of whether a balanced EF1 allocation can always be found for any number of agents; Corollary 5.3 gives a positive answer for the case of two agents. While the round-robin algorithm works when valuations are responsive, the question intriguingly remains open for arbitrary monotonic valuations.

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References

- [Aleksandrov and Walsh, 2018] Martin Aleksandrov and Toby Walsh. Group envy freeness and group Pareto efficiency in fair division with indivisible items. In *KI*, pages 57–72, 2018.
- [Amanatidis *et al.*, 2018] Georgios Amanatidis, Georgios Birmpas, and Evangelos Markakis. Comparing approximate relaxations of envy-freeness. In *IJCAI*, pages 42–48, 2018.
- [Berliant *et al.*, 1992] Marcus Berliant, William Thomson, and Karl Dunz. On the fair division of a heterogeneous commodity. *Journal of Mathematical Economics*, 21(3):201–216, 1992.
- [Bilò *et al.*, 2019] Vittorio Bilò, Ioannis Caragiannis, Michele Flammini, Ayumi Igarashi, Gianpiero Monaco, Dominik Peters, Cosimo Vinci, and William S. Zwicker. Almost envy-free allocations with connected bundles. In *ITCS*, pages 14:1–14:21, 2019.
- [Biswas and Barman, 2018] Arpita Biswas and Siddharth Barman. Fair division under cardinality constraints. In *IJCAI*, pages 91–97, 2018.
- [Bogomolnaia and Jackson, 2002] Anna Bogomolnaia and Matthew O. Jackson. The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230, 2002.
- [Brams and Fishburn, 2000] Steven J. Brams and Peter C. Fishburn. Fair division of indivisible items between two people with identical preferences: Envy-freeness, Pareto-optimality, and equity. *Social Choice and Welfare*, 17(2):247–267, 2000.
- [Brams and Taylor, 1996] Steven J. Brams and Alan D. Taylor. A procedure for divorce settlements. *Mediation Quarterly*, 13(3):191–205, 1996.
- [Brams *et al.*, 2014] Steven J. Brams, D. Marc Kilgour, and Christian Klamler. Two-person fair division of indivisible items: an efficient, envy-free algorithm. *Notices of the AMS*, 61(2):130–141, 2014.
- [Caragiannis *et al.*, 2016] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum Nash welfare. In *EC*, pages 305–322, 2016.
- [Conitzer *et al.*, 2019] Vincent Conitzer, Rupert Freeman, Nisarg Shah, and Jennifer Wortman Vaughan. Group fairness for the allocation of indivisible goods. In *AAAI*, 2019. Forthcoming.
- [Drèze and Greenberg, 1980] Jacques H. Drèze and Joseph Greenberg. Hedonic coalitions: optimality and stability. *Econometrica*, 48(4):987–1003, 1980.
- [Dwork *et al.*, 2012] Cynthia Dwork, Moritz Hardt, Toniann Pitassi, Omer Reingold, and Richard Zemel. Fairness through awareness. In *ITCS*, pages 214–226, 2012.
- [Gal *et al.*, 2017] Ya’akov (Kobi) Gal, Moshe Mash, Ariel D. Procaccia, and Yair Zick. Which is the fairest (rent division) of them all? *Journal of the ACM*, 64(6):39:1–39:22, 2017.
- [Ghodsi *et al.*, 2018] Mohammad Ghodsi, Mohamad Latifi, Arman Mohammadi, Sadra Moradian, and Masoud Seddighin. Rent division among groups. In *COCOA*, pages 577–591, 2018.
- [Husseinov, 2011] Farhad Husseinov. A theory of a heterogeneous divisible commodity exchange economy. *Journal of Mathematical Economics*, 47(1):54–59, 2011.
- [Jafari and Alipour, 2017] Amir Jafari and Sharareh Alipour. On the chromatic number of generalized Kneser graphs. *Contributions to Discrete Mathematics*, 12(2):69–76, 2017.
- [Kilgour and Vetschera, 2018] D. Marc Kilgour and Rudolf Vetschera. Two-player fair division of indivisible items: Comparison of algorithms. *European Journal of Operational Research*, 271(2):620–631, 2018.
- [Lipton *et al.*, 2004] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. On approximately fair allocations of indivisible goods. In *EC*, pages 125–131, 2004.
- [Manurangsi and Suksompong, 2017] Pasin Manurangsi and Warut Suksompong. Asymptotic existence of fair divisions for groups. *Mathematical Social Sciences*, 89:100–108, 2017.
- [Oh *et al.*, 2019] Hoon Oh, Ariel D. Procaccia, and Warut Suksompong. Fairly allocating many goods with few queries. In *AAAI*, 2019. Forthcoming.
- [Plaut and Roughgarden, 2018] Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. In *SODA*, pages 2584–2603, 2018.
- [Segal-Halevi and Nitzan, 2016] Erel Segal-Halevi and Shmuel Nitzan. Envy-free cake-cutting among families. *CoRR*, abs/1607.01517, 2016.
- [Segal-Halevi and Suksompong, 2018] Erel Segal-Halevi and Warut Suksompong. Democratic fair allocation of indivisible goods. In *IJCAI*, pages 482–488, 2018. Extended version: *CoRR*, abs/1709.02564.
- [Su, 1999] Francis Edward Su. Rental harmony: Sperner’s lemma in fair division. *The American Mathematical Monthly*, 106(10):930–942, 1999.
- [Suksompong, 2016] Warut Suksompong. Assigning a small agreeable set of indivisible items to multiple players. In *IJCAI*, pages 489–495, 2016.
- [Suksompong, 2018] Warut Suksompong. Approximate maximin shares for groups of agents. *Mathematical Social Sciences*, 92:40–47, 2018.
- [Suksompong, 2019] Warut Suksompong. Fairly allocating contiguous blocks of indivisible items. *Discrete Applied Mathematics*, 260:227–236, 2019.
- [Todo *et al.*, 2011] Taiki Todo, Runcong Li, Xuemei Hu, Takayuki Mouri, Atsushi Iwasaki, and Makoto Yokoo. Generalizing envy-freeness toward group of agents. In *IJCAI*, pages 386–392, 2011.