

Ordinal Maximin Guarantees for Group Fair Division

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Abstract

We investigate fairness in the allocation of indivisible items among groups of agents using the notion of maximin share (MMS). While previous work has shown that no nontrivial multiplicative MMS approximation can be guaranteed in this setting for general group sizes, we demonstrate that ordinal relaxations are much more useful. For example, we show that if n agents are distributed equally across g groups, there exists a 1-out-of- k MMS allocation for $k = O(g \log(n/g))$, while if all but a constant number of agents are in the same group, we obtain $k = O(\log n / \log \log n)$. We also establish the tightness of these bounds and provide non-asymptotic results for the case of two groups.

1 Introduction

A fundamental problem in society is how to allocate scarce resources among interested parties. The burgeoning field of *fair division* studies how to perform the allocation fairly in scenarios ranging from dividing properties between families to distributing supplies among departments of a university or neighborhoods of a city [Brams and Taylor, 1996; Moulin, 2019]. Often, the resources to be allocated are *indivisible*, meaning that each item is discrete and must be allocated as a whole to one of the parties.

To argue about fairness, one must specify what it means for an allocation to be “fair”. A popular fairness notion when allocating indivisible items is *maximin share fairness* [Budish, 2011]. If the items are allocated among n agents, the *maximin share* (MMS) of an agent is defined as the largest value that the agent can guarantee for herself by dividing the items into n bundles and getting the worst bundle. While an allocation that gives every agent at least her maximin share always exists when $n = 2$, this is no longer the case if $n \geq 3$ [Kurokawa *et al.*, 2018]. A large stream of work has therefore focused on guaranteeing every agent a constant fraction of her maximin share, with the current best constant being marginally above $3/4$ [Akrami and Garg, 2024].

The discussion in the preceding paragraph, like the majority of the fair division literature, assumes that each recipient of a bundle of items corresponds to a single agent. However, in many allocation scenarios, each interested party is in fact a

group of agents. Even though the agents in each group share the same set of items, they do not necessarily share the same views on the items. Indeed, members of the same family, university department, or city neighborhood may have varying desires for different items based on their individual preferences and needs. As a consequence, several researchers have investigated fair division among groups of agents in the last few years [Manurangsi and Suksompong, 2017, 2022; Ghodsi *et al.*, 2018; Segal-Halevi and Nitzan, 2019; Segal-Halevi and Suksompong, 2019; Kyropoulou *et al.*, 2020].¹

The study of maximin share fairness for groups was initiated by Suksompong [2018], who showed, e.g., that for two groups with two agents each, it is possible to guarantee each agent $1/8$ of her MMS. Unfortunately, Suksompong observed that if each group contains three agents, it may already be impossible to give all of them any positive fraction of their MMS.² In light of this, it may appear that maximin share fairness is not useful for group resource allocation unless the groups are extremely small. Nevertheless, while *cardinal* approximations of MMS have been intensively examined in the literature, another type of relaxation, which has also received increasing attention, is *ordinal*. This type of relaxation allows each agent to partition the items into k bundles for some parameter $k > n$ —the corresponding notion is called *1-out-of- k MMS*. For individual fair division, Akrami *et al.* [2023] proved that ordinal MMS fairness can always be satisfied if we set $k = \lceil 4n/3 \rceil$, improving upon prior results by Aigner-Horev and Segal-Halevi [2022] and Hosseini *et al.* [2022b]. Can ordinal approximations of MMS provide general fairness guarantees for groups as well?

1.1 Our Results

We consider a setting where $n = n_1 + \dots + n_g$ agents are partitioned into $g \geq 2$ groups of sizes $n_1, \dots, n_g \geq 1$, respectively. Each agent has an additive and non-negative util-

¹The line of work on *consensus halving* and *consensus 1/k-division* problems (e.g., [Simmons and Su, 2003; Filos-Ratsikas and Goldberg, 2018; Goldberg *et al.*, 2022]) can also be viewed as fair division for groups.

²Suppose there are three items and each agent has utility 1 for two of them and 0 for the remaining item, where the three agents in each group value distinct pairs of items. Then, each agent’s MMS is 1, but every allocation leaves some agent with utility 0.

ity function over the set of items to be allocated. We denote by $p_{\text{MMS}}(n_1, \dots, n_g)$ the smallest integer p such that for g groups of sizes n_1, \dots, n_g , there always exists an allocation that gives every agent at least her 1-out-of- p MMS.

We shall prove asymptotically tight bounds on p_{MMS} . Perhaps surprisingly, we show that the answer differs between the “unbalanced” case where one group is much larger than all other groups and the “balanced” case where this condition does not hold. The first case may be applicable, for instance, when a city has a large central neighborhood in which the majority of its residents live. Formally, we establish the following theorems. Note that the constant 1000 is unimportant, as the two bounds in each theorem are of the same order when $\log(n_1 + 1) = \Theta(\log[(n_2 + 1) \cdots (n_g + 1)])$.

Theorem 1.1 (Upper bounds). *Let $n_1 \geq \dots \geq n_g$ be any positive integers. If $\frac{\log(n_1+1)}{\log[(n_2+1)\cdots(n_g+1)]} \leq 1000$, then*

$$p_{\text{MMS}}(n_1, \dots, n_g) \leq O(\log[(n_1 + 1) \cdots (n_g + 1)]).$$

On the other hand, if $\frac{\log(n_1+1)}{\log[(n_2+1)\cdots(n_g+1)]} > 1000$, then

$$p_{\text{MMS}}(n_1, \dots, n_g) \leq O\left(\frac{\log(n_1 + 1)}{\log\left(\frac{\log(n_1+1)}{\log[(n_2+1)\cdots(n_g+1)]}\right)}\right).$$

Theorem 1.2 (Lower bounds). *Let $n_1 \geq \dots \geq n_g$ be any positive integers and $n = n_1 + \dots + n_g$. If $\frac{\log(n_1+1)}{\log[(n_2+1)\cdots(n_g+1)]} \leq 1000$, then*

$$p_{\text{MMS}}(n_1, \dots, n_g) \geq \Omega(\log[(n_1 + 1) \cdots (n_g + 1)]).$$

On the other hand, if $\frac{\log(n_1+1)}{\log[(n_2+1)\cdots(n_g+1)]} > 1000$, suppose further that $g \leq (\log n)^{1-\delta}$ for some constant $\delta \in (0, 1)$. Then, for any sufficiently large n (depending on δ), we have

$$p_{\text{MMS}}(n_1, \dots, n_g) \geq \Omega\left(\frac{\log(n_1 + 1)}{\log\left(\frac{\log(n_1+1)}{\log[(n_2+1)\cdots(n_g+1)]}\right)}\right).$$

Observe that our upper and lower bounds match in the balanced case, regardless of the relationship between the number of groups and the number of agents—for example, this is true even when $n_1, \dots, n_g \in \Theta(1)$. While our bounds also coincide in the unbalanced case, the corresponding lower bound relies on the condition that $g \in (\log n)^{1-\Omega(1)}$. We remark that this is a relatively mild condition, as unbalancedness already implies that $g \in O(\log n_1) \leq O(\log n)$. Moreover, our upper bounds come with efficient randomized algorithms. To gain more intuition on our bounds, notice that if $n_1 = n_2 = \dots = n_g = n/g$, then $p_{\text{MMS}}(n_1, \dots, n_g) = \Theta(g \log(n/g))$, whereas if $n_2 + \dots + n_g \in \Theta(1)$, then $p_{\text{MMS}}(n_1, \dots, n_g) = \Theta(\log n / \log \log n)$.

In addition to these asymptotic bounds, we also derive concrete non-asymptotic results on p_{MMS} for the case of two groups. When both groups are of the same size, our upper and lower bounds differ by only a factor of 2. Furthermore, our bounds are exactly tight for certain group sizes—for instance, we show that $p_{\text{MMS}}(2, 1) = p_{\text{MMS}}(2, 2) = p_{\text{MMS}}(3, 1) = p_{\text{MMS}}(4, 1) = 3$ and $p_{\text{MMS}}(4, 4) = p_{\text{MMS}}(5, 5) = 4$.

1.2 Additional Related Work

Although fair division has been studied extensively for several decades, the group aspect was only considered quite recently. Manurangsi and Suksompong [2017] assumed that agents’ utilities are drawn from probability distributions and showed that an allocation satisfying another important fairness notion, *envy-freeness*, is likely to exist when there are sufficiently many items. Kyropoulou *et al.* [2020] and Manurangsi and Suksompong [2022] presented worst-case guarantees with respect to relaxations of envy-freeness. In particular, the latter authors showed that when the number of groups is constant, the smallest k such that an “envy-free up to k items” allocation always exists is $k = \Theta(\sqrt{n})$; unlike for MMS as in our work, for envy-freeness the bound is the same no matter how the n agents are distributed into groups. Segal-Halevi and Suksompong [2019] investigated *democratic fairness*, where the objective is to provide fairness guarantees to a fraction of the agents in each group; they also considered ordinal MMS, but the corresponding results are mainly restricted to binary utilities. Segal-Halevi and Nitzan [2019] and Segal-Halevi and Suksompong [2021, 2023] examined group fairness for *divisible* resources modeled as a cake, while Ghodsi *et al.* [2018] analyzed it in rent division.

Besides the model that we consider, group fairness has also been studied in *individual* fair division [Berliant *et al.*, 1992; Todo *et al.*, 2011; Aleksandrov and Walsh, 2018; Benabbou *et al.*, 2019; Conitzer *et al.*, 2019; Aziz and Rey, 2020]. In this setting, each agent receives a bundle of items and there is no sharing; however, fairness is desired not only for individual agents, but also for groups. Some of these authors assumed that the groups are fixed in advance (e.g., ethnic, gender, or socioeconomic groups), whereas others required fairness to hold across all groups that could possibly be formed. Scarlett *et al.* [2023] explored the interplay between individual and group fairness in this model.

As we mentioned earlier, although the majority of work on MMS concerns cardinal approximations, ordinal relaxations have also attracted growing interest among researchers. For individual fair division, Aigner-Horev and Segal-Halevi [2022] established the existence of a 1-out-of- k MMS allocation for $k = 2n - 2$. This was later improved to $k = \lfloor 3n/2 \rfloor$ [Hosseini *et al.*, 2022b], and subsequently to $k = \lceil 4n/3 \rceil$ [Akrami *et al.*, 2023]. Furthermore, ordinal MMS has been investigated in the context of indivisible chore allocation [Hosseini *et al.*, 2022a], cake cutting [Elkind *et al.*, 2021, 2022], as well as land division [Elkind *et al.*, 2023].

2 Preliminaries

We use \log to denote the logarithm with base 2 and \ln to denote the natural logarithm. For any positive integer z , let $[z] := \{1, \dots, z\}$.

Let $g \geq 2$ and n_1, \dots, n_g be positive integers. Let N be a set of $n = n_1 + \dots + n_g$ agents partitioned into g groups, where for each $i \in [g]$, the i -th group is denoted by N_i and contains n_i agents. Let $M = [m]$ be the set of items to be allocated. A *bundle* refers to any (possibly empty) subset of M . Each agent $a \in N$ has an additive utility function $u_a : 2^M \rightarrow \mathbb{R}_{\geq 0}$ over the items in M ; for a single item ℓ , we

sometimes write $u_a(\ell)$ instead of $u_a(\{\ell\})$. For $i \in [g]$ and $j \in [n_i]$, we refer to the j -th agent in N_i as agent (i, j) , and use $u_{i,j}$ to denote the agent's utility function. An *allocation* $\mathcal{A} = (A_1, \dots, A_g)$ is an ordered partition of the items in M such that for each $i \in [g]$, bundle A_i is assigned to group N_i . For $i \in [g]$, we let

$$\beta_i := \frac{\log(n_i + 1)}{\log[(n_2 + 1) \cdots (n_g + 1)]}.$$

2.1 Ordinal MMS

Let p be a positive integer, and denote by $\Pi_p(M)$ the set of all (unordered) partitions of M into p parts. We now introduce our fairness notion of interest in this paper.

Definition 2.1. The $1\text{-out-of-}p$ maximin share of an agent $a \in N$, denoted by $\text{MMS}_a^{1\text{-out-of-}p}$, is defined as

$$\max_{\{B_1, \dots, B_p\} \in \Pi_p(M)} \min_{i \in [p]} u_a(B_i).$$

An allocation (A_1, \dots, A_g) is said to be an $\text{MMS}^{1\text{-out-of-}p}$ allocation if every agent's utility for her group's bundle is at least her $1\text{-out-of-}p$ maximin share. For an agent $a \in N$ and a set of items $S \subseteq M$, we write $\text{MMS}_a^{1\text{-out-of-}p}(S)$ to denote agent a 's $1\text{-out-of-}p$ maximin share when restricted to S .

For any $g \geq 2$ and positive integers n_1, \dots, n_g , we denote by $p_{\text{MMS}}(n_1, \dots, n_g)$ the smallest integer p such that for any g groups containing n_1, \dots, n_g agents respectively, there exists an $\text{MMS}^{1\text{-out-of-}p}$ allocation. We make some observations about p_{MMS} . First, observe that if we add extra agents to existing groups, the problem can only become more difficult, as we still need to satisfy the original agents.

Observation 2.2. For any positive integers n_1, \dots, n_g and n'_1, \dots, n'_g such that $n_i \geq n'_i$ for each $i \in [g]$, we have $p_{\text{MMS}}(n_1, \dots, n_g) \geq p_{\text{MMS}}(n'_1, \dots, n'_g)$.

Similarly, if we remove a group, the problem can only become easier, as the items originally assigned to that group can be allocated arbitrarily among the remaining groups.

Observation 2.3. For any positive integers n_1, \dots, n_g and $i \in [g]$, we have $p_{\text{MMS}}(n_1, \dots, n_g) \geq p_{\text{MMS}}(n_1, \dots, n_i)$.

By considering $g - 1$ identical items, we also have that p needs to be at least the number of groups.

Observation 2.4. For any positive integers n_1, \dots, n_g , we have $p_{\text{MMS}}(n_1, \dots, n_g) \geq g$.

2.2 Covering Designs

When deriving lower bounds, we will make use of *covering designs*, which are extensively studied in combinatorics.

Definition 2.5 (e.g., [Gordon *et al.*, 1996]). For positive integers m, s, t , an (m, s, t) -covering design is a collection $\{S_1, \dots, S_r\}$ of subsets of $[m]$, each of size s , with the property that for any subset $T \subseteq [m]$ of size at most t , there exists $i \in [r]$ such that $T \subseteq S_i$.

Denote by $C(m, s, t)$ the size of the smallest (m, s, t) -covering design. A trivial upper bound on $C(m, s, t)$ can be obtained by observing that the collection of all subsets of size s is sufficient to cover any set of size t .

Observation 2.6. For any positive integers $m \geq s \geq t$, we have $C(m, s, t) \leq \binom{m}{s}$.

The following two upper bounds are known in the literature.

Lemma 2.7 ([Rees *et al.*, 1999, Thm. 5]). For any positive integers m, k, s, t such that $m \geq k + st$, it holds that $C(m, m - k, t) \leq \binom{t + \lceil k/s \rceil}{t}$.

Lemma 2.8 ([Erdős and Spencer, 1974]³). For any positive integers $m \geq s \geq t$, we have $C(m, s, t) \leq \frac{\binom{m}{t}}{\binom{s}{t}} \cdot (1 + \ln \binom{s}{t})$.

2.3 Concentration Bounds

We will also use the following concentration inequalities often referred to as *multiplicative Chernoff bounds*.

Lemma 2.9. Let X_1, \dots, X_k be independent random variables taking values in $[0, 1]$, and let $Z := X_1 + \dots + X_k$. Then, for every $\delta \geq 0$, we have

$$\Pr[Z \leq (1 - \delta)\mathbb{E}[Z]] \leq \exp\left(-\frac{\delta^2}{2}\mathbb{E}[Z]\right) \quad (1)$$

and

$$\Pr[Z \geq (1 + \delta)\mathbb{E}[Z]] \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^{\mathbb{E}[Z]}. \quad (2)$$

3 Upper Bounds

In this section, we derive upper bounds on p_{MMS} by proving Theorem 1.1. We will use the following algorithm for approximating the $\text{MMS}^{1\text{-out-of-}p}$ value.

Lemma 3.1 ([Deuermeyer *et al.*, 1982]). For any agent $a \in N$ and any positive integer p , there is a polynomial-time algorithm that computes a value t_a such that $\text{MMS}_a^{1\text{-out-of-}p} \geq t_a \geq \frac{3}{4} \cdot \text{MMS}_a^{1\text{-out-of-}p}$.

We will also use the following results by Aigner-Horev and Segal-Halevi [2022].⁴

Lemma 3.2 ([Aigner-Horev and Segal-Halevi, 2022]). Suppose that an agent a has utility at most 1 for each item, and $u_a(M) \geq 2p$ for some positive integer p . Then, $\text{MMS}_a^{1\text{-out-of-}p} \geq 1$.

Lemma 3.3 ([Aigner-Horev and Segal-Halevi, 2022]). If $n_1 = \dots = n_g = 1$, there is a polynomial-time algorithm that, given values $(t_a)_{a \in N}$ such that $t_a \leq \text{MMS}_a^{1\text{-out-of-}(2n-2)}$ for all $a \in N$, outputs an allocation that gives each agent $a \in N$ utility at least t_a .

We start by proving a general upper bound of $O(\log[(n_1 + 1) \cdots (n_g + 1)])$, which already implies the first part of Theorem 1.1. The high-level idea is to allocate (roughly) half of the items randomly. We can show that this partial allocation already satisfies the desired MMS guarantee for most of the agents. For the remaining agents, we use the algorithm from Lemma 3.3 to individually assign to them the remaining (roughly half) of the items.

³See, e.g., [Gordon *et al.*, 1995, p. 270].

⁴The first result is stated as Lemma 5.1 in their paper, and the second follows from the same proof as their Theorem 1.4.

Theorem 3.4. For any group sizes n_1, \dots, n_g , let

$$p = 80(\lceil \log(n_1 + 1) \rceil + \dots + \lceil \log(n_g + 1) \rceil).$$

Then, an MMS^{1-out-of-p} allocation always exists. Furthermore, there is a randomized polynomial-time algorithm that finds such an allocation with probability at least 2/3.

Proof. First, we run the algorithm from Lemma 3.1 to find a value t_a for each $a \in N$ such that $\frac{3}{4} \cdot \text{MMS}_a^{\text{1-out-of-}p}(M) \leq t_a \leq \text{MMS}_a^{\text{1-out-of-}p}(M)$. By scaling all utilities of a by a factor of $3/(4t_a)$, we may assume henceforth that $\frac{3}{4} \leq \text{MMS}_a^{\text{1-out-of-}p}(M) \leq 1$. Furthermore, we can assume that each agent's utility for each item is at most 1. Indeed, otherwise we may decrease it to 1, which does not change the MMS^{1-out-of-p} value nor whether a bundle satisfies MMS^{1-out-of-p} for the agent. Our algorithm for finding the desired allocation (A_1, \dots, A_g) works as follows.

- Let $q_1 = \frac{40\lceil \log(n_1 + 1) \rceil}{p}, \dots, q_g = \frac{40\lceil \log(n_g + 1) \rceil}{p}$, and $q = 1 - q_1 - \dots - q_g = \frac{1}{2}$.
- Construct $\tilde{A}_1, \dots, \tilde{A}_g, \tilde{A}$ by assigning each item from M to one of these sets independently with probabilities q_1, \dots, q_g, q , respectively.
- Let $N' = \bigcup_{j \in [g]} \{a \in N_j \mid u_a(\tilde{A}_j) < 1\}$.
- Check whether the following two conditions hold:

$$|N'| \leq 2g \quad (3)$$

and

$$u_a(\tilde{A}) \geq 8g \quad \forall a \in N'. \quad (4)$$

If at least one condition does not hold, the algorithm terminates with failure.

- Run the algorithm from Lemma 3.3 on the agents N' (with $t_a = 1$ for all $a \in N'$) and items \tilde{A} . Let the output be $(\tilde{A}^a)_{a \in N'}$.
- Finally, output (A_1, \dots, A_g) defined by

$$A_j := \tilde{A}_j \cup \bigcup_{a \in N_j \cap N'} \tilde{A}^a.$$

Let us first argue that if the algorithm does not fail, then the output $\mathcal{A} = (A_1, \dots, A_g)$ indeed satisfies MMS^{1-out-of-p}. By definition of N' and our assumption that $\text{MMS}_a^{\text{1-out-of-}p}(M) \leq 1$ for all agents $a \in N$, this is already satisfied for all agents in $N \setminus N'$. For the agents in N' , note that (3) implies that $2|N'| \leq 4g$. Furthermore, Lemma 3.2 together with (4) implies that $\text{MMS}_a^{\text{1-out-of-}4g}(\tilde{A}) \geq 1$ for all $a \in N'$. Thus, the guarantee of Lemma 3.3 ensures that these agents also receive a bundle with utility at least 1 in \mathcal{A} . It follows that \mathcal{A} satisfies MMS^{1-out-of-p}.

Next, we show that the algorithm fails with probability at most 1/3. To this end, we first bound the probability that (3) fails. Consider any agent $a \in N_j$ for some $j \in [g]$. For $\ell \in M$, let X_ℓ denote the random variable $\mathbf{1}[\ell \in \tilde{A}_j] \cdot u_a(\ell)$. Note that $X := \sum_{\ell \in M} X_\ell$ is exactly equal

to $u_a(\tilde{A}_j)$. Furthermore, we have $\mathbb{E}[X] = q_j \cdot u_a(M) \geq q_j \cdot p \cdot \text{MMS}_a^{\text{1-out-of-}p}(M) \geq q_j \cdot p \cdot \frac{3}{4} = 30\lceil \log(n_j + 1) \rceil$. Thus, applying Lemma 2.9 (specifically, (1)), we get

$$\begin{aligned} \Pr[X < 1] &\leq \Pr[X \leq 0.1\mathbb{E}[X]] \\ &\leq \exp(-0.4\mathbb{E}[X]) \\ &\leq \exp(-12\lceil \log(n_j + 1) \rceil) \leq \frac{1}{10n_j}. \end{aligned}$$

The above inequality is equivalent to

$$\Pr[a \in N'] \leq \frac{1}{10n_j}.$$

As a result, we have

$$\mathbb{E}[|N'|] = \sum_{j \in [g]} \sum_{a \in N_j} \Pr[a \in N'] \leq \sum_{j \in [g]} \sum_{a \in N_j} \frac{1}{10n_j} = \frac{g}{10}.$$

Using Markov's inequality, we get

$$\Pr[|N'| > 2g] \leq \frac{1}{20}. \quad (5)$$

Next, we bound the probability that (4) fails. Consider any agent $a \in N_j$ for some $j \in [g]$. For $\ell \in M$, let \tilde{X}_ℓ denote the random variable $\mathbf{1}[\ell \in \tilde{A}] \cdot u_a(\ell)$. Note that $\tilde{X} := \sum_{\ell \in M} \tilde{X}_\ell$ is exactly equal to $u_a(\tilde{A})$. Furthermore, we have $\mathbb{E}[\tilde{X}] = q \cdot u_a(M) \geq q \cdot p \cdot \text{MMS}_a^{\text{1-out-of-}p}(M) \geq q \cdot p \cdot \frac{3}{4} = \frac{3}{8}p \geq 30g$. Thus, applying Lemma 2.9 (specifically, (1)), we get

$$\begin{aligned} \Pr[\tilde{X} < 8g] &\leq \Pr[\tilde{X} \leq 0.3\mathbb{E}[\tilde{X}]] \\ &\leq \exp(-0.24\mathbb{E}[\tilde{X}]) \\ &\leq \exp(-0.09p) \\ &\leq \exp(-7.2 \log[(n_1 + 1) \cdots (n_g + 1)]) \\ &\leq \frac{1}{10(n_1 + 1) \cdots (n_g + 1)} \\ &\leq \frac{1}{10(n_1 + \cdots + n_g)}. \end{aligned}$$

The above inequality is equivalent to

$$\Pr[u_a(\tilde{A}) < 8g] \leq \frac{1}{10(n_1 + \cdots + n_g)}.$$

Using the union bound over all agents, we get

$$\Pr[\exists a \in N, u_a(\tilde{A}) < 8g] \leq \frac{1}{10}. \quad (6)$$

Finally, by combining (5) and (6) via the union bound, we conclude that (3) and (4) hold simultaneously with probability at least $1 - 1/20 - 1/10 > 2/3$. In other words, the algorithm succeeds with probability at least 2/3, as desired. \square

To prove the second part of Theorem 1.1, we again use random assignment in the first stage; however, this time we assign almost all of the items to the first group. We then use the algorithm from Theorem 3.4 to assign the leftover items to the remaining groups. By appealing to the stronger concentration for the upper tail bound when the required value

is much larger than its expectation (i.e., (2) in Lemma 2.9), we arrive at the improved bound stated in the following theorem. Note that this bound implies the desired bound in Theorem 1.1 due to the assumption that $\beta_1 > 1000$. (Recall the definition of β_1 from Section 2.)

Theorem 3.5. *Let n_1, \dots, n_g be group sizes such that $\beta_1 > 1000$, and let*

$$p = 320000 \left(\left\lceil \frac{\log(n_1 + 1)}{\log \beta_1} \right\rceil + \sum_{i=2}^g \lceil \log(n_i + 1) \rceil \right).$$

Then, an MMS^{1-out-of- p} allocation always exists. Furthermore, there is a randomized polynomial-time algorithm that finds such an allocation with probability at least 2/3.

Proof. Similarly to the proof of Theorem 3.4, we may assume for every agent $a \in N$ that $\text{MMS}_a^{\text{1-out-of-}p}(M) \in [3/4, 1]$ and that $u_a(\ell) \leq 1$ for every item $\ell \in M$. Our algorithm for finding the desired allocation (A_1, \dots, A_g) works as follows.

- Let $q = \frac{320 \sum_{i=2}^g \lceil \log(n_i + 1) \rceil}{p}$ and $q_1 = 1 - q$.
- Construct \tilde{A}_1, \tilde{A} by assigning each item from M to one of these sets independently with probabilities q_1, q , respectively.
- Check whether the following two conditions hold:

$$u_a(\tilde{A}_1) \geq 1 \quad \forall a \in N_1 \quad (7)$$

and

$$u_a(\tilde{A}) \geq qp/2 \quad \forall a \in N_2 \cup \dots \cup N_g. \quad (8)$$

If at least one condition does not hold, the algorithm terminates with failure.

- Run the algorithm from Theorem 3.4 ten times on the $g - 1$ groups N_2, \dots, N_g and items \tilde{A} . If any of the runs succeeds, let $(\tilde{A}_2, \dots, \tilde{A}_g)$ be the output of a successful run. Otherwise, the algorithm terminates with failure.
- Finally, output $(\tilde{A}_1, \dots, \tilde{A}_g)$.

Observe that (7) implies that every agent in N_1 receives at least her MMS^{1-out-of- p} (M). Furthermore, Lemma 3.2 together with (8) ensures that the MMS^{1-out-of- $qp/4$} (\tilde{A}) value of every agent in $N_2 \cup \dots \cup N_g$ is at least 1. As a result, if the algorithm succeeds, then its output satisfies MMS^{1-out-of- p} .

Next, we argue that the algorithm succeeds with probability at least 2/3. Since we run the algorithm from Theorem 3.4 ten times, with probability at least $1 - (1/3)^{10}$, at least one run succeeds. As such, it suffices to show that with probability at least $2/3 + (1/3)^{10} < 0.7$, both (7) and (8) hold.

We first bound the probability that (7) holds. Consider any agent $a \in N_1$. For $\ell \in M$, let X_ℓ denote the random variable $\mathbf{1}[\ell \in \tilde{A}] \cdot u_a(\ell)$. Note that $X := \sum_{\ell \in M} X_\ell$ is exactly equal to $u_a(\tilde{A}) = u_a(M) - u_a(\tilde{A}_1) \geq 3p/4 - u_a(\tilde{A}_1)$, where the inequality follows from the assumption that $\text{MMS}_a^{\text{1-out-of-}p}(M) \geq 3/4$. Thus, applying Lemma 2.9

(specifically, (2)) with $\delta = \frac{3p/4-1}{\mathbb{E}[X]} - 1$, we get⁵

$$\begin{aligned} \Pr \left[X \geq \frac{3p}{4} - 1 \right] &= \Pr[X \geq (1 + \delta)\mathbb{E}[X]] \\ &\leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^{\mathbb{E}[X]} \\ &\leq \left(\frac{e}{1 + \delta} \right)^{(1+\delta)\mathbb{E}[X]} \\ &= \left(\frac{e \cdot \mathbb{E}[X]}{\frac{3p}{4} - 1} \right)^{\frac{3p}{4}-1}. \end{aligned}$$

Since $p \geq 320000 \log(n_1 + 1) / \log \beta_1$, we have

$$q \leq 640 \sum_{i=2}^g \frac{\log(n_i + 1)}{p} \leq \frac{1}{500} \log \beta_1 \cdot \frac{1}{\beta_1}.$$

Observe that $\mathbb{E}[X] = q \cdot u_a(M)$. Since $\text{MMS}_a^{\text{1-out-of-}p}(M) \leq 1$, Lemma 3.2 implies⁶ that $u_a(M) < 4p$. Combining the previous two sentences, we get $\mathbb{E}[X] \leq 4qp$. Plugging this into the expression above, we thus have

$$\begin{aligned} \Pr \left[X \geq \frac{3p}{4} - 1 \right] &\leq \left(\frac{4eqp}{\frac{3p}{4} - 1} \right)^{\frac{3p}{4}-1} \\ &\leq (20q)^{\frac{3p}{4}-1} \\ &\leq \left(\frac{1}{25} \log \beta_1 \cdot \frac{1}{\beta_1} \right)^{\frac{3p}{4}-1} \\ &\stackrel{(*)}{\leq} \left(\frac{1}{\beta_1} \right)^{\frac{1}{2}(\frac{3p}{4}-1)} \\ &\leq \left(\frac{1}{\beta_1} \right)^{10 \cdot \frac{\log(n_1 + 1)}{\log \beta_1}} \\ &= \frac{1}{(n_1 + 1)^{10}} \leq \frac{1}{10n_1}, \end{aligned}$$

where for $(*)$ we use the fact that $\frac{1}{25} \cdot \frac{\log x}{x} \leq \frac{1}{\sqrt{x}}$ for all $x \geq 1$.

The above inequality implies that $\Pr[u_a(\tilde{A}_1) < 1] \leq \frac{1}{10n_1}$. Taking the union bound over all $a \in N_1$, we get

$$\Pr[\exists a \in N_1, u_a(\tilde{A}_1) < 1] \leq 0.1. \quad (9)$$

Next, we bound the probability that (8) holds. Consider any agent $a \in N_2 \cup \dots \cup N_g$. For $\ell \in M$, let \tilde{X}_ℓ denote the random variable $\mathbf{1}[\ell \in \tilde{A}] \cdot u_a(\ell)$. Note that $\tilde{X} := \sum_{\ell \in M} \tilde{X}_\ell$ is exactly equal to $u_a(\tilde{A})$. Furthermore, we have $\mathbb{E}[\tilde{X}] = q \cdot u_a(M) \geq q \cdot p \cdot \text{MMS}_a^{\text{1-out-of-}p}(M) \geq \frac{3}{4}qp$. Thus, applying Lemma 2.9 (specifically, (1)), we get

$$\Pr \left[\tilde{X} < \frac{qp}{2} \right] \leq \Pr \left[\tilde{X} \leq \frac{2}{3} \mathbb{E}[\tilde{X}] \right]$$

⁵A later sequence of arguments we make in this proof implies that $\frac{e \cdot \mathbb{E}[X]}{3p/4-1} < 1$, so we have $\delta > 0$.

⁶Otherwise, if $u_a(M) \geq 4p$, then $\text{MMS}_a^{\text{1-out-of-}p}(M) \geq 2 \cdot \text{MMS}_a^{\text{1-out-of-}2p}(M) \geq 2$, a contradiction.

$$\begin{aligned}
&\leq \exp\left(-\frac{\mathbb{E}[\tilde{X}]}{18}\right) \\
&\leq \exp\left(-\frac{qp}{24}\right) \\
&\leq \exp(-10\log[(n_2+1)\cdots(n_g+1)]) \\
&\leq \frac{1}{10(n_2+1)\cdots(n_g+1)} \\
&\leq \frac{1}{10(n_2+\cdots+n_g)}.
\end{aligned}$$

The above inequality is equivalent to

$$\Pr\left[u_a(\tilde{A}) < \frac{qp}{2}\right] \leq \frac{1}{10(n_2+\cdots+n_g)}.$$

Using the union bound over all agents $a \in N_2 \cup \cdots \cup N_g$, we get

$$\Pr\left[\exists a \in N_2 \cup \cdots \cup N_g, u_a(\tilde{A}) < \frac{qp}{2}\right] \leq 0.1. \quad (10)$$

Finally, by combining (9) and (10) via the union bound, we conclude that (7) and (8) hold simultaneously with probability at least $1 - 0.1 - 0.1 > 0.7$. In other words, the algorithm succeeds with probability at least $2/3$, as desired. \square

4 Lower Bounds

In this section, we turn our attention to lower bounds and prove Theorem 1.2. We first present a generic construction based on covering designs, and later apply this construction with more specific parameters to derive our theorem.

Lemma 4.1. *Let $m, p, t_1, \dots, t_g, n_1, \dots, n_g$ be positive integers such that*

1. $t_1 + \cdots + t_g + g > m$;
2. For each $i \in [g]$, at least one of the following holds:
 - $t_i \leq \lfloor m/p \rfloor - 1$;
 - $m - (p-1) > t_i$ and $C(m, m - (p-1), t_i) \leq n_i$.

Then, $p_{\text{MMS}}(n_1, \dots, n_g) > p$.

Proof. For each $i \in [g]$, we define the utilities of the agents in the i -th group N_i for the m items as follows.

- If $t_i \leq \lfloor m/p \rfloor - 1$, let each item have utility 1 for every agent in N_i .
- If $m - (p-1) > t_i$ and $C(m, m - (p-1), t_i) \leq n_i$, then let $\{S_{i,1}, \dots, S_{i,n_i}\}$ be any $(m, m - (p-1), t_i)$ -covering design. For each $j \in [n_i]$, we define the utility function of agent (i, j) by

$$u_{i,j}(\ell) = \begin{cases} \frac{1}{m-(p-1)} & \text{if } \ell \in S_{i,j}; \\ 1 & \text{otherwise.} \end{cases}$$

If both conditions are met, we pick one arbitrarily.

Suppose for contradiction that an $\text{MMS}^{1\text{-out-of-}p}$ allocation exists. We claim that each group N_i must receive at least $t_i + 1$ items. In the first case, group i must receive at least $\lfloor m/p \rfloor \geq t_i + 1$ items. Consider the second case. Each

agent (i, j) 's MMS^{1-out-of-p} is 1, by putting all items from $S_{i,j}$ in one bundle and each remaining item in its own bundle. Now, for any set T of at most t_i items, there exists j such that $T \subseteq S_{i,j}$, and so $u_{i,j}(T) \leq \frac{t_i}{m-(p-1)} < 1$. Hence, group i must again receive at least $t_i + 1$ items in this case. The total number of items that need to be allocated is therefore at least $\sum_{i \in [g]} (t_i + 1) = t_1 + \cdots + t_g + g > m$, a contradiction. It follows that no MMS^{1-out-of-p} allocation exists in this instance, and so $p_{\text{MMS}}(n_1, \dots, n_g) > p$. \square

We now apply Lemma 4.1 using various parameter settings, starting with the balanced case.

Theorem 4.2. *Let $n_1 \geq \cdots \geq n_g \geq 63$ be positive integers such that $\lfloor \frac{1}{6} \log(n_1 + 1) \rfloor \leq \sum_{i=2}^g \lfloor \frac{1}{6} \log(n_i + 1) \rfloor$. Then, for*

$$p = \sum_{i=1}^g \left\lfloor \frac{1}{6} \log(n_i + 1) \right\rfloor,$$

we have $p_{\text{MMS}}(n_1, \dots, n_g) > p$.

Proof. Let $t_i = 2 \lfloor \frac{1}{6} \log(n_i + 1) \rfloor \geq 2$ for all $i \in [g]$. Note that $p = \frac{1}{2}(t_1 + \cdots + t_g)$, and let $m = 2p$. We verify the two conditions in Lemma 4.1. For the first condition, we have $t_1 + \cdots + t_g + g = m + g > m$.

It remains to check the second condition for each $i \in [g]$. To do so, first note that

$$\begin{aligned}
m - (p-1) &= p + 1 > \sum_{i=1}^g \left\lfloor \frac{1}{6} \log(n_i + 1) \right\rfloor \\
&\geq 2 \left\lfloor \frac{1}{6} \log(n_1 + 1) \right\rfloor \\
&\geq 2 \left\lfloor \frac{1}{6} \log(n_i + 1) \right\rfloor = t_i,
\end{aligned}$$

where the second inequality follows from our assumption in the theorem. Let $s_i = \left\lfloor \frac{p+1}{t_i} \right\rfloor \geq 1$. We have $s_i \geq \frac{p+1}{2t_i}$ and $m = (p-1) + (p+1) \geq (p-1) + s_i t_i$. Applying Lemma 2.7, we get

$$\begin{aligned}
C(m, m - (p-1), t_i) &\leq \binom{t_i + \lceil \frac{p-1}{s_i} \rceil}{t_i} \\
&\leq \binom{t_i + \lceil \frac{p-1}{(p+1)/(2t_i)} \rceil}{t_i} \\
&\leq \binom{t_i + 2t_i}{t_i} < 2^{3t_i} \leq n_i + 1,
\end{aligned}$$

where the last inequality is due to the definition of t_i . As a result, Lemma 4.1 implies that $p_{\text{MMS}}(n_1, \dots, n_g) > p$. \square

For the unbalanced case, it will be more convenient to have two separate constructions. First, we consider the case where the smaller groups N_2, \dots, N_g are “not too small”. In this case, we can still use a covering design (from Lemma 2.8) for these groups.

Theorem 4.3. Let $n_1 \geq n_2 \geq \dots \geq n_g$ be positive integers such that $\beta_1 \geq 1000$ and, for each $i \in [g]$, it holds that $n_i \geq 4(\log(n_1 + 1))^4$. Then, for

$$p = \left\lfloor \frac{\log(n_1 + 1)}{10 \log \beta_1} \right\rfloor,$$

we have $p_{\text{MMMS}}(n_1, \dots, n_g) > p$.

Proof. The statement is trivial if $p = 0$, so assume without loss of generality that $p \geq 1$. Let $m = \left\lfloor \frac{(n_1+1)^{1/p}}{e} \right\rfloor \cdot p$ and $t_1 = m - p$. We will show later that $(n_1 + 1)^{1/p} \geq 30$, which means that $m \geq 2p$ and $t_1 \geq 1$. Let $t_i = \lceil \beta_i \cdot p \rceil$ for each $i \in \{2, \dots, g\}$. We verify the two conditions in Lemma 4.1. For the first condition, we have $t_1 + \dots + t_g + g \geq t_1 + (\beta_2 \cdot p + \dots + \beta_g \cdot p) + g = t_1 + p + g = m + g > m$.

It remains to check the second condition for each $i \in [g]$. We do so separately for $i = 1$ and $i \neq 1$. For $i = 1$, it is clear that $m - (p - 1) > t_1$. Also, we have

$$\begin{aligned} C(m, m - (p - 1), t_1) &\leq C(m, m - p, t_1) \\ &\leq \binom{m}{m-p} \\ &= \binom{m}{p} < \left(\frac{em}{p}\right)^p \leq n_1 + 1, \end{aligned}$$

where the second inequality follows from Observation 2.6 and the last inequality from our choice of m .

Now, fix $i \in \{2, \dots, g\}$. It is again clear that $m - (p - 1) \geq p > t_i$. By Lemma 2.8, we have

$$\begin{aligned} C(m, m - (p - 1), t_i) &\leq C(m, m - p, t_i) \\ &\leq \frac{\binom{m}{t_i}}{\binom{m-p}{t_i}} \cdot \left(1 + \ln \binom{m-p}{t_i}\right) \\ &\leq \left(\frac{m-t_i}{m-p-t_i}\right)^{t_i} (1 + t_i \ln m) \\ &\leq \exp\left(t_i \cdot \frac{p}{m-p-t_i}\right) (1 + t_i \ln m), \quad (11) \end{aligned}$$

where for the last inequality we use the well-known bound $1 + x \leq \exp(x)$, which holds for all real numbers x .

We now bound each term separately. Before bounding the first term, observe that

$$(n_1 + 1)^{1/p} \geq (n_1 + 1)^{\frac{10 \log \beta_1}{\log(n_1+1)}} = \beta_1^{10} \geq 30\beta_1,$$

where the last inequality follows from our assumption that $\beta_1 \geq 1000$. From our choice of m , this implies that

$$m \geq 10\beta_1 \cdot p. \quad (12)$$

We can now bound the first term as follows:

$$\begin{aligned} \exp\left(t_i \cdot \frac{p}{m-p-t_i}\right) &\leq \exp\left((\beta_i p + 1) \cdot \frac{p}{m-2p}\right) \\ &\stackrel{(12)}{\leq} \exp\left(\frac{(\beta_i p + 1)p}{8\beta_1 \cdot p}\right) \end{aligned}$$

$$\begin{aligned} &= \exp\left(\frac{\beta_i p + 1}{8\beta_1}\right) \\ &\leq 2 \exp\left(\frac{\beta_i p}{8\beta_1}\right) \\ &\leq 2 \exp\left(\frac{\beta_i \cdot \frac{\log(n_1+1)}{10 \log \beta_1}}{8\beta_1}\right) \\ &= 2 \exp\left(\frac{\log(n_i + 1)}{80 \log \beta_1}\right) \\ &\leq 2 \exp\left(\frac{\log(n_i + 1)}{80}\right) \\ &\leq 2\sqrt{n_i}, \end{aligned}$$

where for the last inequality we use the fact that $n_i \geq 2$, which follows from our assumption on n_i .

Next, we bound the second term. We have

$$\begin{aligned} 1 + t_i \ln m &\leq 1 + p \cdot \ln((n_1 + 1)^{1/p} \cdot p) \\ &= 1 + p \cdot \left(\frac{1}{p} \ln(n_1 + 1) + \ln p\right) \\ &= 1 + \ln(n_1 + 1) + p \ln p \\ &\leq 1 + \ln(n_1 + 1) + (\ln(n_1 + 1))^2 \\ &\leq 2(\ln(n_1 + 1))^2 \leq (\log(n_1 + 1))^2 \leq \sqrt{n_i}/2, \end{aligned}$$

where the last inequality follows from our assumption on n_i .

Plugging the estimates of both terms into (11), we arrive at

$$C(m, m - (p - 1), t_i) \leq n_i.$$

As a result, Lemma 4.1 implies that $p_{\text{MMMS}}(n_1, \dots, n_g) > p$. \square

In the next construction, we do not make any assumption on N_2, \dots, N_g (i.e., they could all be of size 1), and simply use the “identical utilities” construction.

Theorem 4.4. Let n_1 be a positive integer such that $\log(n_1 + 1) \geq 1000(g - 1)$. Then, for

$$p = \left\lfloor \frac{\log(n_1 + 1)}{10 \log\left(\frac{\log(n_1+1)}{g-1}\right)} \right\rfloor,$$

we have $p_{\text{MMMS}}(n_1, 1, \dots, 1) > p$.

Proof. By Observation 2.4, the statement is trivial if $p \leq g$, so assume without loss of generality that $p \geq g + 1$. Let $m = \left\lfloor \frac{p}{g-1} + 1 \right\rfloor \cdot p$, and note that $m > p$. Let $t_1 = m - p$, and $t_i = \lceil p/(g-1) \rceil$ for each $i \in \{2, \dots, g\}$. We verify the two conditions in Lemma 4.1. For the first condition, we have $t_1 + \dots + t_g + g \geq t_1 + p + g = m + g > m$.

It remains to check the second condition for each $i \in [g]$. We do so separately for $i = 1$ and $i \neq 1$. For $i = 1$, it is clear that $m - (p - 1) > t_1$. Also, we have

$$\begin{aligned} C(m, m - (p - 1), t_1) &\leq C(m, m - p, t_1) \\ &\leq \binom{m}{m-p} \end{aligned}$$

$$\begin{aligned}
&= \binom{m}{p} \\
&< \left(\frac{em}{p} \right)^p \\
&= \left(e \left[\frac{p}{g-1} + 1 \right] \right)^p \\
&\leq \left(\frac{10p}{g-1} \right)^p \\
&\leq \left(\frac{\log(n_1+1)}{g-1} \right)^{\frac{\log(n_1+1)}{\log(\frac{\log(n_1+1)}{g-1})}} \\
&= n_1 + 1,
\end{aligned}$$

where the second inequality follows from Observation 2.6, and the last inequality from the definition of p along with our assumption that $\log(n_1+1) \geq 1000(g-1)$.

For $i \in \{2, \dots, g\}$, we simply have $\lfloor m/p \rfloor - 1 = \left\lceil \frac{p}{g-1} + 1 \right\rceil - 1 \geq t_i$; in fact, the inequality is an equality. As a result, Lemma 4.1 implies that $p_{\text{MMS}}(n_1, \dots, n_g) > p$. \square

By appropriately combining the previous three theorems for different regimes of parameters, we arrive at Theorem 1.2.

Proof of Theorem 1.2. We consider the two cases separately.

Case 1: $\beta_1 \leq 1000$. If $\log[(n_1+1) \cdots (n_g+1)] \leq 9000g$, then Observation 2.4 immediately implies the desired bound. Assume therefore that $\log[(n_1+1) \cdots (n_g+1)] > 9000g$. Suppose for contradiction that $n_2 < 63$. Then $n_2, \dots, n_g < 63$ and $\log(n_1+1) > 9000g - 6g > 8000g$. On the other hand, $\beta_1 \leq 1000$ implies that $\log(n_1+1) \leq 1000 \cdot (6g) = 6000g$, a contradiction. Hence, $n_2 \geq 63$.

Let $g' \in \{2, \dots, g\}$ be the largest index such that $n_{g'} \geq 63$. Since $\beta_1 \leq 1000$ and $\log[(n_1+1) \cdots (n_g+1)] > 9000g$, we have $\log[(n_2+1) \cdots (n_g+1)] > 8g$. It follows that

$$\begin{aligned}
&\log[(n_2+1) \cdots (n_{g'}+1)] \\
&\geq \log[(n_2+1) \cdots (n_g+1)] - 6g \\
&\geq \Omega(\log[(n_2+1) \cdots (n_g+1)]) \\
&\geq \Omega(\log[(n_1+1) \cdots (n_g+1)]).
\end{aligned}$$

Applying Theorem 4.2, we get

$$\begin{aligned}
p_{\text{MMS}}(n_2, n_2, n_3, \dots, n_{g'}) \\
&\geq \Omega(\log[(n_2+1)(n_3+1) \cdots (n_{g'}+1)]) \\
&\geq \Omega(\log[(n_1+1)(n_2+1) \cdots (n_g+1)]).
\end{aligned}$$

Finally, combining this with Observations 2.2 and 2.3 yields the desired bound.

Case 2: $\beta_1 > 1000$. This means that $n_1 + 1 > 2^{1000}(n_2 + 1) \cdots (n_g + 1)$, which implies that $n_1 > n_2 + \cdots + n_g$ and therefore $n_1 > n/2$. Let $\delta \in (0, 1)$ be the constant in the theorem statement, i.e., $g \leq (\log n)^{1-\delta}$.

Case 2.1: $\log[(n_2+1) \cdots (n_g+1)] \leq (\log(n_1+1))^{1-\delta/2}$. Since $\beta_1 > 1000$, we have $\log(n_1+1) > 1000(g-1)$. Applying Theorem 4.4 and Observation 2.2, we get

$$p_{\text{MMS}}(n_1, \dots, n_g) \geq \Omega \left(\frac{\log(n_1+1)}{\log \left(\frac{\log(n_1+1)}{g} \right)} \right)$$

$$\begin{aligned}
&\geq \Omega \left(\frac{\log(n_1+1)}{\log \log(n_1+1)} \right) \\
&\geq \Omega \left(\frac{\log(n_1+1)}{\log \left(\frac{\log(n_1+1)}{\log[(n_2+1) \cdots (n_g+1)]} \right)} \right),
\end{aligned}$$

where the last relation holds by the assumption of Case 2.1 and since δ is constant.

Case 2.2: $\log[(n_2+1) \cdots (n_g+1)] > (\log(n_1+1))^{1-\delta/2}$. Let $\tau = \frac{(\log(n_1+1))^{1-\delta/2}}{2g}$, and let $g' \in \{2, \dots, g\}$ be the largest index such that $\log(n_{g'}+1) \geq \tau$; note that g is well-defined due to the assumption of Case 2.2. We have

$$\begin{aligned}
&\log[(n_2+1) \cdots (n_{g'}+1)] \\
&\geq \log[(n_2+1) \cdots (n_g+1)] - \tau g \\
&\geq \frac{1}{2} \log[(n_2+1) \cdots (n_g+1)].
\end{aligned}$$

Recall that $n_1 \geq n/2$. Hence, for any sufficiently large n , we have

$$\begin{aligned}
n_{g'} &\geq 2^\tau - 1 \geq 2^{\Omega((\log n)^{\delta/2})} \geq 4(\log n)^4 \\
&\geq 4(\log(n_1+1))^4,
\end{aligned}$$

where for the second relation we use the assumption $g \leq (\log n)^{1-\delta}$. We can thus apply Theorem 4.3 to conclude that

$$\begin{aligned}
p_{\text{MMS}}(n_1, \dots, n_{g'}) &\geq \Omega \left(\frac{\log(n_1+1)}{\log \left(\frac{\log(n_1+1)}{\log[(n_2+1) \cdots (n_{g'}+1)]} \right)} \right) \\
&= \Omega \left(\frac{\log(n_1+1)}{\log \left(\frac{\log(n_1+1)}{\log[(n_2+1) \cdots (n_g+1)]} \right)} \right).
\end{aligned}$$

Applying Observation 2.3 then yields the desired bound. \square

5 Non-Asymptotic Results

While our bounds on p_{MMS} are already asymptotically tight, in this section, we additionally present some concrete non-asymptotic bounds. Our focus will be on the fundamental case of two groups. We begin with the upper bound.

Theorem 5.1. *Let n_1, n_2, p be positive integers such that*

$$\left(1 - \frac{1}{2^{p-1}} \right)^{n_1} + \left(1 - \frac{1}{2^{p-1}} \right)^{n_2} \geq 1.$$

Then, we have $p_{\text{MMS}}(n_1, n_2) \leq p$.

Before we prove Theorem 5.1, we note that the theorem, together with Bernoulli's inequality, immediately implies the following more explicit bound.

Corollary 5.2. *For any positive integers n_1, n_2 , it holds that $p_{\text{MMS}}(n_1, n_2) \leq 1 + \lceil \log(n_1+n_2) \rceil$.*

For the proof of Theorem 5.1, we need some additional notation.

- For an agent $a \in N$, we write S_a^p to denote the collection of all subsets $S \subseteq M$ such that $u_a(S) \geq \text{MMS}_a^{1\text{-out-of-}p}(M)$.

- We write $\mathcal{P}(S)$ to denote the power set of S , and $T \sim \mathcal{P}(S)$ to denote a (uniformly) random subset of S .
- We write $\mathcal{X}(S)$ to denote the set of all allocations of the items in S to two groups, and write $(T_1, T_2) \sim \mathcal{X}(S)$ to denote a (uniformly) random allocation of S .

We start by establishing the following key lemma, which gives a lower bound on the probability that a random subset of items provides MMS^{1-out-of-p} to an agent.

Lemma 5.3. *For any agent $a \in N$, we have*

$$\Pr_{T \sim \mathcal{P}(M)}[T \in \mathcal{S}_a^p] > 1 - \frac{1}{2^{p-1}}.$$

Proof. Fix any $a \in N$. By definition of MMS^{1-out-of-p}, there exists a partition $\{B_1, \dots, B_p\}$ of M such that $u_a(B_i) \geq \text{MMS}_a^{\text{1-out-of-}p}(M)$ for all $i \in [p]$. Observe that $T \sim \mathcal{P}(M)$ can be equivalently generated as follows:

- Independently sample $(B_i^1, B_i^2) \sim \mathcal{X}(B_i)$ for all $i \in [p]$;
- Independently sample $j_1, \dots, j_p \sim \{1, 2\}$;
- Let $T = \bigcup_{i \in [p]} B_i^{j_i}$.

In other words, we have

$$\begin{aligned} & \Pr_{T \sim \mathcal{P}(M)}[T \in \mathcal{S}_a^p] \\ &= \Pr \left[u_a \left(\bigcup_{i \in [p]} B_i^{j_i} \right) \geq \text{MMS}_a^{\text{1-out-of-}p}(M) \right] \\ &= \mathbb{E} \left[\Pr \left[u_a \left(\bigcup_{i \in [p]} B_i^{j_i} \right) \geq \text{MMS}_a^{\text{1-out-of-}p}(M) \right] \right], \quad (13) \end{aligned}$$

where in the second expression, the probability is taken over $\{(B_i^1, B_i^2) \sim \mathcal{X}(B_i), j_i \sim \{1, 2\}\}_{i \in [p]}$ and in the third expression, the expectation is taken over $\{(B_i^1, B_i^2) \sim \mathcal{X}(B_i)\}_{i \in [p]}$ and the probability over $\{j_i \sim \{1, 2\}\}_{i \in [p]}$.

Consider a fixed collection of $(B_1^1, B_1^2), \dots, (B_p^1, B_p^2)$. We shall lower bound the inner probability above. To this end, first note that the probability does not change when we swap B_i^1 and B_i^2 for some $i \in [p]$, or when we swap (B_i^1, B_i^2) and $(B_{i'}^1, B_{i'}^2)$ for some $i, i' \in [p]$. Thus, we may assume without loss of generality that the following holds:

- $u_a(B_i^1) \geq u_a(B_i^2)$ for all $i \in [p]$;
- $u_a(B_1^2) \leq \dots \leq u_a(B_p^2)$.

We claim that for all collections of $j_1, \dots, j_p \in \{1, 2\}$ such that $(j_1, \dots, j_{p-1}) \neq (2, \dots, 2)$, it must hold that $u_a \left(\bigcup_{i \in [p]} B_i^{j_i} \right) \geq \text{MMS}_a^{\text{1-out-of-}p}(M)$. To see this, let $\ell \in [p-1]$ be an index such that $j_\ell = 1$. We have

$$\begin{aligned} u_a \left(\bigcup_{i \in [p]} B_i^{j_i} \right) &\geq u_a(B_\ell^{j_\ell}) + u_a(B_p^{j_p}) \\ &\geq u_a(B_\ell^1) + u_a(B_p^2) \\ &\geq u_a(B_\ell^1) + u_a(B_\ell^2) \geq \text{MMS}_a^{\text{1-out-of-}p}(M), \end{aligned}$$

proving the claim. From this claim, it follows that

$$\begin{aligned} & \Pr_{j_1, \dots, j_p \sim \{1, 2\}} \left[u_a \left(\bigcup_{i \in [p]} B_i^{j_i} \right) \geq \text{MMS}_a^{\text{1-out-of-}p}(M) \right] \\ &\geq \Pr_{j_1, \dots, j_p \sim \{1, 2\}}[(j_1, \dots, j_{p-1}) \neq (2, \dots, 2)] = 1 - \frac{1}{2^{p-1}}. \end{aligned}$$

Plugging this back into (13), we get

$$\Pr_{T \sim \mathcal{P}(M)}[T \in \mathcal{S}_a^p] \geq 1 - \frac{1}{2^{p-1}}.$$

Finally, to see that the above inequality cannot be an equality, consider the case where $B_1^2 = \dots = B_p^2 = \emptyset$. In this case, which occurs with positive probability, it holds that $\Pr_{j_1, \dots, j_p \sim \{1, 2\}} \left[u_a \left(\bigcup_{i \in [p]} B_i^{j_i} \right) \geq \text{MMS}_a^{\text{1-out-of-}p}(M) \right] \geq 1 - \frac{1}{2^p} > 1 - \frac{1}{2^{p-1}}$. This completes the proof. \square

Lemma 5.3 together with the union bound already implies Corollary 5.2. To obtain the stronger Theorem 5.1, we use the following combinatorial lemma by Kleitman [1966] before applying the union bound.⁷ A collection \mathcal{S} of subsets of M is said to be *monotone* if, for all $S \subseteq S' \subseteq M$ such that $S \in \mathcal{S}$, we have $S' \in \mathcal{S}$ as well.

Lemma 5.4 ([Kleitman, 1966]). *Let $\mathcal{S}_1, \dots, \mathcal{S}_k \subseteq \mathcal{P}(M)$ be monotone collections of subsets of M . Then,*

$$\Pr_{S \sim \mathcal{P}(M)} \left[S \in \bigcap_{i \in [k]} \mathcal{S}_i \right] \geq \prod_{i \in [k]} \left(\Pr_{S \sim \mathcal{P}(M)}[S \in \mathcal{S}_i] \right).$$

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. For an agent $a \in N$, we write \mathcal{X}_a^p to denote the set of allocations that satisfy MMS^{1-out-of-p} for a . Fix an arbitrary instance with two groups N_1, N_2 , and consider the probability that a random allocation is MMS^{1-out-of-p}. We can write this probability as

$$\begin{aligned} & \Pr_{\mathcal{A}=(A_1, A_2) \sim \mathcal{X}(M)}[\forall a \in N, \mathcal{A} \in \mathcal{X}_a^p] \\ &= 1 - \Pr_{\mathcal{A}=(A_1, A_2) \sim \mathcal{X}(M)}[\exists a \in N, \mathcal{A} \notin \mathcal{X}_a^p] \\ &\geq 1 - \Pr_{\mathcal{A}=(A_1, A_2) \sim \mathcal{X}(M)}[\exists a \in N_1, \mathcal{A} \notin \mathcal{X}_a^p] \\ &\quad - \Pr_{\mathcal{A}=(A_1, A_2) \sim \mathcal{X}(M)}[\exists a \in N_2, \mathcal{A} \notin \mathcal{X}_a^p] \\ &= 1 - \Pr_{S \sim \mathcal{P}(M)}[\exists a \in N_1, S \notin \mathcal{S}_a^p] \\ &\quad - \Pr_{S \sim \mathcal{P}(M)}[\exists a \in N_2, S \notin \mathcal{S}_a^p] \\ &= \Pr_{S \sim \mathcal{P}(M)}[\forall a \in N_1, S \in \mathcal{S}_a^p] \\ &\quad + \Pr_{S \sim \mathcal{P}(M)}[\forall a \in N_2, S \in \mathcal{S}_a^p] - 1 \\ &= \Pr_{S \sim \mathcal{P}(M)} \left[S \in \bigcap_{a \in N_1} \mathcal{S}_a^p \right] + \Pr_{S \sim \mathcal{P}(M)} \left[S \in \bigcap_{a \in N_2} \mathcal{S}_a^p \right] - 1, \end{aligned}$$

⁷See Proposition 6.3.1 of Alon and Spencer [2000] for a statement more akin to the form we use below.

$n_1 \downarrow n_2 \rightarrow$	1	2	3	4	5
1	2				
2	3	3			
3	3	[3, 4]	[3, 4]		
4	3	[3, 4]	[3, 4]	4	
5	[3, 4]	[3, 4]	[3, 4]	4	4

Table 1: Bounds on $p_{\text{MMS}}(n_1, n_2)$ for $n_1, n_2 \leq 5$.

where the inequality follows from the union bound.

Notice that the collections \mathcal{S}_a^p are monotone. Applying Lemma 5.4 together with Lemma 5.3, we get

$$\begin{aligned} & \Pr_{\mathcal{A}=(A_1, A_2) \sim \mathcal{X}(M)} [\forall a \in N, \mathcal{A} \in \mathcal{X}_a^p] \\ & > \left(1 - \frac{1}{2^{p-1}}\right)^{n_1} + \left(1 - \frac{1}{2^{p-1}}\right)^{n_2} - 1. \end{aligned}$$

Thus, as long as the right-hand side is non-negative, an MMS_{1-out-of- p} allocation is guaranteed to exist. \square

Next, we turn to lower bounds. For two groups of equal size, we establish the following bound, which is already within a factor of 2 of the upper bound from Corollary 5.2.

Theorem 5.5. *For any positive integer n' , it holds that $p_{\text{MMS}}(n', n') \geq 2 + \left\lfloor \frac{\log n'}{2} \right\rfloor$.*

Proof. If $n' < 4$, the bound is trivial, so we assume that $n' \geq 4$. Let $p = 1 + \left\lfloor \frac{\log n'}{2} \right\rfloor$, $t = p - 1$, and $m = 2t + 1$. By Lemma 4.1 with $g = 2$, $t_1 = t_2 = t$, and $n_1 = n_2 = n'$, it suffices to show that $C(m, m - (p - 1), t) \leq n'$. To see that this is the case, we apply Observation 2.6 to get

$$\begin{aligned} C(m, m - (p - 1), t) & \leq \binom{m}{m - (p - 1)} \\ & = \binom{2p - 1}{p} \leq 2^{2p-2} \leq n', \end{aligned}$$

where the last inequality holds by our choice of p . \square

We summarize the upper and lower bounds for $n_1, n_2 \leq 5$ in Table 1. The bound for $n_1 = n_2 = 1$ follows from the well-known result in the individual setting (e.g., [Kurokawa *et al.*, 2018]). The remaining upper bounds are direct consequences of Theorem 5.1. As for the lower bounds, by Observation 2.2, it suffices to show that $p_{\text{MMS}}(2, 1) \geq 3$ and $p_{\text{MMS}}(4, 4) \geq 4$. The former follows from Theorem 1 of Suksompong [2018]. For the latter, observe that $C(5, 3, 2) \leq 4$ due to the $(5, 3, 2)$ -covering design $\{\{1, 2, 3\}, \{3, 4, 5\}, \{2, 4, 5\}, \{1, 4, 5\}\}$, and apply Lemma 4.1 with $(m, p, g, t_1, t_2, n_1, n_2) = (5, 3, 2, 2, 2, 4, 4)$.

6 Conclusion

In this paper, we have explored group fairness in indivisible item allocation using the popular notion of maximin share (MMS). We showed that ordinal relaxations of MMS, unlike their cardinal counterparts, are useful for providing fairness in this setting. In particular, we derived asymptotically

tight bounds on the ordinal MMS approximations that can be guaranteed—interestingly, these bounds differ depending on whether there is a group whose size is substantially larger than the sizes of all the remaining groups.

Besides obtaining tighter non-asymptotic bounds, an interesting direction is to establish strong guarantees with respect to both MMS and envy-freeness. In *individual* fair division, there always exists an allocation that is envy-free up to one item (EF1), and such an allocation inherently satisfies 1-out-of-($2n - 1$) MMS [Segal-Halevi and Suksompong, 2019, Sec. 2]. In *group* fair division, however, Manurangsi and Suksompong [2022] showed that for constant g , the least k such that the existence of an EF k allocation can be ensured is $k = \Theta(\sqrt{n})$; this cannot imply 1-out-of- ℓ MMS for any $\ell = o(\sqrt{n})$,⁸ so the implied MMS guarantee is significantly weaker than our logarithmic guarantees. An intriguing question is therefore whether optimal EF and MMS guarantees can be made together, or whether trade-offs are inevitable.

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⁸Indeed, for a given agent, the $\Theta(\sqrt{n})$ items that need to be removed from the remaining groups in order to attain envy-freeness could be highly valuable to the agent.

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