

# Asymptotic Analysis of Weighted Fair Division

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## Abstract

Several resource allocation settings involve agents with unequal entitlements represented by weights. We analyze weighted fair division from an asymptotic perspective: if  $m$  items are divided among  $n$  agents whose utilities are independently sampled from a probability distribution, when is it likely that a fair allocation exist? We show that if the ratio between the weights is bounded, a weighted envy-free allocation exists with high probability provided that  $m = \Omega(n \log n / \log \log n)$ , generalizing a prior unweighted result. For weighted proportionality, we establish a sharp threshold of  $m = n/(1 - \mu)$  for the transition from non-existence to existence, where  $\mu \in (0, 1)$  denotes the mean of the distribution. In addition, we prove that for two agents, a weighted envy-free (and weighted proportional) allocation is likely to exist if  $m = \omega(\sqrt{r})$ , where  $r$  denotes the ratio between the two weights.

## 1 Introduction

The fair allocation of scarce resources is a fundamental problem in economics and has received substantial attention recently in computer science [Brams and Taylor, 1996; Robertson and Webb, 1998; Moulin, 2019]. Research in fair division has sought to quantify fairness via precise mathematical definitions, among which two of the most important are envy-freeness and proportionality. An allocation is said to be *envy-free* if each agent values her own bundle at least as much as any other agent’s bundle [Foley, 1967; Varian, 1974]. It is called *proportional* if every agent values her bundle at least  $1/n$  of her value for the entire set of resources, where  $n$  denotes the number of agents [Steinhaus, 1948].

When allocating indivisible items, like houses, cars, or musical instruments, an envy-free or proportional allocation may not exist—this is the case, for example, when all items are valuable and the number of items  $m$  is smaller than  $n$ . In light of this, a line of work has focused on the question of *when* a fair allocation is likely to exist if the agents’ utilities for the items are drawn independently from a probability distribution.<sup>1</sup> For instance, Manurangsi and Suksompong

[2020, 2021] showed that an envy-free allocation exists with high probability<sup>2</sup> provided that  $m = \Omega(n \log n / \log \log n)$ . Not only is this bound tight, but it can also be achieved by a simple round-robin algorithm which lets the agents take turns picking their favorite item from the remaining items until the items run out. The same authors also proved that a proportional allocation is likely to exist as long as  $m \geq n$ .

The aforementioned results, like most of the work in fair division, assume that all agents have the same entitlement to the resource. In recent years, a number of researchers have explored a more general framework where agents may have varying entitlements represented by weights [Farhadi *et al.*, 2019; Aziz *et al.*, 2020, 2023; Chakraborty *et al.*, 2021a, 2024; Hoefer *et al.*, 2024; Springer *et al.*, 2024]. This broader framework allows us to model scenarios such as distributing resources among communities, where larger communities naturally deserve a larger portion of the resource, as well as dividing inheritance, where closer relatives typically receive a greater share of the bequest. Fortunately, both proportionality and envy-freeness can be extended to the weighted setting in an intuitive manner. As an example, if there are three agents with weights 1, 3, and 6, then weighted proportionality requires the first agent to receive at least  $1/10$  of her value for the entire set of items, while weighted envy-freeness stipulates that the second agent should not value the third agent’s bundle more than twice the value of her own bundle.

In addition to its inherent motivation, interest in weighted fair division stems from the fact that it exhibits several differences and brings a range of new challenges compared to its unweighted counterpart. For instance, in the absence of weights, it is trivial to show that the round-robin algorithm guarantees *envy-freeness up to one item (EF1)*, meaning that if an agent envies another agent, then the envy disappears upon removing some item in the latter agent’s bundle. By contrast, while it is possible to extend the round-robin algorithm to incorporate weights and prove that the resulting algorithm ensures *weighted EF1 (WEF1)*, doing so is far less straightforward [Chakraborty *et al.*, 2021a; Wu *et al.*, 2023]. In this paper, we shall analyze weighted fair division from an asymptotic perspective. Do the same relations between  $m$  and  $n$  continue to guarantee the existence of fair allocations

<sup>1</sup>We survey this and another related line of work in Section 1.2.

<sup>2</sup>That is, the probability that it exists converges to 1 as  $n \rightarrow \infty$ .

when agents may have different weights?

## 1.1 Our Results

We assume that each agent’s utility for each item is drawn independently from a non-atomic distribution  $\mathcal{D}$  over  $[0, 1]$ . For most results, we also assume that  $\mathcal{D}$  is *PDF-bounded*, that is, its probability distribution function is bounded between  $\alpha$  and  $\beta$  throughout  $[0, 1]$  for some constants  $\alpha, \beta > 0$ . All of our existence results come with polynomial-time algorithms.

In Sections 3 and 4, we consider the setting where the ratio between the weights is bounded. In Section 3, we show that if  $m = \Omega(n \log n / \log \log n)$ , then the weighted picking sequence algorithm of Chakraborty *et al.* [2021a] produces a weighted envy-free allocation with high probability; this generalizes the corresponding unweighted result by Manurangsi and Suksompong [2021]. Along the way, we provide an alternative proof that the allocation returned by this algorithm always satisfies WEF1; this proof is arguably simpler than existing proofs [Chakraborty *et al.*, 2021a; Wu *et al.*, 2023] and may therefore be of independent interest. In Section 4, we turn our attention to weighted proportionality and prove that, interestingly, the transition from non-existence to existence depends on the mean of the distribution  $\mathcal{D}$ : it occurs at  $m = n/(1 - \mu)$ , where  $\mu \in (0, 1)$  denotes the mean of  $\mathcal{D}$ . This implies that achieving (weighted) proportionality is more difficult than in the unweighted setting, for which the threshold is  $m = n$  [Manurangsi and Suksompong, 2021].

In Section 5, we focus on the case of two agents but allow the ratio  $r \geq 1$  between the agents’ weights to grow; in this case, weighted envy-freeness and weighted proportionality are equivalent. We show that if  $m = \omega(\sqrt{r})$ , then the probability that a weighted envy-free (and therefore weighted proportional) allocation exists approaches 1 as  $r \rightarrow \infty$ . We also establish the tightness of this bound.

## 1.2 Further Related Work

The asymptotic analysis of fair division was initiated by Dickerson *et al.* [2014], who showed that an envy-free allocation is likely to exist if  $m = \Omega(n \log n)$ , but unlikely to exist when  $m = n + o(n)$ . Manurangsi and Suksompong [2020] strengthened these results by exhibiting that existence is likely as long as  $m \geq 2n$  if  $m$  is divisible by  $n$ , but unlikely even when  $m = \Theta(n \log n / \log \log n)$  if  $m$  is not “almost divisible” by  $n$ . The gap in the non-divisible case was closed by Manurangsi and Suksompong [2021], who demonstrated the existence with high probability when  $m = \Omega(n \log n / \log \log n)$  via the round-robin algorithm. The same authors also proved that a proportional allocation is likely to exist when  $m \geq n$ , generalizing earlier results by Suksompong [2016] and Amanatidis *et al.* [2017]. Manurangsi and Suksompong [2017] studied envy-freeness when the agents are partitioned into groups, while Yokoyama and Igarashi [2025] performed an asymptotic analysis of “class envy-free” matchings. Beyond envy-freeness and proportionality, Kurokawa *et al.* [2016] and Farhadi *et al.* [2019] considered *maximin share fairness*—a weaker notion than proportionality—the latter authors also

allowing unequal entitlements. Bai and Gözl [2022] considered an extension where different agents may have different distributions, whereas Bai *et al.* [2022] investigated a “smoothed utility model” in which each agent has a base utility for each item and this utility is “boosted” with some probability. Benadè *et al.* [2024] examined a stochastic model where the agents’ utility functions can be non-additive. Manurangsi and Suksompong [2025] conducted an asymptotic analysis of fair division for chores (i.e., items that yield negative utilities).

While weighted fair allocation has previously been studied for *divisible* items [Segal-Halevi, 2019; Crew *et al.*, 2020; Cseh and Fleiner, 2020], following the broader trend in fair division, it has been intensively examined in the context of *indivisible* items over the last few years. Several authors have proposed and studied variants of envy-freeness, proportionality, and maximin share in the setting with entitlements [Farhadi *et al.*, 2019; Aziz *et al.*, 2020; Babaioff *et al.*, 2021, 2024; Chakraborty *et al.*, 2021a,b, 2024; Wu *et al.*, 2023; Springer *et al.*, 2024; Montanari *et al.*, 2025]. For an extensive overview of weighted fair division, we refer to the survey by Suksompong [2025].

## 2 Preliminaries

Let  $[k] := \{1, 2, \dots, k\}$  for any positive integer  $k$ . We want to allocate a set  $M = [m]$  of indivisible items among a set  $N = [n]$  of agents. Each agent  $i \in N$  has a utility  $u_i(g) \geq 0$  for each item  $g \in M$ , and a positive weight  $w_i \in \mathbb{R}_{>0}$ . Let  $w_{\max}$ ,  $w_{\min}$ , and  $W$  denote the maximum, minimum, and sum of the agents’ weights, respectively. We assume that utilities are *additive*, i.e.,  $u_i(S) = \sum_{g \in S} u_i(g)$  for any subset  $S \subseteq M$ . We refer to any subset of items as a *bundle*. An allocation  $A = (A_1, A_2, \dots, A_n)$  is a partition of  $M$  into  $n$  bundles, where  $A_i$  represents the bundle allocated to agent  $i \in N$ .

We consider a number of fairness notions. An allocation  $A$  is said to be *weighted envy-free (WEF)* if for every pair of agents  $i, j \in N$ , it holds that  $\frac{u_i(A_i)}{w_i} \geq \frac{u_i(A_j)}{w_j}$ . As a relaxation of WEF, an allocation  $A$  is *weighted envy-free up to one item (WEF1)* if for all  $i, j \in N$  with  $A_j \neq \emptyset$ , there exists an item  $g \in A_j$  such that  $\frac{u_i(A_i)}{w_i} \geq \frac{u_i(A_j \setminus \{g\})}{w_j}$ . An allocation  $A$  is said to be *weighted proportional (WPROP)* if  $u_i(A_i) \geq \frac{w_i}{W} \cdot u_i(M)$  for all  $i \in N$ . Note that every allocation that satisfies WEF also satisfies WPROP.

For each agent  $i \in N$  and each item  $g \in M$ , we assume that the utility  $u_i(g)$  is drawn independently from a probability distribution  $\mathcal{D}$  over  $[0, 1]$ . A distribution is *non-atomic* if it assigns zero probability to any single point. Let  $f_{\mathcal{D}}$  denote the probability density function (PDF) of  $\mathcal{D}$ . For  $\alpha, \beta > 0$ , we say that a distribution  $\mathcal{D}$  is  $(\alpha, \beta)$ -*PDF-bounded* if it is non-atomic and  $\alpha \leq f_{\mathcal{D}}(x) \leq \beta$  for all  $x \in [0, 1]$  [Manurangsi and Suksompong, 2021]. A distribution  $\mathcal{D}$  is *PDF-bounded* if it is  $(\alpha, \beta)$ -PDF-bounded for some constants  $\alpha, \beta > 0$ . A random event is said to occur *with high probability* if its probability approaches 1 as  $n \rightarrow \infty$ . We use  $\log$  to denote the natural logarithm (with base  $e$ ).

We now state two lemmas that will be useful for our purposes. In our analysis of the weighted picking sequence al-

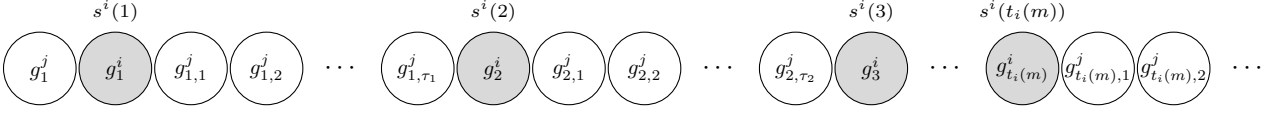


Figure 1: An example of the picks by agents  $i$  and  $j$  in the proof of Proposition 1

gorithm, we will apply Abel's summation formula, which allows us to rewrite a sum of products in a different form.

**Lemma 1** (Abel's summation formula). *For any sequences of real numbers  $(a_1, a_2, \dots, a_n)$  and  $(b_1, b_2, \dots, b_n)$ , we have*

$$\sum_{i=1}^n a_i b_i = a_n \sum_{i=1}^n b_i + \sum_{i=1}^{n-1} \left( (a_i - a_{i+1}) \sum_{i'=1}^i b_{i'} \right).$$

The following Chernoff bound is a standard concentration bound which will be used when we analyze the asymptotic existence of WPROP allocations and the case of two agents.

**Lemma 2** (Chernoff bound). *Let  $X_1, X_2, \dots, X_d$  be independent random variables such that  $X_i \in [0, 1]$  for all  $i \in [d]$ , and let  $X = \sum_{i=1}^d X_i$ . Then, for any  $\delta > 0$ , we have*

1.  $\Pr[X \geq (1 + \delta)\mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2}{2+\delta}\mathbb{E}[X]\right)$ , and
2.  $\Pr[X \leq (1 - \delta)\mathbb{E}[X]] \leq \exp\left(-\frac{\delta^2}{2}\mathbb{E}[X]\right)$ .

### 3 Weighted Envy-Freeness

In this section, we consider weighted envy-freeness. We provide an alternative proof that the weighted picking sequence algorithm of Chakraborty *et al.* [2021a] always outputs a WEF1 allocation. Our proof also lends itself to the asymptotic analysis of WEF, which we present as Theorem 1.

#### 3.1 Alternative Proof of WEF1

We first describe the weighted picking sequence algorithm (Algorithm 1). Define a *step* as an iteration of the while-loop (lines 3–8). Here,  $t_i$  represents the number of items that agent  $i$  has picked so far, and in each step, an agent  $i$  who minimizes  $t_i/w_i$  picks her favorite item from the remaining items.

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#### Algorithm 1 Weighted Picking Sequence Algorithm

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**Require:**  $N, M, (u_i(g))_{i \in N, g \in M}, (w_i)_{i \in N}$

- 1:  $A_i \leftarrow \emptyset$  and  $t_i \leftarrow 0$  for all  $i \in N$
  - 2:  $M_0 \leftarrow M$
  - 3: **while**  $M_0 \neq \emptyset$  **do**
  - 4:  $i^* \leftarrow \operatorname{argmin}_{i \in N} \frac{t_i}{w_i}$
  - 5:  $g^* \leftarrow \operatorname{argmax}_{g \in M_0} u_{i^*}(g)$
  - 6:  $A_{i^*} \leftarrow A_{i^*} \cup \{g^*\}$
  - 7:  $M_0 \leftarrow M_0 \setminus \{g^*\}$
  - 8:  $t_{i^*} \leftarrow t_{i^*} + 1$
  - 9: **return**  $(A_1, A_2, \dots, A_n)$
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Denote by  $t_i(s)$  the number of times agent  $i$  has picked an item up to (and including) step  $s$ . For each  $i \in N$  and each  $k \in [t_i(m)]$ , let  $s^i(k)$  be the step where agent  $i$  picks

her  $k$ -th item, and denote this item by  $g_k^i$ ; for convenience, let  $s^i(0) = 0$ . Finally, let  $A = (A_1, A_2, \dots, A_n)$  be the allocation returned by Algorithm 1.

Chakraborty *et al.* [2021a, Thm. 3.3] gave a rather long algebraic proof that Algorithm 1 always returns a WEF1 allocation. Wu *et al.* [2023, Lem. A.2] provided an alternative proof involving integrals. We present a relatively succinct algebraic proof of this statement via Abel's summation formula.

**Proposition 1** ([Chakraborty *et al.*, 2021a]). *The allocation returned by Algorithm 1 is WEF1.*

*Proof.* Consider any two distinct agents  $i, j \in M$ ; it suffices to show that  $i$  is WEF1 towards  $j$ . Observe that each of  $i$  and  $j$  picks her first item before the other agent picks her second item. Let  $\tau_1 + 1$  be the number of items that  $j$  picks before step  $s^i(2)$ , and denote these items by  $g_1^j, g_{1,1}^j, g_{1,2}^j, \dots, g_{1,\tau_1}^j$ . Note that  $g_1^j$  may be picked either before or after  $g_1^i$ .

For each  $k \in [t_i(m) - 1] \setminus \{1\}$ , let  $\tau_k$  be the number of items that agent  $j$  picks between steps  $s^i(k)$  and  $s^i(k + 1)$ , and for  $\ell \in [\tau_k]$ , let  $g_{k,\ell}^j$  be the  $\ell$ -th item picked by agent  $j$  within this interval. Let  $\tau_{t_i(m)}$  denote the number of items that  $j$  picks after step  $s^i(t_i(m))$ , and for  $\ell \in [\tau_{t_i(m)}]$ , let  $g_{t_i(m),\ell}^j$  be the  $\ell$ -th item picked by  $j$  within this interval. Note that  $\tau_k$  may be zero for some  $k$ . See Figure 1 for an illustration.

We claim that for every  $k \in [t_i(m)]$ ,

$$\frac{\sum_{k'=1}^k \tau_{k'}}{w_j} \leq \frac{k}{w_i}. \quad (1)$$

Indeed, if  $\tau_k \geq 1$ , then (1) holds since agent  $j$  is allowed to pick item  $g_{k,\tau_k}^j$ . If  $\tau_k = 0$ , for  $k = 1$  we have  $\frac{\sum_{k'=1}^1 \tau_{k'}}{w_j} = 0 \leq \frac{1}{w_i}$ , while for  $k \geq 2$  we have  $\frac{\sum_{k'=1}^{k-1} \tau_{k'}}{w_j} \leq \frac{k-1}{w_i}$  from the property for  $k - 1$ , which implies that  $\frac{\sum_{k'=1}^k \tau_{k'}}{w_j} \leq \frac{k-1}{w_i} + \frac{\tau_k}{w_j} < \frac{k-1}{w_i} + \frac{1}{w_i} = \frac{k}{w_i}$ . Thus, (1) holds for all  $k \in [t_i(m)]$ .

For each  $k \in [t_i(m)]$ , let  $\eta_k = 1 - \frac{w_i}{w_j} \sum_{\ell=1}^{\tau_k} \frac{u_i(g_{k,\ell}^j)}{u_i(g_k^i)}$  when  $\tau_k \geq 1$ , and  $\eta_k = 1$  when  $\tau_k = 0$ . Since  $u_i(g_k^i) \geq u_i(g_{k,\ell}^j)$  for all  $\ell \in [\tau_k]$ , we have  $\eta_k \geq 1 - \frac{w_i}{w_j} \cdot \tau_k$  for all  $k$ . Hence, (1) implies that  $\sum_{k'=1}^k \eta_{k'} \geq k - \frac{w_i}{w_j} \sum_{k'=1}^k \tau_{k'} \geq 0$  for all  $k \in [t_i(m)]$ . This means that

$$\begin{aligned} u_i(A_i) - \frac{w_i}{w_j} \cdot u_i(A_j \setminus \{g_1^j\}) \\ = \sum_{k=1}^{t_i(m)} u_i(g_k^i) - \frac{w_i}{w_j} \sum_{k=1}^{t_i(m)} \sum_{\ell=1}^{\tau_k} u_i(g_{k,\ell}^j) = \sum_{k=1}^{t_i(m)} u_i(g_k^i) \cdot \eta_k \end{aligned}$$

$$\begin{aligned}
&= u_i(g_{t_i(m)}^i) \sum_{k=1}^{t_i(m)} \eta_k \\
&\quad + \sum_{k=1}^{t_i(m)-1} \left[ (u_i(g_k^i) - u_i(g_{k+1}^i)) \sum_{k'=1}^k \eta_{k'} \right] \geq 0,
\end{aligned}$$

where the last equality follows from Lemma 1 and the inequality from the fact that  $u_i(g_k^i) \geq u_i(g_{k+1}^i)$  for every  $k$ . Hence, agent  $i$  is WEF1 towards agent  $j$ , as desired.  $\square$

### 3.2 Asymptotic Result

We now present the main result of this section.

**Theorem 1.** *Suppose that  $\mathcal{D}$  is PDF-bounded, and let  $C \geq 1$  be an arbitrary constant. For any weight vector  $(w_1, w_2, \dots, w_n)$  such that  $w_{\max}/w_{\min} \leq C$ , if  $m = \Omega(n \log n / \log \log n)$ , then Algorithm 1 produces a WEF allocation with high probability.*

Before we prove Theorem 1, we introduce some notation. For any  $c \in (0, 1]$ , denote by  $\mathcal{D}_{\leq c}$  the conditional distribution of  $\mathcal{D}$  on  $[0, c]$ . For any positive integer  $k$ , denote by  $\mathcal{D}^{\max(k)}$  the distribution of the maximum of  $k$  independent random variables generated by  $\mathcal{D}$ .

Fix any two distinct agents  $i, j \in N$ . We will use the same notation as in Section 3.1. Let  $X_1^j = u_i(g_1^j)$ . For each  $k \in [t_i(m)]$  and  $\ell \in [\tau_k]$ , let  $X_k = u_i(g_k^i)$  and  $X_{k,\ell}^j = u_i(g_{k,\ell}^j)$ . Lemma A.1 of Manurangsi and Suksompong [2021] yields the following lemma, which gives a convenient description of the distributions of  $X_k$  and  $X_{k,\ell}^j$ .

**Lemma 3** ([Manurangsi and Suksompong, 2021]). *Let  $X_0 = 1$ . Then,  $(X_k)_{k \in [t_i(m)]}$  and  $(X_{k,\ell}^j)_{k \in [t_i(m)], \ell \in [\tau_k]}$  can be generated according to the following process:*

- For each  $k \in [t_i(m)]$ , let  $X_k \sim \mathcal{D}_{\leq X_{k-1}}^{\max(m-s^i(k)+1)}$ ;
- For each  $\ell \in [\tau_k]$ , let  $X_{k,\ell}^j \sim \mathcal{D}_{\leq X_k}$ .

Intuitively, before agent  $i$  picks the item  $g_k^i$  corresponding to  $X_k$ , there are  $m - s^i(k) + 1$  items remaining, and  $i$ 's utility for each of them cannot exceed her utility for the item  $g_{k-1}^i$  corresponding to  $X_{k-1}$ . Moreover,  $i$ 's utility for each item  $g_{k,\ell}^j$  corresponding to  $X_{k,\ell}^j$  picked by  $j$  cannot exceed her utility for  $g_k^i$ , but otherwise  $j$ 's picks are independent of  $i$ 's valuations.

We now proceed to the proof of Theorem 1.

*Proof of Theorem 1.* Suppose that  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded and  $m \geq 10^6 \tilde{\beta} \cdot C \cdot n \log n / \log \log n$ , where  $\tilde{\beta} := \beta/\alpha \geq 1$ .

Recall that for any  $k \in [t_i(m)]$ , we defined

$$\eta_k = 1 - \frac{w_i}{w_j} \sum_{\ell=1}^{\tau_k} \frac{u_i(g_{k,\ell}^j)}{u_i(g_k^i)} = 1 - \frac{w_i}{w_j} \sum_{\ell=1}^{\tau_k} \frac{X_{k,\ell}^j}{X_k}.$$

For each  $k$ , let  $\theta_k = 1 - \frac{w_i}{w_j} \cdot \tau_k$ . By description of the algorithm,  $X_k \geq X_{k,\ell}^j$  holds for all  $k \in [t_i(m)]$  and  $\ell \in [\tau_k]$ , so  $\eta_k \geq \theta_k$  for all  $k$ . From (1), we obtain that for any  $k$ ,

$$\sum_{k'=1}^k \theta_{k'} = k - \frac{w_i}{w_j} \sum_{k'=1}^k \tau_{k'} \geq 0. \quad (2)$$

By Lemma 1 combined with (2) and the fact that  $X_k \geq X_{k+1}$  for every  $k$ , we have

$$\begin{aligned}
&\sum_{k=1}^{t_i(m)} X_k \cdot \theta_k \\
&= X_{t_i(m)} \sum_{k=1}^{t_i(m)} \theta_k + \sum_{k=1}^{t_i(m)-1} (X_k - X_{k+1}) \sum_{k'=1}^k \theta_{k'} \geq 0.
\end{aligned}$$

From this, we get

$$\begin{aligned}
u_i(A_i) - \frac{w_i}{w_j} \cdot u_i(A_j) &= \sum_{k=1}^{t_i(m)} X_k \cdot \eta_k - \frac{w_i}{w_j} \cdot X_1^j \\
&\geq \sum_{k=1}^{t_i(m)} X_k \cdot \eta_k - \frac{w_i}{w_j} \quad (\text{by } X_1^j \leq 1) \\
&\geq \sum_{k=1}^{t_i(m)} X_k (\eta_k - \theta_k) - \frac{w_i}{w_j}.
\end{aligned}$$

Our goal is to demonstrate that, with high probability,  $\sum_{k=1}^{t_i(m)} X_k (\eta_k - \theta_k) - \frac{w_i}{w_j} \geq 0$  holds. To this end, we show that with high probability, there exists a point in the picking sequence where agent  $i$  receives an item of sufficiently high value and has accumulated enough advantage over agent  $j$ . This is formalized in the following lemma.

**Lemma 4.** *Suppose that  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded, and  $m \geq 10^6 \tilde{\beta} \cdot C \cdot n \log n / \log \log n$ . For any distinct agents  $i, j \in N$ , there exists a positive integer  $T$  such that*

- (a)  $t_i(m) \geq T \geq 1$ ;
- (b)  $\Pr[X_T \geq \frac{1}{2}] = 1 - O(\frac{1}{m^3})$ ; and
- (c)  $\Pr\left[\sum_{k=1}^T \sum_{\ell=1}^{\tau_k} \left(1 - \frac{X_{k,\ell}^j}{X_k}\right) \geq 2\right] = 1 - O(\frac{1}{m^3})$ .

We defer the proof of Lemma 4 to Appendix A. Using this lemma, we will show that with probability at least  $1 - O(1/m^3)$ , agent  $i$  does not envy agent  $j$ . We have

$$\begin{aligned}
u_i(A_i) - \frac{w_i}{w_j} u_i(A_j) &\geq \sum_{k=1}^{t_i(m)} X_k (\eta_k - \theta_k) - \frac{w_i}{w_j} \\
&\geq \sum_{k=1}^T X_k (\eta_k - \theta_k) - \frac{w_i}{w_j} \\
&\quad (\text{by } X_k \geq 0 \text{ and } \eta_k \geq \theta_k \text{ for any } k, \text{ and (a) in Lemma 4}) \\
&\geq X_T \sum_{k=1}^T (\eta_k - \theta_k) - \frac{w_i}{w_j} \\
&\quad (\text{by } X_1 \geq X_2 \geq \dots \geq X_T) \\
&= X_T \frac{w_i}{w_j} \sum_{k=1}^T \sum_{\ell=1}^{\tau_k} \left(1 - \frac{X_{k,\ell}^j}{X_k}\right) - \frac{w_i}{w_j} \\
&\geq \frac{w_i}{2w_j} \sum_{k=1}^T \sum_{\ell=1}^{\tau_k} \left(1 - \frac{X_{k,\ell}^j}{X_k}\right) - \frac{w_i}{w_j} \\
&\quad (\text{with probability at least } 1 - O(1/m^3), \text{ by (b) in Lemma 4})
\end{aligned}$$

$$\geq \frac{w_i}{2w_j} \cdot 2 - \frac{w_i}{w_j}$$

(with probability at least  $1 - O(1/m^3)$ , by (c) in Lemma 4)

$$= 0.$$

Finally, by the union bound over all pairs of agents  $i, j$  and the fact that  $m \geq n$ , we have that the probability that the allocation  $A$  is not WEF is at most  $n^2 \cdot O(1/m^3) = O(1/n)$ . This completes the proof of the theorem.  $\square$

## 4 Weighted Proportionality

In this section, we turn our attention to weighted proportionality, and establish a sharp threshold for the transition from non-existence to existence with respect to this notion.

We start with the non-existence.

**Theorem 2.** *Suppose that  $\mathcal{D}$  is a non-atomic distribution with mean  $\mu \in (0, 1)$ , and let  $\varepsilon \in (0, 1/2)$  be a constant. For any  $n$ , there exists a weight vector  $(w_1, w_2, \dots, w_n)$  with  $\frac{w_{\max}}{w_{\min}} \leq \frac{2}{(1-\mu)^2\varepsilon^2}$  such that for any  $m \leq (1-\varepsilon) \cdot \frac{n}{1-\mu}$ , with high probability, no WPROP allocation exists.*

*Proof.* When  $n > m$ , at least one agent receives no item. Since  $\mathcal{D}$  is non-atomic, with probability 1, all agents have positive utilities for all items, in which case no WPROP allocation exists. We thus assume that  $n \leq m \leq (1-\varepsilon) \cdot \frac{n}{1-\mu}$ .

Let  $\delta = (1-\mu)\varepsilon < 1/2$ . For sufficiently large  $n$ , we have  $\lfloor \delta n \rfloor \geq 2$ . Assume that there are  $\lceil (1-\delta)n \rceil$  agents with weight  $\delta/\lceil (1-\delta)n \rceil$  each (called agents of the “first type”), and the remaining  $\lfloor \delta n \rfloor$  agents have weight  $(1-\delta)/\lfloor \delta n \rfloor$  each (called agents of the “second type”). Note that the sum of all weights is  $\delta + (1-\delta) = 1$ , and the ratio between the maximum and minimum weights is

$$\begin{aligned} \frac{w_{\max}}{w_{\min}} &= \frac{1-\delta}{\lfloor \delta n \rfloor} \cdot \frac{\lceil (1-\delta)n \rceil}{\delta} \\ &\leq \frac{1}{\delta n/2} \cdot \frac{n}{\delta} = \frac{2}{\delta^2} = \frac{2}{(1-\mu)^2\varepsilon^2}. \end{aligned}$$

Our choice of  $\delta$  implies that  $(1-\delta)(1-\mu\varepsilon) > 1-\delta-\mu\varepsilon = 1-\varepsilon$ . Rearranging this yields  $\frac{1-\mu\varepsilon}{1-\varepsilon} - \frac{1}{1-\delta} > 0$ . Let  $\gamma = \frac{1}{2} \left( \frac{1-\mu\varepsilon}{1-\varepsilon} - \frac{1}{1-\delta} \right)$ ; by the previous sentence,  $\gamma > 0$ .

Note that  $\mathbb{E}[u_i(M)] = m\mu$  for each  $i \in N$ . Applying Lemma 2, we get

$$\Pr[u_i(M) \leq (1-\gamma)m\mu] \leq \exp\left(-\frac{\gamma^2 m\mu}{2}\right).$$

By the union bound, the probability that there exists  $i \in N$  with  $u_i(M) \leq (1-\gamma)m\mu$  is at most  $n \cdot \exp(-\gamma^2 m\mu/2)$ . Since  $m \geq n$ , we have  $n \cdot \exp(-\gamma^2 m\mu/2) \leq n \cdot \exp(-\gamma^2 n\mu/2)$ , which approaches 0 as  $n \rightarrow \infty$ . Thus, with high probability,  $u_i(M) \geq (1-\gamma)m\mu$  holds for all  $i \in N$ . We assume for the remainder of the proof that this event holds.

For an allocation to satisfy WPROP, each agent of the first type must receive at least one item. Each agent of the second type, given her weight of  $\frac{1-\delta}{\lfloor \delta n \rfloor}$ , must receive a bundle that she values at least  $\frac{1-\delta}{\lfloor \delta n \rfloor} \cdot (1-\gamma)m\mu$ . Since  $u_i(g) \leq 1$  for all  $i \in N$

and  $g \in M$ , for each agent  $i$  of the second type, her bundle  $A_i$  must contain at least  $\frac{1-\delta}{\lfloor \delta n \rfloor} \cdot (1-\gamma)m\mu$  items. Therefore, for a WPROP allocation to exist, the total number of items must be at least  $\lceil (1-\delta)n \rceil + \lfloor \delta n \rfloor \cdot \frac{1-\delta}{\lfloor \delta n \rfloor} \cdot (1-\gamma)m\mu \geq (1-\delta)(n + (1-\gamma)m\mu)$ . Since  $m \leq \frac{1-\varepsilon}{1-\mu} \cdot n$ , this is at least

$$\begin{aligned} &(1-\delta) \left( \frac{1-\mu}{1-\varepsilon} + (1-\gamma)\mu \right) m \\ &\geq (1-\delta) \left( \frac{1-\mu}{1-\varepsilon} + \mu - \gamma \right) m \\ &= (1-\delta) \left( \gamma + \frac{1}{1-\delta} \right) m > m, \end{aligned}$$

a contradiction. We therefore conclude that, with high probability, no WPROP allocation exists.  $\square$

Next, we exhibit the asymptotic existence of weighted proportional allocations.

**Theorem 3.** *Suppose that  $\mathcal{D}$  is a PDF-bounded distribution with mean  $\mu \in (0, 1)$ , and let  $C \geq 1$  and  $\varepsilon \in (0, 1)$  be arbitrary constants. For any weight vector  $(w_1, w_2, \dots, w_n)$  with  $\frac{w_{\max}}{w_{\min}} \leq C$  and any  $m \geq (1+\varepsilon) \cdot \frac{n}{1-\mu}$ , a WPROP allocation exists with high probability. Moreover, such an allocation can be found in polynomial time.*

As WEF is stronger than WPROP, Theorem 1 already implies that a WPROP allocation exists with high probability when  $m = \Omega(n \log n / \log \log n)$ . Hence, to prove Theorem 3, we may restrict our attention to the case  $m = O(n \log n)$ . Let  $\mathcal{D}$  be  $(\alpha, \beta)$ -PDF-bounded, and let  $\tau = 1 - \frac{8(C+1) \log m}{\alpha n}$  and  $\delta = \left(1 + \frac{\varepsilon}{1+\varepsilon} \cdot \frac{1-\mu}{\mu}\right) \tau - 1$ . Note that  $\tau > 0$  and  $\delta > 0$  for sufficiently large  $n$ .

Since  $\mathbb{E}[u_i(M)] = m\mu$  for all  $i \in N$ , we can apply Lemma 2 to get that  $\Pr[u_i(M) \geq (1+\delta)m\mu] \leq \exp\left(-\frac{\delta^2 m\mu}{2+\delta}\right)$ . By the union bound and the assumption that  $m \geq (1+\varepsilon) \cdot \frac{n}{1-\mu} \geq n$ , with high probability, we have  $u_i(M) \leq (1+\delta)m\mu$  for all agents  $i \in N$ . We assume for the remainder of this discussion that this event holds.

Recall that  $W = \sum_{i \in N} w_i$ . For each  $i \in N$ , define  $s_i = \lceil (1+\delta) \frac{w_i}{W} \cdot \frac{\mu m}{\tau} \rceil$ . If we can construct an allocation  $A$  where every agent  $i$  receives at least  $s_i$  items that she values at least  $\tau$  each, then with high probability,  $u_i(A_i) \geq s_i \cdot \tau \geq \frac{w_i}{W} \cdot (1+\delta)m\mu \geq \frac{w_i}{W} \cdot u_i(M)$  for all  $i \in N$ , implying that the allocation  $A$  satisfies WPROP.

Let  $s = (s_1, s_2, \dots, s_n)$ . As  $m \geq (1+\varepsilon) \cdot \frac{n}{1-\mu}$ , we have

$$\begin{aligned} \sum_{i \in N} s_i &\leq \sum_{i \in N} (1+\delta) \frac{w_i}{W} \cdot \frac{\mu m}{\tau} + n \\ &= (1+\delta) \cdot \frac{\mu m}{\tau} + n \\ &\leq (1+\delta) \cdot \frac{\mu m}{\tau} + \frac{1-\mu}{1+\varepsilon} \cdot m \\ &= \left(1 + \frac{\varepsilon}{1+\varepsilon} \cdot \frac{1-\mu}{\mu}\right) \mu m + \frac{1-\mu}{1+\varepsilon} \cdot m = m. \end{aligned}$$

This calculation shows that the total number of required items does not exceed the number of available items.

To find a desired allocation, we use a matching-based approach, which extends an algorithm of Manurangsi and Suksompong [2020] to the weighted setting. Before describing the algorithm, we define a generalized notion of a matching that allows vertices on one side to be matched multiple times.

**Definition 1.** Let  $G = (L \cup R, E)$  be a bipartite graph with  $|L| = n$  and  $|R| = m$ . For a vector of positive integers  $\mathbf{s} = (s_1, s_2, \dots, s_n)$  with  $\sum_{i=1}^n s_i \leq m$ , an  $\mathbf{s}$ -matching in  $G$  is a set of edges  $F \subseteq E$  such that each vertex  $i \in L$  is incident to at most  $s_i$  edges in  $F$  and each vertex in  $R$  is incident to at most one edge in  $F$ . An  $\mathbf{s}$ -matching is called *left-saturating* if every vertex  $i \in L$  is incident to exactly  $s_i$  edges in the matching.

---

**Algorithm 2** Matching-Based Algorithm for WPROP

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**Require:**  $N, M, (u_i(g))_{i \in N, g \in M}, \mathbf{s} = (s_i)_{i \in N}$ , threshold  $\tau$

- 1: **for**  $i \in N$  **do**
- 2:    $M_{\geq \tau}(i) \leftarrow \{j \in M \mid u_i(j) \geq \tau\}$
- 3: Let  $G_{\geq \tau} = (N, M, E_{\geq \tau})$  be the bipartite graph where  $(i, j) \in E_{\geq \tau}$  if and only if  $j \in M_{\geq \tau}(i)$ .
- 4: **if**  $G_{\geq \tau}$  contains a left-saturating  $\mathbf{s}$ -matching **then**
- 5:   **return** any left-saturating  $\mathbf{s}$ -matching in  $G_{\geq \tau}$  (with the unmatched items allocated arbitrarily)
- 6: **else**
- 7:   **return** NULL

---

Our algorithm, described as Algorithm 2, first constructs a bipartite graph  $G_{\geq \tau}$  where an edge exists between an agent and an item if and only if the agent values the item at least the threshold  $\tau$ . The algorithm then determines whether a left-saturating  $\mathbf{s}$ -matching exists in this graph. If so, it simply assigns each matched item to the agent matched with it, and any unmatched item can be assigned arbitrarily. Note that determining whether a left-saturating  $\mathbf{s}$ -matching exists, and finding one in case it does, can be done in polynomial time by creating  $s_i$  copies of each agent  $i$  and computing a maximum matching in the resulting graph.

We now state a lemma that establishes the existence of a left-saturating  $\mathbf{s}$ -matching.

**Lemma 5.** Suppose that  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded, there exists a constant  $C \geq 1$  such that  $\frac{w_{\max}}{w_{\min}} \leq C$ , and  $m = O(n \log n)$ . Set  $\tau = 1 - \frac{8(C+1) \log m}{\alpha n}$  in Algorithm 2. Then, with high probability, Algorithm 2 outputs a left-saturating  $\mathbf{s}$ -matching.

From our earlier discussion, Lemma 5 implies that a WPROP allocation exists (and can be found in polynomial time) with high probability, thereby yielding Theorem 3.

To prove Lemma 5, we first recall basic results from matching theory. For a bipartite graph  $G = (L \cup R, E)$  and a subset  $Y \subseteq L$ , denote by  $N_G(Y)$  the set of vertices in  $R$  that are adjacent to at least one vertex in  $Y$ . A *matching* in a graph is a set of edges no two of which share a vertex. A matching is called *left-saturating* if every vertex in  $L$  is incident to exactly one edge in the matching. We now state Hall's marriage theorem, a classical result in matching theory.

**Lemma 6** (Hall's marriage theorem). Let  $G = (L \cup R, E)$  be a bipartite graph with  $|L| \leq |R|$ . If  $|N_G(Y)| \geq |Y|$  for all subsets  $Y \subseteq L$ , there exists a left-saturating matching in  $G$ .

To extend this result to our setting with  $\mathbf{s}$ -matchings, we construct a new bipartite graph  $G'$  from  $G = (L \cup R, E)$  by replacing each vertex  $i \in L$  with  $s_i$  copies, where each copy is connected to all neighbors of  $i$  in  $G$ . This immediately leads to the following proposition.

**Proposition 2.** Let  $G = (L \cup R, E)$  be a bipartite graph with  $n = |L| \leq |R| = m$ . For any vector of positive integers  $\mathbf{s} = (s_1, s_2, \dots, s_n)$  such that  $\sum_{i=1}^n s_i \leq m$ , if  $|N_G(Y)| \geq \sum_{i \in Y} s_i$  for all subsets  $Y \subseteq L$ , then there exists a left-saturating  $\mathbf{s}$ -matching in  $G$ .

Next, we recall the *Erdős-Rényi random bipartite graph model* [Erdős and Rényi, 1964]. For any  $p \in [0, 1]$ , let  $\mathcal{G}(|L|, |R|, p)$  denote the probability distribution over bipartite graphs with vertex sets  $L$  and  $R$  such that for each pair of vertices  $i \in L$  and  $j \in R$ , the edge  $(i, j)$  exists independently with probability  $p$ . Let  $s_{\max} = \max_{i \in N} s_i$  and  $s_{\min} = \min_{i \in N} s_i$ . The probability that no left-saturating  $\mathbf{s}$ -matching exists can be upper-bounded by the probability that there exists a subset violating the condition in Proposition 2. Based on this insight, we establish the following lemma, whose proof is deferred to Appendix B.

**Lemma 7.** Let  $B \geq 1$  be a constant, and let  $G$  be a bipartite graph sampled from  $\mathcal{G}(n, m, p)$  with  $n \leq m$  and  $\frac{8B \log m}{n} \leq p \leq 1$ . For any vector of positive integers  $\mathbf{s} = (s_1, s_2, \dots, s_n)$  such that  $\sum_{i=1}^n s_i \leq m$  and  $\frac{s_{\max}}{s_{\min}} \leq B$ , with high probability,  $G$  contains a left-saturating  $\mathbf{s}$ -matching.

We can now complete the proof of Lemma 5.

*Proof of Lemma 5.* To apply Lemma 7, we verify two conditions: firstly, that  $\frac{s_{\max}}{s_{\min}} \leq C + 1$ , and secondly, that the graph  $G_{\geq \tau}$  follows the Erdős-Rényi random bipartite graph model. Recall that  $s_i = \lceil (1 + \delta) \frac{w_i}{W} \cdot \frac{\mu m}{\tau} \rceil$ . Since  $s_{\min} \geq 1$  and  $\frac{w_{\max}}{w_{\min}} \leq C$ , we have

$$\begin{aligned} \frac{s_{\max}}{s_{\min}} &\leq \frac{(1 + \delta) \frac{w_{\max}}{W} \cdot \frac{\mu m}{\tau} + 1}{s_{\min}} \\ &\leq \frac{(1 + \delta) \frac{w_{\max}}{W} \cdot \frac{\mu m}{\tau}}{(1 + \delta) \frac{w_{\min}}{W} \cdot \frac{\mu m}{\tau}} + 1 \leq C + 1. \end{aligned}$$

Moreover, since  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded, the events  $\{u_i(j) \geq \tau\}$  occur independently for all pairs  $(i, j) \in N \times M$ , each with probability

$$\begin{aligned} \Pr[u_i(j) \geq \tau] &= \Pr\left[u_i(j) \geq 1 - \frac{8(C+1) \log m}{\alpha n}\right] \\ &\geq \alpha \cdot \frac{8(C+1) \log m}{\alpha n} = \frac{8(C+1) \log m}{n}. \end{aligned}$$

Therefore, by setting  $B = C + 1$  in Lemma 7, we conclude that the graph  $G_{\geq \tau}$  contains a left-saturating  $\mathbf{s}$ -matching with high probability, as desired.  $\square$

## 5 Two Agents

In this section, we focus on the case of  $n = 2$  agents—note that WEF and WPROP are equivalent in this case. We let  $r = w_2/w_1 \geq 1$  and consider the asymptotics as  $r \rightarrow \infty$ . Our main result is that if the number of items grows asymptotically faster than  $\sqrt{r}$ , a WEF allocation is likely to exist.

**Theorem 4.** *Suppose that there are two agents with weights  $(w_1, w_2)$  where  $r := w_2/w_1 \geq 1$ , and that  $\mathcal{D}$  is PDF-bounded. If  $m = \omega(\sqrt{r})$ , then a WEF allocation exists with probability approaching 1 as  $r \rightarrow \infty$ , and such an allocation can be found in polynomial time. The same holds for WPROP.*

*Proof.* Since WEF and WPROP are equivalent for  $n = 2$ , it suffices to focus on WPROP. Suppose that  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded with mean  $\mu \in (0, 1)$ .

We present an allocation algorithm. Set  $p = \frac{\alpha\mu}{2\sqrt{r+1}} \in (0, 1)$ , and define  $\tau$  as the threshold such that  $\Pr[X \leq \tau] = p$  for  $X \sim \mathcal{D}$ . The algorithm allocates each item whose value to agent 2 is less than  $\tau$  to agent 1, and the remaining items to agent 2. Clearly, this algorithm runs in polynomial time. Let  $A = (A_1, A_2)$  denote the resulting allocation. To show that  $A$  is WPROP with probability  $1 - o(1)$ , it suffices to prove that each of the two inequalities  $(r+1)u_1(A_1) < u_1(M)$  and  $(r+1)u_2(A_1) > u_2(M)$  holds with probability  $o(1)$ .

First, consider agent 1. For any sufficiently large  $r$ , we have  $p(r+1) \geq 3$ . When this holds, the union bound yields

$$\begin{aligned} & \Pr[(r+1)u_1(A_1) < u_1(M)] \\ & \leq \Pr\left[u_1(M) > \frac{3}{2}m\mu\right] + \Pr\left[u_1(A_1) < \frac{1}{2}mp\mu\right]. \end{aligned}$$

We may bound each term above via Lemma 2. In particular,  $u_1(M)$  is a sum of  $m$  independent random variables drawn from  $\mathcal{D}$ , so  $\mathbb{E}[u_1(M)] = m\mu$ ; Lemma 2 then implies that  $\Pr[u_1(M) > \frac{3}{2}m\mu] \leq \exp(-\frac{1}{10}m\mu) = o(1)$ . Meanwhile, since each item is independently allocated to agent 1 with probability  $p$ ,  $u_1(A_1)$  is a sum of independent random variables with  $\mathbb{E}[u_1(A_1)] = mp\mu$ , so Lemma 2 implies that  $\Pr[u_1(A_1) < \frac{1}{2}mp\mu] \leq \exp(-\frac{1}{8}mp\mu)$ , which is  $o(1)$  since  $m = \omega(\sqrt{r})$ . Combining these, we get  $\Pr[(r+1)u_1(A_1) < u_1(M)] = o(1)$ , as desired.

Next, consider agent 2. By the union bound, we have

$$\begin{aligned} & \Pr[(r+1)u_2(A_1) > u_2(M)] \\ & \leq \Pr\left[u_2(M) < \frac{1}{2}m\mu\right] + \Pr\left[u_2(A_1) > \frac{m\mu}{2(r+1)}\right]. \end{aligned}$$

For the first term, we can again use Lemma 2 in a similar way as above to conclude that  $\Pr[u_2(M) < \frac{1}{2}m\mu] \leq \exp(-\frac{1}{8}m\mu) = o(1)$ .

We now focus on the second term  $\Pr[u_2(A_1) > \frac{m\mu}{2(r+1)}]$ . For every item  $g \in M$ , define  $X_g$  to be a random variable such that  $X_g = 1$  if  $g \in A_1$ , and  $X_g = 0$  otherwise. Note that each  $X_g$  is an independent random variable with  $\mathbb{E}[X_g] = p$ . Since  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded, we have  $\alpha\tau \leq \Pr[X \leq \tau] = p$ , and so  $\tau \leq \frac{\mu}{2\sqrt{r+1}}$ . By definition of the allocation, every item  $g \in A_1$  satisfies  $u_2(g) \leq \tau$ . Therefore, we have  $u_2(A_1) = \sum_{g \in A_1} u_2(g) \leq \tau \cdot \sum_{g \in M} X_g \leq$

$\frac{\mu}{2\sqrt{r+1}} \sum_{g \in M} X_g$ . Observe that  $\frac{3}{2}\mathbb{E}\left[\sum_{g \in M} X_g\right] = \frac{3}{2}mp = \frac{3m\alpha\mu}{4\sqrt{r+1}} \leq \frac{m}{\sqrt{r+1}}$  since  $\alpha \leq 1$  and  $\mu < 1$ . Using this inequality and the bound on  $u_2(A_1)$ , we obtain

$$\begin{aligned} & \Pr\left[u_2(A_1) > \frac{m\mu}{2(r+1)}\right] \leq \Pr\left[\sum_{g \in M} X_g > \frac{m}{\sqrt{r+1}}\right] \\ & \leq \Pr\left[\sum_{g \in M} X_g > \frac{3}{2}\mathbb{E}\left[\sum_{g \in M} X_g\right]\right] \\ & \leq \exp\left(-\frac{1}{10}\mathbb{E}\left[\sum_{g \in M} X_g\right]\right) = \exp\left(-\frac{1}{10}mp\right), \end{aligned}$$

where we use Lemma 2 for the last inequality. This probability is  $o(1)$  since  $m = \omega(\sqrt{r})$ .

Hence, combining our bounds for both agents, we conclude that  $A$  is WPROP with probability  $1 - o(1)$ .  $\square$

Next, we establish the tightness of the bound in Theorem 4. In particular, the following theorem implies that the probability that no WEF allocation exists is at least constant if  $m = O(\sqrt{r})$ , and approaches 1 if  $m = o(\sqrt{r})$ .

**Theorem 5.** *Suppose that there are two agents with weights  $(w_1, w_2)$  where  $r := w_2/w_1 \geq 1$ , and that  $\mathcal{D}$  is PDF-bounded. With probability at least  $1 - O(m^2/r)$ , no WEF allocation exists. The same holds for WPROP.*

*Proof.* Since WEF and WPROP are equivalent for the case of  $n = 2$ , it suffices to focus on WEF. Suppose that  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded, and let  $Y = \min_{g \in M} u_2(g)$ . If  $1 - \beta m/r \leq 0$ , then  $1 - \beta m^2/r \leq 0 \leq \Pr[Y > m/r]$ . Else, we have

$$\begin{aligned} \Pr\left[Y > \frac{m}{r}\right] &= \prod_{g \in M} \Pr\left[u_2(g) > \frac{m}{r}\right] \\ &\geq \left(1 - \beta \cdot \frac{m}{r}\right)^m \geq 1 - \beta \cdot \frac{m^2}{r}, \end{aligned}$$

where we use the  $(\alpha, \beta)$ -PDF-boundedness of  $\mathcal{D}$  for the first inequality and Bernoulli's inequality for the second.

When  $Y > m/r$ , since any WEF allocation  $(A_1, A_2)$  must give at least one item to agent 1, we have  $r \cdot u_2(A_1) \geq r \cdot Y > m > u_2(A_2)$ . This means that no WEF allocation exists.  $\square$

## 6 Conclusion and Future Work

In conclusion, our work analyzes weighted fair division from an asymptotic perspective and establishes tight or asymptotically tight bounds on the existence of weighted envy-free and weighted proportional allocations. Notably, a larger number of items is required for weighted proportionality than in the unweighted setting, and this number depends on the mean of the distribution from which the utilities are drawn. We also investigated the relationship between the number of items and the weight ratio in the case of two agents.

Since we have considered settings where the weight ratio is bounded and where the number of agents is bounded separately, a natural next step is to obtain a more complete understanding when both parameters are allowed to grow. For

example, if all but one agents have the same (fixed) weight while the last agent’s weight grows, what is the required number of items in terms of the number of agents and the last agent’s weight? Another interesting direction is to extend our results to a more general model where the items are allocated among *groups* of agents rather than individual agents. While asymptotic fair division for groups was previously explored [Manurangsi and Suksompong, 2017], the study thus far has focused on groups with equal entitlements. Permitting different entitlements can help us model applications where the groups have varying importance, e.g., depending on their sizes.

## Acknowledgments

This work was partially supported by JST ERATO under grant number JPMJER2301, by the Singapore Ministry of Education under grant number MOE-T2EP20221-0001, and by an NUS Start-up Grant. We thank the anonymous reviewers for their valuable feedback.

## References

- Georgios Amanatidis, Evangelos Markakis, Afshin Nikzad, and Amin Saberi. Approximation algorithms for computing maximin share allocations. *ACM Transactions on Algorithms*, 13(4):52:1–52:28, 2017.
- Haris Aziz, Hervé Moulin, and Fedor Sandomirskiy. A polynomial-time algorithm for computing a Pareto optimal and almost proportional allocation. *Operations Research Letters*, 48(5):573–578, 2020.
- Haris Aziz, Aditya Ganguly, and Evi Micha. Best of both worlds fairness under entitlements. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, pages 941–948, 2023.
- Moshe Babaioff, Noam Nisan, and Inbal Talgam-Cohen. Competitive equilibrium with indivisible goods and generic budgets. *Mathematics of Operations Research*, 46(1):382–403, 2021.
- Moshe Babaioff, Tomer Ezra, and Uriel Feige. Fair-share allocations for agents with arbitrary entitlements. *Mathematics of Operations Research*, 49(4):2180–2211, 2024.
- Yushi Bai and Paul Gözl. Envy-free and Pareto-optimal allocations for agents with asymmetric random valuations. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI)*, pages 53–59, 2022.
- Yushi Bai, Uriel Feige, Paul Gözl, and Ariel D. Procaccia. Fair allocations for smoothed utilities. In *Proceedings of the 23rd ACM Conference on Economics and Computation (EC)*, pages 436–465, 2022.
- Gerdus Benadè, Daniel Halpern, Alexandros Psomas, and Paritosh Verma. On the existence of envy-free allocations beyond additive valuations. In *Proceedings of the 25th ACM Conference on Economics and Computation (EC)*, page 1287, 2024.
- Steven J. Brams and Alan D. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.
- Mithun Chakraborty, Ayumi Igarashi, Warut Suksompong, and Yair Zick. Weighted envy-freeness in indivisible item allocation. *ACM Transactions on Economics and Computation*, 9(3):18:1–18:39, 2021.
- Mithun Chakraborty, Ulrike Schmidt-Kraepelin, and Warut Suksompong. Picking sequences and monotonicity in weighted fair division. *Artificial Intelligence*, 301:103578, 2021.
- Mithun Chakraborty, Erel Segal-Halevi, and Warut Suksompong. Weighted fairness notions for indivisible items revisited. *ACM Transactions on Economics and Computation*, 12(3):9:1–9:45, 2024.
- Logan Crew, Bhargav Narayanan, and Sophie Spirkl. Disproportionate division. *Bulletin of the London Mathematical Society*, 52(5):885–890, 2020.
- Ágnes Cseh and Tamás Fleiner. The complexity of cake cutting with unequal shares. *ACM Transactions on Algorithms*, 16(3):29:1–29:21, 2020.
- John P. Dickerson, Jonathan Goldman, Jeremy Karp, Ariel D. Procaccia, and Tuomas Sandholm. The computational rise and fall of fairness. In *Proceedings of the 28th AAAI Conference on Artificial Intelligence (AAAI)*, pages 1405–1411, 2014.
- Paul Erdős and Alfréd Rényi. On random matrices. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 8:455–461, 1964.
- Alireza Farhadi, Mohammad Ghodsi, MohammadTaghi Hajiaghayi, Sébastien Lahaie, David Pennock, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. Fair allocation of indivisible goods to asymmetric agents. *Journal of Artificial Intelligence Research*, 64:1–20, 2019.
- Duncan K. Foley. Resource allocation and the public sector. *Yale Economics Essays*, 7(1):45–98, 1967.
- Martin Hoefer, Marco Schmalhofer, and Giovanna Varricchio. Best of both worlds: Agents with entitlements. *Journal of Artificial Intelligence Research*, 80:559–591, 2024.
- David Kurokawa, Ariel D. Procaccia, and Junxing Wang. When can the maximin share guarantee be guaranteed? In *Proceedings of the 30th AAAI Conference on Artificial Intelligence (AAAI)*, pages 523–529, 2016.
- Pasin Manurangsi and Warut Suksompong. Asymptotic existence of fair divisions for groups. *Mathematical Social Sciences*, 89:100–108, 2017.
- Pasin Manurangsi and Warut Suksompong. When do envy-free allocations exist? *SIAM Journal on Discrete Mathematics*, 34(3):1505–1521, 2020.
- Pasin Manurangsi and Warut Suksompong. Closing gaps in asymptotic fair division. *SIAM Journal on Discrete Mathematics*, 35(2):668–706, 2021.
- Pasin Manurangsi and Warut Suksompong. Asymptotic fair division: Chores are easier than goods. In *Proceedings of the 34th International Joint Conference on Artificial Intelligence (IJCAI)*, 2025. Forthcoming.



- Luisa Montanari, Ulrike Schmidt-Kraepelin, Warut Suksompong, and Nicholas Teh. Weighted envy-freeness for sub-modular valuations. *Social Choice and Welfare*, 2025. Forthcoming.
- Hervé Moulin. Fair division in the internet age. *Annual Review of Economics*, 11:407–441, 2019.
- Jack Robertson and William Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. Peters/CRC Press, 1998.
- Erel Segal-Halevi. Cake-cutting with different entitlements: How many cuts are needed? *Journal of Mathematical Analysis and Applications*, 480(1):123382, 2019.
- Max Springer, MohammadTaghi Hajiaghayi, and Hadi Yami. Almost envy-free allocations of indivisible goods or chores with entitlements. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI)*, pages 9901–9908, 2024.
- Hugo Steinhaus. The problem of fair division. *Econometrica*, 16(1):101–104, 1948.
- Warut Suksompong. Asymptotic existence of proportionally fair allocations. *Mathematical Social Sciences*, 81:62–65, 2016.
- Warut Suksompong. Weighted fair division of indivisible items: a review. *Information Processing Letters*, 187:106519, 2025.
- Hal R. Varian. Equity, envy, and efficiency. *Journal of Economic Theory*, 9:63–91, 1974.
- Xiaowei Wu, Cong Zhang, and Shengwei Zhou. Weighted EF1 allocations for indivisible chores. In *Proceedings of the 24th ACM Conference on Economics and Computation (EC)*, page 1155, 2023. Extended version available at arXiv:2301.08090v1.
- Tomohiko Yokoyama and Ayumi Igarashi. Asymptotic existence of class envy-free matchings. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2025. Forthcoming.

## A Proof of Lemma 4

### A.1 Auxiliary Lemmas

Before proving Lemma 4, we establish a key property of the weighted picking sequence algorithm. Specifically, the following lemma shows that at any step  $s$ , the number of picks  $t_i(s)$  made by each agent  $i$  is approximately proportional to her weight  $w_i$ . Recall that  $W = \sum_{i \in N} w_i$ .

**Lemma 8.** *In Algorithm 1, for every step  $s$  and all  $i, j \in N$ , we have*

$$\left| \frac{t_i(s)}{w_i} - \frac{t_j(s)}{w_j} \right| \leq \frac{1}{\min(w_i, w_j)}. \quad (3)$$

Moreover, for every step  $s$  and all  $i \in N$ , we have

$$\left| t_i(s) - \frac{w_i}{W} \cdot s \right| \leq \frac{w_i}{w_{\min}}. \quad (4)$$

*Proof.* We first prove (3) by induction on  $s$ . At  $s = 0$ , we have  $t_i(0) = 0$  for all  $i \in N$ , so  $\left| \frac{t_i(0)}{w_i} - \frac{t_j(0)}{w_j} \right| = 0$  for all  $i, j \in N$ . Assume now that (3) holds for some step  $s \geq 0$ , and let  $i^* = \operatorname{argmin}_{i' \in N} \frac{t_{i'}(s)}{w_{i'}}$ . By description of the algorithm, at step  $s + 1$ , agent  $i^*$  is selected. Thus, we have  $t_{i^*}(s + 1) = t_{i^*}(s) + 1$  and  $t_{i'}(s + 1) = t_{i'}(s)$  for all  $i' \in N \setminus \{i^*\}$ . This implies that for any pair  $i, j \in N \setminus \{i^*\}$ , (3) holds at step  $s + 1$ . Consider  $i^*$  and any  $i' \in N \setminus \{i^*\}$ . If  $\frac{t_{i^*}(s+1)}{w_{i^*}} \geq \frac{t_{i'}(s+1)}{w_{i'}} = \frac{t_{i'}(s)}{w_{i'}}$ , then

$$\left| \frac{t_{i^*}(s+1)}{w_{i^*}} - \frac{t_{i'}(s+1)}{w_{i'}} \right| = \frac{t_{i^*}(s)}{w_{i^*}} + \frac{1}{w_{i^*}} - \frac{t_{i'}(s)}{w_{i'}} \leq \frac{1}{w_{i^*}} \leq \frac{1}{\min(w_{i^*}, w_{i'})},$$

where the first inequality follows from the definition of  $i^*$ . On the other hand, if  $\frac{t_{i^*}(s+1)}{w_{i^*}} < \frac{t_{i'}(s+1)}{w_{i'}}$ , then

$$\left| \frac{t_{i^*}(s+1)}{w_{i^*}} - \frac{t_{i'}(s+1)}{w_{i'}} \right| = \frac{t_{i'}(s)}{w_{i'}} - \frac{t_{i^*}(s)}{w_{i^*}} - \frac{1}{w_{i^*}} \leq \frac{1}{\min(w_{i^*}, w_{i'})} - \frac{1}{w_{i^*}} \leq \frac{1}{\min(w_{i^*}, w_{i'})},$$

where the first inequality follows from the inductive hypothesis. Thus, we have established (3).

Next, we prove (4). From (3), we have

$$\min_{i' \in N} \frac{t_{i'}(s)}{w_{i'}} \leq \frac{t_i(s)}{w_i} \leq \min_{i' \in N} \frac{t_{i'}(s)}{w_{i'}} + \frac{1}{w_{\min}} \quad (5)$$

for all  $i \in N$ . This implies that  $w_i \cdot \min_{i' \in N} \frac{t_{i'}(s)}{w_{i'}} \leq t_i(s) \leq w_i \cdot \left( \min_{i' \in N} \frac{t_{i'}(s)}{w_{i'}} + \frac{1}{w_{\min}} \right)$  for all  $i \in N$ . Summing over all  $i \in N$  and using the fact that  $\sum_{i \in N} t_i(s) = s$ , we get  $W \cdot \min_{i' \in N} \frac{t_{i'}(s)}{w_{i'}} \leq s \leq W \cdot \left( \min_{i' \in N} \frac{t_{i'}(s)}{w_{i'}} + \frac{1}{w_{\min}} \right)$ . It follows that

$$\frac{s}{W} - \frac{1}{w_{\min}} \leq \min_{i' \in N} \frac{t_{i'}(s)}{w_{i'}} \leq \frac{s}{W}.$$

Combining this with (5), we arrive at (4), as desired.  $\square$

We now state useful lemmas from the work of Manurangsi and Suksompong [2021]. These lemmas are presented as Lemma 2.4, Proposition 2.2, and Lemma 2.6 in their work, respectively.

**Lemma 9** ([Manurangsi and Suksompong, 2021]). *Let  $r, c, d$  be any positive integers such that  $r \geq cd$ , and let  $X_1, X_2, \dots, X_r$  be independent random variables in  $[0, 1]$  that are  $(\alpha, \beta)$ -PDF-bounded. Then,*

$$\Pr[X_1 + X_2 + \dots + X_r \geq r - c] \leq 2^r \left( \frac{\beta}{d} \right)^{r - cd}.$$

**Lemma 10** ([Manurangsi and Suksompong, 2021]). *For any  $(\alpha, \beta)$ -PDF-bounded distribution  $\mathcal{D}$  and any  $c \in (0, 1]$ , suppose that we draw  $X$  from  $\mathcal{D}_{\leq c}$ . Then,  $X/c$  is generated from an  $(\alpha/\beta, \beta/\alpha)$ -PDF-bounded distribution.*

**Lemma 11** ([Manurangsi and Suksompong, 2021]). *Let  $T$  be a positive integer, let  $s_1, s_2, \dots, s_T$  be any positive integers, and let  $s_{\min} = \min(s_1, s_2, \dots, s_T)$ . Consider the random variable  $X_0 = 1$ , and  $X_1, X_2, \dots, X_T$  such that  $X_{k+1} \sim \mathcal{D}_{\leq X_k}^{\max(s_{k+1})}$  for each  $k \in \{0, 1, \dots, T-1\}$ . If  $\mathcal{D}$  is  $(\alpha, \beta)$ -PDF-bounded, then for any  $p \in (0, 1)$ , we have*

$$\Pr \left[ X_T \geq 1 - \frac{\beta}{\alpha} \cdot \frac{T \log(T/p)}{s_{\min}} \right] \geq 1 - p.$$

## A.2 Proof of (a) in Lemma 4

Let  $T = 100 \left\lceil \tilde{\beta} \cdot \frac{w_i}{w_j} \cdot \frac{\log m}{\log \log m} \right\rceil \geq 100$ . We will show that  $\frac{t_i(m)}{2} \geq T$ . By Lemma 8, we have

$$\frac{t_i(m)}{2} - T \geq \frac{w_i}{2W} \cdot m - \frac{w_i}{2w_{\min}} - T = w_i \left( \frac{m}{2W} - \frac{1}{2w_{\min}} - \frac{T}{w_i} \right) \geq w_i \left( \frac{m}{2W} - \frac{1}{w_{\min}} - \frac{T}{w_i} \right).$$

We will prove that  $\frac{m}{2W} - \frac{1}{w_{\min}} - \frac{T}{w_i} \geq 0$ . The inequality  $C \geq \frac{w_{\max}}{w_{\min}} \geq \frac{W}{w_{\min} \cdot n}$  implies that  $\frac{m}{4W} \geq \frac{m}{4C \cdot w_{\min} \cdot n}$ . Since the function  $\frac{x}{\log x}$  is non-decreasing for  $x \geq e$  (as its derivative is  $\frac{\log x - 1}{\log^2 x}$ ), and we have  $\frac{m}{n} \geq 10^6 \tilde{\beta} \cdot C \cdot \frac{\log n}{\log \log n}$ , it follows that  $\frac{m/n}{\log(m/n)} \geq 4 \cdot 10^3 \tilde{\beta} \cdot C$  holds for sufficiently large  $n$ . Therefore,

$$\frac{m}{4W} \geq \frac{m}{4C \cdot w_{\min} \cdot n} \geq \frac{10^3 \tilde{\beta}}{w_{\min}} \log \left( \frac{m}{n} \right).$$

This implies that

$$\begin{aligned} \frac{m}{2W} - \frac{1}{w_{\min}} - \frac{T}{w_i} &= \frac{m}{4W} + \frac{m}{4W} - \frac{1}{w_{\min}} - \frac{T}{w_i} \\ &\geq \frac{10^6 \tilde{\beta}}{4w_{\min}} \cdot \frac{\log n}{\log \log n} + \frac{10^3 \tilde{\beta}}{w_{\min}} \log \left( \frac{m}{n} \right) - \frac{1}{w_{\min}} - \frac{T}{w_i} \\ &\geq \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log n}{\log \log n} + \frac{10^3 \tilde{\beta}}{w_{\min}} \log \left( \frac{m}{n} \right) - \frac{T}{w_i} \quad (\text{by } \frac{10^6 \tilde{\beta}}{4w_{\min}} \cdot \frac{\log n}{\log \log n} - \frac{1}{w_{\min}} \geq \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log n}{\log \log n}) \\ &\geq \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log n}{\log \log m} + \frac{10^3 \tilde{\beta}}{w_{\min}} \log \left( \frac{m}{n} \right) - \frac{T}{w_i} \quad (\text{by } m \geq n) \\ &\geq \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log n}{\log \log m} + \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log(m/n)}{\log \log m} - \frac{T}{w_i} \quad (\text{by } \log \log m \geq \log \log n \geq 1) \\ &= \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log m}{\log \log m} - \frac{T}{w_i} \\ &\geq \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log m}{\log \log m} - \frac{100 \tilde{\beta}}{w_j} \cdot \frac{\log m}{\log \log m} - \frac{100}{w_i} \quad (\text{by } T \leq 100 \tilde{\beta} \cdot \frac{w_i}{w_j} \cdot \frac{\log m}{\log \log m} + 100) \\ &\geq \frac{10^3 \tilde{\beta}}{w_{\min}} \cdot \frac{\log m}{\log \log m} - \frac{100 \tilde{\beta}}{w_{\min}} \cdot \frac{\log m}{\log \log m} - \frac{100}{w_{\min}} \quad (\text{by } w_i, w_j \geq w_{\min}) \\ &\geq 0. \end{aligned} \tag{6}$$

Thus, we have  $\frac{t_i(m)}{2} \geq T$ . This establishes that  $T$  and  $s^i(T)$  are well-defined, and  $t_i(m) \geq T \geq 1$ .

## A.3 Proof of (b) in Lemma 4

We show that with probability at least  $1 - O(1/m^3)$ , agent  $i$  receives an item of value at least  $1/2$  in her  $T$ -th pick. By Lemma 3,  $X_1, X_2, \dots, X_T$  are sampled according to  $X_k \sim \mathcal{D}_{\leq X_{k-1}}^{\max(m-s^i(k)+1)}$  for each  $k \in [T]$ .

First, from (6) and Lemma 8, we have

$$\frac{m}{2} - s^i(T) \geq \frac{m}{2} - W \left( \frac{T}{w_i} + \frac{1}{w_{\min}} \right) = W \left( \frac{m}{2W} - \frac{1}{w_{\min}} - \frac{T}{w_i} \right) \geq 0.$$

Thus, we can observe that at any step before the  $T$ -th pick of agent  $i$ , at least half of the items remain available. Specifically, for any  $k \in [T]$ , it holds that  $m - s^i(k) + 1 \geq m - s^i(T) + 1 \geq m - m/2 + 1 \geq m/2$ .

Next, since  $T = 100 \left\lceil \tilde{\beta} \cdot \frac{w_i}{w_j} \cdot \frac{\log m}{\log \log m} \right\rceil$  and  $\frac{w_i}{w_j} \leq C$  where  $C \geq 1$ , we obtain  $T \leq 200 \tilde{\beta} \cdot C \cdot \frac{\log m}{\log \log m}$ . This implies that  $\frac{\beta}{\alpha} \cdot \frac{T \log(Tm^3)}{m/2} \leq \frac{1}{2}$  for sufficiently large  $m$ . Applying Lemma 11 with  $p = 1/m^3$ , we obtain

$$\Pr \left[ X_T \geq \frac{1}{2} \right] \geq \Pr \left[ X_T \geq 1 - \frac{\beta}{\alpha} \cdot \frac{T \log(Tm^3)}{m/2} \right] \geq \Pr \left[ X_T \geq 1 - \frac{\beta}{\alpha} \cdot \frac{T \log(T/p)}{m - s^i(T) + 1} \right] \geq 1 - \frac{1}{m^3},$$

as desired.

#### A.4 Proof of (c) in Lemma 4

Our goal is to show that

$$\Pr \left[ \sum_{k=1}^T \sum_{\ell=1}^{\tau_k} \left( 1 - \frac{X_{k,\ell}^j}{X_k} \right) \geq 2 \right] = 1 - O \left( \frac{1}{m^3} \right).$$

Since agent  $i$  picks item  $g_{T+1}^i$ , by description of Algorithm 1, we have

$$\frac{T}{w_i} \leq \frac{\sum_{k=1}^T \tau_k + 1}{w_j}.$$

It follows that

$$\sum_{k=1}^T \tau_k \geq \frac{w_j}{w_i} \cdot T - 1 \geq \frac{100\tilde{\beta} \log m}{\log \log m} - 1 \geq \frac{80\tilde{\beta} \log m}{\log \log m}.$$

In particular,

$$\frac{8\tilde{\beta}}{\sum_{k=1}^T \tau_k} \leq \frac{\log \log m}{10 \log m} \leq 1.$$

When conditioning on  $X_1 = x_1, \dots, X_T = x_T$ , the random variables  $\frac{X_{1,1}^j}{X_1}, \frac{X_{1,2}^j}{X_1}, \dots, \frac{X_{1,\tau_1}^j}{X_1}, \frac{X_{2,1}^j}{X_2}, \frac{X_{2,2}^j}{X_2}, \dots, \frac{X_{2,\tau_2}^j}{X_2}, \dots, \frac{X_{T,1}^j}{X_T}, \frac{X_{T,2}^j}{X_T}, \dots, \frac{X_{T,\tau_T}^j}{X_T}$  are all independent. By Lemma 10, these random variables are generated from an  $(\alpha/\beta, \beta/\alpha)$ -PDF-bounded distribution. Therefore, applying Lemma 9 with  $r = \sum_{k=1}^T \tau_k$ ,  $c = 2$ , and  $d = \left\lfloor \frac{\sum_{k=1}^T \tau_k}{4} \right\rfloor \geq \frac{\sum_{k=1}^T \tau_k}{8}$ , we get

$$\begin{aligned} & \Pr \left[ \sum_{k=1}^T \sum_{\ell=1}^{\tau_k} \frac{X_{k,\ell}^j}{X_k} \geq \sum_{k=1}^T \tau_k - 2 \mid X_1 = x_1, X_2 = x_2, \dots, X_T = x_T \right] \\ & \leq 2^{\sum_{k=1}^T \tau_k} \cdot \left( \frac{\tilde{\beta}}{\left\lfloor \frac{\sum_{k=1}^T \tau_k}{4} \right\rfloor} \right)^{\sum_{k=1}^T \tau_k - 2 \cdot \left\lfloor \frac{\sum_{k=1}^T \tau_k}{4} \right\rfloor} \\ & \leq 2^{\sum_{k=1}^T \tau_k} \cdot \left( \frac{8\tilde{\beta}}{\sum_{k=1}^T \tau_k} \right)^{\sum_{k=1}^T \tau_k - 2 \cdot \left\lfloor \frac{\sum_{k=1}^T \tau_k}{4} \right\rfloor} \\ & \leq 2^{\sum_{k=1}^T \tau_k} \cdot \left( \frac{8\tilde{\beta}}{\sum_{k=1}^T \tau_k} \right)^{\sum_{k=1}^T \tau_k - 2 \cdot \frac{\sum_{k=1}^T \tau_k}{4}} \quad (\text{by } \frac{8\tilde{\beta}}{\sum_{k=1}^T \tau_k} \leq 1) \\ & = \left( \frac{32\tilde{\beta}}{\sum_{k=1}^T \tau_k} \right)^{\frac{1}{2} \sum_{k=1}^T \tau_k} \\ & \leq \left( \frac{32\tilde{\beta}}{\frac{80\tilde{\beta} \log m}{\log \log m}} \right)^{\frac{1}{2} \cdot \frac{80\tilde{\beta} \log m}{\log \log m}} \\ & = \left( \frac{32 \log \log m}{80 \log m} \right)^{\frac{1}{2} \cdot \frac{80\tilde{\beta} \log m}{\log \log m}} \\ & \leq \left( \frac{\log \log m}{\log m} \right)^{\frac{40\tilde{\beta} \log m}{\log \log m}} \\ & \leq \left( \frac{1}{\sqrt{\log m}} \right)^{\frac{6 \log m}{\log \log m}} \quad (\text{by } \log \log m \leq \sqrt{\log m}) \\ & = \frac{1}{m^3}. \end{aligned}$$

Taking the expectation with respect to all possible values of  $x_1, x_2, \dots, x_T$ , we get

$$\Pr \left[ \sum_{k=1}^T \sum_{\ell=1}^{\tau_k} \frac{X_{k,\ell}^j}{X_k} \geq \sum_{k=1}^T \tau_k - 2 \right] = O\left(\frac{1}{m^3}\right).$$

Hence, with probability at least  $1 - O\left(\frac{1}{m^3}\right)$ , it holds that

$$\sum_{k=1}^T \sum_{\ell=1}^{\tau_k} \left(1 - \frac{X_{k,\ell}^j}{X_k}\right) \geq \sum_{k=1}^T \tau_k - \left(\sum_{k=1}^T \tau_k - 2\right) = 2,$$

completing the proof.

## B Proof of Lemma 7

By Proposition 2, if there is no left-saturating  $\mathbf{s}$ -matching, then there must exist a subset  $Y \subseteq L$  such that  $|N_G(Y)| < \sum_{i \in Y} s_i$ . For any  $k \in [n]$ , let  $\Gamma_k$  be the sum of the  $k$  largest elements of  $\mathbf{s}$ , and  $\gamma_k$  be the sum of the  $k$  smallest elements of  $\mathbf{s}$ . By the union bound, the probability that there is no left-saturating  $\mathbf{s}$ -matching can be bounded as follows:

$$\begin{aligned} & \Pr [\text{No left-saturating } \mathbf{s}\text{-matching exists}] \\ & \leq \Pr \left[ \exists Y \subseteq L \text{ such that } |N_G(Y)| < \sum_{i \in Y} s_i \right] \\ & \leq \sum_{k=1}^n \Pr \left[ \exists Y \subseteq L \text{ such that } |Y| = k, |N_G(Y)| < \sum_{i \in Y} s_i \right] \\ & = \sum_{k=1}^n \Pr \left[ \exists Y \subseteq L, Z \subseteq R \text{ such that } |Y| = k, |Z| = \sum_{i \in Y} s_i - 1, N_G(Y) \subseteq Z \right] \\ & \leq \sum_{k=1}^n \sum_{\substack{Y \subseteq L, Z \subseteq R \\ |Y|=k, |Z|=\sum_{i \in Y} s_i - 1}} \Pr [N_G(Y) \subseteq Z] \\ & = \sum_{k=1}^n \sum_{\substack{Y \subseteq L, Z \subseteq R \\ |Y|=k, |Z|=\sum_{i \in Y} s_i - 1}} (1-p)^{k \cdot (m - \sum_{i \in Y} s_i + 1)} \\ & \leq \sum_{k=1}^n \sum_{\substack{Y \subseteq L, \\ |Y|=k}} \binom{m - \sum_{i \in Y} s_i + 1}{k} \exp \left( -pk \cdot \left( m - \sum_{i \in Y} s_i + 1 \right) \right) \quad (\text{by } (1-p)^{k\ell} \leq \exp(-pk\ell) \text{ for any } \ell \geq 0) \\ & \leq \sum_{k=1}^n \sum_{\substack{Y \subseteq L, \\ |Y|=k}} m^{\min\{\sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1\}} \cdot \exp \left( -pk \cdot \left( m - \sum_{i \in Y} s_i + 1 \right) \right) \\ & = \sum_{k=1}^n \sum_{\substack{Y \subseteq L, \\ |Y|=k}} \exp \left( \min \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \cdot \log m - pk \cdot \left( m - \sum_{i \in Y} s_i + 1 \right) \right) \\ & = \sum_{k=1}^n \sum_{\substack{Y \subseteq L, \\ |Y|=k}} \exp \left( \min \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \cdot \log m \right. \\ & \quad \left. - p \cdot \frac{k}{\sum_{i \in Y} s_i} \cdot \max \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \cdot \min \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \right) \\ & \leq \sum_{k=1}^n \sum_{\substack{Y \subseteq L, \\ |Y|=k}} \exp \left( \left( \log m - p \cdot \frac{k}{\sum_{i \in Y} s_i} \cdot \max \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \right) \cdot \min \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \right). \end{aligned}$$

For  $Y \subseteq L$  with  $|Y| = k$ , observe that  $\max \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \geq \frac{m}{2} \geq \frac{ns_{\min}}{2}$  and  $\frac{k}{\sum_{i \in Y} s_i} \geq \frac{1}{s_{\max}}$ . These inequalities yield

$$\log m - p \cdot \frac{k}{\sum_{i \in Y} s_i} \cdot \max \left\{ \sum_{i \in Y} s_i, m - \sum_{i \in Y} s_i + 1 \right\} \leq \log m - p \cdot \frac{1}{s_{\max}} \cdot \frac{ns_{\min}}{2} \leq \log m - p \cdot \frac{n}{2B} \leq -3 \log m,$$

where we use the assumption  $p \geq 8B \log m/n$  for the last inequality. Moreover, since  $s_i \geq 1$  for all  $i \in N$ , we have  $k \leq \gamma_k$  and  $n - k \leq \Gamma_n - \Gamma_k \leq m - \Gamma_k$  for all  $k \in [n]$ . Combining these bounds, we deduce that

$$\begin{aligned} & \Pr [\text{No left-saturating } s\text{-matching exists}] \\ & \leq \sum_{k=1}^n \sum_{\substack{Y \subseteq L, \\ |Y|=k}} \exp(-3 \log m \cdot \min \{\gamma_k, m - \Gamma_k + 1\}) \\ & \leq \sum_{k=1}^n \binom{n}{k} \exp(-3 \log m \cdot \min \{\gamma_k, m - \Gamma_k + 1\}) \\ & \leq \sum_{k=1}^n n^{\min\{k, n-k\}} \cdot \exp(-3 \log m \cdot \min \{\gamma_k, m - \Gamma_k + 1\}) \\ & = \sum_{k=1}^n \exp(\log n \cdot \min \{k, n - k\}) \cdot \exp(-3 \log m \cdot \min \{\gamma_k, m - \Gamma_k + 1\}) \\ & \leq \sum_{k=1}^n \exp(\log m \cdot \min \{k, n - k\} - 3 \log m \cdot \min \{\gamma_k, m - \Gamma_k + 1\}) \quad (\text{by } \log n \leq \log m) \\ & \leq \sum_{k=1}^n \exp(-2 \log m \cdot \min \{\gamma_k, m - \Gamma_k + 1\}) \quad (\text{by } k \leq \gamma_k \text{ and } n - k \leq m - \Gamma_k + 1) \\ & \leq \sum_{k=1}^n \exp(-2 \log m) \quad (\text{by } \min \{\gamma_k, m - \Gamma_k + 1\} \geq 1) \\ & \leq \frac{1}{n}. \quad (\text{by } n \leq m) \end{aligned}$$

Therefore, the probability that  $G$  contains a left-saturating  $s$ -matching is at least  $1 - 1/n$ , which approaches 1 as  $n \rightarrow \infty$ .