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ABSTRACT

Standard game theory assumes that the structure of the game is common knowledge among players. We relax this assumption by considering extensive games where agents may be unaware of the complete structure of the game. In particular, they may not be aware of moves that they and other agents can make. We show how such games can be represented; the key idea is to describe the game from the point of view of every agent at every node of the game tree. We provide a generalization of Nash equilibrium and show that every game with awareness has a generalized Nash equilibrium. Finally, we extend these results to games with *awareness of unawareness*, where a player i may be aware that a player j can make moves that i is not aware of, and to *subjective games*, where players may have no common knowledge regarding the actual game and their beliefs are incompatible with a common prior.

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1. Introduction

Standard game theory models implicitly assume that all significant aspects of the game (payoffs, moves available, etc.) are common knowledge among the players. Such common knowledge is not necessary for solution concepts such as Nash equilibrium (see Aumann and Brandenburger, 1995); moreover, there are well-known techniques going back to Harsanyi (1968) for transforming a game where some aspects are not common knowledge to one where they are common knowledge. However, these techniques assume that players are at least *aware* of all possible moves in the game. But this is not always a reasonable assumption. For example,

sleazy companies assume that consumers are not aware that they can lodge complaints if there are problems; in a war setting, having technology that an enemy is unaware of (and thus being able to make moves that the enemy is unaware of) can be critical; in financial markets, some investors may not be aware of certain investment strategies (complicated hedging strategies, for example, or tax-avoidance strategies).

In a standard game, a set of strategies is a Nash equilibrium if each agent's strategy is a best response to the other agents' strategies, so each agent i would continue playing its strategy even if i knew what strategies the other agents were using. To understand the relevance of adding the possibility of unawareness to the analysis of games, consider the game shown in Fig. 1. One Nash equilibrium of this game has A playing across _{A} and B playing down _{B} . However, suppose that A is not aware that B can play down _{B} . In that case, if A is rational, A will play down _{A} . Therefore, Nash equilibrium does not seem to be the appropriate solution concept here. Although A would play across _{A} if A knew that B were going to play down _{B} , A cannot even contemplate this possibility, let alone know it.

Our goal is to find appropriate solution concepts for extensive games with possibly unaware players, and more generally, to find ways of representing multiagent systems where some agents may not be aware of features of the system. To do this, we must first find an appropriate representation for such games.

We cannot in general represent what is going on using only one extensive game. The standard representation of a game implicitly assumes that (it is common knowledge that) the modeler and the players all understand the game the same way. This is no longer true once we allow for the possibility of unawareness, since a player's description of the game can now involve only those

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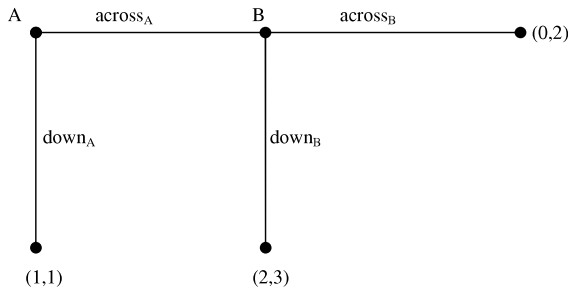


Fig. 1. A simple game.

aspects of the game that he is aware of. Moreover, as pointed out by Rubinstein (1991), “a good model in game theory has to be realistic in the sense that it provides a model for the *perception* of real life social phenomena. It should incorporate a description of the relevant factors involved as perceived by the decision makers. These need not represent the physical rules of the world”. Thus, since we want to allow for players to have different perceptions about reality, the full description of the game with awareness is given by a set of extensive games, one describing the objective or “true” situation and one for each game that at least one of the agents thinks might be the true game in some situation.

Continuing with the game in Fig. 1, the game from the point of view of the type of B that is unaware of the possibility of playing down_B would just include A ’s moves down_A and across_A and the move across_B . In that game, player A is also unaware of the move down_B . By way of contrast, the game from the point of view of the type of B that is aware of down_B would include the move down_B , but may also allow for the possibility that A is not aware that B is aware of this move.

Once we have a modeled (lack of) awareness, the next step is to examine solution concepts. In this paper, we focus on *generalized Nash equilibrium*, a generalization of Nash equilibrium to games of awareness. By considering Nash equilibrium, we do not mean to imply that it is the “right” solution concept. Some variant or refinement may well be more appropriate. Nevertheless, we believe that it is worth studying Nash equilibrium, not only for its own sake, but since an understanding of how Nash equilibrium works in the presence of lack of awareness should guide efforts to construct other solution concepts.

The standard notion of Nash equilibrium consists of a collection of strategies, one for each player. Our generalization consists of a collection of strategies, one for each pair (i, Γ') , where Γ' is a game that agent i considers to be the true game in some situation. Intuitively, the strategy for a player i at Γ' is the strategy i would play in situations where i believes that the true game is Γ' . To understand why we may need to consider different strategies consider, for example, the game of Fig. 1. B would play differently depending on whether or not he was aware of down_B . Roughly speaking, a set of strategies, one for each pair (i, Γ') , is a *generalized Nash equilibrium* if the strategy for (i, Γ') is a best response for player i if the true game is Γ' , given the strategies being used by the other players in Γ' .

We argue that this notion of equilibrium correctly captures our intuitions. We then show that every game with awareness has a generalized Nash equilibrium by associating with a game with awareness a standard game (where agents are aware of all moves) such that there is a one-to-one correspondence between generalized Nash equilibria of the game with awareness and Nash equilibria of the standard game.

For ease of exposition, for most of the paper we focus on games where agents are not aware of their lack of awareness. That is, we do not consider games where one player might be aware that there are moves that another player (or even she herself) might be able to

make, although she is not aware of what they are. Such awareness of unawareness can be quite relevant in practice. For example, in the war setting described above, even if one side cannot conceive of a new technology available to the enemy, they might believe that there is some move available to the enemy without understanding what that particular move is. This, in turn, may encourage peace overtures. To take another example, an agent might delay making a decision because she considers it possible that she might learn about more possible moves, even if she is not aware of what these moves are.

Although, economists usually interpret awareness as “being able to conceive of an event or a proposition”, there are other possible meanings for this concept. For example, awareness may also be interpreted as “understanding the primitive concepts in an event or proposition”, or as “being able to determine if an event occurred or not”, or as “being able to compute the consequences of some fact” (Fagin and Halpern, 1988). If we interpret “lack of awareness” as “unable to compute”, then awareness of unawareness becomes even more significant. Consider a chess game. Although all players understand in principle all the moves that can be made, they certainly do not reason about all the possible plays of the game that could follow from the current board position; this is computationally intractable. A more accurate representation of chess would model this computational unawareness explicitly. We provide such a representation.

Roughly speaking, we capture the fact that player i is aware at a node h in the game tree that there is a move that j can make that she (i) is not aware of by having i ’s subjective representation of the game include a “virtual” move for j at node h . Since i might have only an incomplete understanding of what can happen after this move, i simply describes what she believes could happen after the virtual move, to the extent that she can; if she has no idea what could happen (for example, as in the case of a chess program that has run out of resources to search further down the game tree), she describes her beliefs regarding the payoffs of the game. Our representation can be viewed as a generalization of how chess programs analyze chess games. They explore the game tree up to a certain point, and then evaluate the board position at that point. We can think of the payoffs following a virtual move by j in i ’s subjective representation of a chess game as describing the evaluation of the board from i ’s point of view. This seems like a much more reasonable representation of the game than the standard complete game tree! We describe how chess can be modeled in our framework in more detail in Section 4.

Our framework is flexible enough to deal with games where there is lack of common knowledge about the game being played. In particular, we can deal with lack of common knowledge regarding the utilities, who moves next, the structure of other players’ information sets, and the probability of nature’s moves (even in cases where there is no common prior compatible with the players’ beliefs regarding nature).

As discussed by Heifetz et al. (2011), the standard framework is not expressive enough to model strategic reasoning with awareness; we cannot just remove from the game tree those actions that a player is not aware of. They give an example of an extension of the Bach–Stravinsky game where the players also have the option of going to a Mozart concert; the Mozart concert is the unique equilibrium of the game. As they point out, if only player 1 is aware of the Mozart concert, and he does not tell player 2 about the concert, then the game from player 2’s point of view is just a standard Bach–Stravinsky game, while the game from the point of view of player 1 is the extended version of the game. To capture the possibility of players having different views of the game, new models and solution concepts are needed.

The rest of this paper is organized as follows. In Section 2, we describe how to represent a game with possibly unaware players.

In Section 3, we use our representation to define a generalized notion of Nash equilibrium, and we prove its existence in games with awareness. In Section 4, we describe how we can extend our approach to deal with awareness of unawareness. We compare our work to others in the literature, particularly to that of [Feinberg \(2009\)](#), in Section 5, and conclude in Section 6, where we discuss how to extend our framework to deal with games where there is lack of common knowledge, even if awareness is not an issue.

2. Modeling awareness

The first step in dealing with awareness is modeling it. As we said in the introduction, we assume (throughout most of the paper) that there is underlying objective reality—the game actually being played. Of course, none of the players may be aware of this game. Roughly speaking, we model a game of awareness as a set of extensive games, where each game in the set can be viewed as representing an agent's view of the game, and thus includes only those histories that the agent is aware of. We assume that there are some consistency conditions that these games must satisfy; these assumptions essentially amount to saying that, although the players may not be aware of all moves, they are not completely mistaken about the actual game. (In Section 6, we sketch how the model can be modified to drop the assumption that there is an actual game.) Before giving the formal definitions, we review the definition of an extensive game ([Osborne and Rubinstein, 1994](#)).

A (finite) extensive game is a tuple $(N, M, H, P, f_c, \{\mathcal{I}_i : i \in N\}, \{u_i : i \in N\})$, where

- N is a finite set consisting of the players of the game.
- M is a finite set whose elements are the moves (or actions) available to players (and nature) during the game.¹
- H is a finite set of finite sequences of moves (elements of M) that is closed under prefixes,² so that if $h \in H$ and h' is a prefix of h , then $h' \in H$. Intuitively, each member of H is a *history*. We can identify the nodes in a game tree with the histories in H . Each node n is characterized by the sequence of moves needed to reach n . A *run* in H is a terminal history, one that is not a strict prefix of any other history in H . Let Z denote the set of runs of H . Let $M_h = \{m \in M : h \cdot \langle m \rangle \in H\}$ (where we use \cdot to denote concatenation of sequences); M_h is the set of moves that can be made after history h .
- $P : (H - Z) \rightarrow N \cup \{c\}$ is a function that assigns to each nonterminal history h a member of $N \cup \{c\}$. (We can think of c as representing nature.) If $P(h) = i$, then player i moves after history h ; if $P(h) = c$, then nature moves after h . Let $H_i = \{h : P(h) = i\}$ be the set of all histories after which player i moves.
- f_c is a function that associates with every history for which $P(h) = c$ a probability measure $f_c(\cdot | h)$ on M_h . Intuitively, $f_c(\cdot | h)$ describes the probability of nature's moves once history h is reached.
- \mathcal{I}_i is a partition of H_i with the property that if h and h' are in the same cell of the partition then $M_h = M_{h'}$, i.e., the same set of moves is available at every history in a cell of the partition. Intuitively, if h and h' are in the same cell of \mathcal{I}_i , then h and h' are indistinguishable from i 's point of view; i considers history h' possible if the actual history is h , and vice versa. A cell $I \in \mathcal{I}_i$ is called an (*i*-)information set.

¹ Osborne and Rubinstein did not make M explicit in their definition of an extensive game; we find it convenient to make it explicit here.

² A prefix of a sequence is any initial segment of the sequence. Thus, if $h = \langle m_1, m_2, m_3 \rangle$, then $\langle \rangle$, $\langle m_1 \rangle$, $\langle m_1, m_2 \rangle$ and h are prefixes of h . A prefix of h that is different from h is called a *strict* prefix.

- $u_i : Z \rightarrow \mathbb{R}$ is a payoff function for player i , assigning a real number (i 's payoff) to each run of the game.

In the game of [Fig. 1](#),

- $N = \{A, B\}$, $H = \{\langle \rangle, \langle \text{down}_A \rangle, \langle \text{across}_A \rangle, \langle \text{across}_A, \text{down}_B \rangle, \langle \text{across}_A, \text{across}_B \rangle\}$,
- $P(\langle \rangle) = A$, $P(\langle \text{across}_A \rangle) = B$,
- $I_A = \{\langle \rangle\}$, $I_B = \{\langle \text{across}_A \rangle\}$,
- $u_A(\langle \text{down}_A \rangle) = u_B(\langle \text{down}_A \rangle) = 1$,
- $u_A(\langle \text{across}_A, \text{across}_B \rangle) = 0$, and
- $u_B(\langle \text{across}_A, \text{across}_B \rangle) = 2$.

In this paper, as in most work in game theory, we further assume that players have *perfect recall*: they remember all the actions that they have performed and the information sets they passed through. Formally, we require that

- if h and h' are in the same i -information set and h_1 is a prefix of h such that $P(h_1) = i$, then there is a prefix h'_1 of h' such that h_1 and h'_1 are in the same information set; moreover, if $h_1 \cdot \langle m \rangle$ is a prefix of h (so that m was the action performed when h_1 was reached in h) then $h'_1 \cdot \langle m \rangle$ is a prefix of h' .

As we said, we assume that there is an underlying objective game, which we typically denote Γ . A game Γ^+ based on Γ is an extensive game where all moves available to players in Γ^+ are also available to players in Γ , and the only moves in Γ^+ that are not in Γ are messages by nature that may affect some players' awareness of histories in Γ . Intuitively, Γ^+ represents one subjective view regarding the available moves and what histories are distinguishable. We use nature's messages to represent both random events that may alter a player's awareness and uncertainty a player has about another player's awareness. Formally, $\Gamma^+ = (N^+, M^+, H^+, P^+, f_c^+, \{\mathcal{I}_i^+ : i \in N^+\}, \{u_i^+ : i \in N^+\})$ is a *game based on* $\Gamma = (N, M, H, P, f_c, \{\mathcal{I}_i : i \in N\}, \{u_i : i \in N\})$ if Γ^+ is an extensive game that satisfies the following conditions:

A1. $N^+ = N$.³

We need the following definition for the next two conditions: if $h \in H^+$, then let \bar{h} be the subsequence of h consisting of all the moves in h that are also in M .

A2. If $P^+(h) \in N^+$, then $\bar{h} \in H$, $P^+(h) = P(\bar{h})$ and $M_h^+ \subseteq M_{\bar{h}}$. Moreover, if h and h' are in H^+ , \bar{h} and \bar{h}' are in the same information set for player i in Γ , player i moves at h and h' , and $h \cdot \langle m \rangle \in H^+$, then $h' \cdot \langle m \rangle \in H^+$.

Intuitively, all the moves available to i at h must also be available to i in the objective game Γ . Moreover, if Γ^+ is the game that some player j believes to be the true game and $h \cdot \langle m \rangle$ is one of its histories, then j must be aware that player i can make move m at the information set that contains history \bar{h} in Γ . Since \bar{h} is in the same information set as \bar{h}' in Γ and j is aware of history h' , j must also be aware that i can make move m at history h' , so $h' \cdot \langle m \rangle$ must be a history in Γ^+ .

A3. If $P^+(h) = c$, then either $P(\bar{h}) = c$ and $M_h^+ \subseteq M_{\bar{h}}$, or $P(\bar{h}) \neq c$ and $M_h^+ \cap M = \emptyset$.⁴

³ In a preliminary version of this paper, we allowed N^+ to be a subset of N , but this leads to some problems with our later conditions. For example, if N^+ is a strict subset of N , A5 implies that the player who believes the game is Γ^+ can be aware only of runs of Γ where the only players that move are the ones in N^+ . Since this is quite restrictive, for simplicity we focus here on the case of only unawareness of moves. We discuss in Section 6 how to generalize the model to allow for unawareness of players as well.

⁴ In general we place no requirements on agents' beliefs about nature's moves. There may be applications where it would be reasonable to assume that players understand how nature works on moves that they are aware of. Specifically, we could assume that players are aware of the relative probability of nature's moves that they are aware of; that is, if $a \in M_h^+$, then $f_c(a | \bar{h}) = \frac{f_c^+(a|h)}{\sum_{b \in M_h^+} f_c^+(b|h)}$.

In the case where $M_h^+ \cap M = \emptyset$, the moves in M_h^+ are messages sent by nature to various players that may change their awareness of histories of Γ .⁵ We assume for ease of exposition that each move in $M^+ - M$ has the form $m = (m_i)_{i \in N^+}$, where, intuitively, m_i is a message to i that may have the effect of changing i 's awareness of histories of Γ .⁶

Before giving the next condition, we define what it means for a player to have the same view of the game at two histories. We say that i has the same view in histories h and h' if (1) $P^+(h) = P^+(h') = i$ and \bar{h} and \bar{h}' are in the same information set for player i in Γ , and (2) for every prefix $h_1 \cdot \langle (m_i)_{i \in N^+} \rangle$ of h such that $P^+(h_1) = c$ and $M_{h_1}^+ \cap M = \emptyset$ there exists a prefix $h'_1 \cdot \langle (m'_i)_{i \in N^+} \rangle$ of h' such that $\bar{h}_1 = \bar{h}'_1$, $P^+(h'_1) = c$, $M_{h'_1}^+ \cap M = \emptyset$, and $m_i = m'_i$, and (3) the same condition as (2) holds with the roles of h and h' reversed. Intuitively, condition (1) states that player i cannot distinguish the actual sequence of moves in h and h' , while (2) and (3) state that nature sends exactly the same messages regarding awareness to i in h and h' at the same times.⁷

A4. The histories h and h' are in the same information set for player i in Γ^+ iff i has the same view in both h and h' .

A5. $\{\bar{z} : z \in Z^+\} \subseteq Z$.

A6. For all $i \in N^+$ and runs z in Z^+ , if $\bar{z} \in Z$, then $u_i^+(z) = u_i(\bar{z})$. Thus, a player's utility just depends on the moves made in the objective game. (By A5, we have $\bar{z} \in Z$. We have included the clause "if $\bar{z} \in Z$ " so that A6 is applicable when we consider awareness of unawareness, where we drop A5.)

Conditions A1–A6 require that in a game based on Γ , each history must correspond to a history in Γ , except that there may be additional moves by nature (that can be thought of as changing players' awareness). In addition, by A4, if two histories are in the same information set in a game based on Γ then the corresponding histories in Γ must be in the same information set. On the other hand, it is possible that histories h and h' are in different information sets in a game based on Γ even though \bar{h} and \bar{h}' are in the same information set in Γ . This can happen if, for example, h and h' contain different sequences of messages from nature.

Although these conditions seem to us quite natural, they are not always met. For example, A6 says that agents essentially know the utility function. And A4 essentially says that agents cannot have false beliefs regarding information sets. For example, suppose that h and h' are histories in some game Γ^+ based on Γ . In \bar{h} , some agent j has been granted a patent on a product, and in \bar{h}' , j has not been granted the patent; moreover, i has received a signal regarding whether j has been granted the patent. Thus, \bar{h} and \bar{h}' are in different information sets for i in Γ . However, suppose that i is not aware of the patent issue at all in Γ^+ . Then in Γ^+ , h and h' are in the same information set. This violates A4.

In order to focus specifically on unawareness issues, we initially assume A1–A6. As we said above, in Section 6, we generalize the model and drop these assumptions.

As we have observed, A4 essentially says, among other things, that agents do not have false beliefs about information sets in

the objective game. To understand A4, we must first discuss our view of information sets. As pointed out by Halpern (1997), special attention must be given to the interpretation of information sets in game trees. This issue requires even more care in games with awareness. The standard intuition for information sets is that a player does not know which of the histories in an information set actually occurred. He considers them all possible (although may have beliefs about the relative likelihood of various histories). But this intuition does not apply in games with possibly unaware players. In such games, there may be some histories in an i -information set that i is not aware of; player i cannot consider these histories possible. For example, consider finitely repeated prisoners dilemma where Alice and Bob each move twice before their moves are revealed. Even if Bob is not aware of defection, his information set after Alice's first move in the objective game will still contain the history where Alice defects.

As we said, we interpret an i -information set in a game Γ^+ to be the set of all histories in Γ^+ where player i has the same view. Intuitively, a player's view of the game encodes all the information that i has about the moves he can make, what moves have been made, the other players in the game, his strategy, and so on. We assume that player i 's view is characterized by the sequence of signals that i has received in the course of the game. These signals either come from moves taken in the objective game or from awareness messages sent by nature. The timing of the signals is important as well; this is captured by clauses (2) and (3) in the definition of view. For example, suppose that, as above, in repeated prisoner's dilemma, Bob is not aware that he may defect. But now suppose that he may become aware of defection, not just by observing Alice defect, but by getting a signal from nature. He can then distinguish a history where he gets no signal (and thus does not become aware of defection) from one where he gets a signal before the first play of prisoner's dilemma and from one where he gets a signal after the first play of prisoner's. These three histories would all be in different information sets, because Bob has a different view in each one.

There is a significant distinction between the interpretation of information sets in games of awareness and in standard games. In a standard game, we can think of an information set as representing all the histories that a player considers possible at a given point in the game. As we shall see, this interpretation is inappropriate in a game of awareness, because a player may not be aware of some histories in his information set. We return to this point below.

For the remainder of the paper, we use the following notation: for a game Γ^s , we denote the components of Γ^s with the same superscript s , so that we have M^s , H^s , and so on. Thus, from here on we do not explicitly describe the components of a game.

To see how games based on Γ are used, consider again the game Γ shown in Fig. 1 and suppose that

- players A and B are aware of all histories of the game;
- A is uncertain as to whether B is aware of run $\langle \text{across}_A, \text{down}_B \rangle$ and believes that he is unaware of it with probability p .
- A believes that if B is unaware of $\langle \text{across}_A, \text{down}_B \rangle$, then B believes that the game without $\langle \text{across}_A, \text{down}_B \rangle$ is common knowledge among players; and
- the type of B that is aware of the run $\langle \text{across}_A, \text{down}_B \rangle$ is aware that A is aware of all histories, and he knows that A is uncertain about the game that he (B) considers possible and knows the probability p .⁸

⁵ We place no requirements on an agent's beliefs after getting an awareness-changing message but, as in the work of Ozbay (2007), it might be reasonable to define solution concepts that impose restriction on beliefs. We do not address this issue further in this paper.

⁶ There is no necessary relationship between the messages that i gets from nature (and from the other players) and how i 's awareness changes. It may require several signals from nature, perhaps in combination with signals received from other players, before i 's awareness changes.

⁷ Note that this notion makes sense even if h and h' are histories in different games; we use this fact later.

⁸ In this example, we take "the type of player B that is aware of the run $\langle \text{across}_A, \text{down}_B \rangle$ " to represent both the actual player B and the player B in the mind of player A if A believes that B is aware of $\langle \text{across}_A, \text{down}_B \rangle$.

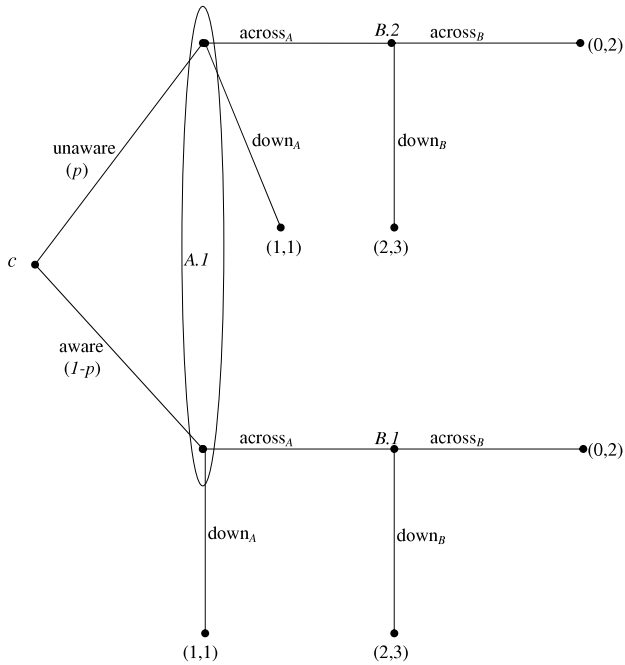


Fig. 2. The game Γ^A .

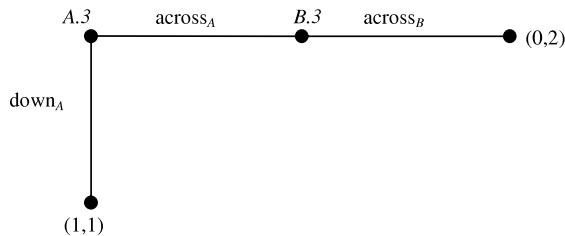


Fig. 3. The game Γ^B .

To model this, we need three games based on Γ . The first is the objective game Γ . When A moves at the node labeled A in Γ , she believes that the actual game is Γ^A , as described in Fig. 2. In Γ^A , nature's initial move captures A's uncertainty about the game B considers possible; nature either makes an awareness move that makes B aware of the possibility of moving down or does not. At the information set labeled A.1, A is aware of all the runs of the objective game. Moreover, at this information set, A believes that the true game is Γ^A . At the node labeled B.1, B is aware of all the runs of the objective game and believes that the true game is Γ ; but at the node labeled B.2, B is not aware that he can play down_B . We thus need to consider a third game based on Γ , the game Γ^B described in Fig. 3. At the nodes labeled A.3 and B.3 in the game Γ^B , neither A nor B is aware of the move down_B . Moreover, both players think the true game is Γ^B .

As this example should make clear, to model a game with possibly unaware players, we need to consider not just one game, but a collection of them. Moreover, we need to describe, at each history in a game, which game the player playing at that history believes is the actual game being played.

To capture these intuitions, we define a *game with awareness* based on $\Gamma = (N, M, H, P, f_c, \{\mathcal{I}_i : i \in N\}, \{u_i : i \in N\})$ to be a pair $\Gamma^* = (\mathcal{G}, \mathcal{F})$, where

- \mathcal{G} is a countable set of finite extensive games based on Γ , of which one is Γ ;
- \mathcal{F} maps an extensive game $\Gamma^+ \in \mathcal{G}$ and a history h in Γ^+ such that $P^+(h) = i$ to a pair (Γ^h, I) , where $\Gamma^h \in \mathcal{G}$ and I is an i -information set in game Γ^h .

If player i moves at h in game $\Gamma^+ \in \mathcal{G}$ and $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, I)$, then Γ^h is the game that i believes to be the true game when the history is h , and I consists of the set of histories in Γ^h that he currently considers possible. Moreover, if $\mathcal{F}(\Gamma^+, h_1) = (\Gamma^+, I_1)$ and $P^+(h_1) = j$, then Γ^h is also the game that player j at history h_1 of Γ^+ believes that i at history h of Γ^+ believes is the actual game. As this example makes clear, the game Γ^+ has a number of interpretations. For simplicity, we simply refer to Γ^h as a possible view of the game from player i 's perspective. In the examples described in Figs. 2 and 3, we have $\mathcal{F}(\Gamma, \langle \rangle) = (\Gamma^A, I)$, where I is the information set labeled A.1 in Fig. 2, and $\mathcal{F}(\Gamma^A, \langle \text{unaware}, \text{across}_A \rangle) = (\Gamma^B, \{\langle \text{across}_A \rangle\})$. Thus, at the node labeled B.2, although player B is playing in the game Γ^A , he believes that he is playing Γ^B and is at B.3. Note that if $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, I)$, then I is perhaps a closer analogue to how information sets are interpreted in standard games than the information set containing h in Γ^+ . See below for further discussion of this point.

Define a game $\Gamma' \in \mathcal{G}$ to be *reachable* from the game $\Gamma^+ \in \mathcal{G}$ by the map \mathcal{F} if there exists a sequence of pairs $(\Gamma_1, h_1), (\Gamma_2, h_2), \dots, (\Gamma_n, h_n)$, where $h_k \in H^k$ for $k = 1, 2, \dots, n$, $\Gamma_1 = \Gamma^+$, $\Gamma_n = \Gamma'$ and $\mathcal{F}(\Gamma_k, h_k) = (\Gamma_{k+1}, \cdot)$ for $k = 1, 2, \dots, n-1$. Note that if $\mathcal{F}(\Gamma', h) = (\Gamma^+, I)$, then Γ^+ does not fully describe i 's view of what is going on. While it describes i 's beliefs about the game being played (and what moves are feasible), it does not describe i 's beliefs about other players' beliefs. To describe these, we must consider the game of awareness $(\mathcal{G}^i, \mathcal{F}^i)$ based on Γ , where \mathcal{G}^i consists of all games in \mathcal{G} reachable from Γ^+ by the mapping \mathcal{F} and \mathcal{F}^i is the restriction of \mathcal{F} to games in \mathcal{G}^i .

Intuitively, \mathcal{G} consists of all games "reachable" from the objective game Γ by applying \mathcal{F} . That is, we start with Γ , and take \mathcal{G} to be the least set such that if $\Gamma^+ \in \mathcal{G}$, h is a history in Γ^+ , and $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, I)$, then $\Gamma^h \in \mathcal{G}$. (Clearly \mathcal{G} contains all games reachable from Γ in this sense, although it may contain more games. Games in \mathcal{G} that are not reachable from Γ turn out not to be relevant for our considerations; although we could have imposed another condition to rule them out, there seems to be no compelling reason to add such a condition. Our results do not require it.)

It may seem that by making \mathcal{F} a function we cannot capture a player's uncertainty about the game being played. However, we can capture such uncertainty by folding it into nature's move. For example, we capture A's uncertainty about whether B is aware of being able to move down_B in the game Γ^A illustrated in Fig. 2 by having nature decide this at the first step. It should be clear that this gives a general approach to capturing such uncertainty.

The mapping \mathcal{F} must satisfy a number of consistency conditions. These conditions capture desirable properties of awareness and of the player's beliefs. In particular, we require \mathcal{F} to satisfy the following constraints, where $\Gamma^+ \in \mathcal{G}$, $h \in H^+$, $P^+(h) = i$, and $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, I)$.

- C1. $h' \in I$ iff h' is a history in H^h where i has the same view as in h .
- C2. If $h' \in H^h$, then there exists $h_1 \in H^+$ such that $\bar{h}' = \bar{h}_1$. Moreover, if $h' \cdot \langle m \rangle \in H^h$ and $m \in M$, then for all $h_1 \in H^+$ such that \bar{h}' and \bar{h}_1 are in the same information set in Γ and $P^h(h') = P^+(h_1)$, we have that $h_1 \cdot \langle m \rangle \in H^+$.

Intuitively, if j 's view of the game is Γ^+ , then the fact that i is aware at $h \in H^+$ of some history \bar{h}' in the objective game means that j believes that i is aware of \bar{h}' at h . Thus, j must be aware of \bar{h}' if his current view of the game is Γ^+ . Moreover, if i is aware that move m is possible at history h' then, in game Γ^+ , j must also be aware that move m is possible at any history h_1 such that the same player moves at h' and h_1 and \bar{h}_1 and \bar{h}' are in the same information set in Γ . This corresponds to a standard property of awareness: if player j is aware that player i is aware of some fact ϕ , then j himself must be aware of ϕ .

- C3. If h' is a history in Γ' , and h'' is a history in Γ'' such that player i has the same view in h' and h'' , then $\mathcal{F}(\Gamma', h') = \mathcal{F}(\Gamma'', h'')$. Properties C3 and A4 together imply that we could have equivalently defined \mathcal{F} as a function from (game, information set) pairs to (game information, information set) pairs.⁹ We use the current definition so as to explicitly identify the assumptions that lead to this equivalence.
- C4. If h' is a prefix of h , $P^+(h') = i$, and $\mathcal{F}(\Gamma^+, h') = (\Gamma^{h'}, I')$, then $\{\bar{h}'_1 : h'_1 \in H^{h'}\} \subseteq \{\bar{h}_1 : h_1 \in H^h\}$. Moreover, $\Gamma^{h'} \neq \Gamma^h$ only if $\{\bar{h}'_1 : h'_1 \in H^{h'}\} \neq \{\bar{h}_1 : h_1 \in H^h\}$.

Intuitively, the player never forgets some history he was previously aware of and he changes the game that he considers possible only if he becomes aware of more moves of the objective game.

To define the next condition, we need a definition. A history h in Γ^+ is *plausible* if for every prefix $h_1 \cdot \langle m \rangle$ of h , if $\mathcal{F}(\Gamma^+, h_1) = (\Gamma^1, I)$ then $m \in M_1^1$. Intuitively, a plausible history is a history h such that, for every player i , if i makes a move m in h , then i is aware of m . Although some player may be aware of histories that are not plausible, only plausible histories can actually be played: each player must be aware of the moves that he actually makes.

- C5. There exists some plausible history in I .

Note that player i might be aware of a history that is not plausible because he is aware that player j can make a certain move a , although j is not aware of a and i knows that j is not aware of a . Thus, such histories may be part of a game from the point of view of i . C5 excludes the possibility that all histories considered possible by player i while moving at h are histories that are not plausible. Note that an information set I' may not contain any plausible history. Such an I' would not be in the range of \mathcal{F} (more precisely, it would not be the second component of a pair in the range of \mathcal{F}).

Note that $\mathcal{F}(\Gamma^+, h)$ may depend on more than just the set of histories of Γ that the player who moves at h is aware of. That is, even if $\{\bar{h}_1 : h_1 \in H^h\} = \{\bar{h}'_1 : h'_1 \in H^{h'}\}$, we may have $\mathcal{F}(\Gamma^+, h) \neq \mathcal{F}(\Gamma^+, h')$. By C3, the game that a player considers possible is determined by the histories he is aware of and how he became aware of them. This is relevant because player i 's beliefs about the game may well be affected by how player j strategically made i aware of various moves. (Although we only have awareness moves by nature in our model, we can capture player j making player i aware of a move by requiring that a certain awareness move of nature always follows a particular move by player j .) We also allow different players to react differently to becoming aware of moves. This extra expressive power allows us to model a situation where, for example, players 2 and 3, who are aware of the same set of histories of Γ and agree on the set of histories of Γ that player 1 is aware of, have different beliefs about the game player 1 considers possible. For another example, consider the case where player 1 is initially unaware that he can move left and that there two different ways he can become aware of being able to move left depending on whether he hears about it from player 2 or player 3. Who he hears about it from may affect his beliefs about the true game, including his beliefs about what other players are aware of.

A standard extensive game Γ can be identified with the game $(\{\Gamma\}, \mathcal{F})$, where for all histories h in an i -information set I in Γ , $\mathcal{F}(\Gamma, h) = (\Gamma, I)$. (This definition of \mathcal{F} is actually forced by C1.) Thus, all players are aware of all the runs in Γ , and agree with each other that the game is Γ . We call this the *canonical representation* of Γ as a game with awareness.

One technical issue: We have assumed that the set \mathcal{G} of games in a game Γ^* with awareness is countable. For our purposes,

this is without loss of generality. We are ultimately interested in what happens in the game Γ , since this is the game actually being played. However, to analyze that, we need to consider what happens in other games in \mathcal{G} . For example, if h is a history in Γ where i moves, we need to understand what happens in the game Γ^h such that $\mathcal{F}(\Gamma, h) = (\Gamma^h, \cdot)$, since Γ^h is the game that i thinks is being played at history h in Γ . It is not hard to see that the set of games we need to consider is the least set \mathcal{G}' such that $\Gamma \in \mathcal{G}'$ and, for every $\Gamma' \in \mathcal{G}'$ and history h in Γ' such that $\mathcal{F}(\Gamma', h) = (\Gamma'', \cdot)$, $\Gamma'' \in \mathcal{G}'$. \mathcal{G}' is guaranteed to be countable, even if \mathcal{G} is not.

3. Local strategies and generalized Nash equilibrium

3.1. Local strategies

In this section, we generalize the notion of Nash equilibrium to games with awareness. To do that, we must first define what a strategy is in a game with awareness. Recall that in a standard game, a *pure* strategy for player i is a function from i -information sets to moves; a *mixed* strategy is a distribution over pure strategies; and a *behavioral* strategy is a function from information sets to distributions over moves. The intuition is that player i 's actions depend on what i knows. A pure strategy can be viewed as a *universal plan*, describing what i will do in every possible situation that can arise; similarly a behavioral strategy prescribes what i will do in every situation, but allows randomization. This makes sense only because i is presumed to know the game tree, and thus to know in advance all the situations that can arise. A mixed strategy can be viewed as the result of doing randomization only at the beginning of the game, and then committing to a pure strategy depending on the outcome of a coin toss.

In games with awareness, these intuitions no longer makes sense. For example, player i cannot plan in advance for what will happen if he becomes aware of something he is initially unaware of. We thus view i 's (pure or behavioral) strategy as a tentative plan, which describes what i will do as long as his belief about what game is being played does not change. But we allow i to change strategies if he becomes aware of more possible moves. We do not consider mixed strategy here, such type of strategy could be seen as a randomization over the set of pure strategies, but such randomization is only made in the beginning of the game or right after the player becomes aware of something he was initially unaware of.

We formalize these intuitions as follows. Let $\mathcal{G}_i = \{\Gamma' \in \mathcal{G} : \text{for some } \Gamma^+ \in \mathcal{G} \text{ and } h \text{ in } \Gamma^+, P^+(h) = i \text{ and } \mathcal{F}(\Gamma^+, h) = (\Gamma', \cdot)\}$. Intuitively, \mathcal{G}_i consists of the games that i views as the real game in some history. Thus, rather than considering a single strategy in a game $\Gamma^* = (\mathcal{G}, \mathcal{F})$ with awareness, we consider a collection $\{\sigma_{i, \Gamma'} : \Gamma' \in \mathcal{G}_i\}$ of what we call *local strategies*, one for each game in \mathcal{G}_i . Intuitively, a local strategy $\sigma_{i, \Gamma'}$ for game Γ' is the strategy that i would use if i were called upon to play and i thought that the true game was Γ' . Thus, the domain of $\sigma_{i, \Gamma'}$ consists of pairs (Γ^+, h) such that $\Gamma^+ \in \mathcal{G}$, h is a history in Γ^+ , $P^+(h) = i$, and $\mathcal{F}(\Gamma^+, h) = (\Gamma', I)$.

Define an equivalence relation \sim_i on pairs (Γ', h) such that $\Gamma' \in \mathcal{G}$ and h is a history in Γ' where i moves such that $(\Gamma_1, h_1) \sim_i (\Gamma_2, h_2)$ if $\mathcal{F}(\Gamma_1, h_1) = \mathcal{F}(\Gamma_2, h_2)$. We can think of \sim_i as defining a *generalized information partition* in Γ^* ; each element in the partition is called a *generalized information set*. Thus, if i is the player that moves at history h in a game Γ^+ and $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, I)$, then there are really three candidates for “ i 's information set at history h in game Γ^+ ”. The first is the information set in Γ^+ that contains h ; this is the set of all histories in Γ^+ where i has the same view as in h . But i may not be aware of all these histories. The second is I ; this is the set of all histories in Γ^h , which is the game that i thinks is being played when at h , where i has the same view

⁹ We thank an anonymous reviewer for this observation.

as in h . Finally, there is the generalized information set containing h ; this is the set of all histories in some game in \mathcal{G} where i has the same view as the one he has at h .

It is easy to check that a \sim_i equivalence class consists of a union of i -information sets in individual games in \mathcal{G} . Moreover, if some element of a \sim_i equivalence class is in the domain of $\sigma_{i,I'}$, then so is the whole equivalence class. Moreover, if two histories (Γ_1, h_1) and (Γ_2, h_2) are in an \sim_i equivalence class that is in the domain of $\sigma_{i,I'}$, then we require that $\sigma_{i,I'}(\Gamma_1, h_1) = \sigma_{i,I'}(\Gamma_2, h_2)$, since at both (Γ_1, h_1) and (Γ_2, h_2) , player i thinks he is in the same information set in Γ' . Thus, the generalized information set plays the same role as an information set in standard games, in the sense that we require players to do the same thing at all histories in a generalized information set.

The following definition summarizes this discussion.

Definition 3.1. Given a game with awareness $\Gamma^* = (\mathcal{G}, \mathcal{F})$, a *local strategy* $\sigma_{i,I'}$ for agent i is a function mapping pairs (Γ^+, h) such that h is a history where i moves in Γ^+ and $\mathcal{F}(\Gamma^+, h) = (\Gamma', I)$ to a probability distribution over $M'_{h'}$, the moves available at a history $h' \in I$, such that $\sigma_{i,I'}(\Gamma_1, h_1) = \sigma_{i,I'}(\Gamma_2, h_2)$ if $(\Gamma_1, h_1) \sim_i (\Gamma_2, h_2)$.

Note that there may be no relationship between the strategies $\sigma_{i,I'}$ for different games Γ' . Intuitively, this is because discovering about the possibility of a different move may cause agent i to totally alter his strategy. We could impose some consistency requirements, but we have not found any that we believe should hold in all games. We believe that all our results would continue to hold in the presence of reasonable additional requirements, although we have not explored the space of such requirements.

We (the modelers) can describe what i does in a game of awareness by a collection of local strategies, one for each game $\Gamma' \in \mathcal{G}_i$, although i himself is not aware of this collection (since i cannot even think about strategies $\sigma_{i,I'}$ for game Γ' that contain histories that he is not be aware of).

3.2. Generalized Nash equilibrium

In a standard game, a strategy profile $\vec{\sigma}$ is a Nash equilibrium if each component σ_i is a best response to σ_{-i} . This makes sense if, intuitively, the strategy profile $\vec{\sigma}$ being used is common knowledge.¹⁰ In games with awareness, the strategy profile cannot in general be common knowledge; if player i is not aware of some component of player j 's strategy, he certainly cannot know that j is using that strategy. We want to define a notion of generalized Nash equilibrium so as to capture the intuition that for every player i , if i believes he is playing game Γ' , then his local strategy $\sigma_{i,I'}$ is a best response to the local strategies of other players in Γ' , and those local strategies are, in a sense, as close to common knowledge as possible. We need to be a bit careful about the interpretation of best response. Nash equilibrium implicitly assumes that players care only about playing best responses after histories that they believe could have happened; that is, at information sets that intersect the equilibrium path nontrivially.¹¹ Similarly, in our notion of generalized Nash equilibrium, no requirements are placed on what

happens at information sets that players do not believe could have happened. More precisely, if at history h of game Γ^+ , player i believes that he is moving at the information set I of game Γ^h and the local strategies of the players at game Γ^h reach I with probability zero, then although player i believes that he is at I , he also believes that I should not have happened and, consequently, i believes that some player acted irrationally. At such an information set I , our notion of generalized Nash equilibrium does not impose any restriction on what i might do, in the same spirit that standard Nash equilibrium does not impose any restriction on information sets that are reached with probability zero. Later on, in this section, we give other arguably more natural interpretations of generalized Nash equilibrium.

Let $B_{i,I'}$ be the set of all local strategies $\sigma_{i,I'}$ for player i at game Γ' . A *generalized strategy profile* of $\Gamma^* = (\mathcal{G}, \mathcal{F})$ is an element of the Cartesian product $\times_{i \in N} \times_{\Gamma' \in \mathcal{G}_i} B_{i,I'}$. That is, a generalized strategy profile $\vec{\sigma}$ consists of a strategy for each agent i and game $\Gamma' \in \mathcal{G}_i$. Intuitively, $\sigma_{i,I'}$ is the strategy that i plays if he believes the game is Γ' . Let $EU_{i,I'}(\vec{\sigma})$ be the expected payoff for i in the game Γ' given that strategy profile $\vec{\sigma}$ is used. Note that the only strategies in $\vec{\sigma}$ that are needed to compute $EU_{i,I'}(\vec{\sigma})$ are the strategies actually used in Γ' ; indeed, all that is needed is the restriction of these strategies to information sets that arise in Γ' . But the strategies used do not necessarily have the form $\sigma_{j,I'}$. If at some history h in Γ' where j moves j believes that the game is actually Γ'' , then j will use the strategy $\sigma_{j,I''}$, not $\sigma_{j,I'}$.

A *generalized Nash equilibrium* of $\Gamma^* = (\mathcal{G}, \mathcal{F})$ is a generalized strategy profile $\vec{\sigma}$ such that for all $\Gamma' \in \mathcal{G}_i$, the local strategy $\sigma_{i,I'}$ is a best response to $\vec{\sigma}_{-(i,I')}$, where $\vec{\sigma}_{-(i,I')}$ is the set of all local strategies in $\vec{\sigma}$ other than $\sigma_{i,I'}$.

Definition 3.2. A generalized strategy profile $\vec{\sigma}^*$ is a *generalized Nash equilibrium* of a game $\Gamma^* = (\mathcal{G}, \mathcal{F})$ with awareness if, for every player i , game $\Gamma' \in \mathcal{G}_i$, and local strategy σ for i in Γ' ,

$$EU_{i,I'}(\vec{\sigma}^*) \geq EU_{i,I'}((\vec{\sigma}^*_{-(i,I')}, \sigma)).$$

Formally, this looks similar in spirit to the definition of Nash equilibrium. But there is a key difference. The definition of Nash equilibrium in standard games implicitly assumes that player i can choose a whole strategy. This is inappropriate in our setting. An agent cannot anticipate that he will become aware of more moves. Essentially, if $\Gamma_1 \neq \Gamma_2$, we are treating player i who considers the true game to be Γ_1 to be a different agent from the version of player i who considers Γ_2 to be the true game. To understand why this is appropriate, suppose that player i considers Γ_1 to be the true game, and then learns about more moves, and so considers Γ_2 to be the true game. At that point, it is too late for player i to change the strategy he was playing when he thought the game was Γ_1 . He should just try to play optimally for what he now considers the true game. Moreover, while player i thinks that the game Γ_1 is the true game, he never considers it possible that he will ever be playing a different game, so that he cannot “prepare himself” for a change in his subjective view of the game.

In a sense, Definition 3.2 requires that $\sigma_{i,I'}^*$ be a best response to $\vec{\sigma}^*_{-(i,I')}$. It may seem strange to require this when $\vec{\sigma}^*_{-(i,I')}$ may contain strategies that involve moves that player i is not aware of in Γ' . Let $\vec{\sigma}^\dagger$ consist of all local strategies $\sigma_{j,I''}$ such that Γ'' does not contain any moves not in Γ' . Then it is easy to see that $\sigma_{i,I'}^*$ is a best response to $\vec{\sigma}^*_{-(i,I')}$ (in the sense of Definition 3.2) iff $\sigma_{i,I'}^*$ is a best response to $\vec{\sigma}^\dagger_{-(i,I')}$. Roughly speaking, this says that what happens in games with moves that player i is not aware of is irrelevant. It is easy to see that $\vec{\sigma}$ is a Nash equilibrium of a standard game iff $\vec{\sigma}$ is a (generalized) Nash equilibrium of the canonical representation of Γ as a game with awareness. Thus, our definition of generalized Nash equilibrium generalizes the standard definition.

¹⁰ Although common knowledge is not necessary for Nash equilibrium (see Aumann and Brandenburger, 1995), it is a sufficient condition. Moreover, as Aumann (1987) shows, a strategy profile $\vec{\sigma}$ is a Nash equilibrium iff there is a model where rationality is common knowledge and it is common knowledge that $\vec{\sigma}$ is played.

¹¹ This does not mean that a player does not care about what happens off the equilibrium path. As pointed out by Burkhard Schipper (private communication, 2012), some Nash equilibria exist only because players do believe in incredible threats off the equilibrium path. Indeed, off the equilibrium path, players may not play best responses and might want to make incredible threats, which, by definition, are not best responses.

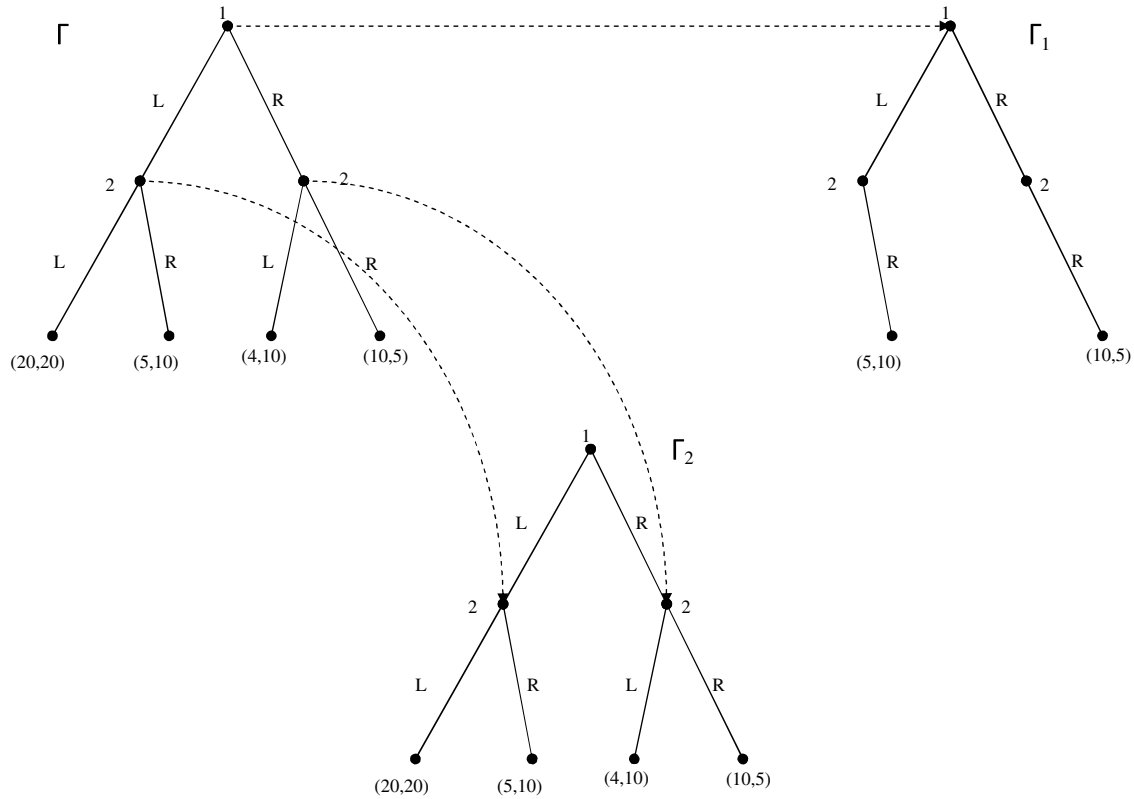


Fig. 4. A game where players may have false beliefs about whether they are on the equilibrium path.

The question still remains as to the appropriateness of generalized Nash equilibrium in games of awareness. Indeed, even in standard extensive games, refinements of Nash equilibrium, such as sequential equilibrium or perfect equilibrium, are considered more appropriate than Nash equilibrium. We are interested in examining such equilibrium refinements in the context of games of awareness, and do so in a companion paper (Rêgo and Halpern, 2012).

Games with awareness introduce subtleties that do not arise in a standard game. In order to understand such situations, we define an information set I of a game Γ^+ to be on the equilibrium path of a generalized Nash equilibrium $\vec{\sigma}$ if the local strategies in $\vec{\sigma}$ that have as domain pairs of the form (Γ^+, \cdot) give I positive probability. If I is not on the equilibrium path, we say that it is off the equilibrium path. In games with awareness, a player may have false beliefs about whether he is (or was) on the equilibrium path. Consider the following game with awareness, where \mathcal{G} consists of the three games Γ , Γ_1 , Γ_2 shown in Fig. 4. Dashed lines depict the function \mathcal{F} ; we omit self-loops of the \mathcal{F} function for better display.

This game has two generalized Nash equilibria in pure strategies, $(\sigma_{1,\Gamma_1}, \sigma_{2,\Gamma_1}, \sigma_{1,\Gamma_2}, \sigma_{2,\Gamma_2})$ and $(\sigma_{1,\Gamma_1}, \sigma_{2,\Gamma_1}, \sigma_{1,\Gamma_2}, \sigma'_{2,\Gamma_2})$, which differ only in player 2's local strategy in Γ_2 ; strategy σ_{1,Γ_1} plays R at the two histories in its domain, $(\Gamma, \langle \rangle)$ and $(\Gamma_1, \langle \rangle)$; σ_{2,Γ_1} plays R at the two histories in its domain, $(\Gamma_1, \langle L \rangle)$ and $(\Gamma_1, \langle R \rangle)$; σ_{1,Γ_2} plays L at $(\Gamma_2, \langle \rangle)$, the only history in its domain; σ_{2,Γ_2} plays L at the two histories in its domain, $(\Gamma_2, \langle L \rangle)$ and $(\Gamma_2, \langle R \rangle)$; and σ'_{2,Γ_2} plays L at $(\Gamma_2, \langle L \rangle)$ and R at $(\Gamma_2, \langle R \rangle)$.

In the first generalized Nash equilibrium, a player 2 has false beliefs about whether he is on the equilibrium path. For example, at history $\langle L \rangle$ in Γ , player 2 falsely believes that he is on the equilibrium path; on the other hand, at history $\langle R \rangle$ in Γ , player 2 falsely believes that he is off the equilibrium path, since he falsely believes that player 1 should have played L in game Γ_2 . The second generalized Nash equilibrium seems strange because at history $\langle R \rangle$ in the objective game Γ , player 2 is on the equilibrium path

but plays a dominated action, because he falsely believes that he is off the equilibrium path. As we said earlier, generalized Nash equilibrium imposes no requirements at or after histories that players believe should not have happened. If we think of players as choosing their strategies at the beginning of the game, then player 2 can be viewed as reasoning as follows: in all equilibria of the game that I think I am playing, player 1 will play L; therefore, I will play L myself after player 1 has played L, and can do anything in the case player 1 plays R, because I think this happens with probability zero.

The reasoning above makes sense because player 2 moves only once in this game. But suppose we change the game by introducing a dummy move for player 2 at the beginning of game Γ , after which the game continues just as before. Moreover, suppose that at this new initial history player 2 believes that the game is Γ_1 , but when he moves for the second time he thinks that it is common knowledge that Γ_2 is being played. As before, player 1 in the objective game believes that the game is Γ_1 ; thus, in equilibrium, player 1 chooses R. When player 2 actually moves, she already saw player 1 choosing R, so she must believe that player 1 made a mistake, since in game Γ_2 , player 1 should have played L. Thus, this example shows that it is possible to reach an information set on the equilibrium path where player 2 moves but player 2 believes that he is at an information set in a different game, one to which he assigns probability 0, and thus there are no constraints on what he must do there in a generalized Nash equilibrium. In a companion paper (Rêgo and Halpern, 2012), we generalize sequential equilibria to games with awareness, and this notion avoid player 2 playing dominated actions on (and off) the equilibrium path.

As we said in the introduction, we study generalized Nash equilibrium here not because we believe it is necessarily the “right” solution concept, but because we believe that an understanding of it will be essential for considering other solution concepts in games with awareness. Not surprisingly, the definition of sequential

equilibrium in games of awareness in Rêgo and Halpern (2012) is very much based on the definition of Nash equilibrium given here.

A generalized Nash equilibrium is perhaps best interpreted as an equilibrium in beliefs. For example, in the game of awareness based on the game in Fig. 1, in the objective game, neither A nor B actually believes that the true game is the game Γ^B in Fig. 3, and thus neither ever plays the strategy associated with that game. However, A believes that B might be playing this game (with probability p) and B knows this, so both A and B have to consider the game Γ^B in their reasoning, and must have beliefs about what strategies will be used in this game.¹² Thus, under this interpretation, in a generalized Nash equilibrium, every player makes a best response to his beliefs in all circumstances. Of course, the question then arises as to where the beliefs are coming from.

As pointed out by Burkhard Schipper (private communication, 2012), in some cases, there is a natural source of beliefs and equilibrium convention. For example, someone might drive on the right-hand side of the road in England because he is unaware of the local rules. When he sees people driving on the left, he becomes aware of the rules. Learning the rules changes the agent's awareness and his knowledge, as well as his beliefs about the equilibrium convention.

In standard games, the usual argument is that players learn what strategies other players are playing over time. Thus, over time, players will learn to play a Nash equilibrium, for example, by playing a best response to their current beliefs. (However, this is not true in general (Nachbar, 1997, 2005).) Arguments involving learning do not lift well to games with awareness, since playing the game repeatedly can make players aware of moves or of other players' awareness, and thus effectively change the game altogether. We do not have a good story for where these beliefs come from in general (any more than there is a general good story about where beliefs might come from in a standard game that is being played for the first time). It may well make sense to make the choice of beliefs part of the solution concept, as is done by Ozbay (2007).

Taking the beliefs as given, the following examples show that generalized Nash equilibrium does the “right” thing in a number of settings.

Example 3.1. Consider the game with awareness shown in Figs. 1–3. We have $\mathcal{G}_A = \{\Gamma^A, \Gamma^B\}$ and $\mathcal{G}_B = \{\Gamma, \Gamma^B\}$. Taking $\text{dom}(\sigma_{i,\Gamma'})$ to denote the domain of the strategy $\sigma_{i,\Gamma'}$, we have

$$\begin{aligned} \text{dom}(\sigma_{A,\Gamma^A}) &= \{(\Gamma, \langle \rangle), (\Gamma^A, \langle \text{unaware} \rangle), (\Gamma^A, \langle \text{aware} \rangle)\}, \\ \text{dom}(\sigma_{B,\Gamma}) &= \{(\Gamma, \langle \text{across}_A \rangle), (\Gamma^A, \langle \text{aware}, \text{across}_A \rangle)\}, \\ \text{dom}(\sigma_{A,\Gamma^B}) &= \{(\Gamma^B, \langle \rangle)\}, \text{ and} \\ \text{dom}(\sigma_{B,\Gamma^B}) &= \{(\Gamma^A, \langle \text{unaware}, \text{across}_A \rangle), (\Gamma^B, \langle \text{across}_A \rangle)\}. \end{aligned}$$

Each of these domains consists of a single generalized information set. If $p < 1/2$, then there exists a generalized Nash equilibrium where $\sigma_{A,\Gamma^A} = \text{across}_A$, $\sigma_{A,\Gamma^B} = \text{down}_A$, $\sigma_{B,\Gamma} = \text{down}_B$, $\sigma_{B,\Gamma^B} = \text{across}_B$. Thus, in the objective game, A plays across_A , B plays down_B , and the resulting payoff vector is $(2, 3)$. On the other hand, if $p > 1/2$, then there exists a generalized Nash equilibrium where $\sigma_{A,\Gamma^A} = \text{down}_A$, $\sigma_{A,\Gamma^B} = \text{down}_A$, $\sigma_{B,\Gamma} = \text{down}_B$, $\sigma_{B,\Gamma^B} = \text{across}_B$. Thus, in the objective game, A plays down_A , and the payoff vector is $(1, 1)$. Intuitively, even though both A and B are aware of all the moves in the objective game, A considers it sufficiently likely that B is not aware of down_B , so A plays down_A . There exists another generalized Nash equilibrium where $\sigma_{A,\Gamma^A} = \text{down}_A$, $\sigma_{A,\Gamma^B} = \text{down}_A$, $\sigma_{B,\Gamma} = \text{across}_B$, and $\sigma_{B,\Gamma^B} = \text{across}_B$ that holds for any value of p . Intuitively, A believes B will play across_B no matter what

he (B) is aware of, and therefore plays down_A ; given that A plays down_A , B cannot improve by playing down_B even if he is aware of that move.¹³

Example 3.2. The following game is due to Feinberg (2005) and was also discussed by Heifetz et al. (2006b) and Meier and Schipper (2012). They considered only normal-form games and, in particular, their example was presented as a normal-form game, so we present it that way too. In the game, there are two players, Alice and Bob. Each player has three possible actions; Alice's actions are a_1 , a_2 , and a_3 , while Bob's actions are b_1 , b_2 , and b_3 . The payoffs of the game are given by the following table:

	b_1	b_2	b_3
a_1	0, 2	3, 3	0, 2
a_2	2, 2	2, 1	2, 1
a_3	1, 0	4, 0	0, 1

Suppose that both players are aware of all actions of the game, but Alice believes that Bob is unaware of action a_3 . Furthermore, Bob knows that Alice believes that he is unaware of action a_3 . To model this game in our framework, we view it as an extensive game where Alice moves first, but Bob does not know what her move is. We need three games in addition to the objective game Γ : the game Γ^A , which describes the game from Alice's point of view, the game Γ^B , which describes the game from Bob's point of view, and the game Γ^{AB} , which describes the game that Alice believes that Bob believes is the actual game. Each of Γ^A , Γ^B , and Γ^{AB} start with an initial “awareness” move of nature, denoted c^A , c^B , c^{AB} . The move c^B makes both players aware of all moves; the move c^A makes only Alice aware of a_3 (but both players are aware of all other moves); the move c^{AB} makes neither player aware of a_3 (but, again, both players are aware of all other moves). Otherwise, the games are the same, except that in Γ^{AB} , Alice does not have the option of making move a_3 .

Γ is given by the extensive representation of the game given in the table above (where Alice moves first). Game Γ^A is given by

- $N^A = \{A, B\}$;
- $M^A = \{c^A, a_1, a_2, a_3, b_1, b_2, b_3\}$;
- $H^A = \{\langle \rangle, \langle c^A \rangle, \langle c^A, a_1 \rangle, \langle c^A, a_2 \rangle, \langle c^A, a_3 \rangle, \langle c^A, a_1, b_1 \rangle, \dots, \langle c^A, a_3, b_3 \rangle\}$;
- $P^A(\langle \rangle) = c$, $P^A(\langle c^A \rangle) = A$, $P^A(\langle c^A, a_1 \rangle) = P^A(\langle c^A, a_2 \rangle) = P^A(\langle c^A, a_3 \rangle) = B$;
- $\mathcal{I}_A^A = \{\{\langle c^A \rangle\}\}$, $\mathcal{I}_B^A = \{\{\langle c^A, a_1 \rangle, \langle c^A, a_2 \rangle, \langle c^A, a_3 \rangle\}\}$;
- u_A^A and u_B^A are given as in the table above.

Γ^B is similar to Γ^A (with the superscript A replaced by B everywhere); finally, Γ^{AB} is similar to Γ^A (with the superscript A replaced by AB everywhere), except that M^{AB} is the subset of M^A consisting of all moves that do not mention a_3 , H^{AB} is the subset of H^A consisting of all histories that do not mention a_3 , and $\mathcal{I}_B^{AB} = \{\{\langle c^{AB}, a_1 \rangle, \langle c^{AB}, a_2 \rangle\}\}$.

The real differences between the games is captured by the \mathcal{F} function.

- $\mathcal{F}(\Gamma, \langle \rangle) = \mathcal{F}(\Gamma^A, \langle c^A \rangle) = \mathcal{F}(\Gamma^B, \langle c^B \rangle) = (\Gamma^A, \{\langle c^A \rangle\})$;
- $\mathcal{F}(\Gamma^{AB}, \langle c^{AB} \rangle) = (\Gamma^{AB}, \{\langle c^{AB} \rangle\})$;
- if $h \in \{\langle a_1 \rangle, \langle a_2 \rangle, \langle a_3 \rangle\}$, then $\mathcal{F}(\Gamma, h) = (\Gamma^B, I_B^B)$;
- if $h \in I_B^B$, then $\mathcal{F}(\Gamma^B, h) = (\Gamma^B, I_B^B)$;
- if $h \in I_B^A$, then $\mathcal{F}(\Gamma^A, h) = (\Gamma^B, I_B^{AB})$;
- if $h \in I_B^{AB}$, then $\mathcal{F}(\Gamma^{AB}, h) = (\Gamma^B, I_B^{AB})$.

This game has two generalized Nash equilibria.

¹² We thank an anonymous referee for stressing this point.

¹³ We did not discuss this latter equilibrium in the preliminary version of this paper.

1. $\sigma_{A, \Gamma^A}(a_2) = 1$, $\sigma_{A, \Gamma^{AB}}(a_2) = 1$, $\sigma_{B, \Gamma^B}(b_1) = 1$, and $\sigma_{B, \Gamma^{AB}}(b_1) = 1$; and
2. $\sigma_{A, \Gamma^A}(a_3) = 1$, $\sigma_{A, \Gamma^{AB}}(a_1) = 1$, $\sigma_{B, \Gamma^B}(b_3) = 1$, and $\sigma_{B, \Gamma^{AB}}(b_2) = 1$.

In the first equilibrium, Alice plays a_2 in both Γ^A and Γ^{AB} (the two games in \mathcal{G}_A), and Bob plays b_1 in both Γ^B and Γ^{AB} (the two games in \mathcal{G}_B). In the second equilibrium, Alice plays a_3 in Γ^A and a_1 in Γ^{AB} , while Bob plays b_3 in Γ^B and b_2 in Γ^{AB} . In the first equilibrium, the actual payoff is (2, 2) and both players (correctly) believe that they will get this payoff. In the second equilibrium, the actual payoff is (0, 1), but Alice believes that she will get a payoff of 4 and that Bob will get a payoff of 0. Since the Nash equilibrium (a_1, b_2) Pareto dominates the Nash equilibrium (a_2, b_1) in the game Γ^{AB} , the second generalized Nash equilibrium is likely to be played, even though both players would benefit from playing the first generalized Nash equilibrium. Thus, even though the objective payoffs of the first equilibrium are better for both players, Alice falsely believes that her payoff in the second equilibrium is better, according to her limited view of the game.

These two generalized Nash equilibria are also the two equilibria in this game according to the models used by Heifetz et al. (2006b) and Feinberg (2005), although, as Heifetz et al. point out, in Feinberg's model, what we (and Heifetz et al.) model as Alice believing that Bob is unaware of a_3 is modeled as Alice being unaware that Bob is aware of a_3 . Since Feinberg does not define a notion of belief in his framework, he needs to use higher-order unawareness to capture the analogue of belief.

The fact that we have the same equilibria as Heifetz et al. is not a coincidence. The intuition behind their equilibrium concept is identical to ours. In fact, if we restrict to normal-form games (which can easily be done; see Barreto, 2008 for details), their model is similar to ours, except that they do not insist that the objective game be part of \mathcal{G} (roughly speaking, for them, there is no objective game) and they do not require that a player must know the set of histories of the objective game he is aware of, and they have state-dependent utilities. There are some subtle differences between our framework and that of Feinberg, although in many cases (such as this one) the equilibria are the same; see Section 5 for further discussion. \square

Example 3.3 (*Finitely Repeated Prisoners' Dilemma with a Possibly Unaware Player*). In prisoners' dilemma, there are two players, Alice and Bob, that act simultaneously, and they have two actions they can perform: cooperate and defect. If both defect, each gets a payoff of 0; if both cooperate, they each get a payoff of 1; if one cooperates and the other defects, the cooperator gets b and the defector gets a , where $a > 1$, $b < 0$, and $a + b < 2$. (The latter requirement makes cooperating better than alternating between defect-cooperate and cooperate-defect.) In *finitely repeated prisoners' dilemma* (FRPD), the basic prisoners' dilemma is played a finite number of times; the payoffs are the same each time it is played. For each player, the payoff for the whole game is the undiscounted sum of the payoff at each period.

It is well known that a standard backward induction argument shows that rational agents playing FRPD will always defect. Kreps et al. (1982) showed that by adding a player who, with very small probability, is irrational and plays tit-for-tat (and thus cooperates as long as the other player does) it is possible to get cooperation in FRPD for almost the whole game. Feinberg (2004) showed that we can replace irrationality by unawareness. Specifically, he considered a variant of FRPD where there is a small probability δ that Bob is not initially aware of the possibility of defection, although Alice is. If Bob is unaware of defection, then the only way that he can learn that defection is a possible move is if he sees Alice defect. Moreover, if Bob does see Alice defect, then he will in fact learn the actual game. Alice knows all this, as does Bob, if he is

aware of defection. Feinberg shows that the small probability of unawareness has the same effect as a small probability that one of the agents is irrational: there is a unique equilibrium (in his sense) of this game, and in this equilibrium, there is cooperation for all but the final k rounds, where k depends only on a , b , and δ (and not the length of the game). In our framework, all generalized Nash equilibria yield the same result. When we model the game in our framework, there are only two games. One describes the objective game; it is also the game from Alice's perspective and the game from Bob's perspective if he is aware of the possibility of defection. The other is the game where Alice and Bob can only cooperate; this is the game from Bob's perspective if he is unaware. The proof of the result is quite similar in spirit to that of Kreps et al. (1982) (as is Feinberg's proof), so we omit details here. \square

Example 3.3 is useful for clarifying how players act on and off an equilibrium path. Suppose that Bob starts cooperating and then Alice defects. At that point, Bob becomes aware of the possibility of defection at all times, so that the game that he considers possible is the true (standard) repeated prisoner's dilemma Γ' , where both players can cooperate and defect. In the game Γ' considered in isolation, the history *(cooperate, defect)* is off the standard Nash equilibrium. However, it is on the equilibrium path in the game of awareness. In the game of awareness, when Bob moves after the history *(cooperate, defect)*, he remembers that initially he was unaware of defection. Therefore, cooperating was a best response (indeed, the only possible response from Bob's point of view) initially.

The equilibrium in **Example 3.3** depends on the assumption that, when Alice defects, Bob knows the whole structure of the game. In more realistic settings, once Bob discovers that there is a move that Alice is aware of that he was not aware of, he may consider it possible that there are other moves that Alice can make that he is unaware of. To model this type of example, we need to extend our representation to games with awareness of unawareness; we do this in Section 4. But before we consider that issue, we show that generalized Nash equilibria always exist.

3.3. The existence of generalized Nash equilibrium

We now show that every game with awareness has at least one generalized Nash equilibrium. The proof adds some insight into the connection between games of awareness and standard games.

We proceed as follows. Given a game $\Gamma^* = (\mathcal{G}, \mathcal{F})$ with awareness, let ν be a probability on \mathcal{G} that assigns each game in \mathcal{G} positive probability. (Here is where we use the fact that \mathcal{G} is countable.) We construct a standard extensive game Γ^ν by essentially "gluing together" all the games $\Gamma' \in \mathcal{G}$, except that we restrict to the histories in Γ' that are plausible (in the sense of C5). Formally, for each $\Gamma' \in \mathcal{G}$, we restrict to the histories $[H'] = \{h \in H' : \text{for every prefix } h_1 \cdot \langle m \rangle \text{ of } h, \text{ if } P'(h_1) = i \in N \text{ and } \mathcal{F}(\Gamma', h_1) = (\Gamma'', I), \text{ then for all } h_2 \in I, h_2 \cdot \langle m \rangle \in H''\}\}$. As we shall see, all the components of Γ^ν are independent of ν except for nature's initial move (as encoded by f_c^ν). In Γ^ν , the set of players is $\{(i, \Gamma') : \Gamma' \in \mathcal{G}_i\}$. The game tree of Γ^ν can be viewed as the union of the pruned game trees of $\Gamma' \in \mathcal{G}$. The histories of Γ^ν have the form $\langle \Gamma' \rangle \cdot h$, where $\Gamma' \in \mathcal{G}$ and $h \in [H']$. The move that a player or nature makes at a history $\langle \Gamma' \rangle \cdot h$ of Γ^ν is the same as the move made at h when viewed as a history of Γ' . The only move in Γ^ν not determined by Γ^* is nature's initial move (at the history $\langle \rangle$), where nature chooses the game $\Gamma' \in \mathcal{G}$ with probability $\nu(\Gamma')$.

Formally, let Γ^ν be a standard game such that

- $N^\nu = \{(i, \Gamma') : \Gamma' \in \mathcal{G}_i\}$;
- $M^\nu = \mathcal{G} \cup_{\Gamma' \in \mathcal{G}} [M']$, where $[M']$ is the set of moves that occur in $[H']$;

- $H^v = \langle \rangle \cup \{ \langle \Gamma' \rangle \cdot h : \Gamma' \in \mathcal{G}, h \in [H'] \}$;
 - $P^v(\langle \rangle) = c$, and
- $$P^v(\langle \Gamma^h \rangle \cdot h') = \begin{cases} (i, \Gamma^h) & \text{if } P^h(h') = i \in N \\ & \text{and } \mathcal{F}(\Gamma^h, h') = (\Gamma^h, \cdot), \\ c & \text{if } P^h(h') = c; \end{cases}$$
- $f_c^v(\Gamma' | \langle \rangle) = v(\Gamma')$ and $f_c^v(\cdot | \langle \Gamma^h \rangle \cdot h') = f_c^h(\cdot | h')$ if $P^h(h') = c$;
 - $\mathcal{I}_{i, \Gamma'}^v$ is just the \sim_i relation restricted to histories $\langle \Gamma'' \rangle \cdot h \in H^v$ where i moves and $\mathcal{F}(\Gamma'', h)$ has the form (Γ', \cdot) ;
 - $u_{i, \Gamma'}^v(\langle \Gamma^h \rangle \cdot z) = \begin{cases} u_i^h(z) & \text{if } \Gamma^h = \Gamma', \\ 0 & \text{if } \Gamma^h \neq \Gamma'. \end{cases}$

Fig. 5 illustrates the game Γ^v corresponding to the original running example described on Section 2. The payoffs are described for the players (A, Γ^A) , (B, Γ) , (A, Γ^B) and (B, Γ^B) , respectively.

Theorem 3.1. For all probability measures v on \mathcal{G}

- Γ^v is a standard extensive game with perfect recall; and
- if v gives positive probability to all games in \mathcal{G} , then $\bar{\sigma}$ is a Nash equilibrium of Γ^v iff $\bar{\sigma}'$ is a generalized Nash equilibrium of Γ^* , where $\sigma_{i, \Gamma'}(\langle \Gamma^h \rangle \cdot h') = \sigma'_{i, \Gamma'}(\Gamma^h, h')$.

Although a Nash equilibrium does not necessarily exist in games with infinitely many players, Γ^v has three special properties: (a) each player has only finitely many information sets, and (b) for each player (i, Γ') , there exists a finite subset $N(i, \Gamma')$ of N^v such that (i, Γ') 's payoff in Γ^v depends only on the strategies of the players in $N(i, \Gamma')$, and (c) Γ^v is a game with perfect recall. This turns out to be enough to show that Γ^v has at least one Nash equilibrium. Thus, we get the following corollary to Theorem 3.1.

Corollary 3.1. Every game with awareness has a generalized Nash equilibrium.

In light of this proof, it is reasonable to ask if we need games of awareness at all. Perhaps we could just model the lack of awareness using a standard game such as that used in the proof. One obvious problem with this approach is that the standard game does not capture all the intuitions we have about awareness. This concern about the conceptual interpretation of Γ^v is particularly relevant when we consider the issue of what the “right” notion of equilibrium for games of awareness is. All that the proof of Theorem 3.1 shows is that there is a one-to-one correspondence between the generalized Nash equilibria of Γ^* and the Nash equilibria of Γ^v . It does not say anything about other solution concepts. (Indeed, as shown in Rêgo and Halpern (2012), the same statement does not hold true if we replace “Nash equilibrium” by “sequential equilibrium”.) Given that it is still far from clear what the right solution concept is for games with awareness, it seems premature to remove considerations of awareness from the game before we have really studied the impact of awareness. Heifetz et al. (2011) discussion of forward induction in the presence of unawareness emphasizes this point.

4. Modeling awareness of unawareness

In this section, we describe how to extend our representation of games with awareness to deal with awareness of unawareness. In an extensive game that represents player i 's subjective view of the game, we want to model the fact that i may be aware of the fact that j can make moves at a history h that i is not aware of. We do this by allowing j to make a “virtual move” at history h . Histories that contain virtual moves are called *virtual histories*. These virtual histories do not necessarily correspond to a history in the objective game Γ (i.e., i may falsely believe that j can make a move at h that he is unaware of), and even if a virtual history does correspond to a history in Γ , the subgame that follows that virtual

history may bear no relationship to the actual subgame that follows the corresponding history in the objective game Γ . Intuitively, the virtual histories describe agent i 's (possibly incomplete and possibly incorrect) view of what would happen in the game if some move she is unaware of is made by agent j . Player j may have several virtual moves available at history h , and may make virtual moves at a number of histories in the extensive game.¹⁴ Note that agent i 's subjective game may include virtual moves for i himself; i may believe that he will become aware of more moves (and may take active steps to try and learn about these moves).

To handle awareness of unawareness, we consider a generalization of the notion of an extensive game based on Γ . We continue to refer to the generalized notion as an extensive game based on Γ . Formally, $\Gamma^+ = (N^+, M^+, H^+, P^+, f_c^+, \{Z_i^+ : i \in N^+\}, \{u_i^+ : i \in N^+\})$ is an extensive game formed by the (standard) finite extensive game $\Gamma = (N, M, H, P, f_c, \{Z_i : i \in N\}, \{u_i : i \in N\})$ if it satisfies conditions A1 and A6, and variants of A2–A5.¹⁵ Before stating these variants, we need to define formally the set of virtual histories of Γ^+ . Intuitively, these are histories that, after removing moves of nature that change players' awareness, do not correspond to histories in H . Thus, a history is virtual if there exists some prefix of that history where either (a) some player made a move not in the objective game, or (b) it was nature's turn to move in the corresponding history in the objective game and nature made a move not available to her in the objective game. As before, \bar{h} is the subsequence of h consisting of all moves in h that are also in M . Formally, the set V^+ of virtual histories of Γ^+ consists of all histories h such that for some prefix $h' \cdot \langle m \rangle$ of h , we have $\bar{h}' \in H$, $P^+(h') \in N^+$ or $P^+(h') = c = P(\bar{h}')$, and $m \in M^+ - M$.

We can now state the variants of A2–A5. They are essentially like the original versions, except we restrict to non-virtual histories.

- If $h \notin V^+$, then $\bar{h} \in H$, and if $P^+(h) \in N^+$, then $P^+(h) = P(\bar{h})$ and $M_h^+ \subseteq M_{\bar{h}} \cup (M^+ - M)$. Moreover, if h and h' are histories in Γ^+ , $h' \notin V^+$, \bar{h} and \bar{h}' are in the same information set for player i in Γ , player i moves at h and h' , $h \cdot \langle m \rangle$ is a history in Γ^+ , then $h' \cdot \langle m \rangle$ is a history in Γ^+ .
- If $P^+(h) = c$ and $h \notin V^+$, then either $P(\bar{h}) = c$ and $M_h^+ \subseteq M_{\bar{h}} \cup (M^+ - M)$, or $P(\bar{h}) \neq c$ and $M_h^+ \cap M = \emptyset$.
- If $h, h' \notin V^+$, then h and h' are in the same information set for player i in Γ^+ iff i has the same view in both h and h' .
- $\{z : z \in (Z^+ - V^+)\} \subseteq Z$. Thus, the non-virtual runs in Z^+ correspond to runs in Z .

A game with awareness of unawareness based on Γ is defined as a pair $\Gamma^* = (\mathcal{G}, \mathcal{F})$ with $\Gamma \in \mathcal{G}$ just as before. If $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, I)$, then C5 must still hold, as well as variants of C1–C4, and new conditions C6–C8, given below.

- If $h \in H^+ - V^+$ and $h' \in H^h - V^h$, then $h' \in I$ iff i has the same view in h and h' .
- If $h' \in H^h - V^h$, then there exists $h_1 \in H^+$ such that $\bar{h}' = \bar{h}_1$. Moreover, if $h' \cdot \langle m \rangle \in H^h - V^h$ and $m \in M$, then for all $h_1 \in H^+ - V^+$ such that $\bar{h}' = \bar{h}_1$ and $P^h(h') = P^+(h_1)$, we have that $h_1 \cdot \langle m \rangle \in H^+$.

¹⁴ In the preliminary version of the paper, we assumed that all virtual moves were terminal moves. This is appropriate if i has no idea at all of what will happen in the game after a virtual move is made. The greater generality we allow here is useful to model situations where player i has some partial understanding of the game. For example, i may know that he can move left after j 's virtual move, no matter what that virtual move is.

¹⁵ We could also relax A1 to allow some “virtual players”. We do not do that here for ease of exposition.

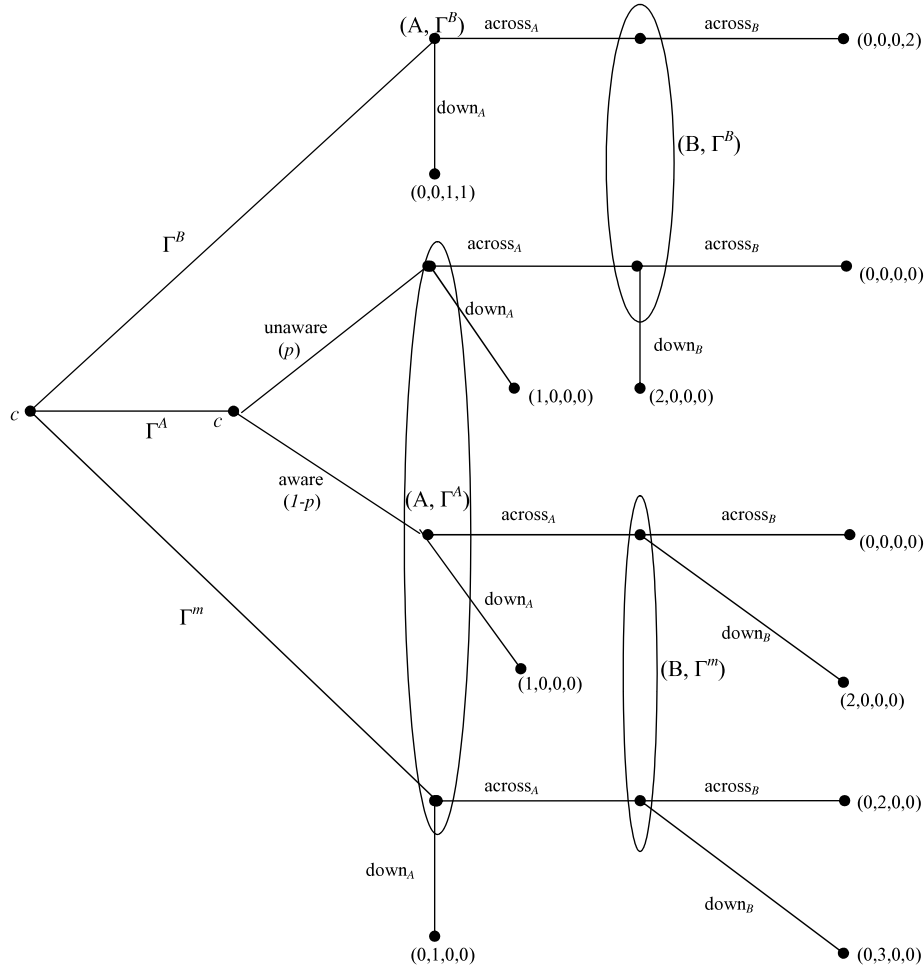


Fig. 5. The game Γ^v corresponding to the game described in Section 2.

- C3'. If h_1 is in the same information set as h and h_2 is a history in I , then $\mathcal{F}(\Gamma^+, h_1) = \mathcal{F}(\Gamma^h, h_2) = \mathcal{F}(\Gamma^+, h)$. Moreover, if h' and h'' are non-virtual histories in Γ' and Γ'' , respectively, and player i has the same view in both h' and h'' , then $\mathcal{F}(\Gamma', h') = \mathcal{F}(\Gamma'', h'')$.
- C4'. If h' is a prefix of h , $P^+(h') = i$, and $\mathcal{F}(\Gamma^+, h') = (\Gamma^{h'}, I')$, then $\{\bar{h}'_1 : h'_1 \in H^{h'}\} \subseteq \{\bar{h}_1 : h_1 \in H^h\}$.
- C6. For all $h' \in I$, we have $M_{h'}^h \subseteq M_h^+$.
- C7. For all histories $h' \in I$, there exists a prefix h'_1 of h' such that $P^h(h'_1) = i$ and $\mathcal{F}(\Gamma^h, h'_1) = (\Gamma', I')$ iff there exists a prefix h_1 of h such that $P^+(h_1) = i$ and $\mathcal{F}(\Gamma^+, h_1) = (\Gamma', I')$. Moreover, $h'_1 \cdot \langle m \rangle$ is a prefix of h' iff $h_1 \cdot \langle m \rangle$ is a prefix of h .
- C8. If $h' \in I$, $h'' \in H^h$, $P^h(h'') = i$, and h'' is a suffix of h' , then $\mathcal{F}(\Gamma^h, h'') = (\Gamma^h, \cdot)$.

C1' and C3' have been weakened so that these restrictions only apply to non-virtual histories of Γ^h . The first part of C2' is the same as the first part of C2; the second is the weakening of the second part of C2 applied only to non-virtual histories. C4' is the first half of C4; the second half of C4 does not hold once we allow virtual moves; player i may consider more virtual moves possible at h than at h' . Then we might have $\Gamma^h \neq \Gamma^{h'}$ even though in both h and h' the player is aware of the same set of moves of the objective game. C6 just says that i does not have moves available at a history in I in Γ^h that he does not also have available at h . Intuitively, if Γ^+ is j 's view of the game, then from j 's point of view, when i moves at h , he (i) cannot mistakenly believe that a move is possible that is actually not possible. (It is not clear what it would mean for

i to have such mistaken beliefs. What would happen if i tried to make the move that he considered possible, but was actually not possible?) It is not hard to check that in games without awareness of unawareness, C6 follows from C1 and C2. However, now that C2 has been weakened to C2', we must mention it explicitly. C7 says that the players remember the set of games that they considered possible and the moves they made. It is also not hard to check that in games without awareness of unawareness, C7 follows from C1, C3 and the fact that Γ is a game with perfect recall. Since C1 and C3 have been weakened to C1' and C3', we must mention it explicitly. C8 says that a player cannot believe that in the future he will consider possible a different game. Note that this does not rule out a situation where a player i realizes at history h' that his view of the game will change at a future history h'' when he receives some additional information. If this is the case, then this should already be described at h' . C8 follows from C2, C3 and C4 in games without awareness of unawareness.

We can now define local strategies, generalized strategy profiles, and generalized Nash equilibrium just as we did for games with awareness. The same technique as that used to prove Corollary 3.1 can be used to prove the following.

Theorem 4.1. *Every game with awareness of unawareness has a generalized Nash equilibrium.*

Example 4.1 (Chess). We are now able to describe in greater detail a representation of chess as a game with awareness of unawareness. We take the objective game Γ to be the standard representation of chess as an extensive game, where all moves

have different labels, and players have perfect information and perfect recall. We want to capture the intuition that, at a given history in the game tree, the player whose move it is explores some possible sequences of moves starting from that history. Thus, if h and h' are two consecutive histories where player i moves, then we expect that the game that i believes he is playing at h' will include all the histories in the game he believed that he was playing at h together with new histories that extend h' that, intuitively, describe the histories that i “explores” at h' .¹⁶ In general, the histories that i is aware of will not be runs (complete histories); i will not have the resources to explore to the end of the game. Nor will they all in general have the same length; for example, i may not bother exploring further histories that seem almost certainly to be leading to bad outcomes. For a history h that player i is aware of that is a complete run of Γ , i 's utility is 1 for a win, 0 for a loss and $1/2$ for a draw; if h is not a complete run of Γ , then it must contain a virtual move. A virtual move made after a history h in chess is used to represent the fact that the player understands that some move can be made in h , but he is unable or unwilling to evaluate the consequences of that move. Thus, a history that i is aware of may contain both actual moves and virtual moves; the actual moves in the history partially specify a board position in the game; the utility assigned by i to such a history is related to the value of this (possibly partially specified) board position for that player; this value could be determined empirically as the relative frequency of victory in past games among all previous games that reach this position, or something “close” to this position. The issue of “closeness” becomes particularly relevant if there is insufficient data about one particular position. Of course, it is precisely in determining “closeness” that a resource-bounded agent is likely to make mistakes, and incorrectly evaluate a position. While capturing this carefully might be nontrivial, such thinking seems to be precisely what is needed to get a realistic model of chess. We remark that this notion of closeness is similar in spirit to the notion of similarity class of nodes in the valuation equilibrium solution concept recently proposed by Jehiel and Samet (2007).

More formally, if $(\mathcal{G}, \mathcal{F})$ is the game of awareness describing chess, for every game $\Gamma^+ \in \mathcal{G}$ if $h \in H_i^+$ is the first time player i moves along the history h , we have that $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, \{h\})$ and $H^h = \{h \cdot h_1 : h_1 \in E\}$, where E is a set of possible sequences of moves (both virtual and actual) that can follow history h and the player who moves at h explores. If $h \in H_i^+$ is the k th time player i moves along the history h and h_1 is the prefix of h where i moved for the $(k-1)$ st time, then $\mathcal{F}(\Gamma^+, h) = (\Gamma^h, \{h\})$ and $H^h = H^{h_1} \cup \{h \cdot h_1 : h_1 \in E\}$, where E is as above. If $h \cdot h_1 \in Z^+$ and $h \cdot h_1 \in Z$ (i.e., if $h \cdot h_1$ describes a complete game of chess), then $u_i(h \cdot h_1) = 1$ if the game ends in a victory for player i , $u_i(h \cdot h_1) = 0$ if it ends in a loss, and $u_i(h \cdot h_1) = .5$ if it ends in a draw. If $h \cdot h_1 \notin Z$, then the sequence $h \cdot h_1$ determines a set of board positions in the game; in this case, $u_i(h \cdot h_1) \in (0, 1)$ can be viewed as describing player i 's evaluation of this board position reached after the history $h \cdot h_1$ has been played. (Note that this formalism allows the same board position reached through different histories to have a different utility; for example, if the history encodes some information about the opponents' playing style, player i might assign different evaluations to the same board position.) \square

A modeler has a great deal of flexibility in games with awareness of unawareness. Given an objective game Γ , any

behavior can be explained as the equilibrium of a game with awareness of unawareness based on Γ with an appropriate choice of virtual moves. We do not believe that this renders games of awareness of unawareness uninteresting. Indeed, we would argue that the distinction between standard games and games with awareness (of unawareness) is not that great.

In real-life settings, we are not handed the objective game Γ any more than we are handed the game of awareness Γ^* . In a complicated scenario, even if we restrict to standard games, it may not be at all clear how to model the game that players think that they are actually playing. The game that the modeler has abstracted may well miss out on some features that are significant from the players' point of view.¹⁷ Unusual behavior of the players can be explained by appropriately modifying a standard game just about as easily it can be explained by adding appropriate virtual moves to a game of awareness. In either case, a modeler must justify the model being used. While it is likely to be more difficult to justify what happens after a virtual move (including the payoffs), there are cases (as in chess) where there seems to be a reasonable way of doing so.

5. Related work

There have been a number of models for unawareness in the literature (see, for example, Fagin and Halpern, 1988; Heifetz et al., 2006a; Modica and Rustichini, 1994, 1999; Dekel et al., 1998). Halpern (2001) and Halpern and Rêgo (2008) showed that in a precise sense all those models are special cases of Fagin and Halpern's (1988) approach where they modeled awareness syntactically by introducing a new modal operator for it. Halpern and Rêgo (2009a) extended Fagin and Halpern's logic of awareness to deal with knowledge of unawareness. In a similar spirit, Grant and Quiggin (2006) proposed a logic to model “the notion that individuals may be aware that there might be unconsidered propositions which they might subsequently discover, or which might be known to others”. To do this, they use two modal operators, c and a ; they interpret $a\phi$ as “the agent is aware of ϕ ” and $c\phi$ as “the agent considers ϕ ”. In the model proposed in Halpern and Rêgo (2009a), it is not possible to model an agent who is uncertain about whether he is aware of all facts. Halpern and Rêgo (2009b); Board and Chung (2009); Sillari (2008) independently pointed out this problem, and provided a solution that involved allowing different languages at different worlds, an idea going back to Modica and Rustichini (1994, 1999). All of these papers focused on logic, and did not analyze the impact of unawareness in a strategic setting.

Feinberg (2004, 2005, 2009) was perhaps the first to consider awareness explicitly in games. The model in Feinberg (2009) is most similar to ours, so we limit our discussion to that model. Feinberg (2009) uses a unified methodology to model lack of awareness in normal-form games, repeated games, games with incomplete information, and extensive games. We focus on the differences between his definition of extensive games with unawareness and ours.

Just as we do, Feinberg takes a game with awareness to be a collection of games. One of these games is taken to be the modeler's game, which is the analogue of our objective game. And, just as in our model, the modeler's game is in a sense the basis of all the other possible views of the game. Each of the other games

¹⁶ In general, we also want to allow imperfect recall. Player i may forget some moves that he has made; although he does not the board position. For ease of exposition, we continue to assume perfect recall, and focus on what i is aware of regarding future play.

¹⁷ If a (standard) game is played repeatedly, a modeler can try to infer what game the players think they are playing (see Penalva-Zuasti and Ryall, 2005 for some preliminary results along these lines), but it does not make sense to consider playing a game of awareness repeatedly. After a player has become aware of certain moves, the original structure of the game no longer describes the player's awareness.

in Feinberg's set is associated with a sequence $\langle v_1, v_2, \dots, v_k \rangle$, where v_j is a node in the game tree $G_{\langle v_1, \dots, v_{j-1} \rangle}$. Roughly speaking, if player i_j moves at node v_j in game $G_{\langle v_1, \dots, v_{j-1} \rangle}$, the game $G_{\langle v_1, v_2, \dots, v_k \rangle}$ can be thought of as the game that player i_1 at node v_1 in game $G_{\langle \rangle}$ believes that player i_2 at node v_2 in game $G_{\langle v_1 \rangle}$ believes that ... player i_k at node v_k in game $G_{\langle v_1, \dots, v_{k-1} \rangle}$ believes is being played. Feinberg places constraints on his games similar in spirit to our conditions C1–C8. We can also associate with each of the games in our framework such a sequence $\langle v_1, \dots, v_k \rangle$, but it is not unique. Feinberg calls the game $G_{\langle \rangle}$ the modeler's game. The modeler's game plays the same role for Feinberg as our objective game; it is the basis for all the other games in the model.

While there are clearly many similarities, there are some differences between our approaches. Feinberg does not describe an extensive game as a set of histories, but as a labeled directed graph. He allows a player to be unaware of some nodes on the game tree without being unaware of all the nodes below them; our approach to modeling lack of awareness in Section 2, which focuses on histories, does not allow this, although when we extend to dealing with awareness of unawareness, we get the same level of expressiveness. Moreover, Feinberg's model does not allow for uncertainty about what the game is. We deal with this by allowing moves by nature that determine a player's uncertainty, as in the game Γ^A in Fig. 2. Feinberg could capture this by associating with each sequence $\langle v_1, \dots, v_n \rangle$ a distribution over games rather than a unique game (and modifying his conditions appropriately to deal with this extension).¹⁸

While these differences seem to us the most significant, there are other more minor differences that we briefly outline below:

- Feinberg's model allow games whose terminal histories do not correspond to a unique terminal history of the objective game. To define utilities for these histories, his condition CD4 states that the utility of such a history is equal to the utility of some terminal history of the objective game that extends it. In our model of games without awareness of unawareness, we do not allow for such games, but we could easily relax conditions A5 and A6 to allow this extra expressive power in the model. We have chosen not to do so, for ease of exposition. In games with awareness of unawareness, we do allow for such histories, but we do not impose any restriction on the utilities of such histories.
- Feinberg does not define a notion of local strategy. Rather, he uses a more standard approach, defining an extended strategy profile to be a collection of standard strategy profiles, one for each game considered possible by some player in the game. While this is similar in spirit to our approach, it is not identical. For example, suppose that player A while moving at information set in the objective game Γ believes that he is playing at information set I^A in Γ^A . Using local strategies, what A does at I and I^A is determined by the local strategy σ_{A, Γ^A} . By way of contrast, Feinberg considers a strategy for player A in game Γ and another in game Γ^A ; his conditions (46) and (47) guarantee that the move that A makes at an information set I in Γ according to his strategy in Γ is the same as the move makes at the corresponding information set I^A in Γ^A according to A's strategy Γ^A . Thus, we can essentially translate back and forth between the two approaches in this regard.

Heifetz et al. (2011) also defined a notion of generalized extensive game to deal with unawareness of moves. There are some subtle differences between their model and ours. First,

they start with a different (although equivalent) formulation of a standard extensive game as a tree. Second, their trees viewed in isolation are extensive games with perfect information, since they do not define information sets in a tree. They define a map π from each decision node n in each tree to a set $\pi(n)$ of nodes in a possibly different tree, just as we define $\mathcal{F}(\Gamma^+, h)$ to be a pair consisting of (Γ^h, I) ; $\pi(n)$ is defined to be agent i 's information set at n (although it does not in general include n). They then add further conditions to ensure that the same set of actions are available at each node in $\pi(n)$, and a condition to ensure that players have perfect recall. One difference between our models is that Heifetz et al. do not have a condition similar to our C5; therefore, their model allows for the possibility that a player considers possible only nodes that could not be reached if players played only actions that they were aware of, which we view as inappropriate. (Of course, an analogue to C5 could be easily added in their model.) Heifetz et al.'s definition of strategy is equivalent to ours, but rather than generalizing Nash equilibrium, they provide a generalization of extensive form rationalizability in their setting.

Sadzik (2005) considers a logic of awareness, knowledge, and probability based on that of Heifetz et al. (2006a), and uses it to give a definition of Bayesian equilibrium in normal-form games with awareness. Heifetz et al. (2006b, 2007) also consider a generalized state-space model with interactive unawareness and probabilistic beliefs and give a definition of Bayesian equilibrium in normal-form games. As we said earlier, their model is similar in spirit to ours (and will often have the same equilibria), although there are some subtle differences. Li (2006a) has also provided a model of unawareness in extensive games, based on her earlier work on modeling unawareness (Li, 2009, 2006b). Although her representation of a game with unawareness is quite similar to ours, her notion of generalized Nash equilibrium is different from ours. Just as we do, she requires that every player i make a best response with respect to his beliefs regarding other player's strategies in the game Γ^i that i considers possible. However, unlike us, she requires that these beliefs satisfy a consistency requirement that implies, for example, that if a player i is aware of the same set of moves for him at both information set I_1 in game Γ_1 and information set I_2 in Γ_2 , and these information sets correspond to the same information set in the objective game Γ , then the local strategies σ_{i, Γ_1} and σ_{i, Γ_2} must agree at these information sets; that is, $\sigma_{i, \Gamma_1}(I_1) = \sigma_{i, \Gamma_2}(I_2)$. Thus, a player's response cannot depend, for example, on how he becomes aware of new information.

Ozbay (2007) proposes a model for games with uncertainty where players may have different awareness regarding a move of nature. He assumes that one of the players is fully aware, and can tell the other player about these moves before the second player moves. Although our model can easily capture this setting, what is interesting about Ozbay's approach is that the second player's beliefs about the probability of these revealed moves of are formed as part of the equilibrium definition. Filiz (2007) uses Ozbay's model in the context of incomplete contracts in the presence of unforeseen contingencies. In this setting, the insurer is assumed to be fully aware of the contingencies, and to decide strategically which contingencies to include in a contract, while the insuree may not be aware of all possible contingencies.

Finally, we remark that our notion of a game with awareness as consisting of the objective game together with description of which game each agent thinks is the actual game at each history has much in common with the intuition behind Gal and Pfeffer (2005) notion of a *Network of Influence Diagrams (NID)*. Formally, NIDs are a graphical language for representing uncertainty over decision-making models. A node in a NID (called a *block* by Gal and Pfeffer) represents an agent's subjective belief about the objective game and what the strategies used by agents depend on. Each node (game) in a NID is associated with a *multiagent influence*

¹⁸ Feinberg can also capture the uncertainty by putting it into the modeler's game, but this would in general require the modeler's game to be infinite, which Feinberg does not allow.

diagram (Koller and Milch, 2001) (MAID), which is a compact representation of a game. A NID has directed edges between nodes labeled by pairs of the form (i, H) , where i is an agent and (in our language) H is a set of histories. Intuitively, if there are edges from a node (game) Γ to a node Γ' in a NID labeled by a pair (i, H) , then H is a set of histories in Γ , there is an agent j that moves at all the histories in H , and in game Γ' , i believes that j believes that Γ' is the true game when moving at a history $h \in H$.

Although Gal and Pfeffer do not try to handle notions of awareness with NIDs, it seems possible to extend them to handle awareness. To do this appropriately, consistency requirements similar to C1–C8 will need to be imposed.

6. Discussion

We have generalized the representation of games to take into account agents who may not be aware of all the moves or all the other agents, but may be aware of their lack of awareness. Moreover, our representation is also flexible enough to deal with subjective games when there is lack of common knowledge about the game, even if awareness is not an issue.

Game theorists have long searched for good approaches to modeling games where there is no common knowledge among players regarding the game being played. Perhaps the most common way of dealing with this is due to Harsanyi (1968), who essentially converted a game without common knowledge to one with common knowledge, by adding types. Our approach is flexible enough to handle such lack of common knowledge directly. In particular, it can model a situation where each player has a completely different conception of what game is actually being played, where that game may have very little relationship to the actual objective game. We now sketch the changes needed to handle lack of common knowledge. We remark that what we do here makes perfect sense even in games where there is full awareness. In fact, recently, Grossi and Turrini (2012) used a particular case of a game with lack of common knowledge to model games where players do not have access to any terminal history of the game.

Formally, we define a *game with lack of common knowledge* exactly as we did for games with awareness, except that \mathcal{G} may contain a countable set of arbitrary finite extensive games with perfect recall. Note that since games in \mathcal{G} are not based on an objective game Γ , they do not have to satisfy any of the conditions A1–A6. We no longer assume that \mathcal{F} satisfies C1–C4, but do assume that it satisfies C5–C8, and the first half of C3', which we denote as C3''.

C3''. If h_1 is in the same information set as h and h_2 is a history in I , then $\mathcal{F}(\Gamma^+, h_1) = \mathcal{F}(\Gamma^h, h_2) = \mathcal{F}(\Gamma^+, h)$.

Despite all the changes to the conditions, the definitions of local strategies and generalized Nash equilibrium, and the theorems and their proofs remain unchanged. Thus, our techniques can deal with highly subjective games as well as awareness.

These generalizations greatly increase the applicability of game-theoretic notions in multiagent systems. In large games involving many agents, agents will almost certainly not be aware of all agents and may well not be aware of all the moves that agents can make. Moreover, as we suggested in the introduction, even in well-understood games like chess, by giving awareness a more computational interpretation, we can provide a more realistic model of the game from the agents' perspective. Although we have focused on generalizing extensive games, as we observed earlier, we can easily extend our framework to deal with normal-form games. The advantage of considering extensive games is that we are able to consider issues of how a player's level of awareness changes over time.

There is clearly much more to be done to understand the role of awareness (and lack of awareness) in multiagent systems. We list some of the many issues here:

- We have assumed perfect recall here. But in long games, it seems more reasonable to assume that agents do not have perfect recall. In a long chess game, typical players certainly do not remember all the moves that have been played and the order in which they were played. There is no difficulty in dropping the perfect-recall requirement from our model. However, the proof of Theorem 4.1 depends critically on the assumption on perfect recall. Wichardt (2008) gives an example showing that there is no Nash equilibrium in behavioral strategies in games of imperfect recall, even without awareness; *a fortiori*, it does not exist in games of awareness. Moreover, as is well known that even in single-agent games, considering agents with imperfect recall leads to a number of conceptual subtleties with regard to information sets (c.f. Halpern, 1997; Piccione and Rubinstein, 1997). We suspect that yet more subtleties will arise when combining imperfect recall with lack of awareness.
- We have focused on (generalized) Nash equilibrium here. While we think that an understanding of these generalized Nash equilibrium will be critical to understanding solution concepts in games with awareness, as we said, we are not convinced that (generalized) Nash equilibrium is necessarily the “right” solution concept. Once we no longer have a common prior, some variant of *rationalizability* (Bernheim, 1984; Pearce, 1984) or a variant of *self-confirming equilibrium* (Fudenberg and Levine, 2003) may be more appropriate. In standard extensive games, *sequential equilibrium* (Kreps and Wilson, 1982) and *perfect equilibrium* (Selten, 1975) are considered more appropriate than Nash equilibrium in any case. Our framework allows for relatively straightforward generalizations of all these solution concepts. However, there are subtleties involved in showing that in all games a (generalized) equilibrium exists. For example, we no longer have a one-to-one correspondence between the generalized sequential equilibria of the game Γ^* and the sequential equilibria of the corresponding standard game Γ^ν . Nevertheless, in a companion paper (Rêgo and Halpern, 2012) we show that these generalized equilibria exist in every game with awareness. However, this still leaves open the question of what is the “right” solution concept; we view that as an important topic for further research.
- We have analyzed situations where agents may be unaware of some moves in the objective game, may be aware of their unawareness, and may have completely false beliefs about the objective game. Of course, there are other cases of interest where additional properties may hold. For example, consider a large geographically-dispersed game where agents interact only with nearby neighbors. In such a game, an agent may be unaware of exactly who is playing the game (although she may realize that there are other agents besides her neighbors, and even realize that the moves made by distant agents may have an indirect effect on her). To model such a situation, we may want to have virtual moves after which the game does not end, and to allow agents to be aware of subsequences of histories in the game. We suspect that a straightforward extension of the ideas in this paper can deal with such situations, but we have not worked out the details.
- There has been a great deal of work on computing Nash equilibria. As we have shown, a generalized Nash equilibrium of a game with awareness is a Nash equilibrium of a standard game. However, this standard game can be rather large. Are there efficient computational techniques for computing generalized Nash equilibrium in interesting special cases?

• If there is little shared knowledge regarding the objective game, the set \mathcal{G} of games can be quite large, or even infinite. Is it important to consider all the iterated levels of unawareness encoded in \mathcal{G} ? Halpern and Moses (1990) showed that, in analyzing coordinated attack, no finite level of knowledge suffices; common knowledge is needed for coordination. Stopping at any finite level has major implications. Rubinstein (1989) considered a variant of the coordinated attack problem with probabilities, and again showed that no finite level suffices (and significant qualitative differences arise if only a finite part of hierarchy of knowledge is considered). On the other hand, Weinstein and Yildiz (2003) provide a condition under which the effect of players' k th order beliefs is exponentially decreasing in k . While we strongly suspect that there are games in which higher-order unawareness will be quite relevant, just as with the Weinstein–Yildiz result, there may be conditions under which higher-order awareness becomes less important, and a simpler representation may suffice. Moreover, it may be possible to use NIDs to provide a more compact representation of games of awareness in many cases of interest (just as Bayesian networks provide a compact representation of probability distributions in many cases of interest), leading to more efficient techniques for computing generalized Nash equilibrium.

We hope to explore some of these issues in forthcoming work.

Appendix. Proofs

Theorem 3.1. For all probability measures ν on \mathcal{G}

- (a) Γ^ν is a standard extensive game with perfect recall;
- (b) if ν gives positive probability to all games in \mathcal{G} , then $\vec{\sigma}$ is a Nash equilibrium of Γ^ν iff $\vec{\sigma}'$ is a generalized Nash equilibrium of Γ^* , where $\sigma_{i,\Gamma'}(\langle \Gamma^h \rangle \cdot h') = \sigma'_{i,\Gamma'}(\Gamma^h, h')$.

Proof. For part (a), suppose that $\langle \Gamma' \rangle \cdot h'_1$ and $\langle \Gamma'' \rangle \cdot h'_1$ are in the same (i, Γ^+) -information set of Γ^ν and that h'_2 is a prefix of h'_1 such that $P^\nu(\langle \Gamma' \rangle \cdot h'_2) = (i, \Gamma^+)$. By definition of Γ^ν , it must be the case that there exist i -information sets I_1 and I_2 in Γ^+ such that $\mathcal{F}(\Gamma', h'_1) = \mathcal{F}(\Gamma'', h'_1) = (\Gamma^+, I_1)$ and $\mathcal{F}(\Gamma', h'_2) = (\Gamma^+, I_2)$. If h_1 is a history in I_1 , C7 (which follows from C1, C3 and the fact that the players have perfect recall in games in \mathcal{G}) implies that there exists a prefix h_2 of h_1 such that $P^+(h_2) = i$, $\mathcal{F}(\Gamma^+, h_2) = (\Gamma^+, I_2)$ and if $h'_2 \cdot \langle m \rangle$ is a prefix of h'_1 , then $h_2 \cdot \langle m \rangle$ is a prefix of h_1 . Applying C7 again, it follows that there exists a prefix h'_2 of h'_1 such that $P^{\Gamma''}(h'_2) = i$ and $\mathcal{F}(\Gamma'', h'_2) = (\Gamma^+, I_2)$ and if $h_2 \cdot \langle m \rangle$ is a prefix of h_1 , then $h'_2 \cdot \langle m \rangle$ is a prefix of h'_1 . Therefore, by definition of Γ^ν , (Γ'', h'_2) and (Γ', h'_2) are in the same information set.

Suppose further that $h'_2 \cdot \langle m \rangle$ is a prefix of h'_1 . Thus, $h_2 \cdot \langle m \rangle$ is a prefix of h_1 , which implies that $h'_2 \cdot \langle m \rangle$ is a prefix of h'_1 . This proves part (a).

For part (b), let $Pr_{\vec{\sigma}}^\nu$ be the probability distribution over the runs in Γ^ν induced by the strategy profile $\vec{\sigma}$ and f_c^ν . $Pr_{\vec{\sigma}}^\nu(z)$ is the product of the probability of each of the moves in z . (It is easy to define this formally by induction on the length of z ; we omit details here.) Similarly, let $Pr_{\vec{\sigma}'}^h$ be the probability distribution over the runs in $\Gamma^h \in \mathcal{G}$ induced by the generalized strategy profile $\vec{\sigma}'$ and f_c^h . Note that if $Pr_{\vec{\sigma}'}^h(z) > 0$, then $z \in [H^h]$. Thus, $\langle \Gamma^h \rangle \cdot z \in H^\nu$.

For all strategy profiles σ and generalized strategy profiles σ' , if $\sigma'_{i,\Gamma'}(\Gamma^h, h') = \sigma_{i,\Gamma'}(\langle \Gamma^h \rangle \cdot h')$, then it is easy to see that for all $z \in Z^h$ such that $Pr_{\vec{\sigma}}^\nu(z) > 0$, we have that $Pr_{\vec{\sigma}}^\nu(\langle \Gamma^h \rangle \cdot z) = \nu(\Gamma^h)Pr_{\vec{\sigma}'}^h(z)$. And since ν is a probability measure such that $\nu(\Gamma^h) > 0$ for all $\Gamma^h \in \mathcal{G}$, we have that $Pr_{\vec{\sigma}}^\nu(\langle \Gamma^h \rangle \cdot z) > 0$ iff $Pr_{\vec{\sigma}'}^h(z) > 0$. Suppose that $\vec{\sigma}$ is a Nash equilibrium of Γ^ν . Suppose, by way of contradiction, that $\vec{\sigma}'$ such that $\sigma'_{i,\Gamma'}(\Gamma^h, h') = \sigma_{i,\Gamma'}(\langle \Gamma^h \rangle \cdot h')$ is not a generalized Nash equilibrium of Γ^* . Thus,

there exists a player i , a game $\Gamma^+ \in \mathcal{G}_i$, and a local strategy s' for player i in Γ^+ such that

$$\sum_{z \in Z^+} Pr_{\vec{\sigma}'}^+(\langle \Gamma^+ \rangle \cdot z) u_i^+(z) < \sum_{z \in Z^+} Pr_{(\vec{\sigma}'_{-(i,\Gamma^+),s'})}^+(\langle \Gamma^+ \rangle \cdot z) u_i^+(z). \quad (A.1)$$

Define s to be a strategy for player (i, Γ^+) in Γ^ν such that $s(\langle \Gamma^h \rangle \cdot h') = s'(\Gamma^h, h')$. Multiplying (A.1) by $\nu(\Gamma^+)$ and using the observation in the previous paragraph, it follows that

$$\begin{aligned} & \sum_{z \in [Z^+]} Pr_{\vec{\sigma}}^\nu(\langle \Gamma^+ \rangle \cdot z) u_i^+(z) \\ & < \sum_{z \in [Z^+]} Pr_{(\vec{\sigma}_{-(i,\Gamma^+),s})}^\nu(\langle \Gamma^+ \rangle \cdot z) u_i^+(z). \end{aligned} \quad (A.2)$$

By definition of $u_{i,\Gamma'}^\nu$, (A.2) holds iff

$$\sum_{z^\nu \in Z^\nu} Pr_{\vec{\sigma}}^\nu(z^\nu) u_{i,\Gamma^+}^\nu(z^\nu) < \sum_{z^\nu \in Z^\nu} Pr_{(\vec{\sigma}_{-(i,\Gamma^+),s})}^\nu(z^\nu) u_{i,\Gamma^+}^\nu(z^\nu). \quad (A.3)$$

Therefore, $\vec{\sigma}$ is not a Nash equilibrium of Γ^ν , a contradiction. The proof of the converse is similar; we leave details to the reader. \square

Corollary 3.1. Every game with awareness has a generalized Nash equilibrium.

Proof. For games with perfect recall, there is a natural isomorphism between mixed strategies and behavioral strategies, so a Nash equilibrium in behavior strategies can be viewed as a Nash equilibrium in mixed strategies (Osborne and Rubinstein, 1994). Moreover, mixed-strategy Nash equilibria of an extensive game are the same as the mixed-strategy Nash equilibria of its normal-form representation. Salonen (2005) showed that there exists a Nash equilibrium in mixed strategies in a normal form games with an arbitrary set N of players if, for each player i , the set S_i of pure strategies of player i is a compact metric space, and the utility functions $u_i : S \rightarrow R$ are continuous for all $i \in N$, where R is the set of real numbers and $S = \prod_{i \in N} S_i$, the set of pure strategies, is endowed with the product topology. Since in Γ^ν , every player has a finite number of pure strategies, S_i is clearly a compact metric space. Moreover, since each player's utility depends only on the strategies of a finite number of other players, it is easy to see that $u_i : S \rightarrow R$ is continuous for each player $i \in N$. It follows that there exists a Nash equilibrium of Γ^ν . Thus, the corollary is immediate from Theorem 3.1. \square

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