

# Welfare Loss in Connected Resource Allocation

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## Abstract

We study the allocation of indivisible goods that form an undirected graph and investigate the worst-case welfare loss when requiring that each agent must receive a connected subgraph. Our focus is on both egalitarian and utilitarian welfare. Specifically, we introduce the concept of *egalitarian* (resp., *utilitarian*) *price of connectivity*, which captures the worst-case ratio between the optimal egalitarian (resp., utilitarian) welfare among all allocations and that among the connected allocations. We provide tight or asymptotically tight bounds on the price of connectivity for various large classes of graphs when there are two agents as well as for paths, stars and cycles in the general case. Many of our results are supplemented with algorithms which find connected allocations with a welfare guarantee corresponding to the price of connectivity.

## 1 Introduction

The allocation of indivisible goods among interested agents with heterogeneous preferences is a fundamental problem in society. Taking a *utilitarian* standpoint, allocating each item to an agent with the highest utility for the item, though economically efficient, can lead to unfair allocations where some agents may have little or no utility for their bundle. To address this issue, the study of *fair division* focuses on allocating resources in a manner that is fair, and possibly ideal in terms of other desirable properties. The concept of fairness can either be represented in the form of axioms which enforce certain fairness criteria, or in our case, as an *egalitarian* objective reflecting the utility obtained by the worst-off agent.

In the real world, there may be constraints on the range of feasible allocations, such as budget or cardinality constraints; see the survey by [Suksompong \[2021\]](#) for a detailed overview. Our paper studies the important constraint of *connectivity* initially proposed by [Bouveret et al. \[2017\]](#), in which each good corresponds to a vertex of a pre-defined graph, and each agent's bundle of items must form a connected subgraph. Under this constraint, we are interested in *quantitative measures* of fairness and efficiency, and we specifically address the following research question:

*When allocating connected subsets of items, what is the worst-case degradation of social welfare resulting from the constraint of connectivity?*

Answers to this question may help central decision makers decide whether the loss of welfare from imposing connectivity constraints outweighs the benefits of having connected allocations. For example, when allocating offices or desks among different research groups, it may be desirable for each research group to receive a connected set of offices/desks, and when partitioning

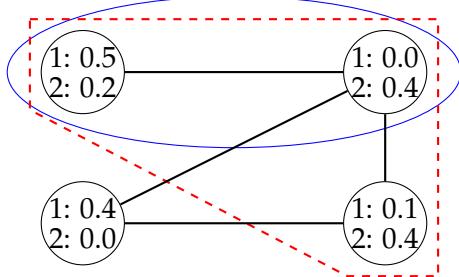


Figure 1: An instance with  $n = 2$  agents and  $m = 4$  items. The numbers in each node indicate the agents’ utilities for the item. The items allocated to agent 2 under the optimal connected egalitarian and utilitarian outcomes are circled by the blue ellipse and red dashed lines, respectively.

players into sports teams, we may want to require that each player knows at least one other player on the same team. Enforcing connected allocations in these scenarios may provide collaborative benefits at the cost of social welfare. Our model and results can also be applied to multi-robot task allocation. In this scenario, robots (items) are to be partitioned into fleets (bundles) which will be assigned to tasks (agents), and an edge between two robots in the graph indicates that they are able to directly communicate with each other. Imposing a connectivity constraint allows for the fleets of robots to be decentralized, without the need of a central server for communication, but this comes at the cost of welfare loss, which our findings help to quantify. Even when connected allocations are mandatory and the central decision maker cannot choose to neglect them, it is still interesting to know how much welfare loss is incurred from the constraint.

Specifically, we quantify the loss of fairness and efficiency, corresponding to two well-studied social welfare functions: the *egalitarian welfare*, defined as the utility of the worst-off agent, and the *utilitarian welfare*, which is the sum of the agents’ utilities. To illustrate our key concepts, consider the instance in Figure 1. Here, the optimal egalitarian and utilitarian welfares are 0.8 and 1.7, respectively, both achieved by giving each item to the agent that values it most. However, the optimal egalitarian welfare among all *connected outcomes* is 0.5, with agent 2’s egalitarian allocation inside the blue ellipse, and the optimal *connected utilitarian* welfare is 1.4, with agent 2’s utilitarian allocation inside the red dashed shape. Thus, due to the connectivity constraint, the optimal egalitarian welfare decreases by a factor of  $0.8/0.5 = 1.6$ , and the optimal utilitarian welfare decreases by a factor of  $1.7/1.4 \approx 1.21$ . This is because the top left and bottom left items are not directly connected, so to give agent 1 both items in a connected allocation, we must also give her the top right item, which she does not value.

## 1.1 Our Results

In this paper, we investigate the potential loss of welfare when a connectivity constraint is imposed on the allocations. To this end, we introduce the concept of *egalitarian (resp., utilitarian) price of connectivity (PoC)* of a graph, which is the worst-case ratio between the optimal egalitarian (resp., utilitarian) welfare from any (possibly disconnected) allocation, and the optimal egalitarian (resp., utilitarian) welfare from a connected allocation, over all possible utility profiles.

We prove tight or asymptotically tight bounds on the price of connectivity for various classes of graphs. Intuitively, a more “connected” graph should have a lower price of connectivity, with a complete graph having an egalitarian/utilitarian PoC of exactly 1. However, in the case of two agents, our results for complete graphs with a non-empty matching removed show that the egalitarian and utilitarian prices of connectivity are strictly higher than 1 even when only one

	Egalitarian PoC	Utilitarian PoC	
Stars	$m - n + 1$ (Theorem 3.12)	$\Omega(n)$	(Theorem 4.6)
Paths	$\begin{cases} m - n + 1, & n \leq m < 2n - 1 \\ n, & 2n - 1 \leq m \end{cases}$ (Theorem 3.13)	$\Omega(n)$	(Theorem 4.7)
Cycles	$\begin{cases} m - n + 1, & n \leq m < 2n - 2 \\ n - 1, & 2n - 2 \leq m < n^2 \\ n, & n^2 \leq m \end{cases}$ (Theorem 3.14)	$\Omega(n)$	(Theorem 4.8)
Any graph	$\leq m - n + 1$ (Lemma 3.1)	$\leq n$	(Proposition 4.5)

Table 1: Table of egalitarian and utilitarian PoC results for any number of agents.

edge is removed. Furthermore, with the exception of the 5-item case, the price of connectivity does not increase further when more disjoint edges are removed. We also supplement this result in the two-agent case with results for another dense class of graphs, that is, complete bipartite graphs. Interestingly, we find that if one side of the graph has two vertices, the utilitarian PoC is a constant, regardless of how many vertices the other side has.

We also consider graphs with low connectivity, finding the egalitarian PoC in the two-agent case for the broad classes of graphs with connectivity 1, including all trees, and graphs with connectivity 2. Although the PoC for graphs with connectivity 1 is lower bounded by the maximum number of disjoint connected subgraphs resulting from the removal of a vertex, the PoC for graphs with connectivity 2 is simply a constant. We also extend to the three-agent case, finding the egalitarian PoC for trees. These results come with algorithms for finding an allocation which guarantees a welfare corresponding to the price of connectivity. Similarly, in the two-agent case, we find the exact utilitarian PoCs for trees and cycles. Unlike in the egalitarian case, the utilitarian PoC for trees does not depend on the maximum degree of the graph. Finally, we give results on stars, paths and cycles for any number of agents, which are summarized in Table 1.

Section 3 gives the egalitarian PoC bounds, while Section 4 presents the utilitarian PoC results.

## 1.2 Related Work

The mathematically rigorous treatment of fair division dates back to the work of [Steinhaus \[1948\]](#) on allocating divisible resources, and has attracted considerable attention since then [[Brams and Taylor, 1996](#); [Moulin, 2003](#); [Thomson, 2016](#)], with a growing interest in the allocation of *indivisible* goods in recent years [[Amanatidis et al., 2023](#)].

The connectivity constraint has been studied by many authors [[Bei et al., 2022](#); [Bilò et al., 2022](#); [Bouveret et al., 2017](#); [Caragiannis et al., 2022](#); [Igarashi, 2023](#); [Lonc, 2023](#); [Lonc and Truszcynski, 2020](#); [Suksompong, 2019](#)], with several of them addressing graphs such as paths, cycles, stars, etc. that we consider in this work. These authors have mostly focused on the existence, approximation and computational complexity of *fair*, connected allocations. One exception is the work by [Igarashi and Peters \[2019\]](#), who studied the problem of finding a connected allocation that is Pareto-optimal. Pareto optimality can be viewed as a qualitative measure of efficiency, differing from the quantitative utilitarian welfare which we study.

The term “price of connectivity (PoC)” was introduced in the work by [Bei et al. \[2022\]](#), and measures the price in terms of a fairness notion called *maximin share*. More precisely, these authors

defined the PoC as the worst-case ratio between the maximin share taken over all possible partitions of the items into  $n$  parts and the maximin share taken over all connected partitions into  $n$  parts. Note that their PoC is only defined with respect to the maximin share of one agent, whilst we define it with respect to the optimal welfare, and thus our definitions involve utilities of all of the agents. When translated to our setting, the (maximin share) PoC results by Bei et al. [2022] imply egalitarian PoC results for identical agents.

The price of connectivity concept is similar in spirit to the *price of fairness* [Bertsimas et al., 2011; Caragiannis et al., 2012], which captures the worst-case welfare loss resulting from fairness constraints. The price of fairness has been widely studied in various fair division settings [Barman et al., 2020; Bei et al., 2021; Celine et al., 2023; Li et al., 2024; Suksompong, 2019]. While these papers typically measure the loss of utilitarian welfare from fairness constraints, part of our work treats fairness as a welfare objective, measuring the loss of egalitarian welfare from connectivity constraints.

## 2 Preliminaries

For any positive integer  $t$ , let  $[t] := \{1, 2, \dots, t\}$ . Denote by  $N = [n]$  the set of  $n$  agents and  $M$  the set of  $m$  indivisible goods. There is a bijection between the goods in  $M$  and the  $m$  vertices of a connected undirected graph  $G$ ; we will refer to goods, items and vertices interchangeably. Each agent  $i \in N$  has a non-negative utility  $u_i(g)$  for each good  $g \in M$ . We assume that utilities are *additive*, i.e.,  $u_i(M') = \sum_{g \in M'} u_i(g)$  for all  $i \in N$  and  $M' \subseteq M$ , as well as normalized, that is,  $u_i(M) = 1$  for all  $i \in N$ . Denote by  $\mathcal{U} = (u_1, u_2, \dots, u_n)$  the *utility profile* of the agents. We refer to a setting with the agents  $N$ , the goods  $M$  and their underlying graph  $G$ , and the utility profile  $\mathcal{U}$  as an *instance*, denoted as  $I = \langle N, G, \mathcal{U} \rangle$ .

A *bundle* is a subset of goods, and is called *connected* if the goods in the bundle form a connected subgraph of  $G$ . An *allocation*  $\mathcal{M} = (M_1, M_2, \dots, M_n)$  is a partition of the goods in  $M$  into  $n$  bundles such that agent  $i \in N$  receives bundle  $M_i$ . Moreover, an allocation or a partition is *connected* if all of its bundles are connected. Denote by  $C(I)$  the set of all connected allocations for instance  $I$ .

In this paper, we quantify the worst-case *egalitarian* and *utilitarian* welfare loss incurred when imposing that each agent must receive a connected bundle. Given an instance  $I$  and an allocation  $\mathcal{M} = (M_1, M_2, \dots, M_n)$  of the instance,

- the *egalitarian welfare* of  $\mathcal{M}$ , denoted as  $\text{SW-egal}(\mathcal{M})$ , is the minimum among the agents' utilities; that is,  $\text{SW-egal}(\mathcal{M}) := \min_{i \in N} u_i(M_i)$ ;
- the *utilitarian welfare* of  $\mathcal{M}$ , denoted as  $\text{SW-util}(\mathcal{M})$ , is the sum of the agents' utilities; that is,  $\text{SW-util}(\mathcal{M}) := \sum_{i \in N} u_i(M_i)$ .

The optimal egalitarian (resp., utilitarian) welfare of instance  $I$ , denoted by  $\text{OPT-egal}(I)$  (resp.,  $\text{OPT-util}(I)$ ), is the maximum egalitarian (resp., utilitarian) welfare over all possible allocations of the instance.

### 2.1 Egalitarian / Utilitarian Price of Connectivity

We now proceed to define the central concept of the paper—the *price of connectivity* (*PoC*)—which captures the largest multiplicative gap between the optimal welfare among all allocations and that among all *connected* allocations.

**Definition 2.1** (Egal-PoC). Given a graph  $G$  and the number of agents  $n$ , the *egalitarian price of connectivity* (*egalitarian PoC*) for  $G$  is defined as<sup>1</sup>

$$\text{Egal-PoC}(G, n) = \sup_{I=\langle N, G, \mathcal{U} \rangle} \frac{\text{OPT-egal}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M})},$$

where the supremum is taken over all possible instances.

The *utilitarian price of connectivity* is similarly defined by replacing OPT-egal with OPT-util and SW-egal with SW-util.

**Definition 2.2** (Util-PoC). Given a graph  $G$  and the number of agents  $n$ , the *utilitarian price of connectivity* (*utilitarian PoC*) for  $G$  is defined as

$$\text{Util-PoC}(G, n) = \sup_{I=\langle N, G, \mathcal{U} \rangle} \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})},$$

where the supremum is taken over all possible instances.

### 3 Egalitarian Price of Connectivity

In this section, we investigate the loss of egalitarian welfare due to the connectivity constraint. Firstly, we remark that given any graph  $G$ , if  $m < n$ , then its optimal egalitarian welfare is always 0 and thus  $\text{Egal-PoC}(G, n) = 1$ . We therefore assume that  $m \geq n$  for the rest of this section.

We start by establishing a general upper bound on the egalitarian PoC for any graph  $G$ . The intuition behind the bound is that for any instance  $I$ , by giving every agent their most preferred item from an optimal egalitarian allocation of the instance, each agent receives a utility of at least  $\frac{\text{OPT-egal}(I)}{m-n+1}$ . This is because each bundle in an optimal egalitarian allocation has at most  $m - n + 1$  goods.

**Lemma 3.1.** *Given any  $G$ ,  $\text{Egal-PoC}(G, n) \leq m - n + 1$ .*

*Proof.* Given any instance  $I$ , let  $B = (B_1, B_2, \dots, B_n)$  be an optimal egalitarian allocation of the instance. If  $\text{OPT-egal}(I) = \min_{i \in N} u_i(B_i) = 0$ , then  $\text{Egal-PoC}(G, n) = 1$ . We thus assume that  $\min_{i \in N} u_i(B_i) > 0$ , which implies that  $1 \leq |B_i| \leq m - n + 1$  for all  $i \in N$ . By giving each agent  $i \in N$  her most preferred good in  $B_i$ , such a (partial) allocation gives the agent a utility of at least

$$\frac{u_i(B_i)}{|B_i|} \geq \frac{\text{OPT-egal}(I)}{|B_i|} \geq \frac{\text{OPT-egal}(I)}{m - n + 1}.$$

We then extend the partial allocation to a complete, connected allocation in an arbitrary way. It follows that  $\text{Egal-PoC}(G, n) \leq m - n + 1$ .  $\square$

We next introduce a reduction technique used in this section in order to simplify our proof for the upper bound on the egalitarian PoC for various graphs. Given any graph  $G$ , denote by  $\mathcal{I}_G$  the set of instances such that each vertex in  $G$  is positively valued by at most one agent. It is worth noting that we do not assume that the agents' utilities are normalized in  $\mathcal{I}_G$ .

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<sup>1</sup>We interpret  $\frac{0}{0}$  in this context to be equal to 1. Note that  $\text{OPT-egal}(I) = 0$  if and only if  $\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M}) = 0$ , because  $G$  is assumed to be connected, and both conditions are equivalent to the condition that there does not exist an allocation where each agent attains positive utility.

**Lemma 3.2.** Let  $\beta \in (0, 1]$ . Fix any graph  $G$ . If for every instance  $\hat{I} \in \mathcal{I}_G$ , there exists a connected allocation  $\hat{\mathcal{M}}$  such that  $\text{SW-egal}(\hat{\mathcal{M}}) \geq \beta \cdot \text{OPT-egal}(\hat{I})$ , then  $\text{Egal-PoC}(G, n) \leq \frac{1}{\beta}$ .

*Proof.* It suffices to show that given any instance  $I = \langle N, G, \mathcal{U} \rangle$ , there always exists a connected allocation with egalitarian welfare at least  $\beta \cdot \text{OPT-egal}(I)$ .

Denote by  $(B_1, B_2, \dots, B_n)$  an optimal egalitarian allocation of instance  $I$ . Construct an instance  $\hat{I}$  using the same graph  $G$  as in instance  $I$  with the following set-up of agents' utilities: for each  $i \in N$ , agent  $i$  values all vertices in bundle  $B_i$  the same as that in instance  $I$ , and all other vertices at 0. It is clear that  $\hat{I} \in \mathcal{I}_G$ . Moreover, as suggested by the statement, there exists a connected allocation  $\hat{\mathcal{M}}$  such that  $\text{SW-egal}(\hat{\mathcal{M}}) \geq \beta \cdot \text{OPT-egal}(\hat{I})$ .

Now consider both instances  $I$  and  $\hat{I}$ . First, it can be verified that both instances have the same optimal egalitarian welfare, i.e.,  $\text{OPT-egal}(I) = \text{OPT-egal}(\hat{I})$ . Next, it follows that the connected complete allocation  $\hat{\mathcal{M}}$  gives an egalitarian welfare of at least  $\beta \cdot \text{OPT-egal}(I)$  in instance  $I$ , completing the proof.  $\square$

Lemma 3.2 suggests that when proving upper bounds on the egalitarian PoC, it suffices to focus only on the instances in which each vertex is valued by at most one agent. The reduction, however, does not run in polynomial time because it requires an allocation that maximizes egalitarian welfare, which is known to be NP-hard even for 2 agents. Our algorithms in this section take reduced instances as their input. If we are given an optimal egalitarian allocation, such as by some oracle, then our algorithms run in polynomial time, as their general structure involves moving down a rooted subtree. We may also use a polynomial-time approximation algorithm (e.g., [Asadpour and Saberi, 2010; Bezákova and Dani, 2005]) to create the reduced input instance and achieve a corresponding approximate solution.

In the remainder of this section, we first consider the important cases in fair division which concern a small number of agents, followed by the general case.

### 3.1 Two Agents

We begin with the case of two agents, addressing a dense class of graphs. Although a complete graph has an egalitarian price of connectivity of 1, we find that even when only one edge is removed from it, the egalitarian PoC becomes strictly higher than 1, meaning that there exists some instance where the optimal egalitarian welfare cannot be achieved by a connected allocation. However, with the exception of the 5-item case, the egalitarian PoC does not increase as more disjoint edges are removed from the graph. Let  $K_m$  denote the complete graph with  $m$  vertices.

**Theorem 3.3.** Let  $G$  be a complete graph with a non-empty matching removed. Then,  $\text{Egal-PoC}(G, 2) = 2$  if  $G$  is

- $K_3$  with an edge removed, or
- $L_5$ , which is  $K_5$  with two disjoint edges removed,

and  $\text{Egal-PoC}(G, 2) = \frac{m-2}{m-3}$  otherwise.

*Proof.* When  $m = 3$ , the egalitarian PoC of 2 for  $K_3$  with an edge removed follows from Theorem 3.13 (for paths).

When  $m = 4$ , the egalitarian PoC of 2 for  $K_4$  with two disjoint edges removed follows from Theorem 3.14 (for cycles). This implies that when the graph is  $K_4$  with a single edge removed, its egalitarian PoC is at most 2. We now give a lower bound example showing that the egalitarian

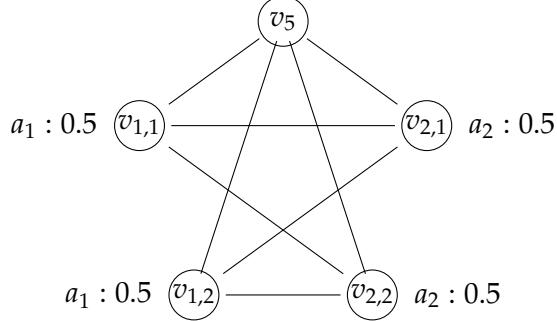


Figure 2: Graph  $L_5$  and utility profile showing that its egalitarian PoC is at least 2.

PoC is also at least 2. Let agent 1 value two vertices without an edge between them at  $1/2$  each, and let agent 2 value the remaining two vertices at  $1/2$  each. We have  $\text{OPT-egal}(I) = 1$  and  $\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M}) = 1/2$ , showing that the egalitarian PoC is at least 2 and concluding the proof for  $m = 4$ . For the remainder of the proof, we suppose that  $m \geq 5$ .

**Lower Bound (for  $m \geq 5$ )** We first show lower bounds. For  $L_5$ , denote the missing edges as  $\{v_{1,1}, v_{1,2}\}$  and  $\{v_{2,1}, v_{2,2}\}$ ; see Figure 2. For  $i \in [2]$ , let agent  $i$  value  $v_{i,1}$  and  $v_{i,2}$  at  $1/2$  each, and let the final vertex  $v_5$  be unvalued. Either agent can obtain both of their positively valued items by taking  $v_5$ , but if one agent does so, the other agent can only obtain one of their valued items. The optimal connected egalitarian welfare is  $1/2$ , showing that the egalitarian PoC is at least 2.

For a graph that is not  $L_5$ , let  $v_1$  and  $v_2$  be two vertices without an edge between them, let agent 1 value each of these vertices at  $1/2$  each, and let agent 2 value the remaining  $m - 2$  vertices at  $\frac{1}{m-2}$  each. The egalitarian-optimal connected allocation has agent 1 receiving both of its valued items along with one of agent 2's valued items, and agent 2 receiving all but one of its items, leading to an egalitarian PoC lower bound of  $\frac{m-2}{m-3}$ .

**Upper Bound (for  $m \geq 5$ )** To prove the upper bound, we show that among all possible instances, the instances described in the lower bound examples maximize  $\frac{\text{OPT-egal}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M})}$ . For simplicity, we may refer to this ratio as the “egalitarian welfare ratio”. We first remark that due to Lemma 3.2, it suffices to consider instances  $I$  where each vertex is positively valued by at most one agent, meaning that  $\text{OPT-egal}(I) = \min_{i \in [2]} u_i(G)$ . Denote the  $k$  missing edges as  $\{v_{1,1}, v_{1,2}\}, \{v_{2,1}, v_{2,2}\}, \dots, \{v_{k,1}, v_{k,2}\}$ . Note that if there exists  $i \in [k]$  such that no agent positively values both  $v_{i,1}$  and  $v_{i,2}$  (i.e., no agent values both vertices of any missing edge), then both agents can receive all of their valued items in a connected allocation as  $v_{i,1}$  is connected to every vertex except  $v_{i,2}$ , and similarly  $v_{i,2}$  is connected to every vertex except  $v_{i,1}$ . For such an instance,  $\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M}) = \text{OPT-egal}(I)$ . Therefore, to maximize the egalitarian welfare ratio, we require that for each  $i \in [k]$ , both  $v_{i,1}$  and  $v_{i,2}$  are positively valued by the same agent, so we assume without loss of generality that agent 1 positively values vertices  $v_{1,1}$  and  $v_{1,2}$ . We can also suppose that each agent positively values at least two items, since otherwise both agents can receive all of their valued items in connected allocations.

We now consider the special case where the graph is  $L_5$ . Under the instance which maximizes the egalitarian welfare ratio, we know that agent 1 positively values  $v_{1,1}$  and  $v_{1,2}$ , and that agent 2 positively values vertices  $v_{2,1}$  and  $v_{2,2}$ . If agent 1 positively values  $v_5$ , then agent 2 can receive both of its valued items by taking agent 1's less-valued item out of  $v_{1,1}$  and  $v_{1,2}$  (suppose without loss of generality that this is  $v_{1,1}$ , i.e.,  $u_1(v_{1,1}) \leq u_1(v_{1,2})$ ). Since agent 1 positively values  $v_5$ , we

see that  $u_1(v_{1,1}) < u_1(G)/2$  and thus agent 1 can obtain over  $u_1(G)/2$  utility by receiving items  $v_{1,2}$  and  $v_5$ , which are connected. We then have  $\frac{\text{OPT-egal}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-egal}(\mathcal{M})} < 2$ . However, the lower bound example gives an egalitarian welfare ratio of 2, implying that this type of instance does not maximize the egalitarian welfare ratio. By a symmetric argument, we see that if agent 2 positively values  $v_5$ , then the egalitarian welfare ratio is strictly lower than 2. Therefore, to maximize the egalitarian welfare ratio,  $v_5$  must be non-valued. Finally, since  $\text{OPT-egal}(I) = \min_{i \in [2]} u_i(G)$ , to maximize the egalitarian welfare ratio, we let  $u_1(v_{1,1}) = u_1(v_{1,2}) = u_2(v_{2,1}) = u_2(v_{2,2})$ , resulting in  $\frac{\text{OPT-egal}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-egal}(\mathcal{M})} = 2$ . By exhaustion of cases, we see that when the graph is  $L_5$ , the egalitarian welfare ratio (and therefore egalitarian PoC) is at most 2, concluding the proof for this special case.

Finally, we consider the general case where the graph is not  $L_5$ . Recall that when the egalitarian welfare ratio is maximized, agent 1 positively values vertices  $v_{1,1}$  and  $v_{1,2}$ , and agent 2 positively values at least two items. We first show that to maximize  $\frac{\text{OPT-egal}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-egal}(\mathcal{M})}$ , every vertex must be positively valued by some agent, as otherwise both agents can receive all of their valued items in connected allocations. To see this, suppose that some vertex (other than  $v_{1,1}$  and  $v_{1,2}$ ) is not valued by any agent. If the graph is  $K_5$  with a single edge removed, then agent 2's valued items must be connected, meaning agent 1 can take the non-valued vertex to obtain all of its valued items. If  $m \geq 6$  and agent 1 values some vertex other than  $v_{1,1}$  and  $v_{1,2}$ , then all of agent 1's valued items are connected, and agent 2 can take the non-valued vertex to obtain all of its valued items. If  $m \geq 6$  and agent 1 only values  $v_{1,1}$  and  $v_{1,2}$ , then either agent 2 values at least three items, or there are at least two non-valued vertices. In either of these cases, both agents can receive all of their valued items in a connected allocation, with agent 1 taking a non-valued vertex and all of its valued items, and agent 2 taking the other non-valued vertex if one exists, or otherwise all of its items (which must be connected since in a complete graph with a matching removed, any subset of three vertices is connected). It follows that to maximize the egalitarian welfare ratio, every vertex must be positively valued by some agent, and we can also see that we should require one agent (say, agent 1) to value only two items, since if both agents value at least three items each, then all of their valued items are connected. In the egalitarian welfare ratio-maximizing instance with agent 1 only valuing  $v_{1,1}$  and  $v_{1,2}$  and agent 2 valuing every other vertex, the allocation where agent 1 obtains  $v_{1,1}$ ,  $v_{1,2}$  and agent 2's least-valued item guarantees an egalitarian social welfare of at least  $\frac{m-3}{m-2} \cdot u_2(G)$ . We have  $\text{OPT-egal}(I) \leq u_2(G)$ , so comparing these terms, we see that the egalitarian PoC is at most  $\frac{m-2}{m-3}$ , matching the lower bound we derived earlier and concluding the proof.  $\square$

Following this, we address another class of dense graphs, complete bipartite graphs, where we find that the egalitarian PoC depends on the number of vertices on the smaller side of the graph.

**Theorem 3.4.** *Let  $G$  be a complete bipartite graph with  $x$  vertices on the smaller side. Then,*

$$\text{Egal-PoC}(G, 2) = \begin{cases} m - 1 & \text{if } x = 1; \\ \frac{x}{x-1} & \text{otherwise.} \end{cases}$$

*Proof.* The case where  $x = 1$  is given by Theorem 3.12 (for stars), so we suppose that  $x \geq 2$  in the remainder of the proof. Denote the two sides of the graph by  $V_1$  and  $V_2$ . The lower bound follows from letting agent 1 value each vertex of  $V_1$  at  $1/|V_1|$ , and agent 2 value each vertex of  $V_2$  at  $1/|V_2|$ . It remains to prove the upper bound.

By Lemma 3.2, it suffices to consider instances where each vertex is positively valued by at most one agent. Since  $G$  is a complete bipartite graph, an agent that receives a vertex on one side

can also receive any subset of vertices on the other side in a connected allocation. As a result, both agents can obtain all of their valued vertices (resulting in an egalitarian welfare of  $\min_{i \in [2]} u_i(G)$ ) if for each  $i \in [2]$ , either  $V_i$  contains a vertex  $v_i$  such that  $u_1(v_i) = u_2(v_i) = 0$ , or  $V_i$  contains vertices  $v_i$  and  $v'_i$  such that  $u_1(v_i) > 0$  and  $u_2(v'_i) > 0$ , or both of those conditions hold for  $V_i$ . Since the optimal egalitarian allocation is trivial for these cases, in the remainder of the proof we suppose without loss of generality that each vertex in  $V_1$  is positively valued by agent 1.

We complete the proof by describing an allocation which guarantees that each agent receives at least  $\frac{x-1}{x}$  of the optimal egalitarian welfare. Let agent 1 receive all but its least-valued vertex in  $V_1$  and all of its positively valued vertices in  $V_2$ . If agent 1 does not value any vertex in  $V_2$ , it receives agent 2's least-valued vertex in  $V_2$ . Also, let agent 2 receive all remaining vertices of the graph. At worst, agent 2 receives all of its valued vertices except its least-valued one, which occurs when it values all vertices of  $V_2$ . Therefore, agent 1 (resp., agent 2) receives a utility of at least  $\frac{|V_1|-1}{|V_1|} \cdot u_1(G)$  (resp.,  $\frac{|V_2|-1}{|V_2|} \cdot u_2(G)$ ). We thus have  $\text{Egal-PoC}(G, 2) \leq \frac{1}{\min\left\{\frac{|V_1|-1}{|V_1|}, \frac{|V_2|-1}{|V_2|}\right\}} = \frac{x}{x-1}$ .  $\square$

We remark that the class of complete graphs with a matching removed can be represented by a *complete k-partite graph*  $K_{n_1, n_2, \dots, n_k}$  where  $n_1, \dots, n_k \in \{1, 2\}$ : vertices are partitioned into  $k$  independent sets of size 1 or 2, and there is an edge between every pair of vertices from different independent sets [Chartrand and Zhang, 2020, p. 42]. This class forms a special case of the *Turán graphs* [Bollobás, 1998, p. 108], but is more general than the *cocktail-party graphs* [Biggs, 1993, p. 17].

We now give results for graphs classified by *vertex connectivity* (sometimes referred to simply as *connectivity*). A graph has connectivity  $k$  if there exist  $k$  vertices whose removal results in the graph being disconnected and  $k$  is the smallest number with this property. We begin with graphs with connectivity 2. We first prove a useful lemma, which states that if a graph admits a bipolar ordering,<sup>2</sup> its egalitarian PoC is at most 2. The proof of the lemma gives an algorithm for finding a connected allocation which guarantees at least half of the optimal egalitarian welfare.

**Lemma 3.5.** *Suppose that  $G$  admits a bipolar ordering. Then,  $\text{Egal-PoC}(G, 2) \leq 2$ .*

*Proof.* By Lemma 3.2, it suffices to consider the case where each item is positively valued by at most one agent. Take the bipolar ordering of the graph vertices, and starting from vertex 1, iteratively build a subgraph in ascending order of bipolar number until some agent  $i$  values it at least  $u_i(G)/2$ . Give the subgraph to agent  $i$  and the remaining subgraph to the other agent  $j$ . Since each vertex is valued by at most one agent, agent  $i$ 's subgraph must be worth less than  $u_j(G)/2$  utility to agent  $j$ ; otherwise agent  $j$  would have received it earlier. Therefore agent  $j$  receives more than  $u_j(G)/2$  utility from its subgraph. Since every vertex assigned with a number  $i \in \{2, \dots, m-1\}$  is adjacent to a vertex with a higher number and a vertex with a lower number, we see that agent  $i$ 's subgraph, which consists of vertices numbered  $1, \dots, k$  for some  $k$ , is connected. Similarly, agent  $j$ 's subgraph, which consists of vertices  $k+1, \dots, m$  is also connected. Both agents receive at least half of their total utility for the graph, which means that the egalitarian price of connectivity is at most 2.  $\square$

We now show the egalitarian PoC for graphs with connectivity 2.

**Theorem 3.6.** *Let  $G$  be a graph with connectivity 2. Then,  $\text{Egal-PoC}(G, 2) = 2$ .*

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<sup>2</sup>A bipolar ordering of a graph  $G = (V, E)$  is a one-to-one assignment of the integers  $1, \dots, m$  to  $V$  such that every vertex assigned with a number  $i \in \{2, \dots, m-1\}$  is adjacent to a vertex with a higher number and a vertex with a lower number.

*Proof.* We first show that if a graph is not 2-linked,<sup>3</sup> its egalitarian PoC is at least 2.

**Lemma 3.7.** *Let  $G$  be a graph that is not 2-linked. Then,  $\text{Egal-PoC}(G, 2) \geq 2$ .*

*Proof.* Since  $G$  is not 2-linked, let  $(a, b)$  and  $(c, d)$  be disjoint pairs of vertices such that there do not exist two disjoint connected subgraphs  $G_1$  and  $G_2$  of  $G$  where  $a, b \in G_1$  and  $c, d \in G_2$ . Let agent 1 value  $a$  and  $b$  at  $1/2$  each, and let agent 2 value  $c$  and  $d$  at  $1/2$  each. If agent 1 receives more than  $1/2$  utility, it must obtain a connected subgraph containing both  $a$  and  $b$ , but after this subgraph is allocated to agent 1, there cannot exist a connected subgraph containing both  $c$  and  $d$  for agent 2. Therefore the connected egalitarian welfare of this instance is  $1/2$ , showing that  $\text{Egal-PoC}(G, 2) \geq 2$ .  $\square$

A graph with connectivity 2 is not 2-linked,<sup>4</sup> and therefore by Lemma 3.7, has an egalitarian PoC of at least 2. Such a graph has a bipolar ordering [Maon et al., 1986], so the upper bound follows from Lemma 3.5.  $\square$

Next, we consider the large class of graphs with connectivity 1, which includes tree graphs. We find that the upper bound depends on the maximum number of disjoint connected subgraphs when a single vertex is removed, and interestingly, the PoC may be lower if the block decomposition of the graph is a path. A *block* is a maximal subgraph with connectivity at least 2 of a graph, and a *cut vertex* is a vertex whose removal disconnects a graph. The *block decomposition* of a graph  $G$  is a bipartite graph with all blocks of  $G$  on one side and all cut vertices of  $G$  on the other side. There is an edge between a block and a cut vertex in the bipartite graph if and only if the cut vertex belongs to the block in  $G$ . See, for example, Figure 3 for a demonstration of block decomposition. For any connected graph  $G$ , the block decomposition of  $G$  is a tree [Bondy and Murty, 2008].

**Theorem 3.8.** *Let  $G$  be a graph with connectivity 1. Denote by  $d$  the maximum number of connected subgraphs after a vertex has been removed from  $G$ . When  $d = 2$ ,  $\text{Egal-PoC}(G, 2) = 2$  if the block decomposition of  $G$  is a path, and  $\text{Egal-PoC}(G, 2) = 3$  otherwise. When  $d \geq 3$ ,  $\text{Egal-PoC}(G, 2) = d$ .*

*Proof.* We start with the lower bound and then show the upper bound.

**Lower Bound** We begin with the case where  $d = 2$ . If the block decomposition of  $G$  forms a path, an egalitarian PoC lower bound of 2 can be found by taking any cut vertex  $v_1$  and labelling the two disjoint pieces resulting from removing  $v_1$  from  $G$  as  $G_1$  and  $G_2$ . The lower bound follows from one agent having 1 utility for  $v_1$ , and the other agent having  $\frac{1}{2}$  utility for  $G_1$  and  $G_2$  each.

We now show that the egalitarian PoC is at least 3 if  $d = 2$  and the block decomposition of  $G$  is not a path (see, e.g., Figure 3). Recall that a block decomposition is a tree and cannot contain any cycles, and that the degree of a block equals the number of cut vertices in it. Every cut vertex in the block decomposition must be adjacent to at most 2 blocks because  $d = 2$ . If there is only one block (with degree 0), then this contradicts  $d = 2$ , and if all blocks have degree 1 or 2, then the block decomposition is a path. Therefore there must exist a block  $B$  with degree at least 3. Let agent 1 value 3 of the block's cut vertices at  $1/3$  each. When these 3 cut vertices are removed in  $G$ , 3 connected subgraphs  $G_1$ ,  $G_2$  and  $G_3$  become disjoint from  $B$ . Let agent 2 value  $G_1$ ,  $G_2$  and  $G_3$  at  $1/3$  each. Agent 2 can only attain more than  $1/3$  utility from a connected subgraph that

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<sup>3</sup>A graph is 2-linked if for any disjoint pairs of vertices  $(a, b)$  and  $(c, d)$ , there exist two disjoint connected subgraphs  $G_1$  and  $G_2$  of  $G$  where  $a, b \in G_1$  and  $c, d \in G_2$ .

<sup>4</sup>This can be seen by letting  $a$  and  $b$  be the vertices which, when removed from  $G$ , result in at least two disjoint connected subgraphs  $G_1, G_2, \dots$ , and by taking vertices  $c \in G_1$  and  $d \in G_2$ .

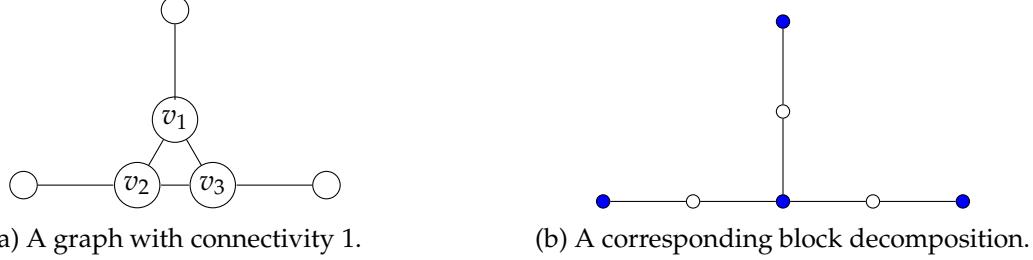


Figure 3: An example in the proof of Theorem 3.8. Figure 3(a) gives a graph which has a maximum number of 2 connected subgraphs after a vertex is removed, i.e.,  $d = 2$ . Vertices  $v_1, v_2, v_3$  are cut vertices. Figure 3(b) demonstrates its block decomposition, which is not a path. Blue vertices correspond to blocks and white vertices correspond to cut vertices.

includes two of agent 1’s valued cut vertices, so the optimal connected egalitarian social welfare of this instance is  $1/3$ . Since the optimal (possibly disconnected) egalitarian welfare for this instance is 1, we find that the PoC in this case is at least 3.

It remains to prove the lower bound for  $d \geq 3$ . Let agent 1 have 1 utility for the vertex  $v_1$ , which when removed, results in  $d$  disjoint connected subgraphs  $G_1, \dots, G_d$ . If agent 2 has  $1/d$  utility for each of  $G_1, \dots, G_d$ , we see that the optimal connected egalitarian welfare is  $1/d$ , proving the lower bound.

**Upper Bound** If  $d = 2$  and the block decomposition of  $G$  forms a path, then  $G$  has a bipolar ordering [Bilò et al., 2022], and therefore by Lemma 3.5,  $G$  has an egalitarian PoC of at most 2. It remains to prove the theorem for the cases where  $d = 2$  and the block decomposition of  $G$  does not form a path, and where  $d \geq 3$ . In particular, we show that  $\text{Egal-PoC}(G, 2) \leq \max\{d, 3\}$ . By Lemma 3.2, it suffices to consider the case where each item in a given instance  $I$  is positively valued by at most one agent, meaning that  $\text{OPT-egal}(I) = \min_{i \in [2]} u_i(G)$ . Let  $\gamma = \max\{d, 3\}$ . To show that  $\text{Egal-PoC}(G, 2) \leq \gamma$ , we give an algorithm which produces a subtree  $T'$  of  $G$  such that  $T'$  is worth at least  $\frac{u_i(G)}{\gamma}$  to some agent  $i$ , and  $G \setminus T'$  is connected and worth at least  $\frac{u_j(G)}{\gamma}$  to the other agent  $j$ . The pseudocode can be found in Algorithm 1.

We now demonstrate the correctness of the algorithm. Firstly, when merging subtrees in lines 3 and 12, we remove the edge connecting one of the constituent subtrees to the root of the tree, to prevent any cycles from being created. The while-loop in line 3 eventually terminates, because after all subtrees are merged where possible, there will be at most  $d$  subtrees, where  $d \leq \gamma$  by the definition of  $\gamma$ . Therefore in line 5, we know such a subtree must exist because the root of  $T$  is positively valued by at most one agent. If the algorithm terminates in line 9, the condition requires that  $T'$  and  $G \setminus T'$  meet the utility requirements, and we know they are both connected because  $T'$  is a subtree of the spanning tree  $T$ .

If the algorithm did not terminate in line 9, then  $u_1(T') \geq \frac{u_1(G)}{\gamma}$  and  $u_2(T') \geq \frac{u_2(G)}{\gamma}$ . This is because if  $u_i(T') \geq \frac{u_i(G)}{\gamma}$  for some agent  $i$  and  $u_j(T') < \frac{u_j(G)}{\gamma}$  for the other agent  $j$ , then  $u_j(T \setminus T') > u_j(G)(1 - \frac{1}{\gamma}) \geq \frac{u_j(G)}{\gamma}$ . Furthermore, all subtrees of  $T'$  are worth strictly lower than  $\frac{u_1(G)}{\gamma}$  to agent 1 and strictly lower than  $\frac{u_2(G)}{\gamma}$  to agent 2, as otherwise we would have set  $T'$  to be a further subtree in line 7. Finally, we know that  $u_1(T \setminus T') < \frac{u_1(G)}{\gamma}$  and  $u_2(T \setminus T') < \frac{u_2(G)}{\gamma}$ , as otherwise we would have been able to allocate  $T \setminus T'$  to one agent and  $T'$  to another agent in line 9.

When the original spanning tree  $T$  is re-rooted in line 11,  $T \setminus T'$  becomes a subtree of  $T$ , and the

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**Algorithm 1:** Allocating a Graph with Connectivity 1 for Two Agents

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**Input:** An instance  $I = \langle [2], G, \mathcal{U} \rangle$ . (Each vertex in  $G$  is valued positively by at most one agent.)

- 1  $\gamma \leftarrow \max\{d, 3\}$ , where  $d$  is the maximum number of connected subgraphs after a vertex is removed.
- 2 Take arbitrary rooted spanning tree  $T$  of  $G$ .
- 3 **while** all subtrees of  $T$  are worth less than  $u_i(G)/\gamma$  for all  $i \in [2]$  **do**
- 4   Take two subtrees of  $T$  which are connected by an edge in  $G$ , and merge them into one connected subtree with that edge.
- 5 Take subtree  $T'$  of  $T$  where  $u_i(T') \geq u_i(G)/\gamma$  for some agent  $i$ .
- 6 **while** there exists a subtree of  $T'$  worth at least  $u_i(G)/\gamma$  to some agent  $i$  **do**
- 7   Set  $T'$  to be such a further subtree.
- 8 **if**  $u_i(T') \geq \frac{u_i(G)}{\gamma}$  for some  $i$  and  $u_j(G \setminus T') \geq \frac{u_j(G)}{\gamma}$  for the other agent  $j$  **then**
- 9     **return**  $T'$  for agent  $i$  and  $G \setminus T'$  for agent  $j$
- 10 **else**
- 11     Root the original spanning tree  $T$  at the root of  $T'$ .
- 12 **while** all subtrees of  $T$  are worth less than  $u_i(G)/\gamma$  for all  $i \in [2]$  **do**
- 13     Take two subtrees of  $T$  which are connected by an edge in  $G$ , and merge them into one connected subtree with that edge.
- 14 Take subtree  $T'$  of  $T$  where  $u_i(T') \geq u_i(G)/\gamma$  for some agent  $i$ .
- 15 **return**  $T'$  for agent  $i$  and  $G \setminus T'$  for the other agent  $j$

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other subtrees of  $T$  are the same as the ‘further subtrees’ of  $T'$  which are worth strictly lower than  $\frac{u_1(G)}{\gamma}$  to agent 1 and strictly lower than  $\frac{u_2(G)}{\gamma}$  to agent 2. As previously explained, the while-loop in line 12 eventually terminates, so in line 14,  $u_i(T') \geq \frac{u_i(G)}{\gamma}$  for some agent  $i$  and  $u_j(T') < \frac{2u_j(G)}{\gamma}$  for the other agent  $j$ . This is because the two subtrees forming  $T'$  were worth less than  $\frac{u_1(G)}{\gamma}$  to agent 1 and less than  $\frac{u_2(G)}{\gamma}$  to agent 2 before merging. Therefore  $u_j(T \setminus T') > u_j(G)(1 - \frac{2}{\gamma}) \geq \frac{u_j(G)}{\gamma}$ .  $\square$

While we have given results for classes of dense and sparse graphs, it remains an intriguing question to give a complete characterization of the egalitarian price of connectivity in the 2-agent case. Even when agents have identical valuations, the egalitarian PoC for 2 agents is still unknown, as seen in Conjecture 3.10 of Bei et al. [2022]. Settling their conjecture could be a crucial step to a complete characterization in our setting where agents have heterogeneous valuations.

### 3.2 Three Agents

Our main result in this subsection shows that when there are three agents, the egalitarian price of connectivity for a tree graph is equal to the maximum number of disjoint connected subgraphs when two vertices are removed, which we denote by  $\delta$ .

**Theorem 3.9.** *Let  $G$  be a tree. Then,  $\text{Egal-PoC}(G, 3) = \delta$ , where  $\delta$  is the maximum number of disjoint connected subgraphs when two vertices are removed from  $G$ .*

We first show that  $\delta$  depends on the degrees of the two highest degree vertices, whose degrees we denote by  $\Delta_1(G)$  and  $\Delta_2(G)$ , as well as whether these two vertices are adjacent to each other.

In case of ties, we favor non-adjacent highest degree vertices.

**Lemma 3.10.** *Let  $G$  be a tree. Then,  $\delta = \Delta_1(G) + \Delta_2(G) - 1$  if there exist two non-adjacent highest degree vertices, and  $\delta = \Delta_1(G) + \Delta_2(G) - 2$  otherwise.*

*Proof.* Observe that removing two vertices with degree  $d_1$  and  $d_2$  results in  $d_1 + d_2 - 1$  disjoint connected subgraphs if the two vertices are non-adjacent, and  $d_1 + d_2 - 2$  disjoint connected subgraphs otherwise. Hence, to obtain the largest number of disjoint connected subgraphs, it is always optimal to choose two highest degree vertices, favoring non-adjacent ones if possible.  $\square$

We now prove the main result for this subsection.

*Proof of Theorem 3.9.* A tree with 3 or 4 vertices must be a path or a star, which will be covered in Theorem 3.12 (for stars) and Theorem 3.13 (for paths). We therefore assume that the tree has at least 5 vertices and is not a star, so that  $\delta \geq 3$  and  $\Delta_2(G) \geq 2$ . Let  $v_1$  and  $v_2$  denote the vertices with degrees  $\Delta_1(G)$  and  $\Delta_2(G)$  whose removal results in  $\delta$  disjoint connected subgraphs.

**Lower Bound** Let agents 1 and 2 have utility 1 for  $v_1$  and  $v_2$  respectively, and let agent 3 have utility  $1/\delta$  for each of the disjoint connected subgraphs resulting from the removal of  $v_1$  and  $v_2$ . It can be verified that (i) the optimal egalitarian welfare of the instance is 1, and (ii) the optimal egalitarian welfare of a connected allocation is  $1/\delta$ , achieved by giving agents 1 and 2 vertices  $v_1$  and  $v_2$ , respectively, and giving agent 3 one of the disjoint connected subgraphs. This shows that  $\text{Egal-PoC}(G, 2) \geq \delta$ .

**Upper Bound** By Lemma 3.2, it suffices to consider instances where each vertex is positively valued by at most one agent, so we have  $\text{OPT-egal}(I) = \min_{i \in [3]} u_i(G)$ . We prove the upper bound by designing Algorithms 2 and 3, which, at a high level, find a subtree  $T'$  of  $G$  such that  $u_i(T') \geq u_i(G)/\delta$  for some agent  $i$  and such that the remaining subtree  $G \setminus T'$  is sufficiently valued and can be divided between the other two agents. Below, we split the algorithm and proof into two cases depending on whether  $\delta = \Delta_1(G) + \Delta_2(G) - 1$  or  $\delta = \Delta_1(G) + \Delta_2(G) - 2$ .

**Case 1:**  $\delta = \Delta_1(G) + \Delta_2(G) - 1$ . We start with a lemma which shows that if we can take a subtree  $T'$  from  $G$  such that:

- $T'$  is worth at least  $u_i(G)/\delta$  to some agent  $i$ , and
- $T'$  is worth at most  $\frac{\Delta_2(G)-1}{\delta} \cdot u_j(G)$  to all agents  $j \neq i$ ; in other words,  $u_j(T^* := G \setminus T') \geq \frac{\Delta_1(G)}{\delta} \cdot u_j(G)$ ,

then we can allocate  $T'$  to agent  $i$  and divide  $G \setminus T'$  between the remaining agents  $[3] \setminus \{i\}$  to complete a connected allocation.

**Lemma 3.11.** *Let  $T^*$  be a subtree of  $G$  such that two agents  $i$  and  $j$  value  $T^*$  at least  $\frac{\Delta_1(G)}{\delta} \cdot u_i(G)$  and  $\frac{\Delta_1(G)}{\delta} \cdot u_j(G)$ , respectively. Then,  $T^*$  can be divided between the two agents so that each agent receives a connected bundle worth at least  $u_i(G)/\delta$  and  $u_j(G)/\delta$ , respectively.*

*Proof.* Theorem 3.8 shows that a tree  $T^*$  can be divided between two agents  $i$  and  $j$  so that each agent receives a connected bundle worth at least  $\frac{u_i(T^*)}{\Delta_1(T^*)}$  and  $\frac{u_j(T^*)}{\Delta_1(T^*)}$ , respectively. Due to  $u_i(T^*) \geq \frac{\Delta_1(G)}{\delta} \cdot u_i(G)$  and  $u_j(T^*) \geq \frac{\Delta_1(G)}{\delta} \cdot u_j(G)$  from the lemma statement and  $\Delta_1(G) \geq \Delta_1(T^*)$ , we can

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**Algorithm 2:** A Subroutine of Tree Allocation for Three Agents in Case 1

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**Input:** An instance  $I = \langle [3], G, \mathcal{U} \rangle$ . (Each vertex in  $G$  is valued positively by at most one agent.)

- 1 Root the tree at  $v_1$ , the vertex with degree  $\Delta_1(G)$ .
- 2  $\delta \leftarrow \Delta_1(G) + \Delta_2(G) - 1$
- 3 Take subtree  $T'$  of  $G$  where  $u_i(T') \geq u_i(G)/\delta$  for some agent  $i$ .
- 4 **while** there exists a branch  $T''$  of  $T'$  such that for some  $i \in [3]$ ,  $u_i(T'') \geq \frac{u_i(G)}{\delta}$  **do**
- 5     $T' \leftarrow T''$
- 6 **return**  $T'$

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divide  $T^*$  between agents  $i$  and  $j$  so that the agents receive a utility of at least  $\frac{\Delta_1(G)}{\delta \cdot \Delta_1(T^*)} \cdot u_i(G) \geq \frac{u_i(G)}{\delta}$  and  $\frac{\Delta_1(G)}{\delta \cdot \Delta_1(T^*)} \cdot u_j(G) \geq \frac{u_j(G)}{\delta}$ , respectively.  $\square$

We now prove the correctness of Algorithm 2 which, given a tree  $G$ , finds such a subtree  $T'$ . Firstly, the subtree found in line 3 always exists because  $v_1$  can only be positively valued by at most one agent, and  $\delta \geq \Delta_1(G)$ . This means that  $T' \neq \emptyset$  before executing the while-loop. By the design of Algorithm 2, when the while-loop terminates and the algorithm returns  $T'$ ,  $u_i(T') \geq \frac{u_i(G)}{\delta}$  for some  $i \in [3]$ . It remains to show  $u_j(T') \leq \frac{\Delta_2(G)-1}{\delta} \cdot u_j(G)$  for all  $j \in [3] \setminus \{i\}$ . Since the while-condition is evaluated as false for  $T'$ , for any branch  $T''$  of  $T'$  and any  $i \in [3]$ ,  $u_i(T'') < \frac{u_i(G)}{\delta}$ . Suppose the root of  $T'$  is vertex  $v$ . As  $T'$  has at most  $\Delta_2(G) - 1$  branches, for all  $i \in [3]$ ,  $u_i(T' \setminus \{v\}) < \frac{\Delta_2(G)-1}{\delta} \cdot u_i(G)$ . If  $u_i(v) = 0$  for all  $i \in [3]$ , meaning that  $u_i(T') = u_i(T' \setminus \{v\}) < \frac{\Delta_2(G)-1}{\delta} \cdot u_i(G)$  for all  $i \in [3]$ , then  $T'$  is as desired. Otherwise, denote by  $i^*$  the agent with  $u_{i^*}(v) > 0$ ; note that the other two agents value vertex  $v$  at 0. If  $u_{i^*}(T') \geq \frac{u_{i^*}(G)}{\delta}$ , then  $u_j(T') = u_j(T' \setminus \{v\}) < \frac{\Delta_2(G)-1}{\delta} \cdot u_j(G)$  for all  $j \in [3] \setminus \{i^*\}$ , as desired. Else,  $u_{i^*}(T') < \frac{u_{i^*}(G)}{\delta} \leq \frac{\Delta_2(G)-1}{\delta} \cdot u_{i^*}(G)$ ; recall that  $\Delta_2(G) \geq 2$ . Thus, some agent  $i \in [3] \setminus \{i^*\}$  values  $T'$  at least  $\frac{u_i(G)}{\delta}$ , and for the agent (denoted as  $k$ ) other than  $i^*$  and  $i$ ,  $u_k(T') = u_k(T' \setminus \{v\}) < \frac{\Delta_2(G)-1}{\delta} \cdot u_k(G)$ , as desired.

**Case 2:**  $\delta = \Delta_1(G) + \Delta_2(G) - 2$ . Similar in spirit to Case 1, Algorithm 3 finds a subtree  $T'$  from  $G$  that is worth at least  $u_i(G)/\delta$  to some agent  $i$  and  $G \setminus T'$  is still valuable enough to both of the remaining agents. Let tree  $G$  be rooted at vertex  $v_1$ , the vertex with degree  $\Delta_1(G)$ . Recall that in this case, the two highest degree vertices are adjacent to each other. This means that any vertex of distance at least two from  $v_1$  has degree at most  $\Delta_2(G) - 1$ ; otherwise we would have two non-adjacent highest degree vertices.

We now prove that Algorithm 3 finds a desirable subtree  $T'$  from  $G$ . To begin, the (first) subtree  $T'$  found in line 3 always exists because  $v_1$  can only be positively valued by at most one agent, and  $\delta \geq \Delta_1(G)$ . We next distinguish cases based on whether the root of  $T'$ , which is not necessarily vertex  $v_2$  and denoted as vertex  $\hat{v}$ , has degree  $\Delta_2(G)$  or not, because we will impose different utility thresholds according to the degrees of the subtrees.

We first address the special case in which line 4 is evaluated as true, i.e.,  $\deg(\hat{v}) = \Delta_2(G)$ , and the inner if-condition is met. As a result, the subtree  $T'$  is returned in line 6. In the following, we give a procedure to compute a desired connected allocation. Give  $T'$  to the agent  $i$  identified in the inner if-condition, so the agent gets a utility of at least  $u_i(G)/\delta$ . By the design of the algorithm,

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**Algorithm 3:** A Subroutine of Tree Allocation for Three Agents in Case 2

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**Input:** An instance  $I = \langle [3], G, \mathcal{U} \rangle$ . (Each vertex in  $G$  is valued positively by at most one agent.)

- 1 Root the tree at  $v_1$ , the vertex with degree  $\Delta_1(G)$ .
- 2  $\delta \leftarrow \Delta_1(G) + \Delta_2(G) - 2$
- 3 Take subtree  $T'$  of  $G$  where  $u_i(T') \geq u_i(G)/\delta$  for some agent  $i$ .
- 4 **if** the root of  $T'$  has degree  $\Delta_2(G)$  **then**
- 5     **if** there exists  $i$  such that  $u_i(T') \geq \frac{u_i(G)}{\delta}$  and  $u_j(T') \leq \frac{\Delta_2(G)-1}{\delta} \cdot u_j(G)$  for all  $j \neq i$  **then**
- 6         **return**  $T'$
- 7     **else**
- 8         Set  $T''$  to be a subtree of  $T'$  where  $u_i(T'') \geq u_i(G)/\delta$  to some agent  $i$ .
- 9          $T' \leftarrow T''$
- 10 **while** there exists a branch  $T''$  of  $T'$  such that for some  $i \in [3]$ ,  $u_i(T'') \geq \frac{u_i(G)}{\delta}$  **do**
- 11      $T' \leftarrow T''$
- 12 **return**  $T'$

---

for all agents  $j \neq i$ , we have  $u_j(T') \leq \frac{\Delta_2(G)-1}{\delta} \cdot u_j(G)$ , meaning that

$$u_j(T^* := G \setminus T') \geq \left(1 - \frac{\Delta_2(G)-1}{\delta}\right) \cdot u_j(G) = \frac{\Delta_1(G)-1}{\delta} \cdot u_j(G).$$

Due to Theorem 3.8 and the fact that  $\Delta_1(T^*) \leq \Delta_1(G) - 1$ , tree  $T^*$  can be divided between agents  $j \in [3] \setminus \{i\}$  such that each agent receives a connected bundle worth at least  $\frac{\Delta_1(G)-1}{\delta \cdot \Delta_1(T^*)} \cdot u_j(G) \geq \frac{u_j(G)}{\delta}$ .

When the inner if-condition is evaluated as false, a further subtree  $T''$  where  $u_i(T'') \geq u_i(G)/\delta$  to some agent  $i$  is selected in line 8. We show that such a subtree always exists. Suppose otherwise, meaning that every subtree is valued strictly less than  $u_\alpha(G)/\delta$  to each agent  $\alpha \in \{i, j, k\}$ , and thus  $u_\alpha(T' \setminus \{\widehat{v}\}) < \frac{\Delta_2(G)-1}{\delta} \cdot u_\alpha(G)$ . If no agent values vertex  $\widehat{v}$  positively, then for all  $i \in [3]$ ,  $u_i(T') = u_i(T' \setminus \{\widehat{v}\}) < \frac{\Delta_2(G)-1}{\delta} \cdot u_i(G)$ . Since some agent  $i \in [3]$  values  $T'$  at least  $u_i(G)/\delta$ , the if-condition would have been evaluated as true for  $T'$ , a contradiction. Hence, some agent  $j$  positively values  $\widehat{v}$ , and  $T'$  is worth over  $\frac{\Delta_2(G)-1}{\delta} \cdot u_j(G)$  to the agent. However, since each vertex can only be positively valued by one agent, and  $\widehat{v}$  is valued by agent  $j$ , then we know that  $T'$  is valued at most  $\frac{\Delta_2(G)-1}{\delta} \cdot u_i(G)$  and  $\frac{\Delta_2(G)-1}{\delta} \cdot u_k(G)$  to the other agents  $i$  and  $k$ . This means that  $T'$  meets the inner if-condition, and would have been returned in line 6, a contradiction.

We now address the general case in the while-loop. Note that all subtrees considered from now on have maximum degree at most  $\Delta_2(G) - 2$ ; otherwise we would have two non-adjacent highest degree vertices and have been in Case 1. We will show that the algorithm terminates when the following set of required conditions on  $T'$  are met:

- $T'$  is worth at least  $u_i(G)/\delta$  to some agent  $i$ , and
- $T'$  is worth at most  $\frac{\Delta_2(G)-2}{\delta} \cdot u_j(G)$  to all agents  $j \neq i$ . Equivalently,  $u_j(T^* := G \setminus T') \geq \left(1 - \frac{\Delta_2(G)-2}{\delta}\right) \cdot u_j(G) = \frac{\Delta_1(G)}{\delta} \cdot u_j(G)$ .

This means that we can allocate  $T'$  to agent  $i$ . Moreover, by Lemma 3.11 proved in Case 1, tree  $T^*$  can be divided between agents  $j \in [3] \setminus \{i\}$  so that each agent receives a connected bundle worth at least  $\frac{\Delta_1(G)}{\delta \cdot \Delta_1(T^*)} \cdot u_j(G) \geq \frac{u_j(G)}{\delta}$ .

We have just shown that  $T' \neq \emptyset$  before the execution of the while-loop. Then, by the design of Algorithm 3, when the while-loop terminates and the algorithm returns  $T'$ ,  $u_i(T') \geq \frac{u_i(G)}{\delta}$  for some  $i \in [3]$ . It remains to show that  $u_j(T') \leq \frac{\Delta_2(G)-2}{\delta} \cdot u_j(G)$  for all  $j \in [3] \setminus \{i\}$ . (The proof is similar to that in Case 1, but we provide the full argument here for the sake of completeness.) Since the while-condition is evaluated as false for  $T'$ , for any branch  $T''$  of  $T'$  and any  $i \in [3]$ ,  $u_i(T'') < \frac{u_i(G)}{\delta}$ . Suppose the root of  $T'$  is vertex  $v$ . As  $T'$  has at most  $\Delta_2(G) - 2$  branches, for all  $i \in [3]$ ,  $u_i(T' \setminus \{v\}) \leq \frac{\Delta_2(G)-2}{\delta} \cdot u_i(G)$ .<sup>5</sup> If  $u_i(v) = 0$  for all  $i \in [3]$ , meaning that  $u_i(T') = u_i(T' \setminus \{v\}) \leq \frac{\Delta_2(G)-2}{\delta} \cdot u_i(G)$  for all  $i \in [3]$ , then  $T'$  is as desired. Otherwise, denote by  $i^*$  the agent with  $u_{i^*}(v) > 0$ ; note that the other two agents value vertex  $v$  at 0. If  $u_{i^*}(T') \geq \frac{u_{i^*}(G)}{\delta}$ , then  $u_j(T') = u_j(T' \setminus \{v\}) \leq \frac{\Delta_2(G)-2}{\delta} \cdot u_i(G)$  for all  $j \in [3] \setminus \{i^*\}$ , as desired. Else,  $u_{i^*}(T') < \frac{u_{i^*}(G)}{\delta}$ . Since some agent  $i \in [3] \setminus \{i^*\}$  values  $T'$  at least  $\frac{u_i(G)}{\delta}$  and  $u_i(v) = 0$ , it must be the case that  $v$  has at least two children. Thus,  $\Delta_2(G) \geq 3$ . This implies that  $u_{i^*}(T') < \frac{u_{i^*}(G)}{\delta} \leq \frac{\Delta_2(G)-2}{\delta} \cdot u_{i^*}(G)$ . Also, for the agent (denoted as  $k$ ) other than  $i^*$  and  $i$ ,  $u_k(T') = u_k(T' \setminus \{v\}) \leq \frac{\Delta_2(G)-2}{\delta} \cdot u_k(G)$ , as desired.  $\square$

### 3.3 Any Number of Agents

We now prove tight egalitarian PoC for any number of agents, beginning with stars.

**Theorem 3.12.** *Let  $n \geq 2$  and  $G$  be a star. Then,  $\text{Egal-PoC}(G, n) = m - n + 1$ .*

*Proof.* First, from Lemma 3.1, we know that the price is at most  $m - n + 1$ . Next, the lower bound example simply has one agent equally valuing  $m - n + 1$  of the leaf vertices, and the remaining agents valuing one distinct vertex each at utility 1.  $\square$

Our next result is for paths.

**Theorem 3.13.** *Let  $n \geq 2$  and  $G$  be a path. Then,*

$$\text{Egal-PoC}(G, n) = \begin{cases} m - n + 1 & \text{if } n \leq m < 2n - 1; \\ n & \text{if } 2n - 1 \leq m. \end{cases}$$

*Proof.* We first provide lower bound examples, followed by matching upper bounds.

**Lower Bound** We begin with the case where  $n \leq m < 2n - 1$ . Consider a path with vertices  $\{1, 2, \dots, m\}$ , labelled one by one from left to right, and the agents have the following utilities:

- Agent 1 values  $m - n + 1$  vertices equally at  $\frac{1}{m-n+1}$  each, and moreover, none of these vertices is adjacent to each other. Note that this can be done since  $n \leq m < 2n - 1$ .
- The remaining  $n - 1$  agents value the  $n - 1$  vertices that are not valued by agent 1. In particular, each of the agents  $\{2, 3, \dots, n\}$  values a single different vertex (of the  $n - 1$  vertices) at 1.

The example shows that  $\text{Egal-PoC}(G, n) \geq m - n + 1$ , since

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<sup>5</sup>We use “ $\leq$ ” instead of “ $<$ ” since  $\Delta_2(G) - 2$  may be equal to 0.

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**Algorithm 4:** Discretized Moving-Knife: Path Allocation for  $n$  Agents

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**Input:** An instance  $I = \langle N, G, \mathcal{U} \rangle$ . (Each vertex in  $G$  is valued by at most one agent.)  
**Output:** A connected allocation  $\mathcal{M}$  with  $\text{SW-egal}(\mathcal{M}) \geq \frac{1}{n} \cdot \text{OPT-egal}(I)$ .

```

1  $\mathcal{M} = (M_1, \dots, M_n) \leftarrow (\emptyset, \dots, \emptyset)$ 
2 while  $|N| \geq 2$  do
3    $B \leftarrow \emptyset$ 
4   Process the vertices along the path from left to right and add them one at a time to
      bundle  $B$  until for some  $i \in N$ ,  $u_i(B) \geq \frac{1}{n} \cdot \text{OPT-egal}(I)$ .
5    $M_i \leftarrow B$ 
6    $N \leftarrow N \setminus \{i\}$ ,  $G \leftarrow G \setminus B$ 
7 Give all remaining vertices to the last agent and update  $\mathcal{M}$  accordingly.
8 return A connected allocation  $\mathcal{M}$ .

```

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- $\text{OPT-egal}(I) = 1$ , by giving each agent all vertices that she values positively;
- $\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M}) = \frac{1}{m-n+1}$ , by giving each agent in  $\{2, 3, \dots, n\}$  the vertex that she values positively, giving agent 1 one vertex that she values positively, and then extending the allocation to be connected and complete.

We now prove the lower bound for the case where  $m \geq 2n - 1$ . Consider a path with vertices  $\{1, 2, \dots, m\}$ , labelled one by one from left to right, and the agents have the following utilities:

- For each agent  $i \in [n-1]$ ,  $u_i(2i) = 1$  and  $u_i(j) = 0$  for all  $j \in [m] \setminus \{2i\}$ .
- For agent  $n$ ,  $u_n(2i-1) = \frac{1}{n}$  for each  $i \in [n]$ . The agent values all other vertices at 0.

The example shows that  $\text{Egal-PoC}(G, n) \geq n$ , because

- $\text{OPT-egal}(I) = 1$ , by giving each agent all vertices that she values positively and then extending it to a complete allocation in an arbitrary way;
- $\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M}) = \frac{1}{n}$ , by giving each agent  $i \in [n-1]$  vertex  $2i$ , agent  $n$  one of her valued vertex, and then extending the allocation to be connected and complete.

**Upper Bound** For  $n \leq m < 2n - 1$ , the upper bound follows from Lemma 3.1, so we consider the case where  $m \geq 2n - 1$ . By Lemma 3.2, it suffices to show that given any instance  $I = \langle N, G, \mathcal{U} \rangle$  in which each vertex of the path  $G$  is positively valued by at most one agent, there always exists a connected allocation such that each agent  $i$  gets utility at least  $\frac{1}{n} \cdot \text{OPT-egal}(I)$ . Our algorithm is a discretized version of the well-known *moving-knife procedure*; the pseudocode can be found in Algorithm 4. Since each vertex is positively valued by at most one agent, whenever a vertex is added to bundle  $B$  in line 4, at most one agent's utility for bundle  $B$  strictly increases. As a result, at any point during the algorithm's run, only one agent may find bundle  $B$  to be worth at least  $\frac{1}{n} \cdot \text{OPT-egal}(I)$ . By the design of the algorithm, each agent removed from the instance in the while-loop receives a contiguous bundle of vertices and gets utility at least  $\frac{1}{n} \cdot \text{OPT-egal}(I)$ . Finally, the last agent values all remaining vertices at least  $\frac{1}{n} \cdot \text{OPT-egal}(I)$ , because otherwise  $\text{OPT-egal}(I) \leq u_i(G) < \frac{n-1}{n} \cdot \text{OPT-egal}(I) + \frac{1}{n} \cdot \text{OPT-egal}(I) = \text{OPT-egal}(I)$ , a contradiction.  $\square$

We finish this section with the egalitarian PoC for cycles.

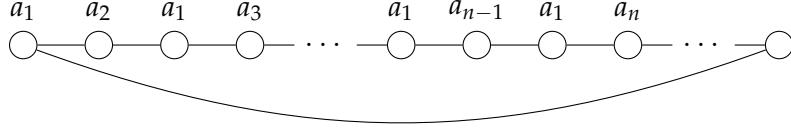


Figure 4: A cycle has  $2n - 2 \leq m < n^2$  vertices. Agent 1 values  $n - 1$  vertices at  $\frac{1}{n-1}$  each, and the remaining agents value their own item at 1.

**Theorem 3.14.** Let  $n \geq 2$  and  $G$  be a cycle. Then,

$$\text{Egal-PoC}(G, n) = \begin{cases} m - n + 1 & \text{if } n \leq m < 2n - 2; \\ n - 1 & \text{if } 2n - 2 \leq m < n^2; \\ n & \text{if } n^2 \leq m. \end{cases}$$

*Proof.* We distinguish three cases as follows.

**Case 1:**  $n \leq m < 2n - 2$ . The upper bound of  $m - n + 1$  follows from Lemma 3.1, so we prove the lower bound. Suppose for an instance  $I$  that agents  $2, 3, \dots, n$  only positively value one item at 1 each, and that agent 1 positively values the remaining  $m - n + 1$  items at  $\frac{1}{m-n+1}$  each, where each of these  $m - n + 1$  items is separated by at least one of the items valued by agents  $2, 3, \dots, n$ . It can be verified that the optimal egalitarian welfare of the instance is 1. The optimal egalitarian welfare of a connected allocation is  $\frac{1}{m-n+1}$ , by giving agents  $2, 3, \dots, n$  their only valued item and agent 1 one of her  $m - n + 1$  positively valued items. This completes the proof for this case.

**Case 2:**  $2n - 2 \leq m < n^2$ . We first prove the lower bound. Suppose for an instance  $I$  that each item is only valued by at most one agent. Suppose that the first  $2n - 2$  vertices of the cycle alternate between an item valued by agent 1 and an item valued by an agent in the sequence  $(2, \dots, n)$ ; see Figure 4. In this example, agent 1 positively values  $n - 1$  items at  $\frac{1}{n-1}$  each, and agents  $2, \dots, n$  each positively value only one item at 1. The example shows that  $\text{Egal-PoC}(G, n) \geq n - 1$  because (i)  $\text{OPT-egal}(I) = 1$ , and (ii)  $\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M}) = \frac{1}{n-1}$  by giving agents  $2, \dots, n$  their only valued item and agent 1 one of her positively valued items, and extending the allocation to be connected and complete.

We now prove the upper bound. By Lemma 3.2, it suffices to show that given any instance  $I = \langle N, G, \mathcal{U} \rangle$  in which each vertex of the cycle is positively valued by at most one agent, there always exists a connected allocation such that each agent gets utility at least  $\frac{\text{OPT-egal}(I)}{n-1}$ . By the pigeonhole principle, at least one agent has at most  $n - 1$  positively valued items. By giving one such agent her most preferred item, she receives a utility of at least  $\frac{\text{OPT-egal}(I)}{n-1}$ . The remaining graph is a path graph with at least  $2n - 3$  vertices, which is to be divided amongst  $n - 1$  agents. From Theorem 3.13, we know that when there are  $n - 1$  agents and  $m \geq 2(n - 1) - 1$  vertices on a path, the egalitarian PoC is at most  $n - 1$  (i.e., the path graph can be divided so that each agent receives at least  $\frac{1}{n-1}$  of the optimal egalitarian welfare). This completes the proof for this case.

**Case 3:**  $m \geq n^2$ . We first prove the lower bound. Suppose for an instance  $I$  that  $m = n^2$ , and that each agent positively values  $n$  items at  $\frac{1}{n}$  each, with no two agents having positive utility for the same item. Thus the optimal egalitarian allocation (which we denote by  $(B_1, B_2, \dots, B_n)$ ) gives each agent all of the items that they positively value, resulting in  $\text{OPT-egal}(I) = 1$ . Now

consider a cycle graph  $G$  where the items from  $B_1, B_2, \dots, B_n$  are connected in the repeating sequence  $(1, 2, \dots, n)^*$ . For any agent (say, agent 1) to receive 2 items that she positively values, her allocation must consist of at least  $n + 1$  items: the 2 items from  $B_1$ , and 1 item from each of  $B_2, \dots, B_n$ . If  $n - 1$  agents each receive 2 items that they positively value, then their combined allocations will have  $n^2 - 1$  items, and thus the  $n$ -th agent can only receive 1 item that she positively values. Therefore, we have  $\max_{\mathcal{M} \in C(I)} \text{SW-egal}(\mathcal{M}) = \frac{1}{n}$  and so  $\text{Egal-PoC}(G, n) \geq n$ . The example can be extended to  $m > n^2$  by adding dummy items which no agent positively values.

The upper bound follows directly from Theorem 3.13, as we can break an arbitrary edge of the cycle and apply the corresponding algorithm in the proof of Theorem 3.13 to the path.  $\square$

## 4 Utilitarian Price of Connectivity

In this section, we present results on the utilitarian price of connectivity, which measures the worst-case loss of utilitarian welfare due to the connectivity constraint.

### 4.1 Two Agents

We begin by computing the utilitarian price of connectivity for instances with two agents. Similar to the egalitarian case, the utilitarian price of connectivity is strictly higher than 1 when just one edge is removed from a complete graph, but does not increase further when additional disjoint edges are removed (with the exception of the 5-item case).

**Theorem 4.1.** *Let  $G$  be a complete graph with a non-empty matching removed. Then,  $\text{Util-PoC}(G, 2) = \frac{4}{3}$  if  $G$  is*

- $K_3$  with an edge removed, or
- $L_5$ , which is  $K_5$  with two disjoint edges removed,

and  $\text{Util-PoC}(G, 2) = \frac{2m-4}{2m-5}$  otherwise.

*Proof.* First note that the optimal utilitarian allocation simply allocates each vertex to the agent that values it most, breaking ties arbitrarily.  $K_3$  with an edge removed is simply a path, which will be covered in Theorem 4.3.  $K_4$  with two disjoint edges removed is a cycle, which has a utilitarian PoC of  $\frac{4}{3}$ , as we will demonstrate in Theorem 4.4. This also implies an upper bound of  $\frac{4}{3}$  for  $K_4$  with only one edge removed, and for a matching lower bound example, let  $\{v_1, v_2\}$  be the missing edge, where  $u_1(v_1) = u_1(v_2) = 1/2$ , and let agent 2 value the remaining two vertices at  $1/2$  each. For the remainder of the proof we assume that  $m \geq 5$ .

**Lower Bound (for  $m \geq 5$ )** We begin with the special case where the graph is  $L_5$ . Denote the two missing edges by  $\{v_{1,1}, v_{1,2}\}$  and  $\{v_{2,1}, v_{2,2}\}$ . Let  $u_1(v_{1,1}) = u_1(v_{1,2}) = 1/2$ , and  $u_2(v_{2,1}) = u_2(v_{2,2}) = 1/2$ . Here,  $\text{OPT-util}(I) = 2$  and  $\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M}) = 3/2$ , showing that the utilitarian PoC is at least  $\frac{4}{3}$ .

For a graph that is not  $L_5$ , denote a missing edge by  $\{v_1, v_2\}$ , and let  $u_1(v_1) = u_1(v_2) = \frac{1}{2}$ . Also, let agent 2 have utility  $\frac{1}{m-2}$  for each of the remaining items. Here,  $\text{OPT-util}(I) = 2$  and  $\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M}) = 1 + \frac{m-3}{m-2} = \frac{2m-5}{m-2}$ , showing that the utilitarian PoC is at least  $\frac{2m-4}{2m-5}$ .

**Upper Bound (for  $m \geq 5$ )** To prove the upper bound, we show that among all possible instances, the instances described in the lower bound examples maximize  $\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M})}$ . For simplicity, we may refer to this ratio as the “utilitarian welfare ratio”. Suppose that there are  $k$  missing edges, denoted as  $\{v_{1,1}, v_{1,2}\}, \{v_{2,1}, v_{2,2}\}, \dots, \{v_{k,1}, v_{k,2}\}$ . Note that if there exists  $i \in [k]$  such that  $u_1(v_{i,1}) \geq u_2(v_{i,1})$  and  $u_1(v_{i,2}) \leq u_2(v_{i,2})$  (or vice versa), then every vertex can be allocated to the agent that values it most (resulting in  $\text{OPT-util}(I) = \max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M})$ ), as  $v_{i,1}$  and  $v_{i,2}$  are connected to every vertex except each other. Therefore, a connected allocation can achieve the optimal utilitarian welfare ratio unless for each  $i \in [k]$ , either  $u_1(v_{i,1}) > u_2(v_{i,1})$  and  $u_1(v_{i,2}) > u_2(v_{i,2})$ , or  $u_1(v_{i,1}) < u_2(v_{i,1})$  and  $u_1(v_{i,2}) < u_2(v_{i,2})$ . We therefore suppose without loss of generality that  $u_1(v_{1,1}) > u_2(v_{1,1})$  and  $u_1(v_{1,2}) > u_2(v_{1,2})$ . Also, note that to maximize  $\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M})}$ , both agents must value at least two vertices strictly more than the other agent, as otherwise we can simply allocate every vertex to an agent that values it most.

We now address the special case where the graph is  $L_5$ . We consider the restricted set of instances where  $u_1(v_{1,1}) > u_2(v_{1,1})$  and  $u_1(v_{1,2}) > u_2(v_{1,2})$ , as well as  $u_1(v_{2,1}) < u_2(v_{2,1})$  and  $u_1(v_{2,2}) < u_2(v_{2,2})$ , because otherwise we would have  $\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M})} = 1$ . Denote by  $v_5$  the remaining vertex. In the optimal (possibly disconnected) allocation, we allocate each vertex to the agent that values it most; suppose without loss of generality that  $v_5$  is valued weakly higher by agent 1. This gives  $\text{OPT-util}(I) = u_1(\{v_{1,1}, v_{1,2}, v_5\}) + u_2(\{v_{2,1}, v_{2,2}\})$ . Also, consider the connected allocation where agent 1 is allocated  $v_{1,1}, v_{1,2}$ , and  $v_5$ , as well as the vertex  $v \in \{v_{2,1}, v_{2,2}\}$  which minimizes  $u_2(v) - u_1(v)$ . Suppose without loss of generality that this vertex is  $v_{2,1}$ . In this allocation, agent 2 only receives vertex  $v_{2,2}$ . This gives  $\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M}) \geq u_1(\{v_{1,1}, v_{1,2}, v_5, v_{2,1}\}) + u_2(v_{2,2})$ . Then,

$$\begin{aligned} \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M})} &\leq \frac{u_1(\{v_{1,1}, v_{1,2}, v_5\}) + u_2(\{v_{2,1}, v_{2,2}\})}{u_1(\{v_{1,1}, v_{1,2}, v_5, v_{2,1}\}) + u_2(v_{2,2})} \\ &= \frac{1 - u_1(\{v_{2,1}, v_{2,2}\}) + u_2(\{v_{2,1}, v_{2,2}\})}{1 - u_1(v_{2,2}) + u_2(v_{2,2})}. \end{aligned}$$

Given an arbitrary (restricted) instance, consider another instance with utility functions  $u'$ , which only differ from  $u$  for vertices  $v_{2,1}$  and  $v_{2,2}$ . Specifically, we have  $u'_1(v_{2,1}) = u'_1(v_{2,2}) = \frac{1}{2}(u_1(\{v_{2,1}, v_{2,2}\}))$  and  $u'_2(v_{2,1}) = u'_2(v_{2,2}) = \frac{1}{2}(u_2(\{v_{2,1}, v_{2,2}\}))$ . Since  $u_2(v_{2,1}) - u_1(v_{2,1}) \leq u_2(v_{2,2}) - u_1(v_{2,2})$ , we have  $u'_2(v_{2,2}) - u'_1(v_{2,2}) \leq u_2(v_{2,2}) - u_1(v_{2,2})$ , and therefore  $1 - u_1(v_{2,2}) + u_2(v_{2,2}) \geq 1 - u'_1(v_{2,2}) + u'_2(v_{2,2})$ . It follows that

$$\begin{aligned} \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M})} &\leq \frac{1 - u_1(\{v_{2,1}, v_{2,2}\}) + u_2(\{v_{2,1}, v_{2,2}\})}{1 - u_1(v_{2,2}) + u_2(v_{2,2})} \\ &\leq \frac{1 - u'_1(\{v_{2,1}, v_{2,2}\}) + u'_2(\{v_{2,1}, v_{2,2}\})}{1 - u'_1(v_{2,2}) + u'_2(v_{2,2})} \\ &= \frac{1 - 2u'_1(v_{2,2}) + 2u'_2(v_{2,2})}{1 - u'_1(v_{2,2}) + u'_2(v_{2,2})} \\ &\leq \frac{4}{3}, \end{aligned}$$

where the last inequality holds because  $\frac{1+2x}{1+x} \leq \frac{4}{3}$  for all  $-1/2 \leq x \leq 1/2$ .

It remains to consider graphs that are not  $L_5$ . For simplicity of notation, we now relabel  $v_{1,1}$  as  $v_1$  and  $v_{1,2}$  as  $v_2$ . Recall that to maximize the utilitarian welfare ratio, we require (without loss of generality)  $u_1(v_1) > u_2(v_1)$  and  $u_1(v_2) > u_2(v_2)$ . We first show that every vertex must be

positively valued by some agent, otherwise every item can be allocated to an agent that values it most. To see this, suppose that some vertex (other than  $v_1$  and  $v_2$ ) is not valued by any agent. If the graph is  $K_5$  with a single edge removed, then the items that are valued by agent 2 more than agent 1 must be connected, meaning agent 1 can take the non-valued vertex to obtain all of the items which it values most. If  $m \geq 6$  and there exists some vertex other than  $v_1$  and  $v_2$  which agent 1 values more than agent 2, then all items which are more valued by agent 1 are connected, and agent 2 can take the non-valued vertex to obtain the remaining items. If  $m \geq 6$  and agent 1 only values  $v_1$  and  $v_2$  more than agent 2, then either agent 2 values at least three items more than agent 1, or there are at least two non-valued vertices. In either of these cases, all items can be allocated to the agents that value them most in a connected allocation, with agent 1 taking a non-valued vertex and all of its higher-valued items, and agent 2 taking the other non-valued vertex if one exists, or otherwise all of its higher-valued items (which must be connected as in a complete graph with a matching removed, any subset of three vertices is connected). We therefore require that every vertex is positively valued by some agent.

Now suppose there exists some vertex  $v$  other than  $v_1$  and  $v_2$  where  $u_1(v) \geq u_2(v)$ . First, consider the case where there are two vertices  $v'_1$  and  $v'_2$  without an edge between them such that  $u_2(v'_1) > u_1(v'_1)$  and  $u_2(v'_2) > u_1(v'_2)$ , and every other vertex is valued weakly higher by agent 1. As seen by our lower bound example, the utilitarian welfare ratio matches the utilitarian PoC lower bound in this specific case. Otherwise, there exists a connected allocation where every vertex is allocated to an agent that values it most: in this allocation, agent 1 obtains  $v, v_1, v_2$  and all other vertices which it values more than agent 2, and agent 2 obtains the remaining vertices. Therefore, apart from the aforementioned special case, the utilitarian welfare ratio is maximized when  $u_1(v_1) > u_2(v_1), u_1(v_2) > u_2(v_2)$ , and  $u_1(v) < u_2(v)$  for each  $v \notin \{v_1, v_2\}$ .

For the class of instances which satisfy these inequalities, consider the connected allocation which gives agent 1 vertices  $v_1, v_2$ , and  $v'$ , where  $v' := \arg \min_{v \in V \setminus \{v_1, v_2\}} (u_2(v) - u_1(v))$ , and agent 2 all of the remaining items. From this allocation, we see that

$$\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} \leq \frac{u_1(\{v_1, v_2\}) + u_2(V \setminus \{v_1, v_2\})}{u_1(\{v_1, v_2, v'\}) + u_2(V \setminus \{v_1, v_2, v'\})}.$$

As in the upper bound proof for the  $L_5$  graph, we consider, for any arbitrary (restricted) instance, another instance with utility functions  $u'$ , which only differ from  $u$  for  $V \setminus \{v_1, v_2\}$ . Specifically, for each  $v \in V \setminus \{v_1, v_2\}$ ,  $u'_1(v) = \frac{u_1(V \setminus \{v_1, v_2\})}{m-2}$  and  $u'_2(v) = \frac{u_2(V \setminus \{v_1, v_2\})}{m-2}$  (i.e., the utilities are “averaged”). Since every vertex in  $V \setminus \{v_1, v_2\}$  now has the same utility under  $u'$ ,  $v'$  continues to be allocated to agent 1. Also,  $v'$  minimizes  $u_2(v') - u_1(v')$ , so we know that  $u_2(v') - u_1(v') \leq u'_2(v') - u'_1(v')$ . Note also that  $u_1(\{v_1, v_2\}) + u_2(V \setminus \{v_1, v_2\}) = u'_1(\{v_1, v_2\}) + u'_2(V \setminus \{v_1, v_2\})$ , and therefore

$$\begin{aligned} u_1(\{v_1, v_2, v'\}) + u_2(V \setminus \{v_1, v_2, v'\}) &= u_1(\{v_1, v_2\}) + u_2(V \setminus \{v_1, v_2\}) - (u_2(v') - u_1(v')) \\ &\geq u'_1(\{v_1, v_2\}) + u'_2(V \setminus \{v_1, v_2\}) - (u'_2(v') - u'_1(v')) \\ &= u'_1(\{v_1, v_2, v'\}) + u'_2(V \setminus \{v_1, v_2, v'\}). \end{aligned}$$

Substituting these expressions, we see that the utilitarian welfare ratio is at most

$$\begin{aligned} \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} &\leq \frac{u_1(\{v_1, v_2\}) + u_2(V \setminus \{v_1, v_2\})}{u_1(\{v_1, v_2, v'\}) + u_2(V \setminus \{v_1, v_2, v'\})} \\ &\leq \frac{u'_1(\{v_1, v_2\}) + (m-2)u'_2(v')}{u'_1(\{v_1, v_2\}) + u'_1(v') + (m-3)u'_2(v')} \end{aligned}$$

$$\begin{aligned}
&= \frac{u'_1(\{v_1, v_2\}) + 1 - u'_2(\{v_1, v_2\})}{u'_1(\{v_1, v_2\}) + \frac{1-u'_1(\{v_1, v_2\})}{m-2} + (m-3)\frac{1-u'_2(\{v_1, v_2\})}{m-2}} \\
&= \frac{u'_1(\{v_1, v_2\}) - u'_2(\{v_1, v_2\}) + 1}{u'_1(\{v_1, v_2\})\frac{m-3}{m-2} - u'_2(\{v_1, v_2\})\frac{m-3}{m-2} + 1} \\
&= \frac{m-2}{m-3} \cdot \frac{u'_1(\{v_1, v_2\}) - u'_2(\{v_1, v_2\}) + 1}{u'_1(\{v_1, v_2\}) - u'_2(\{v_1, v_2\}) + \frac{m-2}{m-3}} \\
&= \frac{m-2}{m-3} \left( 1 - \frac{\frac{1}{m-3}}{u'_1(\{v_1, v_2\}) - u'_2(\{v_1, v_2\}) + \frac{m-2}{m-3}} \right) \\
&\leq \frac{2m-4}{2m-5},
\end{aligned}$$

where the last inequality follows from the fact that  $u'_1(\{v_1, v_2\}) - u'_2(\{v_1, v_2\}) \in [0, 1]$ .  $\square$

We next study complete bipartite graphs. Interestingly, if one side of the graph has two vertices, then the utilitarian PoC is constant, regardless of how many vertices are on the other side.

**Theorem 4.2.** *Let  $G$  be a complete bipartite graph with parts denoted as  $V_1$  and  $V_2$ . Assuming that  $|V_1|, |V_2| \geq 2$ ,  $\text{Util-PoC}(G, 2) = \frac{4}{3}$  if  $|V_1| = 2$  or  $|V_2| = 2$ , and  $\text{Util-PoC}(G, 2) = \frac{2}{2 - \frac{1}{|V_1|} - \frac{1}{|V_2|}}$  otherwise.*

*Proof.* We first provide lower bound examples, followed by upper bound proofs.

**Lower Bound** For the special case where  $|V_1| = 2$  or  $|V_2| = 2$ , suppose without loss of generality that  $|V_1| = 2$ . Let agent 1 have  $\frac{1}{2}$  utility for each item of  $V_1$ , and agent 2 have  $\frac{1}{|V_2|}$  utility for each item of  $V_2$ . The optimal utilitarian connected allocation gives agent 2 each item in  $V_2$  and one item in  $V_1$ , with agent 1 taking the other item of  $V_1$ . This gives a utilitarian PoC lower bound of  $\frac{4}{3}$ .

For the general case where  $|V_1|, |V_2| \geq 3$ , let agent 1 have  $\frac{1}{|V_1|}$  utility for each item in the first part of the graph, and let agent 2 have  $\frac{1}{|V_2|}$  utility for each item in the second part of the graph. Denote the instance as  $I$ . We have  $\text{OPT-util}(I) = 2$  from each agent taking all of their valued items, and  $\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M}) = 2 - \frac{1}{|V_1|} - \frac{1}{|V_2|}$  from each agent taking all but one of their valued items, along with one of the other agent's valued items. This gives a utilitarian PoC lower bound of  $\frac{2}{2 - \frac{1}{|V_1|} - \frac{1}{|V_2|}}$ .

**Upper Bound** To prove the upper bound, we reduce the space of instances to those where agent  $i$  has  $1/|V_i|$  utility for each vertex of  $V_i$ , with each reduction weakly increasing  $\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})}$ , which we refer to as the "utilitarian welfare ratio". Recall that to maximize the utilitarian welfare, we simply give each vertex to an agent that values it most. If for all  $i \in [2]$ , there exists a pair of distinct vertices  $v_i, v'_i \in V_i$  such that  $u_1(v_i) \geq u_2(v_i)$  and  $u_1(v'_i) \leq u_2(v'_i)$ , then we can obtain a connected allocation which attains the optimal utilitarian welfare (of the instance) by giving vertices  $v_1, v_2$  (resp.,  $v'_1, v'_2$ ) to agent 1 (resp., agent 2) and the other vertices to the agent who values it most. Therefore, to maximize the utilitarian welfare ratio, we suppose without loss of generality that for every  $v \in V_1$ ,  $u_1(v) > u_2(v)$ .

Firstly, if  $|V_1| = |V_2| = 2$ , then the graph is a cycle, so we know by Theorem 4.4 that the utilitarian PoC is  $\frac{4}{3}$ . Below, we distinguish cases based on  $|V_2|$ .

**Case 1:**  $|V_2| = 2$ . We first address the case where  $|V_2| = 2$  (and  $|V_1| \geq 3$ ). If there exists some  $v \in V_2$  such that  $u_1(v) \geq u_2(v)$ , then we are able to allocate every vertex to the agent that values it most in a connected allocation. We therefore consider the restricted set of instances where for every  $v \in V_2$ , we have  $u_1(v) < u_2(v)$ . Let  $v' := \arg \min_{v \in V_2} (u_2(v) - u_1(v))$ . Consider the allocation which gives vertices  $V_1$  and  $v'$  to agent 1, and vertex  $V_2 \setminus \{v'\}$  to agent 2, which gives

$$\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} \leq \frac{u_1(V_1) + u_2(V_2)}{u_1(V_1) + u_1(v') + u_2(V_2 \setminus \{v'\})}.$$

Given an arbitrary (restricted) instance, consider another instance with utility functions  $u'$ , which only differ from  $u$  for  $V_2$ . Specifically,  $u'_1(v') = u'_1(V_2 \setminus \{v'\}) = \frac{u_1(V_2)}{2}$  and  $u'_2(v') = u'_2(V_2 \setminus \{v'\}) = \frac{u_2(V_2)}{2}$ . That is, the agents' utilities for the vertices of  $V_2$  are “averaged.” We note that this reduced instance is valid because it respects our earlier utility assumption for this special case. We have  $u_2(v') - u_1(v') \leq u'_2(v') - u'_1(v')$  from the definition of  $v'$ , and  $u_1(V_1) + u_2(V_2) = u'_1(V_1) + u'_2(V_2)$ . Therefore

$$\begin{aligned} u_1(V_1) + u_1(v') + u_2(V_2 \setminus \{v'\}) &= u_1(V_1) + u_2(V_2) - (u_2(v') - u_1(v')) \\ &\geq u'_1(V_1) + u'_2(V_2) - (u'_2(v') - u'_1(v')) \\ &= u'_1(V_1) + u'_1(v') + u'_2(V_2 \setminus \{v'\}). \end{aligned}$$

By substitution, we have

$$\begin{aligned} \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} &\leq \frac{u_1(V_1) + u_2(V_2)}{u_1(V_1) + u_1(v') + u_2(V_2 \setminus \{v'\})} \\ &\leq \frac{u'_1(V_1) + u'_2(V_2)}{u'_1(V_1) + \frac{u'_1(V_2)}{2} + \frac{u'_2(V_2)}{2}} \\ &= \frac{u'_1(V_1) + u'_2(V_2)}{\frac{1}{2} + \frac{u'_1(V_1)}{2} + \frac{u'_2(V_2)}{2}} \\ &= \frac{2 \cdot u'_1(V_1) + 2 \cdot u'_2(V_2)}{1 + u'_1(V_1) + u'_2(V_2)} \\ &= 2 - \frac{2}{1 + u'_1(V_1) + u'_2(V_2)}, \end{aligned}$$

which takes a maximum value of  $\frac{4}{3}$  when  $u'_1(V_1) = u'_2(V_2) = 1$ .

**Case 2:**  $|V_2| \geq 3$ . We address the general case where  $|V_2| \geq 3$  (and  $|V_1| \geq 2$ ). Recall that we assume that for every  $v \in V_1$ ,  $u_1(v) > u_2(v)$ . We first deal with the subcase where there also exists a non-empty set  $V' \subset V_2$  such that for each  $v \in V'$ ,  $u_1(v) \geq u_2(v)$ . Let  $v' := \arg \min_{v \in V_2} (u_1(v) - u_2(v))$ . Consider the connected allocation where agent 2 receives each vertex  $v \in V_2$  where  $u_2(v) > u_1(v)$ , along with vertex  $v'$ , and agent 1 receives every other vertex. This gives

$$\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} \leq \frac{u_1(V_1) + u_1(V') + u_2(V_2 \setminus V')}{u_1(V_1) - u_1(v') + u_1(V') + u_2(v') + u_2(V_2 \setminus V')}.$$

Given an instance which meets our assumptions, consider another instance with utility functions  $u'$  which only differ from  $u$  for  $V_1$ . Specifically,  $u'_1(v') = \frac{u_1(V_1)}{|V_1|}$  and  $u'_1(V_1 \setminus \{v'\}) = (|V_1| - 1) \frac{u_1(V_1)}{|V_1|}$ ,

and  $u'_2(v') = \frac{u_2(V_1)}{|V_1|}$  and  $u'_2(V_1 \setminus \{v'\}) = (|V_1| - 1) \frac{u_2(V_1)}{|V_1|}$  (both agents' utilities for the vertices in  $V_1$  are “averaged”). We note that the reduced instance is valid because it respects our earlier utility assumption for this special case. Note that  $u_1(V_1) + u_1(V') + u_2(V_2 \setminus V') = u'_1(V_1) + u'_1(V') + u'_2(V_2 \setminus V')$  and  $u_1(v') - u_2(v') \leq u'_1(v') - u'_2(v')$ , so we have

$$\begin{aligned} \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} &\leq \frac{u_1(V_1) + u_1(V') + u_2(V_2 \setminus V')}{u_1(V_1) - u_1(v') + u_1(V') + u_2(v') + u_2(V_2 \setminus V')} \\ &\leq \frac{u'_1(V_1) + u'_1(V') + u'_2(V_2 \setminus V')}{(|V_1| - 1) \cdot \frac{u'_1(V_1)}{|V_1|} + u'_1(V') + \frac{u'_2(V_1)}{|V_1|} + u'_2(V_2 \setminus V')}. \end{aligned}$$

We further reduce the (current) instance into one with utility functions  $u''$ , where  $u''_1(V_1) = u'_1(V_1) + (u'_1(V') - u'_2(V'))$ ,  $u''_1(V') = u'_2(V')$ , and  $u''_i(v) = u'_i(v)$  otherwise for  $i \in \{1, 2\}$ . In other words, agent 1 reduces its utility for  $V'$  and increases its utility for  $V_1$  by the same amount. We note again that the reduced instance is valid because it respects our earlier utility assumption for this special case. This gives  $u'_1(V_1) + u'_1(V') + u'_2(V_2 \setminus V') = u''_1(V_1) + u''_1(V') + u''_2(V_2 \setminus V')$  and  $(|V_1| - 1) \cdot \frac{u'_1(V_1)}{|V_1|} + u'_1(V') \geq (|V_1| - 1) \cdot \frac{u''_1(V_1)}{|V_1|} + u''_1(V')$ , and therefore

$$\begin{aligned} \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} &\leq \frac{u'_1(V_1) + u'_1(V') + u'_2(V_2 \setminus V')}{(|V_1| - 1) \cdot \frac{u'_1(V_1)}{|V_1|} + u'_1(V') + \frac{u'_2(V_1)}{|V_1|} + u'_2(V_2 \setminus V')} \\ &\leq \frac{u''_1(V_1) + u''_2(V_2)}{(|V_1| - 1) \cdot \frac{u''_1(V_1)}{|V_1|} + \frac{u''_2(V_1)}{|V_1|} + u''_2(V_2)} \\ &= \frac{u''_1(V_1) + u''_2(V_2)}{(|V_1| - 1) \cdot \frac{u''_1(V_1)}{|V_1|} + \frac{1}{|V_1|} + (|V_1| - 1) \cdot \frac{u''_2(V_2)}{|V_1|}} \\ &\leq \frac{2}{2 - \frac{1}{|V_1|}}, \end{aligned}$$

where the last inequality is equivalent to  $u''_1(V_1) + u''_2(V_2) \leq 2$ , which holds. If  $|V_1| = 2$ , the expression  $\frac{2}{2 - \frac{1}{|V_1|}}$  becomes  $4/3$ . Else, the expression is strictly less than  $\frac{2}{2 - \frac{1}{|V_1|} - \frac{1}{|V_2|}}$ .

We now address the remaining subcase of **Case 2**:  $|V_2| \geq 3$ , and for every  $v \in V_2$ ,  $u_2(v) > u_1(v)$ . Note that if  $|V_1| = 2$ , the instance has already been covered in **Case 1**, so we suppose that  $|V_1| \geq 3$ . Let  $v'_1 := \arg \min_{v \in V_1} (u_1(v) - u_2(v))$  and  $v'_2 := \arg \min_{v \in V_2} (u_2(v) - u_1(v))$ . Consider the allocation where agent 1 receives  $v'_2$  and  $V_1 \setminus \{v'_1\}$ , and agent 2 receives  $v'_1$  and  $V_2 \setminus \{v'_2\}$ . This gives

$$\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} \leq \frac{u_1(V_1) + u_2(V_2)}{u_1(v'_2) + u_1(V_1 \setminus \{v'_1\}) + u_2(v'_1) + u_2(V_2 \setminus \{v'_2\})}.$$

We now reduce the instance by setting  $u'_1(v'_1) = \frac{u_1(V_1)}{|V_1|}$  and  $u'_1(V_1 \setminus \{v'_1\}) = (|V_1| - 1) \frac{u_1(V_1)}{|V_1|}$ ,  $u'_2(v'_1) = \frac{u_2(V_1)}{|V_1|}$  and  $u'_2(V_1 \setminus \{v'_1\}) = (|V_1| - 1) \frac{u_2(V_1)}{|V_1|}$ ,  $u'_1(v'_2) = \frac{u_1(V_2)}{|V_2|}$  and  $u'_1(V_2 \setminus \{v'_2\}) = (|V_2| - 1) \frac{u_1(V_2)}{|V_2|}$ , and  $u'_2(v'_2) = \frac{u_2(V_2)}{|V_2|}$  and  $u'_2(V_2 \setminus \{v'_2\}) = (|V_2| - 1) \frac{u_2(V_2)}{|V_2|}$ . We note that the reduced instance is valid because it respects our earlier utility assumption for this special case. We have  $u_1(V_1) + u_2(V_2) = u'_1(V_1) + u'_2(V_2)$  and  $(u_1(v'_1) - u_2(v'_1)) + (u_2(v'_2) - u_1(v'_2)) \leq (u'_1(v'_1) - u'_2(v'_1)) + (u'_2(v'_2) - u'_1(v'_2))$  from the definitions of  $v'_1$  and  $v'_2$ , so

$$\frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} \leq \frac{u_1(V_1) + u_2(V_2)}{u_1(v'_2) + u_1(V_1 \setminus \{v'_1\}) + u_2(v'_1) + u_2(V_2 \setminus \{v'_2\})}$$

$$\begin{aligned}
&\leq \frac{u'_1(V_1) + u'_2(V_2)}{\frac{u'_1(V_2)}{|V_2|} + (|V_1| - 1)\frac{u'_1(V_1)}{|V_1|} + \frac{u'_2(V_1)}{|V_1|} + (|V_2| - 1)\frac{u'_2(V_2)}{|V_2|}} \\
&= \frac{u'_1(V_1) + u'_2(V_2)}{\frac{1}{|V_1|} + \frac{1}{|V_2|} + (u'_1(V_1) + u'_2(V_2))(1 - \frac{1}{|V_1|} - \frac{1}{|V_2|})} \\
&= \frac{1}{\frac{1/|V_1|+1/|V_2|}{u'_1(V_1)+u'_2(V_2)} + 1 - \frac{1}{|V_1|} - \frac{1}{|V_2|}} \\
&\leq \frac{2}{2 - \frac{1}{|V_1|} - \frac{1}{|V_2|}}
\end{aligned}$$

for  $|V_1|, |V_2| \geq 3$ . In the third line, we substitute  $\frac{u'_2(|V_1|)}{|V_1|} = \frac{1-u'_2(V_2)}{|V_1|}$  and  $\frac{u'_1(|V_2|)}{|V_2|} = \frac{1-u'_1(V_1)}{|V_2|}$ , and in the last line, we set  $u'_1(V_1)$  and  $u'_2(V_2)$  to their maximum value of 1.

By exhaustion of cases and applying welfare ratio-increasing reductions, we have proven the utilitarian PoC upper bound.  $\square$

We next find that the utilitarian PoC for trees is only dependent on the number of items. This contrasts with the egalitarian PoC, which is lower bounded by the maximum degree of the tree (Theorem 3.8).

**Theorem 4.3.** *Let  $G$  be a tree. Then, Util-PoC( $G, 2$ ) =  $\frac{2(m-1)}{m}$ .*

*Proof.* We start by presenting the tightness example. Color the vertices of the tree in two colors (say, red and blue) so that no two adjacent vertices have the same color. For an instance  $I$ , let agent 1 only value the red vertices, and agent 2 only value the blue vertices. The value of each vertex  $v$  is  $\frac{\deg(v)}{m-1}$ . Since there are  $m-1$  edges, and each edge is adjacent to exactly one red and one blue vertex, each agent's total value is 1. In this example, OPT-util( $I$ ) = 2. We claim that for all connected allocations  $\widehat{\mathcal{M}} \in C(I)$ , SW-util( $\widehat{\mathcal{M}}$ )  $\leq \frac{m}{m-1}$ . Consider a connected allocation—it partitions the tree into two connected components, with exactly one edge connecting the two components. Each edge connects a red vertex and a blue vertex, so it adds exactly  $\frac{1}{m-1}$  to the utilitarian welfare; the only exception is the edge connecting the two parts, which adds utility at most  $\frac{2}{m-1}$ . In total, SW-util( $\widehat{\mathcal{M}}$ )  $\leq \frac{m}{m-1}$ . Thus, the price of connectivity is at least  $\frac{2}{m-1} = \frac{2(m-1)}{m}$ .

We now show that the price is at most  $\frac{2(m-1)}{m}$ . Without loss of generality, assume that the tree is rooted. Denote  $\Delta_v := u_2(v) - u_1(v)$  for all  $v \in V$ . We can write the utilitarian welfare of the optimal allocation and the optimal connected allocation as functions of  $\Delta_v$  as follows:

$$\text{OPT-util}(I) = \sum_{v \in V} \max(u_1(v), u_2(v)) = \sum_{v \in V} \frac{u_1(v) + u_2(v) + |u_2(v) - u_1(v)|}{2} = 1 + \frac{\sum_{v \in V} |\Delta_v|}{2}.$$

For the optimal connected allocation  $\mathcal{M}$ , let  $\Delta := \max_T |\sum_{i \in T} \Delta_i|$ , where the sum is taken over all subtrees  $T$  such that both  $T$  and  $G \setminus T$  are connected. We claim that SW-util( $\mathcal{M}$ ) =  $1 + \Delta$ . To see this, observe that the maximum connected utilitarian welfare can be obtained by first allocating all vertices to agent 1, and then giving a connected subtree  $T$  to agent 2 such that  $G \setminus T$  remains connected.

In order to show that  $\frac{\text{OPT-util}}{\text{SW-util}} \leq \frac{2(m-1)}{m}$ , it therefore suffices to show that

$$\frac{2(m-1)}{m} \geq \frac{1 + \frac{\sum_{v \in V} |\Delta_v|}{2}}{1 + \Delta}$$

$$\begin{aligned}
&\iff (2m-2)(1+\Delta) \geq m + \frac{m}{2} \sum_{v \in V} |\Delta_v| \\
&\iff (2m-2)\Delta \geq \frac{m}{2} \left( \sum_{v \in V} |\Delta_v| \right) - (m-2) \\
&\iff \Delta \geq \frac{\frac{m}{2} (\sum_{v \in V} |\Delta_v|) - (m-2)}{2m-2}.
\end{aligned}$$

It remains to prove the last inequality. Suppose for the sake of contradiction that  $\Delta < S := \frac{\frac{m}{2} (\sum_{v \in V} |\Delta_v|) - (m-2)}{2m-2}$ . We claim that for each vertex  $v \in V$ ,  $|\Delta_v| < \deg(v) \cdot S$ . Denote by  $T_1, \dots, T_{\deg(v)}$  the  $\deg(v)$  subtrees emanating from  $v$ . By definition of  $\Delta$ , for all  $i \in [\deg(v)]$ , it holds that  $|\sum_{j \in T_i} \Delta_j| \leq \Delta < S$ . We have

$$\begin{aligned}
|\Delta_v| &= \left| \Delta_v + \sum_{i \in [\deg(v)-1]} \sum_{j \in T_i} \Delta_j - \sum_{i \in [\deg(v)-1]} \sum_{j \in T_i} \Delta_j \right| \\
&\leq \left| \Delta_v + \sum_{i \in [\deg(v)-1]} \sum_{j \in T_i} \Delta_j \right| + \sum_{i \in [\deg(v)-1]} \left| \sum_{j \in T_i} \Delta_j \right| \\
&= \left| \sum_{j \in T_{\deg(v)}} \Delta_j \right| + \sum_{i \in [\deg(v)-1]} \left| \sum_{j \in T_i} \Delta_j \right| \\
&< \deg(v) \cdot S,
\end{aligned}$$

where the first inequality follows from the triangle inequality, and the second equality from the fact that  $\sum_{v \in V} \Delta_v = 0$ . Summing this up over all vertices  $v \in V$ , we have

$$\begin{aligned}
\sum_{v \in V} |\Delta_v| &< S \cdot \sum_{v \in V} \deg(v) = \frac{\frac{m}{2} (\sum_{v \in V} |\Delta_v|) - (m-2)}{2m-2} \cdot 2(m-1) = \frac{m}{2} \left( \sum_{v \in V} |\Delta_v| \right) - (m-2) \\
&\iff m-2 < \frac{m-2}{2} \sum_{v \in V} |\Delta_v| \\
&\iff 2 < \sum_{v \in V} |\Delta_v|.
\end{aligned}$$

Since  $\sum_{v \in V} |\Delta_v| \leq \sum_{v \in V} (u_1(v) + u_2(v)) = 2$ , we have reached a contradiction.  $\square$

Next, we provide tight bounds for the case of cycles. Again, the utilitarian PoC is only dependent on the total number of items.

**Theorem 4.4.** *Let  $G$  be a cycle. Then, Util-PoC( $G, 2$ ) =  $\frac{2k}{k+1}$ , where  $k = \lfloor \frac{m}{2} \rfloor$ .*

*Proof.* We distinguish between odd and even length cycles.

**Even Cycles ( $m = 2k$ )** The cycle contains  $m = 2k$  vertices  $\{1, 2, \dots, 2k-1, 2k\}$ . Consider an instance  $I$  with the following valuations:

- agent 1 values odd vertices equally, i.e.,  $u_1(1) = u_1(3) = \dots = u_1(2k-1) = 1/k$ .
- agent 2 values even vertices equally, i.e.,  $u_2(2) = u_2(4) = \dots = u_2(2k) = 1/k$ .

The example shows that  $\text{Egal-PoC}(G, 2) \geq \frac{2k}{k+1}$  because

- $\text{OPT-util}(I) = 2$ , by giving odd vertices to agent 1 and even vertices to agent 2, and
- $\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M}) = 1/k + 1 = (k+1)/k$ , by giving vertex 1 to agent 1 and the remaining vertices to agent 2.

We now show that the price is at most  $\frac{2k}{k+1}$ . Denote by  $\Delta_i := u_2(i) - u_1(i)$  for all  $i \in [2k]$ . We use  $\Delta_i$ 's to represent the optimal welfare (of an instance  $I$ ) and the optimal welfare among connected allocations as follows:

- $\text{OPT-util}(I) = \sum_{i \in [2k]} \max(u_1(i), u_2(i)) = \sum_{i \in [2k]} \frac{u_1(i) + u_2(i) + |u_2(i) - u_1(i)|}{2} = 1 + \frac{\sum_{i \in [2k]} |\Delta_i|}{2}$ .
- Denote by  $\Delta := \max_{P \in \text{all paths}} \sum_{i \in P} \Delta_i$ . Then,  $\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M}) = 1 + \Delta$ . This is because we can assume without loss of generality that agent 1 gets all of the vertices at the beginning and next we give a path to agent 2 to increase the welfare.

We want to show that

$$\begin{aligned} \frac{2k}{k+1} &\geq \frac{\text{OPT-util}(I)}{\max_{\mathcal{M} \in C(I)} \text{SW-util}(\mathcal{M})} = \frac{1 + \frac{\sum_{i \in [2k]} |\Delta_i|}{2}}{1 + \Delta} \\ \iff 2k + 2k \cdot \Delta &\geq k + 1 + \frac{k+1}{2} \sum_{i \in [2k]} |\Delta_i| \\ \iff \Delta &\geq \frac{1 - k + \frac{k+1}{2} \sum_{i \in [2k]} |\Delta_i|}{2k}, \end{aligned}$$

meaning that we are done if the last inequality holds. Suppose for the sake of contradiction that  $\Delta < S := \frac{1 - k + \frac{k+1}{2} \sum_{i \in [2k]} |\Delta_i|}{2k}$ . This means that  $|\Delta_i| < S$  for all  $i \in [2k]$  because the allocation with any vertex alone is connected. That is,

$$\begin{aligned} \sum_{i \in [2k]} |\Delta_i| &< 2k \cdot S = 2k \cdot \frac{1 - k + \frac{k+1}{2} \sum_{i \in [2k]} |\Delta_i|}{2k} = 1 - k + \frac{k+1}{2} \sum_{i \in [2k]} |\Delta_i| \\ \iff k - 1 &< \frac{k-1}{2} \sum_{i \in [2k]} |\Delta_i| \\ \iff 2 &< \sum_{i \in [2k]} |\Delta_i|. \end{aligned}$$

Since  $\sum_{i \in [2k]} |\Delta_i| \leq \sum_{i \in [2k]} (u_1(i) + u_2(i)) = 2$ , we have reached a contradiction.

**Odd Cycles ( $m = 2k + 1$ )** The cycle contains  $m = 2k + 1$  vertices  $\{1, 2, \dots, 2k - 1, 2k, 2k + 1\}$ . The same example as in the previous case, with the exception that vertex  $2k + 1$  is of value 0 to both agents, shows that the utilitarian PoC is at least  $\frac{2k}{k+1}$ . We next show that the price is at most  $\frac{2k}{k+1}$  using a similar argument as before. Note that there must exist consecutive  $\Delta_i, \Delta_{i+1}$  with the same sign (where  $\Delta_{2k+2} = \Delta_1$ ). Since the allocation with these two vertices together in one part is connected, we have  $|\Delta_i| + |\Delta_{i+1}| = |\Delta_i + \Delta_{i+1}| < S$ . Repeating the same argument, we end up with

$$\sum_{i \in [2k+1]} |\Delta_i| < ((2k+1) - 1) \cdot S = 1 - k + \frac{k+1}{2} \sum_{i \in [2k+1]} |\Delta_i|$$

$$\begin{aligned} &\iff k - 1 < \frac{k - 1}{2} \sum_{i \in [2k+1]} |\Delta_i| \\ &\iff 2 < \sum_{i \in [2k+1]} |\Delta_i|, \end{aligned}$$

which leads to the same contraction as in the previous case.  $\square$

## 4.2 Any Number of Agents

We now move to the case of general  $n$ . In this section, all of our utilitarian PoC bounds are asymptotically tight, with the lower bounds matching (as  $m \rightarrow \infty$ ) the following upper bound for any connected graph.

**Proposition 4.5.** *Given any instance  $I = \langle N, G, \mathcal{U} \rangle$ ,  $\text{Util-PoC}(G, n) \leq n$ .*

*Proof.* Given any instance  $I$ , the optimal utilitarian welfare  $\text{OPT-util}(I)$  is at most  $n$ , as each agent has a total utility of 1 for the entire set of items. Under a connected graph, consider an allocation  $\mathcal{M}$  which awards some agent all of the items that they positively value. Such an allocation gives a connected utilitarian welfare  $\text{SW-util}(\mathcal{M})$  of at least 1. Dividing these welfare terms, we see that the price of connectivity for utilitarian welfare is at most  $n$ .  $\square$

We begin by proving a lower bound for star graphs, which depends on the relation between  $n$  and  $m$ . Here, one agent only values the center vertex, and each of the remaining agents has a distinct set of leaf vertices which they value equally, and the cardinality of these leaf vertex sets differs by at most 1.

**Theorem 4.6.** *Let  $n \geq 2$  and  $G$  be a star graph. Then,  $\text{Util-PoC}(G, n) \geq \frac{nc(c+1)}{c^2 + 2nc + n - c - m}$  for  $m - 1 = c(n - 1) + d$ , where  $c \in \mathbb{Z}^+$  and  $d \in \{0, \dots, n - 2\}$ . This is  $\Omega(n)$  for large  $c$  (i.e., large  $m$ ).*

*Proof.* Suppose that each item is positively valued by at most one agent, leading to  $\text{OPT-util}(I) = n$ . Also suppose that agent 1 only positively values the center vertex. Let  $c, d \in \mathbb{Z}^+$  be such that  $m - 1 = c(n - 1) + d$  and  $d < n - 1$ . Here, we are dividing the  $m - 1$  leaf vertices amongst the  $n - 1$  agents, such that each agent gets exactly  $c$  vertices, and  $d$  is the remainder term.

Let  $d$  of the  $n - 1$  agents each equally and positively value  $c + 1$  of the leaf vertices, and  $n - 1 - d$  agents each equally and positively value  $c$  of the leaf vertices. This gives

$$\max_{\mathcal{M} \in \mathcal{C}(I)} \text{SW-util}(\mathcal{M}) = 1 + \frac{d}{c + 1} + \frac{n - d - 1}{c} = \frac{c^2 + 2nc + n - c - m}{c(c + 1)}.$$

Dividing  $\text{OPT-util}(I)$  by this term gives the price of connectivity lower bound of

$$\frac{nc(c + 1)}{c^2 + 2nc + n - c - m}.$$

$\square$

We next move to the path graph lower bound, constructing an instance where each item is valued by exactly one agent and is listed in the repeating sequence  $(1, 2, \dots, n)^*$ , where the number denotes the agent that values the item.

**Theorem 4.7.** *Let  $n \geq 2$  and  $G$  be a path graph. Then,  $\text{Util-PoC}(G, n) \geq \frac{c^2n + cn}{c^2 + cn - d + n + 1}$  for  $m = cn + d$ , where  $c \in \mathbb{Z}^+$  and  $d \in \{0, \dots, n - 1\}$ . This is  $\Omega(n)$  for large  $c$  (i.e., large  $m$ ).*

*Proof.* For the proof, we consider instances where each item is valued by only one agent, and each agent has an equal valuation for each item that they positively value. Thus the optimal utilitarian welfare is  $n$ . We also suppose that the items are arranged along the path in the repeating sequence  $(1, 2, \dots, n)^*$ , where the numbers of the sequence represent the agent that positively values the item. As a result, for an agent to receive  $k$  of its valued items in a connected bundle, the bundle must contain a total of  $n(k - 1) + 1$  items.

If  $m = cn + d$ , where  $d \in \{0, \dots, n - 1\}$ , then agents  $1, \dots, d$  positively value  $c + 1$  of the items each, and agents  $d + 1, \dots, n$  positively value  $c$  of the items each. We construct the optimal connected allocation, beginning with the allocation which allocates the first  $n$  items to the agents which value them. Every time we wish to increase the utility of some agent in  $\{1, \dots, d\}$  by  $\frac{1}{c+1}$ , or some agent in  $\{d + 1, \dots, n\}$  by  $\frac{1}{c}$ , we must allocate  $n$  more items to that agent. Thus we achieve the optimal connected allocation by simply allocating all of the remaining items to agent  $n$ . This allocation clearly maximizes the number of items allocated to agents who positively value them, and is therefore optimal.

In the optimal connected allocation, agent  $n$  receives 1 utility,  $d$  agents receive  $\frac{1}{c+1}$  utility, and  $n - d - 1$  agents receive  $\frac{1}{c}$  utility. The utilitarian welfare of this connected allocation is therefore  $1 + \frac{d}{c+1} + \frac{n-d-1}{c}$ . Dividing the optimal utilitarian welfare by this value gives our lower bound of  $\frac{c^2n+cn}{c^2+cn-d+n+1}$ , which is  $\Omega(n)$  for large  $c$  (i.e., large  $m$ ).  $\square$

Lastly, we give lower bounds for cycle graphs.

**Theorem 4.8.** Let  $n \geq 2$  and  $G$  be a cycle graph. Then, when  $m = cn + d$ , where  $c \in \mathbb{Z}^+$  and  $d \in \{0, \dots, n - 1\}$ ,

$$\text{Util-PoC}(G, n) \geq \begin{cases} \frac{c^2n+cn}{c^2+cn+n+1} & \text{if } d = 0; \\ \frac{c^2n+cn}{c^2+cn+c+n-3} & \text{if } d = 1 \text{ and } c \geq 2; \\ \frac{n}{n-\frac{1}{2}} & \text{if } d = 1 \text{ and } c = 1; \\ \frac{c^2n+cn}{c^2+cn+c-d+n-1} & \text{if } d \geq 2. \end{cases}$$

This is  $\Omega(n)$  for large  $c$  (i.e., large  $m$ ).

*Proof.* As in Theorem 4.7, we consider instances where each item is valued by only one agent, and each agent has an equal valuation for each item that they positively value, leading to an optimal utilitarian welfare of  $n$ . Again we arrange items in sequences where the number in the sequence corresponds to the agent that values the item. Since we are now dealing with cycle graphs, we assume the last item in the sequence is connected to the first item.

For  $d = 0$ , the price of connectivity is the same as in Theorem 4.7; the proof holds verbatim for a cycle where the items are arranged in the repeating sequence  $(1, 2, \dots, n)^*$ .

For  $d = 1$ , consider the sequence  $(1, 2, \dots, n, 1, 2, \dots, n, \dots, 1, 2, \dots, n, n - 1)$ . Note that the ‘remainder’ item is valued by agent  $n - 1$ . Here, there are two possible optimal connected allocations depending on  $c$ .

- If  $c = 1$ , the optimal connected allocation gives agent  $n - 1$  just one of its two valued items, and each other agent their one valued item, giving a utilitarian welfare of  $n - \frac{1}{2}$ .
- For  $c \geq 2$ , suppose we give agent  $n - 1$  three of its valued items in the sequence  $(n - 1, n, n - 1, 1, 2, \dots, n - 1)$ , leaving behind a path graph with the sequence  $(n, \underbrace{1, 2, \dots, n}_{c-2 \text{ times}}, 1, 2, \dots, n - 2)$ . By using similar reasoning as in the proof of Theorem 4.7, we see that this can be

optimally allocated by giving agent  $n$  all of its  $c - 1$  remaining valued items, and agents  $1, \dots, n - 2$  one valued item each. Overall, the utilitarian welfare is  $\frac{3}{c+1} + \frac{c-1}{c} + \frac{n-2}{c}$ .

We now show that for  $d = 1$ , any other connected allocation results in strictly lower utilitarian welfare. The subcase where  $c = 1$  is trivial, so suppose that  $c \geq 2$ . We first show that it is not optimal for agent  $n - 1$  to receive zero or one of its valued items. Suppose this is the case. To obtain the optimal connected utilitarian welfare, we remove all of agent  $(n - 1)$ 's items from the graph and reconnect the remaining vertices to form a cycle graph with the repeating sequence  $(1, 2, \dots, n - 2, n)^*$ . Observe from the proof of Theorem 4.7 that a cycle graph with a repeating sequence and no remainder vertices has the same PoC as a path graph with the same vertices. Therefore it is strictly better to give agent  $n - 1$  one of its valued vertices and allocate the remaining path graph to the other agents, than to give agent  $n - 1$  nothing and allocate the remaining cycle graph to the other agents. Applying the result of Theorem 4.7, we see that the optimal utilitarian welfare of the latter allocation is  $1 + \frac{1}{c+1} + \frac{n-2}{c}$ . However,  $\frac{3}{c+1} + \frac{c-1}{c} + \frac{n-2}{c} > 1 + \frac{1}{c+1} + \frac{n-2}{c}$ , so from our optimal allocation example, it is strictly better to give agent  $n - 1$  two of its valued items. It remains to consider connected allocations which allocate to agent  $n - 1$  at least two of its valued items, and it suffices to consider those that begin and end with agent  $(n - 1)$ 's valued items. Under such an allocation, at least one of the following three sequences is contained within agent  $(n - 1)$ 's bundle:  $(n - 1, n, 1, 2, \dots, n - 1)$ ,  $(n - 1, 1, 2, \dots, n - 1)$ , and  $(n - 1, n, n - 1)$ .

If the sequence  $(n - 1, n, 1, 2, \dots, n - 1)$  is at the end of agent  $n - 1$ 's bundle, un-allocating the outside item from agent  $n - 1$  will 'free up' items  $(n, 1, 2, \dots, n - 1)$ , which can be appended to another agent's bundle. This transformation increases the utilitarian welfare by  $\frac{1}{c} - \frac{1}{c+1}$ , and is repeated until only  $(n - 1, 1, 2, \dots, n - 1)$  and/or  $(n - 1, n, n - 1)$  are at the ends of agent  $(n - 1)$ 's bundle. Since these two sequences overlap in the original cycle graph, if they are both at the ends of agent  $(n - 1)$ 's bundle, then the bundle must be the union of those sequences, which is our optimal allocation sequence for agent  $n - 1$ . The only two remaining cases involve agent  $n - 1$  receiving either  $(n - 1, 1, 2, \dots, n - 1)$  or  $(n - 1, n, n - 1)$  for their entire bundle.

If agent  $n - 1$  only receives  $(n - 1, n, n - 1)$ , then the remaining path graph is a repeating  $(1, 2, \dots, n)^*$  sequence with  $(c - 1)n + (n - 2)$  items. Using the same idea as in the proof of Theorem 4.7, we see that the optimal connected allocation gives agent  $n$  all of its  $c - 1$  remaining preferred items, and agents  $1, \dots, n - 2$  one item each. This gives a utilitarian welfare of  $\frac{2}{c+1} + \frac{c-1}{c} + \frac{n-2}{c}$ , which is less than the utilitarian welfare of our optimal connected allocation.

If agent  $n - 1$  only receives  $(n - 1, 1, 2, \dots, n - 1)$ , then the remainder path graph can be permuted into a repeating  $(1, 2, \dots, n)^*$  sequence with  $(c - 1)n + 1$  items. It is clear that the optimal connected allocation here has weakly lower utilitarian welfare than the previous case where agent  $n - 1$  only receives  $(n - 1, n, n - 1)$ .

By exhaustion of cases, we have proven the optimality of our connected allocations. For  $c \geq 2$ , dividing the optimal welfare of  $n$  by  $\frac{3}{c+1} + \frac{c-1}{c} + \frac{n-2}{c}$  gives our lower bound of  $\frac{c^2n+cn}{c^2+cn+n+c-3}$ .

For  $d \geq 2$ , consider the repeating sequence  $(1, 2, \dots, n)^*$ . Here, agents  $1, \dots, d$  have  $\frac{1}{c+1}$  utility for each of their positively valued items, whilst agents  $d + 1, \dots, n$  have  $\frac{1}{c}$  utility for each of their valued items. In this cycle graph, an optimal connected allocation allocates to agent  $n$  all of its valued items. To see this, we note that when one of agent  $n$ 's valued items is unallocated from agent  $n$ , the item sequence  $(1, 2, \dots, n)^*$  is 'freed', and another agent may append at most one of its valued items to its own bundle. Since agent  $n$  has  $\frac{1}{c}$  utility for each of its valued items, this does not increase the utilitarian welfare. Now with agent  $n$  receiving all of its valued items, the remaining path graph is  $(1, 2, \dots, d, 1, 2, \dots, n - 1)$ . Agents  $d + 1, \dots, n - 1$  have only one of their valued items remaining, so they receive that item under the optimal connected allocation. Finally, the path graph  $(1, 2, \dots, d, 1, 2, \dots, d)$  remains, and it is clear that the optimal connected allocation has one

agent (say, agent 1) receiving 2 of its valued items, and the remaining  $d - 1$  agents receiving only one of its valued items. The optimal connected utilitarian welfare is  $1 + \frac{n-d-1}{c} + \frac{d+1}{c+1}$ . Dividing the optimal utilitarian welfare of  $n$  by this value, we obtain the lower bound of  $\frac{c^2n+cn}{c^2+cn+c-d+n-1}$ .  $\square$

## 5 Discussion

In this work, we have introduced, for the allocation of indivisible goods on a graph, the concepts of egalitarian and utilitarian price of connectivity. These quantify the worst-case welfare loss due to connectivity constraints (i.e., each agent must receive a connected subgraph of items). We have studied the price of connectivity for various classes of graphs, including graphs with connectivity 1 and 2, complete graphs with a matching removed, complete bipartite graphs, stars, paths, and cycles, and produced tight or asymptotically tight bounds.

There is still room to extend the current results further. For the case of two agents, while we have addressed classes of dense and sparse graphs, a complete characterization of the egalitarian PoC for arbitrary graphs is still unknown, and the utilitarian PoC for graphs with connectivity 1 and 2 is also open. Furthermore, it would be interesting to generalize the existing egalitarian PoC result for trees and three agents to graphs with connectivity 1, and to also supplement this result with utilitarian PoC results.

As noted in the survey by [Suksompong \[2021\]](#), in addition to the connectivity constraint studied in this paper, other constraints have been studied in the fair division literature. For instance, a cardinality constraint could require the number of items received by each agent to be the same (or similar) [[Biswas and Barman, 2018](#)], and matroid constraints restrict the possible bundles received by each agent to a specific set of bundles [[Dror et al., 2023](#)].

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