

Reforming an Unfair Allocation by Exchanging Goods

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Abstract

Fairly allocating indivisible goods is a frequently occurring task in everyday life. Given an initial allocation of the goods, we consider the problem of reforming it via a sequence of exchanges to attain fairness in the form of envy-freeness up to one good (EF1). We present a vast array of results on the complexity of determining whether it is possible to reach an EF1 allocation from the initial allocation and, if so, the minimum number of exchanges required. In particular, we uncover several distinctions based on the number of agents involved and their utility functions. Furthermore, we derive essentially tight bounds on the worst-case number of exchanges needed to achieve EF1.

1 Introduction

Fair division is a research area that studies how to allocate scarce resources to interested agents in a fair manner. The theory developed in this area has been applied to several tools for fairly allocating resources in practice, including Spliddit [Goldman and Procaccia, 2014], Course Match [Budish *et al.*, 2017], and Kajibuntan [Igarashi and Yokoyama, 2023].

A ubiquitous fair division problem is the allocation of *indivisible goods*, such as books, furniture, paintings, and human resources. When allocating indivisible goods, a prominent fairness notion is *envy-freeness up to one good (EF1)*, which requires that if an agent envies another agent, there must exist a good in the latter agent’s bundle whose removal would make the envy disappear. An EF1 allocation always exists and can be found, for example, by the *round-robin algorithm*, which lets the agents pick their favorite goods in a round-robin fashion. The allocation returned by the round-robin algorithm is also *balanced*, meaning that the numbers of goods that any pair of agents receive differ by at most one.

The fair division literature typically assumes that there is a set of unallocated goods and the objective is to allocate them fairly. In certain scenarios, however, an existing (possibly unfair) allocation of the goods is already in place, and the goal is to “reform” it in order to arrive at a fair allocation. This is the case, for instance, in the allocation of personnel among teams in an organization. As the personnel experience individual

growth or decline, and as the needs of the teams evolve, these changes can necessitate a reevaluation and potential reformation of the current allocation by the organization leadership. Another example is the distribution of a museum’s exhibits across its branches—the museum director may decide to adjust the distribution so as to ensure fairness based on the most recent valuations. Such scenarios fall under the umbrella of *control problems*, which have been studied extensively in computational social choice [Faliszewski and Rothe, 2016].

In this work, we shall allow agents to *exchange* pairs of goods in the reformation process, and use EF1 as our fairness criterion. Exchanges preserve the size of each agent’s bundle, thereby ensuring that any cardinality constraints remain fulfilled. Note that capacity constraints are prevalent in fair division applications and have accordingly received interest in the literature [Biswas and Barman, 2018; Wu *et al.*, 2021; Hummel and Hetland, 2022; Shoshan *et al.*, 2023]. Naturally, given an initial allocation, we wish to reach an EF1 allocation using a small number of exchanges. However, it is sometimes impossible to reach an EF1 allocation via *any* finite number of exchanges, so we start by exploring whether the corresponding decision problem can be answered efficiently. Since this problem is equivalent to determining whether an EF1 allocation with a certain size vector exists in a given instance, it is meaningful independently of exchange considerations.¹ We also investigate other fundamental questions in this setting. Namely, if it is possible to reach an EF1 allocation, can we efficiently determine the smallest number of exchanges required to achieve this goal? And how many exchanges might we need to make in the worst case in order to attain EF1?

1.1 Our Results

In our model, there is an initial allocation of a set of goods. As is often assumed in fair division, each agent has an additive utility function over the goods. At each step of the reformation process, two agents can exchange a pair of goods with each other to obtain another allocation, and the goal is

¹When an EF1 allocation is not guaranteed to exist in some instances due to cardinality requirements, an approach taken by previous work is to relax the EF1 condition (e.g., Wu *et al.*, 2021). However, this leads to unnecessarily weak guarantees in instances where EF1 can be attained.

to reach an EF1 allocation at the end of the process. More details on our model are provided in Section 2.

In Section 3, we consider the decision problem of determining whether a given initial allocation can be reformed into an EF1 allocation. As mentioned earlier, this problem is equivalent to determining whether an EF1 allocation with a given size vector exists, so we focus on the latter decision problem instead. We demonstrate interesting distinctions in the complexity based on the number of agents and their utility functions. Specifically, in the case of two agents, the problem can be solved in polynomial time if the agents have identical utilities, but becomes (weakly) NP-complete otherwise. For three or more agents, the problem is NP-complete even with identical utilities; however, it can be solved efficiently when the utilities are binary provided that the number of agents is constant. Finally, for an arbitrary (non-constant) number of agents, the problem is strongly NP-hard even for identical *or* binary utilities, but can be solved in polynomial time if the utilities are identical *and* binary. The results of this section are summarized in Table 1.

Having determined the reformability of the initial allocation, we next explore in Section 4 the problem of deciding whether the optimal (i.e., minimum) number of exchanges required to reach an EF1 allocation is at most some given number k . For (a) two agents with identical utilities, (b) a constant number of agents with binary utilities, and (c) any number of agents with identical binary utilities, we show that this problem can be solved in polynomial time, just like the corresponding decision problem in Section 3. For the remaining scenarios, since deciding whether an allocation is reformable is already NP-complete (from Section 3), we instead focus on the special case where the allocation is balanced—an EF1 allocation is guaranteed to be reachable in this case (see Proposition 2.2). We show that the problem for this special case remains NP-complete.

Finally, in Section 5, instead of considering specific instances, we derive worst-case bounds on the number of exchanges necessary to reach an EF1 allocation. We assume that each of the n agents possesses s goods—this again ensures that an EF1 allocation is reachable. We show that roughly $s(n - 1)/2$ exchanges always suffice for instances with general utilities or with binary utilities; moreover, our bound is essentially tight for any n and s , and exactly tight when $n = 2$ as well as when s is divisible by n . For instances with identical binary utilities, we show that an essentially tight bound for the number of exchanges is $sn/4$ for even n and $s(n - 1)(n + 1)/4n$ for odd n .

1.2 Additional Related Work

As mentioned earlier, the majority of work in fair division assumes that there is no initial allocation of the resources—we now discuss the key exceptions and their differences from our model. Boehmer *et al.* [2022] studied the problem of *discarding* goods from an initial allocation in order to reach an envy-free or EF1 allocation. As it is possible to reallocate the goods in several practical situations, discarding them can be unnecessarily wasteful for the agents involved. In a similar vein, Dorn *et al.* [2021] investigated deleting goods to attain another fairness notion called *proportionality*; they assumed

that agents have ordinal preferences (rather than cardinal utilities) over the goods, and considered both the settings with and without an initial allocation.² Aziz *et al.* [2019] focused on reallocating goods to make agents better off, but did not delve into the aspect of fairness. Igarashi *et al.* [2024] aimed to transition from an initial allocation to a target allocation, both of which are EF1, while maintaining EF1 throughout the process. Chandramouleeswaran *et al.* [2024] examined *transferring* goods starting from a “near-EF1” allocation with the goal of reaching an EF1 allocation. Segal-Halevi [2022] considered the reallocation of a *divisible* good and explored the trade-off between guaranteeing a minimum utility for every agent and ensuring each agent a certain fraction of her original utility. Chevaleyre *et al.* [2007] also strived to reach fair allocations but via exchanges with money.

Further afield, the idea of improving an initial allocation has also been examined when each agent receives only one good, a setting sometimes known as a *housing market*. Gourvès *et al.* [2017] assumed an underlying social network and allowed beneficial exchanges between agents who are neighbors in the network—their work led to a series of follow-up papers on similar models [Huang and Xiao, 2020; Li *et al.*, 2021; Müller and Bentert, 2021; Ito *et al.*, 2023]. Damamme *et al.* [2015] also considered exchanges but without an underlying graph structure, while Brandt and Wilczynski [2024] used *Pareto-optimality* as their target notion. Note that the papers mentioned so far in this paragraph did not have fairness as their objective. Ito *et al.* [2022] incorporated fairness in the form of envy-freeness into this setting—starting with an envy-free allocation, they let each agent exchange her current good with a preferred unassigned good as long as the exchange keeps the allocation envy-free.

2 Preliminaries

Let $N = \{1, \dots, n\}$ be a set of $n \geq 2$ agents, and M be a set of $m \geq 1$ goods typically denoted by g_1, \dots, g_m . A *bundle* is a (possibly empty) subset of goods. An *allocation* $\mathcal{A} = (A_1, \dots, A_n)$ is an ordered partition of M into n bundles such that bundle A_i is allocated to agent $i \in N$. An (*allocation*) *size vector* $\vec{s} = (s_1, \dots, s_n)$ is a vector of non-negative integers such that $\sum_{i \in N} s_i = m$. We say that an allocation \mathcal{A} has size vector \vec{s} if $|A_i| = s_i$ for all $i \in N$. A size vector \vec{s} is *balanced* if $|s_i - s_j| \leq 1$ for all $i, j \in N$, and an allocation is *balanced* if it has a balanced size vector.

Each agent $i \in N$ has an additive *utility function* $u_i : 2^M \rightarrow \mathbb{R}_{\geq 0}$ that maps bundles to non-negative real numbers; additivity means that $u_i(M') = \sum_{g \in M'} u_i(\{g\})$ for all $i \in N$ and $M' \subseteq M$. We write $u_i(g)$ instead of $u_i(\{g\})$ for a single good $g \in M$. The utility functions are *identical* if $u_i = u_j$ for all $i, j \in N$ —we shall use u to denote the common utility function in this case. The utility functions are *binary* if $u_i(g) \in \{0, 1\}$ for all $i \in N$ and $g \in M$. Agent

²When there is no initial allocation, Dorn *et al.* [2021] considered deleting goods so that a proportional allocation of the remaining goods exists. In an earlier work, Aziz *et al.* [2016] examined discarding or adding goods to achieve envy-freeness, also in the absence of an initial allocation and under ordinal preferences.

utilities	general	identical	binary	identical & binary
$n = 2$	wNP-c (Theorem 3.3)	P (Theorem 3.2)	P (Theorem 3.7)	P (Theorem 3.7)
constant $n \geq 3$	wNP-c (Theorem 3.5)	wNP-c (Theorem 3.5)	P (Theorem 3.7)	P (Theorem 3.7)
general n	sNP-c (Theorem 3.8)	sNP-c (Theorem 3.8)	sNP-c (Theorem 3.9)	P (Theorem 3.10)

Table 1: Computational complexity of REFORMABILITY, the problem of deciding whether an EF1 allocation with a given size vector exists in a given instance. The top row represents the class of utility functions considered. The leftmost column represents the number of agents. “sNP-c” and “wNP-c” stand for strongly NP-complete and weakly NP-complete respectively.

i is *EF1 towards agent j* in an allocation $\mathcal{A} = (A_1, \dots, A_n)$ if either $A_j = \emptyset$ or there exists a good $g \in A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{g\})$. An allocation \mathcal{A} is *EF1* if every agent is EF1 towards every other agent in \mathcal{A} . A (*fair division*) *instance* \mathcal{I} consists of a set of agents N , a set of goods M , and the agents’ utility functions $(u_i)_{i \in N}$.

An *exchange* involves an agent i giving one good from her bundle to agent j and simultaneously receiving one good from agent j ’s bundle. We say that allocations $\mathcal{A} = (A_1, \dots, A_n)$ and $\mathcal{B} = (B_1, \dots, B_n)$ can be *reached via an exchange* if there exist distinct $i, j \in N$, $g \in A_i$, and $g' \in A_j$ such that $B_i = (A_i \cup \{g'\}) \setminus \{g\}$, $B_j = (A_j \cup \{g\}) \setminus \{g'\}$, and $B_k = A_k$ for all $k \in N \setminus \{i, j\}$. An allocation \mathcal{B} can be *reached* from an allocation \mathcal{A} if there exist a non-negative integer T and a sequence of allocations $(\mathcal{A}^0, \mathcal{A}^1, \dots, \mathcal{A}^T)$ such that $\mathcal{A}^0 = \mathcal{A}$, $\mathcal{A}^T = \mathcal{B}$, and for each $t \in \{0, \dots, T-1\}$, \mathcal{A}^t and \mathcal{A}^{t+1} can be reached via an exchange. The *optimal number of exchanges* required to reach \mathcal{B} from \mathcal{A} is the smallest T across all such sequences of allocations—if no such T exists (i.e., \mathcal{B} cannot be reached from \mathcal{A}), then the optimal number of exchanges is defined to be ∞ .³ Observe that two allocations can be reached from each other if and only if they share the same size vector.

Proposition 2.1. *Given an instance, let \mathcal{A} and \mathcal{B} be allocations in the instance. Then, \mathcal{B} can be reached from \mathcal{A} if and only if \mathcal{A} and \mathcal{B} have the same size vector.*

Proof. Note that every exchange preserves the size vector of the allocation, since each agent involved in the exchange gives away one good and receives one good in return, while other agents retain their bundles.

(\Rightarrow) If \mathcal{B} can be reached from \mathcal{A} , then there exist a non-negative integer T and a sequence of allocations $(\mathcal{A}^0, \mathcal{A}^1, \dots, \mathcal{A}^T)$ such that $\mathcal{A}^0 = \mathcal{A}$, $\mathcal{A}^T = \mathcal{B}$, and for each $t \in \{0, \dots, T-1\}$, \mathcal{A}^t and \mathcal{A}^{t+1} can be reached via an exchange. For each $t \in \{0, \dots, T-1\}$, \mathcal{A}^t and \mathcal{A}^{t+1} have the same size vector. Therefore, the whole sequence of allocations, including \mathcal{A} and \mathcal{B} , have the same size vector.

(\Leftarrow) Assume that \mathcal{A} and \mathcal{B} have the same size vector; we shall create a sequence of allocations from \mathcal{A} to \mathcal{B} with the desired properties. We first remedy the goods in agent 1’s bundle. If $A_1 = B_1$, then all the goods in agent 1’s bundle are correct and we are done. Otherwise, since $|A_1| = |B_1|$, we

³The optimal number of exchanges can be viewed as the distance between the two allocations in the implicit *exchange graph*, where the allocations are vertices and the edges connect allocations that can be reached via an exchange [Igarashi et al., 2024]. When there is no path connecting two vertices of a graph, it is common to define the distance between them as infinity.

must have $|A_1 \setminus B_1| = |B_1 \setminus A_1| > 0$. Perform an exchange between a good $g \in A_1 \setminus B_1$ and a good $g' \in B_1 \setminus A_1$. This creates a new allocation where the number of wrong goods in agent 1’s bundle decreases by one. By repeating this procedure, we eventually arrive at an allocation with agent 1’s bundle remedied. We then remedy the goods in the bundles of agents 2, 3, …, n in the same manner until every agent has her own bundle in \mathcal{B} . Note that when the goods in agent i ’s bundle are remedied, there is no exchange of goods involving agents 1 to $i-1$ anymore, and so the bundles of agents 1 to $i-1$ remain correct. This shows that \mathcal{B} can be reached from \mathcal{A} . \square

Next, we state a simple proposition that characterizes the existence of EF1 allocations based on the size vector.

Proposition 2.2. *Let $\vec{s} = (s_1, \dots, s_n)$, and let $m = \sum_{i=1}^n s_i$.*

- (a) *If \vec{s} is balanced, then every instance with n agents and m goods admits an EF1 allocation with size vector \vec{s} .*
- (b) *If \vec{s} is not balanced, then there exists an instance with n agents and m goods that does not admit any EF1 allocation with size vector \vec{s} .*

Proof. (a) An EF1 allocation can be guaranteed by allowing agents to pick their favorite goods in a round-robin fashion, with agents with higher s_i (if any) starting before those with lower s_i , until each agent i has s_i goods.

(b) Let \mathcal{I} be an instance with m goods such that $u_i(g) = 1$ for all $i \in N$ and $g \in M$. Let \mathcal{A} be any allocation with size vector \vec{s} . Since \vec{s} is not balanced, there exist distinct $i, j \in N$ such that $s_j - s_i \geq 2$. Then, we have $|A_j| = s_j > 0$, so $A_j \neq \emptyset$. Furthermore, $u_i(A_i) = s_i < s_j - 1 = u_i(A_j \setminus \{g\})$ for all $g \in A_j$. This shows that agent i is not EF1 towards agent j , and so \mathcal{A} is not EF1. Therefore, \mathcal{I} does not admit an EF1 allocation with size vector \vec{s} . \square

Finally, we introduce an NP-hard decision problem called the **BALANCED MULTI-PARTITION** problem, which we will use later in the proofs of several results. In **BALANCED MULTI-PARTITION**, we are given positive integers p, q, K and a multiset of positive integers $X = \{x_1, \dots, x_{pq}\}$ such that $K < x_j \leq 2K$ for all $j \in \{1, \dots, pq\}$, and the sum of all the integers in X is $p(q+1)K$. The problem is to decide whether X can be partitioned into multisets X_1, \dots, X_p of equal cardinalities and sums, i.e., for each $i \in \{1, \dots, p\}$, the cardinality of X_i is q and the sum of all the integers in X_i is $(q+1)K$. The NP-hardness of this problem is based on a reduction from the equal-cardinality version of the NP-hard problem **PARTITION** [Garey and Johnson, 1979, p. 223].

Proposition 2.3. For any fixed $p \geq 2$, BALANCED MULTI-PARTITION is NP-hard.

Proof. We shall prove NP-hardness via a series of reductions from the equal-cardinality version of PARTITION. In this version, we are given positive integers q, K' and a multiset of positive integers $W = \{w_1, \dots, w_{2q}\}$ such that the sum of the integers in W is $2K'$. The problem is to decide whether W can be partitioned into multisets W_1 and W_2 of equal cardinalities and sums. This problem is known to be NP-hard [Garey and Johnson, 1979, p. 223].

Let an instance of the equal-cardinality version of PARTITION be given, and let $p \geq 2$ be a fixed integer. If some integer in W is more than K' , then W cannot be partitioned into the desired multisets; therefore, we assume that every integer in W is at most K' . Define a multiset $W^1 = \{w_j \mid j \in \{2q+1, \dots, 2q+(p-2)\}\}$ such that $w_j = K'$ for all $w_j \in W^1$; define a multiset $W^0 = \{w_j \mid j \in \{2q+(p-2)+1, \dots, pq\}\}$ such that $w_j = 0$ for all $w_j \in W^0$; and define $W' = W \cup W^1 \cup W^0$. Essentially, we are adding $p-2$ copies of the number K' and sufficiently many copies of the number 0 so that the total number of elements in W' is pq . Note that every integer in W' is at most K' , and the sum of all the elements in W' is pK' . We claim that W' can be partitioned into multisets W'_1, \dots, W'_p of equal cardinalities and sums (i.e., each W'_i has cardinality q and sum K') if and only if W can be partitioned into multisets W_1 and W_2 of equal cardinalities and sums.

(\Leftarrow) If we are given a partition into multisets W_1 and W_2 , let $W'_1 = W_1$, let $W'_2 = W_2$, and let W'_i contain one element from W^1 and $q-1$ elements from W^0 for each $i \in \{3, \dots, p\}$. Each of W'_1, \dots, W'_p has q elements with sum K' . This gives a desired partition of W' .

(\Rightarrow) Assume that we are given a partition into multisets W'_1, \dots, W'_p . If some W'_i contains at least two elements in W^1 , then the sum of W'_i is more than K' , which is not possible. Therefore, every W'_i contains at most one element in W^1 . Furthermore, for each W'_i containing some element in W^1 , if it contains some element in W , then its sum would exceed K' , which is again not possible. Therefore, there are $p-2$ of the W'_i such that each of them contains one element from W^1 and $q-1$ elements from W^0 . This means that two of the W'_i contain exactly the elements in W . These two W'_i induce the desired partition into W_1 and W_2 of W .

Now, define an instance of BALANCED MULTI-PARTITION as follows. Let $K = K' + q$, and let $X = \{x_1, \dots, x_{pq}\}$ be such that $x_j = w_j + K + 1$ for all $j \in \{1, \dots, pq\}$. For each $j \in \{1, \dots, pq\}$, since $0 \leq w_j \leq K'$, we have $K < x_j \leq K' + K + 1 \leq 2K$. The sum of all integers in X is $pK' + pq(K+1) = p(q+1)K$. It is clear that X can be partitioned into multisets X_1, \dots, X_p of equal cardinalities and sums if and only if W' can be partitioned into multisets W'_1, \dots, W'_p of equal cardinalities and sums, since the difference between x_j and w_j is the same for all j . Note that the reductions in this proof can all be done in polynomial time. This establishes the NP-hardness of BALANCED MULTI-PARTITION. \square

3 Reformability of Allocations

We start by investigating the decision problem of whether a given initial allocation can be reformed into an EF1 allocation. By Proposition 2.1, this reformation is possible if and only if there exists an EF1 allocation with the same size vector as the initial allocation. Therefore, in the rest of this section, we shall equivalently focus on the problem of deciding the existence of an EF1 allocation with a given size vector—this problem can be of interest independently of reformation considerations, e.g., when space constraints are present.

Now, Proposition 2.2 tells us that an EF1 allocation with a balanced size vector always exists. This means that the only time when we may have difficulties in ascertaining whether an EF1 allocation exists is when the given size vector is *not* balanced. In fact, as some of our proofs in this section show, the decision problem is NP-hard even when the sizes of the agents' bundles differ by *exactly two* (e.g., in Theorem 3.3).

We discuss the cases of two agents, a constant number of agents, and a general number of agents separately. For each of these cases, we explore how the hardness of the decision problem varies across different classes of utility functions. Our results are summarized in Table 1.

For convenience, we refer to as REFORMABILITY the problem of deciding whether an EF1 allocation with a given size vector exists in a given instance. Note that REFORMABILITY is in NP regardless of the number of agents, as we can verify in polynomial time whether a given allocation satisfies the condition by simply checking its size vector and comparing the bundles of every pair of agents for EF1.

3.1 Two Agents

For two agents, interestingly, the computational complexity of the problem turns out to be different depending on whether the agents have identical utilities or not. We begin our discussion with the case of identical utilities.

For two agents with identical utilities, we first provide a simple characterization for the existence of a desired EF1 allocation based on the size vector and the utilities of the goods. We show that an EF1 allocation with a given size vector exists if and only if the agent with fewer goods (say, agent 1) is EF1 towards the other agent (say, agent 2) in the allocation where agent 1 receives the most valuable goods. Note that the condition in the lemma only requires checking that agent 1 is EF1 towards agent 2; in particular, it does not require checking that agent 2 is EF1 towards agent 1.

Lemma 3.1. Given an instance with two agents with identical utilities, let $\vec{s} = (s_1, s_2)$ be a size vector with $s_1 \leq s_2$. Assume that the goods g_1, \dots, g_m are arranged in non-increasing order of utility, and let $M_0 = \{g_1, \dots, g_{s_1}\}$. Then, there exists an EF1 allocation with size vector \vec{s} if and only if agent 1 is EF1 towards agent 2 in the allocation $(M_0, M \setminus M_0)$.

Proof. We say in this proof that for any nonempty set $M' \subseteq M$, the good $g_i \in M'$ is the *most valuable* good in M' if g_i is the good with the smallest index in M' ; likewise, g_i is the *least valuable* good in M' if g_i is the good with the largest index in M' . Note that the most (resp. least) valuable good in

M' is the one with the highest (resp. lowest) utility among all the goods in M' , with ties broken by index.

(\Rightarrow) Let (A_1, A_2) be an EF1 allocation with size vector \vec{s} . Let g and g' be the most valuable good in A_2 and $M \setminus M_0$ respectively. Since M_0 is the set containing the s_1 most valuable goods, we have $u(M_0) \geq u(A_1)$. Since (A_1, A_2) is an EF1 allocation, we have $u(A_1) \geq u(A_2 \setminus \{g\})$. Moreover, since $M \setminus M_0$ is the set containing the s_2 least valuable goods, we have $u(A_2 \setminus \{g\}) \geq u((M \setminus M_0) \setminus \{g'\})$. Combining the three inequalities, we get $u(M_0) \geq u((M \setminus M_0) \setminus \{g'\})$. It follows that agent 1 is EF1 towards agent 2 in the allocation $(M_0, M \setminus M_0)$.

(\Leftarrow) Suppose that agent 1 is EF1 towards agent 2 in the allocation $(M_0, M \setminus M_0)$. If agent 2 is also EF1 towards agent 1 in $(M_0, M \setminus M_0)$, then we are done; therefore, assume that agent 2 envies agent 1 by more than one good. For notational simplicity, let $h_j = g_{s_1+j}$ for $j \in \{1, \dots, s_1\}$, so that the goods arranged in non-increasing order of utility are $g_1, g_2, \dots, g_{s_1}, h_1, h_2, \dots, h_{s_1}, g_{2s_1+1}, \dots, g_m$. Let $A_1^1 = M_0 = \{g_1, \dots, g_{s_1}\}$ and $A_2^1 = M \setminus M_0 = \{h_1, \dots, h_{s_1}\} \cup \{g_{2s_1+1}, \dots, g_m\}$.

Let $t = 1$. In the allocation (A_1^t, A_2^t) , agent 1 is EF1 towards agent 2, but agent 2 envies agent 1 by more than one good. Since g_t is the most valuable good in A_1^t , we have $u(A_2^t) < u(A_1^t \setminus \{g_t\})$. Let $A_1^{t+1} = (A_1^t \cup \{h_t\}) \setminus \{g_t\}$ and $A_2^{t+1} = (A_2^t \cup \{g_t\}) \setminus \{h_t\}$ be the bundles after exchanging g_t and h_t . Then, we have

$$\begin{aligned} u(A_1^{t+1}) &= u((A_1^t \cup \{h_t\}) \setminus \{g_t\}) \\ &\geq u(A_1^t \setminus \{g_t\}) \\ &> u(A_2^t) \\ &= u((A_2^t \cup \{h_t\}) \setminus \{g_t\}) \geq u(A_2^{t+1} \setminus \{g_t\}), \end{aligned}$$

so agent 1 is EF1 towards agent 2 in (A_1^{t+1}, A_2^{t+1}) . If agent 2 is also EF1 towards agent 1, then (A_1^{t+1}, A_2^{t+1}) is an EF1 allocation and we are done. Otherwise, agent 2 envies agent 1 by more than one good, and we increment t by 1 and repeat the discussion in this paragraph.

If we still have not found an EF1 allocation after $t = s_1$, then agent 1 is EF1 towards agent 2 in $(A_1^{s_1+1}, A_2^{s_1+1})$, where $A_1^{s_1+1} = \{h_1, \dots, h_{s_1}\} \subseteq A_2^1$ and $A_2^{s_1+1} = \{g_1, \dots, g_{s_1}\} \cup \{g_{2s_1+1}, \dots, g_m\} \supseteq A_1^1$, and g_1 is the most valuable good in $A_2^{s_1+1}$. This implies that

$$u(A_1^{s_1+1}) \leq u(A_2^1) < u(A_1^1 \setminus \{g_1\}) \leq u(A_2^{s_1+1} \setminus \{g_1\}),$$

which means that agent 1 is *not* EF1 towards agent 2 in $(A_1^{s_1+1}, A_2^{s_1+1})$. This is a contradiction; therefore, (A_1^t, A_2^t) must be EF1 for some $t \in \{1, \dots, s_1\}$. \square

Since the condition in Lemma 3.1 can be checked in polynomial time, we can derive the following result.

Theorem 3.2. REFORMABILITY is in P for two agents with identical utilities.

Proof. Without loss of generality, let the size vector be (s_1, s_2) with $s_1 \leq s_2$. Arrange the goods g_1, \dots, g_m in non-increasing order of utility (which can be done in polynomial

time), and let M_0 be the set of s_1 goods with the highest utilities. By Lemma 3.1, there exists an EF1 allocation with size vector (s_1, s_2) if and only if agent 1 is EF1 towards agent 2 in the allocation $(M_0, M \setminus M_0)$. The latter condition can be checked in polynomial time. \square

While deciding whether an EF1 allocation with a given size vector exists can be done efficiently for two agents with identical utilities, we remark here that deciding whether an envy-free allocation exists is NP-hard for two agents with identical utilities even if we allow any size vector—this follows directly from a reduction from PARTITION.⁴

We now proceed to general utilities. Lemma 3.1 assumes identical utilities, and there is no obvious way to generalize it to non-identical utilities. In fact, perhaps surprisingly, we show that the decision problem becomes NP-hard when we drop the assumption of identical utilities. The proof follows from a reduction from BALANCED MULTI-PARTITION with $p = 2$, an NP-hard problem by Proposition 2.3.

Theorem 3.3. REFORMABILITY is weakly NP-complete for two agents.

Proof. Clearly, this problem is in NP. The “weak” aspect is demonstrated later in Lemma 3.4, which says that there exists a pseudopolynomial-time algorithm that solves this problem for any constant number of agents. Therefore, it suffices to show that this problem is NP-hard.

To demonstrate NP-hardness, we shall reduce from the NP-hard problem BALANCED MULTI-PARTITION with $p = 2$ (see Proposition 2.3). Let a BALANCED MULTI-PARTITION instance be given with $p = 2$. Without loss of generality, assume that $q \geq 2$. Let $Y = \{y_1, \dots, y_{2q+2}\}$ be a multiset such that $y_j = x_j$ for $j \in \{1, \dots, 2q\}$, $y_{2q+1} = 2K$, and $y_{2q+2} = 0$. We claim that Y can be partitioned into two multisets Y_1 and Y_2 of equal cardinalities (i.e., of size $q + 1$ each) with sums $(q + 3)K$ and $(q + 1)K$ respectively if and only if X can be partitioned into two multisets X_1 and X_2 of equal cardinalities and sums. If the latter condition holds, then let Y_1 (resp. Y_2) contain the corresponding elements in X_1 (resp. X_2), and let $y_{2q+1} \in Y_1$ and $y_{2q+2} \in Y_2$ —this gives an appropriate partition of Y . Conversely, if the former condition holds, then we show that X can be partitioned appropriately. Note that if $y_{2q+1} \in Y_2$, then there are at least $q - 1 > 0$ integers in $\{y_1, \dots, y_{2q}\}$ that are also in Y_2 . Since every integer in $\{y_1, \dots, y_{2q}\}$ is more than K , the sum of Y_2 will be more than $(q - 1)K + 2K = (q + 1)K$, which is a contradiction. This means that $y_{2q+1} \in Y_1$. Similarly, if $y_{2q+2} \in Y_1$, then there are exactly $q + 1$ integers in $\{y_1, \dots, y_{2q}\}$ that are in Y_2 . The sum of Y_2 will be more than $(q + 1)K$, which is a contradiction. Hence, $y_{2q+2} \in Y_2$. Now, this means that $\{y_1, \dots, y_{2q}\}$ must be partitioned into two multisets of equal cardinalities (i.e., of size q each) with sum $(q + 1)K$ each. This induces an appropriate partition of X .

Next, define a fair division instance as follows. There are $n = 2$ agents and a set of goods $M = \{g_1, \dots, g_{2q+6}\}$.

⁴If we require both agents to receive the same number of goods, the problem for envy-freeness remains NP-hard by a reduction from the equal-cardinality version of PARTITION.

Agent 2's utility is such that $u_2(g_j) = y_j$ for $j \in \{1, \dots, 2q+2\}$, $u_2(g_{2q+3}) = u_2(g_{2q+4}) = 0$, and $u_2(g_{2q+5}) = u_2(g_{2q+6}) = 2K$. Agent 1's utility is such that $u_1(g) = u_2(g) + 4K$ for all $g \in M$. The size vector \vec{s} is $(q+2, q+4)$.

This reduction can be done in polynomial time. We claim that there exists an EF1 allocation with size vector \vec{s} in this instance if and only if Y can be partitioned into two multisets Y_1 and Y_2 of equal cardinalities (i.e., of size $q+1$ each) with sums $(q+3)K$ and $(q+1)K$ respectively.

(\Leftarrow) Let J'_1 and J'_2 be a partition of $\{1, \dots, 2q+2\}$ of equal cardinalities such that $\sum_{j \in J'_1} y_j = (q+3)K$ and $\sum_{j \in J'_2} y_j = (q+1)K$. Let $A_1 = \{g_j \mid j \in J'_1\} \cup \{g_{2q+5}\}$ and $A_2 = M \setminus A_1$ be the two agents' bundles respectively. From agent 1's perspective, agent 1's bundle has utility $((q+3)K + 2K) + (q+2)(4K) = (5q+13)K$, agent 2's bundle has utility $((q+1)K + 2K) + (q+4)(4K) = (5q+19)K$, and a most valuable good in agent 2's bundle (e.g., g_{2q+6}) has utility $6K$, so agent 1 is EF1 towards agent 2. From agent 2's perspective, agent 2's bundle has utility $(q+1)K + 2K = (q+3)K$, agent 1's bundle has utility $(q+3)K + 2K = (q+5)K$, and a most valuable good in agent 1's bundle (e.g., g_{2q+5}) has utility $2K$, so agent 2 is EF1 towards agent 1. Accordingly, (A_1, A_2) is an EF1 allocation with size vector $(q+2, q+4)$.

(\Rightarrow) Let (A_1, A_2) be an EF1 allocation with size vector \vec{s} . From agent 1's perspective, $u_1(M) = (10q+32)K$ and a most valuable good (e.g., g_{2q+5}) has utility $6K$. For agent 1 to be EF1 towards agent 2, we must have $u_1(A_1) \geq ((10q+32)K - 6K)/2 = (5q+13)K$ and $u_2(A_1) = u_1(A_1) - (q+2)(4K) \geq (q+5)K$. This means that $u_1(A_2) = u_1(M) - u_1(A_1) \leq (5q+19)K$ and $u_2(A_2) = u_1(A_2) - (q+4)(4K) \leq (q+3)K$. On the other hand, from agent 2's perspective, $u_2(M) = (2q+8)K$ and a most valuable good has utility $2K$. For agent 2 to be EF1 towards agent 1, we must have $u_2(A_2) \geq ((2q+8)K - 2K)/2 = (q+3)K$ and $u_1(A_2) = u_2(A_2) + (q+4)(4K) \geq (5q+19)K$. This means that $u_2(A_1) = u_2(M) - u_2(A_2) \leq (q+5)K$ and $u_1(A_1) = u_2(A_1) + (q+2)(4K) \leq (5q+13)K$. By combining these inequalities, we conclude that these inequalities are tight, i.e., agent 1's utilities for both agents' bundles are exactly $(5q+13)K$ and $(5q+19)K$ respectively so that the sum is $(10q+32)K$, and agent 2's utilities for both agents' bundles are exactly $(q+5)K$ and $(q+3)K$ respectively so that the sum is $(2q+8)K$. Additionally, both agents must each have a most valuable good worth $6K$ and $2K$ to them respectively. Without loss of generality, we may assume that $g_{2q+5} \in A_1$ and $g_{2q+6} \in A_2$ (note that g_{2q+1} is also a most valuable good, but we use g_{2q+5} and g_{2q+6} for simplicity).

Since $g_{2q+6} \in A_2$, there are $q+3$ goods in $A_2 \setminus \{g_{2q+6}\}$ and $u_2(A_2 \setminus \{g_{2q+6}\}) = (q+3)K - 2K = (q+1)K$. These goods are chosen from $M_1 = \{g_1, \dots, g_{2q+1}\}$ and $M_0 = \{g_{2q+2}, g_{2q+3}, g_{2q+4}\}$. Recall from the construction that $u_2(g) > K$ for all $g \in M_1$, and $u_2(g) = 0$ for all $g \in M_0$. If $A_2 \setminus \{g_{2q+6}\}$ contains at least $q+1$ goods from M_1 , then $u_2(A_2 \setminus \{g_{2q+6}\}) > (q+1)K$, a contradiction. Therefore, $A_2 \setminus \{g_{2q+6}\}$ contains at most q goods from M_1 , and at least 3 goods from M_0 . Since $|M_0| = 3$, we must have $M_0 \subseteq A_2$. Note that $u_2(M_0) = 0$, so $u_2((A_2 \setminus \{g_{2q+6}\}) \setminus M_0) = (q+1)K$. Therefore, the q goods

from M_1 in agent 2's bundle have a total utility of $(q+1)K$. These goods, together with g_{2q+2} , induce the set Y_2 with cardinality $q+1$ and sum $(q+1)K$. Then, $Y_1 = Y \setminus Y_2$ and Y_2 give a required partition of Y . \square

The proof of Theorem 3.3 suggests that the problem is NP-hard even when the sizes of the two agents' bundles differ by *exactly two*. Note that this problem is in P when the sizes of the agents' bundles differ by *at most one* (in fact, every such instance is a Yes-instance by Proposition 2.2).

For two agents with binary utilities, we shall show later that the decision problem is in P (see Theorem 3.7).

3.2 Constant Number of Agents

Next, we discuss the complexity of the decision problem for a *constant number of agents*. In this case, we devise a pseudopolynomial-time algorithm for deciding the existence of an EF1 allocation with a given size vector. This algorithm uses dynamic programming to check for such an allocation.

Lemma 3.4. *Let an instance with n agents and a size vector be given, where n is a constant. Suppose that the utility of each good is an integer, and let $R = \max_{i \in N} u_i(M)$. Then, there exists an algorithm running in time polynomial in m and R that decides whether the instance admits an EF1 allocation with the size vector.*

Proof. The algorithm uses dynamic programming. We construct a table with m columns and L rows, where L will be specified later. The index of each row is represented by a tuple containing $a_{i,j}$, $b_{i,j}$, and c_i for each $i, j \in N$, i.e., $(a_{1,1}, a_{1,2}, \dots, a_{n,n}, b_{1,1}, b_{1,2}, \dots, b_{n,n}, c_1, \dots, c_n)$. The value of $a_{i,j}$ is the utility of agent j 's bundle from agent i 's perspective, i.e., $a_{i,j} = u_i(A_j)$; the value of $b_{i,j}$ is the utility of a most valuable good in agent j 's bundle from agent i 's perspective, i.e., $b_{i,j} = \max_{g \in A_j} u_i(g)$ (note that this value is zero if $A_j = \emptyset$); and the value of c_i is the number of goods in agent i 's bundle. Note that $a_{i,j}, b_{i,j} \in \{0, \dots, R\}$ and $c_i \in \{0, \dots, m\}$, so there are $L = (R+1)^{2n^2}(m+1)^n$ rows, which is polynomial in m and R . An entry in column q represents whether an allocation involving $\{g_1, \dots, g_q\}$ is possible for the tuple representing the row, and is either positive or negative.

Initialize every entry to negative. Consider the n possibilities of adding g_1 to each of the agents' bundles respectively, and set the corresponding entries in the first column of the table to positive. In particular, for each $j \in N$, the row represented by the tuple such that $a_{i,j} = b_{i,j} = u_i(g_1)$ and $c_j = 1$ for all $i \in N$, and zero for all other values in the tuple, has the entry (in the first column) set to positive.

Now, for each $q \in \{2, \dots, m\}$ in ascending order, for each positive entry in column $q-1$, consider the n possibilities of adding g_q into each of the n agents' bundles respectively, and set the corresponding entry for each of these possibilities in column q to positive. Once this procedure is done, consider all positive entries in column m . If some positive entry corresponds to an EF1 allocation with the required size vector, then the instance admits such an EF1 allocation; otherwise, no such allocation exists.

Since n is a constant, the number of entries in the table is polynomial in m and R . At each column, there is a polynomial number of rows with positive entries, and hence the update is polynomial. Finally, checking for a feasible EF1 allocation at the last column can also be done in polynomial time. \square

We now move to *polynomial-time* algorithms that determine the existence of an EF1 allocation with a given size vector. Recall that such an algorithm exists for two agents with identical utilities (Theorem 3.2). However, it turns out that such an algorithm does not exist for three or more agents with identical utilities, unless $P = NP$. In particular, we establish the NP-hardness of the decision problem via a reduction from BALANCED MULTI-PARTITION with $p = 2$, an NP-hard problem by Proposition 2.3.

Theorem 3.5. REFORMABILITY is weakly NP-complete for $n \geq 3$ agents with identical utilities, where n is a constant.

Proof. Clearly, this problem is in NP. The “weak” aspect is demonstrated in Lemma 3.4. Therefore, it suffices to show that this problem is NP-hard.

To show NP-hardness, we shall reduce from the NP-hard problem BALANCED MULTI-PARTITION with $p = 2$ (see Proposition 2.3). Let a BALANCED MULTI-PARTITION instance with $p = 2$ be given. Define a fair division instance as follows. There are $n \geq 3$ agents with identical utilities, and a set of goods $M = \{g_1, \dots, g_{2q}, h_1, \dots, h_n\}$ such that $u(g_j) = x_j$ for $j \in \{1, \dots, 2q\}$ and $u(h_k) = (q+1)K$ for $k \in \{1, \dots, n\}$. The size vector \vec{s} is such that $s_1 = s_2 = q+1$ and $s_k = 1$ for all $k \in \{3, \dots, n\}$. This reduction can be done in polynomial time. We claim that there exists an EF1 allocation with size vector \vec{s} in this instance if and only if X can be partitioned into multisets X_1, X_2 of equal cardinalities and sums.

(\Leftarrow) Let (X_1, X_2) be such a partition. Define an allocation such that agent k receives h_k for $k \in N$, agent 1 additionally receives the q goods corresponding to the integers in X_1 , and agent 2 additionally receives the q goods corresponding to the integers in X_2 . We show that this allocation is EF1. The utilities of agent 1’s and agent 2’s bundles are $2(q+1)K$ each, and the utilities of the other agents’ bundles are $(q+1)K$ each, so agents 1 and 2 do not envy anyone else. Therefore, it remains to check that agent k is EF1 towards agents 1 and 2 for $k \in \{3, \dots, n\}$. Upon the removal of the single good h_1 (resp. h_2) from agent 1’s (resp. agent 2’s) bundle, the remaining bundle has utility $(q+1)K$, so agent k is EF1 towards agent 1 (resp. agent 2). Therefore, the allocation is EF1, as desired.

(\Rightarrow) Let (A_1, \dots, A_n) be an EF1 allocation with size vector $(q+1, q+1, 1, \dots, 1)$. If agent 1’s bundle has at least two goods from $\{h_1, \dots, h_n\}$, then her bundle without the most valuable good has utility more than $(q+1)K$ since her bundle also contains other goods with positive utility. Agent 3, having a bundle of utility at most $(q+1)K$, will not be EF1 towards agent 1, contradicting the assumption that the allocation is EF1. Therefore, agent 1’s bundle has at most one good from $\{h_1, \dots, h_n\}$; likewise for agent 2’s bundle. This means that every agent receives *exactly* one good from

$\{h_1, \dots, h_n\}$. Having established this, agent 3’s bundle has a utility of $(q+1)K$, and agent 3 is EF1 towards agent 1. This means that agent 1’s bundle without a most valuable good (say, some h_k) must have utility at most $(q+1)K$. The same argument can be used to show the same statement for agent 2’s bundle. This means that the goods $\{g_1, \dots, g_{2q}\}$ must be divided between agents 1 and 2 with each agent receiving a utility of *exactly* $(q+1)K$. Such a division of $\{g_1, \dots, g_{2q}\}$ induces a partition of X into two multisets of equal cardinalities and sums, as desired. \square

Since the decision problem is NP-hard even for identical utilities, it must also be NP-hard for general utilities. We now consider another class of utilities: binary utilities. When there are n agents, every good g belongs to one of 2^n types of goods represented by the vector $(u_1(g), \dots, u_n(g))$. For the purpose of determining whether an EF1 allocation exists, it suffices to consider different goods of the same type as *indistinguishable*. We say that two allocations are in the same equivalence class if the number of goods of each type that each agent has is the same in both allocations. If \mathcal{A} is an EF1 allocation, then all allocations in the same equivalence class as \mathcal{A} are also EF1 and are reachable from \mathcal{A} . We shall proceed with a result which enumerates all (essentially equivalent) EF1 allocations in time polynomial in the number of goods, provided that the number of agents is a constant.

Lemma 3.6. Let an instance with n agents with binary utilities and a size vector be given, where n is a constant. Then, there exists an algorithm running in time polynomial in m that outputs all equivalent classes of EF1 allocations with the size vector.

Proof. An agent’s bundle can be represented by a 2^n -vector where each component of the vector is the number of goods of that type in her bundle. Since the number of goods of each type is an integer between 0 and m , there are $m+1$ possible values for each component, and hence at most $(m+1)^{2^n}$ possible vectors to represent each agent’s bundle. Allocations in an equivalence class can be represented by an ordered collection of n such vectors—one for each agent—and there are at most $((m+1)^{2^n})^n$ such collections. Since $((m+1)^{2^n})^n$ is polynomial in m whenever n is a constant, there is at most a polynomial number of possible equivalence classes of allocations in the instance. For each of these equivalence classes of allocations, we can check whether an allocation in the equivalence class is EF1 and has the required size vector in polynomial time, and output the equivalence class if so. Therefore, the overall running time is polynomial in m , as claimed. \square

Lemma 3.6 implies that the decision problem can be solved efficiently for binary utilities.

Theorem 3.7. REFORMABILITY is in P for a constant number of agents with binary utilities.

Proof. Use the algorithm as described in Lemma 3.6 to enumerate all equivalence classes with an EF1 allocation with the size vector, and output “Yes” if and only if such an equivalence class is found. Note that if some allocation in an equivalence class is EF1, then all allocations in the same equivalence class are also EF1. \square

3.3 General Number of Agents

For any constant number of agents, the problem of determining the existence of an EF1 allocation with a given size vector is *weakly NP-hard* by Theorem 3.5 (even for identical utilities). For a general number of agents, the pseudopolynomial-time algorithm as described in Lemma 3.4 does not work, since that algorithm is at least exponential in the number of agents. Therefore, the decision problem for a general number of agents might not be *weakly NP-hard*. We show that the problem is indeed *strongly NP-hard*, even for identical utilities, by a reduction from 3-PARTITION, a strongly NP-hard problem [Garey and Johnson, 1979, p. 224].

Theorem 3.8. REFORMABILITY is strongly NP-complete for identical utilities.

Proof. Clearly, this problem is in NP. Therefore, it suffices to show that it is strongly NP-hard.

To this end, we shall reduce from 3-PARTITION. In 3-PARTITION, we are given positive integers q and K , and a multiset $X = \{x_1, \dots, x_{3q}\}$ of positive integers of total sum qK . The problem is to decide whether X can be partitioned into multisets X_1, \dots, X_q of equal cardinalities and sums, i.e., for each $i \in \{1, \dots, q\}$, $|X_i| = 3$ and the sum of all the integers in X_i is exactly K . This decision problem is known to be strongly NP-hard, even if $K/4 < x_j < K/2$ for every $j \in \{1, \dots, 3q\}$ [Garey and Johnson, 1979, p. 224].

Let an instance of 3-PARTITION be given. Define a fair division instance as follows. There are $n = q + 1$ agents with identical utilities, and a set of goods $M = \{g_1, \dots, g_{3q+6}\}$ such that $u(g_j) = x_j$ for $j \in \{1, \dots, 3q\}$ and $u(g_k) = K/5$ for $k \in \{3q + 1, \dots, 3q + 6\}$. The size vector \vec{s} is such that $s_j = 3$ for all $j \in \{1, \dots, q\}$ and $s_{q+1} = 6$. This reduction can be done in polynomial time. We claim that there exists an EF1 allocation with size vector \vec{s} in this instance if and only if there exists a partition of X into multisets X_1, \dots, X_q of equal cardinalities and sums.

(\Leftarrow) Let such a partition be given. Define an allocation such that each of the first q agents receives the three goods corresponding to each multiset, and agent $q + 1$ receives $\{g_{3q+1}, \dots, g_{3q+6}\}$. Note that agents 1 to q have bundles worth K each, and agent $q + 1$ has a bundle worth $6K/5$. Since each of the goods from agent $(q + 1)$'s bundle is worth exactly $K/5$, every agent is EF1 towards agent $q + 1$. Accordingly, the allocation is EF1.

(\Rightarrow) Let an EF1 allocation with size vector \vec{s} be given. Note that every good is worth at least $K/5$, so agent $(q + 1)$'s bundle without a most valuable good is worth at least K . If agent $q + 1$ receives a bundle worth more than $6K/5$, then some agent receives a bundle worth less than K , and will not be EF1 towards agent $q + 1$. Therefore, agent $q + 1$ must receive a bundle worth at most $6K/5$. The only way this is possible is when agent $q + 1$ receives $\{g_{3q+1}, \dots, g_{3q+6}\}$. Now, since agents 1 to q are EF1 towards agent $q + 1$, these agents must each receive a bundle worth at least K . The only way this is possible is when each of them receives a bundle worth exactly K . This induces the desired partition. \square

The reduction in Theorem 3.8 requires us to check whether the set of agents holding the goods with utilities equal to the

integers in the partition problem is EF1 towards the remaining agents. This stands in contrast to the reduction in Theorem 3.5, which entails checking whether the complementary set of agents is EF1 towards the remaining agents.

For binary utilities, the decision problem for a *constant* number of agents is in P (Theorem 3.7). The crucial reason is that in this case, the number of different types of goods is also a constant, which allows us to enumerate all the (essentially equivalent) EF1 allocations in polynomial time (Lemma 3.6). This is no longer possible when the number of agents is non-constant. In fact, we show that the decision problem is strongly NP-hard for a general number of agents with binary utilities. To this end, we reduce from GRAPH k -COLORABILITY, which is strongly NP-hard for any fixed $k \geq 3$ [Garey and Johnson, 1979, p. 191].

Theorem 3.9. REFORMABILITY is strongly NP-complete for binary utilities.

Proof. Clearly, this problem is in NP. Therefore, it suffices to show that it is strongly NP-hard.

To this end, we shall reduce from GRAPH k -COLORABILITY with $k \geq 3$. In GRAPH k -COLORABILITY, we are given a graph $G = (V, E)$ and a positive integer k , and the problem is to decide whether G is k -colorable, i.e., whether each of the vertices in V can be assigned one of k colors in such a way that no two adjacent vertices are assigned the same color. This decision problem is known to be strongly NP-hard for any fixed $k \geq 3$ [Garey and Johnson, 1979, p. 191].

Let an instance of GRAPH k -COLORABILITY be given with a fixed $k \geq 3$, where $V = \{v_1, \dots, v_p\}$ and $E = \{e_1, \dots, e_q\}$. Define a fair division instance as follows. There are $n = q + k$ agents where the first q agents are called *edge agents* and the last k agents are called *color agents*. There are $m = kp$ goods. Each color agent assigns zero utility to every good. For $r \in \{1, \dots, q\}$, if $e_r = \{v_i, v_j\}$, then the r^{th} edge agent assigns a utility of 1 to each of g_i and g_j , and zero utility to every other good. Note that only the first p goods correspond to vertices and are valuable to the edge agents whose corresponding edges are incident to the vertices; the remaining $(k - 1)p$ goods are not valuable to any agent. The size vector \vec{s} is such that $s_r = 0$ for each edge agent r and $s_c = p$ for each color agent c . This reduction can be done in polynomial time. We claim that there exists an EF1 allocation with size vector \vec{s} in this instance if and only if G is k -colorable.

(\Leftarrow) Let a proper k -coloring of G be given. For $t \in \{1, \dots, p\}$, if vertex v_t is assigned the color c , then allocate g_t to the color agent c . Since there are p such goods and each color agent is supposed to have p goods in her bundle, it is possible to allocate all of these goods. Subsequently, allocate the remaining goods arbitrarily among the color agents until every color agent has exactly p goods. We claim that this allocation is EF1. Every color agent assigns zero utility to every good and is thus EF1 towards every other agent. Each edge agent assigns a utility of 1 to only two goods, so we only need to check that these two goods are in different bundles. Indeed, these two goods correspond to vertices which are adjacent to each other in G , and proper coloring of G implies

that the vertices are of different colors, so the corresponding goods are in different color agents' bundles. Therefore, the allocation is EF1.

(\Rightarrow) Let an EF1 allocation with size vector \vec{s} be given. For $t \in \{1, \dots, p\}$, if the good g_t is with color agent c , assign v_t to color c . We claim that this coloring is a proper k -coloring of G . Since there are k color agents, at most k colors are used. Therefore, it suffices to check that adjacent vertices are assigned different colors. Let $v_i, v_j \in V$ be adjacent vertices. Then, there exists an edge $e_r = \{v_i, v_j\}$. The edge agent r assigns a utility of 1 to each of g_i and g_j . Since agent r 's bundle is empty, g_i and g_j must be in different (color agents') bundles in order for agent r to be EF1 towards every other agent. This implies that v_i and v_j are assigned different colors. \square

Even though the decision problem is strongly NP-hard for identical *or* binary utilities, we prove next that it can be solved in polynomial time for identical *and* binary utilities. Indeed, this can be done by checking whether the total number of valuable goods is within a certain threshold which can be computed in polynomial time. This threshold is in fact the sum over all agents $j \in N$ of the minimum between s_j and $1 + \min_{i \in N} s_i$. If the number of valuable goods is within this threshold, then we can distribute the valuable goods in a round-robin fashion first, ensuring EF1. Conversely, if the number of valuable goods is beyond this threshold, then some agent i with the minimum s_i will not be EF1 towards another agent who receives at least $s_i + 2$ valuable goods.

Theorem 3.10. REFORMABILITY is in P for identical binary utilities.

Proof. Let \vec{s} be the given size vector. Let $s_0 = \min_{i \in N} s_i$ be the size of the smallest bundle, and $n_0 = |\{i \in N \mid s_i = s_0\}|$ be the number of agents with exactly s_0 goods in their bundles. We claim that an EF1 allocation with size vector \vec{s} exists if and only if the number of valuable goods is at most $s_0n + n - n_0$. Note that checking whether the number of valuable goods is at most $s_0n + n - n_0$ can be done in polynomial time.

If an EF1 allocation with size vector \vec{s} exists, then an agent with bundle size s_0 receives at most s_0 valuable goods. For this agent to be EF1 towards every other agent, every other agent can only receive at most $s_0 + 1$ valuable goods. Since there are $n - n_0$ agents with bundle size at least $s_0 + 1$ and n_0 agents with bundle size exactly s_0 , the total number of valuable goods is at most $(n - n_0)(s_0 + 1) + n_0s_0 = s_0n + n - n_0$.

Conversely, if the number of valuable goods is at most $s_0n + n - n_0$, then we can allocate the valuable goods in a round-robin fashion up to the bundle size of each agent, followed by the non-valuable goods. Note that every agent's bundle size is at least s_0 . If there are at most s_0n valuable goods, then these valuable goods can be distributed fairly with the difference in the number of valuable goods between agents being at most one, and hence the allocation is EF1. Otherwise, the first s_0n valuable goods can be distributed so that every agent receives s_0 of them. Since there are $n - n_0$ agents with bundle size at least $s_0 + 1$ and at most $n - n_0$

valuable goods left, the remaining valuable goods can be arbitrarily allocated to these agents so that each of these agents receives at most one more valuable good. Then, each agent receives s_0 or $s_0 + 1$ valuable goods, and so the allocation is EF1. \square

4 Optimal Number of Exchanges

In this section, we consider the complexity of computing the *optimal number of exchanges* required to reach an EF1 allocation from an initial allocation.

Recall that the decision problem in Section 3 is to determine whether there exists an EF1 allocation that can be reached from a given initial allocation. This is equivalent to determining whether the optimal number of exchanges to reach an EF1 allocation is finite or not. We have established a few scenarios where there exist polynomial-time algorithms for this task: (a) two agents with identical utilities (Theorem 3.2), (b) any constant number of agents with binary utilities (Theorem 3.7), and (c) any number of agents with identical binary utilities (Theorem 3.10). For these scenarios, we can run the respective polynomial-time algorithms to decide whether such an EF1 allocation exists—if none exists, then the optimal number of exchanges is ∞ . Therefore, for the proofs in this section pertaining to these scenarios, we proceed with the assumption that such an EF1 allocation exists. We will show that the problem of computing the optimal number of exchanges for these scenarios is also in P.⁵

For the remaining scenarios where the decision problem in Section 3 is NP-hard, it is NP-hard to even decide whether the optimal number of exchanges to reach an EF1 allocation is finite or not. Therefore, for these scenarios, we shall focus on the special case where the given size vector is *balanced*, so that the optimal number of exchanges is guaranteed to be finite (see Proposition 2.2). Even with this assumption, we will show that the computational problem for these scenarios remains NP-hard.

For convenience, we refer to as OPTIMAL EXCHANGES the problem of deciding—given an instance, an initial allocation in the instance, and a number k —whether the optimal number of exchanges required to reach an EF1 allocation is at most k . Note that the optimal number of exchanges in the scenarios mentioned above are finite. By the proof of Proposition 2.1, the optimal number of exchanges in these scenarios is polynomial in the number of agents and the number of goods. As a result, OPTIMAL EXCHANGES is in NP regardless of the number of agents, as we can easily verify whether an exchange path starting from the given initial allocation indeed reaches an EF1 allocation using at most k exchanges.

4.1 Two Agents

We begin with the case of two agents. For two agents with identical utilities, we show that there exists a polynomial-time algorithm that computes the optimal number of exchanges. This algorithm performs the exchanges until an EF1 allocation is reached, while keeping track of the number of exchanges required. The algorithm is “greedy” in the sense that

⁵Our algorithms can be modified to compute an optimal sequence of exchanges as well.

at each step, it performs an exchange involving the most valuable good from the agent whose bundle has the higher utility, and the least valuable good from the other agent. We demonstrate that this choice is the best in terms of the number of exchanges required to reach an EF1 allocation.

Theorem 4.1. OPTIMAL EXCHANGES is in P for two agents with identical utilities.

Proof. We show that we can compute the optimal number of exchanges in polynomial time. We assume that an EF1 allocation with the given size vector exists. If the initial allocation \mathcal{A} is EF1, we are done. Otherwise, assume without loss of generality that agent 2 has a higher utility than agent 1 in \mathcal{A} . Let $\vec{s} = (s_1, s_2)$ be the size vector. By rearranging the labels of the goods, assume that the goods are in non-increasing order of utility, i.e., $u(g_1) \geq u(g_2) \geq \dots \geq u(g_m)$. The algorithm proceeds as follows: repeatedly exchange the most valuable good in agent 2's bundle with the least valuable good in agent 1's bundle until agent 1 is EF1 towards agent 2. The optimal number of exchanges required is then the number of exchanges made in this algorithm.

First, we claim that in each exchange, a good from $\{g_1, \dots, g_{s_1}\}$ in agent 2's bundle is always exchanged with a good from $\{g_{s_1+1}, \dots, g_m\}$ in agent 1's bundle. Suppose on the contrary that this is not true. Let \mathcal{A}' be the allocation just before we make the exchange that violates this claim. The only way for the claim to be violated is when $A'_1 = \{g_1, \dots, g_{s_1}\}$ and $A'_2 = \{g_{s_1+1}, \dots, g_m\}$. If $s_1 \leq s_2$, then by Lemma 3.1, there does not exist an EF1 allocation with size vector \vec{s} —this would contradict our assumption that an EF1 allocation with size vector \vec{s} exists. Otherwise, $s_1 > s_2$, and every good in A'_1 has a higher utility than every good in A'_2 , so agent 1 is EF1 towards agent 2. This would contradict our assumption that the algorithm has not terminated. Hence, the claim is true.

By the claim in the previous paragraph, the total number of exchanges made is at most $\min\{s_1, s_2\} \leq m$. Each exchange can be performed in polynomial time, and so the algorithm terminates in polynomial time. We show next that an EF1 allocation is obtained when the algorithm terminates. It suffices to show that agent 2 is EF1 towards agent 1 in the final allocation. To this end, we show that agent 2 is EF1 towards agent 1 at every step of the algorithm. Let the initial allocation be $\mathcal{A}^0 = \mathcal{A}$, and let \mathcal{A}^t be the allocation after t steps of the algorithm. Note that \mathcal{A}^0 satisfies the condition that agent 2 is EF1 towards agent 1, since agent 2 has a higher utility than agent 1 in \mathcal{A} . We show that if \mathcal{A}^t has the property that agent 2 is EF1 towards agent 1 and agent 1 is not EF1 towards agent 2, then \mathcal{A}^{t+1} has the property that agent 2 is EF1 towards agent 1. Suppose that $g \in A_2^t$ is exchanged with $h \in A_1^t$; note that g is a good with the highest utility in A_2^t . This means that $u(A_1^t) < u(A_2^t \setminus \{g\})$. Then,

$$\begin{aligned} u(A_2^{t+1}) &\geq u(A_2^t \setminus \{h\}) \\ &= u(A_2^t \setminus \{g\}) \\ &> u(A_1^t) \\ &\geq u(A_1^t \setminus \{h\}) \\ &= u(A_1^{t+1} \setminus \{g\}), \end{aligned}$$

showing that agent 2 is EF1 towards agent 1 in \mathcal{A}^{t+1} .

Finally, we show that the optimal number of exchanges required to reach an EF1 allocation is at least the number of exchanges made in this algorithm. Let T be the number of exchanges made in this algorithm. For each $t \in \{1, \dots, T\}$, let $g^t \in A_2$ (resp. $h^t \in A_1$) be the good in agent 2's bundle (resp. agent 1's bundle) that is exchanged at the t^{th} step of the algorithm. Note that $u(g^1) \geq \dots \geq u(g^T) \geq u(h^T) \geq \dots \geq u(h^1)$. Also, we have $u(A_1^{T-1}) < u(A_2^{T-1} \setminus \{g^T\})$, where g^T is a good with the highest utility in agent 2's bundle in \mathcal{A}^{T-1} . Suppose on the contrary that only $k \leq T - 1$ exchanges are required to reach an EF1 allocation. Since \mathcal{A} is not EF1, we have $1 \leq k < T$. Let (B_1, B_2) be the EF1 allocation after the k exchanges. The utility of B_1 is upper-bounded by the utility of A_1 after adding k goods of the highest utility from A_2 and removing k goods of the lowest utility from A_1 , so we have

$$\begin{aligned} u(B_1) &\leq u((A_1 \cup \{g^1, \dots, g^k\}) \setminus \{h^1, \dots, h^k\}) \\ &= u(A_1) + \sum_{t=1}^k (u(g^t) - u(h^t)) \\ &\leq u(A_1) + \sum_{t=1}^{T-1} (u(g^t) - u(h^t)) \\ &= u((A_1 \cup \{g^1, \dots, g^{T-1}\}) \setminus \{h^1, \dots, h^{T-1}\}) \\ &= u(A_1^{T-1}). \end{aligned}$$

On the other hand, the utility of B_2 without the most valuable good is lower-bounded by the utility of A_2 after adding k goods of the lowest utility from A_1 and removing $k+1 \leq T$ goods of the highest utility from A_2 , so we have

$$\begin{aligned} u(B_2 \setminus \{g\}) &\geq u((A_2 \cup \{h^1, \dots, h^k\}) \setminus \{g^1, \dots, g^{k+1}\}) \\ &= u(A_2) - u(g^1) - \sum_{t=1}^k (u(g^{t+1}) - u(h^t)) \\ &\geq u(A_2) - u(g^1) - \sum_{t=1}^{T-1} (u(g^{t+1}) - u(h^t)) \\ &= u((A_2 \cup \{h^1, \dots, h^{T-1}\}) \setminus \{g^1, \dots, g^T\}) \\ &= u(A_2^{T-1} \setminus \{g^T\}) \end{aligned}$$

for every $g \in B_2$. This gives the inequality $u(B_1) \leq u(A_1^{T-1}) < u(A_2^{T-1} \setminus \{g^T\}) \leq u(B_2 \setminus \{g\})$ for all $g \in B_2$. Hence, agent 1 is not EF1 towards agent 2 in (B_1, B_2) , contradicting the assumption that (B_1, B_2) is EF1. It follows that at least T exchanges are required to reach an EF1 allocation. \square

However, if the utilities are not identical, then computing the optimal number of exchanges is NP-hard, even for balanced allocations. We show this by modifying the construction from the NP-hardness proof of Theorem 3.3 in determining the existence of an EF1 allocation with a given size vector.

Theorem 4.2. OPTIMAL EXCHANGES is NP-complete for two agents, even when the initial allocation is balanced.

Proof. Clearly, this problem is in NP. To demonstrate NP-hardness, we modify the construction from the proof of Theorem 3.3. Recall that we have $Y = \{y_1, \dots, y_{2q+2}\}$ with $K < y_j \leq 2K$ for $j \in \{1, \dots, 2q\}$, $y_{2q+1} = 2K$, and $y_{2q+2} = 0$. A fair division instance \mathcal{I}' is defined with $n = 2$ agents and a set of goods $M' = \{g_1, \dots, g_{2q+6}\}$. Agent 2's utility is such that $u_2(g_j) = y_j$ for $j \in \{1, \dots, 2q+2\}$, $u_2(g_{2q+3}) = u_2(g_{2q+4}) = 0$, and $u_2(g_{2q+5}) = u_2(g_{2q+6}) = 2K$. Agent 1's utility is such that $u_1(g) = u_2(g) + 4K$ for all $g \in M$. The size vector \vec{s}' is $(q+2, q+4)$. In Theorem 3.3, it was proven that there exists an EF1 allocation with size vector \vec{s}' in this instance if and only if Y can be partitioned into two multisets Y_1 and Y_2 of equal cardinalities (i.e., of size $q+1$ each) with sums $(q+3)K$ and $(q+1)K$ respectively. Both problems were proven to be NP-hard.

Define a new fair division instance \mathcal{I} as follows. There are $n = 2$ agents and a set of goods $M = \{g_1, \dots, g_{4q+12}\}$. For $j \in \{1, \dots, 2q+6\}$, the utility of g_j for each agent is identical to that in the original fair division instance \mathcal{I}' . For $j \in \{2q+7, \dots, 4q+12\}$, we have $u_i(g_j) = 0$ for $i \in \{1, 2\}$. The size vector \vec{s} is $(2q+6, 2q+6)$. In the initial allocation \mathcal{A} , agent 1 has $A_1 = \{g_{2q+7}, \dots, g_{4q+12}\}$ and agent 2 has $A_2 = \{g_1, \dots, g_{2q+6}\}$. This reduction can be done in polynomial time. We claim that the optimal number of exchanges required to reach an EF1 allocation from \mathcal{A} is at most $q+2$ in \mathcal{I} if and only if there exists an EF1 allocation with size vector \vec{s}' in \mathcal{I}' .

(\Leftarrow) Let (A'_1, A'_2) be an EF1 allocation with size vector \vec{s}' in \mathcal{I}' . Note that $A'_1 \subseteq A_2$ and $|A'_1| = q+2$. In \mathcal{A} , exchange the $q+2$ goods from A'_1 with any $q+2$ goods in A_1 . This requires a total of $q+2$ exchanges. The new allocation has exactly the same goods as that in (A'_1, A'_2) along with other goods with zero utility, and so is EF1. Therefore, the optimal number of exchanges to reach an EF1 allocation from \mathcal{A} is at most $q+2$.

(\Rightarrow) Suppose that an EF1 allocation \mathcal{B} is reached from \mathcal{A} after $t \leq q+2$ exchanges in \mathcal{I} . We may assume that every good is not exchanged more than once. By the same reasoning as in the proof of Theorem 3.3, we must have $u_1(B_1) \geq (5q+13)K$ and $u_2(B_1) \leq (q+5)K$. Since t goods are transferred from A_2 , we have $u_1(B_1) = u_2(B_1) + t(4K) \leq (q+5)K + (q+2)(4K) = (5q+13)K$. This means that the inequalities for $u_1(B_1)$ are tight, and we have $u_1(B_1) = 5q+13$ and $t = q+2$. Letting $M_1 = A_2 \cap B_1$, we have $|M_1| = q+2$ and $M_1 \subseteq A_2 \subseteq M'$. Since (B_1, B_2) is an EF1 allocation, the allocation that removes all goods with zero utility is also EF1, namely, $(M_1, M' \setminus M_1)$. This induces an EF1 allocation with size vector \vec{s}' in \mathcal{I}' . \square

4.2 Constant Number of Agents

While a polynomial-time algorithm to compute the optimal number of exchanges exists for two agents with identical utilities, this is not the case for three or more agents unless P = NP. Indeed, we establish the NP-hardness of this problem via a reduction from BALANCED MULTI-PARTITION with $p \geq 2$, an NP-hard problem by Proposition 2.3.

Theorem 4.3. OPTIMAL EXCHANGES is NP-complete for $n \geq 3$ agents with identical utilities, even when the initial allocation is balanced.

Proof. Clearly, this problem is in NP. To demonstrate NP-hardness, we shall reduce from the NP-hard problem BALANCED MULTI-PARTITION (see Proposition 2.3). Let a BALANCED MULTI-PARTITION instance be given with $p \geq 2$. Define a fair division instance as follows. There are $n = p+1$ agents with identical utilities, and a set of goods $M = \{g_1, \dots, g_{n(pq+q+2)}\}$ such that $u(g_j) = x_j$ for all $j \in \{1, \dots, pq\}$, $u(g_j) = K$ for all $j \in \{pq+1, \dots, pq+q+2\}$, and $u(g) = 0$ for the remaining goods g . Note that every good in $M_1 = \{g_1, \dots, g_{pq}\}$ has utility more than K and at most $2K$, every good in $M_2 = \{g_{pq+1}, \dots, g_{pq+q+2}\}$ has utility exactly K , and every good in $M \setminus (M_1 \cup M_2)$ has zero utility. The size vector \vec{s} is such that $s_i = pq+q+2$ for all $i \in N$. In the initial allocation \mathcal{A} , agent n has $M_1 \cup M_2$, while the remaining agents have the remaining goods. This reduction can be done in polynomial time. We claim that the optimal number of exchanges required to reach an EF1 allocation from \mathcal{A} is at most pq if and only if X can be partitioned into multisets X_1, \dots, X_p of equal cardinalities and sums.

(\Leftarrow) Let (X_1, \dots, X_p) be such a partition. For each $j \in \{1, \dots, pq\}$, if x_j is in X_i for some $i \in \{1, \dots, p\}$, exchange g_j in agent n 's bundle with a zero-utility good in agent i 's bundle. After pq exchanges, each agent $i \in \{1, \dots, p\}$ has q goods corresponding to the integers in X_i along with other goods with zero utility, and the total utility of these goods is $(q+1)K$. Meanwhile, agent n has M_2 along with other goods with zero utility. There are $q+2$ goods with utility K each, and so the utility of agent n 's bundle without a most valuable good is $(q+1)K$. This shows that the resulting allocation is EF1. Therefore, the optimal number of exchanges required to reach an EF1 allocation from \mathcal{A} is at most pq .

(\Rightarrow) Suppose that the optimal number of exchanges required to reach an EF1 allocation from \mathcal{A} is at most pq . Let \mathcal{B} be one such EF1 allocation after these exchanges. Since at most pq exchanges were made, agent n has at least $|M_1 \cup M_2| - pq = q+2$ goods from $M_1 \cup M_2$ in \mathcal{B} . Every good in $M_1 \cup M_2$ has utility at least K , so the utility of agent n 's bundle in \mathcal{B} without a most valuable good is at least $(q+1)K$. For every agent to be EF1 towards agent n , they must each have a utility of at least $(q+1)K$ in \mathcal{B} . The total utility of agent 1 to agent p 's bundles is therefore at least $p(q+1)K$, which is exactly the utility of M_1 . Since every good in M_1 has a higher utility than every good in M_2 , this implies that the only possibility is that all the goods in M_1 go to agents 1 to p , leaving M_2 (along with other goods with zero utility) with agent n . This means that agent n 's bundle in \mathcal{B} without a most valuable good has utility exactly $(q+1)K$, and that the goods in M_1 must be split among agents 1 to p so that every agent receives a utility of $(q+1)K$ from M_1 . Furthermore, none of these agents can receive more than q goods from M_1 ; otherwise, if some agent receives at least $q+1$ goods from M_1 , then the utility of her bundle is more than $(q+1)K$, which leaves another agent with utility less than $(q+1)K$, a contradiction. Therefore, agent 1 to p each receives exactly q goods from M_1 . Hence, it is possible to partition the goods in M_1 into p bundles so that each bundle has exactly q goods and the utility of each bundle is exactly $(q+1)K$. This induces a partition of X into p multisets of equal cardinalities and sums. \square

Next, we consider binary utilities. We have shown that deciding whether the optimal number of exchanges to reach an EF1 allocation is finite can be done in polynomial time (Theorem 3.7). We now establish that the same is true for computing this exact number. To this end, we first prove that finding the optimal number of exchanges between two equivalence classes of allocations can be done efficiently.

Lemma 4.4. *Let an instance with n agents with binary utilities be given, where n is a constant. Then, there exists an algorithm running in time polynomial in m that computes the optimal number of exchanges required to reach some allocation in a given equivalence class from another given allocation.*

Proof. If the size vectors of the given allocation and an allocation in the given equivalence class are different, then the optimal number of exchanges is ∞ . Therefore, we may henceforth assume that the size vectors are the same.

For each agent i and each type of good c in the given initial allocation, we have an n -vector such that the j^{th} component of this vector represents the number of goods of type c that need to be moved from agent i in the initial allocation to agent j in some final allocation in the given equivalence class under some exchange. Since the number of goods of each type is an integer between 0 and m , there are $m+1$ possible numbers for each component, and hence at most $(m+1)^n$ possible vectors to represent this information. The movement of goods from the initial allocation to the final allocation can be represented by an ordered collection of such vectors over all agents and over all types of goods in each agent's initial allocation. Since there are n agents and 2^n types of goods, there are at most $((m+1)^n)^{n \cdot 2^n}$ such collections. Since $((m+1)^n)^{n \cdot 2^n}$ is polynomial in m whenever n is a constant, there are at most a polynomial number of such movements between the two allocations. For each of these movements, we can verify in polynomial time whether it indeed gives some final allocation in the equivalence class. We thus have an enumeration of all such feasible movements in polynomial time.

For each feasible movement between the initial allocation \mathcal{A} and the final allocation \mathcal{B} , define a directed graph where the vertices are the agents and each edge e_g represents a good such that if $g \in A_i \cap B_j$, then $e_g = (i, j)$. Igarashi *et al.* [2024, Prop. 4.1] showed that the number of exchanges required to reach \mathcal{B} from \mathcal{A} is $m - c^*$, where c^* is the maximum number of disjoint cycles in the item exchange graph. Note that each cycle consists of a subset of the agents in some order, so the number of cycle types, L , is at most $(n+1)!$, which is a constant. We have an L -vector such that the k^{th} component of this vector represents the number of cycles in the item exchange graph of cycle type k . Since the number of cycles of each cycle type is an integer between 0 and m , there are $m+1$ possible numbers for each component, and hence at most $(m+1)^L$ possible vectors to represent the cycles in the graph, which is polynomial in m . We can enumerate all such vectors, consider only those vectors that represent the item exchange graph, and output the maximum number of disjoint cycles from these vectors as c^* in polynomial time. Then, calculating $m - c^*$ gives the optimal number of exchanges for that feasible movement of goods. Finally, the minimum

optimal number of exchanges across all feasible movements is the quantity we desire. \square

This lemma, together with Lemma 3.6, yields the result.

Theorem 4.5. *OPTIMAL EXCHANGES is in P for a constant number of agents with binary utilities.*

Proof. We use the algorithm in Lemma 3.6 to enumerate all the possible equivalence classes of EF1 allocations with the given size vector in polynomial time. For each equivalence class of allocations, we use the algorithm in Lemma 4.4 to compute the optimal number of exchanges required to reach some allocation from this equivalence class from the given initial allocation in polynomial time. The smallest such number will then answer the decision problem of OPTIMAL EXCHANGES. \square

4.3 General Number of Agents

Let us now consider a non-constant number of agents. We have shown that computing the optimal number of exchanges required to reach an EF1 allocation is NP-hard, even for identical utilities (Theorem 4.3). We thus consider binary utilities.

Although this problem belongs to P when the number of agents is a constant (Theorem 4.5), we show that it is NP-hard for a general number of agents, even for the special case where the initial allocation is balanced. To this end, we reduce from EXACT COVER BY 3-SETS (X3C), an NP-hard problem [Garey and Johnson, 1979, p. 221].

Theorem 4.6. *OPTIMAL EXCHANGES is NP-complete for binary utilities, even when the initial allocation is balanced.*

Proof. Clearly, this problem is in NP. To demonstrate NP-hardness, we shall reduce from X3C. In X3C, we are given positive integers p and q , a set $X = \{x_1, \dots, x_{3q}\}$, and a collection $C = \{Y_1, \dots, Y_p\}$ of three-element subsets of X , i.e., for each $j \in \{1, \dots, p\}$, $|Y_j| = 3$ and $Y_j \subseteq X$. The problem is to decide whether there exists an exact cover for X in C , i.e., whether there exists $D \subseteq C$ such that $|D| = q$ and for all $x \in X$, there exists $Y \in D$ such that $x \in Y$. This decision problem is known to be NP-hard [Garey and Johnson, 1979, p. 221].

Let an X3C instance be given. Note that if there exists an exact cover D for X , and if some $x' \in X$ appears in exactly one $Y' \in C$, then Y' must be in D , and moreover, the other two elements in $Y' \setminus \{x'\}$ must not appear in any other three-element sets in D . In this case, we can reduce the problem further by considering the set $X \setminus Y'$ and the collection $\{Y_j \in C \mid Y_j \cap Y' = \emptyset\}$ instead. Therefore, assume without loss of generality that each $x \in X$ appears in at least two three-element sets in C .

Define a fair division instance as follows. There are $n = 3q + 1$ agents with binary utilities, and a set of goods $M = \{g_{i,j} \mid i \in \{1, \dots, 3q+1\}, j \in \{1, \dots, p\}\}$. For notational simplicity, let $h_j = g_{3q+1,j}$ for all $j \in \{1, \dots, p\}$; the good h_j is associated with Y_j . The size vector is $\vec{s} = (p, \dots, p)$. In the initial allocation $\mathcal{A} = (A_1, \dots, A_{3q+1})$, we have $A_i = \{g_{i,j} \mid j \in \{1, \dots, p\}\}$ for all $i \in N$. Agent $3q+1$ is a special agent and assigns zero utility to every good. For each non-special agent $i \in \{1, \dots, 3q\}$, let n_i

be the number of three-element subsets in C that contain x_i , i.e., $n_i = |\{Y \in C \mid x_i \in Y\}|$. By assumption, we have $n_i \geq 2$. Then, agent i values exactly $n_i - 2$ goods in A_i , e.g., $u_i(g_{i,j}) = 1$ for $j \in \{1, \dots, n_i - 2\}$ and $u_i(g_{i,j}) = 0$ for $j \in \{n_i - 1, \dots, p\}$. Agent i also values the goods associated with any Y_j that contains x_i , i.e., $u_i(h_j) = 1$ if and only if $x_i \in Y_j$. Agent i assigns zero utility to every other good not mentioned. Note that in the initial allocation, from each non-special agent i 's perspective, the utility of agent i 's bundle is $n_i - 2$, the utility of agent $(3q + 1)$'s bundle is n_i , and the utilities of the other agents' bundles are zero. This reduction can be done in polynomial time. We claim that the optimal number of exchanges required to reach an EF1 allocation from \mathcal{A} is at most q if and only if there exists an exact cover for X in C .

(\Leftarrow) Let D be an exact cover for X . For each $Y_j \in D$, we perform one exchange as follows: select any $x_i \in Y_j$ arbitrarily, and exchange $g_{i,p}$ in agent i 's bundle with h_j in agent $(3q + 1)$'s bundle. Note that there are exactly q exchanges, since $|D| = q$. We claim that the final allocation is EF1. Since agent $3q + 1$ does not value any good, she is EF1 towards every other agent. Therefore, we only need to consider agent i 's envy for $i \in \{1, \dots, 3q\}$. Note that there exists $j \in \{1, \dots, p\}$ such that $x_i \in Y_j$ and $Y_j \in D$. This means that h_j is moved to some non-special agent's bundle in an exchange (possibly agent i). Regardless of whom h_j goes to, agent i 's utility for her own bundle is at least $n_i - 2$, and agent i 's utility for agent $(3q + 1)$'s bundle is exactly $n_i - 1$, so agent i is EF1 towards agent $3q + 1$. Furthermore, if h_j goes to some agent $i' \neq i$, then agent i 's utility for the bundle of agent i' is 1, so agent i is EF1 towards agent i' . Every other non-special agent's bundle yields zero utility to agent i . This shows that agent i is EF1 towards every other agent. Accordingly, the final allocation is EF1.

(\Rightarrow) Suppose that after at most q exchanges, an EF1 allocation is reached. Let $i \in \{1, \dots, 3q\}$. The valuable goods from agent i 's perspective are with agent i or with agent $3q + 1$. Since agent i 's utility for her own bundle in \mathcal{A} is $n_i - 2$ and her utility for agent $(3q + 1)$'s bundle is n_i , some valuable good from agent $(3q + 1)$'s bundle needs to be moved to another agent's bundle (possibly i 's) in an exchange. Now, each good in agent $(3q + 1)$'s bundle is valuable to exactly three agents. Since the movement of each good in agent $(3q + 1)$'s bundle can only resolve the envy for at most three agents, at least q goods need to be moved to make agents 1 to $3q$ EF1. This means that exactly q exchanges are made; moreover, each good h_j moved from agent $(3q + 1)$'s bundle is associated with three distinct agents. The set of q goods moved from agent $(3q + 1)$'s bundle induces an exact cover D with cardinality q . \square

Finally, we consider identical binary utilities. We show that for this restricted class of utilities, the computational problem can be solved efficiently regardless of whether the size vector of the initial allocation is balanced or not. To show this, we demonstrate that the greedy algorithm allows an EF1 allocation to be reached using the smallest number of exchanges.

Theorem 4.7. OPTIMAL EXCHANGES is in P for identical binary utilities.

Proof. Let $\mathcal{A} = (A_1, \dots, A_n)$ be the given allocation. As mentioned at the beginning of Section 4, we assume that an EF1 allocation can be reached from \mathcal{A} . By the proof of Theorem 3.10, there must be at most $s_0n + n - n_0$ valuable goods, where $s_0 = \min_{i \in N} |A_i|$ and $n_0 = |\{i \in N \mid |A_i| = s_0\}|$. Suppose that there are $m_1 \leq s_0n + n - n_0$ valuable goods. An EF1 allocation requires every agent to receive at least $F := \lfloor m_1/n \rfloor$ valuable goods and at most $F + 1$ valuable goods. Since an EF1 allocation with the same size vector as \mathcal{A} exists, every agent must have at least F goods in \mathcal{A} , i.e., $s_0 \geq F$. Let N_0 be the set of agents who have at most F valuable goods in the initial allocation, and N_1 be the set of agents who have at least $F + 1$ valuable goods in the initial allocation. Note that $N = N_0 \cup N_1$. Let $c_0 = \sum_{i \in N_0} (F - u(A_i))$ and $c_1 = \sum_{i \in N_1} (u(A_i) - (F + 1))$. We claim that the optimal number of exchanges required to reach an EF1 allocation is $\max\{c_0, c_1\}$. Note that this value can be computed in polynomial time, so it suffices to prove the claim.

First, we show that the optimal number of exchanges required to reach an EF1 allocation is at least $\max\{c_0, c_1\}$. Each agent $i \in N_0$ needs to receive at least $F - u(A_i) \geq 0$ valuable goods in order to arrive at a bundle with utility at least F . In receiving these valuable goods, agent i must give away the same number of non-valuable goods from her bundle in A_i —note that this is possible since every agent has at least F goods. Therefore, there exist at least $F - u(A_i)$ valuable goods from other agents' bundles that should go to agent i 's bundle and at least $F - u(A_i)$ non-valuable goods from agent i 's bundle that should go to other agents' bundles. Summing up over all $i \in N_0$, we have that at least $\sum_{i \in N_0} 2(F - u(A_i)) = 2c_0$ goods are in the wrong hands. Since each exchange places at most two goods in correct hands, the number of exchanges required is at least $2c_0/2 = c_0$. By an analogous argument on the agents in N_1 , we have that the number of exchanges required is at least c_1 . This proves that the optimal number of exchanges required to reach an EF1 allocation is at least $\max\{c_0, c_1\}$.

Next, we describe an algorithm that allows us to reach an EF1 allocation with at most $\max\{c_0, c_1\}$ exchanges. The algorithm is as follows: repeatedly exchange a valuable good from an agent with the highest utility with a non-valuable good from an agent with the lowest utility, until every agent has at least F valuable goods and at most $F + 1$ valuable goods. We show that this ending will always be reached. Suppose on the contrary that this is not the case, and consider the final allocation just before the algorithm cannot proceed further. Since every agent has at least F goods in \mathcal{A} , it must be possible that every agent receives at least F valuable goods in the final allocation, and so $F = s_0$. This means that some agent has more than $F + 1$ valuable goods in the final allocation, and every agent who has F goods in \mathcal{A} has F valuable goods in the final allocation. Then, the number of valuable goods is $m_1 > Fn_0 + (F + 1)(n - n_0) = s_0n + n - n_0$, which is a contradiction. This shows that it is possible to reach the desired ending.

Now, we are ready to show that the optimal number of exchanges required is at most $\max\{c_0, c_1\}$. If $c_0 \geq c_1$, then the first c_1 exchanges involve exchanging valuable goods from agents in N_1 with non-valuable goods from agents in

N_0 . At this point, every agent in N_1 has exactly $F + 1$ valuable goods, and every agent in N_0 has at most F valuable goods. Call this allocation (B_1, \dots, B_n) . We have that $\sum_{i \in N_0} (F - u(B_i)) = c_0 - c_1$. If $|N_1| < c_0 - c_1$, then after $|N_1|$ further exchanges, every agent has at most F valuable goods and some agent has fewer than F valuable goods, contradicting the assumption that $F = \lfloor m_1/n \rfloor$. Therefore, we must have $|N_1| \geq c_0 - c_1$. Now, after $c_0 - c_1$ further exchanges, every agent in N_0 has exactly F valuable goods and every agent in N_1 has between F and $F + 1$ valuable goods (inclusive), giving an EF1 allocation. Hence, if $c_0 \geq c_1$, then the optimal number of exchanges required to reach an EF1 allocation is at most $c_1 + (c_0 - c_1) = c_0$. By an analogous argument, if $c_0 < c_1$, then the optimal number of exchanges required to reach an EF1 allocation is at most $\max\{c_0, c_1\}$. \square

5 Worst-Case Bounds

In this section, instead of instance-specific optimization, we turn our attention to the *worst-case* number of exchanges required to reach an EF1 allocation from some initial allocation. Since an EF1 allocation may not always be reachable (as can be seen from Section 3), we shall focus on the special case where the number of goods in each agent's bundle is the same. We say that a size vector $\vec{s} = (s_1, \dots, s_n)$ is *s-balanced* for a positive integer s if $s_i = s$ for all $i \in N$, and an allocation is *s-balanced* if it has an *s-balanced* size vector. We shall consider the worst-case number of exchanges starting from an *s-balanced* allocation for n agents.

5.1 General Utilities

Given n and s , let $f(n, s)$ be the smallest integer such that for every instance with n agents and ns goods and every *s-balanced* allocation \mathcal{A} in the instance, there exists an EF1 allocation that can be reached from \mathcal{A} using at most $f(n, s)$ exchanges. We shall examine the bounds for $f(n, s)$.

We first derive an upper bound for $f(n, s)$. At a high level, we use an algorithm by Biswas and Barman [2018] to find an EF1 allocation under cardinality constraints such that every agent retains roughly s/n of her goods from her original bundle. The algorithm also distributes the goods in each agent's initial bundle to the other agents as evenly as possible in order to maximize the number of goods that can be exchanged one-to-one, thereby minimizing the total number of exchanges. One can check that roughly $s(n - 1)/2$ exchanges are required to reach this EF1 allocation from the initial allocation.

Theorem 5.1. *Let n and s be positive integers, and let $q = \lfloor s/n \rfloor$ and $r = s - qn$ be the quotient and remainder when s is divided by n respectively. Then,*

$$f(n, s) \leq \begin{cases} s(n - 1)/2 & \text{if } r = 0; \\ s(n - 1)/2 + r(n - 3)/2 + 1 & \text{otherwise.} \end{cases}$$

Moreover, we have $f(2, s) \leq (s - r)/2$ for all s .

Proof. Let \mathcal{A} be an *s-balanced* allocation. It suffices to find an EF1 *s-balanced* allocation \mathcal{B} such that the optimal number

of exchanges to reach \mathcal{B} from \mathcal{A} is at most the expression given in the theorem statement.

When $n = 2$, allocate the goods in A_1 to the two agents in a round-robin fashion with agent 1 going first, and allocate the goods in A_2 to the two agents in a round-robin fashion with agent 2 going first. Call this new allocation \mathcal{B} . Note that \mathcal{B} is clearly *s-balanced*. We have $A_i \cap B_{3-i} = (s - r)/2$ for $i \in \{1, 2\}$, so the optimal number of exchanges required to reach \mathcal{B} from \mathcal{A} is (exactly) $(s - r)/2$. To see that \mathcal{B} is EF1, observe that agent 1 does not envy agent 2 with respect to the goods chosen from A_1 and is EF1 towards agent 2 with respect to the goods chosen from A_2 , so agent 1 is EF1 towards agent 2 in \mathcal{B} ; likewise, agent 2 is EF1 towards agent 1 in \mathcal{B} . This shows that $f(2, s) \leq (s - r)/2$.

When $n \geq 3$, we shall find an EF1 *s-balanced* allocation \mathcal{B} by generalizing the method for two agents. We define $n + r$ categories of goods $C_1, \dots, C_n, D_1, \dots, D_r$ as follows. For $i \in N$, category C_i contains qn goods arbitrarily selected from A_i only; note that r goods remain unselected in A_i . Next, we form D_w recursively as follows: let $w \in \{1, \dots, r\}$ be the smallest index such that D_w does not have n goods yet, let $i \in N$ be the smallest index such that A_i still has unselected goods, arbitrarily select a good in A_i , and add it to D_w . At the end of this process, every category C_i has exactly qn goods from A_i , and every category D_w has exactly n goods from consecutive agents' bundles, say, $A_{i_w}, A_{i_w+1}, \dots, A_{j_w}$.

We now proceed to form \mathcal{B} using the algorithm by Biswas and Barman [2018] which finds an EF1 allocation under cardinality constraints. In particular, there exists an EF1 allocation $\mathcal{B} = (B_1, \dots, B_n)$ such that $|C_i \cap B_j| = |C_i|/n = q$ for all $i, j \in N$ and $|D_w \cap B_j| = |D_w|/n = 1$ for all $w \in \{1, \dots, r\}, j \in N$. Also, \mathcal{B} is *s-balanced* because $|B_j| = qn + r = s$ for all $j \in N$. We shall bound the number of exchanges required to reach \mathcal{B} from \mathcal{A} .

For each unordered pair of distinct $i, j \in N$, exchange the q goods from $C_i \cap B_j$ (which are in A_i) with the q goods from $C_j \cap B_i$ (which are in A_j). This requires a total of $qn(n - 1)/2$ exchanges. Call this intermediate allocation $\mathcal{A}' = (A'_1, \dots, A'_n)$. At this point, the only goods that are possibly in the wrong bundles in \mathcal{A}' (as compared to \mathcal{B}) are the goods in all the D_w , and there are at most rn such goods. For each $i \in N$, let $X_i = A'_i \cap (D_1 \cup \dots \cup D_r)$.

If $r = 0$, then $\mathcal{A}' = \mathcal{B}$, and we are done since the total number of exchanges is $qn(n - 1)/2 = s(n - 1)/2$. Else, $r > 0$. Consider the directed graph where the vertices are the agents and each edge e_g represents a good $g \in M$ such that if $g \in A'_i \cap B_j$, then $e_g = (i, j)$. Igarashi et al. [2024, Prop. 4.1] showed that the number of exchanges required to reach \mathcal{B} from \mathcal{A}' is $m - c^*$, where c^* is the maximum possible cardinality of a partition of the edges of the graph into (directed) circuits. In \mathcal{A}' , qn^2 goods from all the C_i are in the correct bundle by the previous process, and each of the edges representing these goods has its own circuit, say, (i, i) if the good is in A'_i . We shall show that the edges representing the rn goods in all the D_w can be partitioned into at least $2r - 1$ disjoint circuits. This will give at least $qn^2 + (2r - 1) = sn - (rn - 2r + 1)$ as the cardinality of one such partition of the edges of the graph into circuits. Accordingly, $c^* \geq sn - (rn - 2r + 1)$, and the number of exchanges required to reach \mathcal{B} from \mathcal{A}'

is $m - c^* \leq rn - 2r + 1$. Then, the number of exchanges required to reach \mathcal{B} from \mathcal{A} (via \mathcal{A}') is at most $qn(n-1)/2 + (rn - 2r + 1) = s(n-1)/2 + r(n-3)/2 + 1$, establishing the theorem.

Let $w \in \{1, \dots, r\}$ be given. We shall show that there exists a cycle formed with a subset of the edges representing the goods in D_w . The goods in D_w come from consecutive agents' bundles in \mathcal{A}' , say, agents i_w to j_w . Every agent receives exactly one good from D_w in \mathcal{B} ; in particular, agents i_w to j_w each receives exactly one good from D_w . Consider the good g in $D_w \cap B_{i_w}$. If g is in X_{i_w} , then the edge $e_g = (i_w, i_w)$ is a desired cycle. Otherwise, g belongs to some agent $i' \in \{i_w + 1, \dots, j_w\}$ in \mathcal{A}' . Then, the edge e_g is (i', i_w) . Next, we consider the good g' in $D_w \cap B_{i'}$, and find the agent that has g' in \mathcal{A}' . The edge representing g' then points to i' from that agent. By repeating this, we eventually find a cycle formed with some of these edges and with a subset of the agents i_w to j_w as vertices. Let $M_w \subseteq D_w$ be the set of goods that are represented by the edges in this cycle. Note that each X_i for $i \in \{i_w, \dots, j_w\}$ contains at most one good in M_w , and each X_i for $i \in N \setminus \{i_w, \dots, j_w\}$ does not contain any good in M_w .

Now, consider the goods represented by the edges of the r cycles—one for each w . Note that these cycles are disjoint since the sets M_w are pairwise disjoint. Let $M_0 = \bigcup_{w=1}^r M_w$. We claim that $|M_0| < 2n$. Since the r goods in X_1 are entirely contained in D_1 , we have $|X_1 \cap M_1| \leq 1$ and $|X_1 \cap M_w| = 0$ for $w \in \{2, \dots, r\}$, which implies that $|\bigcup_{w=1}^r (X_1 \cap M_w)| \leq 1$. Now, for each $i \in N \setminus \{1\}$, the r goods in X_i can only be contained in at most two D_w —to see this, observe that if the r goods are contained in $D_{w'}, D_{w'+1}$, and $D_{w'+2}$, then $D_{w'+1} \subseteq X_i$, which implies that $r = |X_i| \geq |D_{w'+1}| = n$, a contradiction. Thus, we have $|X_i \cap M_w| \leq 1$ for all $w \in \{1, \dots, r\}$, and $|X_i \cap M_w| = 1$ for at most two w , and so $|\bigcup_{w=1}^r (X_i \cap M_w)| \leq 2$. Since $M_0 = \bigcup_{i \in N} \bigcup_{w=1}^r (X_i \cap M_w)$, we have $|M_0| \leq 1 + (n-1) \cdot 2 < 2n$, proving the claim.

Finally, consider the edges representing the rn goods in all the D_w . We have shown that fewer than $2n$ of these edges can be used to form r disjoint circuits (in fact, cycles). There are more than $rn - 2n = (r-2)n$ edges remaining. Since we can always require every circuit to have length at most n , there exists a partition of the remaining edges into more than $r-2$ disjoint circuits, i.e., at least $r-1$ disjoint circuits. The total number of circuits in this partition is at least $r+(r-1) = 2r-1$. This completes the proof. \square

If no good is involved in more than one exchange, then $s(n-1)/2$ exchanges means that a total of $s(n-1) = m(1 - 1/n)$ goods are exchanged. When n is large, the fraction of goods involved in the exchanges becomes close to 1. While this bound might not seem impressive, we show next that it is, in fact, already essentially tight. Specifically, we establish a lower bound for $f(n, s)$ by constructing an instance (with binary utilities) and an s -balanced allocation \mathcal{A} in the instance such that roughly $s(n-1)/2$ exchanges are necessary to reach an EF1 allocation from \mathcal{A} .

Theorem 5.2. *Let n and s be positive integers, and let $q = \lfloor s/n \rfloor$ and $r = s - qn$ be the quotient and remainder when s*

is divided by n respectively. Then,

$$f(n, s) \geq \begin{cases} s(n-1)/2 & \text{if } r = 0; \\ s(n-1)/2 - (n-r)/2 & \text{otherwise.} \end{cases}$$

Proof. Let $M = \{g_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq s\}$ be the set of goods such that each good $g_{i,j}$ is worth 0 to agent i and worth 1 to all agents except i . We have $u_i(M) = s(n-1)$. We claim that an EF1 allocation requires every agent to receive a bundle worth at least $s - q - \lceil r/n \rceil$ from her perspective. To see this, suppose on the contrary that some agent i receives a bundle worth less than $s - q - \lceil r/n \rceil$ to her. For the allocation to be EF1, every other agent receives a bundle worth at most $s - q - \lceil r/n \rceil$ to agent i . Then, we must have $u_i(M) < n(s - q - \lceil r/n \rceil)$. When $r = 0$, it holds that $\lceil r/n \rceil = 0$ and $n(s - q - \lceil r/n \rceil) = n(s - q) = sn - s = s(n-1)$. When $r > 0$, it holds that $\lceil r/n \rceil = 1$ and $n(s - q - \lceil r/n \rceil) = n(s - q - 1) = sn - (qn+r) - (n-r) = s(n-1) - (n-r) \leq s(n-1)$. In both cases, we have $u_i(M) < s(n-1) = u_i(M)$, a contradiction.

Let \mathcal{A} be the allocation such that $A_i = \{g_{i,j} \mid 1 \leq j \leq s\}$ for every i . In order to reach an EF1 allocation, each agent must give away at least $s - q - \lceil r/n \rceil$ goods from her bundle in order to receive from the other agents the same number of valuable goods from her perspective. The total number of goods that are currently in the wrong hands across all agents is at least $n(s - q - \lceil r/n \rceil)$, and the optimal number of exchanges required to reach an EF1 allocation is at least half of this number, since each exchange places at most two goods in the correct hands. When $r = 0$, the optimal number of exchanges required is at least $n(s - q - \lceil r/n \rceil)/2 = s(n-1)/2$. When $r > 0$, the optimal number of exchanges required is at least $n(s - q - \lceil r/n \rceil)/2 = s(n-1)/2 - (n-r)/2$. \square

For two agents, Theorems 5.1 and 5.2 give a tight bound of $f(2, s) = (s-r)/2 = m/4 - r/2 = \lfloor m/4 \rfloor$. This means that in the worst-case scenario, the number of exchanges required to reach an EF1 allocation is roughly one-quarter of the total number of goods between the two agents, or equivalently, roughly half of the goods need to be exchanged between the two agents to reach an EF1 allocation.

Theorems 5.1 and 5.2 also give a tight bound of $f(n, s) = s(n-1)/2$ whenever s is divisible by n . By observing the proof of Theorem 5.1, we can achieve an EF1 allocation with $f(n, s)$ exchanges without involving each good in more than one exchange. This means that a $(1 - 1/n)$ fraction of all goods need to be exchanged in the worst-case scenario. Intuitively, this happens when each agent only values the goods in the bundle of every agent except her own in the initial allocation, and therefore needs to ensure that these goods are evenly distributed among all agents including herself.

Define $f_{\text{bin}}(n, s)$ as the smallest integer such that for every *binary* instance with n agents and ns goods and every s -balanced allocation \mathcal{A} in the instance, there exists an EF1 allocation that can be reached from \mathcal{A} using at most $f_{\text{bin}}(n, s)$ exchanges. The proof of Theorem 5.2 uses a binary instance, which means that the lower bound of the theorem works for f_{bin} as well. Clearly, the upper bound of Theorem 5.1 works for f_{bin} , so the discussion in the preceding paragraphs also applies to binary instances too.

5.2 Identical Binary Utilities

Given n and s , let $f_{\text{id,bin}}(n, s)$ be the smallest integer such that for every instance with n agents with *identical binary* utilities and ns goods and every s -balanced allocation \mathcal{A} in the instance, there exists an EF1 allocation that can be reached from \mathcal{A} using at most $f_{\text{id,bin}}(n, s)$ exchanges. We show that $f_{\text{id,bin}}(n, s)$ is roughly $sn/4$ for even n and $s(n-1)(n+1)/4n$ for odd n —note that this is approximately half of the bound $f(n, s)$ for arbitrary utilities, which is roughly $s(n-1)/2$ (see Section 5.1). The upper bounds (of $sn/4$ and $s(n-1)(n+1)/4n$ respectively) correspond to the case where half of the agents have all the valuable goods while the remaining half have all the non-valuable goods.

Theorem 5.3. *Let n and s be positive integers. If n is even, then*

$$\frac{n}{2} \left\lfloor \frac{s}{2} \right\rfloor \leq f_{\text{id,bin}}(n, s) \leq \frac{sn}{4}.$$

If n is odd, then

$$\frac{n+1}{2} \left\lfloor \frac{s(n-1)}{2n} \right\rfloor \leq f_{\text{id,bin}}(n, s) \leq \frac{s(n-1)(n+1)}{4n}.$$

Proof. Recall that the proof of Theorem 4.7 provides a way to compute the optimal number of exchanges to reach an EF1 allocation from a given initial allocation. To recap, let m_1 be the total number of valuable goods, $F = \lfloor m_1/n \rfloor$ be the minimum number of valuable goods each agent must receive in an EF1 allocation, N_0 be the set of agents who have at most F valuable goods in the initial allocation, N_1 be the set of agents who have at least $F+1$ valuable goods in the initial allocation, $c_0 = \sum_{i \in N_0} (F - u(A_i))$, and $c_1 = \sum_{i \in N_1} (u(A_i) - (F+1))$. The optimal number of exchanges is $\max\{c_0, c_1\}$.

We first prove the lower bounds for $f_{\text{id,bin}}(n, s)$ by providing an explicit initial allocation and showing that the optimal number of exchanges to reach an EF1 allocation is at least $\lfloor [n/2]s/n \rfloor \cdot \lceil n/2 \rceil$, which corresponds to the lower bounds for both even and odd n . In the initial allocation, $\lfloor n/2 \rfloor$ agents have s valuable goods each and the remaining $\lceil n/2 \rceil$ agents have s non-valuable goods each. There are a total of $m_1 = \lfloor n/2 \rfloor \cdot s$ valuable goods, and $F = \lfloor [n/2]s/n \rfloor$. The value of c_0 is $\sum_{i \in N_0} (F - u(A_i)) = \sum_{i \in N_0} (\lfloor [n/2]s/n \rfloor - 0) = \lfloor [n/2]s/n \rfloor \cdot \lceil n/2 \rceil$. Since $\max\{c_0, c_1\} \geq c_0$, the lower bounds follow.

We now prove the upper bounds for $f_{\text{id,bin}}(n, s)$. Let an s -balanced allocation $\mathcal{A} = (A_1, \dots, A_n)$ be given, and let $n_0 = |N_0|$ and $n_1 = |N_1|$. We first derive upper bounds for c_0 and c_1 . Note that $m_1 \leq sn_1 + \sum_{i \in N_0} u(A_i)$. We have

$$\begin{aligned} c_0 &= \sum_{i \in N_0} (F - u(A_i)) \\ &= (n - n_1)F - \sum_{i \in N_0} u(A_i) \\ &\leq (n - n_1)\frac{m_1}{n} - \sum_{i \in N_0} u(A_i) \\ &= \left(1 - \frac{n_1}{n}\right)m_1 - \sum_{i \in N_0} u(A_i) \end{aligned}$$

$$\begin{aligned} &\leq \left(1 - \frac{n_1}{n}\right) \left(sn_1 + \sum_{i \in N_0} u(A_i)\right) - \sum_{i \in N_0} u(A_i) \\ &\leq sn_1 - \frac{sn_1^2}{n} + \sum_{i \in N_0} u(A_i) - \sum_{i \in N_0} u(A_i) \\ &= \frac{sn_1}{n}(n - n_1). \end{aligned}$$

On the other hand, $m_1 \geq \sum_{i \in N_1} u(A_i)$. We have

$$\begin{aligned} c_1 &= \sum_{i \in N_1} (u(A_i) - (F+1)) \\ &= \sum_{i \in N_1} u(A_i) - n_1 \left(\left\lfloor \frac{m_1}{n} \right\rfloor + 1\right) \\ &\leq \sum_{i \in N_1} u(A_i) - n_1 \left(\frac{m_1}{n}\right) \\ &\leq \sum_{i \in N_1} u(A_i) - \frac{n_1}{n} \sum_{i \in N_1} u(A_i) \\ &= \frac{1}{n}(n - n_1) \sum_{i \in N_1} u(A_i) \\ &\leq \frac{1}{n}(n - n_1)sn_1 \\ &= \frac{sn_1}{n}(n - n_1). \end{aligned}$$

We have thus shown that $\max\{c_0, c_1\} \leq n_1(n - n_1)s/n$. Therefore, the optimal number of exchanges is at most $n_1(n - n_1)s/n$, which is a quadratic expression in n_1 . When n is even, $n_1(n - n_1)s/n$ attains a maximum value at $n_1 = n/2$, and this value is $sn/4$. When n is odd, $n_1(n - n_1)s/n$ attains a maximum value at $n_1 = (n+1)/2$ and $n_1 = (n-1)/2$, and this value is $s(n-1)(n+1)/4n$. The upper bounds for $f_{\text{id,bin}}(n, s)$ follow. \square

6 Conclusion and Future Work

In this paper, we have studied the reformability of unfair allocations and the number of exchanges required in the reformation process. We demonstrated several distinctions in the complexity of these problems based on the number of agents and their utility functions, and showed that the number of exchanges required to reach an EF1 allocation is relatively high in the worst case.

While our worst-case bounds in Section 5.1 are already exactly tight in certain scenarios and almost tight generally, an open question is to tighten them for more than two agents when the number of goods in each agent's bundle is not divisible by the number of agents. Additionally, although these bounds also work for binary utilities, one could try to derive bounds for *identical* utilities. We provide some insights for identical (but not necessarily binary) utilities in Appendix A. Another interesting direction is to require each exchange to be beneficial for both agents involved—in Appendix B, we prove that the problem of deciding whether a given initial allocation can be reformed into an EF1 allocation using only beneficial exchanges is NP-complete for binary utilities. One could also consider the model of *transferring* goods instead

of exchanging them—an EF1 allocation is always reachable from any allocation in this model, so it will be interesting to determine the optimal number of exchanges needed for this goal. Finally, beyond EF1, one could consider reforming an allocation using other notions as fairness benchmarks.

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A Worst-Case Bounds for Identical Utilities

We continue the discussion from Section 5 on worst-case bounds, and focus on identical utilities in this appendix.

Given n and s , let $f_{\text{id}}(n, s)$ be the smallest integer such that for every instance with n agents with *identical* utilities and ns goods and every s -balanced allocation \mathcal{A} in the instance, there exists an EF1 allocation that can be reached from \mathcal{A} using at most $f_{\text{id}}(n, s)$ exchanges.

A tight bound for two agents is an immediate consequence of our previous results.

Theorem A.1. *Let s be a positive integer. Then, $f_{\text{id}}(2, s) = \lfloor s/2 \rfloor$.*

Proof. The lower bound follows from Theorem 5.3, while the upper bound follows from Theorem 5.1. \square

For three or more agents, we conjecture that $f_{\text{id}}(n, s)$ is roughly $sn/4$, like $f_{\text{id,bin}}(n, s)$. However, proving this turns out to be surprisingly challenging. We shall present a result using a slightly weaker fairness notion in the case of three agents.

We say that agent i is *weak-EF1 towards agent j* in an allocation $\mathcal{A} = (A_1, \dots, A_n)$ if $u_i(A_i) \geq u_i(A_j) - \max_{g \in M} u_i(g)$; note the condition $g \in M$ as opposed to $g \in A_j$ for EF1. An allocation \mathcal{A} is *weak-EF1* if every agent is weak-EF1 towards every other agent in \mathcal{A} . Weak-EF1 is the fairness notion originally considered by Lipton *et al.* [2004] (although their algorithm satisfies EF1), and weak-EF1 and EF1 are equivalent when the utilities are binary. Since we consider identical utilities, we use u instead of u_i . Without loss of generality, we may divide all utilities by $\max_{g \in M} u(g)$. Then, the utility of each good is in $[0, 1]$, and the condition for agent i to be weak-EF1 towards agent j is $u(A_i) \geq u(A_j) - 1$.

Given n and s , let $\tilde{f}_{\text{id}}(n, s)$ be the smallest integer such that for every instance with n agents with identical utilities and ns goods, and every s -balanced allocation \mathcal{A} in the instance, there exists a weak-EF1 allocation that can be reached from \mathcal{A} using at most $\tilde{f}_{\text{id}}(n, s)$ exchanges. We shall determine the value of $\tilde{f}_{\text{id}}(3, s)$.

We describe an algorithm \mathfrak{A} that performs a sequence of exchanges of goods starting from an initial allocation \mathcal{A}^0 . For each t starting from 0, we begin with the allocation $\mathcal{A}^t = (A_1^t, \dots, A_n^t)$. If \mathcal{A}^t is weak-EF1, then we are done

and the algorithm terminates. Otherwise, we perform an exchange of goods between two agents to reach the allocation $\mathcal{A}^{t+1} = (A_1^{t+1}, \dots, A_n^{t+1})$. For each agent k , let g_k^t and h_k^t be a good of the highest utility and a good of the lowest utility in agent k 's bundle, A_k^t , respectively. Let i_t be an agent with the most valuable bundle, i.e., $i_t = \arg \max_{k \in N} u(A_k^t)$, and j_t be an agent with the least valuable bundle, i.e., $j_t = \arg \min_{k \in N} u(A_k^t)$; we may resolve ties arbitrarily. Note that agent j_t is not weak-EF1 towards agent i_t —otherwise, \mathcal{A}^t is weak-EF1—and hence $i_t \neq j_t$. We then exchange $g_{i_t}^t$ with $h_{j_t}^t$ to form \mathcal{A}^{t+1} , i.e., $A_{i_t}^{t+1} = (A_{i_t}^t \setminus \{g_{i_t}^t\}) \cup \{h_{j_t}^t\}$, $A_{j_t}^{t+1} = (A_{j_t}^t \setminus \{h_{j_t}^t\}) \cup \{g_{i_t}^t\}$, and $A_k^{t+1} = A_k^t$ for all $k \in N \setminus \{i_t, j_t\}$. Subsequently, we increment t by 1 and repeat the procedure.

To establish our result, we prove a series of lemmas on properties of this algorithm.

Lemma A.2. *Let \mathcal{A}^t be an allocation which is not weak-EF1. Then, $u(g_{i_t}^t) > u(h_{j_t}^t)$.*

Proof. If $u(g_{i_t}^t) \leq u(h_{j_t}^t)$, then

$$u(A_{i_t}^t) - 1 \leq u(A_{i_t}^t) \leq s \cdot u(g_{i_t}^t) \leq s \cdot u(h_{j_t}^t) \leq u(A_{j_t}^t),$$

so agent j_t is weak-EF1 towards agent i_t , and therefore \mathcal{A}^t is weak-EF1, a contradiction. Hence, $u(g_{i_t}^t) > u(h_{j_t}^t)$. \square

Lemma A.3. *Let \mathcal{A}^t be an allocation which is not weak-EF1. Then, in \mathcal{A}^{t+1} ,*

- agent i_t is weak-EF1 towards every agent; and
- every agent is weak-EF1 towards agent j_t .

Proof. Let $k \in N \setminus \{i_t, j_t\}$. Note that $u(A_k^{t+1}) = u(A_k^t)$.

Since $i_t = \arg \max_{\ell \in N} u(A_\ell^t)$, we have

$$\begin{aligned} u(A_{i_t}^{t+1}) &= u((A_{i_t}^t \setminus \{g_{i_t}^t\}) \cup \{h_{j_t}^t\}) \\ &\geq u(A_{i_t}^t \setminus \{g_{i_t}^t\}) \\ &\geq u(A_{i_t}^t) - 1 \geq u(A_k^t) - 1 = u(A_k^{t+1}) - 1, \end{aligned}$$

showing that agent i_t is weak-EF1 towards agent k .

Similarly, since $j_t = \arg \min_{\ell \in N} u(A_\ell^t)$, we have

$$\begin{aligned} u(A_k^{t+1}) &= u(A_k^t) \\ &\geq u(A_{j_t}^t) \\ &= u((A_{j_t}^{t+1} \cup \{h_{j_t}^t\}) \setminus \{g_{i_t}^t\}) \\ &\geq u(A_{j_t}^{t+1} \setminus \{g_{i_t}^t\}) \geq u(A_{j_t}^{t+1}) - 1, \end{aligned}$$

showing that agent k is weak-EF1 towards agent j_t .

Finally, since agent j_t is not weak-EF1 towards agent i_t , we have $u(A_{j_t}^t) < u(A_{i_t}^t) - 1$. Thus,

$$\begin{aligned} u(A_{i_t}^{t+1}) &= u((A_{i_t}^t \setminus \{g_{i_t}^t\}) \cup \{h_{j_t}^t\}) \\ &\geq u(A_{i_t}^t \setminus \{g_{i_t}^t\}) \\ &\geq u(A_{i_t}^t) - 1 \\ &> u(A_{j_t}^t) \\ &= u((A_{j_t}^{t+1} \cup \{h_{j_t}^t\}) \setminus \{g_{i_t}^t\}) \end{aligned}$$

$$\begin{aligned} &\geq u(A_{j_t}^{t+1} \setminus \{g_{i_t}^t\}) \\ &\geq u(A_{j_t}^{t+1}) - 1, \end{aligned}$$

showing that agent i_t is weak-EF1 towards agent j_t . \square

For each $t \geq 0$, call i_t a *strong agent* and j_t a *weak agent*. Let $I^0 = J^0 = \emptyset$, and for each $t \geq 0$, let $I^{t+1} = I^t \cup \{i_t\}$ be the set of strong agents up to round t , and $J^{t+1} = J^t \cup \{j_t\}$ be the set of weak agents up to round t .

Lemma A.4. *Let $t \geq 0$ be given such that $\mathcal{A}^0, \dots, \mathcal{A}^t$ are not weak-EF1. Then, $I^{t+1} \cap J^{t+1} = \emptyset$.*

Proof. Suppose on the contrary that there exists an agent k such that $k \in I^{t+1} \cap J^{t+1}$. Let t_p be the smallest index such that $k \in I^{t_p+1}$, and t_q be the smallest index such that $k \in J^{t_q+1}$. Then, we have $k = i_{t_p} = j_{t_q}$. Note that $t_p \neq t_q$, since $i_{t'} \neq j_{t'}$ for all t' .

Suppose first that $t_p < t_q$. We show by induction that agent k is weak-EF1 towards every agent in $\mathcal{A}^{t'+1}$ for all $t' \in \{t_p, \dots, t\}$. The base case of $t' = t_p$ is true by Lemma A.3 since $k = i_{t_p}$. For the inductive step, suppose that agent k is weak-EF1 towards every agent in $\mathcal{A}^{t'+1}$ for some $t' \in \{t_p, \dots, t-1\}$. Then, agent k cannot be $j_{t'+1}$. If agent k is $i_{t'+1}$, then agent k is weak-EF1 towards every agent in $\mathcal{A}^{t'+2}$ by Lemma A.3, making the inductive statement true. If agent k is not $i_{t'+1}$, then agent k does not take part in the exchange going from $\mathcal{A}^{t'+1}$ to $\mathcal{A}^{t'+2}$.

- Agent k is weak-EF1 towards agent $i_{t'+1}$ in $\mathcal{A}^{t'+2}$ since agent k is weak-EF1 towards $i_{t'+1}$ in $\mathcal{A}^{t'+1}$ by the inductive hypothesis, and agent $i_{t'+1}$'s utility of her own bundle decreases after the exchange by Lemma A.2.
- Agent k is weak-EF1 towards agent $j_{t'+1}$ in $\mathcal{A}^{t'+2}$ by Lemma A.3.
- Agent k is weak-EF1 towards every other agent in $\mathcal{A}^{t'+2}$ since their bundles did not change from $\mathcal{A}^{t'+1}$.

Overall, these show that agent k is weak-EF1 towards every agent in $\mathcal{A}^{t'+2}$, proving the inductive statement. Since agent k is weak-EF1 towards every agent in $\mathcal{A}^{t'+1}$ for all $t' \in \{t_p, \dots, t\}$, agent k can never be j_{t_q} . This shows that $t_p < t_q$ is false.

Therefore, we must have $t_p > t_q$. The argument for this case is similar to that for the previous case. We show by induction that every agent is weak-EF1 towards agent k in $\mathcal{A}^{t'+1}$ for all $t' \in \{t_q, \dots, t\}$. The base case of $t' = t_q$ is true by Lemma A.3 since $k = j_{t_q}$. For the inductive step, suppose that every agent is weak-EF1 towards agent k in $\mathcal{A}^{t'+1}$ for some $t' \in \{t_q, \dots, t-1\}$. Then, agent k cannot be $i_{t'+1}$. If agent k is $j_{t'+1}$, then every agent is weak-EF1 towards agent k in $\mathcal{A}^{t'+2}$ by Lemma A.3, making the inductive statement true. If agent k is not $j_{t'+1}$, then agent k does not take part in the exchange going from $\mathcal{A}^{t'+1}$ to $\mathcal{A}^{t'+2}$.

- Agent $i_{t'+1}$ is weak-EF1 towards agent k in $\mathcal{A}^{t'+2}$ by Lemma A.3.

- Agent $j_{t'+1}$ is weak-EF1 towards agent k in $\mathcal{A}^{t'+2}$ since agent $j_{t'+1}$ is weak-EF1 towards agent k in $\mathcal{A}^{t'+1}$ by the inductive hypothesis, and agent $j_{t'+1}$'s utility of her own bundle increases after the exchange by Lemma A.2.

- Every other agent is weak-EF1 towards agent k in $\mathcal{A}^{t'+2}$ since their bundles did not change from $\mathcal{A}^{t'+1}$.

Overall, these show that every agent is weak-EF1 towards agent k in $\mathcal{A}^{t'+2}$, proving the inductive statement. Since every agent is weak-EF1 towards agent k in $\mathcal{A}^{t'+1}$ for all $t' \in \{t_q, \dots, t\}$, agent k can never be i_{t_p} . This yields the desired contradiction. \square

Lemma A.5. *Let $t \geq 0$ be given such that $\mathcal{A}^0, \dots, \mathcal{A}^t$ are not weak-EF1. Then,*

- for any $i \in I^{t+1}$, $u(A_i^0) \geq \dots \geq u(A_i^{t+1})$; and
- for any $j \in J^{t+1}$, $u(A_j^0) \leq \dots \leq u(A_j^{t+1})$.

Proof. At every time step $t' \in \{0, \dots, t\}$, an agent $i \in I^{t+1}$ cannot be a weak agent by Lemma A.4. Therefore, agent i either takes part in the exchange from $\mathcal{A}^{t'}$ to $\mathcal{A}^{t'+1}$ as a strong agent $i_{t'}$ or does not take part in the exchange. The utility of agent i 's bundle either decreases in the former case due to Lemma A.2 or remains the same in the latter case. An analogous argument holds for $j \in J^{t+1}$. \square

Lemma A.6. *Each good is not exchanged more than once in algorithm \mathfrak{A} .*

Proof. Suppose on the contrary that some good g is exchanged more than once. We first consider the case where g is in a strong agent's bundle in \mathcal{A}^0 and is exchanged for the first time at round t , i.e., $g = g_{i_t}^t$. After its first exchange, the good is now with agent j_t . By Lemma A.4, $j_t \notin I^{t'}$ for any $t' > t$. Since the good is exchanged again, it must be that $g = h_{j_{t'}}^{t'}$ for some $t' > t$, where $j_{t'} = j_t$. Then, we have

$$\begin{aligned} &u(A_{i_{t'}}^t) - 1 \\ &\geq u(A_{i_{t'}}^{t'}) - 1 && \text{(by Lemma A.5 on } i_{t'} \in I^{t'+1} \text{)} \\ &> u(A_{j_{t'}}^{t'}) && \text{(since } j_{t'} \text{ is not weak-EF1 towards } i_{t'} \text{)} \\ &\geq s \cdot u(g) && \text{(since } g \text{ is the least valuable good in } A_{j_{t'}}^{t'} \text{)} \\ &\geq u(A_{i_t}^t) && \text{(since } g \text{ is the most valuable good in } A_{i_t}^t \text{)} \\ &> u(A_{i_t}^t) - 1, \end{aligned}$$

which means that agent j_t should have exchanged goods with agent $i_{t'}$ at round t instead of with agent i_t . This contradiction shows that a good in a strong agent's bundle in \mathcal{A}^0 cannot be exchanged more than once.

Analogously, we now consider the case where g is in a weak agent's bundle in \mathcal{A}^0 and is exchanged for the first time at round t , i.e., $g = h_{j_t}^t$. After its first exchange, the good is now with agent i_t . By Lemma A.4, $i_t \notin J^{t'}$ for any $t' > t$. Since the good is exchanged again, it must be that $g = g_{i_{t'}}^{t'}$ for some $t' > t$, where $i_{t'} = i_t$. Then, we have

$$u(A_{j_{t'}}^{t'})$$

$$\begin{aligned}
&\leq u(A_{j_t}^{t'}) && \text{(by Lemma A.5 on } j_{t'} \in J^{t'+1}) \\
&< u(A_{i_{t'}}^{t'}) - 1 && \text{(since } j_{t'} \text{ is not weak-EF1 towards } i_{t'}) \\
&< u(A_{i_{t'}}^{t'}) \\
&\leq s \cdot u(g) && \text{(since } g \text{ is the most valuable good in } A_{i_{t'}}^{t'}) \\
&\leq u(A_{j_t}^t), && \text{(since } g \text{ is the least valuable good in } A_{j_t}^t)
\end{aligned}$$

which means that agent i_t should have exchanged goods with agent $j_{t'}$ at round t instead of with agent j_t . This contradiction shows that a good in a weak agent's bundle in \mathcal{A}^0 also cannot be exchanged more than once. \square

Lemma A.7. *Algorithm \mathfrak{A} terminates in finite time.*

Proof. Since each good is not exchanged more than once by Lemma A.6, at most $\lfloor m/2 \rfloor$ pairs of goods can be exchanged, and the algorithm terminates by round $\lfloor m/2 \rfloor$. \square

Since the algorithm terminates in finite time by Lemma A.7, there exists $T \geq 0$ such that $\mathcal{A}^0, \dots, \mathcal{A}^T$ are not weak-EF1 but \mathcal{A}^{T+1} is weak-EF1. Let $I = I^{T+1}$ be the set of strong agents and $J = J^{T+1}$ be the set of weak agents. By Lemma A.4, I and J are disjoint sets of agents. Therefore, at each round $t \in \{0, \dots, T\}$ of the algorithm, some agent $i_t \in I$ exchanges a good with some agent $j_t \in J$.

We derive a bound on the number of steps that \mathfrak{A} takes in the case of two agents.

Lemma A.8. *For $n = 2$ agents with s goods each, algorithm \mathfrak{A} terminates after at most $\lfloor s/2 \rfloor$ rounds.*

Proof. The statement is clear when $s = 1$, so we assume that $s \geq 2$. Suppose on the contrary that after $T = \lfloor s/2 \rfloor$ rounds, the allocation \mathcal{A}^T is still not weak-EF1. Without loss of generality, assume that $1 \in I$ and $2 \in J$. Then, the most valuable $\lfloor s/2 \rfloor$ goods from agent 1's bundle A_1^0 are exchanged with the least valuable $\lfloor s/2 \rfloor$ goods from agent 2's bundle A_2^0 to reach \mathcal{A}^T . Let $B_1 \subseteq A_1^0$ and $B_2 \subseteq A_2^0$ be the sets of goods from the respective bundles that are exchanged between the two agents, and let $C_1 = A_1^0 \setminus B_1$ and $C_2 = A_2^0 \setminus B_2$. Note that all these sets are disjoint by Lemma A.6. Let g be an arbitrary good in C_1 , and let $C'_1 = C_1 \setminus \{g\}$. We have $|B_1| = |B_2| = \lfloor s/2 \rfloor$, $|C_1| = |C_2| = \lceil s/2 \rceil$, and $|C'_1| \leq \lfloor s/2 \rfloor$.

Now, $u(B_1) \geq u(C'_1)$ since the goods with the highest values from A_1^0 are exchanged and B_1 has at least as many goods as C'_1 . Also, $u(C_2) \geq u(B_2)$ since the goods with the lowest values from A_2^0 are exchanged and C_2 has at least as many goods as B_2 . Therefore, we have

$$\begin{aligned}
u(A_2^T) &= u(B_1 \cup C_2) \\
&\geq u(C'_1 \cup B_2) = u(A_1^T \setminus \{g\}) \geq u(A_1^T) - 1,
\end{aligned}$$

which shows that agent 2 is weak-EF1 towards agent 1 in \mathcal{A}^T . On the other hand, agent 1 is also weak-EF1 towards agent 2 in \mathcal{A}^T due to Lemma A.3 applied on \mathcal{A}^{T-1} . This shows that \mathcal{A}^T is weak-EF1, a contradiction. \square

We now come to our main lemma, which bounds the number of steps that \mathfrak{A} takes for three agents. For convenience of the analysis, we focus on the case where s is divisible by 3.

Lemma A.9. *Let s be a positive integer divisible by 3. For $n = 3$ agents with s goods each, algorithm \mathfrak{A} terminates after at most $2s/3$ rounds.*

Proof. Suppose on the contrary that after $T = 2s/3$ rounds, the allocation \mathcal{A}^T is still not weak-EF1. Note that $T > 0$, so $I^T, J^T \neq \emptyset$. If $|I^T| = |J^T| = 1$, then after at most $\lfloor s/2 \rfloor$ rounds, the agent $i \in I^T$ and the agent $j \in J^T$ are weak-EF1 towards each other by Lemma A.8, while the agent $k \in N \setminus \{i, j\}$ is weak-EF1 towards everyone and vice versa since agent k does not partake in the exchanges. Since $\lfloor s/2 \rfloor \leq 2s/3$, the allocation $\mathcal{A}^{T'}$ is weak-EF1 for some $T' \leq 2s/3$, contradicting our assumption. Therefore, we must have $I^T \cup J^T = N$.

Case 1: $|I^T| = 1$. We consider the allocation \mathcal{A}^T relative to \mathcal{A}^0 . Without loss of generality, let $1 \in I^T$. Let $B_{1,2} \subseteq A_1^0$ and $B_2 \subseteq A_2^0$ be the sets of goods in the respective bundles that are exchanged between agents 1 and 2, $B_{1,3} \subseteq A_1^0$ and $B_3 \subseteq A_3^0$ be the sets of goods in the respective bundles that are exchanged between agents 1 and 3, and let $C_1 = A_1^0 \setminus (B_{1,2} \cup B_{1,3})$, $C_2 = A_2^0 \setminus B_2$, and $C_3 = A_3^0 \setminus B_3$. Note that all these sets are disjoint by Lemma A.6. Let $x = |B_{1,2}| = |B_2|$ and $y = |B_{1,3}| = |B_3|$. We have $x + y = 2s/3$, $|C_2| = s - x$, $|C_3| = s - y$, and $|C_1| = s - x - y = s/3$. Without loss of generality, let $x \geq y$. Note that $x \leq s/2$, since otherwise agent 2 will be weak-EF1 towards agent 1 by Lemma A.8 and does not need to exchange more goods with agent 1.

In \mathcal{A}^0 , we have $A_1^0 = B_{1,2} \cup B_{1,3} \cup C_1$, $A_2^0 = B_2 \cup C_2$, and $A_3^0 = B_3 \cup C_3$. In \mathcal{A}^T , we have $A_1^T = B_2 \cup B_3 \cup C_1$, $A_2^T = B_{1,2} \cup C_2$, and $A_3^T = B_{1,3} \cup C_3$. Since the algorithm always exchanges the most valuable goods from agent 1's bundle and the least valuable goods from agent 2's and agent 3's bundles, we have $u(B_{1,2})/x \geq u(C_1)/(s/3)$, $u(B_{1,3})/y \geq u(C_1)/(s/3)$, $u(C_2)/(s-x) \geq u(B_2)/x$, and $u(C_3)/(s-y) \geq u(B_3)/y$. By Lemma A.2, we have $u(B_{1,2}) \geq u(B_2)$ and $u(B_{1,3}) \geq u(B_3)$.

Since $x \geq y$, we have $s/3 \leq x \leq s/2$ and hence $s/6 \leq y \leq s/3$. Let $\alpha = (6x-s)/(3x+s) = 2 - 3s/(3x+s) = 2 - s/(s-y)$. Since $s/3 \leq x \leq s/2$, we have $1/2 \leq \alpha \leq 4/5$.

Let $\alpha_2 = \alpha(s-x)/x$ and $\alpha_3 = (1-\alpha)(s-y)/y$. Since $1/3 \leq x/s \leq 1/2$, we have $\alpha \geq x/s$, which implies that $\alpha_2 = \alpha(s-x)/x = \alpha s/x - \alpha \geq 1 - \alpha$. On the other hand, the derivative of $\alpha_2 = \alpha(s-x)/x$ with respect to x is

$$\begin{aligned}
&\left(\frac{6x-s}{3x+s} \right) \left(-\frac{s}{x^2} \right) + \left(\frac{9s}{(3x+s)^2} \right) \left(\frac{s-x}{x} \right) \\
&= \left(\frac{s}{x(3x+s)} \right) \left(\frac{s-6x}{x} + \frac{9s-9x}{3x+s} \right) \\
&= \left(\frac{s}{x(3x+s)} \right) \left(\frac{(s+9x)(s-3x)}{x(3x+s)} \right).
\end{aligned}$$

When the derivative of α_2 with respect to x is equal to 0, we get $x = -s/9$ or $x = s/3$. It can be verified that α_2 attains a local maximum at $x = s/3$. For $x \in [s/3, s/2]$, the maximum value of α_2 is hence equal to 1 at $x = s/3$. Together, we have $1 - \alpha \leq \alpha_2 \leq 1$.

Now,

$$\alpha_3 = (1-\alpha) \frac{s-y}{y}$$

$$= \left(1 - \left(2 - \frac{s}{s-y}\right)\right) \frac{s-y}{y} = -\frac{s-y}{y} + \frac{s}{y} = 1.$$

We shall show that $\alpha u(A_2^T) + (1-\alpha)u(A_3^T) \geq u(A_1^T)$. We have

$$\begin{aligned} & \alpha u(A_2^T) + (1-\alpha)u(A_3^T) \\ &= \alpha(u(B_{1,2}) + u(C_2)) + (1-\alpha)(u(B_{1,3}) + u(C_3)) \\ &\geq \alpha u(B_{1,2}) + \frac{\alpha(s-x)}{x} u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + \frac{(1-\alpha)(s-y)}{y} u(B_3) \\ &= \alpha u(B_{1,2}) + \alpha_2 u(B_2) + (1-\alpha)u(B_{1,3}) + \alpha_3 u(B_3) \\ &= (\alpha + \alpha_2 - 1)u(B_{1,2}) + (1-\alpha_2)u(B_{1,2}) + \alpha_2 u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + u(B_3) \\ &\geq (\alpha + \alpha_2 - 1)u(B_{1,2}) + (1-\alpha_2)u(B_2) + \alpha_2 u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + u(B_3) \\ &= (\alpha + \alpha_2 - 1)u(B_{1,2}) + u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + u(B_3) \\ &\geq (\alpha + \alpha_2 - 1) \frac{3x}{s} u(C_1) + u(B_2) \\ &\quad + (1-\alpha) \frac{3y}{s} u(C_1) + u(B_3) \\ &= \frac{3}{s} ((\alpha + \alpha_2)x - x + (1-\alpha)y) u(C_1) \\ &\quad + u(B_2) + u(B_3). \end{aligned}$$

Since $\alpha_2 x = \alpha(s-x)$ implies $(\alpha + \alpha_2)x = \alpha s$ and $y = \alpha_3 y = (1-\alpha)(s-y)$ implies $(1-\alpha)y = (1-\alpha)s - y$, the expression $(\alpha + \alpha_2)x - x + (1-\alpha)y$ simplifies to $\alpha s - x + (1-\alpha)s - y$, which gives $s - x - y$. Using the fact that $x + y = 2s/3$, the expression simplifies to $s/3$. Therefore,

$$\begin{aligned} & \alpha u(A_2^T) + (1-\alpha)u(A_3^T) \\ &\geq \frac{3}{s} \left(\frac{s}{3}\right) u(C_1) + u(B_2) + u(B_3) \\ &= u(C_1) + u(B_2) + u(B_3) = u(A_1^T). \end{aligned}$$

Let $j \in \arg \max_{k \in \{1,2,3\}} u(A_k^T)$. Since $\alpha u(A_2^T) + (1-\alpha)u(A_3^T) \geq u(A_1^T)$ for some $\alpha \in (0, 1)$, we may assume that $j \in J^T$. Suppose without loss of generality that $j = 2$. Note that agent 2 is weak-EF1 towards every other agent in \mathcal{A}^T . Agent 1 is weak-EF1 towards every other agent in \mathcal{A}^T by Lemma A.3. Let $t < T$ be the round that agent 2 exchanges a good with agent 1 for the final time, i.e., agent 2 exchanges a good with agent 1 going from \mathcal{A}^t to \mathcal{A}^{t+1} . Then, by Lemma A.3, agent 3 is weak-EF1 towards agent 2 in \mathcal{A}^{t+1} . Since the utility of agent 3's bundle does not decrease thereafter and agent 2's bundle remains the same thereafter, agent 3 is weak-EF1 towards agent 2 in \mathcal{A}^T . Then, agent 3 is weak-EF1 towards every other agent in \mathcal{A}^T . This shows that \mathcal{A}^T is weak-EF1, contradicting the original assumption.

Case 2: $|I^T| = 2$. We consider the allocation \mathcal{A}^T relative to \mathcal{A}^0 . Without loss of generality, let $1 \in J^T$. Let $B_{1,2} \subseteq A_1^0$ and $B_2 \subseteq A_2^0$ be the sets of goods in the respective bundles that are exchanged between agents 1 and 2, $B_{1,3} \subseteq A_1^0$ and

$B_3 \subseteq A_3^0$ be the sets of goods in the respective bundles that are exchanged between agents 1 and 3, and let $C_1 = A_1^0 \setminus (B_{1,2} \cup B_{1,3})$, $C_2 = A_2^0 \setminus B_2$, and $C_3 = A_3^0 \setminus B_3$. Note that all these sets are disjoint by Lemma A.6. Let $x = |B_{1,2}| = |B_2|$ and $y = |B_{1,3}| = |B_3|$. We have $x + y = 2s/3$, $|C_2| = s - x$, $|C_3| = s - y$, and $|C_1| = s - x - y = s/3$. Without loss of generality, let $x \geq y$. Note that $x \leq s/2$, since otherwise agent 1 will be weak-EF1 towards agent 2 by Lemma A.8 and does not need to exchange more goods with agent 2.

In \mathcal{A}^0 , we have $A_1^0 = B_{1,2} \cup B_{1,3} \cup C_1$, $A_2^0 = B_2 \cup C_2$, and $A_3^0 = B_3 \cup C_3$. In \mathcal{A}^T , we have $A_1^T = B_2 \cup B_3 \cup C_1$, $A_2^T = B_{1,2} \cup C_2$, and $A_3^T = B_{1,3} \cup C_3$. Since the algorithm always exchanges the least valuable goods from agent 1's bundle and the most valuable goods from agent 2's and agent 3's bundles, we have $u(B_{1,2})/x \leq u(C_1)/(s/3)$, $u(B_{1,3})/y \leq u(C_1)/(s/3)$, $u(C_2)/(s-x) \leq u(B_2)/x$, and $u(C_3)/(s-y) \leq u(B_3)/y$. By Lemma A.2, we have $u(B_{1,2}) \leq u(B_2)$ and $u(B_{1,3}) \leq u(B_3)$.

Since $x \geq y$, we have $s/3 \leq x \leq s/2$. Let $\alpha = (6x-s)/(3x+s) = 2 - 3s/(3x+s) = 2 - s/(s-y)$, $\alpha_2 = \alpha(s-x)/x$, and $\alpha_3 = (1-\alpha)(s-y)/y$. By the same reasoning as in Case 1, we have $1/2 \leq \alpha \leq 4/5$, $1 - \alpha \leq \alpha_2 \leq 1$, and $\alpha_3 = 1$.

We shall show that $\alpha u(A_2^T) + (1-\alpha)u(A_3^T) \leq u(A_1^T)$. We have

$$\begin{aligned} & \alpha u(A_2^T) + (1-\alpha)u(A_3^T) \\ &= \alpha(u(B_{1,2}) + u(C_2)) + (1-\alpha)(u(B_{1,3}) + u(C_3)) \\ &\leq \alpha u(B_{1,2}) + \frac{\alpha(s-x)}{x} u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + \frac{(1-\alpha)(s-y)}{y} u(B_3) \\ &= \alpha u(B_{1,2}) + \alpha_2 u(B_2) + (1-\alpha)u(B_{1,3}) + \alpha_3 u(B_3) \\ &= (\alpha + \alpha_2 - 1)u(B_{1,2}) + (1-\alpha_2)u(B_{1,2}) + \alpha_2 u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + u(B_3) \\ &\leq (\alpha + \alpha_2 - 1)u(B_{1,2}) + (1-\alpha_2)u(B_2) + \alpha_2 u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + u(B_3) \\ &= (\alpha + \alpha_2 - 1)u(B_{1,2}) + u(B_2) \\ &\quad + (1-\alpha)u(B_{1,3}) + u(B_3) \\ &\leq (\alpha + \alpha_2 - 1) \frac{3x}{s} u(C_1) + u(B_2) \\ &\quad + (1-\alpha) \frac{3y}{s} u(C_1) + u(B_3) \\ &= \frac{3}{s} ((\alpha + \alpha_2)x - x + (1-\alpha)y) u(C_1) \\ &\quad + u(B_2) + u(B_3). \end{aligned}$$

By the same reasoning as in Case 1, we have $(\alpha + \alpha_2)x - x + (1-\alpha)y = s/3$, and therefore, $\alpha u(A_2^T) + (1-\alpha)u(A_3^T) \leq u(A_1^T)$.

Let $i \in \arg \min_{k \in \{1,2,3\}} u(A_k^T)$. Since $\alpha u(A_2^T) + (1-\alpha)u(A_3^T) \leq u(A_1^T)$ for some $\alpha \in (0, 1)$, we may assume that $i \in I^T$. Suppose without loss of generality that $i = 2$. Note that every agent is weak-EF1 towards agent 2 in \mathcal{A}^T . Every agent is weak-EF1 towards agent 1 in \mathcal{A}^T by Lemma A.3. Let $t < T$ be the round that agent 2 exchanges a good with

agent 1 for the final time, i.e., agent 2 exchanges a good with agent 1 going from \mathcal{A}^t to \mathcal{A}^{t+1} . Then, by Lemma A.3, agent 2 is weak-EF1 towards agent 3 in \mathcal{A}^{t+1} . Since the utility of agent 3's bundle does not increase thereafter and agent 2's bundle remains the same thereafter, agent 2 is weak-EF1 towards agent 3 in \mathcal{A}^T . Then, every agent is weak-EF1 towards agent 3 in \mathcal{A}^T . This shows that \mathcal{A}^T is weak-EF1, contradicting the original assumption. \square

We are now ready to show the result on $\tilde{f}_{\text{id}}(n, s)$ for three agents.

Theorem A.10. *Let s be a positive integer divisible by 3. Then, $\tilde{f}_{\text{id}}(3, s) = 2s/3$.*

Proof. The lower bound of $\tilde{f}_{\text{id}}(3, s)$ follows from Theorem 5.3—note that weak-EF1 and EF1 are equivalent for binary utilities. The upper bound follows from Lemma A.9. \square

B Beneficial Exchanges

Let us say that an exchange is *beneficial* if the two agents involved in the exchange strictly benefit from the exchange, i.e., if the goods $g \in A_i$ and $g' \in A_j$ are exchanged, then $u_i(g') > u_i(g)$ and $u_j(g) > u_j(g')$. In this appendix, we investigate the decision problem of whether a given initial allocation can be reformed into an EF1 allocation using *only* beneficial exchanges. For convenience, we refer to this problem as **BENEFICIAL EXCHANGES**.

We show that **BENEFICIAL EXCHANGES** is NP-complete, even for binary utilities, using a reduction from **MINIMUM k -COVERAGE**. In **MINIMUM k -COVERAGE**, we are given positive integers k, ℓ, p, q such that $k \leq q$ and $\ell \leq p$, a set $X = \{x_1, \dots, x_q\}$, and a collection $C = \{Y_1, \dots, Y_p\}$ of subsets of X . The problem is to decide whether there exists a set $I \subseteq \{1, \dots, p\}$ of indices such that $|I| = \ell$ and $|\bigcup_{i \in I} Y_i| \leq k$. This decision problem is known to be NP-hard [Vinterbo, 2002].

Theorem B.1. ***BENEFICIAL EXCHANGES** is NP-complete for binary utilities.*

Proof. For membership in NP, observe that in a sequence of beneficial exchanges for binary utilities, each good $g \in M$ can only be part of at most one exchange. Indeed, if good g is part of at least two beneficial exchanges, then it must be received by some agent i (and hence worth 1 to i) and be given away by agent i (and hence worth 0 to i), which is impossible. Therefore, such a sequence consists of at most $m/2$ exchanges, and can be used as a certificate for polynomial-time verification.

It remains to show that the problem is NP-hard. Let an instance of **MINIMUM k -COVERAGE** be given. Define an instance of **BENEFICIAL EXCHANGES** as follows. There are $n = 2p + q + k - \ell$ agents and $m = 2n$ goods. We shall label the agents $a_{1,1}, \dots, a_{1,q}, a_{2,1}, \dots, a_{2,k}, a_{3,1}, \dots, a_{3,p}, a_{4,1}, \dots, a_{4,p-\ell}$; we use $u_{i,j}$ for the utility of agent $a_{i,j}$. For each agent $a_{i,j}$, there are two goods $g_{i,j}^0$ and $g_{i,j}^1$ that are both in agent $a_{i,j}$'s bundle in the initial allocation. The valuable goods for the agents are as follows:

- For $i \in \{1, \dots, q\}$, $u_{1,i}(g_{2,j}^1) = 1$ for all $j \in \{1, \dots, k\}$. Additionally, if $x_i \in Y_j$ for some $j \in \{1, \dots, p\}$, then $u_{1,i}(g_{3,j}^0) = u_{1,i}(g_{3,j}^1) = 1$.
- For $i \in \{1, \dots, k\}$, $u_{2,i}(g_{1,j}^1) = 1$ for all $j \in \{1, \dots, q\}$.
- For $i \in \{1, \dots, p\}$, $u_{3,i}(g_{4,j}^1) = 1$ for all $j \in \{1, \dots, p - \ell\}$.
- For $i \in \{1, \dots, p - \ell\}$, $u_{4,i}(g_{3,j}^1) = 1$ for all $j \in \{1, \dots, p\}$.

All other goods not mentioned above are worth 0 to the respective agents. This reduction can be done in polynomial time.

In the initial allocation, every agent has zero utility for her own bundle, and the only agents who are possibly not EF1 are agents $a_{1,i}$, who envy $a_{3,j}$ if $x_i \in Y_j$. By construction, the only possible beneficial exchanges are between $g_{1,i}^1$ in agent $a_{1,i}$'s bundle and $g_{2,j}^1$ in agent $a_{2,j}$'s bundle, or between $g_{3,i}^1$ in agent $a_{3,i}$'s bundle and $g_{4,j}^1$ in agent $a_{4,j}$'s bundle.

We claim that the initial allocation can be reformed into an EF1 allocation via only beneficial exchanges if and only if there exists a set $I \subseteq \{1, \dots, p\}$ of indices such that $|I| = \ell$ and $|\bigcup_{i \in I} Y_i| \leq k$.

(\Leftarrow) Suppose that there exists a set $I \subseteq \{1, \dots, p\}$ of indices such that $|I| = \ell$ and $|\bigcup_{i \in I} Y_i| \leq k$.

- Let $I' = \{1, \dots, p\} \setminus I$. Since $|I'| = p - \ell$, there exists a bijection $\sigma : I' \rightarrow \{1, \dots, p - \ell\}$. For each $i' \in I'$, exchange $g_{3,i'}^1$ in agent $a_{3,i'}$'s bundle with $g_{4,\sigma(i')}^1$ in agent $a_{4,\sigma(i')}$'s bundle.
- Let $J = \{j \mid x_j \in \bigcup_{i \in I} Y_i\}$. Since $|J| \leq k$, there exists an injection $\phi : J \rightarrow \{1, \dots, k\}$. For each $j \in J$, exchange $g_{1,j}^1$ in agent $a_{1,j}$'s bundle with $g_{2,\phi(j)}^1$ in agent $a_{2,\phi(j)}$'s bundle.

We now show that the new allocation is EF1. It is easy to see that the allocation is EF1 for agents $a_{2,i}$, $a_{3,i}$, and $a_{4,i}$, since every other agent has at most one of their valuable goods. Therefore, it suffices to show that the allocation is EF1 for agents $a_{1,j}$.

- If $j \in J$, then agent $a_{1,j}$ has the valuable good $g_{2,\phi(j)}^1$ in her bundle, so her utility of her own bundle is at least 1. Since her utility of every other agent's bundle is at most 2, agent $a_{1,j}$ is EF1 towards every other agent.
- If $j \notin J$, then $x_j \notin \bigcup_{i \in I} Y_i$, and so $x_j \notin Y_i$ for all $i \in I$. Note that the valuable goods for $a_{1,j}$ are possibly in the form $g_{2,i}^1$, $g_{3,i}^0$, and $g_{3,i}^1$. Suppose on the contrary that $a_{1,j}$ is not EF1 towards some agent. This agent must have two such goods in the final allocation. The only way for this to happen is when there exists i^* such that agent a_{3,i^*} has both g_{3,i^*}^0 and g_{3,i^*}^1 . This means agent a_{3,i^*} had not exchanged any goods, and so $i^* \notin I'$. This implies that $i^* \in I$. Since $x_j \notin Y_i$ for all $i \in I$, we must have $u_{1,j}(g_{3,i^*}^0) = u_{1,j}(g_{3,i^*}^1) = 0$. This contradicts the assumption that $a_{1,j}$ is not EF1 towards agent a_{3,i^*} . Therefore, $a_{1,j}$ is EF1 towards every agent.

(\Rightarrow) Suppose that the initial allocation can be reformed into an EF1 allocation via only beneficial exchanges. Consider one such sequence of beneficial exchanges. Let $I' \subseteq \{1, \dots, p\}$ be the set of all indices i' such that agent $a_{3,i'}$ exchanged a good with another agent in this sequence. Since $a_{3,i'}$ can only exchange a good with some $a_{4,i''}$ once, and there are only $p - \ell$ agents of the form $a_{4,i''}$, we have $|I'| \leq p - \ell$. Therefore, $I_0 := \{1, \dots, p\} \setminus I'$ has cardinality at least ℓ , and I_0 contains indices i such that agent $a_{3,i}$ retains her original bundle from the initial allocation.

We claim that $|\bigcup_{i \in I_0} Y_i| \leq k$. Let $J \subseteq \{1, \dots, q\}$ be the set of all indices j such that agent $a_{1,j}$ exchanged a good with another agent in this sequence. Since $a_{1,j}$ can only exchange a good with some $a_{2,j''}$ once, and there are only k agents of the form $a_{2,j''}$, we have $|J| \leq k$. Therefore, $J' := \{1, \dots, q\} \setminus J$ has cardinality at least $q - k$, and J' contains all indices j' such that agent $a_{1,j'}$ retains her original bundle from the initial allocation; these agents have utility 0. Since the final allocation is EF1, these agents do not envy agents $a_{3,i}$ by more than one good for each $i \in I_0$. Therefore, we must have $u_{1,j'}(g_{3,i}^0) = u_{1,j'}(g_{3,i}^1) = 0$, which implies that $x_{j'} \notin Y_i$ for all $j' \in J'$ and $i \in I_0$. This means that $x_{j'} \notin \bigcup_{i \in I_0} Y_i$ for every $j' \in J'$. Therefore, at least $q - k$ of the x_j 's are not in $\bigcup_{i \in I_0} Y_i$, which shows that $\bigcup_{i \in I_0} Y_i$ has cardinality at most $q - (q - k) = k$, as claimed.

Finally, take any subset $I \subseteq I_0$ with cardinality ℓ . The proof is completed by noting that $\bigcup_{i \in I} Y_i \subseteq \bigcup_{i \in I_0} Y_i$. \square

While we have shown that a sequence of beneficial exchanges must be of polynomial length for binary utilities, the same statement in fact holds for general utilities. Indeed, for any sequence of beneficial exchanges, for each good g_{t_1} in some agent i 's initial bundle, it is exchanged with another good g_{t_2} , which is subsequently exchanged with another good g_{t_3} , and so on, until some g_{t_k} in agent i 's final bundle. Since we must have $u_i(g_{t_1}) < \dots < u_i(g_{t_k})$ due to the exchanges being beneficial, it must hold that $k \leq m$, and so there are at most $m - 1$ exchanges starting from g_{t_1} . Since there are m goods in total and each exchange involves two goods, the maximum number of exchanges in the sequence is $m(m - 1)/2$. Hence, by Theorem B.1, we have NP-completeness for general utilities as well.