National University of Singapore
School of Computing
IT5005 Artificial Intelligence

Propositional Logic 1

1. Verify the following logical equivalences.

   (a) $\neg(P \vee \neg Q) \vee (\neg P \wedge \neg Q) \equiv \neg P$

   Solution:

$$\begin{aligned}
\neg(P \vee \neg Q) \vee (\neg P \wedge \neg Q) &\equiv (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) && \text{De Morgan's law} \\
&\equiv \neg P \wedge (Q \vee \neg Q) && \text{distributivity of } \wedge \text{ over } \vee \\
&\equiv \neg P \wedge \textbf{true} \\
&\equiv \neg P
\end{aligned}$$

   (b) $(P \wedge \neg(\neg P \vee Q)) \vee (P \wedge Q) \equiv P$

   Solution:

$$\begin{aligned}
(P \wedge (\neg(\neg P \vee Q))) \vee (P \wedge Q) &\equiv (P \wedge (P \wedge \neg Q)) \vee (P \wedge Q) && \text{De Morgan's law} \\
&\equiv ((P \wedge P) \wedge \neg Q) \vee (P \wedge Q) && \text{Associativity of } \wedge \\
&\equiv (P \wedge \neg Q) \vee (P \wedge Q) \\
&\equiv P \wedge (\neg Q \vee Q) && \text{distributivity of } \wedge \text{ over } \vee \\
&\equiv P \wedge \textbf{true} \\
&\equiv P
\end{aligned}$$

2. State whether the following statements are SAT (satisfiable), VALID (tautology), or UNSAT (contradiction).

   (a) $(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q)$

   Solution: SAT.

P	Q	$(P \Rightarrow Q) \land (P \Rightarrow \neg Q)$
true	true	false
true	false	false
false	true	true
false	false	true

(b) $P \land (P \Rightarrow \neg Q) \land Q$

Solution: UNSAT.

P	Q	$P \land (P \Rightarrow \neg Q) \land Q$
true	true	false
true	false	false
false	true	false
false	false	false

(c) $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \land Q) \Rightarrow R)$

Solution: VALID.

P	Q	R	$(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \land Q) \Rightarrow R)$
true	true	true	true
true	true	false	true
true	false	true	true
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

3. Prove or disprove that the following sentence is a tautology via truth table method.

$$(P \lor Q) \land (R \lor \neg P) \Rightarrow (R \lor Q)$$

Solution:

P	Q	R	$(P \vee Q) \wedge (R \vee \neg P)$	$(R \vee Q)$	$(P \vee Q) \wedge (R \vee \neg P) \Rightarrow (R \vee Q)$
T	T	T	T	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	F	T

Table 1: Truth Table

4. Prove that the following sentence is a tautology without truth table enumeration.

$$(\neg P \vee \neg Q \vee R) \wedge (\neg R \vee S \vee T) \Rightarrow (\neg P \vee \neg Q \vee S \vee T)$$

Solution

The given sentence can be written as

$$((P \wedge Q) \Rightarrow R) \wedge (R \Rightarrow (S \vee T)) \Rightarrow ((P \wedge Q) \Rightarrow (S \vee T)) \tag{1}$$

From Rule of Transitivity, the above sentence is a tautology.

5. Convert the following sentences to CNF

(a) $\neg((P \Rightarrow Q) \wedge \neg R)$

Solution:

$$\begin{aligned}
\neg((P \Rightarrow Q) \wedge \neg R) &\equiv \neg((\neg P \vee Q) \wedge \neg R) && \text{implication elimination} \\
&\equiv \neg(\neg P \vee Q) \vee \neg\neg R && \text{De Morgan's law} \\
&\equiv \neg(\neg P \vee Q) \vee R && \text{double negation elimination} \\
&\equiv (\neg\neg P \wedge \neg Q) \vee R && \text{De Morgan's law} \\
&\equiv (P \wedge \neg Q) \vee R && \text{double negation elimination} \\
&\equiv R \vee (P \wedge \neg Q) && \text{commutativity of } \vee \\
&\equiv (R \vee P) \wedge (R \vee \neg Q) && \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}$$

(b) $(X_1 \land Y_1) \lor (X_2 \land Y_2))$

Solution:

$$\begin{aligned}
(X_1 \land Y_1) \lor (X_2 \land Y_2)) &\equiv ((X_1 \land Y_1) \lor X_2) \land ((X_1 \land Y_1) \lor Y_2) && \text{distr. of } \lor \text{ over } \land \\
&\equiv (X_2 \lor (X_1 \land Y_1)) \land (Y_2 \lor (X_1 \land Y_1)) && \text{commutativity of } \lor \\
&\equiv (X_2 \lor X_1) \land (X_2 \lor Y_1) \land (Y_2 \lor X_1) \land (Y_2 \lor Y_1) && \text{distr. of } \lor \text{ over } \land
\end{aligned}$$

6. Which of the following statements are $True$?

(a) $False \models True$

Solution

Because False has no models, ie., $M(False) = \emptyset$. From set theory, empty set is a subset of every set. Consequently, $False$ entails every sentence.

$M(True) = \mathbb{U}$, where $\mathbb{U}$ is an universal set. Every set is a subset of universal set. Consequently, $True$ is entailed by every sentence.

(b) $(A \land B) \models (A \Leftrightarrow B)$

Solution

$M(A \land B) = \{A = True, B = True\}$ and $M(A \Leftrightarrow B) = \{\{A = True, B = True\}, \{A = False, B = False\}\}$. $M(A \land B) \subseteq M(A \Leftrightarrow B)$. Hence the statement is $True$.