National University of Singapore
School of Computing
IT5005 Artificial Intelligence

Probabilistic Reasoning over Time (MC and HMM)

1. Three vectors that represent probability distribution of a discrete random variable X are given. Identify valid probability distributions. If the vector does not represent a probability distribution, convert it to a valid probability distribution.

   (a)
$$\mathbf{P}(X) = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix}$$

   (b)
$$\mathbf{P}(X) = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.5 \end{bmatrix}$$

   (c)
$$\mathbf{P}(X) = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

2. Identify the valid transition matrices for a Markov chain.

   (a)
$$T_1 = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.3 & 0.2 & 0.5 \\ 0.1 & 0.7 & 0.2 \end{bmatrix}$$

   (b)
$$T_2 = \begin{bmatrix} 0.1 & 0.2 & 0.8 \\ 0.1 & 0.2 & 0.7 \\ 0.0 & 0.7 & 0.3 \end{bmatrix}$$

3. Markov model can be extended in several ways. One such extension is shown in the Figure 1. Obtain the expression for joint distribution of first n variables.

4. Let X be the random variable that represent the stock price of a company. X can take on one of two possible values:
$$High(H), Low(L)$$

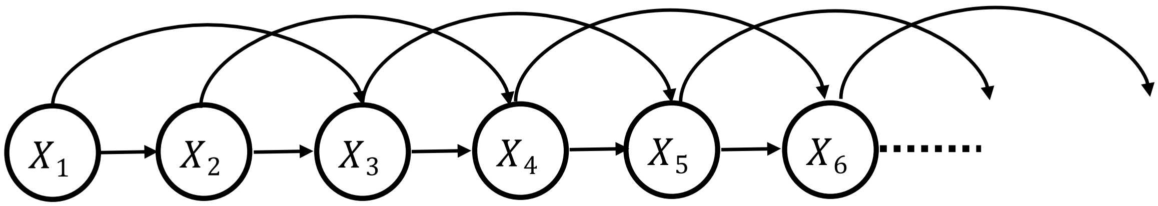

Figure 1: Markov Chain

Figure 2: State Transition Model

on any given day. Further, assume that X follows a first order Markov model that is shown in Figure 2.

Also, note that that the stock price of the company on the first day can be any of the two values with equal probability. Given these information, answer the questions below:

(a) Write the state transition model (transition matrix) and prior distribution.

(b) The sequence of stock prices of the company for ten days is

$$L \rightarrow L \rightarrow L \rightarrow H \rightarrow L \rightarrow H \rightarrow H \rightarrow H \rightarrow H \rightarrow L$$

Given the sequence, find the probability of stock price being high on day 11.

(c) For the above sequence, calculate the probability of stock prices for the first five days?

(d) Draw the trellis of the Markov model for three time steps and calculate the probability distribution of the stock price being high on day 3 i.e., $P(X_3 = H)$?

(e) Find the probability distribution of the stock price on Day 4.

(f) What is the stationary distribution of the stock price i.e., $P(X)$ at time $t \rightarrow \infty$?

(g) Does the stationary distribution depend on the initial probability distribution? Further, assume that you are more of a long-term investor, will you choose to invest in the company?

5. Consider a hidden Markov model with two hidden states $X = 1, 2$, and two possible output symbols $E = A, B$. The probability of starting in states $X = 1$ and $X = 2$ are 0.49 and 0.51, respectively. If X_t is the state at time t, the state transition probabilities are

$$\begin{aligned} P(X_{t+1} = 1 | X_t = 1) &= 1 \\ P(X_{t+1} = 1 | X_t = 2) &= 1 \end{aligned}$$

and the output probabilities are

$$\begin{aligned} P(E_t = A | X_t = 1) &= 1 \\ P(E_t = B | X_t = 2) &= 1 \end{aligned}$$

What is the sequence of three output symbols that has the highest probability of being generated from this HMM model? Explain your answer.

6. Let the set of output states be e, f, g and the set of hidden states be $s1, s2, s3$ where $s3$ is the stop state. Some data has been gathered regarding sequences of output states and their corresponding hidden states as follows: "$e/s1, f/s1$"; "$e/s2, f/s2$"; "$f/s1, g/s2$"; "$g/s2, e/s1$".

   Assume T is the transition distribution, where each element T_{ij} represents a transition from state i to state j; O is the emission distribution, where each element $O_{o,j}$ represents the probability of output o in state j; and π_i is the initial probability distribution of states. State the maximum likelihood estimates for all possible π_i, T_{ij}, $O_{o,j}$ given the data.

7. Let X_t be the hidden state at time t and it takes the values from the set S, A, B, C, D, where D is the $STOP$ state and the set of possible outputs be $E_1 = d$, $E_2 = p$, $E_3 = o$. You can assume 0 as the $START$ state. The following probabilities are given:

   $\pi_S = P(S|0) = 1$, $\pi_A = P(A|0) = 0$, $\pi_B = P(B|0) = 0$, $\pi_C = P(C|0) = 0$

   The transition and emission probabilities are shown in Table 1 and 2, respectively.

Table 1: Transition Probabilities

T_{ij}	$x_j = S$	$x_j = A$	$x_j = B$	$x_j = C$	$x_j = D$
$x_i = S$	0.3	0.2	0.1	0.1	0.3
$x_i = A$	0.25	0.1	0.05	0.15	0.45
$x_i = B$	0.05	0.4	0.15	0.2	0.2
$x_i = C$	0.5	0.2	0.2	0.05	0.05

Calculate the following:

(a) $\mathbf{P}(X_1|E_1)$

(b) $\mathbf{P}(X_2|E_1, E_2)$

(c) $\mathbf{P}(X_3|E_1, E_2, E_3)$

Table 2: Emission Probabilities

O_{ij}	$e_j = d$	$e_j = p$	$e_j = o$
$x_i = S$	0.6	0.2	0.2
$x_i = A$	0.1	0.5	0.4
$x_i = B$	0.2	0.1	0.7
$x_i = C$	0.3	0.4	0.3