NUS | Computing
National University of Singapore

IT5005 Artificial Intelligence

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AY2025/2026: Semester 1

Probabilistic Reasoning: Tutorial

Q1a: Show that $ToothAche$ and $Catch$ are independent given $Cavity$

$ToothAche$	$Cavity$	$Catch$	$P(ToothAche, Cavity, Catch)$
t	t	t	0.108
t	t	f	0.012
t	f	t	0.016
t	f	f	0.064
f	t	t	0.072
f	t	f	0.008
f	f	t	0.144
f	f	f	0.576

Show that $ToothAche$ and $Catch$ are independent given $Cavity$

(Or)

Show that

$$\boldsymbol{P}(ToothAche|Catch, Cavity) = \boldsymbol{P}(ToothAche|Cavity)$$

(Or)

Show that

$$\boldsymbol{P}(Catch|ToothAche, Cavity) = \boldsymbol{P}(Catch|Cavity)$$

(Or)

Show that

$$\boldsymbol{P}(Catch, Toothache|\ Cavity) = \boldsymbol{P}(Catch|Cavity)\ \boldsymbol{P}(ToothAche|Cavity)$$

$\boldsymbol{P}(ToothAche|Catch, Cavity) = \boldsymbol{P}(ToothAche|Cavity)$

$ToothAche$	$Cavity$	$Catch$	$P(ToothAche, Cavity, Catch)$
t	t	t	0.108
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t	f	t	0.016
t	f	f	0.064
f	t	t	0.072
f	t	f	0.008
f	f	t	0.144
f	f	f	0.576

Four cases:

1. $Catch = t\ or\ catch, Cavity = t\ or\ cavity$
2. $Catch = t\ or\ catch, Cavity = f\ or\ \neg cavity$
3. $Catch = f\ or\ \neg catch,\ Cavity = t\ or\ cavity$
4. $Catch = f\ or\ \neg catch, Cavity = f\ or\ \neg cavity$

Evaluate LHS term $\boldsymbol{P}(ToothAche|Catch, Cavity)$ and RHS term $\boldsymbol{P}(ToothAche|Cavity)$
Show that they are equal

We show this for Case 1: $catch, cavity$
Remaining cases are homework

LHS: Left Hand Side
RHS: Right Hand Side

$\boldsymbol{P}(ToothAche|Catch, Cavity) = \boldsymbol{P}(ToothAche|Cavity)$

Case 1: LHS

$$\begin{aligned}\boldsymbol{P}(ToothAche|catch, cavity) &= [P(ToothAche = t|catch, cavity) \quad P(ToothAche = f|catch, cavity)] \\ &= \left[\frac{P(ToothAche=t,catch,cavity)}{P(catch,cavity)} \quad \frac{P(ToothAche=f,catch,cavity)}{P(catch,cavity)}\right] \\ &= \alpha[a \quad b]\end{aligned}$$

Bayes rule

where

$$a = P(ToothAche = t, catch, cavity)$$
$$b = P(ToothAche = f, catch, cavity)$$
$$\alpha = \frac{1}{P(catch,cavity)}$$

Normalization constant;
no need to calculate $P(catch, cavity)$

$$= \frac{1}{a+b}$$

so that a and b represent probabilities

$$\begin{aligned}\boldsymbol{P}(ToothAche|catch, cavity) &= \alpha[0.108 \quad 0.072] \\ &= \frac{1}{0.108+0.072}[0.108 \quad 0.072] \\ &= [0.60 \quad 0.40]\end{aligned}$$

$ToothAche$	$Cavity$	$Catch$	$P(ToothAche, Cavity, Catch)$
t	t	t	0.108
t	t	f	0.012
t	f	t	0.016
t	f	f	0.064
f	t	t	0.072
f	t	f	0.008
f	f	t	0.144
f	f	f	0.576

$$\boldsymbol{P}(ToothAche|Catch, Cavity) = \boldsymbol{P}(ToothAche|Cavity)$$

Case 1: RHS

$$\boldsymbol{P}(ToothAche|\, cavity) = [P(ToothAche = t|\, cavity) \quad P(ToothAche = f|cavity)]$$

$$= \left[\frac{P(ToothAche=t, cavity)}{P(cavity)} \quad \frac{P(ToothAche=f, cavity)}{P(cavity)}\right]$$ Bayes rule

$$= \alpha[a \quad b]$$

where

$$a = P(ToothAche = t, cavity)$$
$$b = P(ToothAche = f, cavity)$$
$$\alpha = \frac{1}{P(cavity)}$$

Normalization constant;
no need to calculate $P(catch, cavity)$

$$= \frac{1}{a+b}$$ so that a and b represent probabilities

$ToothAche$	$Cavity$	$Catch$	$P(ToothAche, Cavity, Catch)$
t	t	t	0.108
t	t	f	0.012
t	f	t	0.016
t	f	f	0.064
f	t	t	0.072
f	t	f	0.008
f	f	t	0.144
f	f	f	0.576

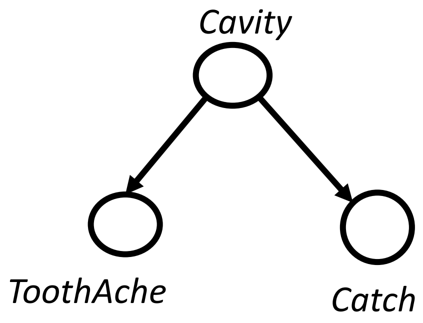
$$\boldsymbol{P}(ToothAche|\, cavity) = \alpha[P(ToothAche = t, cavity) \quad P(ToothAche = f\ cavity)]$$

$$= \alpha\left[\sum_{Catch=\{t,f\}} P(ToothAche = t, Catch, cavity) \quad \sum_{Catch=\{t,f\}} P(ToothAche = f, Catch\ cavity)\right]$$

$$= \alpha[0.108 + 0.012 \quad 0.072 + 0.008]$$
$$= \alpha[0.12 \quad 0.08]$$
$$= \frac{1}{0.12+0.08}[0.12 \quad 0.08] = \frac{1}{0.2}[0.12 \quad 0.08] = [0.6 \quad 0.4]$$

Q1b: Show that

$$\boldsymbol{P}(ToothAche, Cavity, Catch) = \boldsymbol{P}(Cavity)\boldsymbol{P}(ToothAche|Cavity)\boldsymbol{P}(Catch|Cavity)$$



$\Pr(Cavity\ =\ t)$
0.2

$ToothAche$	$Cavity$	$Catch$	$P(ToothAche, Cavity, Catch)$
t	t	t	0.108
t	t	f	0.012
t	f	t	0.016
t	f	f	0.064
f	t	t	0.072
f	t	f	0.008
f	f	t	0.144
f	f	f	0.576

$Cavity$	$\Pr(ToothAche{=}t\,|\,Cavity)$
t	0.6
f	0.1

$Cavity$	$\Pr(Catch{=}t\,|\,Cavity)$
t	0.9
f	0.2

Verify that LHS and RHS are equal for all eight assignment of values

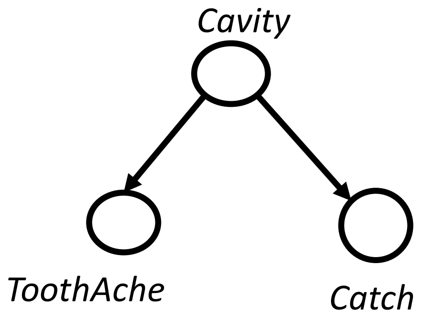
Q1b: Show that

$$\boldsymbol{P}(ToothAche, Cavity, Catch) = \boldsymbol{P}(Cavity)\boldsymbol{P}(ToothAche|Cavity)\boldsymbol{P}(Catch|Cavity)$$

$ToothAche$	$Cavity$	$Catch$	$P(ToothAche, Cavity, Catch)$
t	t	t	0.108
t	t	f	0.012
t	f	t	0.016
t	f	f	0.064
f	t	t	0.072
f	t	f	0.008
f	f	t	0.144
f	f	f	0.576

$$P(toothAche, cavity, catch) = 0.108$$
$$\mathrm{P}(cavity)P(toothAche|cavity)P(catch|cavity) = 0.2 * 0.6 * 0.9$$
$$= 0.108$$

$$P(\neg toothAche, \neg cavity, \neg catch) = 0.576$$
$$\mathrm{P}(\neg cavity)P(\neg toothAche|\neg cavity)P(\neg catch|\neg cavity)$$
$$= 0.8 * 0.9 * 0.8$$
$$= 0.576$$

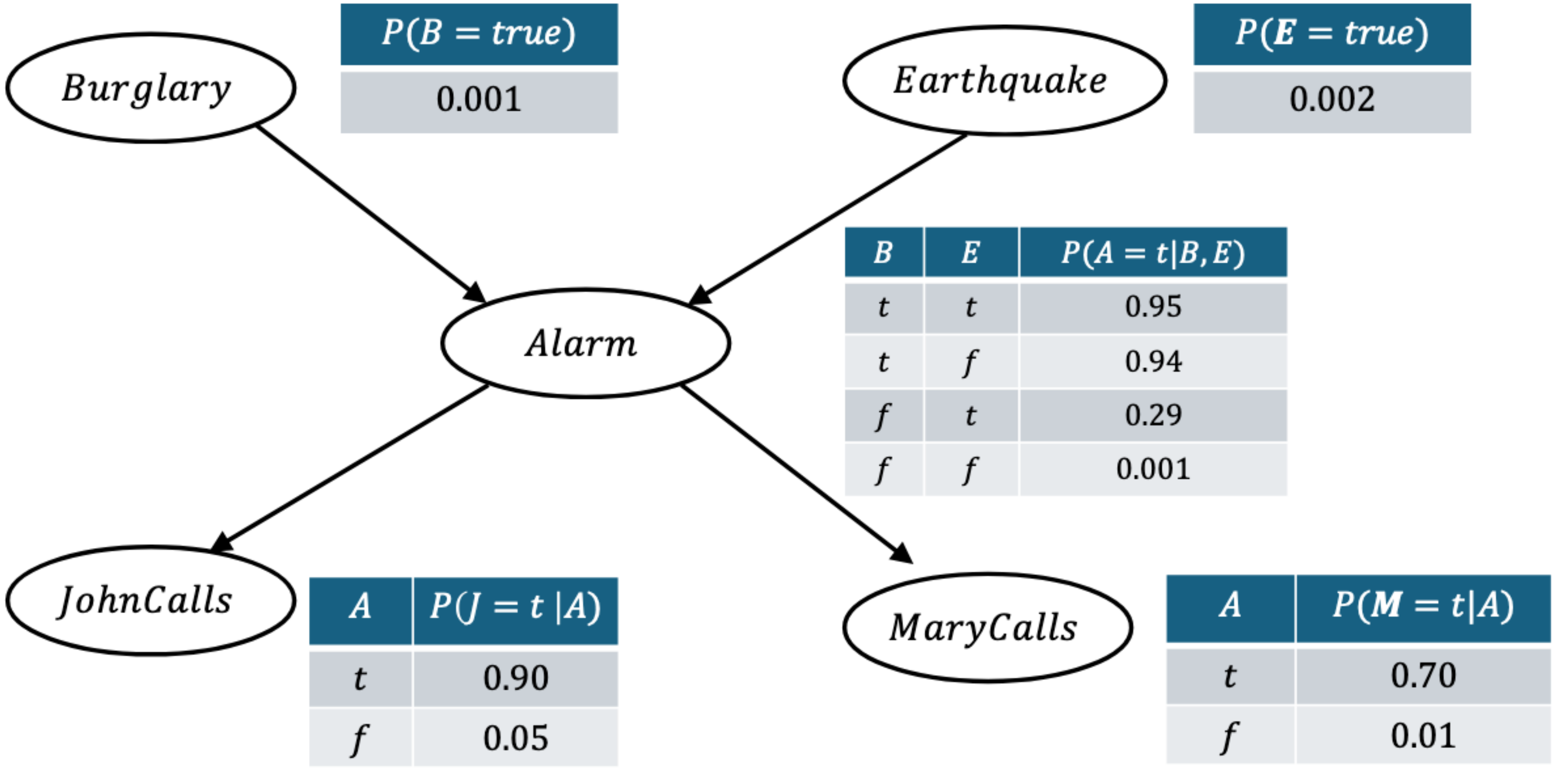


$\Pr(Cavity = t)$
0.2

$Cavity$	$\Pr(ToothAche{=}t\|Cavity)$
t	0.6
f	0.1

$Cavity$	$\Pr(Catch{=}t\|Cavity)$
t	0.9
f	0.2

Q2: Calculate $\boldsymbol{P}(J|a)$ and $\boldsymbol{P}(J|b)$



$P(B = true)$
0.001

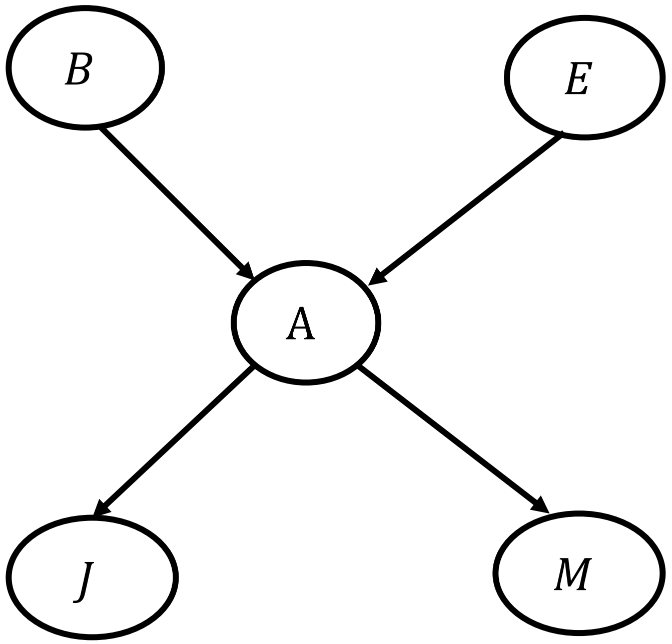
$P(\boldsymbol{E} = true)$
0.002

B	E	$P(A = t\|B, E)$
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	$P(J = t\ \|A)$
t	0.90
f	0.05

A	$P(\boldsymbol{M} = t\|A)$
t	0.70
f	0.01

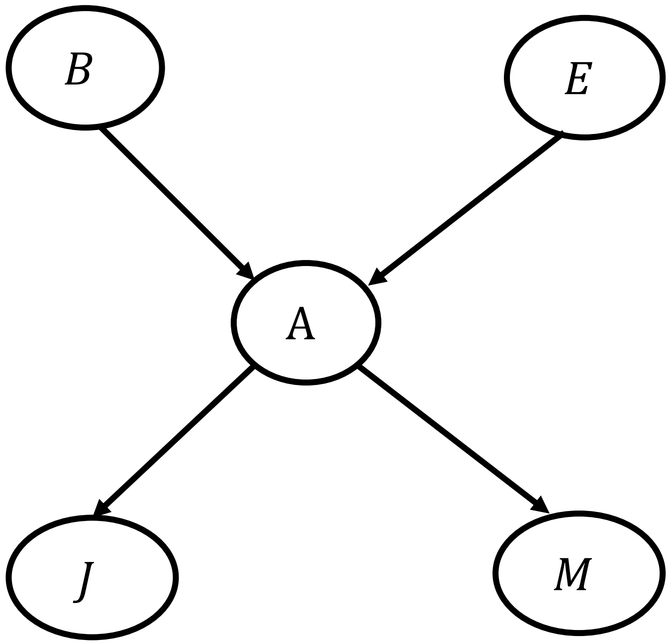
Query: $\boldsymbol{P}(J|a)$



- Query Variable: J
- Evidence: $A = a$
- Hidden Variables: B, E, M
- Remove irrelevant variables
  - $M \perp J \mid A$
  - $B \perp J \mid A$
  - $E \perp J \mid A$
    - i.e., M, B, E are irrelevant to J given A
  - We are only left with $P(J|a)$
- $\boldsymbol{P}(J|a)$ = [0.9 0.1]

Query: $\boldsymbol{P}(J|b)$

- **Query: $\boldsymbol{P}(J|b)$**
  - Query Variable: J
  - Evidence Variable: $\mathrm{B} = b$
  - Hidden Variables: E, A, M
- Any variable that is not ancestor of evidence/query variable is irrelevant to query
  - Remove $MaryCalls$

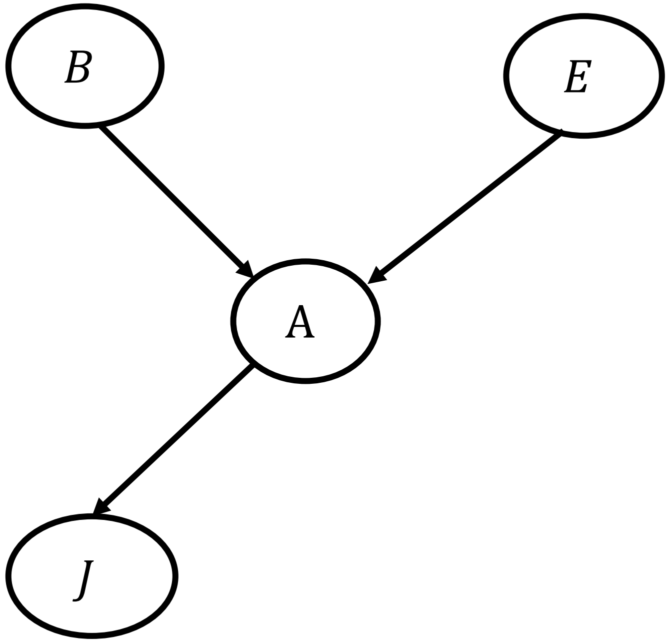


=1

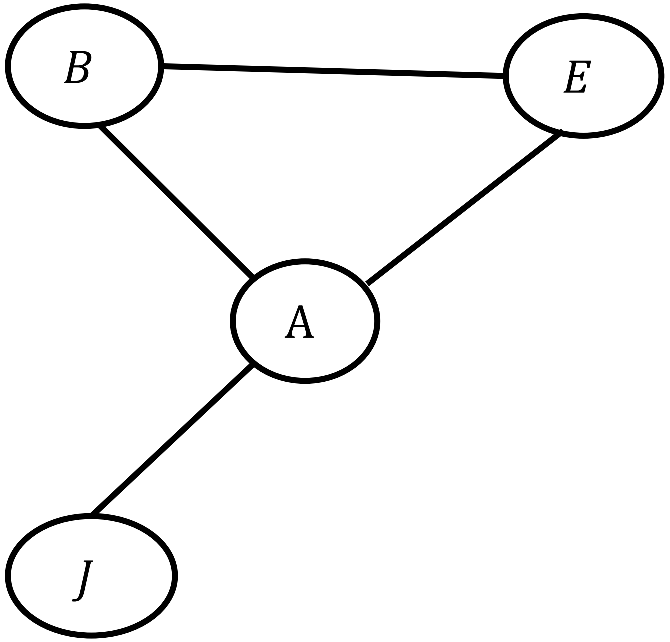
$$P(j|b) = \alpha P(j,b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(j|a) \sum_m P(m|a)$$

Query: $\boldsymbol{P}(J|b)$

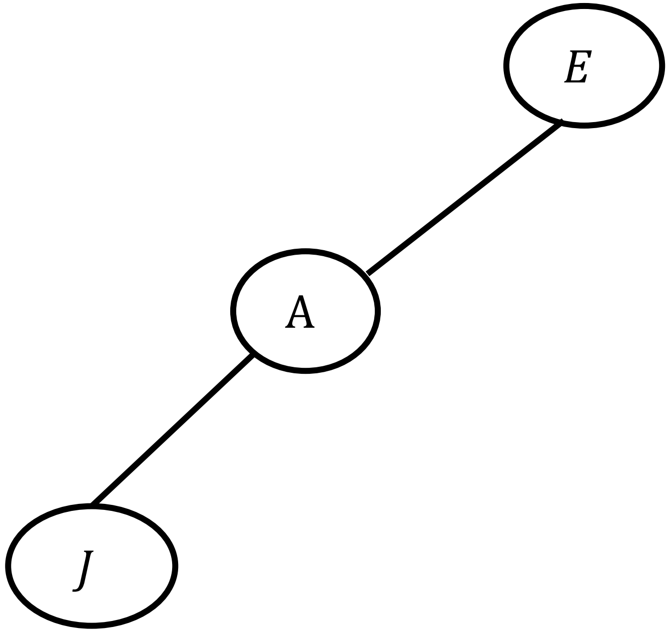
After removing descendant variables



Moralized ancestor graph

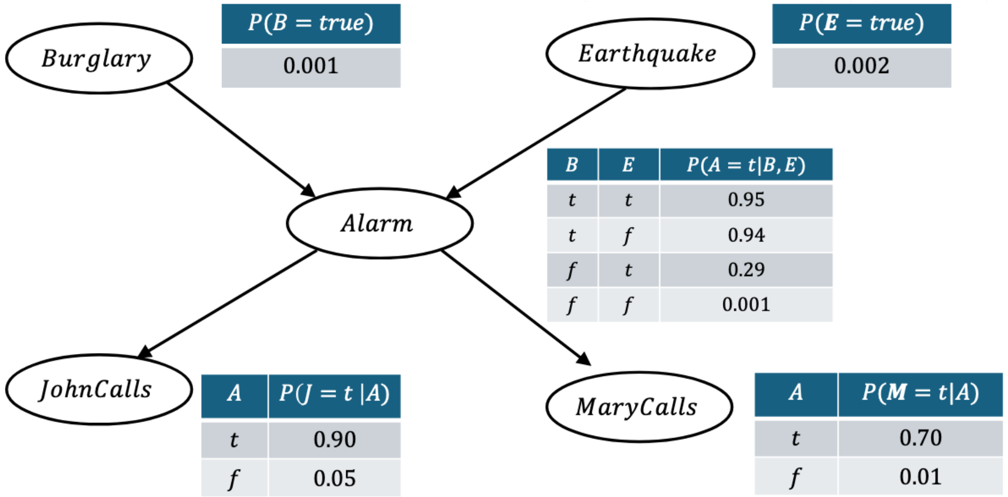


Removing evidence



Both A and E are relevant for the query

Query: $\boldsymbol{P}(J|b)$



$P(B = true)$
0.001

$P(E = true)$
0.002

B	E	$P(A = t\|B, E)$
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	$P(J = t\|A)$
t	0.90
f	0.05

A	$P(M = t\|A)$
t	0.70
f	0.01

$MaryCalls$ is irrelevant for the query

m is absent

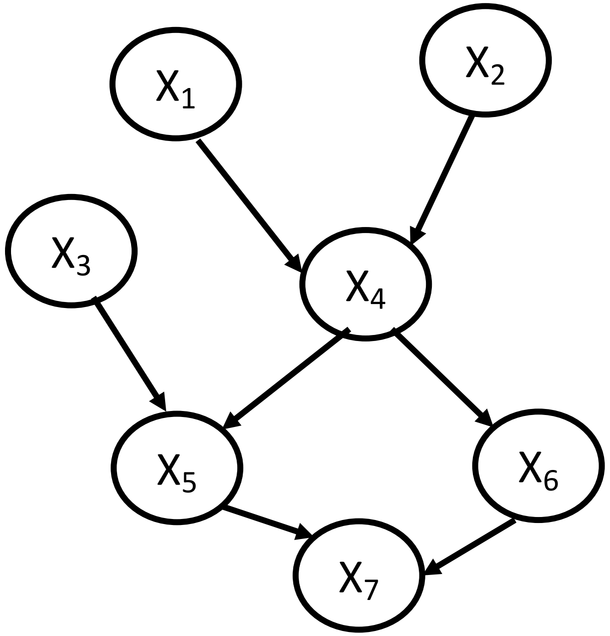
$$p = P(j, b)$$
$$= \sum_e \sum_a P(j, b, a, e) \quad \text{marginalization}$$
$$= \sum_e \sum_a P(a|b, e)P(j|a)P(b)P(e) \quad \text{factoring}$$
$$= P(b) \sum_e P(e) \sum_a P(a|b, e)P(j|a)$$
$$= P(b)\Big(P(e)\big(P(a|b, e)P(j|a) + P(\neg a|b, e)P(j|\neg a)\big) + P(\neg e)\big(P(a|b, \neg e)P(j|a) + P(\neg a|b, \neg e)P(j|\neg a)\big)\Big)$$

$$q = P(\neg j, b)$$
$$= \sum_e \sum_a P(\neg j, b, a, e) \quad \text{marginalization}$$
$$= \sum_e \sum_a P(a|b, e)P(\neg j|a)P(b)P(e) \quad \text{factoring}$$
$$= P(b) \sum_e P(e) \sum_a P(a|b, e)P(\neg j|a)$$
$$= P(b)\Big(P(e)\big(P(a|b, e)P(\neg j|a) + P(\neg a|b, e)P(\neg j|\neg a)\big) + P(\neg e)\big(P(a|b, \neg e)P(\neg j|a) + P(\neg a|b, \neg e)P(\neg j|\neg a)\big)\Big)$$

```
>>> p = 0.001*(0.002*(0.95*0.9+0.05*0.05)+0.998*(0.94*0.9+0.06*0.05))
>>> q = 0.001*(0.002*(0.95*0.1+0.05*0.95)+0.998*(0.94*0.1+0.06*0.95))
>>> alpha  =1/(p+q)
>>> alpha*p
0.8490170000000001
>>> alpha*q
0.150983
```

$$\boldsymbol{P}(J|b) = \alpha\boldsymbol{P}(J, b) = [0.849 \quad 0.151]$$

Q3: Independence



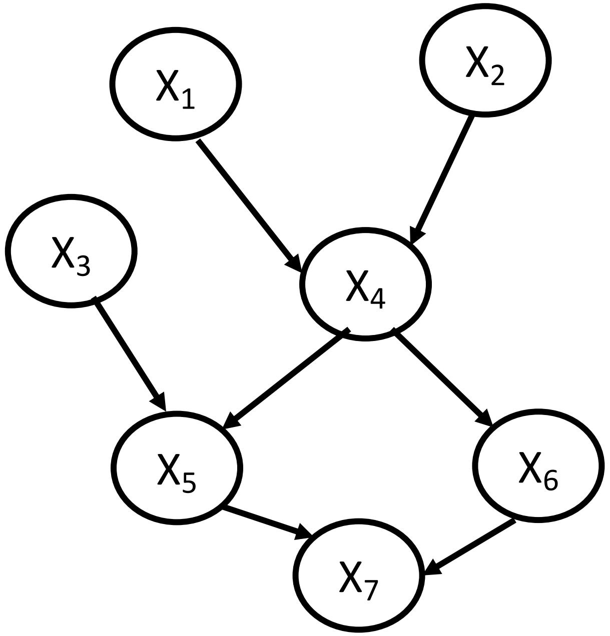
a: $X_1 \perp X_2$, $i.e.$, is X_1 independent of X_2?

b: $X_1 \perp X_2 | X_4$, $i.e.$, is X_1 independent of X_2 given X_4?

c. $X_3 \perp X_7 | X_5$, Is X_3 independent of X_7 given X_5?

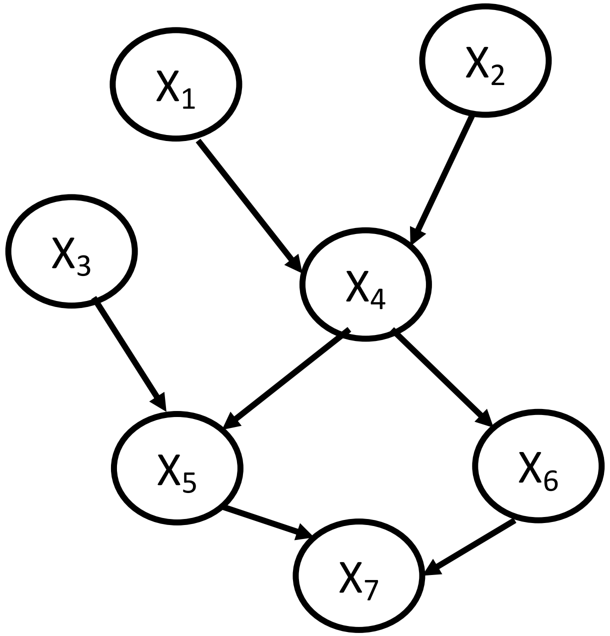
$Q3a$: $X_1 \perp X_2$, $i.e.$, is X_1 independent of X_2?

Yes

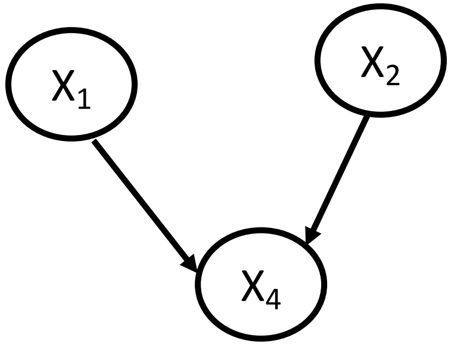


1. After removing descendant variables

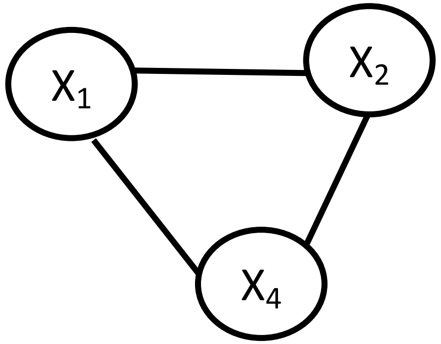
$Q3b$: $X_1 \perp X_2 | X_4$, $i.e.$, is X_1 independent of X_2 given X_4?



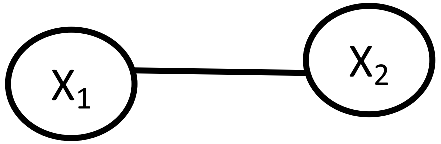
1. After removing descendant variables



2. Moralized ancestor graph

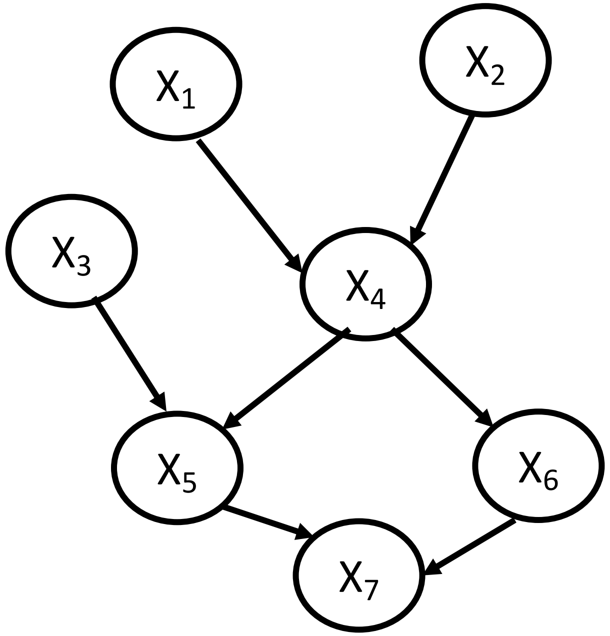


3. After removing evidence

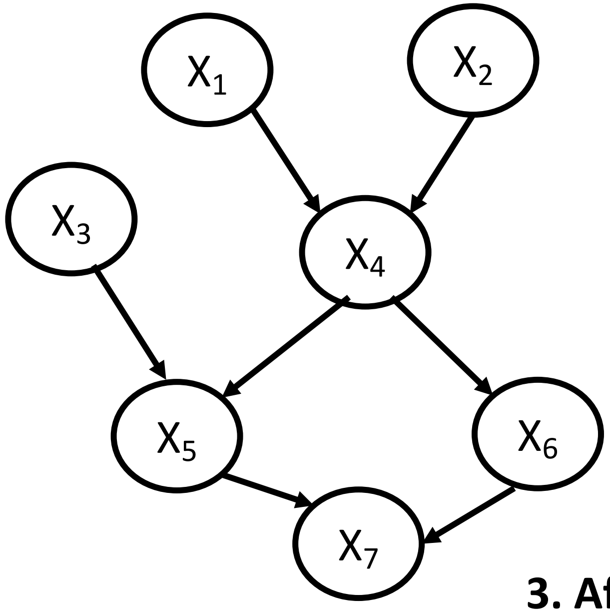


X_1 and X_2 are not guaranteed to be independent given X_4

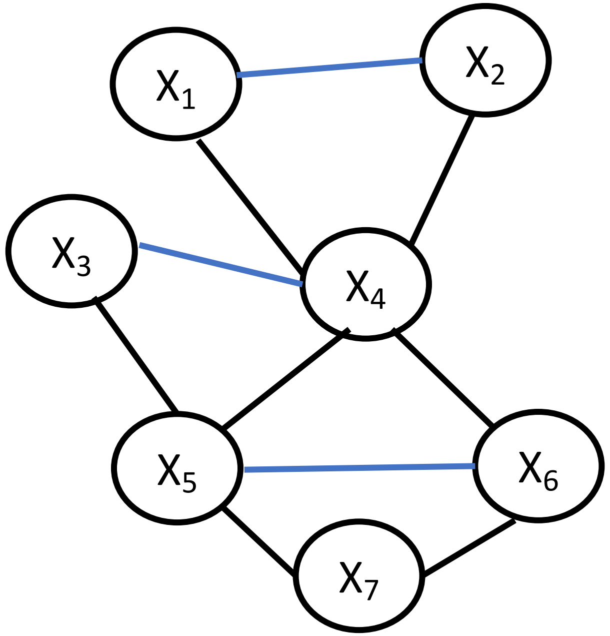
$Q3c.\ X_3 \perp X_7 | X_5$, Is X_3 independent of X_7 given X_5?



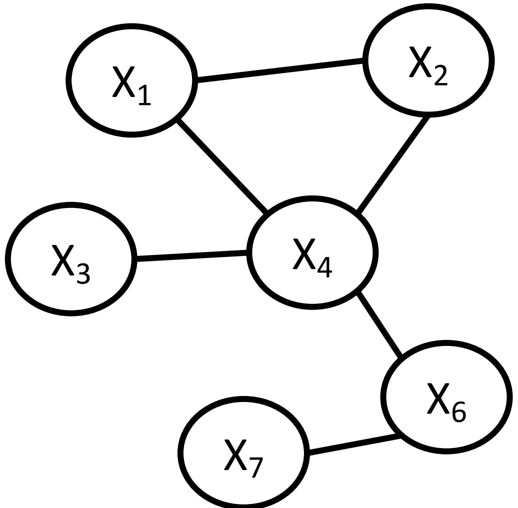
1. After removing descendant variables



2. Moralized ancestor graph



3. After removing evidence

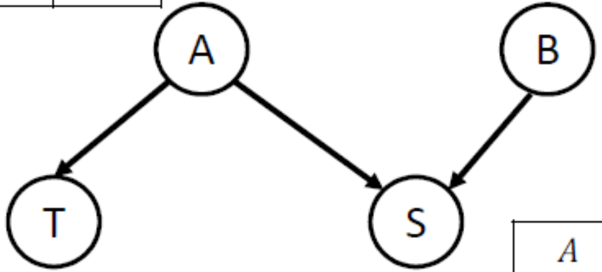


X_3 and X_7 are not guaranteed to be independent given X_5

Q4: Consider the Bayesian network shown below:

A	$\Pr(A)$
a	0.1

B	$\Pr(B)$
b	0.5



A	$P(T=t\|A)$
a	1
$\neg a$	0.2

A	B	$P(S=s\|A,B)$
a	b	1
a	$\neg b$	0.8
$\neg a$	b	1
$\neg a$	$\neg b$	0

a. Compute the probability $P(\neg a, \neg t, b, s)$

b. What is the probability that the patient has disease A given that they have symptom S and test T returns positive?

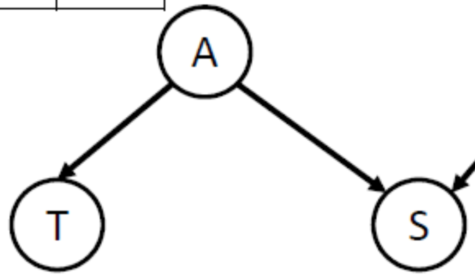
c. Suppose that both diseases become more likely as person ages. Add a new variable and the related arcs to the Bayesian network to reflect this new knowledge. Note that you only need to draw the revised Bayesin network and the CPTs need not be defined for the new variable.

Q4a: Compute $P(\neg a, \neg t, b, s)$

$$
\begin{aligned}
P(\neg a, \neg t, b, s) &= P(\neg t|\neg a)P(s|\neg a, b)P(\neg a)P(b) \\
&= 0.8 * 1 * 0.9 * 0.5 = 0.36
\end{aligned}
$$

A	$\Pr(A)$
a	0.1

B	$\Pr(B)$
b	0.5



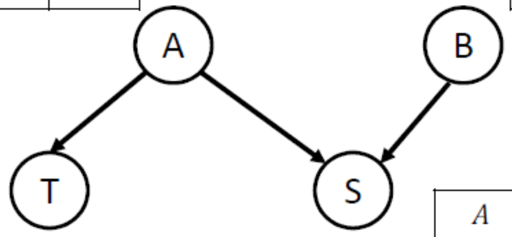
A	$P(T = t\|A)$
a	1
$\neg a$	0.2

A	B	$P(S = s\|A, B)$
a	b	1
a	$\neg b$	0.8
$\neg a$	b	1
$\neg a$	$\neg b$	0

Q4b. What is the probability that the patient has disease A given that they have symptom S and test T returns positive?

A	$\Pr(A)$
a	0.1

B	$\Pr(B)$
b	0.5



A	$P(T=t\|A)$
a	1
$\neg a$	0.2

A	B	$P(S=s\|A,B)$
a	b	1
a	$\neg b$	0.8
$\neg a$	b	1
$\neg a$	$\neg b$	0

$$P(a|s,t) = \frac{P(a,s,t)}{P(s,t)} = \frac{\sum_b P(a,s,t,b)}{\sum_{a,b} P(a,s,t,b)} = \frac{\sum_b P(t|a)P(s|a,b)P(a)P(b)}{\sum_{a,b} P(t|a)P(s|a,b)P(a)P(b)} = \frac{P(t|a)P(a)\sum_b P(s|a,b)P(b)}{\sum_a \sum_b P(t|a)P(s|a,b)P(a)P(b)}$$

$$= \frac{P(t|a)P(a)\big(P(s|a,b)P(b)+P(s|a,\neg b)P(\neg b)\big)}{P(t|a)P(s|a,b)P(a)P(b)+P(t|\neg a)P(s|\neg a,b)P(\neg a)P(b)+P(t|a)P(s|a,\neg b)P(a)P(\neg b)+P(t|\neg a)P(s|\neg a,\neg b)P(\neg a)P(\neg b)}$$

$=0.5$

Q4c. Suppose that both diseases become more likely as person ages. Add a new variable and the related arcs to the Bayesian network to reflect this new knowledge. Note that you only need to draw the revised Bayesin network and the CPTs need not be defined for the new variable.