Non-convergence Analysis of Randomized Direct Search

AMA613

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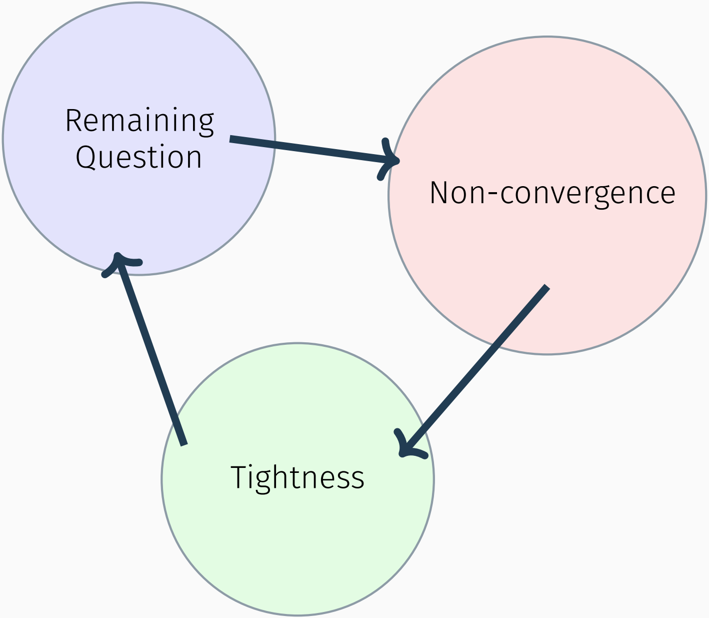
Big Picture



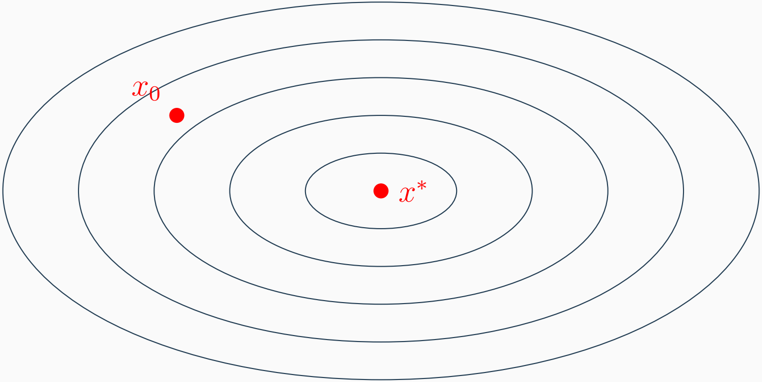
Remaining question

Brief review of last talk

In last talk

1. Derivative-free optimization (DFO)
   - A branch of optimization
   - Do not use derivatives (only use function evaluations)
2. Randomized direct search (RDS)
   - Make decisions based on simple comparisons of function values
   - Choose direction sets randomly

Illustration of how RDS works



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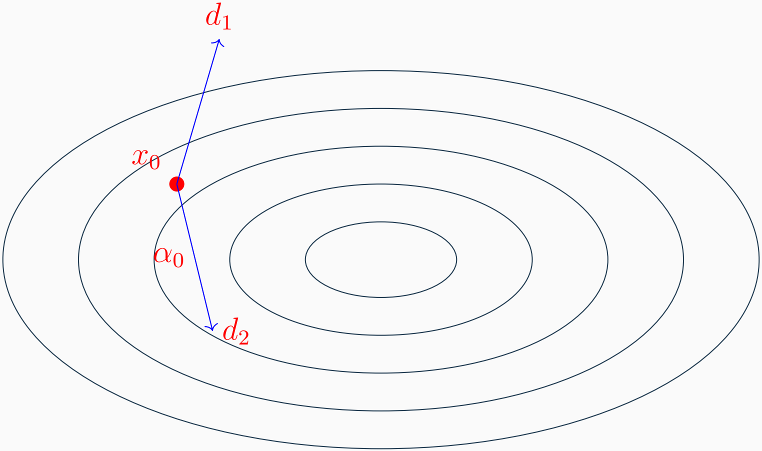

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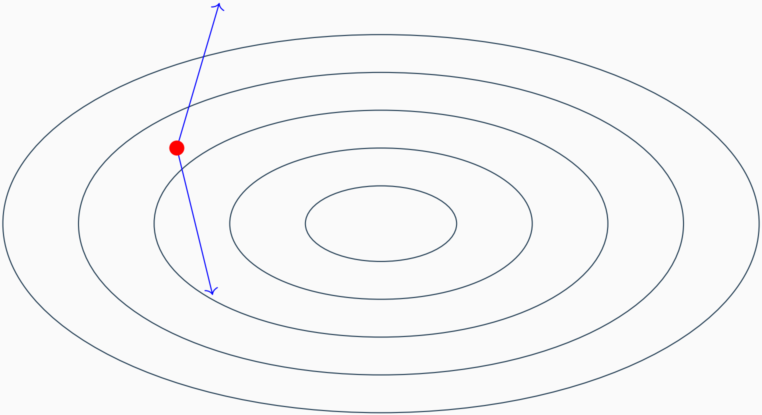

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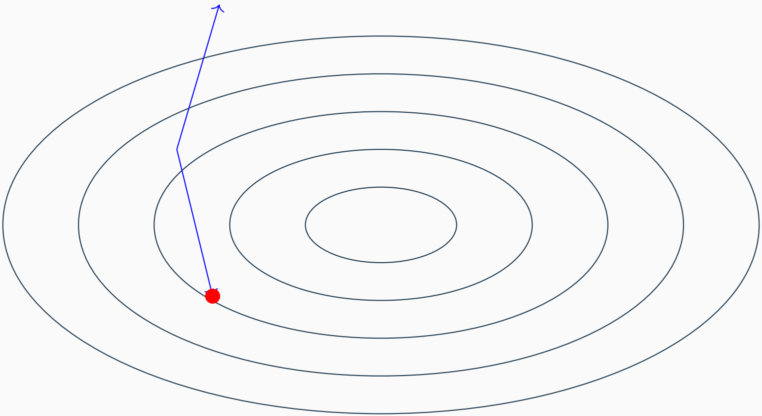

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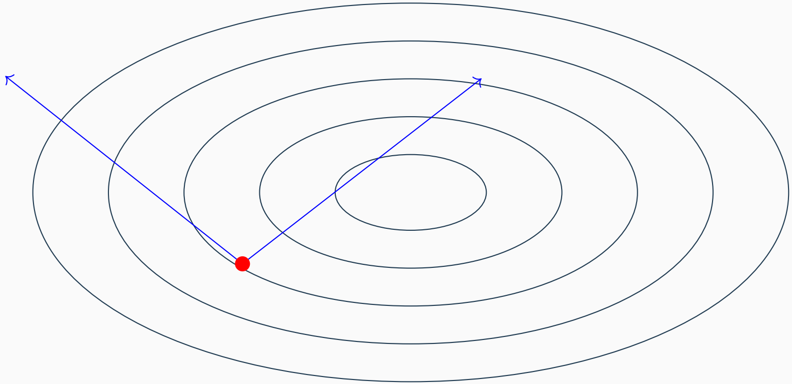

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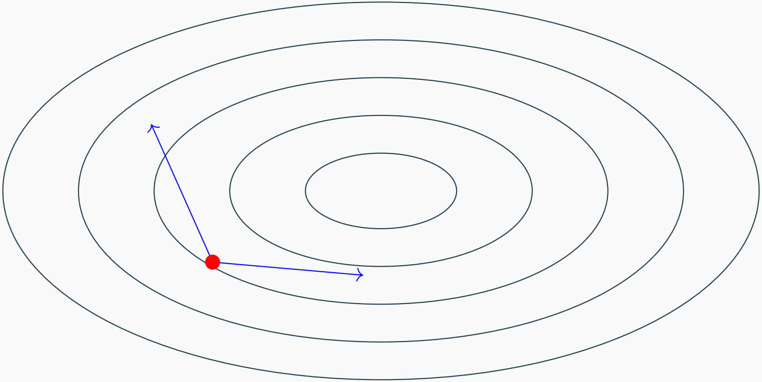

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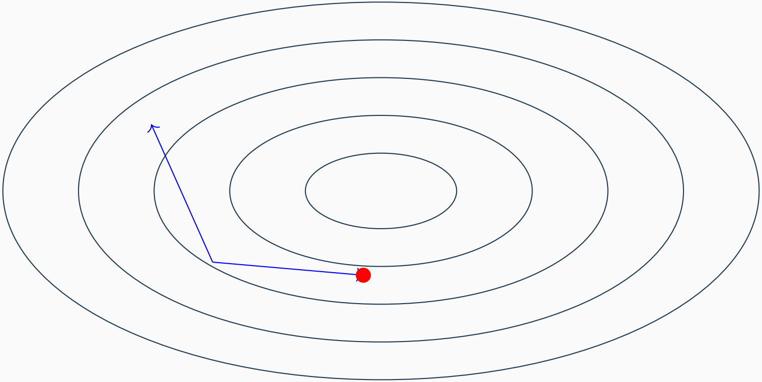

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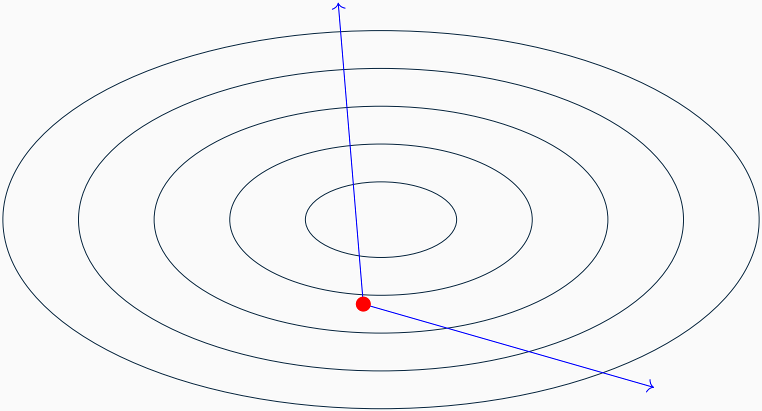

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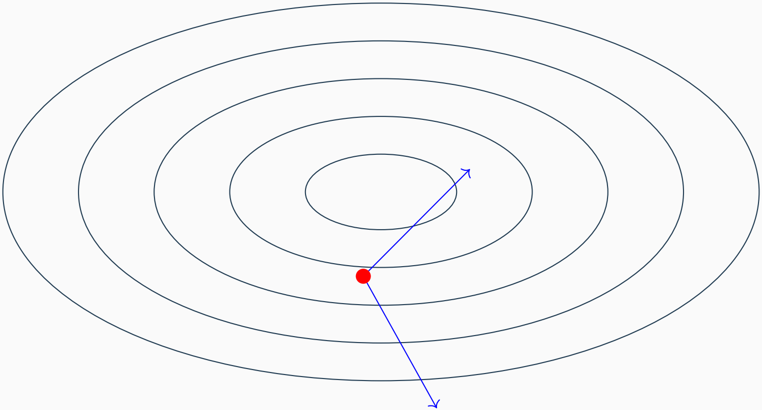

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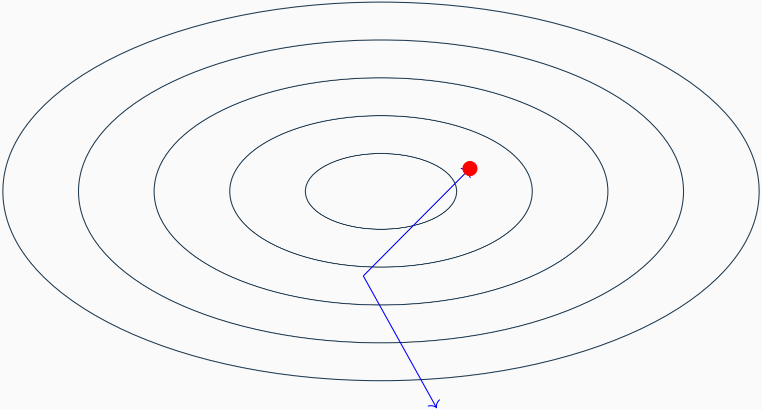

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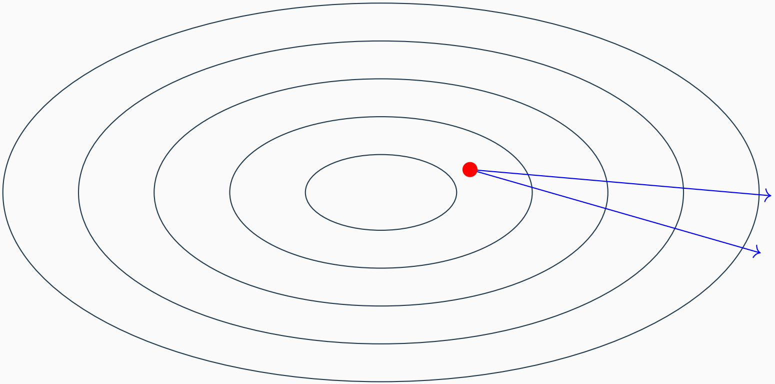

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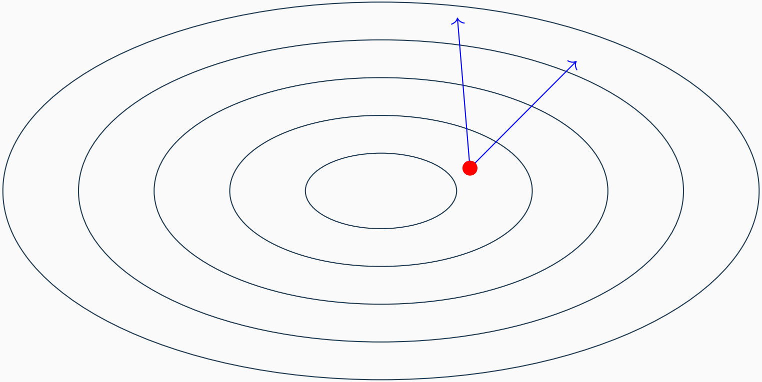

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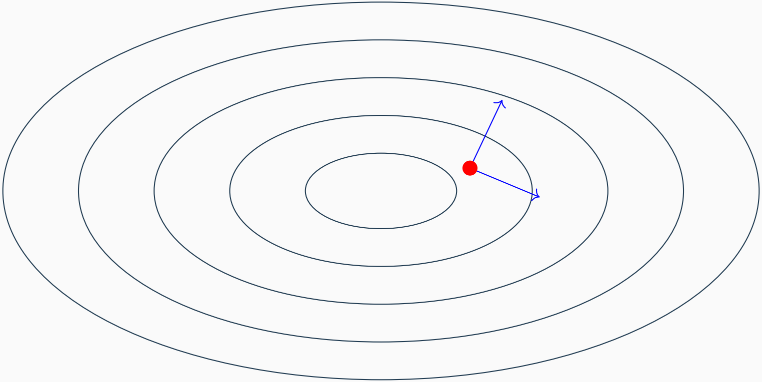

Remaining question

Recall that

Theorem (Gratton et al. 2015)

If $D_k = \{d_1, \ldots, d_m\}$, *where* $d_i \overset{\text{i.i.d.}}{\sim}$ *Unif*$(\mathcal{S}^{n-1})$, *then RDS is convergent if*

$$m \; > \; \log_2 \left(1 - \frac{\log \theta}{\log \gamma} \right).$$

Reminder: θ *is shrinking factor,* γ *is expanding factor.*

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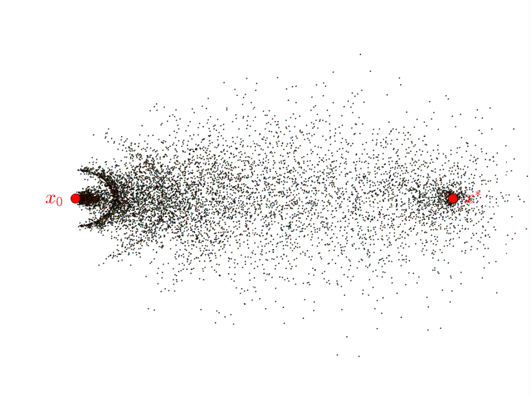

Non-convergence

Motivation: non-convergence analysis matters

Many well-known algorithms have non-convergence analysis.

- S. Reddi, S. Kale, and S. Kumar. On the convergence of Adam and beyond. In Y. Bengio, Y. LeCun, T. Sainath, I. Murray, M. Ranzato, and O. Vinyals, editors, *International Conference on Learning Representations (ICLR 2018)*. Curran Associates, Inc., 2018.
- C. Chen, B. He, Y. Ye, and X. Yuan. The direct extension of ADMM for multi-block convex minimization problems is not necessarily convergent. *Math. Program.*, 155:57-79, 2016.
- W. Mascarenhas. The divergence of the BFGS and Gauss Newton methods. *Math. Program.*, 147:253-276, 2014.
- ...

Practically meaningful: guide the choice of algorithmic parameters

Recall cosine measure

Definition (Cosine measure)

Cosine measure for a finite set of nonzero vectors $D \subseteq \mathbb{R}^n$ w.r.t. a given vector $v \in \mathbb{R}^n$:

$$\mathrm{cm}(D, v) \; = \; \max_{d \in D} \frac{d^\top v}{\|d\|\|v\|}.$$

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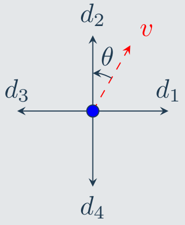
$$\mathrm{cm}(D, v) \ = \ \cos\theta$$



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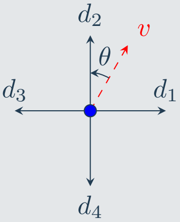

Measure the ability that "D approximates v"

Establishment of non-convergence

p-probabilistically κ-descent

$$\mathbb{P}\left(\mathrm{cm}\left(D_k, -g_k\right) \geq \kappa \mid D_0, \ldots, D_{k-1}\right) \geq p \quad \text{for each } k \geq 0.$$

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q-probabilistically ascent

$$\mathbb{P}\left(\mathrm{cm}\left(D_k, -g_k\right) > 0 \mid D_0, \ldots, D_{k-1}\right) \leq q \quad \text{for each } k \geq 0.$$

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$$\begin{tikzcd}
\mathrm{cm}\left(D_k,-g_k\right) \le 0 \arrow[rr] & & \text{No descent direction} \arrow[d, "f \text{ convex}"] \\
\alpha_k \text{ shrinks} & & f(x_k+\alpha_k d_k) \geq f(x_k)\ \forall d_k \in D_k \arrow[ll]
\end{tikzcd}$$

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non-convergence for convex functions

$\Downarrow$

non-convergence in general

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$$x_0 \;\notin\; \bar{\mathcal{B}}(x^*, \textstyle\sum_{k=0}^{\infty} \alpha_k)$$

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⬇

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⬇

$$\mathbb{P}\,(\text{Convergence}) \;<\; 1$$

Key for analysis

- Define indicator function $Y_k = \mathbb{1}_{\{\mathrm{cm}(D_k, -g_k) > 0\}}$
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Under q-probabilistically ascent assumption, can we find a constant ζ such that

$$\mathbb{P}\left(\sum_{k=1}^{\infty} U_k < \zeta \right) \; > \; 0?$$

Main results

Assumption

$$\mathbb{P}\left(\mathrm{cm}\left(D_k, -g_k\right) \leq 0 \mid D_0, \ldots, D_{k-1}\right) \;\geq\; q \;>\; q_0 \quad \text{for each } k \geq 0,$$

where $q_0 = 1 - p_0 = \log\gamma / \log(\theta^{-1}\gamma)$.

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Note that $\sum_{k=1}^{\infty} U_k = \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \gamma^{Y_i}\theta^{1-Y_i} \geq \theta/(1-\theta)$

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Tightness

Almost zero gap

Let $D_k = \{d_1, \ldots, d_m\}$, where $d_i \overset{\text{i.i.d.}}{\sim}$ Unif$(\mathcal{S}^{n-1})$.

Recall that RDS is convergent if

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Natural question:

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Answer: NO

Example

We assume

- $f : \mathbb{R}^n \to \mathbb{R}$ be L-smooth and strongly convex,
- $\theta = 1/2$ and $\gamma = 2$, $\Rightarrow q_0 = 1/2$
- $D_k = \{g_k/\|g_k\|\}$ or $D_k = \{-g_k/\|g_k\|\}$ with probability $1/2$, respectively,

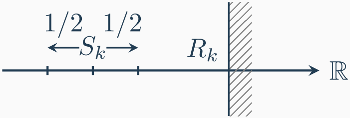
then we have

$$\mathbb{P}\left(\lim_{k\to\infty} \|g_k\| = 0\right) \;=\; 1.$$

Transformed to a special random walk problem

Two stochastic processes
$(R_k)_{k\in\mathbb{N}}$ uniformly upper bounded, $(S_k)_{k\in\mathbb{N}}$ a special random walk satisfying:

When $S_k < R_k$,



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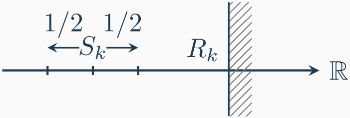

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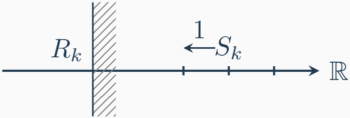

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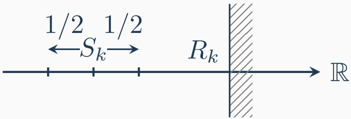

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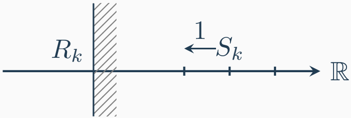

Does $\|g_k\| \to 0$?

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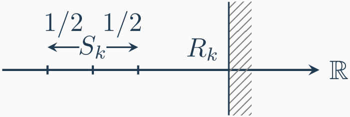

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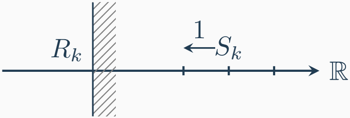

Does S_k go beyond the “wall” R_k i.o. with probability 1?

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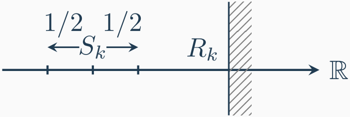

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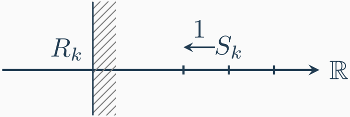

Does S_k go beyond the “wall” R_k i.o. with probability 1? YES

Call for ideas!

Two remaining interesting questions

- RDS converges or not when $\log_2(1 - \log\theta / \log\gamma) \in \mathbb{N}_+$,

  especially when $\gamma = 1/\theta = 2$ and $m = 1$.
- Estimate the CDF of $\sum_{k=1}^{\infty} U_k$

$$F(x) = \mathbb{P}\left(\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}\gamma^{Y_i}\theta^{1-Y_i} \le x\right),$$

where $Y_i \overset{\text{i.i.d.}}{\sim} \text{Bern}(p)$.

Thank you!

References I

- Conn, A. R., Scheinberg, K., and Vicente, L. N. (2009). *Introduction to Derivative-Free Optimization*. Vol. 8. MOS-SIAM Ser. Optim. Philadelphia: SIAM.
- Durrett, R. (2010). *Probability: Theory and Examples*. Fourth. Camb. Ser. Stat. Probab. Math. Cambridge: Cambridge University Press.
- Gratton, S. et al. (2015). “Direct search based on probabilistic descent”. *SIAM J. Optim.* 25, pp. 1515–1541.