Randomized Direct Search for Derivative-free Optimization

AMA613

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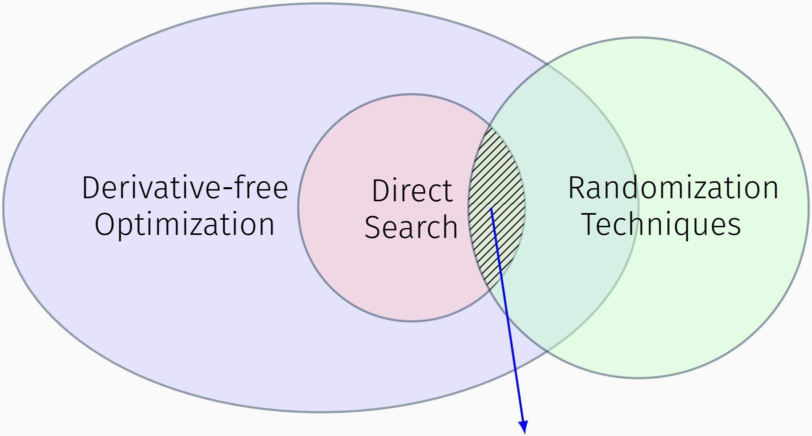
Big Picture



Our final goal in this talk
Randomized Direct Search

What is DFO and Why?

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- A branch of optimization
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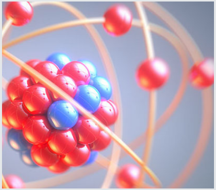
Example

$$x \in \mathbb{R}^n \longrightarrow \boxed{f : \mathbb{R}^n \to \mathbb{R}} \longrightarrow f(x)$$

Wide Application and Popularity

Applications



Nuclear Physics

Machine Learning

Cosmology

Wide Application and Popularity

- Powell's conjugate direction method (1964)

An efficient method for finding the minimum of a function of several variables without calculating derivatives
MJD Powell
The computer journal, 1964 • academic.oup.com
☆ Save Cite Cited by 5969 Related articles All 4 versions

- Nelder-Mead simplex method (1965)

A simplex method for function minimization
JA Nelder, R Mead - The computer journal, 1965 - academic.oup.com
☆ Save Cite Cited by 37851 Related articles All 16 versions Web of Science: 20486

Basic Assumptions

In this talk, we solve the unconstrained problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

where

- ∇f is Lipschitz continuous with constant ν, cannot be evaluated,
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The fewer, the better.

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- Sample points based on a finite direction set
- Make decisions by simple comparison (no explicit models)

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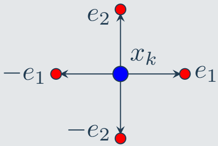
$$D = \{e_1, -e_1, e_2, -e_2\}$$



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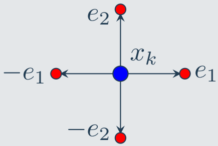

In this talk, direction sets only contain unit vectors in $\mathbb{R}^n$.

Algorithm 1: Direct Search with sufficient decrease

Input: $x_0 \in \mathbb{R}^n$, $\alpha_0 \in (0, \infty)$, $0 < \theta < 1 \leq \gamma$.

for $k = 0, 1, \ldots$ **do**

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Everything almost clear, except "select D_k".

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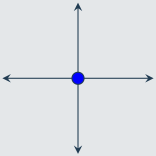

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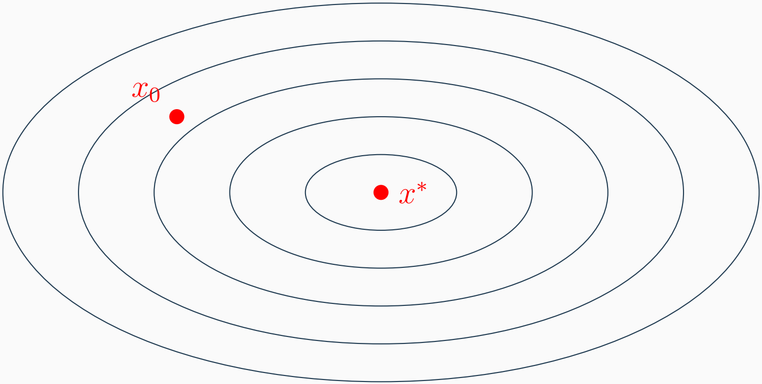

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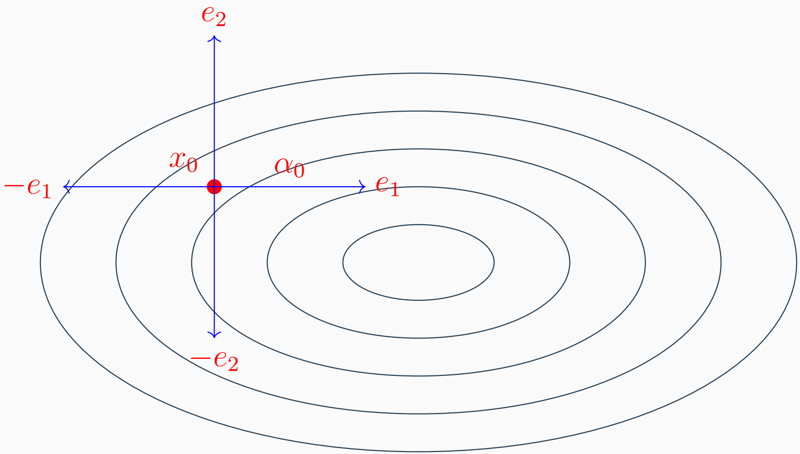

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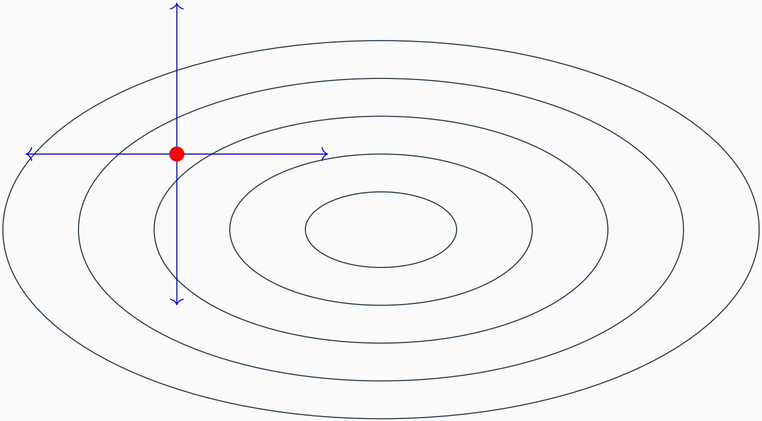

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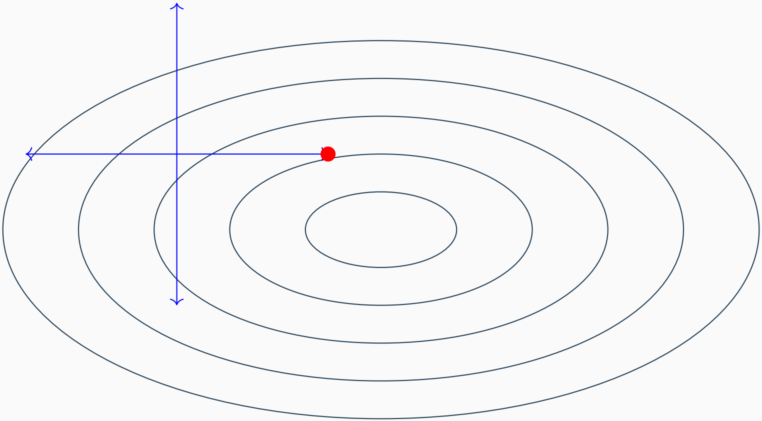

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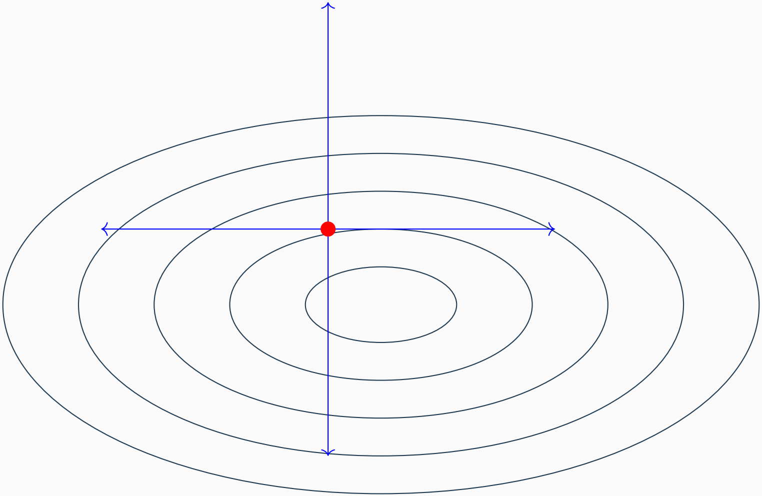

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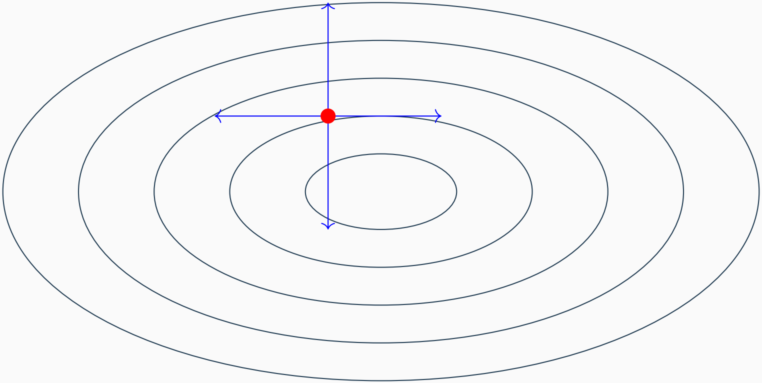

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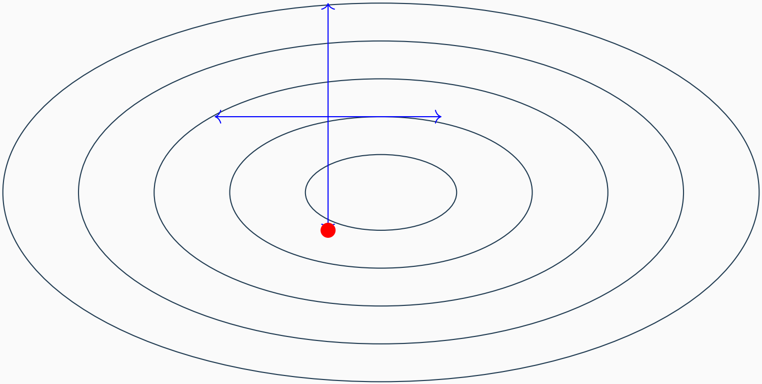

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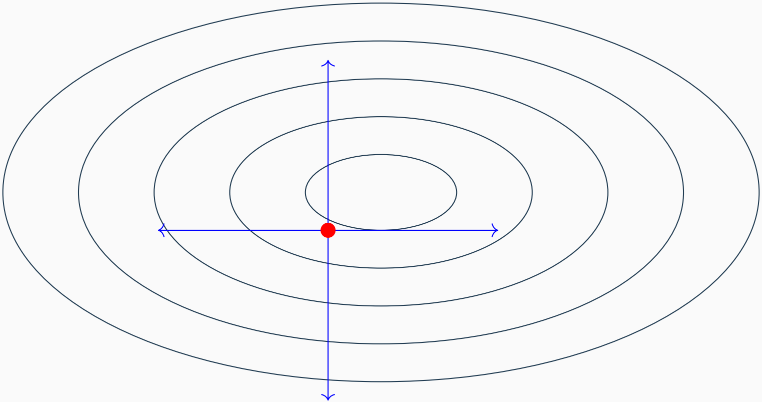

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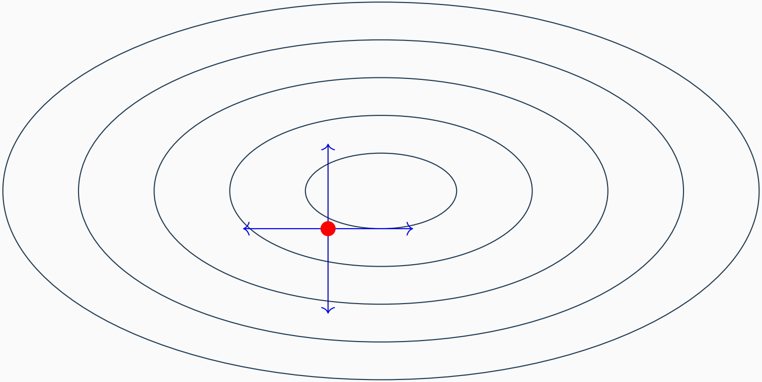

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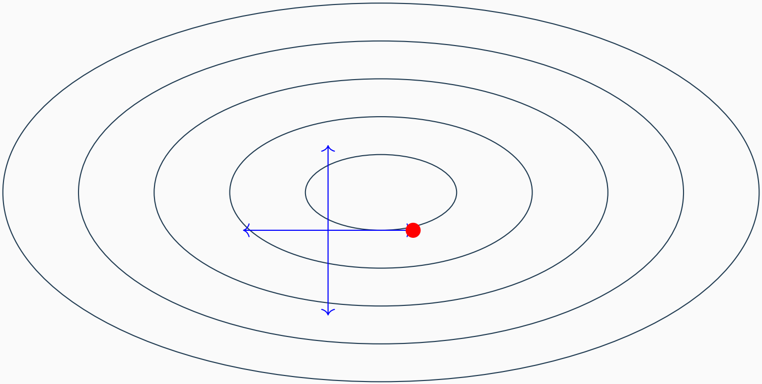

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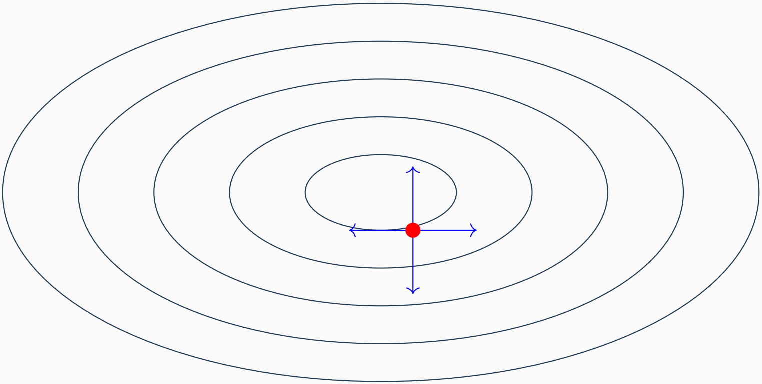

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Cosine measure

Cosine measure for a finite set of nonzero vectors $D \subseteq \mathbb{R}^n$:

$$\mathrm{cm}(D) \;=\; \min_{v \in \mathbb{R}^n \setminus \{0\}} \mathrm{cm}(D, v) \;=\; \min_{v \in \mathbb{R}^n \setminus \{0\}} \max_{d \in D} \frac{d^\top v}{\|d\| \|v\|}.$$

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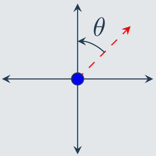
Example

$$\mathrm{cm}(D) \;=\; \cos\theta \;=\; \frac{\sqrt{2}}{2}$$



Convergence of Deterministic Direct Search

Theorem

If we have

$$\mathrm{cm}(D_k) \;\geq\; \kappa > 0 \qquad \textit{for all } k \geq 0,$$

then

$$\liminf_{k\to\infty} \|g_k\| \;=\; 0.$$

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Each direction set D_k should be a "good enough" PSS.

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Idea

Reduce $|D|$ from $\mathcal{O}(n)$ to $\mathcal{O}(1)$ by randomization techniques

Algorithm 2: Randomized Direct Search

Input: $x_0 \in \mathbb{R}^n$, $\alpha_0 \in (0, \infty)$, $0 < \theta < 1 \leq \gamma$.

for $k = 0, 1, \dots$ **do**

  Randomly generate $\mathfrak{D}_k \subset \mathbb{R}^n$.

  if $f(x_k + \alpha_k d) < f(x_k) - \rho(\alpha_k)$ for some $d \in \mathfrak{D}_k$ **then**

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Notations for random variables or random vectors

$D_k \Rightarrow \mathfrak{D}_k$, $d \Rightarrow \mathfrak{d}$, $x_k \Rightarrow X_k$, $\alpha_k \Rightarrow A_k$, $g_k \Rightarrow G_k$

Assumptions on Randomness

Actually, what we need is not $\mathrm{cm}(D_k) \geq \kappa$ but $\mathrm{cm}(D_k, -G_k) \geq \kappa$.

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Definition (p-probabilistically κ-descent)

$(\mathfrak{D}_k)_{k\geq 0}$ is said to be p-probabilistically κ-descent, if

$$\mathbb{P}\left(\mathrm{cm}(\mathfrak{D}_k, -G_k) \geq \kappa \mid \mathfrak{D}_0, \ldots, \mathfrak{D}_{k-1}\right) \;\geq\; p \quad \text{for each } k \geq 0.$$

Intuition:

each $\mathfrak{D}_k$ is “good enough with lower-bounded probability”,

no matter what happened before

Convergence of Probabilistic Direct Search

Theorem (Gratton et al. 2015)

If $(\mathfrak{D}_k)_{k\geq 0}$ *is* p*-probabilistically* κ*-descent with*

$$p = \log\theta / \log(\gamma^{-1}\theta),$$

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Complexity: $\mathcal{O}(n\varepsilon^{-2})$ with overwhelmingly high probability

Typical Choice in Practice

Corollary (Gratton et al. 2015)

If $\mathfrak{D}_k = \{\mathfrak{d}_1, \ldots, \mathfrak{d}_m\}$, where $\mathfrak{d}_1, \ldots, \mathfrak{d}_m$ are independent random vectors uniformly distributed on the unit sphere in $\mathbb{R}^n$, then the algorithm is convergent if

$$m \; > \; \log_2\left(1 - \frac{\log\theta}{\log\gamma}\right).$$

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"We must know. We will know."

Takeaways

- Try DFO solvers when you are dealing with tough problems!
- Try randomization techniques to reduce complexity!
- Try scanning the following two QR codes!

My Personal Website

Video on DFO

Thank you!

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