OptTrot

Optimized Trotterized circuit library

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1 Introduction

In combinatorics, Sperner's lemma is about a specific coloring result of triangulation, Sperner coloring. It was published in 1928 by Emanuel Sperner[1], and there have been various studies and applications. Since, by choosing a proper coloring function and triangulation space, we can adopt the lemma in various fields, and it yields many non-trivial results. For example using the lemma, we can prove well-known equivalent theorems in algebraic topology and set theory[1]. However, the lemma has lot of potential not only for the theoretical results, but also for a practical computational aspect.

In this assignment, focusing on computational method, we overlook an application of Sperner's lemma in root finding method on complex plane. In the right Fig (**??**), we can see a basic process of the algorithm. When 3 colors guarantees the existence of roots inside the triangle. We can make a finer sub-triangles to converges the root. Even we reduce the triangle arbitrary size, by the lemma there always exists a rainbow triangle containing a root. More precisely, using the lemma, we can simply prove Fundamental Theorem of Algebra(FTA)[2]. In addition, we can find such roots for the given polynomial, using a same process of the proof.

2 Terminology and basic theorems

Let S be n-simplex determined by $n+1$ number of vertices, $\{v_i\}_{i=1}^{n+1}$ in $\mathbb{R}^n$.

Definition 1 ***Sperner coloring***

A given simplex S of triangulation P with V_p inner vertices,

- *$v \in \partial S$ are all distinct.*

[1] These are *Brouwer fixed point theorem*, and *KKM theorem*.

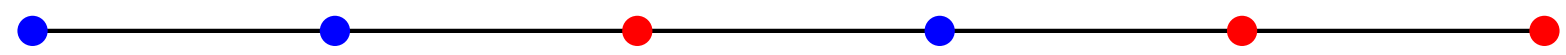
- *v in facet of S are colored one of the corner vertices of the same facet.*
- *$v \in V_p$ vertex could have any color of vertex color in ∂S.*

Lemma 1 ***Sperner's lemma***

About any Sperner's colored triangulation of the given n-simplex S, (Weak): There is a rainbow n-simplex in P. (Strong): There are odd number of rainbow n-simplex in P.

Proof

$n = 1$ case: n-simplex be a line segment. Suppose that the two end colors to be -1:blue, and 1:red.



If we set a function $f : V \to \{-1, 1\}$, then by the intermediate value theorem, there are odd number of root so that the odd number of rainbow triangle exist.

$n > 1$ case: In the general case, the proof use double counting method. Let

- $R :=$ # of rainbow n-simplices of P.
- $Q :=$ # of n-simplices of P having all $[1, \ldots n]$ as its color, but $n + 1$.
- $X :=$ # of $(n-1)$-simplices of P having all of $[1, \ldots, n]$ as its color that contained in ∂S
- $Y :=$ # of $(n-1)$-simplices of P having all of $[1, \ldots, n]$ as its color that contained in the interior of S

Each P and Q attached to ∂S has $[1, \ldots, n]$ color exactly has one X. Furthermore, P and Q in interior of S, each Y lies between two elements in (P,Q) or (Q, Q).

Thus, the next hold true

$$R + 2Q = X + 2Y \tag{1}$$

However, X is odd, since it is R of $n - 1$ case, thus R is odd and by the induction it holds for all $n > 1$. Done.

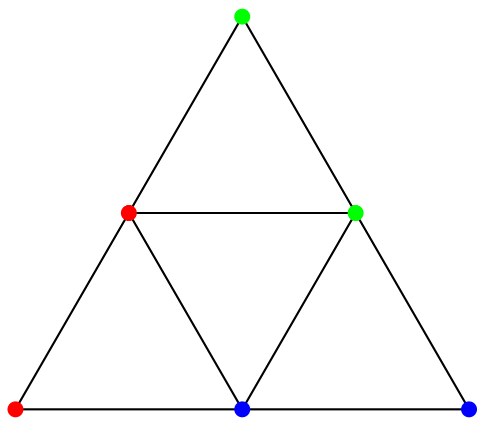


Figure 1: Example of Sperner's lemma in $\mathbb{R}^2$.

Huang generalized the lemma with permutation function N[2], as

Definition 2 *Let $v_i, i \in [k]$ be non-negative integers,*

$$N(v_1, v_2, \dots, v_k) = \begin{cases} 1 & \textit{It is a permutation and \# of cross is even.} \\ -1 & \textit{It is a permutation and \# of cross is odd.} \\ 0 & \textit{Not a permuation.} \end{cases} \tag{2}$$

Lemma 2 ***Generalized Sperner's lemma*** *Let C be a labeled k-chain of k-simplices of labels $\{0, \dots k\}$. Then,*

$$N(C) = (-1)^k N(\partial(C)) \tag{3}$$

If we let C be a triangulized n-simplex of Sperner's coloring,

3 Roots of complex polynomial

3.1 Fundamental theorem of algebra

Theorem 1 ***FTA*** *Every polynomial of degree n in $\mathbb{C}$ has exactly n roots on $\mathbb{C}$.*

To prove FTA, we need two concepts from complex analysis.

Definition 3 ***Winding Number[3]***

Winding number of contour C about point z_0 is a positive integer $N(C, z_0)$ that

$$N(C, z_0) = \frac{1}{2\pi i} \oint_C \frac{dz}{z - z_0} \tag{4}$$

which represents a number that curve pass around a point z_0.

For example, $f(z) = z^n$ about origin $N(f, 0) = n$. Using $z = re^{i\theta}$, $z^n = r^n e^{in\theta}$. Cauchy's argument theorem

Theorem 2 ***Argument theorem[4]***

For a given meromorphic function, f, on complex plane, next identity hold for a given closed contour, C

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = Z - P \tag{5}$$

where, Z is a # of roots inside C and P is a # of poles in the C.

If the meromorphic function, f, was a finite degree complex polynomial, since, every finite degree polynomial has no pole,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = Z \tag{6}$$

3.2 Proof of FTA using Sperner's lemma

Let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ be a monic polynomial where $a_i \in \mathbb{C}$.

3.2.1 Existence of n number of rainbow simplex

For a large enough disk, D whose center is origin of complex plane, we can make a triangulation of D, treating $\mathbb{C}$ as $\mathbb{R}^2$. Since, D is large near the boundary $p(z) \sim z^n$. Each vertex is colored by labeling with $\phi(z) = j$ if $z \in R_j, j \in \{0, 1, 2\}$,

$$R_j := \{z \in \mathbb{C} \,|\, \frac{2\pi}{3} j \leq \arg(z) \leq \frac{2\pi}{3}(j+1)\} \tag{7}$$

Now, z^n winds the boundary of P along origin n times. Consequently, there exist exactly n number of (0, 1)-simplex on ∂P. By the definition of N, Def (2), only (0, 1)-simplex increases N value, so that

$$N(\partial P) = n \tag{8}$$

By Lemma 2, $N(P) = n$, consequently there are exactly n number of rainbow simplex in P.

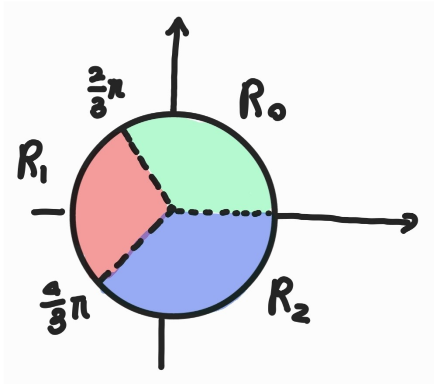


Figure 2: Phase of $R_j, j \in \{0, 1, 2\}$

3.2.2 Rainbow simplex to root

Since, P is fine triangulation of D, for the rainbow simplex in interior of P and their vertex z_j, z_{j+1}, z_{j+2}, $|p(z_q) - p(z_p)| < \frac{1}{k}, \forall p, q \in j, j+1, j+2$, for arbitrary $k > 0$.

Claim $|p(z_p)| < \frac{2}{\sqrt{3}k}$

Proof

WOR, $z_0 \in R_0$ and so do z_1, z_2 to be in different $R_i, i \in \{0, 1, 2\}$, then since R_i are separated by $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, at least, two of three, z_0, z_1, z_2 have phase greater or equal than $\pi/3$. Let such points z_0 and z_1. The subspace spanned by z_1 with minimizing distance to z_0, then the maximum distance is a $|z_0| \sin(\pi/3)$.

$$|z_0| \sin(\frac{\pi}{3}) \leq |z_0| |\arg(z_0) - \arg(z_1)| \leq |z_0 - z_1| \frac{1}{k} \tag{9}$$

Thus, $|z_0| \leq 2/(\sqrt{3}k)$.

Polynomial has continuity everywhere, the finer triangulation makes z_p converge to specific point. Since, every finite degree polynomial has no pole everywhere, the converged points are all root of the polynomial. Done.

3.3 Root finding

From the above proof, we found that the proper 3 colors, Eq (7), to be converged to root by finer triangulation. During the proof, we used arbitrary finite degree polynomial, however, we can use any complex valued function to find a root. Since, the convergence only requires 3 colored points and triangulation of the domain. Any given domain and complex valued function we can search roots by the above method in the rainbow simplex to root.

The remained part is how to find a proper triangulation method for the search, and verify the result whether it is a pole or root. The verification could be archived by theorem 2, however, triangulation methods are various and pros and cons also differ by each method. The selection of the method requires additional

Summary the Sperner's based root finding algorithm is

- f: complex valued function.
- $\epsilon > 0$: given tolerance.

1. Find 3 color points by Eq (7).
2. Make triangulation of the triangle consist of the three points.
3. Mark the inner vertex by calculating the $f(z)$ and Eq (7).
4. Go to 1 and repeat until the $|f(z_i) - f(z_j)| < \epsilon$ for all vertex of a rainbow triangle.
5. Verify the cell point using Theorem 2.

4 Limitation and further application

In fact, the above method had been noticed for a long time, at least before 2005, however, there has been no practical implementation algorithm based on Sperner's lemma. In 2005, Huang mentioned the practical aspect of the method. By him, Sperner's lemma based method requires too much cost to achieve a desired tolerance error. It is not only required for triangulation of the given region, but also we have to calculate all the function value on the triangulation vertices.

Despite the fact of the inefficient, the algorithm does not require separated calculation of real and imaginary part calculation. For example, there was a study about root-finding algorithm of general complex valued function with given domain by triangulation[5]. The method required real and imaginary part of the function separately to generate the Re(f) =0 and Im(f)=0 line. The candidate points are overlapped points of the two lines. Since, Sperner's lemma based algorithm does not have to calculate those two zero lines, combine those two method could improve root finding routine for general functions.

5 Conclusion

In this assignment, we explored how Sperner's lemma can be used to find roots of polynomials on the complex plane. This approach shows how combinatorial theories like Sperner's lemma can be applied beyond theoretical studies, extending into practical computational algorithms. Although using Sperner's lemma for root finding in complex polynomials is innovative, it is not very practical due to the high computational resources it requires. However, this method offers a unique perspective because it doesn't require splitting the function into real and imaginary parts. This could potentially simplify some computational processes.

Future studies could look into combining Sperner's lemma with other numerical methods to create more efficient algorithms. This might help overcome the current limitations and make the lemma more useful for practical applications.

References

[1] Emanuel Sperner. Neuer beweis für die invarianz der dimensionszahl und des gebietes. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 6:265–272, 1928.

[2] Jonathan Huang. On the sperner lemma and its applications. 2004.

[3] Eric W Weisstein. Contour winding number.

[4] Yong Xu, Gang Chen, Jie Chen, and Li Qiu. Argument principle and integral relations: Hidden links and generalized forms. *IEEE Transactions on Automatic Control*, 68(3):1831–1838, 2023.

[5] Piotr Kowalczyk. Complex root finding algorithm based on delaunay triangulation. *ACM Trans. Math. Softw.*, 41(3), jun 2015.