Continuous-Time Dynamics

- Most general/generic for smooth system:

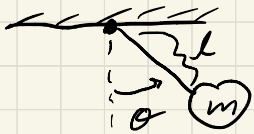
$$\dot{x} = \underbrace{f}_{\text{"dynamics"}}(\underbrace{x}_{\text{"state"} \in \mathbb{R}^n}, \underbrace{u}_{\text{"input"} \in \mathbb{R}^m})$$

- For a mechanical system $x = \begin{bmatrix} q \\ v \end{bmatrix}$

"configuration" (not always a vector)

"velocity"

- Example (Pendulum):



$$\boxed{ml^2\ddot{\theta} + mgl\sin(\theta) = \underbrace{\tau}_{u}}$$

$$q = \theta, \quad v = \dot{\theta}$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) + \frac{1}{ml^2}u \end{bmatrix}}_{f(x,u)}$$

$$q \in S^1 \text{ (circle)}, \quad x \in S^1 \times \mathbb{R} \text{ (cylinder)}$$

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* Control-Affine Systems

$$\dot{x} = \underbrace{f_0(x)}_{\text{drift}} + \underbrace{B(x)}_{\text{input Jacobian}}u$$

- most systems can be put in this form
- Pendulum:

$$f_0(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

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* Manipulator Dynamics:

$$\underbrace{M(q)}_{\text{"Mass Matrix"}}\dot{v} + \underbrace{C(q,v)}_{\substack{\text{"Dynamic Bias"} \\ \text{(Coriolis + Gravity)}}} = \underbrace{B(q)}_{\text{"Input Jacobian"}}u$$

$$\underbrace{\dot{q} = G(q)\,v}_{\text{"Velocity Kinematics"}} \qquad \dot{x} = f(x,u) = \begin{bmatrix} G(q)\,v \\ -M(q)^{-1}(B(q)u - C) \end{bmatrix}$$

- Pendulum:

$$M(q) = ml^2, \quad C(q,v) = gl\sin(\theta), \quad B = I, \quad G = I$$

- All mechanical systems can be written in this form.

- This is just a way of re-writing the Euler-Lagrange equation for:

$$L = \frac{1}{2}v^T M(q)\,v - U(q)$$

(Bonus Points for showing this)

* Linear Systems

$$\dot{x} = A(t)x + B(t)u$$

- Called "time invariant" if $A(t) = A$, $B(t) = B$
- Called "time varying" otherwise
- Super important in control
- We often approximat nonlinear systems with linear ones:

$$\dot{x} = f(x,u) \Rightarrow A = \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial u}$$

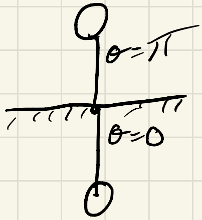
Equilibria:

- A point where the system will "remain at rest"

$$\Rightarrow \dot{x} = f(x,u) = 0$$

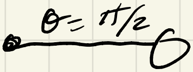
- Algebraically, roots of the dynamics
- Pendulum:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \dot{\theta} = 0 \\ \theta = 0, \pi \end{matrix}$$



* First Control Problem:

- Can I move the equilibria?



$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -g/\ell \sin(\pi/2) + 1/m\ell^2 \, u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{m\ell^2} u = \frac{g}{\ell} \underbrace{\sin(\pi/2)}_{= 1} \Rightarrow u = mg\ell$$

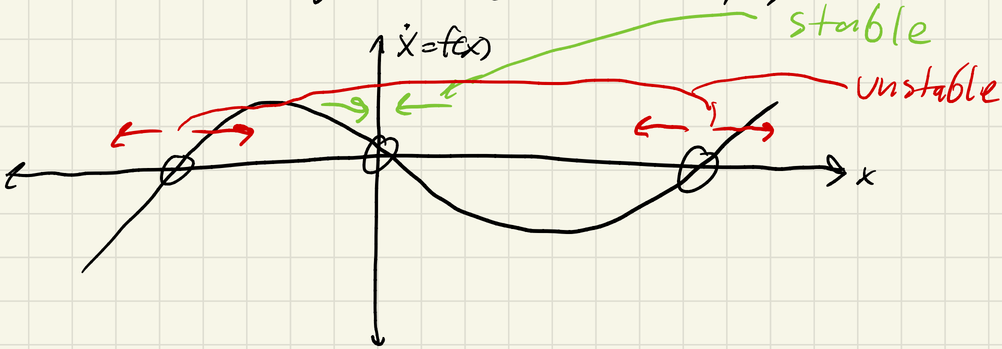
- In general, we get a root-finding problem in u:

$$f(x^*, u) = 0$$

Stability of Equilibria:

- When will we stay "near" an equilibrium point under perturbations.

- Look at a 1D system ($x \in \mathbb{R}$)



$\frac{\partial f}{\partial x} < 0 \Rightarrow$ stable , $\frac{\partial f}{\partial x} > 0 \Rightarrow$ unstable

- In higher dimensions:

  $\frac{\partial f}{\partial x}$ is a Jacobian matrix

  - Take an eigendecomposition $\Rightarrow$ decouple into n 1D systems

$$\mathrm{Re}\left[\mathrm{eig}\left(\frac{\partial f}{\partial x}\right)\right] < 0 \Rightarrow \text{stable}$$

otherwise $\Rightarrow$ unstable

- Pendulum:

$$f(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -g/l\cos(\theta) & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=\pi} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} \Rightarrow \mathrm{eig}\left(\frac{\partial f}{\partial x}\right) = \pm\sqrt{\frac{g}{l}}$$

$\Rightarrow$ unstable

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=0} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \Rightarrow \mathrm{eig}\left(\frac{\partial f}{\partial x}\right) = \pm i\sqrt{\frac{g}{l}}$$

- Pure imaginary case is called "marginally stable"

  $\Rightarrow$ Undamped oscillations

- Add damping (e.g. $u = -K_d \dot{\theta}$) results in strictly negative real part.