# Continuous-Time Dynamics

- Most general/generic for smooth system:

$$\dot{x} = f(x, u)$$

"input" $\in \mathbb{R}^m$

"dynamics"   "state" $\in \mathbb{R}^n$

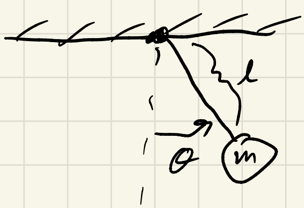
- For a mechanical system:

$$x = \begin{bmatrix} q \\ v \end{bmatrix}$$

*) "configuration" (not always a vector)

"velocity"

- Example (Pendulum):



$$m l^2 \ddot{\theta} + m g l \sin(\theta) = \underbrace{\tau}_{u}$$

$$q = \theta, \quad v = \dot{\theta}$$

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) + \frac{1}{ml^2} u \end{bmatrix}}_{f(x,u)}$$

$q \in S^1$ (circle), $\quad x \in S^1 \times \mathbb{R}$ (cylinder)

* Control-Affine Systems

$$\dot{x} = \underbrace{f_0(x)}_{\text{"drift"}} + \underbrace{B(x)}_{\text{"input Jacobian"}} u$$

- most systems can be put in this form
- Pendulum:

$$f_0(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix}, \quad B(x) \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

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* Manipulator Dynamics:

$$\underbrace{M(q)}_{\text{"Mass matrix"}}\dot{v} + \underbrace{C(q,v)}_{\substack{\text{"Dynamic Bias"} \\ \text{(Coriolis + Gravity)}}} = \underbrace{B(q)}_{\text{"Input Jacobian"}} u + F$$

"external forces"

$$\underbrace{\dot{q} = G(q)v}_{\text{"Velocity Kinematics"}} \qquad \dot{x} = f(x,u) = \begin{bmatrix} G(q)v \\ M(q)^{-1}(B(q)u - C) \end{bmatrix}$$

- Pendulum:

$$M(q) = ml^2, \quad C(q,v) = gl\sin(\theta), \quad B = I, \quad G = I$$

- All mechanical systems can be written like this
- This is just a way of re-writing the Euler-Lagrange equation for:

$$L = \underbrace{\tfrac{1}{2}v^T M(q) v}_{\text{kinetic energy}} - \underbrace{U(q)}_{\text{Potential energy}}$$

$ Linear Systems

$$\dot{x} = A(t)x + B(t)u$$

- Called "time invariant" if $A(t) = A$, $B(t) = B$
- Called "time varying" otherwise
- Super important in control
- We often approximate nonlinear systems with linear ones:

$$\dot{x} = f(x,u) \Rightarrow A = \frac{\partial f}{\partial x}, \quad B = \frac{\partial f}{\partial u}$$

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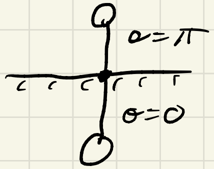
Equilibria:

- A point where the system will "remain at rest"

$$\Rightarrow \dot{x} = f(x,u) = 0$$

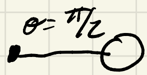
- Algebraically, roots of the dynamics
- Pendulum:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{\ell}\sin(\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \dot{\theta} = 0 \\ \theta = 0, \pi \end{matrix}$$



## * First Control Problem

- Can I move the equilibria?




$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ -g/l \sin(\pi/2) + 1/ml^2 u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{ml^2} u = \frac{g}{l} \underbrace{\sin(\pi/2)}_{1} \Rightarrow u = mgl$$

- In general, we get a root-finding problem in $u$:

$$f(x^*, u) = 0$$

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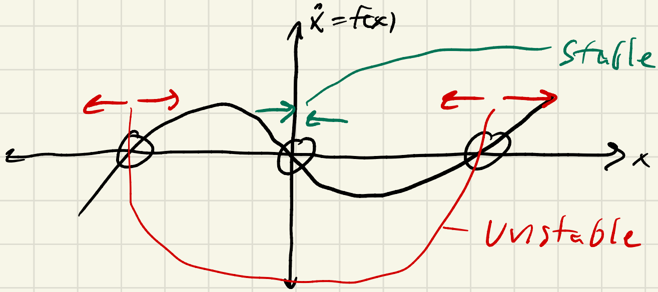
### Stability of Equilibria:

- When will we stay "near" an equilibrium point under perturbations.

- Look at a 1D system $(x \in \mathbb{R})$




$$\frac{\partial f}{\partial x} < 0 \Rightarrow \text{stable} \quad , \quad \frac{\partial f}{\partial x} > \Rightarrow \text{unstable}$$

- In higher dimensions:

$\frac{\partial f}{\partial x}$ is a Jacobian matrix

- Take an eigendecomposition $\Rightarrow$ decouple into n 1D systems

$$\mathrm{Re}\left[\mathrm{eig}\left(\frac{\partial f}{\partial x}\right)\right] < 0 \Rightarrow \text{stable}$$

$$\text{otherwise} \Rightarrow \text{unstable}$$

- Pendulum:

$$f(x) = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin(\theta) \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -g/l\cos(\theta) & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=\pi} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} \Rightarrow \mathrm{eig}\left(\frac{\partial f}{\partial x}\right) = \pm\sqrt{\frac{g}{l}}$$

$$\Rightarrow \text{unstable}$$

$$\left.\frac{\partial f}{\partial x}\right|_{\theta=0} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \Rightarrow \mathrm{eig}\left(\frac{\partial f}{\partial x}\right) \pm i\sqrt{\frac{g}{l}}$$

- Pure imaginary case is called "marginally stable"

$\Rightarrow$ undamped oscillations

- Add damping (e.g. $u = -K_d\dot{\theta}$) results in strictly negative real part.