Last Time:

- DDP details
- Constraints

Today:

- Minimum/free time problems
- Direct Tajectory Optimization
- Sequential Quadratic Programming
- Direct Collocation

✱ Handling Free / Minimum Time Problems:

$$\min_{x(t),\,u(t),\,T} \quad J = \int_0^T 1 \, dt$$

$$\text{s.t.} \quad \dot{x} = f(x, u)$$

$$x(T) = x_{goal}$$

$$u_{min} \le u(t) \le u_{max}$$

} minimum time problem

- We don't want to change the number of knot points
- Make h (time step) from RK a control input:

$$x_{n+1} = f(x_n, \tilde{u}_n), \quad \tilde{u}_n = \begin{bmatrix} u_n \\ h_n \end{bmatrix}$$

- Also scale the cost by h e.g.

$$J(x,u) = \sum_{n=1}^{N-1} h_n \ell(x_n, u_n) + \ell_N(x_N)$$

- Always nonlinear/nonconvex even if the dynamics are linear.

- Requires constraints on h. Otherwise the solver can "cheat physics" by making h very large or negative to exploit discretization error.

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* Direct Traj Opt

- Basic strategy: Discretize and "transcribe" continuous-time optimal control problem into a nonlinear program (NLP):

"Standard NLP"

$$\begin{cases} \min\limits_{x} & f(x) \leftarrow \text{objective (cost) function} \\ \text{s.t.} & c(x) = 0 \leftarrow \text{dynamics constraints} \\ & d(x) \leq 0 \leftarrow \text{other constraints} \end{cases}$$

- All functions assumed C^2 smooth

- Lots of off-the-shelf large-scale NLP solvers can be used to solve these problems.

- Most common: IPOPT (free), SNOPT (commercial), KNITRO (commercial).

- Common solution strategy: Sequential Quadratic Programming (SQP)

* Sequential Quadratic Programming

- Strategy: Use 2nd-order Taylor expansion of the Lagrangian and linearize $c(x)$, $d(x)$ to approximate NLP as a QP:

$$\min_{\Delta x} \quad f(x) + g^T \Delta x + \frac{1}{2} \Delta x^T H \Delta x$$

$$\text{s.t.} \quad c(x) + C\Delta x = 0$$

$$d(x) + D\Delta x \leq 0$$

where $H = \frac{\partial^2 L}{\partial x}$, $g = \frac{\partial L}{\partial x}$, $C = \frac{\partial c}{\partial x}$, $D = \frac{\partial d}{\partial x}$

$$L(x, \lambda, \mu) = f(x) + \lambda^T c(x) + \mu^T d(x)$$

- Solve QP to compute primal-dual search direction:

$$\Delta z = \begin{bmatrix} \Delta x \\ \Delta \lambda \\ \Delta \mu \end{bmatrix}$$

- Perform line search with merit function

- With only equality constraints, reduces to Newton's method on the Lagrangian:

$$\begin{bmatrix} H & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} -\frac{\partial L}{\partial x} \\ -c(x) \end{bmatrix}$$

KKT System

- Think of SQP as generalization of Newton to inequality constrained problems.

- Can use any QP solver to solve sub-problems but good implementations typically warm start using previous QP iteration.

- For good performance on traj opt problems taking advantage of sparsity in KKT system is crucial. SNOPT (sparse Nonlinear Optimizer) does this well.

- If inequalities are convex (e.g. conic) can generalize SQP to SCP (sequential convex programming) where inequalities are passed directly to the sub-problem solver.

- SCP is still an active research area.

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* Direct Collocation:

- So far we've used explicit RK methods:

$$\dot{x} = f(x,u) \rightarrow x_{next} = f(x_k, u_k)$$

- This makes sense if you're doing rollouts

- However, in a direct method where dynamics are enforced as equality constraints between knot points, this doesn't matter:

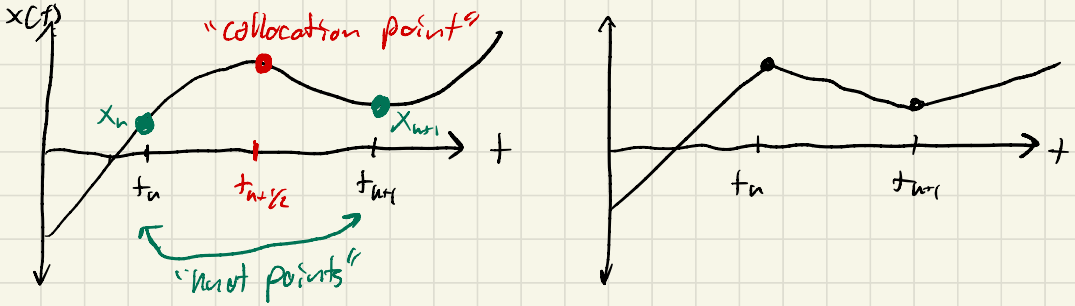
$$c_k(x_k, u_k, x_{k+1}, u_{k+1}) = 0$$

- Collocation methods represent trajectories with polynomial splines and enforce dynamics on spline derivatives.

- Classic DIRCOL algorithm uses cubic splines for state trajectories and piecewise linear interpolation for $u(t)$.

- Very high-order polynomials are sometimes used (e.g. spacecraft trajectories) but not common.

- DIRCOL Spline Approximations:



$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$

$$\Downarrow$$

$$\dot{x}(t) = c_1 + 2c_2 t + 3c_3 t^2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & h & h^2 & h^3 \\ 0 & 1 & 2h & 3h^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_n \\ \dot{x}_n \\ x_{n+1} \\ \dot{x}_{n+1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/h^2 & -2/h & 3/h^2 & -1/h \\ 2/h^3 & 1/h^2 & -2/h^3 & 1/h^2 \end{bmatrix} \begin{bmatrix} x_n \\ \dot{x}_n \\ x_{n+1} \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Evaluate at $t_{k+1/2}$:

$$x_{k+1/2} = x(t_k + h/2) = \frac{1}{2}(x_k + x_{k+1}) + \frac{h}{8}(\dot{x}_k - \dot{x}_{k+1})$$

$$= \frac{1}{2}(x_k + x_{k+1}) + \frac{h}{8}(f(x_k, u_k) - f(x_{k+1}, u_{k+1}))$$

(continuous time)

$$\dot{x}_{k+1/2} = \dot{x}(t_k + h/2) = -\frac{3}{2h}(x_k - x_{k+1}) - \frac{1}{4}(\dot{x}_k + \dot{x}_{k+1})$$

$$= -\frac{3}{2h}(x_k - x_{k+1}) - \frac{1}{4}(f(x_k, u_k) + f(x_{k+1}, u_{k+1}))$$

$$u_{k+1/2} = u(t_k + h/2) = \frac{1}{2}(u_k + u_{k+1})$$

- We can now enforce dynamics constraints!

$$c_i(x_k, u_k, x_{k+1}, u_{k+1}) =$$

$$f(x_{k+1/2}, u_{k+1/2}) - \left[-\frac{3}{2h}(x_k - x_{k+1}) - \frac{1}{4}(f(x_k, u_k) + f(x_{k+1}, u_{k+1}))\right]$$

$$= 0$$

(continuous dynamics)

- Note that only x_k, u_k are decision variables (not $x_{k+1/2}$, $u_{k+1/2}$)
- Called "Hermite-Simpson" integration.
- Achieves 3rd order integration accuracy like RK3
- Requires fewer dynamics calls than explicit RK3!

Explicit RK3:

$$f_1 = f(x_n, u_n)$$
$$f_2 = f(x_n + \tfrac{1}{2}h f_1, u_n)$$
$$f_3 = f(x_n + 2h f_1 - h f_2, u_n)$$

$$x_{n+1} = x_n + \frac{h}{6}(f_1 + 4f_2 + f_3)$$

$\Rightarrow$ 3 dynamics evals per step

Hermite-Simpson:

$$f(x_{n+1/2}, u_{n+1/2}) + \tfrac{3}{2h}(x_n - x_{n+1}) - \tfrac{1}{4}(f(x_n, u_n) + f(x_{n+1}, u_{n+1})) = 0$$

These get reused at adjacent steps!

$\Rightarrow$ Only 2 evals per time step!

- Since dynamics calls often dominate computational cost, this can be a ~50% savings.

- Example:
  - Acrobot w/ DIRCOL
  - Warm starting with dynamically infeasible guess can help a lot!