1. Is the matrix A^TA positive semi-definite? (hint, for A^TA to be positive semi-definite, $x^TA^TAx \geq 0$ for all x.

   (a) yes
   (b) no

   Solution: Yes, no matter what type of matrix A is. $x^TA^TAx = (Ax)^T(Ax) = \|Ax\|_2^2 \geq 0$.

2. Is the following linear system controllable?

$$x_{k+1} = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u_k \tag{1}$$

   (a) yes
   (b) no

   Solution: Yes, calculate the controllability matrix and see that it's rank 2.

3. Below is a finite-horizon LQR problem,

$$\begin{aligned} \min_{x_{1:N}, u_{1:N-1}} \quad & \sum_{i=1}^{N-1} \left[\frac{1}{2} x_i^T Q x_i + \frac{1}{2} u_i^T R u_i \right] + \frac{1}{2} x_N^T Q_f x_N && (2) \\ \text{st} \quad & x_1 = x_{\text{IC}}, && (3) \\ & x_{i+1} = A x_i + B u_i \quad \text{for } i = 1, 2, \ldots, N-1, && (4) \end{aligned}$$

   where $Q \succeq 0$, $R \succ 0$, and $[A, B]$ is controllable. We can solve this problem with either convex optimization that solves for (x, u) directly, or a Ricatti recursion that solves for an optimal policy $u_i = -K_i x_i$. Will these two solutions be equivalent?

   (a) yes
   (b) no

   Solution: Yes they are equivalent. This is the classic finite horizon LQR problem.

4. If we modify the finite-horizon LQR problem to incorporate bound constraints on the controls, we get the following problem:

$$\begin{aligned} \min_{x_{1:N}, u_{1:N-1}} \quad & \sum_{i=1}^{N-1} \left[\frac{1}{2} x_i^T Q x_i + \frac{1}{2} u_i^T R u_i \right] + \frac{1}{2} x_N^T Q_f x_N && (5) \\ \text{st} \quad & x_1 = x_{\text{IC}}, && (6) \\ & x_{i+1} = A x_i + B u_i \quad \text{for } i = 1, 2, \ldots, N-1, && (7) \\ & u_{min} \leq u_i \leq u_{max} \quad \text{for } i = 1, 2, \ldots, N-1. && (8) \end{aligned}$$

   If we take the Ricatti solution $(K_{1:N-1})$ from the original finite-horizon LQR problem, and apply the following policy:

$$u_i = \max(u_{min}, \min(-K_i x_i, u_{max})). \tag{9}$$

   Will we recover a solution to the new optimization problem? You should think about the theory as well as implementing it yourself and trying. First figure out what $\max(a, \min(u, b))$ does when $a < b$, making sure to use max.() and min.() if you're doing this in Julia.

   (a) yes
   (b) no

   Solution: No they are not. First, $\max(a, \min(u, b))$ clamps u to be within $[a, b]$. Clamping the $u's$ on the rollout will satisfy primal feasibility of the optimization problem, but is no longer optimal. This is because when we did our dynamic programming backwards recursion to get our $P's$ and $K's$, we get the recursion as the result of unconstrained minimizations of the action value function with respect to u, which is no longer valid when there are constraints on u.

Solution: For questions 5 and 6, you should check out the following resources:

- `https://www.stat.cmu.edu/~ryantibs/convexopt/` (lectures 1,2).
- `https://web.stanford.edu/~boyd/cvxbook/` (chapters 1-3)
- `https://dcp.stanford.edu/` (go through all the tabs)

After you feel comfortable with convex functions, keep in mind that if any function $g(x)$ is convex, then $g(Ax+b)$ is also convex.

5. Which of the following cost functions are convex in $x \in \mathbb{R}^N$? ($Q \succ 0$)

   (a) $c^T x$
   (b) $-c^T x$
   (c) $\sum_i x_i$
   (d) $\|x\|_1$
   (e) $\|x\|_2$
   (f) $\|x\|_2^2$
   (g) $\|Ax - b\|_1$
   (h) $x^T Q x$
   (i) $(Ax - b)^T Q (Ax - b)$

   Solution: They are all convex.

6. Which of the following constraints are convex in $x \in \mathbb{R}^N$? ($Q \succ 0$)

   (a) $Ax = b$
   (b) $Ax \leq b$
   (c) $\|x\|_2 \geq 3$
   (d) $\|Ax - b\|_1 \leq 3$
   (e) $\|Ax - b\|_2 \leq 3$
   (f) $\|Ax - b\|_2 = 3$
   (g) $\|x\|_2^2 \leq 3$
   (h) $x^T Q x \leq 3$
   (i) $x^T Q x = 3$
   (j) $x^T Q x \geq 3$
   (k) $x^T Q x + q^T x \leq 3$

   Solution: All but c, f, i, j are convex. Remember that any constraint $g(x) \leq 0$ is convex if g is convex.