1. Which of these ordinary differential equations is linear in the variables $x = [x_1, x_2]^T$ and $u \in \mathbb{R}$? (there are two)

   (a) $\dot{x} = \begin{bmatrix} .2 & .4 \\ .5 & -.1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$

   (b) $\dot{x} = \begin{bmatrix} x_1 & -x_2 \\ 3x_2 & -x_1 \end{bmatrix} x + \begin{bmatrix} u \\ 2 \end{bmatrix} u$

   (c) $\dot{x} = \begin{bmatrix} u_1 & -u_2 \\ 3u_2 & -u_1 \end{bmatrix} x + \begin{bmatrix} x \\ 2 \end{bmatrix} u$

   (d) $\dot{x} = \begin{bmatrix} \sin t & \cos t \\ t^2 & \sqrt{t} \end{bmatrix} x + \begin{bmatrix} e^t \\ 1 \end{bmatrix} u$

   (e) $\dot{x} = \begin{bmatrix} x_1 & \cos x_2 \\ t^2 & u_1 \end{bmatrix} x + \begin{bmatrix} 1 \\ x_2 \end{bmatrix} u$

   Solution: (a) and (d). In order for something in the form $\dot{x} = Ax + Bu$ to be linear, the matrices A and B must **not** be a function of $x$ or $u$. In (a) and (d), the matrices are just a function of $t$.

2. Which of these are accurate descriptions of a first-order Taylor series of a function $f(x)$ at a point $\bar{x}$ where $f(x) : \mathbb{R} \to \mathbb{R}$ (there are two)

   (a) $f(\bar{x}) \approx f(x) + \left[\frac{\partial f}{\partial x}\big|_x\right](x - \bar{x})$

   (b) $f(\bar{x} + \Delta x) \approx f(\bar{x}) + \left[\frac{\partial f}{\partial x}\big|_{\bar{x}}\right]\Delta x$

   (c) $f(x) \approx f(\bar{x}) + \left[\frac{\partial f}{\partial x}\big|_{\bar{x}}\right](x - \bar{x})$

   (d) $f(\bar{x} + \Delta x) \approx f(\bar{x}) + \left[\frac{\partial f}{\partial x}\big|_{\Delta x}\right]\Delta x$

   Solution: (b) and (c). These are the equivalent statements of the first order Taylor series. If we define $x = \bar{x} + \Delta x$ and substitute, they are equivalent.

3. If we have a continuous time dynamical system described by $\dot{x} = Ax$, which of the following statements about the eigenvalues of A tells us that the system is stable?

   (a) the eigenvalues of A all have negative real parts

   (b) the eigenvalues of A are all real

   (c) the eigenvalues of A are all imaginary

   (d) the eigenvalues of A all have a modulus less than 1

   Solution: (a). For continuous time systems, the eigenvalues must have negative real parts for stability. To get more insight into this, watch recitation 1 and see how the matrix exponential is used to solve these ODE's, and how the eigenvalues of A change the behavior of this solution.

4. If we have a discrete time dynamical system described by $x_{k+1} = Ax_k$, which of the following statements about the eigenvalues of A tells us that the system is stable?

   (a) the eigenvalues of A all have negative real parts

   (b) the eigenvalues of A are all real

   (c) the eigenvalues of A are all imaginary

   (d) the eigenvalues of A all have a modulus less than 1

Solution: (d). For discrete time systems, the modulus of the eigenvalues must be less than 1. To show this, imagine we take an eigendecomposition of A that's the following:

$$A = S\Lambda S^{-1} \tag{1}$$

where $S$ has the eigenvectors as columns, and $\Lambda$ is a diagonal matrix with the eigenvalues in the diagonals. Now let's take a look at a simulation where $x_1 = Ax_0$:

$$x_2 = A(Ax_0) \tag{2}$$

$$x_2 = S\Lambda S^{-1} S\Lambda S^{-1} x_0 \tag{3}$$

$$x_2 = S\Lambda^2 S^{-1} x_0 \tag{4}$$

We see from this that $x_k = S\Lambda^k S^{-1} x_0$, where $\Lambda^k$ is just taking each eigenvalue in the diagonal to the power of $k$. If the modulus of each eigenvalue is less than 1, it will go to zero as $\lambda^\infty$, and if the modulus is greater than 1, it will go to $\infty$.