1. We have the following optimization problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \frac{1}{2}x^T P x + p^T x \\ \text{subject to} & Cx = d \end{array} \tag{1}$$

where $P \succ 0$ (Q is positive definite), are the KKT conditions linear in the primal and dual variables?

(a) yes

(b) no

Solution: Yes, the Lagrangian is

$$\mathcal{L}(x, \lambda) = \frac{1}{2}x^T P x + p^T x + \lambda^T (Cx - d), \tag{2}$$

Which gives us our stationarity condition, and primal feasibility:

$$\nabla_x \mathcal{L} = Px + p + C^T x = 0 \tag{3}$$

$$Cx - d = 0 \tag{4}$$

Which are linear in x and λ.

2. For the optimization problem shown in (1), how many Newton steps would it take to solve for the optimal primal and dual variables?

(a) 1

(b) 2

(c) can't tell without more information

Solution: 1, since the above KKT conditions are linear in x and λ, we can solve for them in a linear system:

$$\begin{bmatrix} P & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -p \\ d \end{bmatrix} \tag{5}$$

Since this function is linear, that means that any first order Taylor series is exact, so 1 Newton step from any initial starting point will approximate the KKT conditions exactly, meaning only one Newton step is needed to find the solution.

3. For **any** square matrix $G \in \mathbb{R}^{N \times N}$ (not guaranteed to be symmetric), is $0.5(G + G^T)$ symmetric?

(a) yes

(b) no

Solution: Yes, if we take the transpose of this, we see that the expression is unchanged: $0.5(G + G^T)^T = 0.5 * (G^T + G)$.

4. For a **symmetric** matrix V, does $V = 0.5(V + V^T)$?

(a) yes

(b) no

Solution: Yes, if V is symmetric, that means $V = V^T$. Using this, we see that $0.5(V+V^T) = 0.5(V+V) = V$.

5. For **any** square matrix $G \in \mathbb{R}^{N \times N}$ (not guaranteed to be symmetric), does $x^T G x = x^T \left[0.5(G + G^T)\right] x$?

(a) yes

(b) no

Solution: Yes, to show this we are going to remember that the transpose of any scalar is equal to itself. This means that $(x^T G x)^T = x^T G x$, so $(x^T G x)^T = x^T G^T x = x^T G x$. Using this, let's work it out:

$$\begin{aligned} x^T \left[0.5(G + G^T)\right] &= 0.5x^T G x + 0.5x^T G^T x && (6) \\ &= 0.5x^T G x + 0.5x^T G x && (7) \\ &= x^T G x && (8) \end{aligned}$$

It is important to note that even when $G \neq 0.5(G + G^T)$, $x^T G x = x^T \left[0.5(G + G^T)\right] x$.

6. Can any quadratic form $x^T G x$ be equivalently represented with $x^T V x$ where G **is not** a symmetric matrix, but V **is** a symmetric matrix?

   (a) yes

   (b) no

   Solution: Yes, as we just showed in 5, $x^T G x = x^T \left[0.5(G + G^T)\right] x$ for any matrix G. This means that for this problem, $V = 0.5(G + G^T)$ which is symmetric (see q3).

7. A symmetric matrix R is positive semi-definite, with one zero eigenvalue (and a corresponding null space with dimension one). If our cost function is $J(u) = u^T R u$, is there a non-zero vector u that has a cost of 0?

   (a) yes

   (b) no

   Solution: Yes, if u is in the null space of R, then $Ru = 0$, and $J(u) = u^T 0 = 0$.