OPTIMISATION OF THE METROPOLIS ALGORITHM FOR A POTTS MODEL

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ABSTRACT

In this project, a fast implementation of the metropolis algorithm for the 4-states Potts model on a 3D cubic lattice was written. The algorithm was split into four independent modules. For each of these modules, multiple optimisations were applied, resulting in up to 5x speedup. On top of the modules a install-time search routine was implemented to auto tune the module-composition for different sizes. The measurements were performed on two different processors (Intel Core i7 Haswell / Intel Core 2 Wolfdale).

1. INTRODUCTION

1.1. Motivation

By studying the Potts model, questions about ferromagnetic as well as other phenomena of a solid state can be investigated. The Potts model is a generalisation of the Ising model. It is not just used in statistical physics, but also in electrical engineering (signal processing) and in biology (neural networks). Our goal is to investigate numerous optimisation techniques for the 4-state 3D Potts model and provide a framework with the optimal implementation.

1.2. Related work

The Potts model is a very well - studied problem. It was defined by Renfrey Potts in 1951 and thus named Potts model, even though equivalent models were studied already earlier (Ashkin - Teller, 1943). Today (06.2014) google scholar lists 185'000 entries for the Potts model, confirming the popularity and importance of this model. Concerning implementations, we did not find any work that has a similar approach as this project. Many implementations are specialized for certain cases (e.g. 2D) or modified to solve a different problem (e.g. neural networks).

2. BACKGROUND: POTTS MODEL AND THE METROPOLIS ALGORITHM

This section briefly describes the Potts model and introduces the Metropolis Algorithm, which can be used to simulate a Potts system.

2.1. Potts Model

The n-state Potts model describes an ensemble of spins on a lattice, where n different states are available for each spin. These spins interact only with their nearest neighbours. The energy of the entire system is given by:

$$E = J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$$

where J is an interaction constant, σ_i^z the z-component of the spin at lattice site i and $\langle i, j \rangle$ the set of all nearest-neighbour-pairs. If $J < 0$ the spins tend to order ferromagnetically (i.e. they are preferably aligned). The 2-state Potts model on a 2D square lattice is identical to the famous Ising model. For this work the 4-states Potts model on an 3D cubic lattice with periodic boundary conditions was chosen.
The mean properties of an observable A (e.g. energy or magnetisation) can be measured in the following way:

$$\langle A \rangle_\Omega = \sum_{\omega \in \Omega} A(\omega) \cdot p(\omega)$$

where Ω denotes the configuration space and

$$p(\omega) = \exp\left(-\frac{E(\omega)}{k_b T}\right)$$

the canonical probability to find a certain configuration ω in Ω. Due to the large configuration space (increases exponentially with the number of spins) this sum cannot be calculated directly in a fast manner. For this reason, one needs the metropolis algorithm.

2.2. Metropolis Algorithm

The metropolis algorithm is a Markov chain Monte Carlo method [1] that allows the calculation of expectation values without having to generate all configurations ω. After generating a starting state ω_0 one does many local spin updates to change the system and generate the next state ω_1. One has make sure that successive states are not correlated, otherwise the error estimation is too small.[1] For the Potts model, the algorithm consists of the following steps:

1. Select a random spin σ_i^z from the current configuration ω_s and change it randomly by $\Delta\sigma_i^z = \pm 1$ to obtain the configuration ω_s^*
2. Calculate the energy difference ΔE_i between the two states and $\frac{p(\omega_s^*)}{p(\omega_s)} = \exp\left(-\frac{\Delta E_i}{k_b T}\right)$
3. Accept the new configuration with the probability $p_{accept} = min(1, \frac{p(\omega_s *)}{p(\omega_s)})$
4. Repeat step 1 to 3 m times until ω_s and ω_{s+m} are (somewhat) decorrelated
5. Measure functions of interest on the state ω_{s+m}
6. If the error in the measurements is too large, repeat from step 1

3. MODULAR OPTIMISATIONS AND AUTO-TUNING

The following section describes the implementation of the Potts model, specifically the splitting into four modules, optimisations performed on those modules and finally autotuning to select the best-performing modules.

3.1. Modular Structure

Splitting the implementation into four modules (with their respective interfaces) allowed for an efficient workflow. It became possible to optimize parts of the implementation completely independently, which is a great benefit when working in a team. The four modules are as follows:

- **`SIM`:** Contains the high-level aspects of the metropolis algorithm: Computing probabilities, accepting or rejecting an update, and also calls to the other modules (i.e. requesting random numbers or getting / setting spins).
- **`GRID`:** Takes care of the boundary conditions, i.e. it computes the nearest neighbours of a selected spin and interfaces to the `MATRIX` module.
- **`MATRIX`:** Contains the explicit data format of the system (e.g. `std::vector` or `C array`).
- **`RNG`:** Provides the random numbers. All implementations use a mersenne twister engine.

In the following parts, the different optimisation techniques used for each module will briefly be mentioned.

3.2. `SIM` optimisations

- **Probability Precomputation:** Since there is a finite number of possible energy differences ΔE, the probabilities $p_i = \exp\left(-\frac{\Delta E_i}{k_B T}\right)$ of accepting a spin change are precomputed. This may increase memory usage, but gets rid of any floating point computation.
- **Interleaving:** The calls to the random number generator (picking the spin location) and the actual computation is interleaved amongst two steps. This allows prefetching, i.e. the data for the next step can be loaded during the current step.
- **Explicit Prefetching:** Instead of leaving it up to the Compiler and/or Hardware, the prefetching is done explicitly.

3.3. `GRID` optimisations

A **lookup table** (one array for each space direction) was used to get the nearest neighbour indices efficiently. This method is useful mainly because it replaces the potentially expensive check for the boundary condition.

3.4. `MATRIX` optimisations

The main concern with the `MATRIX` module storing the system in a memory - efficient way. Also, since the access to spins is random, the `MATRIX` is the only way we can hope to achieve locality (between one position and its nearest neighbours). The following optimisations have been done:

- **Compression:** Since each spin can only be in one of four states, it can be described using only 2 `bits`. This can be used to shrink the system's size in memory by a factor of 4 (compared to using one `byte` per spin).
- **Z - order:** Bit interleaving in the three indices is used to increase locality amongst a spin and its nearest neighbours.

3.5. `RNG` optimisations

- **Economic use:** Since many of the random numbers used only are a few `bits` long, the `std::mt19937` implementation (which uses 32`bit` for each random number at least) can be optimized greatly by using all the bits in each generated random number (with some overhead for splitting up the random numbers).
- **MKL:** The underlying Mersenne twister engine was exchanged for the MKL implementation [3] (whilst

[1] A common method is to keep some correlation between successive states, because complete decorrelation is difficult to achieve without wasting steps. This can, however, be corrected in a post-processing step (see [2] for possible methods).

still using the method of economic use described before).

3.6. Autotuning

Many of the optimisations seen in sections 3.2 - 3.5 perform well only under **certain conditions** (i.e. at certain system sizes and temperatures). For example, using Z - order might be great at large sizes, but the overhead for computing the index is too large for smaller ones.
The modular structure of the code provides the ideal grounds for tackling this problem using an **auto-tuning** approach:
For a fixed number of sizes, the optimized combination of modules is searched either by doing a full sweep over all the combinations, or - much faster - by **iteratively exchanging** one module whilst keeping the others fixed.
That means one starts with a certain combination of modules and then tries all possible combinations by only exchanging the first one. The fastest version amongst those is kept for the next step, in which the second module is being exchanged, and so on.
This process is repeated until convergence is reached, that is until a full iteration (trying to exchange each modules) does not yield any faster combinations. However, this does **not guarantee** that the process has found the **global minimum**, since it might be stuck at a local one.
This installation routine creates a header file containing template specialisations describing the best combination for each measured size. Listing 1 shows a snippet of this header.

Listing 1: example for a template specialisation

```
template<>
struct opt<150> {
   template<int S>
   using impl = greschd_v3_sim::impl<S, S,
       S, addon::mkl_mt_rng, msk_v1_pbc,
       msk_v2_dynamic_zip>;
};
```

For an arbitrary size N, the implementation will then choose the appropriate specialisation (that is, the one with size closest to N) by use of type traits, as indicated in Listing 2 (`sizes` is a `constexpr` array containing all measured sizes; for brevity some edge cases are not shown).

Listing 2: size_helper (partial), part of the type traits

```
template<int S, int N, bool B>
struct size_helper {
   static constexpr int size =
       size_helper<S, N - 1, S >= (sizes[N]
       + sizes[N - 1])/ 2 >::size;
};
template<int S, int N>
struct size_helper<S, N, true> {
   static constexpr int size = sizes[N +
       1];
};
```

Making an installation that also varies according to temperature was also discussed, but dismissed because the temperature of the system may be varied during one measurement cycle. This could lead to a large overhead for switching between versions.

4. EXPERIMENTAL RESULTS

This section describes the different measurements that were performed and their results. After introducing the platforms that were used and some notation, the impact on runtime for each optimisation is being studied. Finally, the results of screening all module combinations and the performance of the autotuned code is shown.

4.1. Experimental setup

The two platforms used for measurements are shown in Table 1. They will be referenced by their architecture codename (Wolfdale / Haswell).

	Haswell	**Wolfdale**
Architecture	Intel Core i7	Intel Core 2
Frequency	2.4 GHz	2.4 GHz
OS	Ubuntu 13.10	Ubuntu 14.04
Compiler	gcc 4.8.1	gcc 4.8.1
Flags	-std=c++11 -DNDEBUG -O3 -march=core-avx2	-std=c++11 -DNDEBUG -O3 -march=core2

Table 1: Platforms used for the measurements

For all measurements, the **rdtsc** command was used to determine the runtime. In the roofline measurements, the memory traffic was measured with **perfplot** [4]. Additionally, VTune, perf and gprof were used for profiling during development.
Measurements showing the **splitting** of runtime amongst the different **modules** is achieved by starting / stopping the time measurement between different tasks in the algorithm. This may to some extent influence the total runtime by inhibiting out-of-order computation.
In the following sections, combinations of modules where one module is being exchanged will be discussed. For those measurements, one can look at both decrease in total runtime and decrease in **runtime caused** by that **specific module**. However, the former is always influenced by the choice

of all the other modules, which makes the latter a better measure for the effectivity of an optimisation.
Runtime of `GRID` and `MATRIX` is measured together, but split into accessing selected spin itself (single spin, s.s.) and accessing its nearest neighbours (n.n.).
The exact configuration of modules used in a measurement is noted using a **four - digit index** corresponding to (SIM, GRID, MATRIX, RNG) as listed in Table 2 (with optimisations as described in Sec. 3).
As an example, (0132) would correspond to a baseline `SIM` with boundary lookup table in `GRID`, Z-order `MATRIX` and economic use & MKL `RNG`.

	digit	optimisations
SIM:		
	0	baseline
	1	probability precomputation
	2	1 & interleaving
	3	2 & explicit prefetching
GRID:		
	0	baseline
	1	boundary lookup table
MATRIX:		
	0	baseline (`std::vector`)
	1	C array
	2	compressed
	3	Z - order
	4	compressed & Z - order
RNG:		
	0	baseline (STL mt)
	1	economic use
	2	1 & MKL engine

Table 2: List of Modules

4.2. Results: SIM optimisation

Figure 1 shows the runtime partition of the single spin update. All modules except `SIM` are the same in both configurations. The baseline-module computes the **Boltzmann factors** each time, while the second module **precomputes** these probabilities. The amount of cycles needed for the SIM part is reduced on both systems. On Haswell, this shows an overall speedup of 2x for a system with side length $N = 20$. Considering only the part of the runtime caused by the `SIM` module, however, the speedup is **roughly 6x**.

4.3. Results: GRID optimisation

Storing the boundary condition in a lookup table **doesn't show** a **very significant speedup** in terms of total runtime. However, nearest neighbour access - the part that is influenced most by the improvement - is sped up by roughly 2x for sidelength $N = 20$ (see Fig. 2).

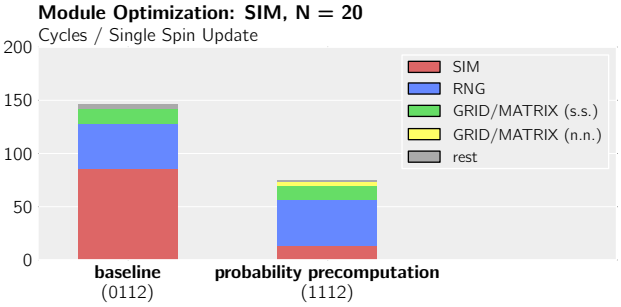

Fig. 1: Speedup by probability precomputation, on Haswell

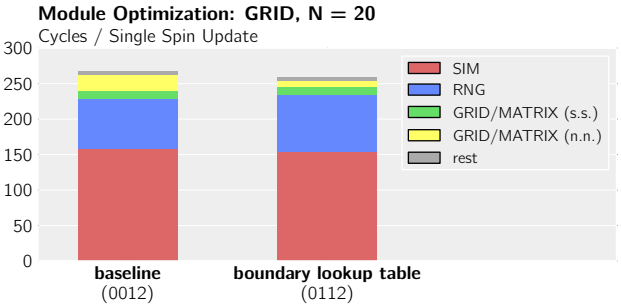

Fig. 2: Influence on runtime by using a boundary lookup table (`GRID` optimisation), on Wolfdale

4.4. Results: MATRIX optimisation

The impact compression and Z-order have on the runtime is clearly **size - dependent**: For a small system ($N = 20, 8000$ spins, on Wolfdale), the influence of compression is negligible and the overhead for using Z - order is larger than the gain (see Fig. 3).
At larger sizes, as the runtime for accessing the spins increases, compression and Z-order both become **significant improvements** (see Fig. 4). This improvement is most noticeable when the system starts running **out of last level cache**. At those sizes, the compressed version might still fit into LLC and hence perform much better.
This effect can be observed directly in Fig. 5, comparing C array and compressed `MATRIX` modules. Whilst the two modules are along the same line in the roofline plot, the system size at which the performance starts dropping is different (C array: $N \approx 100$, compressed: $N \approx 200$).
On Haswell, it can be observed that even for large sizes ($N = 1000$, 10^9 spins), accessing the nearest neighbours takes next to no time (see Fig. 6). This might be due to the **prefetching** in Haswell, which could load the neighbours into Cache as soon as the position of the single spin is known. However, this would still **cost** memory bandwidth, but go **undetected** by this measuring scheme since it happens out-of-order (see Sec. 4.1).

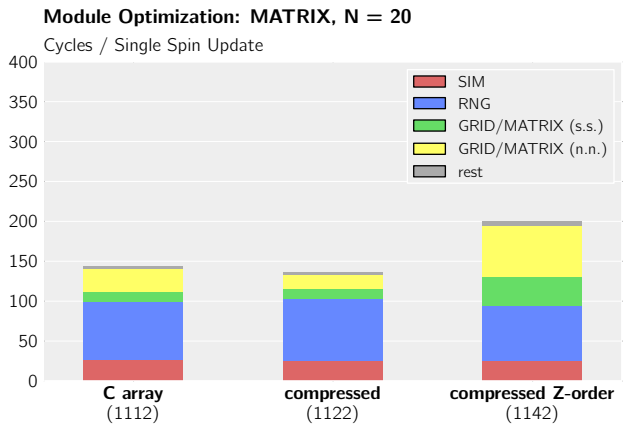

Fig. 3: Different `MATRIX` optimisations for $N = 20$, on Wolfdale

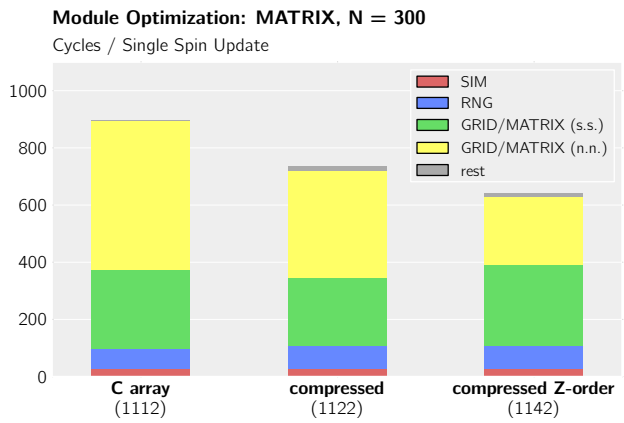

Fig. 4: Different `MATRIX` optimisations for $N = 300$, on Wolfdale

4.5. Results: RNG optimisation

The impact of economic use of random numbers and the MKL Mersenne - twister engine (as explained in Sec. 3.5) are shown in Fig. 7.

It can be seen that the **main speedup** comes from **economic use**, whilst changing to the MKL engine has virtually no effect on runtime.

For a system of size $N = 20$ (8000 spins), these optimisations amount to an overall speedup of roughly $2 - 2.5$x (on Haswell). Considering only the part of total runtime caused by the `RNG` module, the speedup is **roughly 6x**, and hence similar to the speedup seen for `SIM`.

4.6. Results: Autotuning

A full screen of all the runtimes of all the module combinations has been done to determine which perform best at various sizes and temperatures.

Fig. 8 and 9 show which module combination performed

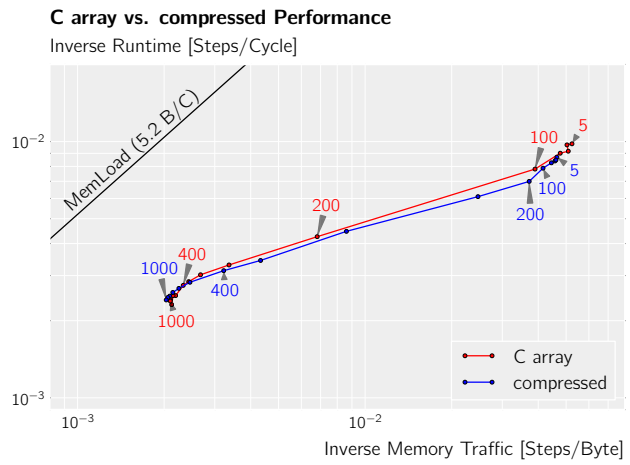

Fig. 5: Performance of C array and compressed `MATRIX` modules, for various sizes, on Haswell

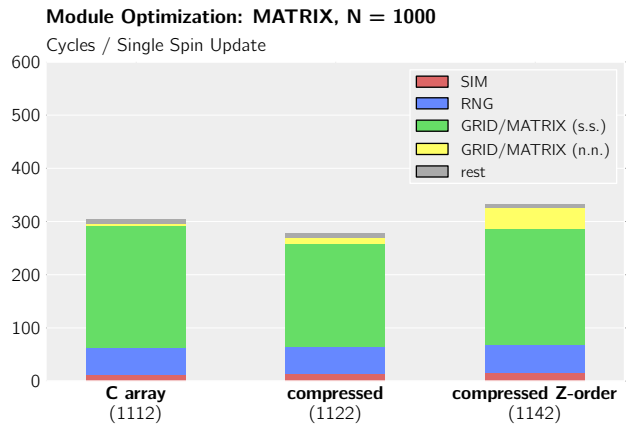

Fig. 6: Different `MATRIX` optimisations for $N = 1000$, on Haswell

best, using the indexing introduced in Sec. 4.1. A few points are worth noting:

Firstly, the module combinations used on the two platforms are sometimes quite different from each other and do not have too much of a pattern. This indicates that it might be quite **difficult** to actually **predict** which module works best.

Secondly, it has to be noted that for some sizes, the **difference** between fastest and second - fastest module combinations is very **small** - sometimes even within measurement error. So it might be possible to construct a neater arrangement of modules by allowing those second - best combinations.

Thirdly, whilst the temperature does have some impact on which module is used, the **dominant factor** seems to be the **size**. This justifies not optimising w.r.t. temperature in the installation procedure (see Sec. 3.6).

Lastly, the region $N \in [150, 400]$ (on Wolfdale) / $N \in$

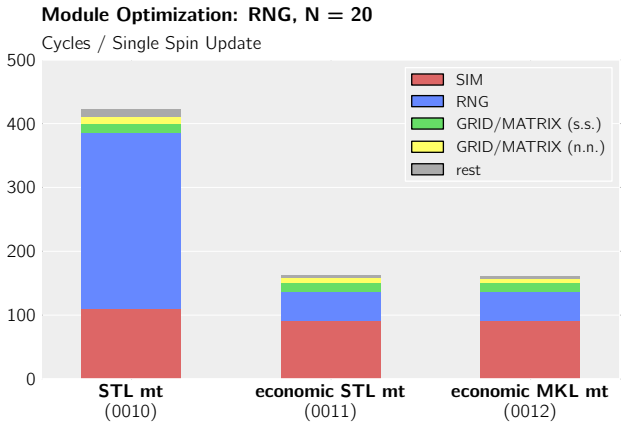

Fig. 7: Runtime using different RNG modules, on Haswell

$[200, 400]$ (on Haswell) is of particular interest. This region corresponds to the system size **running out of LLC**. Accordingly, the MATRIX module used in this region is the compressed version (third digit of the index $= 2$).

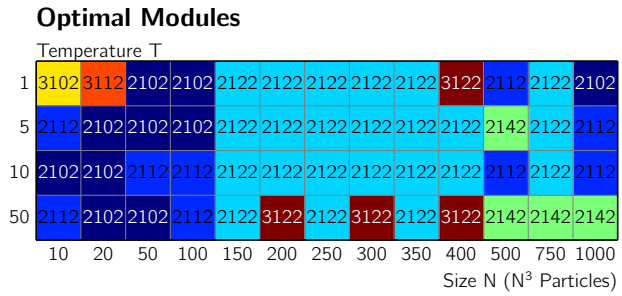

Fig. 8: Best - performing modules (marked by index) for different sizes and temperatures, on Wolfdale

The performance of the final version, using the **fast auto-tuning procedure** described in Sec. 3.6, is shown in Fig. 10. Compared to the baseline implementation, the overall **speedup** ranges between $\mathbf{2-5x}$, depending on system size. The speedup tends to be **bigger** for **smaller sizes**, where the vastly improved SIM and RNG modules takes up a significant part of the runtime.

5. CONCLUSIONS

The modular interface and auto-tuning routine help the project to remain expandable. It is possible to write additional modules and run the installation again with little effort. The most successful optimisation techniques were precompuation, economic use of random bits and data compression for both platforms. There are a few open questions worth investigating: What is the runtime penalty for the modular interface? How exactly does Haswell manage to reduce the lookup runtime of the nearest neighbours even at large sizes

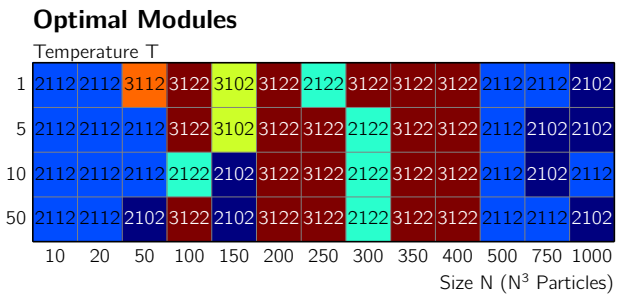

Fig. 9: Best - performing modules (marked by index) for different sizes and temperatures, on Haswell

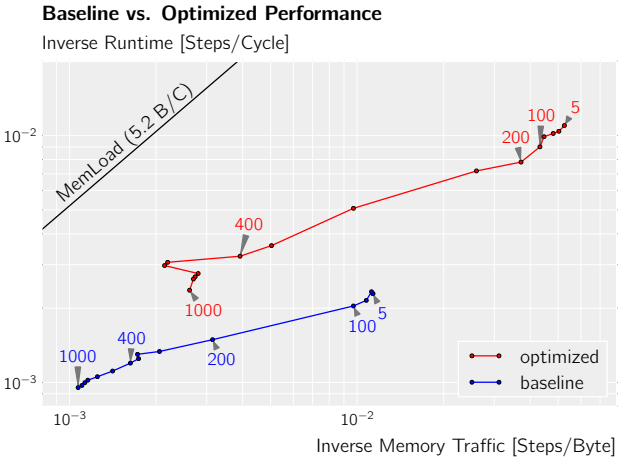

Fig. 10: Final (installed) version of the code vs. baseline implementation, on Haswell

(see Fig. 6)? Where could one use SSE/AVX for the project (we used SSE for Z-order index computation, but measured no speedup there)? The project was a valuable experience in optimisation and teamwork.

6. REFERENCES

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